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Asymptotic Approximations of Gravity Waves in Water

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Master of Science in Physics and Mathematics
Submission date: June 2009
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Problem Description

A study of of gravitational waves in water. Deriving and studying various PDE's describing these phenomena.

Assignment given: 14. January 2009
Supervisor: Helge Holden, MATH

Preface

With this thesis, under the course code TMA4900, I complete my five year of studies of industrial mathematics at the Norwegian University of Science and Technology (NTNU). Writing this paper has been fun and revealing, but also challenging at times. It has been very rewarding to be able to apply the theoretical knowledge I have accumulated over the past five years to model a real life phenomena, and then to see how the mathematics can describe a physical system so perfectly. Perhaps the most interesting has been to follow the paths of the minds of great mathematicians and to (re)discover the hidden tricks and reasonings applied throughout every paper.

The topic of water wave theory is, however, extremly large and it has therefore been difficult to find a place to stop writing. The more I read and learned, the more interesting the theory became and the more I wanted to include in the thesis. I do feel that much has yet to be told, but with the time and resources available one cannot summarise the last hundred and fifty years of work in just one thesis. I therefore chose to focus on something I liked the most; the derivation of equations describing water waves belonging to a special class, the completely integrable equations.

Acknowledgements

I would like to thank my supervisor Professor Helge Holden for all the help given and for keeping me on the right path to the insight of water wave theory. Thank you, Marja, Herbert and Johan, for your comments and proofreads. And last, but absolutely not the least, Karen should be thanked; for your support and patience. Thank you.

Arne Kristian Jansen
June 12, 2009
Trondheim

Abstract

The governing equations for waves propagating in water are derived by use of conservation laws. The equations are then cast onto dimensionless form and two important parameters are obtained. Approximations by use of asymptotic expansions in one or both of the parameters are then applied on the governing equations and we show that several different completely integrable equations, with different scaling transformations and at different order of approximations, can be derived. More precisely, the Korteweg-de Vries, Kadomtsev-Petviashvili and Boussinesq are obtained at first order, while the Camassa-Holm, Degasperis-Procesi, nonlinear Schrödinger and the Davey-Stewartson equations are obtained at second order. We discuss shortly some of the properties for each of the obtained equations.

Contents

Chapter 1

Introduction

Little by little, one travels far.

J.R.R. Tolkien

The topic of mathematical modelling of water waves is immense; Indeed, each of the equations we derive in this thesis would be worth at least one thesis each and several (for example the KdV equation or the NLS equation) has hundreds of papers and whole books dedicated to themselves. It is thus quite clear we cannot embrace every bit of this topic and have to choose what part to focus on. We have chosen to focus on the derivation, by use of asymptotic expansions, of some of the most studied (water wave) equations, the completely integrable equations. These will then be be derived in detail, directly from the basic governing equations for water waves, more precisely from the governing equations for inviscid incompressible fluids. For each equation we also shortly describe some of the most prominent properties and solutions.

The thesis is mainly divided into three parts. In the next chapter we will describe some of the preliminary theory needed to understand the later work, more precisely the theory of asymptotic approximations will be described and a very short introduction to solitons will be given.

Next we derive the governing equations of water waves, by use of some basic conservation laws and state the boundary conditions we impose. After having obtained a set of equations which can be used to describe the wave propagation, we do a dimension analysis and rewrite the set in dimensionless variables. This also helps us find two very important parameters, which will be used when later making the approximations. These chapters might seem a bit basic and very thourough, but as this was our first encounter with the theory of water waves, we needed to ensure a firm basic.

Having found the governing equations and recast them on non-dimensional form we use the theory of asymptotic expansions to derive different completely integrable equations. These equations do all describe wave propagation in water, but all in either different regions in space and time or under different assumptions (e.g. amplitude modulation or not). Thus even though the equations are derived from and approximate the same basic set, the scaling and the order of approximation determines which equation will be obtained.

The notation will be defined as they occur for the first time throughout the thesis and will also be clear of the context. However, we will here mention some remarks on the notation we have adopted. We mainly use the standard Leibniz notation for differentiation, it will be more convenient to use the subscript notation in running text, i.e. $u_x = \partial u/\partial x$ (and similar for ordinary differentiation) and also note that vectors are expressed in bold face (e.g. the velocity vector is expressed as $\mathbf{u}$). We will work in a standard right-handed rectangular Cartesian frame of reference. The z-axis is taken upwards, with $z = 0$ as the free surface at rest and the displacement from rest is denoted by $\eta(x, y, t)$, see figure 1.1. Thus x and y will be the horizontal axes and in case of one-dimensional waves, we will use the x-axis as the wave propagation direction. The depth is given by $h(x, y, t)$, so that the bottom is at $z = -h(x, y, t)$. Note that for the sake of generality a time dependency is included in the depth function. This is only necessary if the bottom is moving, i.e. underwater land slides or earth quakes, which is beyond the scope of this thesis and the time dependency will therefore later be omitted. For simplicity, the bottom will be taken to be horizontal, which means the depth will be constant, denoted by $h = h_0$ (or we could equally have used a mean depth instead).

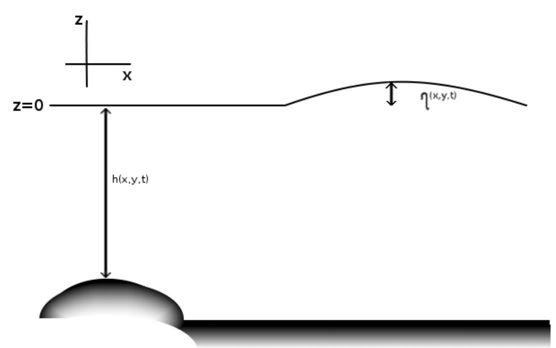

Figure 1.1: Schematic view of coordinate system. The y-direction is perpendicular to the x- and z-axis.

Chapter 2

Preliminary Theory

> It is the theory that decides
> what can be observed.
>
> Albert Einstein

In this chapter we first make a short introduction to asymptotic analysis, which will be the main tool when later making the approximations on the governing equations. Next we state some of the properties of solitons, which we use when later deriving soliton solutions.

2.1 Asymptotic Analysis

In this section we describe the basic concepts of asymptotic analysis. The whole topic of asymptotic analysis is too large and extensive to be completely described here and we therefore refer to the book of Miller [56] or the book of Zeytounian [70] for details and a more comprehensive handling of the subject.

Asymptotic analysis is used to describe the behaviour of a function in a limiting situation. The function might for example be constant in the limit, e.g. the function $f(x) = 1/(2 + x^2)$, for $x \to 0$ would in the limit behave as the constant $1/2$. Or, it might for example behave similar to another function, e.g. $f(x) = \sin x$ for $x \to 0$ would behave like the linear function $g(x) = x$ in the limit.

Often, at least in this thesis, the limit is taken on a parameter emerging from the problem at hand. For a function $f(x)$ which depends on a parameter ϵ, we write $f(x; \epsilon)$. To have a consistent way of determine when and how the function is bounded by another function we will use the common Landau notation (also known as the big- and small-"oh" notation) with the symbols $O(\cdot)$ and $o(\cdot)$.

We define $O(\cdot)$ in the following way: Let $f(x)$ and $g(x)$ be two (complex) functions defined on some set D and let x_0 be a (complex) number in the closure of D. We then write

$$f(x) = O(g(x)), \quad x \to x_0 \quad \text{from} \quad D$$

if there exists a number $\delta > 0$ and $M > 0$ such that

$$|f(x)| \leq M|g(x)|, \qquad x \in D \quad \text{where} \quad 0 < |x - x_0| < \delta.$$

Note that in this thesis x_0 usually is taken to be zero.

$o(\cdot)$ is defined as follows: Let $f(x)$, $g(x)$ and x_0 be defined as earlier. We then write

$$f(x;\epsilon) = o(g(x;\epsilon)) \quad x \to x_0 \quad \text{from} \quad D$$

if we for any given $\epsilon > 0$, however small, can find a $\delta(\epsilon) > 0$ such that

$$|f(x)| \leq \epsilon|g(x)|, \qquad x \in D \quad \text{where} \quad 0 < |x - x_0| < \delta(\epsilon).$$

The function $g(x)$ (or in our case $g(x;\epsilon)$) is often called the gauge function. In this thesis the gauge function will be in terms of the powers of the limiting parameter, i.e. $1, \epsilon, \epsilon^2, \epsilon^3$ and so on. Note that the limit point x_0 does not necessarily have to finite, it might take on any complex number (as long it is inside the domain of f and g) or even infinity. However, in the latter case some additional definitions need to be set. In this thesis the limit will always be 0 and for details on how to handle other limits we refer to the book of Miller [56].

If we have a sequence of functions $\{f^{(n)}(x)\}_{n=0}^{\infty}$ and for each $n > m$ we have $f^{(n)}(x) = o\left(f^{(m)}(x)\right)$, then $\{f^{(n)}(x)\}_{n=0}^{\infty}$ is called an *asymptotic sequence*. Note that $f^{(n)}$ means the n'th function in the sequence and not the n'th power.

The sum $\sum_{n=0}^{N} a^{(n)} f^{(n)}(x)$ is an asymptotic series (expansion) of $f(x)$ if

$$f(x) - \sum_{n=0}^{N} a^{(n)} f^{(n)}(x) = o\left(f^{(N)}(x)\right) \qquad \text{for} \quad x \to x_0,$$

where N might be infinity. We could equally have written

$$f(x) - \sum_{n=0}^{N} a^{(n)} f^{(n)}(x) = O\left(f^{(N+1)}(x)\right) \qquad \text{for} \quad x \to x_0.$$

A more compact way of writing the asymptotic expansion is (and letting N be infinity)

$$f(x) \sim \sum_{n=0}^{\infty} a^{(n)} f^{(n)}(x) \qquad \text{for} \quad x \to x_0.$$

Note that we have not imposed any convergence criterion on the series, and they might therefore converge extremely slowly or even be divergent (and hence the notation $\sim$ and not $=$). As mentioned, we have in this thesis one or more parameters which we take the limit on and the asymptotic approximation will thus be written as

$$f(x;\epsilon) \sim \sum_{n=0}^{N} f^{(n)}(x;\epsilon) \qquad \text{for} \quad \epsilon \to 0, \tag{2.1}$$

where x is fixed (or $O(1)$). In many cases, and also in this thesis, $f_n(x;\epsilon)$ can be expressed in a separable and thus simpler form as

$$f_n(x;\epsilon) = \epsilon^n b^{(n)}(x).$$

If (2.1) holds for every x (in the domain of $f(x;\epsilon)$) the expansion is said to be *uniformly valid*. Otherwise the expansion is said to be *non-uniform*. Unfortunately, in the case of water waves, the latter will be mostly true. That is, the asymptotic expansion will only be valid in a certain region of space and time (determined by the scaling) and will break down outside these regions. This means that any equations which are obtained by asymptotic approximations cannot be used uniformly over the whole domain but only in their respective regions (often long time behaviour in far field situations).

A very important note is that the asymptotic representation is by no means unique. That is, different functions can have the same asymptotic representation. For example if

$$f(x;\epsilon) \sim \sum_{n=0}^{\infty} f^{(n)}(x;\epsilon) \qquad \text{for} \quad \epsilon \to 0,$$

and $g(x;\epsilon) = o\left(f^{(n)}(x;\epsilon)\right)$ for every n, then the function $h(x;\epsilon) = f(x;\epsilon) + g(x;\epsilon)$ can also be represented by

$$h(x;\epsilon) \sim \sum_{n=0}^{\infty} f^{(n)}(x;\epsilon) \qquad \text{for} \quad \epsilon \to 0.$$

A consequence is that one can begin from quite different starting points and end up with the same result. A simple example is the sine and tangent function, which both have the same asymptotic representation at leading order. That is, $\sin\epsilon \sim \epsilon$ and $\tan\epsilon \sim \epsilon$ for $\epsilon \to 0$ and ϵ can thus be used to describe *both* these equations in the limit. Thus, the equations we derive in the context of water waves (which will be equations asymptotic equivalent to the basic governing equations) may equally be derived from a quite different physical problem.

A final note; The tool of asymptotic analysis can be used to solve differential equations, to approximate solutions of algebraic and transcendent equations or to approximate integrals. See [50] or [56] for details on this.

2.2 Solitons

All the equations we derive (except for the linear case) emit "soliton" solutions. This type of solutions was coined by Zabusky and Kruskal in 1968 [68], after a numerical experiment where solitary waves with particle-like properties were observed. That is, they observed two solitary waves interacting (colliding) nonlineary with each other. After the collision the original waveforms again emerged and retained their shapes and identity and continued without loss of speed. (A similar type of wave, this time a single wave and thus no collision, was observed in a canal over hundred years earlier, more precisely in 1834 by John Scott Russel. This is the actual beginning of nonlinear water wave theory and leads to the Boussinesq equations, the KdV equation and up to the experiments of Zabusky and Kruskal).

The observations and the theory developed to explain this behaviour resulted in a new direction in mathematics, called soliton theory. This thesis is by no means meant to be a paper on solitons and we therefore refer to the book of Drazin and Johnson [25] for a more complete handling of the subject.

An exact definition of a soliton is difficult to find and might differ from which area in mathematics (or physics) we are working on. Drazin and Johnson [25] ascribe the following three attributes to solitons

1. They are of permanent form;
2. They are localised within a region;
3. They can interact with other solitons and emerge from the collision unchanged (except perhaps for a phase shift),

which we will use when we later derive soliton solutions for some of the equations.

We especially use the first and second property. That is, for a solution $u(x,t)$ we will assume the permanent form by introducing the transformation $\xi = x - ct$, which results in a wave travelling in the positive x-direction with speed c.

The second requirement is that any solution $u(x,t)$ and its derivatives vanishes for $|x| \to \infty$. This will then give us additional boundary conditions to determine variables (typically constants of integration) that might emerge.

The third property will not be pursued in this paper.

Note that for a (soliton) wave to have a permanent form and to be able to collide and again regain its identity, some quantities have to be preserved. This is one of the key observations in soliton theory and much work has been made to find the conserved quantities and explain this behaviour. The study of this is, however, beyond the scope of this thesis.

Chapter 3

The governing equations

> The campaign is over. It's time for the work of governing to begin.
>
> Tom Daschle

In this chapter we derive the governing equations for water waves by using the principles of conservation of mass and conservation of momentum. To make a complete description of the moving fluid we will use the velocity $\mathbf{u} = \{u, v, w\}$, the mass density ρ and the fluid pressure p, all generally functions of time and space, while assuming that any thermodynamical effects can be neglected.

We first derive the governing equations in terms of the velocity directly, with the following assumptions; All functions in the problem are continous, any thermodynamical effects can be neglected, any viscosity effects can be neglected, the fluid is incompressible and no fluid is created or destroyed (no source or sink). Furthermore, we impose two boundary conditions at the free surface; a dynamical condition and a kinematic condtion. At the bottom a kinematic condition is imposed. With these assumptions and boundary conditions we obtain a complete set of equations describing, in terms of the velocity, pressure and the free surface variable η, wave propagation in water.

After having obtained the governing equations in this case, we make an additional assumption of irrotational flow. The velocity potential ϕ then exists and we also derive the governing equations in terms of this, with imposing the same type of boundary conditions. It turns out that we in this case obtain the Laplace equation in ϕ and that the dynamic free surface condition can be altered.

The derivation of the basic governing equations of this section can be found in any number of books about water waves. We follow the work of Johnson [41], Stoker [62] and Dingemans [23]. For the governing equations in terms of the velocity potential we also refer to Whitham [67], Dingemans [23], Kinsman [47], Mei *et al.* [55].

It should be noted that an altogether different approach of derivation exists. Deriving the water wave equations by a variational formulation has also been common in the last forty years (since the paper [52] of Luke in 1967). Using a variational formulation is well suited when, for example, studying the energy in the system (via the Hamiltonian). This approach can thus also be found in any number of books about water waves (cf. Dingemans [23], Whitham [67]), but we will not pursue it here.

3.1 Conservation of mass

We look at a volume V, which is in and totally occupied by the fluid. V is bounded by the surface S, where we denote the outward normal vector of S by $\mathbf{n}$. We let the volume V be fixed for some chosen inertial frame and any fluid in motion can cross the surface S. We assume that no mass is created or destroyed and together with the assumption of the fluid being incompressible, the total mass inside V must be constant for all time.

We let $\mathrm{d}V$ be a volume element of V and $\rho = \rho(\mathbf{x}, t)$ the density of the fluid, where $\mathbf{x} = \{x, y, z\}$ and t the time variable. With this, the total mass inside the volume can be expressed as

$$\int_V \rho(\mathbf{x}, t)\,\mathrm{d}V,$$

where $\int_V V\,\mathrm{d}$ is a triple integral over V.

Any change of mass over time can mathematically be expressed by the time derivative and if the mass is conserved we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho(\mathbf{x}, t)\,\mathrm{d}V = 0.$$

Since we know that we have no creation or destruction of mass, the sole cause of change of mass inside the volume is due to the flow in or out of V. Using the outward normal vector $\mathbf{n}$ of S, the net rate of mass flow out can be expressed in terms of the fluid velocity $\mathbf{u} = \{u, v, w\}$ as

$$\int_S \rho\,\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S,$$

where $\int_S \mathrm{d}S$ is the double integral over the surface S. Thus the change of mass in V is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho(\mathbf{x}, t)\,\mathrm{d}V = -\int_S \rho\,\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S,$$

where the minus sign is due to the *outward* normal $\mathbf{n}$. We can apply the divergence theorem (also known as the Gauss' theorem) on the integral on the right side to make it an integral over the volume rather than the surface and rearrange the terms to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho(\mathbf{x},t)\,\mathrm{d}V + \int_V \nabla(\rho\mathbf{u})\,\mathrm{d}V = 0.$$

As mentioned earlier, V is fixed and the only dependency on t is through ρ and if we assume ρ to be a continuous function we can take the differentiation inside the first integral[1]. Combining the integrals we thus have

$$\int_V \frac{\partial\rho}{\partial t} + \nabla(\rho\mathbf{u})\,\mathrm{d}V = 0.$$

This is the equation of mass conservation on integral form. Another assumption of the whole integrand to be continous will lead to that the integrand itself always must be zero[2]. Since we have V arbitrary, we thus obtain

$$\frac{\partial\rho}{\partial t} + \nabla(\rho\mathbf{u}) = 0, \tag{3.1}$$

which is the equation for conservation of mass on differential form.

Using the chain rule, $\nabla(\rho\mathbf{u})$ can be expressed as

$$\nabla(\rho\mathbf{u}) = \mathbf{u}\nabla\rho + \rho\nabla\mathbf{u},$$

which substituted into (3.1) gives

$$\frac{\partial\rho}{\partial t} + \mathbf{u}\nabla\rho + \rho\nabla\mathbf{u} = 0.$$

Defining the material derivative (also known as the convective derivative) by

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla,$$

the former equation can be written in a more compact form

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\mathbf{u} = 0. \tag{3.2}$$

If we assume the fluid is incompressible, which means that the density ρ is constant and $D\rho/Dt = 0$, then the conservation of mass equation reduces to the simple expression

$$\nabla\cdot\mathbf{u} = 0. \tag{3.3}$$

[1] Note that additional requirements on the integrand then apply, see for example the paper of Flanders [30] or a book on calculus (Adams [1]p. 815).

[2] This is easily verified by assuming the (continous) integrand to be non-zero at some point in the domain. Continuity then requires that the integrand also is non-zero in some region around the point and we can focus the integration over that region, which would lead to the above equation not holding anymore.

3.2 Equations of motion

The equation of motion, also known as Euler's equation, is derived from one of the most fundamental principles in classical physics; The conservation of momentum or also known as Newtons second law of motion. The principle was stated by Newton in 1687 in his famous work "Principia Mathematica" [59] and reads

> *Mutationem motus proportionalem esse vi motrici impressae, et fieri secundum lineam rectam qua vis illa imprimitur.*

and can be expressed mathematically as

$$\mathbf{F} = m\mathbf{a}, \tag{3.4}$$

or as $\mathbf{F} = \rho\mathbf{a}$ if we only look at a fluid element of unit volume, where ρ is the mass (fluid) density.

In this thesis there are two types of relevant forces; a local force, the interaction between the fluid elements themselves (and might therefore be different for each element) and secondly a body force, where the source of the force is external and should approximately be the same for each fluid element. Examples of body forces acting on a fluid could be gravity and the Coriolis effect, while examples of local forces would be surface tension, pressure and friction forces due to viscosity. We will mainly look at cases where gravity the $\mathbf{g}$ and pressure the p are the main forces, while the the other forces are either not present or may be neglected. The case of surface tension is just a slight extension of the present work and will therefore also be included.

As with the conservation of mass we look at a volume V, bounded by the surface S and fixed in our frame of reference and which is in and totally occupied by the fluid. We denote the body forces, for now unknown, by $\mathbf{F}$ and look at the case where the pressure p is the only local force. The total force acting on the fluid in V can be expressed as

$$\int_V \rho\mathbf{F}\,\mathrm{d}V - \int_S p\mathbf{n}\,\mathrm{d}S,$$

where $\mathbf{n}$ is the outward normal of S. The divergence theorem can be applied on the second integral which yields

$$\int_V \rho\mathbf{F} - \nabla p\,\mathrm{d}V.$$

The rate of change of momentum of the fluid in V can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\mathbf{u}\,\mathrm{d}V$$

and the rate of flow of momentum across S into V is

$$-\int_S \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\,\mathrm{d}S. \tag{3.5}$$

Using the integral identity

$$\int_S \mathbf{a}(\mathbf{b}\cdot\mathbf{c})\,\mathrm{d}S = \int_V (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}(\nabla\cdot\mathbf{b})\,\mathrm{d}V,$$

equation (3.5) can be rewritten as

$$-\int_S \rho\mathbf{u}(\mathbf{u}\cdot\mathbf{n})\,\mathrm{d}S = -\int_V \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{u}(\nabla\cdot\mathbf{u})\rho\,\mathrm{d}V.$$

Combining these results, Newton Second law (3.4) then states

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\mathbf{u}\,\mathrm{d}V = \int_V \rho\mathbf{F} - \nabla p\,\mathrm{d}V - \int_V \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{u}(\nabla\cdot\mathbf{u})\rho\,\mathrm{d}V,$$

which can be simplified by taking $\mathrm{d}/\mathrm{d}t$ through the integral sign (again note that additional requirements on the integrand then apply, see Flanders [30]) and rearranging terms

$$\int_V \frac{\partial}{\partial t}(\rho\mathbf{u}) + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{u}(\nabla\cdot\mathbf{u})\rho\,\mathrm{d}V = \int_V (\rho\mathbf{F} - \nabla p)\,\mathrm{d}V.$$

The chain rule can be applied on the first term on the left side

$$\int_V \rho\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\frac{\partial\rho}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{u}(\nabla\cdot\mathbf{u})\rho\,\mathrm{d}V = \int_V (\rho\mathbf{F} - \nabla p)\,\mathrm{d}V,$$

and then collecting similar terms

$$\int_V \rho\left(\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) + \mathbf{u}\left(\frac{\partial\rho}{\partial t} + (\nabla\cdot\mathbf{u})\rho\right)\mathrm{d}V = \int_V (\rho\mathbf{F} - \nabla p)\,\mathrm{d}V.$$

The first parenthesis we recognize as the material derivative of u, while the second parenthesis is the conservation of mass derived previously, equation (3.1), and is thus zero. This yields

$$\int_V \rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} - \rho\mathbf{F} + \nabla p\,\mathrm{d}V = 0.$$

Again the volume is V is arbitrary and we assume the integrand to be continuous, which gives

$$\rho\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} - \rho\mathbf{F} + \nabla p = 0. \tag{3.6}$$

This equation is called Euler's equation. In case where the only body force is gravity, $\mathbf{g} = \{0, 0, -g\}$, equation (3.6) becomes

$$\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} - \rho \mathbf{g} + \nabla p = 0.$$

Note that it is conventional to divide[3] Euler's equation throughout with ρ, which we do when writing Euler's equation on component form

$$\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \quad \frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \quad \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g.$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \right).$$

3.3 Boundary conditions

When modelling water waves, one usually has two types of boundaries; The first is the free surface, where we have interaction between water and air. The second boundary is the bottom, which is assumed to be a solid surface (i.e. zero flow through this boundary). The bottom boundary is assumed to be known at all times, while the free surface boundary (where the wave motion often is the most prominent) will be unknown (except, of course, at the initial state or if it is at rest).

Note that a characteristic of a fluid surface (which can be both a free surface or a solid surface) is that any fluid particle on the surface will remain on the surface (cf. Dingemans [23]).

3.3.1 Free surface conditions

We denote the surface elevation by $\eta(x, y, t)$ and we can thus write the free surface as $z - \eta(x, y, t) = 0$.

Kinematic condition

As the name suggests, the kinematic conditions are conditions containing the velocity of the fluid. At the free surface we have

$$\frac{\mathrm{D}}{\mathrm{D}t}(z - \eta(x, y, t)) = 0, \qquad \text{at} \quad z = \eta(x, y, t),$$

[3]This is possible as long as ρ is non-zero everywhere, which is the case for water.

due to a surface fluid element will remain on the surface. Expanding the paranthesis and writing out the material derivative, the kinematic free surface condition reads

$$w - \frac{\partial \eta}{\partial t} - \left(u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} \right) = 0, \qquad \text{at} \quad z = \eta(x, y, t), \tag{3.7}$$

where we have used $z_t = w$.

Dynamic condition

To find the pressure we must solve Euler's equation at the surface with $p = p_{\mathrm{a}}$ on the free surface, where p_{a} is the (constant) surface pressure. Note that in case where the surface tension cannot be neglected, the pressure at the surface can no longer be regarded as constant and the dynamic condition must therefore be altered accordingly. This will be handled shortly.

3.3.2 Bottom condition

If no viscosity is present (or may be neglected) the body surface can be regarded as a surface of the fluid. Then, by denoting the bottom surface as $z - h(x, y, t) = 0$, the bottom condition can be written as

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(z - h(x, y, t)\right) = 0.$$

Similar to earlier this can be expanded, which yields

$$w - \frac{\partial h}{\partial t} - \left(u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} \right) = 0 \qquad \text{on} \quad z = -h(x, y, t); \tag{3.8}$$

the general bottom boundary condition. If the bottom is non-moving (no dependency on t) the condition reduces to

$$w - \left(u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} \right) = 0 \qquad \text{on} \quad z = -h(x, y, t). \tag{3.9}$$

Often the bottom is assumed to be horizontal, $h(x, y, t) = h_0$, and the boundary condition at the bottom is then described by the simple equation

$$w = 0 \qquad \text{on} \quad z = -h_0. \tag{3.10}$$

3.3.3 Including surface tension

When including effects due to surface tension, but still neglecting any viscosity effects, the dynamic boundary condition must be modified; The surface pressure can no longer be regarded as constant over the whole free surface. Writing p_1 as the pressure right above the fluid and p_2 as the pressure right below the fluid surface, the pressure difference can be expressed in terms of the surface tension as (cf. Johnson [41] or Dingemans [23])

$$p_1 - p_2 = \frac{\Gamma}{R},$$

where $1/R$ is the mean curvature (also called the Gaussian curvature)

$$\frac{1}{R} = \frac{1}{\kappa_1} + \frac{1}{\kappa_2},$$

and κ_1, κ_2 are the principal radii of curvature. The parameter Γ is the coefficient of surface tension and usually depends on the temperature. We assumed any thermodynamical effects to be negligible and thus treat the temperature as constant, and hence also Γ. It can be shown that (Adams [1, p. 675])

$$\frac{1}{R} = \nabla \cdot \frac{\nabla S}{|\nabla S|}, \tag{3.11}$$

where S is the surface given by $S(\mathbf{x}, t) = 0$. In our case the surface is expressed by $z - \eta(x, y, t) = 0$ and (3.11) therefore becomes

$$\frac{1}{R} = -\frac{\left(1+\eta_y^2\right)\eta_{xx} + \left(1+\eta_x^2\right)\eta_{yy} - 2\eta_{xy}\eta_x\eta_y}{\left(1+\eta_x^2+\eta_y^2\right)^{\frac{3}{2}}},$$

where the subscripts denote the partial derivatives. We denote the surface tension as

$$\mathcal{T} = \Gamma\frac{\left(1+\eta_y^2\right)\eta_{xx} + \left(1+\eta_x^2\right)\eta_{yy} - 2\eta_{xy}\eta_x\eta_y}{\left(1+\eta_x^2+\eta_y^2\right)^{\frac{3}{2}}} \tag{3.12}$$

and the dynamic boundary condition must be altered to include this. Thus the dynamic free surface condition now reads

$$p = p_{\mathrm{a}} - \mathcal{T} \qquad \text{on} \quad z = \eta(x, y, t).$$

3.4 Irrotational flow

Making an additional assumption of irrotationality, the governing equations can be rewritten in a simpler form. The vorticity $\boldsymbol{\omega}$ of the flow can be expressed in terms of the fluid velocity as

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}.$$

If $\boldsymbol{\omega} = 0$ we say the flow is irrotational and there then exists a scalar function ϕ such that $\mathbf{u} = \nabla\phi$, where the function ϕ is called the velocity potential of the flow.

In the following we retrace the steps from the previous part and alter the equations accordingly.

3.4.1 Conservation of mass

The derivation of the equation of mass conservation will be the same as for the rotational case (see the derivation which leads to equations (3.2) - (3.3)). We will finally come to the equation $\nabla \cdot \mathbf{u} = 0$ (when one also assumes constant density) and then using $\mathbf{u} = \nabla\phi$, the conservation of mass can be expressed in terms of the Laplace equation

$$\nabla^2\phi = 0.$$

3.4.2 Equations of motion

The equations of motion are in the case of irrotational flow derived in the exact same manner as earlier and are (now expressed in terms of ϕ)

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\partial\phi}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \quad \frac{\mathrm{D}}{\mathrm{D}t}\frac{\partial\phi}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \quad \frac{\mathrm{D}}{\mathrm{D}t}\frac{\partial\phi}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g.$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \left(\frac{\partial\phi}{\partial x}\frac{\partial}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial}{\partial y} + \frac{\partial\phi}{\partial z}\frac{\partial}{\partial z}\right),$$

with the additional condition

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{e}_1 + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{e}_2 + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{e}_3 = 0,$$

where $\mathbf{e}_i$ are the unit vectors (in this case along the x, y and z axis, respectively). Note that this last equation actually gives three separate conditions which must simultaneously be fulfilled (and which, of course, can also be expressed in terms of ϕ).

3.4.3 Boundary conditions

The same boundary conditions as earlier are imposed. The derivation of the kinematic conditions are done in the same manner, while the dynamic free surface condition can be derived by using Bernoulli's equation.

Free surface conditions

Kinematic condition This condition is derived in exactly in the same manner as earlier (i.e. equation (3.7)), but expressed in terms of the velocity potential it instead becomes

$$\frac{\partial \phi}{\partial z} - \frac{\partial \eta}{\partial t} - \left(\frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \eta}{\partial y} \right) = 0, \qquad \text{on} \quad z = \eta(x, y, t). \tag{3.13}$$

Dynamic boundary condition Unlike earlier we will in this case use Bernoulli's equation for unsteady incompressible irrotational inviscid flow when deriving the dynamic boundary condition. Bernoulli's equation reads

$$z = -\frac{1}{g}\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(u^2 + v^2 + w^2\right) + f(t) + \int \frac{1}{\rho}\,\mathrm{d}p \right],$$

where the derivation of this equation can be found in Appendix A. The unknown function $f(t)$ can be absorbed into ϕ, by setting $\Phi_t(\mathbf{x}, t) = \phi_t(\mathbf{x}, t) + f(t)$ (cf. Dingemans [23] or Kinsman [47]). We will, however, still write ϕ for the velocity potential while assuming the $f(t)$ function being eliminated. Bernoulli's equation then reads

$$z = -\frac{1}{g}\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(u^2 + v^2 + w^2\right) + \int \frac{1}{\rho}\,\mathrm{d}p \right].$$

Evaluation of this at the free surface, i.e. at $z = \eta(x, y, t)$, yields

$$\eta = -\frac{1}{g}\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(u^2 + v^2 + w^2\right) + \int \frac{1}{\rho}\,\mathrm{d}p \right], \tag{3.14}$$

which is the dynamic free surface condition. When the fluid density ρ can be regarded as constant this equation simplifies to ($z = \eta(x, y, t)$)

$$\eta = -\frac{1}{g}\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(u^2 + v^2 + w^2\right) + \frac{p}{\rho} \right]. \tag{3.15}$$

If the atmospheric pressure at the free surface also can be regarded as constant, we have $p - p_{\mathrm{a}} = 0$ (at the surface), where p_{a} is the atmospheric pressure. We then can take $p = 0$ at the free surface and any pressure measured or calculated in the fluid will then be the pressure excess of the atmospheric pressure. In this case the dynamic free surface condition reduces to

$$g\eta = -\frac{\partial \phi}{\partial t} - \frac{1}{2}\left(u^2 + v^2 + w^2\right) \qquad z = \eta(x, y, t).$$

It is possible to use this equation, together with the kinematic free surface condition (3.13), to eliminate $\eta(x, y, t)$ from the equation, which would mean that η only appears in the unknown domain of the problem. For the sake of simplicity we assume the density to be constant (and thus use the condition (3.15)). The elimination of η is obtained as follows: First apply the material derivative on (3.15)

$$\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right) g\eta = -\left(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla\right)\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}\left[u^2+v^2+w^2\right] + \frac{1}{\rho}\right].$$

and then use the kinematic free surface condition (3.13) together with

$$\mathbf{u}\cdot\nabla\frac{\partial\phi}{\partial t} = \frac{1}{2}\frac{\partial\mathbf{u}^2}{\partial t}$$

to obtain

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{p}{\rho} + \frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} + \frac{\partial\mathbf{u}^2}{\partial t} + \frac{1}{2}\mathbf{u}\cdot\nabla\mathbf{u}^2 = 0 \qquad \text{on} \quad z = \eta(x,y,t).$$

If the pressure is constant (at the surface) we instead have

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} + \frac{\partial\mathbf{u}^2}{\partial t} + \frac{1}{2}\mathbf{u}\cdot\nabla\mathbf{u}^2 = 0 \qquad \text{on} \quad z = \eta(x,y,t).$$

Note that this is a condition on ϕ alone as the $\mathbf{u}$ terms can be converted to ϕ by $\mathbf{u} = \nabla\phi$, which means

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} + \left(\frac{\partial}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\right)|\nabla\phi|^2 = 0 \qquad \text{on} \quad z = \eta(x,y,t).$$

Note also that in general the motion of the air above the water is coupled with the water motion, see for example the book of Kinsman [47] for details on this.

Bottom condition

Kinematic condition At the bottom (solid) surface $B(\mathbf{x},t) = 0$ any fluid particle can only move tangentially, which can be expressed by ϕ and the outward normal vector $\mathbf{n}$ of B as

$$\frac{\partial\phi}{\partial\mathbf{n}} = 0, \qquad \text{on} \quad z = -h(x,y,t),$$

which is the kinematic boundary condition at the bottom. Although the boundary condition can be written on this form, it is often more convenient to write it in the same manner as earlier, i.e. equations (3.8) - (3.10), where one might replace w with ϕ_z.

3.5 Summary

We have in the previous sections derived the general equations for wave propagation in a fluid under the assumptions of the fluid being inviscid, incompressible, with constant atmospheric pressure p_{a}, all functions being continuous and gravity as the the only body force. The equations were then obtained by use of the principles of mass conservation and conservation of momentum, with prescribed boundary conditions at the free surface and at the bottom.

Both the cases of rotational and irrotational flow were considered and we now make a summary of the obtained equations.

3.5.1 Rotational flow

We state the full set of governing equations for waves in an inviscid incompressible fluid;

The conservation of mass

$$\text{Compressible:} \quad \frac{\partial \rho}{\partial t} + \nabla(\rho \mathbf{u}) = 0$$
$$\text{Incompressible:} \quad \nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

The conservation of momentum, including a general term $\mathbf{F}$ for other body forces,

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} = -\rho \mathbf{g} - \frac{1}{\rho}\nabla p + \rho \mathbf{F},$$

or on component form, without any other body forces,

$$\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \quad \frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \quad \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g,$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \right).$$

In addition we have the following boundary conditions: The kinematic free surface condition

$$w - \frac{\partial \eta}{\partial t} - \left(u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} \right) = 0, \qquad \text{on} \quad z = \eta(x, y, t),$$

the dynamic free surface condition (with p_{a} as the constant surface pressure)

$$\text{Neglecting surface tension:} \quad p = p_{\mathrm{a}} \qquad \text{on} \quad z = \eta(x, y, t),$$
$$\text{Including surface tension:} \quad p = p_{\mathrm{a}} - \mathcal{T} \qquad \text{on} \quad z = \eta(x, y, t),$$

where

$$\mathcal{T} = \Gamma \frac{\left(1+\eta_y^2\right)\eta_{xx} + \left(1+\eta_x^2\right)\eta_{yy} - 2\eta_{xy}\eta_x\eta_y}{\left(1+\eta_x^2+\eta_y^2\right)^{3/2}}.$$

Finally we have the kinematic condition at the bottom

$$\begin{aligned}
&\text{Uneven, moving:} \quad w - \frac{\partial h}{\partial t} - \left(u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}\right) = 0, \quad \text{on} \quad z = -h(x,y,t),\\
&\text{Uneven, non-moving:} \quad w - \left(u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}\right) = 0, \quad \text{on} \quad z = -h(x,y),\\
&\text{Horizontal, non-moving:} \quad w = 0, \quad \text{on} \quad z = -h_0.
\end{aligned}$$

This is the basic set of equations used to describe the wave motion in water. Note that we still need certain initial conditions to complete the description. Note also that the problem is non-trivial, as the unknown η appears in the boundary.

3.5.2 Irrotational flow

When the flow is assumed to be irrotational and the fluid to have constant density the governing equations become, expressed in terms of the velocity potential:

The Laplace equation (due to incompressibility $\nabla \cdot \mathbf{u} = 0$ and $\mathbf{u} = \nabla\phi$)

$$\nabla^2\phi = 0$$

inside the domain $-h(x,y,t) < z < \eta(x,y,t)$, with

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\mathbf{e}_1 + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\mathbf{e}_2 + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{e}_3 = 0,$$

the dynamic free surface condition

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(u^2+v^2+w^2\right) + \frac{p_{\mathrm{a}}}{\rho} + g\eta = 0, \qquad \text{on} \quad z = \eta(x,y,t),$$

the kinematic free surface condition

$$\frac{\partial \phi}{\partial z} - \frac{\partial \eta}{\partial t} - \left(\frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \eta}{\partial y}\right) = 0, \qquad \text{on} \quad z = \eta(x,y,t),$$

and finally the kinematic condition at the bottom

$$\begin{aligned}
&\text{Uneven, moving:} \quad \frac{\partial \phi}{\partial z} - \frac{\partial h}{\partial t} - \left(\frac{\partial \phi}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial h}{\partial y}\right) = 0, \quad \text{on} \quad z = -h(x,y,t),\\
&\text{Uneven, non-moving:} \quad \frac{\partial \phi}{\partial z} - \left(\frac{\partial \phi}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial h}{\partial y}\right) = 0, \quad \text{on} \quad z = -h(x,y),\\
&\text{Horizontal, non-moving:} \quad \frac{\partial \phi}{\partial z} = 0, \quad \text{on} \quad z = -h_0.
\end{aligned}$$

The kinematic free surface boundary condition can be combined with the dynamic free surface condition to produce

$$\frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} + \left(\frac{\partial}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\right)|\nabla\phi|^2 = 0 \qquad \text{on} \quad z = \eta(x,y,t),$$

which has the unknown variable η only appearing in the boundary.

Note how we in the case of irrotational flow do not solve Euler's equation directly, but solve Laplace's equation and obtain the pressure through the boundary conditions instead.

Chapter 4

Dimension analysis and scaling

> Man knows that the world is not made on a human scale; he wishes that it were.
>
> Andre Malraux

In this chapter we first do a dimension analysis and rewrite the equations on a dimensionless form. The importance of this cannot be stressed enough. Firstly such an analysis will reveal if the equations derived earlier are consistent, i.e. that the terms can be combined in the way they are. Secondly it gives us additional information about which terms in the equations might be negligible or which terms might dominate the equations (under certain circumstances). And last, but not the least, we will obtain two parameters which will help us identify, in a rather precise way, how to make the approximations and achieve the desired equations.

After obtaining a set of equations on non-dimensional form, we observe that additional scaling are needed and present the non-dimensional scaled equations.

4.1 Non-dimensional variables

Following Johnson [41] we use the natural length and time scales that appear in the problem to non-dimensionalize the equations. There is no unique way of defining the non-dimensional variables, but often the problem at hand will give a hint of what should be done. Writing all the variables in a dimension table can also be of great help, which we have done in table 4.1 (see [50] or [58] for more details on dimension tables and the use of these).

	$\mathbf{u}$	p	$\mathbf{x}$	ρ	t	η	h	g	λ
s	-1	-2	0	0	1	0	0	-2	0
m	1	-1	1	-3	0	1	1	1	1
kg	0	1	0	1	0	0	0	0	0

Table 4.1: Dimension table for the governing equations

The table can help finding which quantities can be combined to make the variables dimensionless. As is readily seen, many combinations exists, and we have to make some choices. We will therefore use the experience of our predecessors when we determine which combinations are best suited for our cause (cf. Dullin *et al.* [27, 26], Johnson [41, 16], Dingemans [24], Mei *et al.* [55]).

There are two length scales that are directly at hand; The typical (or maximum) water depth at equilibrium, denoted by h_0, and λ as the typical wave length of the wave (usually at the surface). These will be used as the vertical length scale and the horizontal length scale, respectively.

A typical speed of the horizontal wave propagation can be found to be $U = \sqrt{gh_0}$ (cf. Whitham [67] or Kinsman [47]) and it is this we will use as the velocity scale for the horizontal velocity components. We will see, when writing the equation of mass conservation on a non-dimensional form, we have to require the vertical velocity scale to be h_0U/λ for this equation to be consistent.

Having a typical speed U and a typical length λ for the wave propagation it is readily seen that by combining those a typical time can be obtained as λ/U, which will be the time scale.

We use for the surface wave the typical (or maximum) amplitude a as scale. We could also have chosen h_0 (Dingemans [23]) or λ (Choi [14]) as the scaling factor, but we choose to follow Johnson and Constantin [41, 16], such that a second parameter is obtained in a more direct way.

For the pressure we use the hydrostatic pressure at the bottom (at typical depth h_0) as pressure scale, i.e. $\rho g h_0$.

Introducing primes to denote the non-dimensional variables we have

$$x' = \frac{x}{\lambda}, \quad y' = \frac{y}{\lambda}, \quad z' = \frac{z}{h_0}, \quad t' = \frac{U}{\lambda}t, \quad h' = \frac{h}{h_0}$$
$$u' = \frac{u}{U}, \quad v' = \frac{v}{U}, \quad w' = \frac{\lambda}{h_0 U}w, \quad \eta' = \frac{\eta}{a}.$$

while the pressure can be rewritten as

$$p = p_{\mathrm{a}} + \rho g z + \rho g h_0 p', \tag{4.1}$$

where p' is the non-dimensional variable, p_{a} the (constant) atmospheric pressure, $\rho g z$ the hydrostatic pressure distribution at depth z. This means the pressure p' measures the deviation from the hydrostatic pressure distribution, i.e. $p' \neq 0$ only when a passage of a wave occurs. The rewriting of the pressure in this way will be clear when writing the third (vertical) component of Euler's equation in a non-dimensional form, see Appendix B.

We include only the calculations for the equation of conservation of mass, as it should confirm the choice of the vertical velocity scale:

$$\begin{aligned} 0 &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \\ &= \frac{U}{\lambda}\frac{\partial u'}{\partial x'} + \frac{U}{\lambda}\frac{\partial v'}{\partial y'} + \frac{h_0 U}{\lambda h_0}\frac{\partial w'}{\partial z'} \\ &= \frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} + \frac{\partial w'}{\partial z'}, \end{aligned}$$

where we now see that we have chosen the correct vertical velocity scale and the non-dimensional conservation of mass equation is consistent. The calculations for the other governing equations can be found in Appendix B, where it is clear how two important parameters, ϵ and δ, emerge.

To ease the notation we will from this point on ignore the primes and write the (non-dimensional) variables as usual, e.g. the first horizontal variable on non-dimensional form would then simply be x. The full set of governing equations on non-dimensional form is thus

$$\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x}, \quad \frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{\partial p}{\partial y}, \quad \delta^2\frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{\partial p}{\partial z} \tag{4.2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{4.3}$$

with on the free surface $z = \epsilon\eta(x, y, t)$

$$w = \epsilon\left[\frac{\partial \eta}{\partial t} + \left(u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}\right)\right] \tag{4.4}$$

$$\eta = p \tag{4.5}$$

and on the bottom $z = -h(x, y, t)/h_0$

$$w = \frac{\partial h}{\partial t} + \left(u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}\right), \tag{4.6}$$

where

$$\begin{aligned} \frac{\mathrm{D}}{\mathrm{D}t} &= \frac{\partial}{\partial t} + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right) \\ \delta &= \frac{h_0}{\lambda} \qquad \epsilon = \frac{a}{h_0}. \end{aligned}$$

If the surface tension cannot be neglected, equation (4.5) must be replaced by

$$p = \epsilon\eta - \epsilon\left(\frac{\Gamma}{\rho g\lambda^2}\right)\left[\frac{(1+\epsilon^2\delta^2\eta_y^2)\eta_{xx} + (1+\epsilon^2\delta^2\eta_x^2)\eta_{yy} - 2\epsilon^2\delta^2\eta_x\eta_y\eta_{xy}}{(1+\epsilon^2\delta^2\eta_x^2+\epsilon^2\delta^2\eta_y^2)^{3/2}}\right],$$

where we follow Johnson [41] and set $\delta^2 W_\mathrm{e} = \frac{\Gamma}{\rho g\lambda^2}$ and thus have $W_\mathrm{e} = \frac{\Gamma}{\rho g h_0^2}$ and

$$p = \epsilon\left(\eta - \delta^2 W_\mathrm{e}\tilde{\mathcal{T}}\right) \qquad \text{on} \quad z = \epsilon\eta(x,y,t),$$

with

$$\tilde{\mathcal{T}} = \frac{(1+\epsilon^2\delta^2\eta_y^2)\eta_{xx} + (1+\epsilon^2\delta^2\eta_x^2)\eta_{yy} - 2\epsilon^2\delta^2\eta_x\eta_y\eta_{xy}}{(1+\epsilon^2\delta^2\eta_x^2+\epsilon^2\delta^2\eta_y^2)^{3/2}}. \tag{4.7}$$

W_e is a Weber number[1] and is used to measure the contribution of the surface tension.

Note how we by these calculations have obtained two new parameters, $\epsilon = a/h_0$ and $\delta = h_0/\lambda$, which can be used control the type of water-wave problem. Thus for $\epsilon \ll 1$ (which implies $h_0 \gg a$) the amplitude is said to be small and we are in the small amplitude regime. For $\delta \ll 1$ (which implies $h_0 \gg \lambda$) the water is said to shallow and we then are in the regime of shallow water (also known as the long wave regime). Correspondingly for large amplitude and deep water (short waves) we have, $\epsilon \gg 1$ and $\delta \gg 1$, respectively. In this thesis we will mainly look at the former regime, i.e. at small amplitudes in shallow water.

Also note that if the bottom is horizontal, $h(x,y,t) = h_0$ it would be described by $z = -1$. This assumption will be made throughout this thesis.

The corresponding equations for irrotational flow are (again omitting the primes)

$$\delta^2\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \frac{\partial^2\phi}{\partial z^2} = 0$$

inside the domain $-h(x,y,t) < z < \epsilon\eta(x,y,t)$, with

$$\delta^2\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0, \quad \frac{\partial u}{\partial z} - \delta^2\frac{\partial w}{\partial x} = 0, \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

and the free surface conditions ($z = \epsilon\eta(x,y,t)$)

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \frac{1}{\delta^2}\left(\frac{\partial\phi}{\partial z}\right)^2\right] + \epsilon\eta = 0$$

$$\frac{\partial\phi}{\partial z} - \epsilon\delta^2\left[\frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\eta}{\partial y}\right] = 0,$$

[1] This is strictly speaking an inverse Weber number, which means that in our case $W_\mathrm{e} \ll 1$ stands for little contribution from the surface tension, in contrary to the common literature (e.g. White [66]).

and finally the kinematic condition at the bottom ($z = -h(x, y, t)$)

$$\frac{\partial \phi}{\partial z} - \delta^2 \left(\frac{\partial h}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial h}{\partial y} \right) = 0.$$

4.2 Scaling

Having recast all the equations on a non-dimensional form we observe how the free surface conditions (equation (4.4) and equation (4.5)) give that w and p are proportional to ϵ (i.e. the amplitude) here. This means that when $\epsilon \to 0$, we will have $p \to 0$ (when the surface comes to rest, the pressure vanishes). Similar yields for w, both which are to be expected. To also have these variables of order unity, we rescale by (introducing primes to denote the scaled variables)

$$p' = \frac{p}{\epsilon}, \quad \mathbf{u}' = \frac{1}{\epsilon}\mathbf{u},$$

which gives

$$\frac{\mathrm{D}u'}{\mathrm{D}t} = -\frac{\partial p'}{\partial x}, \quad \frac{\mathrm{D}v'}{\mathrm{D}t} = -\frac{\partial p'}{\partial y}, \quad \delta^2 \frac{\mathrm{D}w'}{\mathrm{D}t} = -\frac{\partial p'}{\partial z},$$
$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + \frac{\partial w'}{\partial z} = 0,$$

with

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \epsilon \left(u'\frac{\partial}{\partial x} + v'\frac{\partial}{\partial y} + w'\frac{\partial}{\partial z} \right)$$

and boundary conditions

$$w' = \frac{1}{\epsilon}\frac{\partial h}{\partial t} + \left(u'\frac{\partial h}{\partial x} + v'\frac{\partial h}{\partial y} \right) \qquad \text{on} \quad z = -h(x, y, t)$$
$$w' = \frac{\partial \eta}{\partial t} - \epsilon \left(u'\frac{\partial \eta}{\partial x} + v'\frac{\partial \eta}{\partial y} \right) \qquad \text{on} \quad z = \epsilon\eta(x, y, t)$$
$$p' = \eta + \delta^2 W_{\mathrm{e}} \left[\frac{(1 + \epsilon^2\delta^2\eta_y^2)\eta_{xx} + (1 + \epsilon^2\delta^2\eta_x^2)\eta_{yy} - 2\epsilon^2\delta^2\eta_x\eta_y\eta_{xy}}{(1 + \epsilon^2\delta^2\eta_x^2 + \epsilon^2\delta^2\eta_y^2)^{3/2}} \right],$$

or $p' = \eta$ in the last equation if the surface tension is neglected.

Correspondingly for the irrotational case we then have

$$\delta^2 \left(\frac{\partial^2 \phi'}{\partial x^2} + \frac{\partial^2 \phi'}{\partial y^2} \right) + \frac{\partial^2 \phi'}{\partial z^2} = 0$$

inside the domain $-h(x, y, t) < z < \epsilon\eta(x, y, t)$, with

$$\delta^2 \frac{\partial w'}{\partial y} - \frac{\partial v'}{\partial z} = 0, \quad \frac{\partial u'}{\partial z} - \delta^2 \frac{\partial w'}{\partial x} = 0, \quad \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} = 0$$

with the free surface conditions ($z = \epsilon\eta(x, y, t)$)

$$\frac{\partial\phi}{\partial t} + \frac{\epsilon}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \frac{1}{\delta^2}\left(\frac{\partial\phi}{\partial z}\right)^2\right] + \eta = 0$$

$$\frac{\partial\phi}{\partial z} - \delta^2\left[\frac{\partial\eta}{\partial t} - \epsilon\left(\frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\eta}{\partial y}\right)\right] = 0,$$

and finally the kinematic condition at the bottom ($z = -h(x, y, t)$)

$$\frac{\partial\phi}{\partial z} - \delta^2\left(\frac{1}{\epsilon}\frac{\partial h}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial h}{\partial y}\right) = 0$$

From this point on we again ignore the primes and will assume that the equations are non-dimensional and scaled in the way we have done here. It is of course possible to retrieve the original variables, e.g. the (dimensional) vertical velocity will be $\epsilon(h_0\sqrt{gh_0}/\lambda) \cdot w$.

Chapter 5

Approximate equations

> If you cannot do great things, do small things in a great way.
>
> Napoleon Hill

In section 2.1 we described the tool of asymptotic analysis and in the following we will use this tool on the governing equations derived in chapter 3. This will lead to new approximate equations describing water wave propagation. Depending on how far we take the asymptotic expansion, in which parameter (we have directly at hand the parameters ϵ and δ) the expansion is taken and how we (re)scale the variables, different equations can be obtained.

We will mainly make the asymptotic expansion in terms of the parameter ϵ as it goes to zero and thus making an assumption of small amplitude, while letting δ be fixed. Letting q represent the function we assume can be expressed as an asymptotic series (say p or u) we write

$$q(\mathbf{x}, t; \epsilon) \sim \sum_{n=0}^{\infty} \epsilon^n q^{(n)}(\mathbf{x}, t). \tag{5.1}$$

In some cases (e.g. the Camassa-Holm equation) it is necessary to assume shallow water (long wave) in terms of $\delta \to 0$ and we can then make the double asymptotic expansion

$$q(\mathbf{x}, t; \epsilon, \delta) \sim \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \epsilon^i \delta^{2j} q^{(i,j)}(\mathbf{x}, t)$$

instead.

In the next sections we will derive different sets of equations at different orders of the asymptotic expansions. We begin with the simplest case, the leading order equations.

5.1 Leading order

The leading order equations are the most simple[1] equations to be obtained from an asymptotic expansion and in this case will lead to the same set of governing equations as the linearised water wave equations. We develop the asymptotic expansions in terms of $\epsilon \to 0$, letting δ be fixed. Thus at leading order we have

$$q(\mathbf{x}, t; \epsilon) \sim q^{(0)}$$

which put into the governing equations and collecting only terms of order $O(1)$ yields the set

$$\frac{\partial u^{(0)}}{\partial t} = -\frac{\partial p^{(0)}}{\partial x}, \quad \frac{\partial v^{(0)}}{\partial t} = -\frac{\partial p^{(0)}}{\partial y}, \quad \delta^2 \frac{\partial w^{(0)}}{\partial t} = -\frac{\partial p^{(0)}}{\partial z}, \tag{5.2}$$

$$\frac{\partial u^{(0)}}{\partial x} + \frac{\partial v^{(0)}}{\partial y} + \frac{\partial w^{(0)}}{\partial z} = 0, \tag{5.3}$$

with on the free surface $z = 0$

$$w^{(0)} = \frac{\partial \eta^{(0)}}{\partial t} \tag{5.4}$$

$$p^{(0)} = \eta^{(0)} + \delta^2 W_{\mathrm{e}} \left(\frac{\partial^2 \eta^{(0)}}{\partial x^2} + \frac{\partial^2 \eta^{(0)}}{\partial y^2} \right) \tag{5.5}$$

and on $z = -h(x, y)$

$$w^{(0)} = u^{(0)} \frac{\partial h^{(0)}}{\partial x} + v^{(0)} \frac{\partial h^{(0)}}{\partial y}. \tag{5.6}$$

Note how the free surface at this order is on $z = 0$ and the boundary is thus independent of η (the problem is no longer non-trivial). These equations are equivalent to the linearised water wave equations on water of any depth.

5.1.1 Linear shallow water equations

If we should impose the condition of shallow water, $\delta \to 0$, additional terms in the governing equations can be neglected (at leading order of ϵ). Note that the same can also be achieved by a rescaling of the equations in such a way that δ^2 would be replaced by ϵ (or a power of ϵ), as will be done later (the KdV equation among them). The system then becomes

$$\frac{\partial u^{(0)}}{\partial t} = -\frac{\partial p^{(0)}}{\partial x}, \quad \frac{\partial v^{(0)}}{\partial t} = -\frac{\partial p^{(0)}}{\partial y}, \quad \frac{\partial p^{(0)}}{\partial z} = 0, \tag{5.7}$$

$$\frac{\partial u^{(0)}}{\partial x} + \frac{\partial v^{(0)}}{\partial y} + \frac{\partial w^{(0)}}{\partial z} = 0 \tag{5.8}$$

[1] Simple in this case does not refer to the complexity of the equations, but in comparison to higher order asymptotic expansions.

with on the free surface $z = 0$

$$w^{(0)} = \frac{\partial \eta^{(0)}}{\partial t} \tag{5.9}$$

$$p^{(0)} = \eta^{(0)} \tag{5.10}$$

and on the bottom $z = -h(x, y)$

$$w^{(0)} = u^{(0)} \frac{\partial h^{(0)}}{\partial x} + v^{(0)} \frac{\partial h^{(0)}}{\partial y} \tag{5.11}$$

This set of equations is often referred to as the linearised shallow water wave equations. It is possible to obtain a very classical result from these equations in the following way:

We solve the vertical component of Euler's equation (third equation in (5.7)) and obtain, after using (5.10),

$$p^{(0)} = \eta^{(0)}$$

for $-1 \leq z \leq 0$. This and the first component of Euler's equation (5.7) gives us

$$\frac{\partial u^{(0)}}{\partial t} = -\frac{\partial \eta^{(0)}}{\partial x}, \tag{5.12}$$

for $-1 \leq z \leq 0$. We obtain similar for the second horizontal component

$$\frac{\partial v^{(0)}}{\partial t} = -\frac{\partial \eta^{(0)}}{\partial y}, \tag{5.13}$$

for $-1 \leq z \leq 0$.

From the equation of conservation of mass (5.8) we have

$$\frac{\partial w^{(0)}}{\partial z} = -\left(\frac{\partial u^{(0)}}{\partial x} + \frac{\partial v^{(0)}}{\partial y} \right) \tag{5.14}$$

and by integrating with respect to z, we easily find the leading order of the vertical velocity to be

$$w^{(0)} = -z \left(\frac{\partial u^{(0)}}{\partial x} + \frac{\partial v^{(0)}}{\partial y} \right) + C(x, y, t).$$

where $C(x, y, t)$ is a constant of integration. Using the bottom condition (5.11) we can determine $C(x, y, t)$ and obtain

$$w^{(0)} = -(z + 1) \left(\frac{\partial u^{(0)}}{\partial x} + \frac{\partial v^{(0)}}{\partial y} \right) \qquad -1 \leq z < 0. \tag{5.15}$$

To make this equation also satisfy the kinematic free surface condition (5.9) we require

$$\left(\frac{\partial u^{(0)}}{\partial x} + \frac{\partial v^{(0)}}{\partial y} \right) = -\frac{\partial \eta^{(0)}}{\partial t}$$

We have thus obtained the equations for the leading order and in summary these are

$$\begin{aligned} &p^{(0)} = \eta^{(0)}, \quad \frac{\partial u^{(0)}}{\partial t} = -\frac{\partial \eta^{(0)}}{\partial x}, \\ &\frac{\partial v^{(0)}}{\partial t} = -\frac{\partial \eta^{(0)}}{\partial y}, \quad w^{(0)} = -(z+1) \left(\frac{\partial u^{(0)}}{\partial x} + \frac{\partial v^{(0)}}{\partial y} \right), \end{aligned} \tag{5.16}$$

all valid in the domain $-1 \leq z \leq 0$, where the last equation is valid on the free surface only if the condition

$$\left(\frac{\partial u^{(0)}}{\partial x} + \frac{\partial v^{(0)}}{\partial y} \right) = -\frac{\partial \eta^{(0)}}{\partial t}, \tag{5.17}$$

also is met.

The classical result is then obtained by combining the expressions for the horizontal velocities in (5.16) with (5.17) to obtain the (two-dimensional) wave equation

$$\frac{\partial^2 \eta^{(0)}}{\partial t^2} - c \left(\frac{\partial^2 \eta^{(0)}}{\partial x^2} + \frac{\partial^2 \eta^{(0)}}{\partial y^2} \right) = 0$$

valid on $z = 0$ and wave speed $c = 1$. This can then be solved by the method of spherical means (cf. McOwen [54, chapter 3.2]).

In the one-dimensional case we have (omitting the y-dependency) for $z = 0$ and wave speed $c = 1$

$$\frac{\partial^2 \eta^{(0)}}{\partial t^2} - c \frac{\partial^2 \eta^{(0)}}{\partial x^2} = 0 \tag{5.18}$$

If initial conditions were given, use of d'Alemberts formula[2] would give a solution of the form

$$\eta^{(0)}(x,t) = \frac{1}{2} \left(g(x+ct) + g(x-ct) \right) + \frac{1}{2c} \int_{x-ct}^{x+ct} h(\xi, t) \, \mathrm{d}\xi,$$

where $g(x,t)$ and $h(x,t)$ are given by the initial conditions. More generally the solution would be on the form

$$\eta^{(0)}(x,t) = f(x-ct) + \tilde{f}(x+ct),$$

[2]For derivation and discussion of d'Alemberts formula we refer to McOwen [54].

where f is the wave profile and $\mp$ refers to to the right and left moving wave, respectively. Thus looking for a solution of a wave travelling in the positive x-direction would be of form $\eta^{(0)}(x,t) = f(x-ct)$, which is partially the motivation when we later derive, among others, the KdV equation under the assumption of a wave travelling in one direction.

5.2 First order

In this section we keep terms of first order in the asymptotic expansions. In a single asymptotic expansion in ϵ this means we keep all terms of order $O(\epsilon)$ and write

$$q \sim q^{(0)} + \epsilon q^{(1)}, \tag{5.19}$$

while for a double expansion in ϵ and δ^2 (independently of each other) terms of order $O(\epsilon)$, $O(\delta^2)$ and $O(\epsilon\delta^2)$ will be kept

$$q \sim q^{(0,0)} + \epsilon q^{(1,0)} + \delta^2 q^{(0,1)} + \epsilon\delta^2 q^{(1,1)},$$

where $q^{(i)}$ and $q^{(i,j)}$ represents any of u, v, w, p, η.

Using the single asymptotic expansion (5.19) on the governing equations and collecting terms of order $O(\epsilon)$ (we already have the leading order equations from the previous section) we obtain the following set of equations

$$\frac{\mathrm{D}u^{(1)}}{\mathrm{D}t} = -\frac{\partial p^{(1)}}{\partial x}, \quad \frac{\mathrm{D}v^{(1)}}{\mathrm{D}t} = -\frac{\partial p^{(1)}}{\partial y}, \quad \delta^2\frac{\mathrm{D}w^{(1)}}{\mathrm{D}t} = -\frac{\partial p^{(1)}}{\partial z}, \tag{5.20}$$

$$\frac{\partial u^{(1)}}{\partial x} + \frac{\partial v^{(1)}}{\partial y} + \frac{\partial w^{(1)}}{\partial z} = 0, \tag{5.21}$$

with on the free surface $z = \epsilon\eta^{(0)}(x,y,t)$

$$w^{(1)} = \frac{\partial \eta^{(1)}}{\partial t} - u^{(0)}\frac{\partial \eta^{(0)}}{\partial x} - v^{(0)}\frac{\partial \eta^{(0)}}{\partial y} \tag{5.22}$$

$$p^{(1)} = \eta^{(1)} + \delta^2 W_{\mathrm{e}}\mathcal{T}^{(1)} \tag{5.23}$$

and on the bottom $z = -h(x,y)$

$$w^{(1)} = 0. \tag{5.24}$$

Here

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \epsilon\left(u^{(0)}\frac{\partial}{\partial x} + v^{(0)}\frac{\partial}{\partial y} + w^{(0)}\frac{\partial}{\partial z}\right)$$

and

$$\mathcal{T}^{(1)} = \frac{\partial^2\eta^{(1)}}{\partial x^2} + \frac{\partial^2\eta^{(1)}}{\partial y^2}.$$

Notice how the first order equations (5.20), (5.22) and the boundary domain $z = \epsilon\eta^{(0)}$ all are coupled with terms of leading order, which means we first have to solve the leading order equations, before being able to solve the first order equations. Also note that at this order non-linearities have emerged.

In the following we use this set, with appropriate scaling, to obtain some of the most famous equations in water wave theory; The Korteweg-de Vries equation, the Kadomtsev-Petviashvili equation and the Boussinesq equation.

5.2.1 Korteweg - de Vries equation

The Korteweg-de Vries (KdV) equation is one of the most arch-typical equations in nonlinear water wave theory and was first derived by Korteweg and his student de Vries in 1895 [48]. It describes waves propagating in one direction (we choose to look at waves travelling in the positive x-direction and neglect any y dependency) over a horizontal bottom. There are several ways in obtaining the KdV equation and its occurrence is not only limited to water waves (it can for example be obtained as a continuum limit of the Toda lattice [64]).

The trick of obtaining the KdV equation is to balance the nonlinear effects with the dispersion effects (cf. Johnson [41, 42]), i.e. to balance the parameters ϵ and δ. Note however that assuming any relation between the two parameters, say $\delta^2 = O(\epsilon)$ is not the correct way of obtaining the balance, since then $\delta \to 0$ for $\epsilon \to 0$ and the KdV balance would then occur under very strict conditions (in very shallow water with very small amplitude), which is in direct contradiction to observations. We thus need an approach where we do not make any assumption on a functional relation between δ and ϵ.

The balancing is instead obtained by the following rescaling

$$x \to \frac{\delta}{\epsilon^{1/2}} x, \quad t \to \frac{\delta}{\epsilon^{1/2}} t,$$

which should be read as x being replaced by $(\delta/\sqrt{\epsilon})x$ and correspondingly for t. To be consistent with the equation of mass conservation (see equation (5.21) and the calculations which lead to the non-dimensional form in section 4) we also have the scaling

$$w \to \frac{\epsilon^{1/2}}{\delta} w.$$

Using th new scaled variables directly in the original governing equations yields

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + \epsilon\left(u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} \\
&\epsilon\left[\frac{\partial w}{\partial t} + \epsilon\left(u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}\right)\right] = -\frac{\partial p}{\partial z} \\
&\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0
\end{aligned}
$$

with on the free surface $z = \epsilon\eta$

$$
\begin{aligned}
\eta &= p + \epsilon W_{\mathrm{e}}\frac{\partial^2 \eta}{\partial x^2} \\
w &= \frac{\partial \eta}{\partial t} + \epsilon u\frac{\partial \eta}{\partial x}
\end{aligned}
$$

and on the horizontal bottom $z = -1$

$$
w = 0.
$$

We observe the transformation gives δ^2 being replaced by ϵ (without assuming any functional relation between them). As mentioned earlier we only look at waves propagating the the positive x-direction, which can be expressed as

$$
\xi = x - t \tag{5.25}
$$

and we define a new (slow) time variable by

$$
\tau = \epsilon t. \tag{5.26}
$$

With these transformations we observe that

$$
\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} \quad \text{and} \quad \frac{\partial}{\partial t} = -\frac{\partial}{\partial \xi} + \epsilon\frac{\partial}{\partial \tau}. \tag{5.27}
$$

Searching for solutions in terms of asymptotic expansions, we have earlier derived the equations at leading order (equations (5.16)) and using the transformations and scaling introduced here these equations become

$$
p^{(0)} = \eta^{(0)}, \quad \frac{\partial u^{(0)}}{\partial \xi} = \frac{\partial \eta^{(0)}}{\partial \xi}, \quad w^{(0)} = -(z+1)\frac{\partial u^{(0)}}{\partial \xi} \tag{5.28}
$$

and we can directly see that this leads to the set of equations

$$
p^{(0)} = \eta^{(0)}, \quad u^{(0)} = \eta^{(0)}, \quad w^{(0)} = -(z+1)\frac{\partial \eta^{(0)}}{\partial \xi}.
$$

on the domain $-1 \leq z \leq 0$.

Note that no restrictions are made on the $\eta^{(0)}$ function and it is therefore arbitrary. At first order we still have the problems of having an unknown variable in the boundary. To simplify this, we Taylor expand the free surface conditions[3] around the surface at rest $z = 0$. After inserting the asymptotic expansion (5.19) the dynamic free surface condition reads

$$p^{(0)} + \epsilon p^{(1)} = \eta^{(0)} + \epsilon\eta^{(1)} + \epsilon W_{\mathrm{e}} \frac{\partial^2 \eta^{(0)}}{\partial x^2} + O\left(\epsilon^2\right) \qquad \text{on} \quad z = \epsilon\eta^{(0)}. \tag{5.29}$$

We can Taylor expand the pressure around the free surface at rest $z = 0$ with a perturbation z_0

$$p(x, z, t)|_{z=z_0} = p(x, 0, t) + \frac{\partial}{\partial z} p(x, 0, t) z_0 + O\left(z_0^2\right).$$

Letting the perturbation be described by $z_0 = \epsilon\eta^{(0)}$ and inserting the Taylor expansion into (5.29) yields

$$p^{(0)} + \epsilon\eta^{(0)} \frac{\partial p^{(0)}}{\partial z} + \epsilon p^{(1)} = \eta^{(0)} + \epsilon\eta^{(1)} + \epsilon W_{\mathrm{e}} \frac{\partial^2 \eta^{(0)}}{\partial x^2} + O\left(\epsilon^2\right), \tag{5.30}$$

now on $z = 0$. Similarly we can Taylor expand the vertical velocity $w^{(0)}$ and the kinematic free surface condition becomes

$$\begin{aligned} w^{(0)} + \epsilon\eta^{(0)} \frac{\partial w^{(0)}}{\partial z} + \epsilon w^{(1)} = &-\frac{\partial \eta^{(0)}}{\partial \xi} - \epsilon \frac{\partial \eta^{(1)}}{\partial \xi} \\ &+ \epsilon \left(\frac{\partial \eta^{(0)}}{\partial \tau} + u^{(0)} \frac{\partial \eta^{(0)}}{\partial \xi} \right) + O\left(\epsilon^2\right), \end{aligned} \tag{5.31}$$

also on $z = 0$.

Using the transformations (5.25) - (5.27) in the governing first order equations (5.20) - (5.23) and replacing the boundary conditions with the conditions above we obtain the following set of governing equations at first order

$$-\frac{\partial p^{(1)}}{\partial \xi} = -\frac{\partial u^{(1)}}{\partial \xi} + \frac{\partial u^{(0)}}{\partial \tau} + u^{(0)} \frac{\partial u^{(0)}}{\partial \xi} + w^{(0)} \frac{\partial u^{(0)}}{\partial z} \tag{5.32}$$

$$\frac{\partial p^{(1)}}{\partial z} = \frac{\partial w^{(0)}}{\partial \xi} \tag{5.33}$$

$$\frac{\partial u^{(1)}}{\partial \xi} + \frac{\partial w^{(1)}}{\partial z} = 0, \tag{5.34}$$

[3] For the Taylor expansion to be valid we need to impose additional conditions on the functions. However, as Johnson points out, any convergence requirements are unnecessary as the functions need only to satisfy the conditions laid down for asymptotic validity as $\epsilon \to 0$ (cf. Johnson [41, p. 140]).

with on the free-surface $z = 0$

$$p^{(1)} + \eta^{(0)} \frac{\partial p^{(0)}}{\partial z} = \eta^{(1)} + W_{\mathrm{e}} \frac{\partial^2 \eta^{(0)}}{\partial x^2} \tag{5.35}$$

$$w^{(1)} + \eta^{(0)} \frac{\partial w^{(0)}}{\partial z} = -\frac{\partial \eta^{(1)}}{\partial \xi} + \frac{\partial \eta^{(0)}}{\partial \tau} + u^{(0)} \frac{\partial \eta^{(0)}}{\partial \xi} \tag{5.36}$$

and finally the bottom condition $z = -1$

$$w^{(1)} = 0. \tag{5.37}$$

From the leading order solutions we have $u^{(0)} = p^{(0)}$, which differentiating with respect to z and using $p_z^{(0)} = 0$ gives

$$\frac{\partial u^{(0)}}{\partial z} = \frac{\partial p^{(0)}}{\partial z} = 0. \tag{5.38}$$

In addition we have an expression for the vertical velocity of leading order (third equation in (5.28)), which can be differentiated with respect to ξ

$$\frac{\partial w^{(0)}}{\partial \xi} = -(z+1) \frac{\partial^2 \eta^{(0)}}{\partial \xi^2}. \tag{5.39}$$

The first order expression for the pressure $p^{(1)}$ can be obtained by substituting (5.39) into (5.33),

$$\frac{\partial p^{(1)}}{\partial z} = \frac{\partial w^{(0)}}{\partial \xi} = -(z+1) \frac{\partial^2 \eta^{(0)}}{\partial \xi^2},$$

and then integrate the result with respect to z

$$\begin{aligned} p^{(1)} &= -\int (z+1) \frac{\partial^2 \eta^{(0)}}{\partial \xi^2} \, \mathrm{d}z \\ &= -\left(\frac{1}{2} z^2 + z \right) \frac{\partial^2 \eta^{(0)}}{\partial \xi^2} + C_1(\xi, \tau) \\ &= -\frac{1}{2} \left(z^2 + 2z - 2W_{\mathrm{e}} \right) \frac{\partial^2 \eta^{(0)}}{\partial \xi^2} + \eta^{(1)}, \end{aligned} \tag{5.40}$$

where we have used the first order dynamic boundary condition (5.35) in the last step, together with (5.38) to determine the integration constant $C_1(\xi, \tau)$.

We continue with deriving an expression for $w^{(1)}$; The conservation of mass (5.34) together with Euler's equation (5.32) and using (5.38) yields

$$\frac{\partial w^{(1)}}{\partial z} = -\frac{\partial u^{(1)}}{\partial \xi} = -\frac{\partial p^{(1)}}{\partial \xi} - \frac{\partial u^{(0)}}{\partial \tau} - u^{(0)} \frac{\partial u^{(0)}}{\partial \xi}$$

where we can use (5.40) (after differentiating it with respect to ξ) for the pressure term

$$\frac{\partial w^{(1)}}{\partial z} = \frac{1}{2}\left(z^2 + 2z - 2W_\mathrm{e}\right)\frac{\partial^3 \eta^{(0)}}{\partial \xi^3} - \frac{\partial \eta^{(1)}}{\partial \xi} - \frac{\partial u^{(0)}}{\partial \tau} - u^{(0)}\frac{\partial u^{(0)}}{\partial \xi}.$$

Similar to earlier this can be integrated with respect to z

$$\begin{aligned} w^{(1)} &= \int \frac{1}{2}\left(z^2 + 2z - 2W_\mathrm{e}\right)\frac{\partial^3 \eta^{(0)}}{\partial \xi^3} - \frac{\partial \eta^{(1)}}{\partial \xi} - \frac{\partial \eta^{(0)}}{\partial \tau} - \eta^{(0)}\frac{\partial \eta^{(0)}}{\partial \xi}\,\mathrm{d}z \\ &= \frac{1}{3}\left(z^3 + 3z^2 - 6W_\mathrm{e} z\right)\frac{\partial^3 \eta^{(0)}}{\partial \xi^3} - \left(\frac{\partial \eta^{(1)}}{\partial \xi} + \frac{\partial \eta^{(0)}}{\partial \tau} + \eta^{(0)}\frac{\partial \eta^{(0)}}{\partial \xi}\right) z + C_2(\xi,\tau), \end{aligned}$$

where we can find the constant of integration $C_2(\xi,\tau)$ from the bottom boundary condition (5.37) and have

$$C_2(\xi,\tau) = -\left(\frac{2}{3} - 3W_\mathrm{e}\right)\frac{\partial^3 \eta^{(0)}}{\partial \xi^3} - \left(\frac{\partial \eta^{(1)}}{\partial \xi} + \frac{\partial \eta^{(0)}}{\partial \tau} + \eta^{(0)}\frac{\partial \eta^{(0)}}{\partial \xi}\right).$$

The expression for $w^{(1)}$ thus becomes

$$\begin{aligned} w^{(1)} = \frac{1}{3}&\left(z^3 + 3z^2 - 3W_\mathrm{e}(2z-1) - 2\right)\frac{\partial^3 \eta^{(0)}}{\partial \xi^3} \\ &- \left(\frac{\partial \eta^{(1)}}{\partial \xi} + \frac{\partial \eta^{(0)}}{\partial \tau} + \eta^{(0)}\frac{\partial \eta^{(0)}}{\partial \xi}\right)(z+1). \end{aligned} \tag{5.41}$$

Using this in equation (5.36), at the surface $z = 0$, yields

$$\begin{aligned} \left(W_\mathrm{e} - \frac{2}{3}\right)\frac{\partial^3 \eta^{(0)}}{\partial \xi^3} - \frac{\partial \eta^{(1)}}{\partial \xi} - \frac{\partial \eta^{(0)}}{\partial \tau} - \eta^{(0)}\frac{\partial \eta^{(0)}}{\partial \xi} + \eta^{(0)}\frac{\partial w^{(0)}}{\partial z} \\ = -\frac{\partial \eta^{(1)}}{\partial \xi} + \frac{\partial \eta^{(0)}}{\partial \tau} + u^{(0)}\frac{\partial \eta^{(0)}}{\partial \xi}, \end{aligned}$$

which, by using $\eta^{(0)}_\xi = -w^{(0)}_z$ (from the conservation of mass), $u^{(0)} = \eta^{(0)}$ and rearranging terms becomes

$$2\frac{\partial \eta^{(0)}}{\partial \tau} + 3\eta^{(0)}\frac{\partial \eta^{(0)}}{\partial \xi} + \left(\frac{2}{3} - W_\mathrm{e}\right)\frac{\partial^3 \eta^{(0)}}{\partial \xi^3} = 0. \tag{5.42}$$

This is the Korteweg-de Vries equation which describes the leading-order contribution to the surface wave. Note that $\eta^{(1)}$ is unknown at this order.

Remarks

As pointed out earlier, the KdV equation was first derived (or at least published) in 1895, but it did not immediately obtain much attention; It was believed to be just one more equation, among several other, describing water waves. First sixty years later, in 1955, after the famous numerical experiment by Fermi, Pasta and Ulam (the FPU-experiment [29]) did the KdV equation mark itself as an equation of more importance than first assumed. Ten years after the FPU-experiment Zabusky and Kruskal [68] studied the FPU-problem from a continuum perspective and discovered not only that the KdV equation was the limiting equation but also that solitary waves (which they named "solitons") dominate the asymptotic solution of the KdV equation. These discoveries intensified the studies of the KdV equation and its solutions and as a consequence in the following years several new mathematical tools were developed, among these the Inverse Scattering Transform, Hirota's method, the Miura transformation and the Lax equation.

An important property of the KdV equation is that it is completely integrable, as shown by Miura, Gardner and Kruskal in [57]. The integrability property was also confirmed again the next year when Lax derived the Lax equation (Lax [51])

$$\frac{\partial}{\partial t}L = [L, A] = LA - AL$$

and found that writing (in this case L is the Sturm-Liouville operator)

$$L = -\frac{\partial^2}{\partial x^2} + \eta, \quad A = 4\frac{\partial^3}{\partial x^3} - 3\left(\eta\frac{\partial}{\partial x} + \frac{\partial \eta}{\partial x}\right),$$

the KdV equation also had an infinite of conserved quantities (cf. the paper of Lax [51] or the book of Babelon *et al.* [3, chapter 11]).

The exclusion of surface tension results only in a minor modification in the derived KdV equation. The modification will be in the dynamic surface condition at first order which then becomes (see equation (5.30))

$$p^{(1)} + \eta^{(0)}\frac{\partial p^{(0)}}{\partial z} = \eta^{(1)}$$

and the pressure at first order instead becomes (equation (5.40))

$$p^{(1)} = -\frac{1}{2}\left(z^2 + 2z\right)\frac{\partial^2 \eta^{(0)}}{\partial \xi^2} + \eta^{(1)}$$

Following exactly the same procedure we see that the KdV equation with surface tension excluded reads

$$2\frac{\partial \eta^{(0)}}{\partial \tau} + \eta^{(0)}\frac{\partial \eta^{(0)}}{\partial \xi} + \frac{3}{2}\frac{\partial^3 \eta^{(0)}}{\partial \xi^3} = 0,$$

and note that only the third derivative term has been modified.

A special solution of the KdV equation is the soliton, a solitary wave that keeps its shape over time and can interact (nonlineary) with other waves and still retain its original shape afterwards. The soliton solution of the KdV equation is easily found by first assuming that we initially have a fixed wave form given by $f(x)$ and then only look at right propagating waves (with speed c, say). We can thus write $\eta(x,t) = f(x-ct)$ and define $\xi = x - ct$. Applying this on the KdV equation (where we have neglected the surface tension for simplicity) it reduces to the ordinary differential equation

$$-2c\frac{\mathrm{d}f}{\mathrm{d}\xi} + \frac{3}{2}\frac{\mathrm{d}f^2}{\mathrm{d}\xi} + \frac{2}{3}\frac{\mathrm{d}^3 f}{\mathrm{d}\xi^3} = 0,$$

which can be integrated once with respect to ξ

$$\frac{2}{3}\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} = 2cf - \frac{3}{2}f^2 + C_1,$$

where C_1 is a constant of integration. This can be integrated once more, after multiplying throughout with f_ξ, which yields

$$\frac{1}{3}\left(\frac{\mathrm{d}f}{\mathrm{d}\xi}\right)^2 = cf^2 - \frac{1}{2}f^3 + C_1 f + C_2,$$

where C_2 is another constant of integration. We now impose the following (boundary) conditions: f, f_x and $f_{xx} \to 0$ as $|x| \to \infty$ (which is equivalent to saying that the wave form is localised in one region only). If we had chosen periodic boundary conditions in x, we would obtain cnoidal solutions instead (cf. Dingemans [24]). These conditions yield $C_1 = C_2 = 0$ and

$$2\left(\frac{\mathrm{d}f}{\mathrm{d}\xi}\right)^2 = f^2\left(2c - f\right),$$

where we see that a real solution only exists if $2c - f > 0$. We introduce a change of variables by

$$q = \sqrt{\frac{2c}{f}} \quad \Rightarrow \quad \frac{\mathrm{d}q}{\mathrm{d}x} = \frac{\mathrm{d}q}{\mathrm{d}f}\frac{\mathrm{d}f}{\mathrm{d}x} = -\frac{1}{2}\sqrt{2c}f^{-3/2}\frac{\mathrm{d}f}{\mathrm{d}x}.$$

which inserted into the former equation becomes

$$2\left(-\frac{2}{\sqrt{2c}}f^{3/2}\frac{\mathrm{d}q}{\mathrm{d}\xi}\right)^2 = f^2\left(2c - f\right),$$

or by some minor manipulations

$$\frac{4}{c}\left(\frac{\mathrm{d}q}{\mathrm{d}\xi}\right)^2 = q^2 - 1.$$

By taking the square root on both sides and then rearrange terms, this can be integrated as

$$\sqrt{\frac{4}{c}}\int \frac{1}{\sqrt{q^2-1}}\,\mathrm{d}q = \pm\int \mathrm{d}\xi,$$

which is equal to

$$\operatorname{arccosh} q = \pm\sqrt{\frac{c}{4}}\xi + C_3,$$

where C_3 is another (arbitrary) constant of integration. Solving first for q, back substituting both for f and $x - ct$ and define $x_0 = C_3$ we thus have

$$f(x-ct) = 2c\,\mathrm{sech}^2\left[\frac{\sqrt{c}}{2}(x-ct) + x_0\right].$$

This is the solitary wave (soliton) solution of the KdV equation. Note that the last integration constant x_0 only determines the position of the initial peak (and thus plays no great role here) and that the solution exists for all c.

Other forms of the KdV equation have been derived (but not necessarily in the asymptotic approach we have used here) and among these we mention the generalized KdV equation (Boyd [9])

$$\frac{\partial u}{\partial t} + u\frac{\partial^2 u}{\partial x^2} - \frac{\partial^5 u}{\partial x^5} = 0,$$

the so-called modified KdV equation (Calogero and Degasperis [11])

$$\frac{\partial u}{\partial t} \pm 6u^2\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

and the nearly concentric KdV equation (or Johnson's equation as it first appeared in a paper by Johnson [39] and also see Johnson [41]p. 214-216)

$$\frac{\partial}{\partial x}\left(2\frac{\partial H}{\partial R} + \frac{1}{R}H + 3H\frac{\partial H}{\partial x} + \frac{1}{3}\frac{\partial^3 H}{\partial x^3}\right) + \frac{1}{R^2}\frac{\partial^2 H}{\partial \Theta^2} = 0.$$

In the next section we derive a similar equation to the KdV equation, but then including the y-dependency.

5.2.2 Kadomtsev-Petviashvili equation

In this section we relax the restrictions on the KdV equation a bit. We still require the waves to travel in the (say) positive x-direction, but we include an additional spacial dimension. The dependency on the second horizontal variable y, however, will be weakly compared to the x-direction. This will lead to the Kadomtsev-Petviashvili (KP) equation. It was originally derived by Kadomtsev and Petviashvili [45] (and later a more formal consistent derivation by Freeman and Davey in [32]). It has also been shown that the KP equation arises is as a model for sound waves in ferromagnetic media (Turitsyn *et al.* [65]).

The scaling and transformations introduced here are similar to the KdV derivation

$$\xi = x - t, \quad \tau = \epsilon\tau, \quad Y = \sqrt{\epsilon}y, \quad V = \frac{1}{\sqrt{\epsilon}}v,$$

where the scaling of the second horizontal velocity component is due to consistency with the conservation of mass equation. This also leads to the same transformation of the differential operators as in (5.27). The equations are scaled to replace δ^2 by ϵ

$$x \to \frac{\delta}{\epsilon^{1/2}}x, \quad y \to \frac{\delta}{\epsilon^{1/2}}y, \quad t \to \frac{\delta}{\epsilon^{1/2}}t,$$

as was done for the KdV equation. Similar to earlier we assume a horizontal bottom, $h(x, y) = h_0$. We will, however, for simplicity neglect the surface tension in this derivation which means the governing equations become

$$-\frac{\partial p}{\partial \xi} = -\frac{\partial u}{\partial \xi} + \epsilon\left(\frac{\partial u}{\partial \tau} + u\frac{\partial u}{\partial \xi} + \epsilon V\frac{\partial u}{\partial Y} + w\frac{\partial u}{\partial z}\right) \tag{5.43}$$

$$-\frac{\partial p}{\partial Y} = -\frac{\partial V}{\partial \xi} + \epsilon\left(\frac{\partial V}{\partial \tau} + u\frac{\partial V}{\partial \xi} + \epsilon V\frac{\partial V}{\partial Y} + w\frac{\partial V}{\partial z}\right) \tag{5.44}$$

$$-\frac{\partial p}{\partial z} = \epsilon\left[-\frac{\partial w}{\partial \xi} + \epsilon\left(\frac{\partial w}{\partial \tau} + u\frac{\partial w}{\partial \xi} + \epsilon V\frac{\partial w}{\partial Y} + w\frac{\partial w}{\partial z}\right)\right] \tag{5.45}$$

$$\frac{\partial u}{\partial \xi} + \epsilon\frac{\partial V}{\partial Y} + \frac{\partial w}{\partial z} = 0, \tag{5.46}$$

with the free surface boundary conditions (on $z = \epsilon\eta(\xi, Y, \tau)$)

$$p = \eta \tag{5.47}$$

$$w = -\frac{\partial \eta}{\partial \xi} + \epsilon\left(\frac{\partial \eta}{\partial \tau} + u\frac{\partial \eta}{\partial \xi} + \epsilon V\frac{\partial \eta}{\partial Y}\right) \tag{5.48}$$

and the bottom condition (on $z = -1$)

$$w = 0. \tag{5.49}$$

Again we note that δ^2 has been replaced by ϵ and that the second horizontal velocity terms are all of order ϵ.

The solutions we seek are in the same manner as earlier in terms of asymptotic expansion for $\epsilon \to 0$, keeping δ fixed. The leading order terms (5.16) will remain mainly unchanged after the scaling and transformations (except they may now also depended on Y, in addition to ξ and τ)

$$\begin{aligned} p^{(0)} &= \eta^{(0)}, \quad u^{(0)} = \eta^{(0)}, \\ \frac{\partial V^{(0)}}{\partial \xi} &= \frac{\partial \eta^{(0)}}{\partial Y}, \quad w^{(0)} = -(z+1)\frac{\partial \eta^{(0)}}{\partial \xi}, \end{aligned} \tag{5.50}$$

all valid in the domain $-1 \leq z \leq 0$. Note how the vertical velocity component at this order $w^{(0)}$ is independent of $V^{(0)}$ and that we also obtain relations like

$$\frac{\partial V^{(0)}}{\partial \xi} = \frac{\partial p^{(0)}}{\partial Y} = \frac{\partial u^{(0)}}{\partial Y}, \tag{5.51}$$

which are needed when we later determine $w^{(1)}$.

For the first order equations (5.20) - (5.23) we obtain

$$-\frac{\partial p^{(1)}}{\partial \xi} = -\frac{\partial u^{(1)}}{\partial \xi} + \frac{\partial u^{(0)}}{\partial \tau} + u^{(0)}\frac{\partial u^{(0)}}{\partial \xi} + w^{(0)}\frac{\partial u^{(0)}}{\partial z} \tag{5.52}$$

$$-\frac{\partial p^{(1)}}{\partial Y} = -\frac{\partial V^{(1)}}{\partial \xi} + \frac{\partial V^{(0)}}{\partial \tau} + u^{(0)}\frac{\partial V^{(0)}}{\partial \xi} + w^{(0)}\frac{\partial V^{(0)}}{\partial z} \tag{5.53}$$

$$-\frac{\partial p^{(1)}}{\partial z} = -\frac{\partial w^{(0)}}{\partial \xi} \tag{5.54}$$

$$\frac{\partial u^{(1)}}{\partial \xi} + \frac{\partial V^{(0)}}{\partial Y} + \frac{\partial w^{(1)}}{\partial z} = 0, \tag{5.55}$$

with the free surface boundary conditions (on $z = 0$)

$$p^{(1)} + \eta^{(0)}\frac{\partial p^{(0)}}{\partial z} = \eta^{(1)} \tag{5.56}$$

$$w^{(1)} + \eta^{(0)}\frac{\partial w^{(0)}}{\partial z} = -\frac{\partial \eta^{(1)}}{\partial \xi} + \frac{\partial \eta^{(0)}}{\partial \tau} + u^{(0)}\frac{\partial \eta^{(0)}}{\partial \xi} \tag{5.57}$$

and the bottom condition (on $z = -1$)

$$w = 0, \tag{5.58}$$

where we again have Taylor expanded boundary conditions around the free surface at rest (see equations (5.30) and (5.31) for the KdV equation).

From the leading order equations (5.50) we will obtain the same results as in the KdV derivation, equations (5.38) - (5.40). That is, we obtain

$$\begin{aligned} \frac{\partial u^{(0)}}{\partial z} &= \frac{\partial p^{(0)}}{\partial z} = 0, \\ \frac{\partial w^{(0)}}{\partial \xi} &= -(z+1)\frac{\partial^2 \eta^{(0)}}{\partial \xi^2}, \\ p^{(1)} &= -\frac{1}{2}\left(z^2 + 2z\right)\frac{\partial^2 \eta^{(0)}}{\partial \xi^2} + \eta^{(1)}. \end{aligned}$$

From the last equation we have

$$\frac{\partial p^{(1)}}{\partial \xi} = -\frac{1}{2}\left(z^2 + 2z\right)\frac{\partial^3 \eta^{(0)}}{\partial \xi^3} + \frac{\partial \eta^{(1)}}{\partial \xi}, \tag{5.59}$$

which we will need when we determine the expression for $w^{(1)}$. The conservation of mass (5.55) together with both of the horizontal components of Euler's equation (5.52) - (5.53) and then using (5.39), (5.51) and (5.59) yields

$$\begin{aligned}
\frac{\partial w^{(1)}}{\partial z} &= -\frac{\partial u^{(1)}}{\partial \xi} - \frac{\partial V^{(0)}}{\partial Y} \\
&= -\frac{\partial p^{(1)}}{\partial \xi} - \frac{\partial u^{(0)}}{\partial \tau} - u^{(0)}\frac{\partial u^{(0)}}{\partial \xi} - w^{(0)}\frac{\partial u^{(0)}}{\partial z} - \frac{\partial V^{(0)}}{\partial Y} \\
&= \frac{1}{2}\left(z^2 + 2z\right)\frac{\partial^3 \eta^{(0)}}{\partial \xi^3} - \frac{\partial \eta^{(1)}}{\partial \xi} - \frac{\partial u^{(0)}}{\partial \tau} - u^{(0)}\frac{\partial u^{(0)}}{\partial \xi} - \frac{\partial V^{(0)}}{\partial Y} \\
&= \frac{1}{2}\left(z^2 + 2z\right)\frac{\partial^3 \eta^{(0)}}{\partial \xi^3} - \frac{\partial \eta^{(1)}}{\partial \xi} - \frac{\partial \eta^{(0)}}{\partial \tau} - \eta^{(0)}\frac{\partial \eta^{(0)}}{\partial \xi} - \frac{\partial V^{(0)}}{\partial Y}
\end{aligned}$$

This can be integrated once with respect to z

$$\begin{aligned}
w^{(1)} = &\frac{1}{3}\left(z^3 + 3z^2\right)\frac{\partial^3 \eta^{(0)}}{\partial \xi^3} \\
&- \left(\frac{\partial \eta^{(1)}}{\partial \xi} + \frac{\partial \eta^{(0)}}{\partial \tau} + \eta^{(0)}\frac{\partial \eta^{(0)}}{\partial \xi} + \frac{\partial V^{(0)}}{\partial Y}\right) z + C(\xi, Y, \tau),
\end{aligned}$$

where we can find $C(\xi, Y, \tau)$ from the bottom boundary condition (5.58), which gives

$$\begin{aligned}
w^{(1)} = &\frac{1}{3}\left(z^3 + 3z^2 - 2\right)\frac{\partial^3 \eta^{(0)}}{\partial \xi^3} \\
&- \left(\frac{\partial \eta^{(1)}}{\partial \xi} + \frac{\partial \eta^{(0)}}{\partial \tau} + \eta^{(0)}\frac{\partial \eta^{(0)}}{\partial \xi} + \frac{\partial V^{(0)}}{\partial Y}\right)(z + 1).
\end{aligned}$$

At the surface $z = 0$ we have from the kinematic condition (5.57)

$$\begin{aligned}
-\frac{\partial \eta^{(1)}}{\partial \xi} + \frac{\partial \eta^{(0)}}{\partial \tau} + u^{(0)}\frac{\partial \eta^{(0)}}{\partial \xi} = &-\frac{2}{3}\frac{\partial^3 \eta^{(0)}}{\partial \xi^3} - \frac{\partial \eta^{(1)}}{\partial \xi} \\
&- \frac{\partial \eta^{(0)}}{\partial \tau} - \eta^{(0)}\frac{\partial \eta^{(0)}}{\partial \xi} - \frac{\partial V^{(0)}}{\partial Y} + \eta^{(0)}\frac{\partial w^{(0)}}{\partial z}
\end{aligned}$$

with using $u^{(0)} = \eta^{(0)}$, $w_z^{(0)} = -\eta_\xi^{(0)}$ and rearranging terms becomes

$$2\frac{\partial \eta^{(0)}}{\partial \tau} + 3\eta^{(0)}\frac{\partial \eta^{(0)}}{\partial \xi} + \frac{2}{3}\frac{\partial^3 \eta^{(0)}}{\partial \xi^3} + \frac{\partial V^{(0)}}{\partial Y} = 0.$$

Differentiating this equation once with respect to ξ and using $V_\xi^{(0)} = \eta_Y^{(0)}$ we obtain

$$\frac{\partial}{\partial \xi}\left(2\frac{\partial \eta^{(0)}}{\partial \tau} + 3\eta^{(0)}\frac{\partial \eta^{(0)}}{\partial \xi} + \frac{2}{3}\frac{\partial^3 \eta^{(0)}}{\partial \xi^3}\right) + \frac{\partial^2 \eta^{(0)}}{\partial Y^2} = 0. \tag{5.60}$$

This equation describes the surface elevation $\eta^{(0)}$ in two dimensions and is often referred to as the Kadomtsev-Pitviashvili (KP) equation[4], or as the two-dimensional Korteweg-de Vries equation (2D KdV). Similar as for the KdV equation, the KP equation is valid only at leading order and $\eta^{(1)}$ is unknown.

Remarks

Firstly we note how the one-dimensional KdV equation is retrieved if the dependency on Y should be omitted and equation (5.60) then reduces to the KdV equation (5.42) derived in the previous section, with slightly different coefficients[5].

It is common to refer to equation (5.60) as the KP-II equation, because of the sign of the η_{YY} term. If this term should be negative, the equation is referred to as KP-I. The latter case is usually used when the surface tension is strong (cf. Debnath [20] and references therein).

The one-soliton (more precisely the line or plane soliton) solution of the KP equation can be found in a similar manner as for the KdV equation. Firstly assume that we have travelling wave described by $\eta(x, y, t) = f(\mathbf{k} \cdot \mathbf{x} - ct)$, where $\mathbf{k} = \{k_1, k_2\} = \{a, -ab\}$ and $\mathbf{x} = \{x, y\}$. This yields the soliton solution (cf. Infeld & Rowlands [36, chapter 8.3.1], Biondini & Chakravarty [6])

$$\eta(x, y, t) = \frac{1}{2}a^2 \operatorname{sech}^2\left[\frac{1}{2}a(x - by - \frac{ct}{a} - x_0)\right]$$

For other explicit solutions of the KP equation (both I and II) we refer to Alexander *et al.* [2], Foka & Sung [31], Zhou [71].

5.2.3 Boussinesq equation

In the previous sections we derived water wave equations where we a priori assumed the solutions to propagate in one predefined direction. We relax this assumption and include the possibility of waves travelling in both directions and will so arrive at the Boussinesq equation. For the sake of simplicity we will neglect any y-dependencies, the surface tension $/W_e \approx 0)$ and assume a

[4] Actually more common would be the KP-I equation, see under Remarks.

[5] This is of no importance as a simple scaling would lead to any desirable coefficients.

horizontal bottom $h(x, y) = h_0$. See Dingemans [24, sections 5.4.2 - 5.4.3] for details on the two-dimensional Boussinesq equation (also see Johnson [40]) and the effects of inclusion of surface tension. The Boussinesq equations in their original form are derived in Appendix C, where one also has to make the assumption of irrotationality.

We rescale the variables by (these are in fact the same as the KdV and KP scaling in the previous sections)

$$x \to \frac{\delta}{\epsilon^{1/2}} x, \quad t \to \frac{\delta}{\epsilon^{1/2}} t$$

and to be consistent with the equation of mass conservation (see equation (5.21)) we also rescale w by

$$w \to \frac{\epsilon^{1/2}}{\delta} w.$$

The governing equations will therefore become the same as in the case of the KdV equation

$$\frac{\partial u}{\partial t} + \epsilon \left(u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} \tag{5.61}$$

$$\epsilon \left[\frac{\partial w}{\partial t} + \epsilon \left(u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \right) \right] = -\frac{\partial p}{\partial z} \tag{5.62}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{5.63}$$

$$\tag{5.64}$$

with on the free surface $z = \epsilon \eta(x, t)$

$$\eta = p \tag{5.65}$$

$$w = \frac{\partial \eta}{\partial t} + \epsilon u \frac{\partial \eta}{\partial x} \tag{5.66}$$

and on the horizontal bottom $z = -1$

$$w = 0. \tag{5.67}$$

At leading order this also yields the same equations as derived earlier

$$\begin{aligned} p^{(0)} &= \eta^{(0)}, \quad & \frac{\partial u^{(0)}}{\partial t} &= -\frac{\partial \eta^{(0)}}{\partial x}, \\ w^{(0)} &= -(z+1)\frac{\partial u^{(0)}}{\partial x}, \quad & \frac{\partial u^{(0)}}{\partial x} &= -\frac{\partial \eta^{(0)}}{\partial t}, \end{aligned} \tag{5.68}$$

all valid in the domain $-1 \leq z \leq 0$. Note how this leads to the classical wave equation (also see (5.18))

$$\frac{\partial^2 \eta(0)}{\partial t^2} - \frac{\partial^2 \eta(0)}{\partial x^2} = 0. \tag{5.69}$$

The first order equation for the pressure p is found in the same manner as for (5.40) and thus reads[6]

$$p^{(1)} = -\left(\frac{1}{2}z^2 + z\right)\frac{\partial}{\partial t}\frac{\partial u^{(0)}}{\partial x} + \eta^{(1)},$$

which using in the horizontal component of Euler's equation, equation (5.61), yields

$$\begin{aligned}\frac{\partial u^{(1)}}{\partial t} + u^{(0)}\frac{\partial u^{(0)}}{\partial x} + w^{(0)}\frac{\partial u^{(0)}}{\partial z} &= -\frac{\partial}{\partial x}\left(-\left(\frac{1}{2}z^2 + z\right)\frac{\partial}{\partial t}\frac{\partial u^{(0)}}{\partial x} + \eta^{(1)}\right)\\ &= \left(\frac{1}{2}z^2 + z\right)\frac{\partial}{\partial t}\frac{\partial^2 \eta^{(0)}}{\partial x^2} - \frac{\partial \eta^{(1)}}{\partial x}.\end{aligned}$$

If we differentiate this with respect to x

$$\frac{\partial}{\partial x}\frac{\partial u^{(1)}}{\partial t} = \left(\frac{1}{2}z^2 + z\right)\frac{\partial}{\partial t}\frac{\partial^3 u^{(0)}}{\partial x^3} - \frac{\partial^2 \eta^{(1)}}{\partial x^2} - \frac{\partial}{\partial x}\left(u^{(0)}\frac{\partial u^{(0)}}{\partial x}\right)$$

and then use it in the equation for conservation of mass, (5.63),

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial w^{(1)}}{\partial z} &= -\frac{\partial}{\partial t}\frac{\partial u^{(1)}}{\partial x}\\ &= -\left(\frac{1}{2}z^2 + z\right)\frac{\partial}{\partial t}\frac{\partial^3 \eta^{(0)}}{\partial x^3} + \frac{\partial^2 \eta^{(1)}}{\partial x^2} + \frac{\partial}{\partial x}\left(\eta^{(0)}\frac{\partial \eta^{(0)}}{\partial x}\right),\end{aligned}$$

we obtain after integrating with respect to z

$$\frac{\partial w^{(1)}}{\partial t} = -\left(\frac{1}{6}z^3 + \frac{1}{2}z^2\right)\frac{\partial}{\partial t}\frac{\partial^3 u^{(0)}}{\partial x^3} + \frac{\partial^2 \eta^{(1)}}{\partial x^2}z + \frac{\partial}{\partial x}\left(u^{(0)}\frac{\partial u^{(0)}}{\partial x}\right)z + C(x,t),$$

where $C(x,t)$ is a constant of integration and is determined by the bottom boundary condition (5.67) ($z = -1$). Thus

$$C(x,t) = \frac{1}{3}\frac{\partial}{\partial t}\frac{\partial^3 u^{(0)}}{\partial x^3} + \frac{\partial^2 \eta^{(1)}}{\partial x^2} + \frac{\partial}{\partial x}\left(u^{(0)}\frac{\partial u^{(0)}}{\partial x}\right),$$

which yields

$$\begin{aligned}\frac{\partial w^{(1)}}{\partial t} = &-\left(\frac{1}{6}z^3 + \frac{1}{2}z^2 - \frac{1}{3}\right)\frac{\partial}{\partial t}\frac{\partial^3 u^{(0)}}{\partial x^3}\\ &+ (z+1)\frac{\partial^2 \eta^{(1)}}{\partial x^2} + (z+1)\frac{\partial}{\partial x}\left(u^{(0)}\frac{\partial u^{(0)}}{\partial x}\right).\end{aligned} \tag{5.70}$$

[6]Note how we in this case cannot replace $u^{(0)}$ with $\eta^{(0)}$ and do not have a second derivative in x as for the KdV equation.

Before continuing we need to Taylor expand the boundary conditions, so that we can evaluate them on $z = 0$ rather than $z = \epsilon\eta^{(0)}$, as was done with the KdV equation. Thus (also see equation (5.31) and its derivation) the kinematic free surface condition (5.22) becomes at this order

$$w^{(1)} + \eta^{(0)}\frac{\partial w^{(0)}}{\partial z} = \frac{\partial \eta^{(1)}}{\partial t} + u^{(0)}\frac{\partial \eta^{(0)}}{\partial x}.$$

Differentiating with respect to t and then using (5.70) to substitute for $w_t^{(1)}$ (remember $z = 0$)

$$-\frac{1}{3}\frac{\partial}{\partial t}\frac{\partial^3 u^{(0)}}{\partial x^3} + \frac{\partial^2 \eta^{(1)}}{\partial x^2} + \frac{\partial}{\partial x}\left(u^{(0)}\frac{\partial u^{(0)}}{\partial x}\right) + \frac{\partial}{\partial t}\left(\eta^{(0)}\frac{\partial w^{(0)}}{\partial z}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \eta^{(1)}}{\partial t} + u^{(0)}\frac{\partial \eta^{(0)}}{\partial x}\right)$$

and we obtain

$$\frac{\partial^2 \eta^{(1)}}{\partial t^2} - \frac{\partial^2 \eta^{(1)}}{\partial x^2} + \frac{\partial}{\partial t}\left(u^{(0)}\frac{\partial \eta^{(0)}}{\partial x}\right) - \frac{\partial}{\partial x}\left(u^{(0)}\frac{\partial u^{(0)}}{\partial x}\right) - \frac{\partial}{\partial t}\left(\eta^{(0)}\frac{\partial w^{(0)}}{\partial z}\right) + \frac{1}{3}\frac{\partial^3}{\partial x^3}\frac{\partial u^{(0)}}{\partial t} = 0.$$

Expanding the parenthesise and using the relations we found at leading order (5.68) yields

$$\frac{\partial^2 \eta^{(1)}}{\partial t^2} - \frac{\partial^2 \eta^{(1)}}{\partial x^2} - \frac{\partial \eta^{(0)}}{\partial x}\frac{\partial \eta^{(0)}}{\partial x} - u^{(0)}\frac{\partial^2 u^{(0)}}{\partial x^2} - \frac{\partial u^{(0)}}{\partial x}\frac{\partial u^{(0)}}{\partial x} - u^{(0)}\frac{\partial^2 u^{(0)}}{\partial x^2} - \frac{\partial u^{(0)}}{\partial x}\frac{\partial u^{(0)}}{\partial x} - \eta^{(0)}\frac{\partial^2 \eta^{(0)}}{\partial x^2} - \frac{1}{3}\frac{\partial^4 \eta^{(0)}}{\partial x^4} = 0,$$

which by some simple manipulation (mainly by use of the chain rule) can be rewritten as

$$\frac{\partial^2 \eta^{(1)}}{\partial t^2} - \frac{\partial^2 \eta^{(1)}}{\partial x^2} - \frac{\partial^2}{\partial x^2}\left[\frac{1}{2}\left(\eta^{(0)}\right)^2 + \left(u^{(0)}\right)^2\right] - \frac{1}{3}\frac{\partial^4 \eta^{(0)}}{\partial x^4} = 0.$$

Thus at the current order, $O(\epsilon)$, we have for $\eta \sim \eta^0 + \epsilon\eta^{(1)}$

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 \eta}{\partial x^2} - \epsilon\frac{\partial^2}{\partial x^2}\left[\frac{1}{2}\eta^2 + \left(\int_{-\infty}^{x}\frac{\partial \eta}{\partial t}\,\mathrm{d}x\right)^2\right] - \frac{\epsilon}{3}\frac{\partial^4 \eta}{\partial x^4} = O\left(\epsilon^2\right), \qquad (5.71)$$

where we used equation (5.68) to define

$$u^{(0)} = -\int_{-\infty}^{x}\frac{\partial \eta^{(0)}}{\partial t}\,\mathrm{d}x,$$

with the condition of $u^{(0)} \to 0$ as $x \to -\infty$. Equation (5.71) is the Boussinesq equation and describes the surface elevation for waves propagating in both directions..

With some additional scaling and transformations, see Appendix D, this can be transformed to the more conventional equation

$$\frac{\partial^2 H}{\partial t^2} - \frac{\partial^2 H}{\partial X^2} - 3\frac{\partial^2}{\partial X^2}(H)^2 - \frac{\partial^4 H}{\partial X^4} = 0. \tag{5.72}$$

Note however that by all the transformations and scalings the connection to the original variables has now become quite obscured.

Remarks

Firstly we note the Boussinesq approximation here has terms of both leading order and first order (readily seen from equation (5.71)), which means we first have to solve the leading order equation (5.69) (with given initial conditions) before solving (5.71).

The soliton solutions of the Boussinesq equation can be obtained in a similar manner as for the KdV equation. As mentioned earlier the Boussinesq equation emits waves in both directions but after a long enough time the waves will have travelled far enough from each other such that we can consider only one of them. We therefore consider only the right running wave and write $\xi = x - ct$, which applied on the Boussinesq equation (5.72) yields the ordinary differential equation

$$\frac{\partial^2}{\partial \xi^2}\left[(c^2-1)f - 3f^2 - \frac{\partial^2 f}{\partial \xi^2}\right] = 0$$

This can be integrated twice

$$(c^2-1)f - 3f^2 - \frac{\partial^2 f}{\partial \xi^2} = C_1\xi + C_2,$$

where C_1 and C_2 are constants of integration. It is readily seen that we must have $C_1 = 0$, otherwise f and f_{xx} would not be bounded for $|\xi| \to \infty$. We can multiply throughout with f_x and then integrate to obtain

$$\frac{1}{2}\left(\frac{\partial f}{\partial \xi}\right)^2 = -f^3 + \frac{1}{2}(c^2-1)f^2 - C_2 f + C_3,$$

where C_3 is another constant of integration. Imposing the boundary conditions f, f_x and $f_{xx} \to 0$ for $|\xi| \to \infty$ give $C_2 = C_3 = 0$. It is also readily seen that real solutions are only obtained if the right hand side is positive, which means $f^2(-2f + c^2 - 1) \geq 0$ (and thus $(-2f + c^2 - 1) \geq 0$). Introducing a new variable by

$$q = \sqrt{\frac{c^2 - 1}{f}}$$

the former equation can be rewritten as

$$4\left(\frac{\mathrm{d}q}{\mathrm{d}\xi}\right)^2 = (c^2 - 1)(q^2 - 1)$$

which can be integrated (after taking the square root on both sides) and we thus obtain

$$\operatorname{arccosh} q = \pm\frac{\sqrt{c^2 - 1}}{2}\xi.$$

Solving first for q and then returning to the original variables yields

$$f(x,t) = (c^2 - 1)\operatorname{sech}^2\left[\frac{\sqrt{c^2 - 1}}{2}(x - ct)\right],$$

which is the soliton solution of the Boussinesq equation. Note the similarity to the KdV-soliton solution in the KdV-section.

The derivation of the Boussinesq equation followed the derivation of the KdV equation closely, except that we in this case did not make any a priori assumptions on the propagation direction. It might therefore not be a big surprise that the KdV equation can be obtained directly from the Boussinesq equation by assuming $\xi = x \pm t$ and $\tau = \epsilon t$ (cf. Dingemans [24], Johnson [41]).

5.3 Second order

In this section we will go yet another order higher in the asymptotic expansions and include second order terms in the calculations. However, we do not state the general equations at this order, as has been done at leading and first order; Stating the general second order equations will not be of any real help, as each of the derivations in this section differ too much from each other.

In the first derivation we make a double asymptotic expansion in both of ϵ and δ (after necessary scaling and transformations), independently of each other and with the assumption of one-directional waves. This leads then to two similar equations, the Camassa-Holm equation and the Degasperis-Procesi equation.

In the derivation of the last equations, we again return to a single asymptotic expansion in ϵ, but now with the additional assumption of a modulated amplitude. In this way we will obtain the nonlinear Schrödinger equation and its two-dimensional sibling, the Davey-Stewartson equations.

5.3.1 Camassa-Holm equation

In aim of this section is to derive the Camassa-Holm (CH) equation. The CH equation was first implied by Fokas and Fuchssteiner in 1981 [34] but did not immediately obtain much attention. First when it later was directly derived in the Hamiltonian for Euler's equation by Camassa and Holm in 1993 [12] (and some additional properties in a following paper, Camassa *et al.* [13]) did it obtain more attention (especially since it was proven to be completely integrable for all values of a parameter κ). The CH equation is derived in one spacial dimension, under the assumption of zero surface tension and a horizontal bottom, and reads

$$\frac{\partial u}{\partial t} + 2\kappa \frac{\partial u}{\partial x} - \frac{\partial}{\partial t}\frac{\partial^2 u}{\partial x^2} + 3u\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + u\frac{\partial^3 u}{\partial x^3}, \tag{5.73}$$

where κ is a constant related to the critical shallow water speed (Camassa *et al.* [13]). The asymptotic derivation in the regime of water waves was made by Johnson in [42]. He later also derived a version where the CH equation is applied on water with an ambient underlying flow (Johnson [43] and also see Ivanov [38]). Furthermore, there exists extensions of the CH equation into higher spacial dimensions. We will not pursue this here, but rather refer to Johnson [42] (which version is unsure if is integrable) or Kraenkel & Zenchuk [49] (which is proven to be integrable).

It has been pointed out the CH equation also is relevant to nonlinear waves in cylindrical hyper-elastic rods (Dai [18], then with $\kappa = 0$) and to the motion of a special non-Newtonian fluid (Busuioc [10]). As far as the author is known, the CH equation has not (yet) emerged in other areas of physics.

A last note before we begin the derivation; It is important to retrieve exactly the same ratio between the coefficients as stated in (5.73), since we would otherwise loose the integrability property of the equation (cf. Dullin *et al.* [27]).

Similar to the KdV and KP equations, we look at one-directional (right-running) waves. We do, however, look at a different region in space and time, determined by the scaling

$$\xi = \sqrt{\epsilon}(x - t), \quad \tau = \epsilon\sqrt{\epsilon}t, \quad W = \frac{1}{\sqrt{\epsilon}}w.$$

The last transformation is to be consistent with the mass conservation equation. The corresponding differential operators are

$$\frac{\partial}{\partial x} = \sqrt{\epsilon}\frac{\partial}{\partial \xi} \quad \text{and} \quad \frac{\partial}{\partial t} = -\sqrt{\epsilon}\frac{\partial}{\partial \xi} + \epsilon\sqrt{\epsilon}\frac{\partial}{\partial \tau},$$

which applied on the governing equations yields

$$\epsilon\frac{\partial u}{\partial \tau} - \frac{\partial u}{\partial \xi} + \epsilon\left(u\frac{\partial u}{\partial \xi} + W\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial \xi} \tag{5.74}$$

$$\epsilon\delta^2\left[-\frac{\partial W}{\partial \xi} + \epsilon\left(\frac{\partial W}{\partial \tau} + u\frac{\partial W}{\partial \xi} + W\frac{\partial W}{\partial z}\right)\right] = -\frac{\partial p}{\partial z} \tag{5.75}$$

$$\frac{\partial u}{\partial \xi} + \frac{\partial W}{\partial z} = 0, \tag{5.76}$$

with on the free surface $z = \epsilon\eta$

$$\eta = p \tag{5.77}$$

$$W = -\frac{\partial \eta}{\partial \xi} + \epsilon\frac{\partial \eta}{\partial \tau} + \epsilon u\frac{\partial \eta}{\partial \xi} \tag{5.78}$$

and on the horizontal bottom $z = -1$

$$W = 0. \tag{5.79}$$

It is possible to rewrite the free surface conditions (5.77) and (5.78) in a similar way as earlier, by expanding them in their Taylor series

$$\eta = \sum_{k=0}^{\infty}(\epsilon\eta)^k\frac{\partial^k p}{\partial z^k} \tag{5.80}$$

$$\sum_{k=0}^{\infty}(\epsilon\eta)^k\frac{\partial^k W}{\partial z^k} = -\frac{\partial \eta}{\partial \xi} + \epsilon\frac{\partial \eta}{\partial \tau} + \epsilon u\frac{\partial \eta}{\partial \xi}, \tag{5.81}$$

which are now evaluated on $z = 0$ instead.

We search for solutions by means of asymptotic expansions, now with both $\epsilon \to 0$ and $\delta \to 0$, independently. That is, we search for solutions as

$$q(\xi, z, \tau; \epsilon, \delta) \sim \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\epsilon^i\delta^{2j}q^{(i,j)}(\xi, z, \tau),$$

$$\eta(\xi, \tau; \epsilon, \delta) \sim \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\epsilon^i\delta^{2j}q^{(i,j)}(\xi, \tau),$$

where q represents any of u, W or p. For example for the expansion in η up to $i = j = 1$, we have

$$\eta \sim \eta^{(0,0)} + \epsilon\eta^{(1,0)} + \delta^2\eta^{(0,1)} + \epsilon\delta^2\eta^{(1,1)}.$$

To ease the notation a bit we will write $\eta^{(0)} = \eta^{(0,0)}$ and similar for the other functions.

At leading order there is no change compared to the earlier derivations and we arrive at (also see the derivation of the single asymptotic expansion at leading order, which leads the the equations in (5.16))

$$p^{(0)} = \eta^{(0)} = u^{(0)}, \quad W^{(0)} = -(z+1)\frac{\partial \eta^{(0)}}{\partial \xi}, \tag{5.82}$$

all valid in the domain $-1 \leq z \leq 0$.

To ease the notation further, we follow[7] Ivanov [38] and collect all first order terms into one term when writing

$$u \sim \eta^{(0)} + E_1(u) \qquad \text{and} \quad W \sim -(z+1)\frac{\partial \eta^{(0)}}{\partial \xi} + E_1(W). \tag{5.83}$$

Here $E_1(u)$ and $E_1(W)$ denote the first order terms in the asymptotic expansion of u and W and might thus include terms of order $O(\epsilon)$, $O\left(\delta^2\right)$ or $O\left(\epsilon\delta^2\right)$. Inserting the first equation in (5.83) into the first component of Euler's equation (equation (5.74)) and then collecting all terms of first order in u (thus terms like $\epsilon E_1(u)$ and $E_1(u)E_1(u)$ are neglected) we obtain

$$\frac{\partial}{\partial \xi} E_1(u) = \epsilon \left(\frac{\partial \eta^{(0)}}{\partial \tau} + \eta^{(0)} \frac{\partial \eta^{(0)}}{\partial \xi} \right), \tag{5.84}$$

which is, at this point, not of much help. We therefore continue by rearranging the terms in the conservation of mass (equation (5.76)) and integrate

$$W = -\int \frac{\partial u}{\partial \xi} \, \mathrm{d}z. \tag{5.85}$$

Substituting with (5.83) on both sides, we then obtain

$$\begin{aligned} -(z+1)\frac{\partial \eta^{(0)}}{\partial \xi} + E_1(W) &= -\int \frac{\partial}{\partial \xi}\left(\eta^{(0)} + E_1(u)\right) \mathrm{d}z \\ &= -(z+1)\frac{\partial \eta^{(0)}}{\partial \xi} - \int \frac{\partial}{\partial \xi} E_1(u) \, \mathrm{d}z \\ &= -(z+1)\frac{\partial \eta^{(0)}}{\partial \xi} - \int \epsilon \left(\frac{\partial \eta^{(0)}}{\partial \tau} + \eta^{(0)} \frac{\partial \eta^{(0)}}{\partial \xi} \right) \mathrm{d}z, \\ &= -(z+1)\frac{\partial \eta^{(0)}}{\partial \xi} - \epsilon(z+1)\left(\frac{\partial \eta^{(0)}}{\partial \tau} + \eta^{(0)} \frac{\partial \eta^{(0)}}{\partial \xi} \right) \end{aligned}$$

[7] Note that Ivanov's notation can be confusing as he does not explicitly show what order in the expansion we are working with, i.e. he writes η where we have $\eta^{(0)}$ and similar for the higher orders.

where we have used the bottom boundary condition (5.79) in the first and third step and the equation we found for $E_1(u)_\xi$ (5.84) in the second step. Thus

$$E_1(W) = -\epsilon(z+1)\left(\frac{\partial\eta^{(0)}}{\partial\tau} + \eta^{(0)}\frac{\partial\eta^{(0)}}{\partial\xi}\right), \tag{5.86}$$

which means

$$W \sim -(z+1)\left(\frac{\partial\eta^{(0)}}{\partial\xi} + \epsilon\frac{\partial\eta^{(0)}}{\partial\tau} + \epsilon\eta^{(0)}\frac{\partial\eta^{(0)}}{\partial\xi}\right). \tag{5.87}$$

This can then be used in the kinematic free surface condition (5.78), on $z = \epsilon\eta^{(0)}$, to obtain[8]

$$\begin{aligned}&-(\epsilon\eta^{(0)}+1)\left(\frac{\partial\eta^{(0)}}{\partial\xi} + \epsilon\frac{\partial\eta^{(0)}}{\partial\tau} + \epsilon u^{(0)}\frac{\partial\eta^{(0)}}{\partial\xi}\right)\\ &\quad= -\frac{\partial\eta^{(0)}}{\partial\xi} + \epsilon\frac{\partial\eta^{(0)}}{\partial\tau} + \epsilon u^{(0)}\frac{\partial\eta^{(0)}}{\partial\xi} + O\left(\epsilon^2, \epsilon\delta^2, \delta^4\right)\end{aligned}$$

which means at leading order we have an equation for $\eta^{(0)}$

$$\frac{\partial\eta^{(0)}}{\partial\tau} = -\frac{3}{2}\eta^{(0)}\frac{\partial\eta^{(0)}}{\partial\xi} + O\left(\epsilon, \delta^2\right). \tag{5.88}$$

Substituting this into the expression we found for $E_1(u)$, i.e. equation (5.84), yields

$$\begin{aligned}\frac{\partial}{\partial\xi}E_1(u) &= \epsilon\left(\frac{\partial\eta^{(0)}}{\partial\tau} + \eta^{(0)}\frac{\partial\eta^{(0)}}{\partial\xi}\right)\\ &= \epsilon\left(-\frac{3}{2}\eta^{(0)}\frac{\partial\eta^{(0)}}{\partial\xi} + \eta^{(0)}\frac{\partial\eta^{(0)}}{\partial\xi} + O\left(\epsilon, \delta^2\right)\right)\\ &= -\frac{\epsilon}{4}\frac{\partial}{\partial\xi}(\eta^{(0)})^2 + O\left(\epsilon^2, \epsilon\delta^2\right)\end{aligned}$$

which can, if we assume $\eta^{(0)} = 0$ for $|\xi| \to \infty$, be integrated to obtain

$$E_1(u) = -\frac{\epsilon}{4}(\eta^{(0)})^2.$$

Note how $E_1(u)$ only depends on ϵ and no δ^2 is present here. We thus obtain (from (5.83))

$$u \sim \eta^{(0)} - \frac{\epsilon}{4}(\eta^{(0)})^2$$

[8] We could correspondingly used the modified surface condition (5.81) by expanding up to the same order, i.e. $k = 1$, and we would then have evaluated on $z = 0$ instead.

and similar for W

$$W \sim -(z+1)\left(\frac{\partial \eta^{(0)}}{\partial \xi} - \frac{\epsilon}{2}\eta^{(0)}\frac{\partial \eta^{(0)}}{\partial \xi}\right).$$

Having found the first order equations for u and W, we continue searching for a first order expression for p. We write

$$p \sim p^{(0)} + E_1(p),$$

where $E_1(p)$ might have terms of order $O(\epsilon)$, $O(\delta^2)$ and $O(\epsilon\delta^2)$. The second component of Euler's equation, equation (5.75), yields

$$\begin{aligned} -\frac{\partial}{\partial z}\left(p^{(0)} + E_1(p)\right) &= \epsilon\delta^2\left[-\frac{\partial W^{(0)}}{\partial \xi} + O(\epsilon)\right] \\ &= \epsilon\delta^2(z+1)\frac{\partial^2 \eta^{(0)}}{\partial \xi^2} + O\left(\epsilon^2\delta^2\right) \end{aligned}$$

where we have used the second equation in (5.82) for $W^{(0)}$. The first term on the left side is zero due to $p^{(0)} = \eta^{(0)}$ and $\eta_z^{(0)} = 0$. Integrating with respect to z gives

$$\begin{aligned} E_1(p) &= -\int \epsilon\delta^2(z+1)\frac{\partial^2 \eta^{(0)}}{\partial \xi^2}\,\mathrm{d}z + O\left(\epsilon^2\delta^2\right) \\ &= -\epsilon\delta^2(\frac{1}{2}z^2 + z)\frac{\partial^2 \eta^{(0)}}{\partial \xi^2} + C(\xi,\tau) + O\left(\epsilon^2\delta^2\right), \end{aligned}$$

where we find the constant of integration from the modified free surface condition (5.77) and thus have

$$E_1(p) = -\epsilon\delta^2(\frac{1}{2}z^2 + z)\frac{\partial^2 \eta^{(0)}}{\partial \xi^2} + E_1(\eta) + O\left(\epsilon^2\delta^2\right),$$

which means

$$p \sim -\epsilon\delta^2(\frac{1}{2}z^2 + z)\frac{\partial^2 \eta^{(0)}}{\partial \xi^2} + E_1(\eta).$$

Here $E_1(\eta)$ is the first order terms of η (from $\eta \sim \eta^{(0)} + E_1(\eta)$) and might contain terms of $O(\epsilon)$, $O(\delta)$ and $O(\epsilon\delta^2)$. We now have the asymptotic expansions of u, W and p up to first order. Both u and W can be found at this order, as we have an equation for $\eta^{(0)}$. p, however, also depends on $E_1(\eta)$ and cannot be determined completely at this order.

The same procedure can then be repeated to obtain the next order corrections. We will arrive at the following equations (Ivanov [38], Johnson

[42, 44])

$$u \sim \eta - \frac{\epsilon}{4}\eta^2 - \epsilon\delta^2\left(\frac{1}{2}z^2 + z + \frac{1}{6}\right)\frac{\partial^2\eta}{\partial\xi^2} \tag{5.89}$$

$$\begin{aligned} W \sim &-(z+1)\left[\frac{\partial\eta}{\partial\xi} - \frac{\epsilon}{2}\eta\frac{\partial\eta}{\partial\xi}\right] \\ &+ \frac{\epsilon\delta^2}{6}\left(z^3 + 3z^2 + z - 1\right)\frac{\partial^3\eta}{\partial\xi^3} \end{aligned} \tag{5.90}$$

$$p \sim \eta + \epsilon\delta^2\left(\frac{1}{2}z^2 + z\right)\frac{\partial^2\eta}{\partial\xi^2}, \tag{5.91}$$

where $\eta(\xi,\tau) \sim \eta^{(0)} + \epsilon\eta^{(1,0)} + \delta^2\eta^{(0,1)} + \epsilon\delta^2\eta^{(1,1)}$ satisfies

$$\begin{aligned} 2\frac{\partial\eta}{\partial\tau} = &-3\eta\frac{\partial\eta}{\partial\xi} - \frac{1}{3}\delta^2\frac{\partial^3\eta}{\partial\xi^3} + \frac{3}{4}\epsilon\eta^2\frac{\partial\eta}{\partial\xi} \\ &- \frac{1}{12}\epsilon\delta^2\left(23\frac{\partial\eta}{\partial\xi}\frac{\partial^2\eta}{\partial\xi^2} + 10\eta\frac{\partial^3\eta}{\partial\xi^3}\right) + O\left(\epsilon^2,\delta^4\right). \end{aligned} \tag{5.92}$$

In what follows (as proposed by Johnson [42]) it is necessary to introduce a new parameter λ, defined by looking at u at a specific depth, say z_0 (with $-1 \le z_0 \le 0$). We can then define $\hat{u} = u(\xi, z_0, \tau)$ and invert equation (5.89) to obtain

$$\eta \sim \hat{u} + \frac{\epsilon}{4}\hat{u}^2 - \epsilon\delta^2\lambda\frac{\partial^2\hat{u}}{\partial\xi^2}, \tag{5.93}$$

where the new parameter λ is given by

$$\lambda = -\left(\frac{1}{2}z_0^2 + z_0 + \frac{1}{6}\right). \tag{5.94}$$

Since $-1 \le z_0 \le 0$, we have λ bounded as

$$-\frac{1}{6} \le \lambda \le \frac{1}{3}.$$

It is possible to use the asymptotic expansion of η in terms of $\hat{u}$, equation (5.93), in the asymptotic expression we found for u, equation (5.92). Substituting into the left hand side of (5.92) yields

$$\begin{aligned} 2\frac{\partial}{\partial\tau}\left(\hat{u} + \frac{\epsilon}{4}\hat{u}^2 - \epsilon\delta^2\lambda\frac{\partial^2\hat{u}}{\partial\xi^2}\right) = &\, 2\frac{\partial\hat{u}}{\partial\tau} + \epsilon\hat{u}\frac{\partial\hat{u}}{\partial\tau} \\ &- 2\epsilon\delta^2\lambda\frac{\partial^2}{\partial\xi^2}\frac{\partial\hat{u}}{\partial\tau} + O\left(\epsilon^2,\delta^4\right). \end{aligned} \tag{5.95}$$

From equation (5.88) and the relation $\eta \sim \hat{u}$ at lowest order we can write

$$\frac{\partial\hat{u}}{\partial\tau} = -\frac{3}{2}\hat{u}\frac{\partial\hat{u}}{\partial\xi} + O\left(\epsilon,\delta^2\right),$$

which can, with keeping at the same order of approximation, be used in the second and third term of (5.95)

$$
\begin{aligned}
2\frac{\partial \hat{u}}{\partial \tau} + \epsilon \hat{u}\frac{\partial \hat{u}}{\partial \tau} - 2\epsilon\delta^2\lambda \frac{\partial^2}{\partial \xi^2}\frac{\partial \hat{u}}{\partial \tau} = 2\frac{\partial \hat{u}}{\partial \tau} &- \frac{3}{2}\epsilon \hat{u}^2 \frac{\partial \hat{u}}{\partial \xi} \\
&+ 3\epsilon\delta^2\lambda\left(3\frac{\partial \hat{u}}{\partial \xi}\frac{\partial^2 \hat{u}}{\partial \xi^2} + \hat{u}\frac{\partial^3 \hat{u}}{\partial \xi^3}\right) \\
&+ O\left(\epsilon^2, \delta^4\right).
\end{aligned}
$$

We then continue with calculating the right hand side of (5.92), by substituting η with (5.93)

$$
\begin{aligned}
&-3\left(\hat{u} + \frac{\epsilon}{4}\hat{u}^2 - \epsilon\delta^2\lambda\frac{\partial^2 \hat{u}}{\partial \xi^2}\right)\frac{\partial}{\partial \xi}\left(\hat{u} + \frac{\epsilon}{4}\hat{u}^2 - \epsilon\delta^2\lambda\frac{\partial^2 \hat{u}}{\partial \xi^2}\right) \\
&\quad + \frac{3}{4}\epsilon\left(\hat{u} + \frac{\epsilon}{4}\hat{u}^2 - \epsilon\delta^2\lambda\frac{\partial^2 \hat{u}}{\partial \xi^2}\right)^2\frac{\partial}{\partial \xi}\left(\hat{u} + \frac{\epsilon}{4}\hat{u}^2 - \epsilon\delta^2\lambda\frac{\partial^2 \hat{u}}{\partial \xi^2}\right) \\
&\quad - \frac{1}{3}\delta^2\frac{\partial^3}{\partial \xi^3}\left(\hat{u} + \frac{\epsilon}{4}\hat{u}^2 - \epsilon\delta^2\lambda\frac{\partial^2 \hat{u}}{\partial \xi^2}\right) \\
&\quad - \frac{1}{12}\epsilon\delta^2\left(23\frac{\partial \hat{u}}{\partial \xi}\frac{\partial^2 \hat{u}}{\partial \xi^2} + 10\hat{u}\frac{\partial^3 \hat{u}}{\partial \xi^3}\right) \\
&= -3\hat{u}\frac{\partial \hat{u}}{\partial \xi} - \frac{9}{4}\epsilon\hat{u}^2\frac{\partial \hat{u}}{\partial \xi} + 3\epsilon\delta^2\lambda\hat{u}\frac{\partial^3 \hat{u}}{\partial \xi^3} + 3\epsilon\delta^2\lambda\frac{\partial \hat{u}}{\partial \xi}\frac{\partial^2 \hat{u}}{\partial \xi^2} \\
&\quad + \frac{3}{4}\epsilon\hat{u}^2\frac{\partial \hat{u}}{\partial \xi} \\
&\quad - \frac{1}{3}\delta^2\frac{\partial^3 \hat{u}}{\partial \xi^3} - \frac{1}{12}\epsilon\delta^2\frac{\partial^3}{\partial \xi^3}\hat{u}^2 \\
&\quad - \frac{1}{12}\epsilon\delta^2\left(23\frac{\partial \hat{u}}{\partial \xi}\frac{\partial^2 \hat{u}}{\partial \xi^2} + 10\hat{u}\frac{\partial^3 \hat{u}}{\partial \xi^3}\right) \\
&\quad + O\left(\epsilon^2, \delta^4\right).
\end{aligned}
$$

Thus combining of the left and right hand side yields

$$
\begin{aligned}
&2\frac{\partial \hat{u}}{\partial \tau} - \frac{3}{2}\epsilon\hat{u}^2\frac{\partial \hat{u}}{\partial \xi} + 3\epsilon\delta^2\lambda\left(3\frac{\partial \hat{u}}{\partial \xi}\frac{\partial^2 \hat{u}}{\partial \xi^2} + \hat{u}\frac{\partial^3 \hat{u}}{\partial \xi^3}\right) \\
&= -3\hat{u}\frac{\partial \hat{u}}{\partial \xi} - \frac{6}{4}\epsilon\hat{u}^2\frac{\partial \hat{u}}{\partial \xi} + 3\epsilon\delta^2\lambda\hat{u}\frac{\partial^3 \hat{u}}{\partial \xi^3} + 3\epsilon\delta^2\lambda\frac{\partial \hat{u}}{\partial \xi}\frac{\partial^2 \hat{u}}{\partial \xi^2} \\
&\quad - \frac{1}{3}\delta^2\frac{\partial^3 \hat{u}}{\partial \xi^3} - \frac{1}{12}\epsilon\delta^2\frac{\partial^3}{\partial \xi^3}\hat{u}^2 \\
&\quad - \frac{1}{12}\epsilon\delta^2\left(23\frac{\partial \hat{u}}{\partial \xi}\frac{\partial^2 \hat{u}}{\partial \xi^2} + 10\hat{u}\frac{\partial^3 \hat{u}}{\partial \xi^3}\right) \\
&\quad + O\left(\epsilon^2, \delta^4\right),
\end{aligned}
$$

which should[9] be reduced to

$$\begin{aligned}\frac{\partial \hat{u}}{\partial \tau} &= -\frac{3}{2}\hat{u}\frac{\partial \hat{u}}{\partial \xi} - \frac{1}{6}\delta^2\frac{\partial^3 \hat{u}}{\partial \xi^3} \\ &\quad - \frac{1}{2}\epsilon\delta^2\left[\left(\frac{29}{12} + 6\lambda\right)\frac{\partial \hat{u}}{\partial \xi}\frac{\partial^2 \hat{u}}{\partial \xi^2} + \frac{5}{6}\hat{u}\frac{\partial^3 \hat{u}}{\partial \xi^3}\right] \\ &\quad + O\left(\epsilon^2, \delta^4\right),\end{aligned} \tag{5.96}$$

after using $(\hat{u}^2)_{\xi\xi\xi} = 6\hat{u}_\xi\hat{u}_{\xi\xi} + 2\hat{u}\hat{u}_{\xi\xi\xi}$. Note that we at this point can retrieve the KdV equation by neglecting terms of order $O(\epsilon)$ for fixed δ.

In order to obtain the CH equation we need to return to the original variables by introducing $T = \sqrt{\epsilon}t$ and $X = \sqrt{\epsilon}x$ and keep the scaling only in ϵ. Thus

$$T = \frac{1}{\epsilon}\tau, \quad X = \xi + \frac{1}{\epsilon}\tau,$$

where we have the corresponding differential operators

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial X}, \quad \epsilon\frac{\partial}{\partial \tau} = \frac{\partial}{\partial T} + \frac{\partial}{\partial X}.$$

These scale transformations applied to (5.96) yields

$$\begin{aligned}\frac{\partial \hat{u}}{\partial T} + \frac{\partial \hat{u}}{\partial X} &= -\frac{3}{2}\epsilon\hat{u}\frac{\partial \hat{u}}{\partial X} - \frac{1}{6}\epsilon\delta^2\frac{\partial^3 \hat{u}}{\partial X^3} \\ &\quad - \frac{1}{2}\epsilon^2\delta^2\left[\left(\frac{29}{12} + 6\lambda\right)\frac{\partial \hat{u}}{\partial X}\frac{\partial^2 \hat{u}}{\partial X^2} + \frac{5}{6}\hat{u}\frac{\partial^3 \hat{u}}{\partial X^3}\right] \\ &\quad + O\left(\epsilon^3, \epsilon\delta^4\right),\end{aligned} \tag{5.97}$$

where we have multiplied throughout with ϵ. As it turns out, no value of λ gives the CH equation and we therefore need an additional (arbitrary) parameter μ. This is obtained by the simple trick of adding and subtracting the same quantity into the former equation. That is, we can add

$$\left(\epsilon\delta^2\mu\frac{\partial^2}{\partial X^2}\frac{\partial \hat{u}}{\partial T} - \epsilon\delta^2\mu\frac{\partial^2}{\partial X^2}\frac{\partial \hat{u}}{\partial T}\right) \tag{5.98}$$

on the left side of (5.97), for arbitrary μ. But first we observe that from (5.97) we obtain

$$\frac{\partial \hat{u}}{\partial T} = -\frac{\partial \hat{u}}{\partial X} - \frac{3}{2}\epsilon\hat{u}\frac{\partial \hat{u}}{\partial X} + O\left(\epsilon^2, \epsilon\delta^2\right),$$

[9] We did not get the same coefficient as Johnson and Ivanov in front of the uu_{xxx} term. In our case we obtained the coefficient 1 while it should be 5/6. We believe there is a small calculation error in our case, as both Johnson and Ivanov did get the same result and will therefore continue with their results instead.

which we can use on the first term in (5.98). We are able to use this approximation here, as the neglected terms are multiplied by $\epsilon\delta^2$ and will be of the same order as the earlier approximations. Equation (5.98) can thus be written as

$$\left[\epsilon\delta^2\mu\frac{\partial^2}{\partial X^2}\left(-\frac{\partial\hat{u}}{\partial X}-\frac{3}{2}\epsilon\hat{u}\frac{\partial\hat{u}}{\partial X}\right)-\epsilon\delta^2\mu\frac{\partial^2}{\partial X^2}\frac{\partial\hat{u}}{\partial T}\right]+O\left(\epsilon^3,\epsilon\delta^4\right)$$

which added on the left hand side of (5.97) and rearranging terms yields

$$\begin{aligned}&\frac{\partial}{\partial T}\left(\hat{u}-\frac{1}{2}\epsilon\delta^2\mu\frac{\partial^2\hat{u}}{\partial X^2}\right)\\&\quad=-\frac{\partial\hat{u}}{\partial X}-\frac{3}{2}\epsilon\hat{u}\frac{\partial\hat{u}}{\partial X}+\epsilon\delta^2\left(\frac{1}{2}\mu-\frac{1}{6}\right)\frac{\partial^3\hat{u}}{\partial X^3}\\&\quad-\frac{1}{2}\epsilon^2\delta^2\left[\left(\frac{29}{12}+6\lambda-\frac{9}{2}\mu\right)\frac{\partial\hat{u}}{\partial X}\frac{\partial^2\hat{u}}{\partial X^2}+\left(\frac{5}{6}-\frac{3}{2}\mu\right)\hat{u}\frac{\partial^3\hat{u}}{\partial X^3}\right]\\&\quad+O\left(\epsilon^3,\epsilon\delta^4\right).\end{aligned} \tag{5.99}$$

Comparing this equation with the CH equation (5.73), clearly reveals that these are quite similar, except for the coefficients and that the CH equation does not have an u_{XXX} term. The latter is easily remedied by introducing a simple transformation by $u' = u - c$, for an arbitrary constant c, which applied on the CH equation (5.73) yields

$$\frac{\partial u'}{\partial t}+2\tilde{\kappa}\frac{\partial u'}{\partial x}-\frac{\partial}{\partial t}\frac{\partial^2 u'}{\partial x^2}+3u'\frac{\partial u'}{\partial x}=2\frac{\partial u'}{\partial x}\frac{\partial^2 u'}{\partial x^2}+u'\frac{\partial^3 u'}{\partial x^3}+c\frac{\partial^3 u'}{\partial x^3},$$

and we have thus obtained the desired term with an arbitrary coefficient and with a new $\tilde{\kappa}$ defined by $\tilde{\kappa} = \kappa + 3c$. Alternatively we could perform the transformation $X' = X - T$ (which is equivalent to following the CH equation in a moving frame) on (5.99) and gives $\hat{u}_{XXX}$ being replaced by $\hat{u}_{XXT}$. The invariance of the CH equation under these transformations is described by Ivanov [38], Dullin *et al.* [27].

The question now is to find suitable values for the parameters λ and μ such that equation (5.99) becomes *the* Camassa-Holm equation (5.73). In [38] Ivanov uses a "reversed engineering" approach. That is, after obtaining equation (5.99) he scales (with arbitrary constants) the CH equation (5.73) and finds which scaling and transformations are needed to transform the CH equation to be of the form (5.99). He then compares the coefficients on the transformed equation with (5.99) and finds the requirements on the parameters μ and λ. This is also partly done in the approach applied by Johnson in [42]. In a paper by Dullin *et al.* [27] it was pointed out that certain ratios between the terms in the CH equation are crucial for it to be completely integrable (and is also used by Johnson in [42]). These ratios are

$$C\left(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\right) : C\left(u\frac{\partial^3 u}{\partial x^3}\right) = 2:1,$$
$$C\left(\frac{\partial}{\partial t}\frac{\partial^2 u}{\partial x^2}\right) C\left(u\frac{\partial u}{\partial x}\right) : C\left(u\frac{\partial^3 u}{\partial x^3}\right) = 3:1,$$

where $C(*)$ stands for the coefficient of the term $*$ in the scaled equation. We use these ratios to determine the parameters more precisely.

To have the ratio $2:1$ between the terms $\hat{u}_X\hat{u}_{XX}$ and $\hat{u}\hat{u}_{XXX}$ we require in equation (5.99)

$$\frac{29}{12} + 6\lambda - \frac{9}{2}\mu = 2\cdot\left(\frac{5}{6} - \frac{3}{2}\mu\right),$$

which means

$$\lambda = \frac{1}{4}\mu - \frac{1}{8}. \tag{5.100}$$

Furthermore, to have the ration $3:1$ between the $\hat{u}_{XXT}\cdot\hat{u}\hat{u}_X$ and $\hat{u}\hat{u}_{XXX}$ terms, we also require

$$\frac{1}{2}\epsilon\delta^2\mu\cdot\frac{3}{2}\epsilon = -3\cdot\frac{1}{2}\epsilon^2\delta^2\left(\frac{5}{6} - \frac{3}{2}\mu\right),$$

which means

$$\mu = \frac{5}{6}.$$

Thus from equation (5.100) we obtain $\lambda = 1/12$, which again from equation (5.94) yields[10] $z_0 = -1 + 1/\sqrt{2}$.

With these values, equation (5.99) becomes (after dropping higher order terms in ϵ and δ^2)

$$\begin{aligned}\frac{\partial}{\partial T}\left(\hat{u} - \frac{5}{12}\epsilon\delta^2\frac{\partial^2\hat{u}}{\partial X^2}\right) &= -\frac{\partial\hat{u}}{\partial X} - \frac{3}{2}\epsilon\hat{u}\frac{\partial\hat{u}}{\partial X} + \frac{1}{4}\epsilon\delta^2\frac{\partial^3\hat{u}}{\partial X^3}\\ &\quad + \frac{1}{2}\epsilon^2\delta^2\left[\frac{5}{6}\frac{\partial\hat{u}}{\partial X}\frac{\partial^2\hat{u}}{\partial X^2} + \frac{5}{12}\hat{u}\frac{\partial^3\hat{u}}{\partial X^3}\right],\end{aligned} \tag{5.101}$$

which is the desired CH equation.

It is possible to rewrite this equation on the standard CH-form (i.e. equation (5.73)) if we apply the following scaling and transformations (Johnson [42, 44])

$$X' = \frac{1}{2}\sqrt{\frac{5}{3}}\left(X - \frac{3}{5}T\right), \quad \hat{u} = \sqrt{\frac{5}{3}}U(X', T; \epsilon, \delta)$$

[10] As we have a quadric equation in z_0, we actually obtain two solutions. The second solution is not applicable, however, since the z_0 value is then below -1, i.e. outside the domain.

(which will replace the U_{XXX} term with a U_{XXT} term and give the standard coefficients) and then remove the ϵ and δ by the additional scaling

$$X' \to \delta\sqrt{\epsilon}X', \quad T \to \delta\sqrt{\epsilon}T, \quad U \to \frac{1}{\epsilon}U.$$

(Alternatively to this last scaling, one could instead have set $\epsilon = \delta = 1$). With this (5.101) becomes the CH equation on standard form

$$\frac{\partial U}{\partial T} + 2\kappa\frac{\partial U}{\partial X'} + 3U\frac{\partial U}{\partial X'} - \frac{\partial}{\partial T}\frac{\partial^2 U}{\partial X'^2} = 2\frac{\partial U}{\partial X'}\frac{\partial^2 U}{\partial X'^2} + U\frac{\partial^3 U}{\partial X'^3},$$

an equation describing the horizontal velocity at depth $z_0 = -1 + 1/\sqrt{2}$ and with $2\kappa = 4/5\sqrt{3/5}$.

Note that the surface elevation η can be found, at this order of approximation, by

$$\eta \sim \sqrt{\frac{5}{3}}\left(U + \sqrt{\frac{5}{3}}\frac{\epsilon}{4}U^2 - \frac{1}{5}\epsilon\delta^2\lambda\frac{\partial^2 U}{\partial X'^2}\right), \tag{5.102}$$

or alternatively by solving equation (5.92) directly.

Remarks

First of all we note that this equation differs from the earlier equations, by that it does not describe the evaluation of the free surface, but the horizontal velocity of a wave at a certain depth, more precisely at $z = z_0 = -1 + 1/\sqrt{2}$. As mentioned, it is possible to instead retrieve a description of the surface evaluation (i.e. η) by means of the asymptotic equivalence relation (5.93) or solving equation (5.92) directly. The CH equation also differs from the earlier equations by that it captures the phenomena of wave breaking, i.e. its solution remains bounded while its slope becomes unbounded in finite time (cf. Constantin *et al.* [17]).

Note that if we make a cruder approximation, by neglecting the terms of order $O\left(\epsilon^2\delta^2\right)$, the CH equation (5.101) reduces to

$$\frac{\partial U}{\partial T} + 2\kappa\frac{\partial U}{\partial X'} + 3\epsilon U\frac{\partial \hat{U}}{\partial X'} - \epsilon\delta^2\frac{\partial}{\partial T}\frac{\partial^2 U}{\partial X'^2} = 0,$$

which is a Benjamin-Bona-Mahoney (BBM) equation (Benjamin *et al.* [4]). This equation is also known as the regularised long wave (RLW) equation, as coined by Benjamin *et al.* in the same paper. The BBM equation is yet another equation describing wave propagation in shallow water, but unlike the other equations derived here, it is not completely integrable (Duzhin & Tsujishita [28]).

The CH equation, however, is completely integrable (as we mentioned in the beginning of this section), which was shown by Camassa and Holm in their original paper [12]. Later this as been confirmed several times over, by other means of integrability tests (cf. Constantin [15] or Fuchssteiner [33]). In particular (for smooth solutions) the quantities

$$\int u \,\mathrm{d}x, \quad \int \left[u^2 + \left(\frac{\partial u}{\partial x}\right)^2 \right] \mathrm{d}x, \quad \int \left[u^3 + u \left(\frac{\partial u}{\partial x}\right)^2 \right] \mathrm{d}x$$

are all time independent. Defining the potential $m = u - u_{xx}$ the Lax pair can be written as (Camassa & Holm [12], Constantin [15])

$$\begin{aligned} \frac{\partial^2 \psi}{\partial x^2} &= \left[\frac{1}{4} + \sigma(m+\kappa)\right]\psi \\ \frac{\partial \psi}{\partial t} &= \left(\frac{1}{2\sigma} - u\right)\frac{\partial \psi}{\partial x} + \frac{1}{2}\frac{\partial u}{\partial x}\psi \end{aligned}$$

and the infinite of conserved quantities follow from the Lax equation.

A special solution of the CH equation which as gained much attention is the peakon solution, which is obtained when $\kappa = 0$. Peakons are peaked wave forms (the solution is continuous, while its derivative is not) with similar properties as solitons (e.g. can interact nonlineary with each other and retain their identity afterwards). The multipeakon (multiple peakons) solution is given by (Camassa & Holm [12])

$$u(x,t) = \sum_{i=1}^{n} p_i(t) \exp(-|x - q_i(t)|),$$

where $p_i(t)$ and $q_i(t)$ satisfy

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} q_i(t) &= \sum_{j=1}^{n} p_j(t) \exp(-|q_i - q_j|), \\ \frac{\mathrm{d}}{\mathrm{d}t} p_i(t) &= \sum_{j=1}^{n} p_i p_j \operatorname{sgn}(q_i - q_j) \exp(-|q_i - q_j|). \end{aligned}$$

Also see Holden & Raynaud [35] for details about multipeakon solutions for the CH equation. Since $\kappa \neq 0$ in the case of water waves, this type of solutions are not applicable here.

The soliton solution of the CH equation, however, is applicable for water waves (as κ will be arbitrary in this case). Firstly assume a right-propagating wave by the transformation $\xi = x - ct$. Used on the standard form of the CH equation (equation (5.73)) yields the ordinary differential equation

$$-\frac{\partial u}{\partial \xi} + c\frac{\partial^3 u}{\partial \xi^3} + 3u\frac{\partial u}{\partial \xi} + 2\kappa\frac{\partial u}{\partial \xi} = 2\frac{\partial u}{\partial \xi}\frac{\partial^3 u}{\partial x^3} + u\frac{\partial^3 u}{\partial \xi^3},$$

which by integrating once and imposing u decaying at infinity gives

$$-cu + c\frac{\partial^2 u}{\partial \xi^2} + \frac{3}{2}u^2 + 2\kappa u = u\frac{\partial^2 u}{\partial \xi^2} + \frac{1}{2}\left(\frac{\partial u}{\partial \xi}\right)^2.$$

This can be integrated once more, after multiplying throughout with u_ξ and once more using the assumption of u (and its derivatives) decaying. Then rearranging terms and dividing out the common factor half yields

$$\left(\frac{\partial u}{\partial \xi}\right)^2 = u^2\frac{(c - 2\kappa - u)}{(c - u)}.$$

This gives unfortunately no explicit solutions for u for arbitrary κ (setting $\kappa = 0$ will then lead to the peakon solution, which is not a proper solution here). However, Camassa *et al.* [13] showed this equation can be integrated to obtain the implicit solution

$$\left(\frac{F - \gamma}{F + \gamma}\right)^\gamma \left(\frac{F + \gamma}{F - \gamma}\right) = \exp(-\xi),$$

where

$$F = \sqrt{\frac{c - u}{c - 2\kappa - u}}, \quad \gamma = \sqrt{\frac{c}{c - 2\kappa}}.$$

In the same paper Camassa *et al.* showed that in the limit $c - \kappa \to 0$ (which implies $\gamma \to \infty$) the equation reduces to

$$u = (c - \kappa)\,\mathrm{sech}^2\left[\sqrt{\frac{c - 2\kappa}{2^c}}(x - ct)\right] + O\left((c - \kappa)^2\right).$$

Johnson [42] then showed that in the case for water waves, one can use the assumption of $\epsilon \to 0$ to make the asymptotic approximation (he then also retains all coefficients with ϵ and δ in the soliton derivation) and obtains

$$\hat{u} \sim a\,\mathrm{sech}^2(\beta\xi) = a\,\mathrm{sech}^2(\beta(x - ct)),$$

where

$$\beta \sim \frac{1}{2\delta}\sqrt{\frac{a}{2\kappa}}\left(1 - \frac{\epsilon a}{2\kappa}\right).$$

and $a > 0$ is the arbitrary amplitude and $c \sim 2\kappa + \epsilon a$. We recognize this then as a soliton solution of the CH equation. Note that the soliton solution can also be found in terms of η by means of (5.102) (cf. Johnson [42]).

Finally we note that the CH equation is a special case of the wider family of equations given by

$$\frac{\partial u}{\partial t} - \frac{\partial^2}{\partial x^2}\frac{\partial u}{\partial t} + (b + 1)u\frac{\partial u}{\partial x} = b\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + u\frac{\partial^3 u}{\partial x^3}$$

where b is arbitrary. The CH equation is retrieved by setting $b = 2$ (Holm *et al.* [21]). Only two values of n results in a completely integrable equation (Ivanov [37]), one of them being the CH equation and the other the Degasperis-Procesi equation, which will be the topic of the next section.

5.3.2 Degasperis-Procesi Equation

Being in the family of completely integrable equations, the CH equation derived in the previous section obtained much attention. Degasperis and Procesi studied this equation (Degasperis & Procesi [22]) and asked themselves if any other completely integrable equations existed of a similar form as the CH equation. They discovered this was the case and found the equation to be

$$\frac{\partial}{\partial t}\left(u-\frac{\partial^2 u}{\partial x^2}\right)+2\kappa\frac{\partial u}{\partial x}+4u\frac{\partial u}{\partial x}=3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}+u\frac{\partial^3 u}{\partial x^3}, \tag{5.103}$$

where κ is an arbitrary real parameter. This equation is after its discovery coined the Degasperis-Procesi (DP) equation. However, the connection of the DP equation to be an approximate model for shallow water wave propagation was discovered only later (Johnson [44], Dullin *et al.* [27], Constantin *et al.* [17], Ivanov [38]). The aim of this section is thus to obtain the DP equation from the governing equations for water waves.

As it turns out, most of the work to obtain the DP equation has already been made when we derived the CH equation in the previous section. We will see that only the final stages of the previous derivation need to be changed to obtain the DP equation. For the CH equation we obtained the following equation (see the derivation which leads up to equation (5.99) in the previous section)

$$\begin{aligned}
&\frac{\partial}{\partial T}\left(\hat{u}-\frac{1}{2}\epsilon\delta^2\mu\frac{\partial^2\hat{u}}{\partial X^2}\right)\\
&\quad=-\frac{\partial\hat{u}}{\partial X}-\frac{3}{2}\epsilon\hat{u}\frac{\partial\hat{u}}{\partial X}+\epsilon\delta^2\left(\frac{1}{2}\mu-\frac{1}{6}\right)\frac{\partial^3\hat{u}}{\partial X^3}\\
&\quad-\frac{1}{2}\epsilon^2\delta^2\left[\left(\frac{29}{12}+6\lambda-\frac{9}{2}\mu\right)\frac{\partial\hat{u}}{\partial X}\frac{\partial^2\hat{u}}{\partial X^2}+\left(\frac{5}{6}-\frac{3}{2}\mu\right)\hat{u}\frac{\partial^3\hat{u}}{\partial X^3}\right]\\
&\quad+O\left(\epsilon^3,\epsilon\delta^4\right).
\end{aligned} \tag{5.104}$$

The question then is: does there exist any values of the parameters μ and λ such that this equation becomes the DP equation (5.103). Using a similar approach as in the previous section, we compare the ratios between the terms in the DP equation (5.103) and observe that

$$\begin{aligned}
C\left(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2}\right):C\left(u\frac{\partial^3 u}{\partial x^3}\right)&=3:1,\\
C\left(\frac{\partial}{\partial t}\frac{\partial^2 u}{\partial x^2}\right)C\left(u\frac{\partial u}{\partial x}\right):C\left(u\frac{\partial^3 u}{\partial x^3}\right)&=4:1.
\end{aligned}$$

Using these ratios we are able to determine the free parameters μ and λ.

From the first ratio comparison we obtain

$$\frac{29}{12} + 6\lambda - \frac{9}{2}\mu = 3 \cdot \left(\frac{5}{6} - \frac{3}{2}\mu\right),$$

which means

$$\lambda = \frac{1}{72}.$$

From the second ratio 4 : 1 we require

$$\frac{1}{2}\epsilon\delta^2\mu \cdot \frac{3}{2}\epsilon = -4 \cdot \frac{1}{2}\epsilon^2\delta^2 \left(\frac{5}{6} - \frac{3}{2}\mu\right),$$

which yields

$$\mu = \frac{20}{27}.$$

Thus, with $\lambda = 1/72$ and $\mu = 20/27$, equation (5.104) becomes the DP equation

$$\begin{aligned}\frac{\partial}{\partial T}\left(\hat{u} - \frac{10}{27}\epsilon\delta^2\frac{\partial^2\hat{u}}{\partial X^2}\right) = -\frac{\partial\hat{u}}{\partial X} - \frac{3}{2}\epsilon\hat{u}\frac{\partial\hat{u}}{\partial X} + \frac{11}{54}\epsilon\delta^2\frac{\partial^3\hat{u}}{\partial X^3} \\ + \frac{5}{12}\epsilon^2\delta^2\left[\frac{\partial\hat{u}}{\partial X}\frac{\partial^2\hat{u}}{\partial X^2} + \frac{1}{3}\hat{u}\frac{\partial^3\hat{u}}{\partial X^3}\right],\end{aligned}$$

where we have neglected higher order terms. From equation[11] (5.94) we see that this equation describes the (horizontal) wave velocity at depth $z_0 = -1 + \sqrt{23}/6$.

Remarks

As the derivation of the DP equation follows the derivation of the CH equation closely, we can expect them to have similar properties. For example, the DP equation describes the (horizontal) velocity at a certain depth, is completely integrable and captures the phenomena of wave breaking (cf. Constantin *et al.* [17]).

In addition (some of) the solutions are similar. The DP equation also has multipeakon (multiple peaked solitons) solutions, as was shown by Degasperis *et al.* in [21]. These solutions are valid only for $\kappa = 0$, and are described by

$$u(x,t) = \sum_{i=1}^{n} m_i(t)e^{-|x-x_i(t)|},$$

[11] Again this equation yields two values for z_0, where only one is valid (the other value is outside the domain).

where the functions m_i and x_i satisfy

$$\frac{\partial}{\partial t}x_i = \sum_{j=1}^{n} m_j e^{-|x_i - x_j|},$$
$$\frac{\partial}{\partial t}m_i = 2m_i \sum_{j=1}^{n} m_j \operatorname{sgn}(x_i - x_j) e^{-|x_i - x_j|}.$$

We note that in our case $\kappa = 1/2$, which means that the multipeakon solutions are not applicable for water waves. However, Matsuno showed in [53] that for $\kappa > 0$ (as is our case) the solutions of the DP equation are smooth. Should $\kappa \to 0$ (which of course is unlikely for the case of water waves), Matsuno also showed, in the same paper, that they would in the limit converge to the peakon solutions.

Introducing the potential $m = u - u_{xx}$ a convenient rewriting of the DP equation (on the form (5.103)) is

$$\frac{\partial m}{\partial t} + u\frac{\partial m}{\partial x} + 3\frac{\partial u}{\partial x}m = 0, \quad m = u - \frac{\partial^2 u}{\partial x^2}.$$

An important note here is that we now have obtained two *linear* equations. Using these equations, the Lax pair of the DP equation can be written as (Degasperis *et al.* [21])

$$\frac{\partial^3}{\partial x^3}\psi = \frac{\partial}{\partial x}\psi + \lambda m(x,t)\psi$$
$$\frac{\partial}{\partial t}\psi = \frac{1}{\lambda}\frac{\partial^2}{\partial x^2}\psi - u(x,t)\frac{\partial}{\partial x}\psi + \frac{\partial}{\partial x}u(x,t)\psi,$$

where the function $\psi(x,t,\lambda)$ is the common solution of these two equations and λ is the auxiliary complex parameter. The infinite number of conserved quantities then follows from the Lax equation.

5.3.3 Nonlinear Schrödinger equation

The aim of this section is to derive an equation describing the evolution of the envelope of modulated harmonic wave groups. This means we will have a non-constant amplitude described by the (complex) function $A(x,t)$ and initial data given by

$$\eta(x,0) = A(x,0)\exp(ikx) + \text{c.c},$$

where k is the wave number and c.c the complex conjugate. As it turns out, the equation obtained is a nonlinear version of the Schrödinger equation. The Schrödinger equation originally appeared in 1926, in the area of theoretical physics (more precisely quantum mechanics) and is named after its discoverer Schrödinger [61]. The nonlinear Schrödinger (NLS) equation does emerge in the fields of optics (cf. Sulem & Sulem [63]), quantum mechanics (as a special case of the nonlinear Schrödinger field) and, of course, in the field of water waves. The connection of the NLS equation being an equation describing water wave modulation was shown by Zakharov in 1968 [69], where he also describes the Hamiltonian structure of water waves.

There exist several approaches in obtaining the NLS equation in the context of water waves; The NLS equation can be derived directly from the governing equations without any assumption of irrotationality (cf. Johnson [44], Miller [56]). Another approach is described by Dingemans, where he first assumes a modulated surface wave, then expands the dispersion relation and finally uses operator correspondences to obtain the NLS equation (Dingemans [24]p. 888 - 890). Or, one can begin with the governing equations, with the assumption of irrotationality (Johnson [41] Debnath [20]), which is the approach we will follow here.

We repeat the (non-dimensional) governing equations in terms of the velocity potential (and neglect for now the y-dependency)

$$\delta^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0, \tag{5.105}$$

where the free-surface conditions, $z = \epsilon\eta$, are given by

$$\frac{\partial \phi}{\partial t} + \frac{\epsilon}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{1}{\delta^2}\left(\frac{\partial \phi}{\partial z}\right)^2\right] + \eta = 0 \tag{5.106}$$

$$\frac{\partial \phi}{\partial z} - \delta^2\left(\frac{\partial \eta}{\partial t} + \epsilon \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x}\right) = 0 \tag{5.107}$$

and the kinematic condition at the horizontal bottom, $z = -1$, by

$$\frac{\partial \phi}{\partial z} = 0. \tag{5.108}$$

As the NLS equation describes modulated wave propagation, we seek solutions which represents an amplitude modulation of a harmonic wave, where we scale and transform the variables by

$$\zeta = \epsilon\left(x - c_{\mathrm{g}} t\right), \quad \xi = x - c_{\mathrm{p}} t, \quad \tau = \epsilon^2 t.$$

Here ζ describes the modulated wave, while the carrier wave is described by ξ. The variables $c_{\mathrm{p}} = c_{\mathrm{p}}(k)$ and $c_{\mathrm{g}} = c_{\mathrm{g}}(k)$ are yet to be determined (but the choice of names should give a good clue of what these represent). The corresponding differential operators are

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \epsilon\frac{\partial}{\partial \zeta}, \quad \frac{\partial}{\partial t} = \epsilon^2\frac{\partial}{\partial \tau} - \left(\epsilon c_{\mathrm{g}}\frac{\partial}{\partial \zeta} + c_{\mathrm{p}}\frac{\partial}{\partial \xi}\right).$$

Using these transformations on the governing equations (5.105) - (5.108) we obtain

$$\frac{\partial^2\phi}{\partial z^2} + \delta^2\left(\frac{\partial^2\phi}{\partial\xi^2} + 2\epsilon\frac{\partial}{\partial\zeta}\frac{\partial\phi}{\partial\xi} + \epsilon^2\frac{\partial^2\phi}{\partial\zeta^2}\right) = 0,$$

with on free-surface $z = \epsilon\eta$

$$\epsilon^2\frac{\partial\phi}{\partial\tau} + \eta - \left(\epsilon c_{\mathrm{g}}\frac{\partial\phi}{\partial\zeta} + c_{\mathrm{p}}\frac{\partial\phi}{\partial\xi}\right) + \frac{\epsilon}{2}\left[\frac{1}{\delta^2}\left(\frac{\partial^2\phi}{\partial z^2}\right)^2 + \left(\frac{\partial\phi}{\partial\xi} + \epsilon\frac{\partial\phi}{\partial\zeta}\right)^2\right] = 0$$

$$\frac{\partial\phi}{\partial z} - \delta^2\left[\epsilon^2\frac{\partial\eta}{\partial\tau} - \left(\epsilon c_{\mathrm{g}}\frac{\partial\eta}{\partial\zeta} + c_{\mathrm{p}}\frac{\partial\eta}{\partial\xi}\right) + \epsilon\left(\frac{\partial\phi}{\partial\xi} + \epsilon\frac{\partial\phi}{\partial\zeta}\right)\left(\frac{\partial\eta}{\partial\xi} + \epsilon\frac{\partial\eta}{\partial\zeta}\right)\right] = 0$$

and on the horizontal bottom $z = -1$

$$\frac{\partial\phi}{\partial z} = 0.$$

As has been done several times now, we seek an asymptotic solution for the parameter $\epsilon \to 0$ and δ fixed. Thus

$$\phi(\xi,\zeta,z,\tau;\epsilon) \sim \sum_{n=0}^{\infty}\epsilon^n\phi^{(n)}(\xi,\zeta,z,\tau), \quad \eta(\xi,\zeta,\tau;\epsilon) \sim \sum_{n=0}^{\infty}\epsilon^n\eta^{(n)}(\xi,\zeta,\tau),$$

which we assume are periodic functions in ξ.

It is readily seen that at leading order we obtain

$$\frac{\partial^2\phi^{(0)}}{\partial z^2} + \delta^2\frac{\partial^2\phi^{(0)}}{\partial\xi^2} = 0 \tag{5.109}$$

$$\eta^{(0)} - c_{\mathrm{p}}\frac{\partial\phi^{(0)}}{\partial\xi} = 0 \qquad \text{on} \quad z = 0 \tag{5.110}$$

$$\frac{\partial\phi^{(0)}}{\partial z} + \delta^2 c_{\mathrm{p}}\frac{\partial\eta^{(0)}}{\partial\xi} = 0 \qquad \text{on} \quad z = 0 \tag{5.111}$$

$$\frac{\partial\phi^{(0)}}{\partial z} = 0 \qquad \text{on} \quad z = -1 \tag{5.112}$$

We seek a solution as a harmonic wave of wave number k and velocity c_{p}. That is, we seek for

$$\phi^{(0)} = f^{(0)}(\zeta,\tau) + \Phi^{(0)}(\zeta,z,\tau)E + \text{c.c}, \quad \eta^{(0)} = A^{(0)}(\zeta,\tau)E + \text{c.c}, \tag{5.113}$$

where $E = \exp(ik\xi)$, c.c denotes the complex conjugate of E and $f^{(0)}(\zeta,\tau)$ accommodates for the mean drift (for details see for example Johnson [41] p. 139-146). In terms of this, the Laplace equation (5.109) becomes

$$\frac{\partial^2\Phi^{(0)}}{\partial z^2} = \delta^2 k^2\Phi^{(0)},$$

where the general solution of this equation is found as

$$\Phi^{(0)} = C_1(\zeta,\tau)e^{\delta k z} + C_2(\zeta,\tau)e^{-\delta k z}.$$

Using the boundary condition (5.112) ($z = -1$) yields

$$C_1(\zeta,\tau) = -C_2(\zeta,\tau)e^{2\delta k},$$

which substituted back into the general solution gives

$$\begin{aligned}\Phi^{(0)} &= C_1(\zeta,\tau)e^{\delta k z} + C_2(\zeta,\tau)e^{-\delta k z} \\ &= -C_2(\zeta,\tau)e^{\delta k}\left(e^{\delta k(z+1)} + e^{-\delta k(z+1)}\right).\end{aligned}$$

Writing $F^{(0)}(\zeta,\tau) = -2e^{\delta k}C_2(\zeta,\tau)$ and using the definition of the hyperbolic function cosh, the general solution of $\Phi^{(0)}$ can be written as

$$\Phi^{(0)} = F^{(0)}(\zeta,\tau)\cosh\delta k(z+1). \tag{5.114}$$

Using (5.113) in the dynamic surface condition (5.110) (remember that $z = 0$) yields

$$A^{(0)}E - c_{\mathrm{p}}\frac{\partial}{\partial\xi}\left(f^{(0)} + \Phi^{(0)}E\right) + \mathrm{c.c} = A^{(0)}E - ikc_{\mathrm{p}}\Phi^{(0)}E + \mathrm{c.c} = 0,$$

where we can use (5.114) to obtain

$$F^{(0)} = \frac{A^{(0)}}{ikc_{\mathrm{p}}\cosh\delta k} = -\frac{iA^{(0)}}{kc_{\mathrm{p}}}\operatorname{sech}\delta k. \tag{5.115}$$

Thus the general solution of $\Phi^{(0)}$ is

$$\Phi^{(0)} = -\frac{iA^{(0)}}{kc_{\mathrm{p}}}\frac{\cosh\delta k(z+1)}{\cosh\delta k}, \tag{5.116}$$

where the wave amplitude $A^{(0)} = A^{(0)}(\zeta,\tau)$ is yet to be determined. It is also possible to obtain an alternative expression for $\Phi^{(0)}$ by using (5.113) in the kinematic surface condition (5.111) and then solve for (say) $F^{(0)}$, which we combine with (5.115) and then solve for c_{p}. We then find

$$c_{\mathrm{p}} = \frac{\tanh\delta k}{c_{\mathrm{p}}\delta k}, \tag{5.117}$$

which inserted into (5.116) yields

$$\Phi^{(0)} = -\frac{iA^{(0)}c_{\mathrm{p}}\delta k}{k\tanh\delta k}\frac{\cosh\delta k(z+1)}{\cosh\delta k} = -iA^{(0)}c_{\mathrm{p}}\delta\frac{\cosh\delta k(z+1)}{\sinh\delta k}.$$

We continue with the next order, $O(\epsilon)$. At this order the boundary at the free surface again is non-trivial, as it contains η explicitly. We therefore Taylor expand the free surface conditions around $z=0$, similar as to what we did in the KdV and CH derivation (see for example equations (5.29) - (5.31) in the KdV section). The governing equations at order $O(\epsilon)$ are

$$\frac{\partial^2\phi^{(1)}}{\partial z^2}+\delta^2\left(\frac{\partial^2\phi^{(1)}}{\partial\xi^2}+2\frac{\partial}{\partial\zeta}\frac{\partial\phi^{(0)}}{\partial\xi}\right)=0 \tag{5.118}$$

with on the free surface (now $z=0$, due to the Taylor expansions)

$$\begin{aligned}\eta^{(1)}-c_{\mathrm{g}}\frac{\partial\phi^{(0)}}{\partial\zeta}-c_{\mathrm{p}}\left(\frac{\partial\phi^{(1)}}{\partial\xi}+\eta^{(0)}\frac{\partial}{\partial\xi}\frac{\partial}{\partial z}\phi^{(0)}\right)&\\+\frac{1}{2}\left[\frac{1}{\delta^2}\left(\frac{\partial}{\partial z}\phi^{(0)}\right)^2+\left(\frac{\partial}{\partial\xi}\phi^{(0)}\right)^2\right]&=0\end{aligned} \tag{5.119}$$

$$\frac{\partial\phi^{(1)}}{\partial z}+\eta^{(0)}\frac{\partial^2\phi^{(0)}}{\partial z^2}+\delta^2\left[c_{\mathrm{g}}\frac{\partial\eta^{(0)}}{\partial\zeta}+c_{\mathrm{p}}\frac{\partial\eta^{(1)}}{\partial\xi}-\frac{\partial\phi^{(0)}}{\partial\xi}\frac{\partial\eta^{(0)}}{\partial\xi}\right]=0 \tag{5.120}$$

and on the bottom $z=-1$

$$\frac{\partial\phi^{(1)}}{\partial z}=0. \tag{5.121}$$

Due to the non-linearities which now have emerged, higher harmonics like E^2 (and its complex conjugate) is generated. Similarly will at still higher orders of ϵ higher harmonics of E be generated (e.g. at order $O(\epsilon^2)$ harmonics such as E^3 will be generated). We therefore follow Debnath [20] (or similarly Johnson [41]) when we write

$$\phi^{(m)}=\sum_{n=0}^{m+1}\Phi^{(m,n)}E^n+\text{c.c},\qquad \eta^{(m)}=\sum_{n=0}^{m+1}A^{(m,n)}E^n+\text{c.c}, \tag{5.122}$$

where $m=1,2,3,...$ and the functions $\Phi^{(m,n)}=\Phi^{(m,n)}(\zeta,z,\tau)$ and $A^{(m,n)}=A^{(m,n)}(\zeta,\tau)$ are to be determined in what follows. To ease the notation a bit we will write $A^{(0)}=A^{(0,0)}$ and similarly for the other functions.

Using (5.122) in the Laplace equation at first order of the asymptotic expansion, equation (5.118), we obtain three separate equations (one for each order in E, i.e. E^0, E^1 and E^2)

$$\frac{\partial^2\Phi^{(1,0)}}{\partial z^2}=0,\qquad \frac{\partial^2\Phi^{(1,2)}}{\partial z^2}-4\delta^2k^2\Phi^{(1,2)}=0 \tag{5.123}$$

and

$$\frac{\partial^2\Phi^{(1,1)}}{\partial z^2}-\delta^2k^2\Phi^{(1,1)}+2i\delta^2k\frac{\partial\Phi^{(0)}}{\partial\zeta}=0.$$

The solutions of these equations are readily found (in similar fashion as the solution at leading order) and are after using the bottom boundary condition (5.121)

$$\Phi^{(1,0)} = F^{(1,0)}, \quad \Phi^{(1,2)} = F^{(1,2)} \cosh 2\delta k(z+1)$$
$$\Phi^{(1,1)} = F^{(1,1)} \cosh \delta k(z+1) - i\delta(z+1)\frac{\partial F^{(0,0)}}{\partial \zeta} \sinh \delta k(z+1),$$

where $F^{(1,n)} = F^{(1,n)}(\zeta, \tau)$ are arbitrary (determined by given initial conditions). We can now insert the expansions (5.122), together with these solutions, into the two first-order free surface conditions, which will lead to six different equations. These calculations are fairly straightforward but rather lengthy. We therefore only calculate in detail the first of the six equations and state the rest in their final form (cf. Debnath [20] or Johnson [41]).

Inserting (5.113) and (5.122) into the dynamic first order free surface condition (5.119) we obtain

$$\begin{aligned}
&\left(A^{(1,0)} + A^{(1,1)}E + \overline{A^{(1,1)}}E^{-1} + A^{(1,2)}E^2 + \overline{A^{(1,2)}}E^{-2}\right) \\
&\quad - c_{\mathrm{g}}\frac{\partial}{\partial \zeta}\left(f^{(0)} + \Phi^{(0)}E + \overline{\Phi^{(0)}}E^{-1}\right) \\
&\quad - c_{\mathrm{p}}\frac{\partial}{\partial \xi}\left(\Phi^{(1,0)} + \overline{\Phi^{(1,0)}} + \Phi^{(1,1)}E + \overline{\Phi^{(1,1)}}E^{-1} + \Phi^{(1,2)}E^2 + \overline{\Phi^{(1,2)}}E^{-2})\right) \\
&\quad - c_{\mathrm{p}}\left(A^{(0)}E + \overline{A^{(0)}}E^{-1}\right) \cdot \frac{\partial}{\partial \xi}\frac{\partial}{\partial z}\left(f^{(0)} + \Phi^{(0)}E + \overline{\Phi^{(0)}}E^{-1}\right) \\
&\quad + \frac{1}{2\delta^2}\left[\frac{\partial}{\partial z}\left(f^{(0)} + \Phi^{(0)}E + \overline{\Phi^{(0)}}E^{-1}\right)\right]^2 \\
&\quad + \frac{1}{2}\left[\frac{\partial}{\partial \xi}\left(f^{(0)} + \Phi^{(0)}E + \overline{\Phi^{(0)}}E^{-1}\right)\right]^2 = 0,
\end{aligned}$$

where the bar denotes the complex conjugate of the respective function. This actually gives us three separate equations, as the whole equation will be zero only if all terms of each order of E is zero (for arbitrary ξ). For example, we collect all terms of order E^0 and then set it equal to zero to obtain

$$\begin{aligned}
A^{(1,0)} - c_{\mathrm{g}}\frac{\partial}{\partial \zeta}f^{(0)} - ikc_{\mathrm{p}}\frac{\partial}{\partial z}\left(\Phi^{(0)}\overline{A^{(0)}} - A^{(0)}\overline{\Phi^{(0)}}\right) \\
+ \frac{1}{\delta^2}\frac{\partial \Phi^{(0)}}{\partial z}\frac{\partial \overline{\Phi^{(0)}}}{\partial z} + \Phi^{(0)}\overline{\Phi^{(0)}} = 0
\end{aligned}$$

We note that from (5.114) we obtain

$$\frac{\partial}{\partial z}\Phi^{(0)} = \delta k F^{(0)} \sinh \delta k(z+1)$$

which leads to, after substituting the expressions for $\Phi^{(0)}$ and $\Phi^{(1,0)}$ (equations (5.114) and (5.123)) and their complex conjugates,

$$A^{(1,0)} - c_\mathrm{g}\frac{\partial}{\partial\zeta} f^{(0)} - i\delta k^2 c_\mathrm{p}\left(F^{(0)}\overline{A^{(0)}} - A^{(0)}\overline{F^{(0)}}\right)\sinh\delta k + k^2 F^{(0)}\overline{F^{(0)}}\left(\sinh^2\delta k + \cosh^2\delta k\right) = 0.$$

This is then the first of the six equations we obtain.

Similar calculations yields for the other orders of E and we will obtain one such equation for each order of E (in this case one for each of E^0, E^1 and E^2). It is obvious that even though the calculations are straightforward, they become cumbersome and lengthy. We therefore omit the other calculations (as they are principally the same as what we already have done) and state all the final equations directly. The three equations obtained from the dynamic free surface condition (5.119) are

$$E^0: \quad A^{(1,0)} + i\delta k^2 c_\mathrm{p}\left(A^{(0)}\overline{F^{(0)}} + \overline{A^{(0)}}F^{(0)}\right)\sinh\delta k + k^2 F^{(0)}\overline{F^{(0)}}\left(\sinh^2\delta k + \cosh^2\delta k\right) - c_\mathrm{g}\frac{\partial f^{(0)}}{\partial\zeta} = 0 \tag{5.124}$$

$$E^1: \quad A^{(1,1)} - c_\mathrm{g}\frac{\partial F^{(0)}}{\partial\zeta}\cosh\delta k - ikc_\mathrm{p}\left(F^{(1,1)}\cosh\delta k - i\delta\frac{\partial F^{(0)}}{\partial\zeta}\sinh\delta k\right) = 0 \tag{5.125}$$

$$E^2: \quad A^{(1,2)} - \frac{1}{2}k^2\left(F^{(0)}\right)^2 - ic_\mathrm{p}k\left(2F^{(1,2)}\cosh 2\delta k + \delta k A^{(0)}F^{(0)}\sinh\delta k\right) = 0. \tag{5.126}$$

We can similarly use the kinematic free surface condition (5.120) to obtain three additional equations

$$E^0: \quad \left(A^{(0)}\overline{F^{(0)}} + \overline{A^{(0)}}F^{(0)}\right)\delta^2 k^2\cosh\delta k = \delta^2 k^2\left(A^{(0)}\overline{F^{(0)}} + \overline{A^{(0)}}F^{(0)}\right)\cosh\delta k \tag{5.127}$$

$$E^1: \quad \delta k F^{(1,1)}\sinh\delta k - i\delta\frac{\partial F^{(0)}}{\partial\zeta}\left(\sinh\delta k + \delta k\cosh\delta k\right) = -\delta^2\left(c_\mathrm{g}\frac{\partial A^{(0)}}{\partial\zeta} + ikc_\mathrm{p}A^{(1,1)}\right) \tag{5.128}$$

$$E^2: \quad 2\delta k F^{(1,2)}\sinh 2\delta k + \delta^2 k^2 A^{(0)}F^{(0,0)}\cosh\delta k = -\delta^2\left(2ikc_\mathrm{p}A^{(1,2)} + k^2 A^{(0)}F^{(0)}\cosh\delta k\right). \tag{5.129}$$

Note how (5.127) is identically satisfied. It is possible to use these equations to determine $A^{(m,n)}$, which is the next task at hand.

Inserting the expression (5.115) for $F^{(0)}$ into the first equation we found at E^0 (5.124) we find

$$\begin{aligned} A^{(1,0)} &= c_{\mathrm{g}} \frac{\partial f^{(0)}}{\partial \zeta} + \delta k \left(-A^{(0)} \overline{A^{(0)}} \operatorname{sech} \delta k + \overline{A^{(0)}} A^{(0)} \operatorname{sech} \delta k \right) \sinh \delta k \\ &\quad - \frac{1}{c_{\mathrm{p}}^2} A^{(0)} \overline{A^{(0)}} \operatorname{sech}^2 \delta k \left(\sinh^2 \delta k + \cosh^2 \delta k \right) = 0, \end{aligned}$$

which means, after using (5.117) for c_{p} and the relation $\sinh 2\delta k = 2 \sinh \delta k \cosh \delta k$, should[12] become

$$A^{(1,0)} = -\frac{2\delta k}{\sinh 2\delta k} A^{(0)} \overline{A^{(0)}} + c_{\mathrm{g}} \frac{\partial f^{(0)}}{\partial \zeta}.$$

The first expression for E^1, equation (5.125), leads directly to

$$A^{(1,1)} = c_{\mathrm{g}} \frac{\partial F^{(0)}}{\partial \zeta} \cosh \delta k + ikc_{\mathrm{p}} \left(F^{(1,1)} \cosh \delta k - i\delta \frac{\partial F^{(0)}}{\partial \zeta} \sinh \delta k \right).$$

It is noted that if use this expression, together with the expressions for c_{p} and $F^{(0)}$ found earlier, in (5.128) an expression for c_{g} can be obtained as

$$c_{\mathrm{g}} = \frac{1}{2} c_{\mathrm{p}} \left(1 + 2\delta k \operatorname{cosech} 2\delta k \right),$$

if we also assume[13] $F_{\zeta}^{(0)} \neq 0$ (because we divide by this throughout before obtaining the expression for c_{g}). This is then the group velocity for water waves (cf. Dingemans [23]chapter 2.1.2).

Following the same procedure, we use equations (5.126), (5.129) and solve for $F^{(1,2)}$ and $A^{(1,2)}$ (again after substituting for $F^{(0)}$) which yields

$$\begin{aligned} F^{(1,2)} &= -\frac{3i}{4} \frac{\delta^2 k c_{\mathrm{p}} \left(A^{(0)} \right)^2}{\sinh^4 \delta k} \\ A^{(1,2)} &= \frac{\delta k \cosh \delta k}{2 \sinh^3 \delta k} \left(2\cosh^2 \delta k + 1 \right) \left(A^{(0)} \right)^2, \end{aligned}$$

where $A^{(0)}$ is yet to be determined.

[12] There seems to be a trick we have missed here as we cannot seem to eliminate the $\left(\sinh^2 \delta k + \cosh^2 \delta k \right)$ factor, while both Johnson and Debnath have managed so. We will continue on with their results.

[13] This actually means we have to assume $\partial A^{(0)} / \partial \zeta \neq 0$.

To obtain the required equation it is needed to go yet another order higher in the asymptotic expansion (up to $O\left(\epsilon^2\right)$). The governing equations at this order are

$$\frac{\partial^2 \phi^{(2)}}{\partial z^2} + \delta^2 \frac{\partial^2 \phi^{(2)}}{\partial \xi^2} + 2\delta^2 \frac{\partial}{\partial \zeta}\frac{\partial \phi^{(1)}}{\partial \xi} + \delta^2 \frac{\partial^2 \phi^{(0)}}{\partial \zeta^2} = 0 \tag{5.130}$$

with at the (modified) conditions on the free surface $z = 0$

$$\frac{\partial \phi^{(0)}}{\partial \tau} + \eta^{(2)} - \left(c_{\mathrm{g}} \frac{\partial \phi^{(1)}}{\partial \zeta} + c_{\mathrm{p}} \frac{\partial \phi^{(2)}}{\partial \xi}\right) + \frac{1}{2}\left[\frac{1}{\delta^2}\left(\frac{\partial^2 \phi^{(1)}}{\partial z^2}\right)^2 + \left(\frac{\partial \phi^{(1)}}{\partial \xi} + \frac{\partial \phi^{(0)}}{\partial \zeta}\right)^2\right] = 0$$

$$\frac{\partial \phi^{(2)}}{\partial z} - \delta^2 \left[\frac{\partial \eta^{(0)}}{\partial \tau} - \left(c_{\mathrm{g}} \frac{\partial \eta^{(1)}}{\partial \zeta} + c_{\mathrm{p}} \frac{\partial \eta^{(2)}}{\partial \xi}\right) + \epsilon \left(\frac{\partial \phi}{\partial \xi} + \epsilon \frac{\partial \phi}{\partial \zeta}\right)\left(\frac{\partial \eta}{\partial \xi} + \epsilon \frac{\partial \eta}{\partial \zeta}\right)\right] = 0$$

and on the bottom $z = -1$

$$\frac{\partial \phi^{(2)}}{\partial z} = 0.$$

The procedure is similar to what we already have followed; Insert the expansions (5.122) and obtain equations for the $\Phi^{(n,m)}$ that arise. For example, from equation (5.130) we obtain the following equations at order E^0 and E^1

$$\frac{\partial^2 \Phi^{(2,0)}}{\partial z^2} + \delta^2 \frac{\partial^2 f^{(0)}}{\partial \zeta^2} = 0, \quad \frac{\partial^2 \Phi^{(2,1)}}{\partial z^2} - \delta^2 k^2 \Phi^{(2,1)} + 2ik\delta^2 \frac{\partial \Phi^{(1,1)}}{\partial \zeta} + \delta^2 \frac{\partial^2 \Phi^{(0)}}{\partial \zeta^2} = 0.$$

Similar yields for higher orders of E (now up to E^3). Next we need to find the solutions of the new $\Phi^{(m,n)}$ and then use the surface conditions to obtain a new set of equations, also obtained from each order of E. The earlier solutions found for $F^{(m,n)}$ and $A^{(m,n)}$ can then be used and an equation in terms of $A^{(0)}$ can be found. A more detailed calculation can be found in for example Debnath [20]. At the end we remain with an equation for $A^{(0)}$, which reads

$$-2ikc_{\mathrm{p}} \frac{\partial A^{(0)}}{\partial \tau} + \alpha \frac{\partial^2 A^{(0)}}{\partial \zeta^2} + \beta A^{(0)} \left|A^{(0)}\right|^2 = 0,$$

with

$$\alpha = c_{\mathrm{g}}^2 - (1 - \delta k \tanh \delta k)\,\mathrm{sech}^2\,\delta k = -k c_{\mathrm{p}} \frac{\mathrm{d}^2\omega}{\mathrm{d}k^2}, \qquad \omega(k) = k c_{\mathrm{p}} \tag{5.131}$$

$$\beta = \frac{k^2}{c_{\mathrm{p}}^2}\left[\frac{1}{2}\left(1 + 9\coth^2\delta k - 13\,\mathrm{sech}^2\,\delta k - 2\tanh^4\delta k\right) - \frac{1}{1-c_{\mathrm{g}}^2}\left(2c_{\mathrm{p}} + c_{\mathrm{g}}\,\mathrm{sech}^2\,\delta k\right)^2\right]. \tag{5.132}$$

This is the NLS equation, describing the amplitude modulation of a harmonic wave in water at leading order. It is possible to normalise the equation by the scaling (provided $\beta \neq 0$)

$$\tau \to -2kc_{\mathrm{p}}\tau, \quad \zeta \to \sqrt{\alpha}\zeta, \quad A^{(0)} \to \frac{1}{\sqrt{\beta}}A^{(0)},$$

so that it becomes on a more common and simpler form

$$i\frac{\partial A^{(0)}}{\partial \tau} + \frac{\partial^2 A^{(0)}}{\partial \zeta^2} + A^{(0)}\left|A^{(0)}\right|^2 = 0. \tag{5.133}$$

Remarks

First we note that the equation here does differ quite from the earlier equations by that it does not describe the surface elevation η nor the velocity u but the *envelope* of a group of waves on the surface.

It is possible to make additional approximation by assuming shallow water by letting $\delta \to 0$, or similarly infinity depth by letting $\delta \to \infty$. One can then use approximations like $\mathrm{sech}\,\delta k \to 1$ for $\delta \to 0$ (or $\mathrm{sech}\,\delta k \to 0$ for $\delta \to \infty$). Note however that the validity of the approximations then must be questioned. For example for shallow water (long waves), $\delta \to 0$ we obtain

$$-2ik\delta^2\frac{\partial A^{(0)}}{\partial \tau} + \delta^4 k^2 \frac{\partial^2 A^{(0)}}{\partial \zeta^2} - \frac{9}{2}A^{(0)}\left|A^{(0)}\right|^2 = 0,$$

while for deep water (short waves), $\delta \to \infty$, we instead have

$$-2i\sqrt{\frac{k}{\delta}}\frac{\partial A^{(0)}}{\partial \tau} + \frac{1}{4\delta k}\frac{\partial^2 A^{(0)}}{\partial \zeta^2} + 4k^3\delta A^{(0)}\left|A^{(0)}\right|^2 = 0.$$

Note that the relative sign of $A_{xx}^{(0)}$ and $A^{(0)}|A^{(0)}|^2$ is important for the characteristics of the solution (thus the transformation $\tau \to -2kc_{\mathrm{p}}\tau$ applied when normalising does not change the characteristics of the solution, it simply gives the complex conjugate equation). When the term $A^{(0)}|A^{(0)}|^2$ is positive the solutions are "self-focusing" while for the negative case they are "defocussing" (Peregrine [60]). Thus in the case of water waves, the solutions are "self-focusing". For a more complete handling on the different solutions of the NLS equation we refer to the paper of Peregrine [60] or the book of Kamvissis *et al.* [46]

Another property the NLS equation shares with the other equations is soliton solutions. A soliton solution for a self-focusing NLS equation (5.133) can be found by first assuming a travelling wave solution on the form (cf. Sulem & Sulem [63])

$$A = e^{i(ax-ct)}\Psi(\xi),$$

where a and c are arbitrary real constants, $\xi = x - ct$ and Ψ assumed to be a real function and decaying at infinity (as is one of the soliton properties). Inserting the travelling wave assumption into the NLS equation yields

$$e^{i(ax-ct)}\left(c\Psi - ic\frac{\partial\Psi}{\partial\xi}\right)+e^{i(ax-ct)}\left(-a^2\Psi + 2ia\frac{\partial\Psi}{\partial\xi} + \frac{\partial^2\Psi}{\partial\xi^2}\right)+e^{i(ax-ct)}\Psi^3 = 0,$$

where we can divide out the common exponential factor and choose $a = c/2$ to eliminate the Ψ_ξ terms. Thus

$$\frac{\partial^2\Psi}{\partial\xi^2} - r\Psi + \Psi^3 = 0,$$

where $r = a^2 - c$. Multiplying throughout with Ψ_ξ, integrating and again imposing the condition of Ψ and its derivatives vanishing at infinity (which eliminates any constants of integration) yields

$$\left(\frac{\partial\Psi}{\partial\xi}\right)^2 = r\Psi^2 - \frac{1}{2}\Psi^4,$$

or by dividing by Ψ^4

$$\left(\frac{\partial q}{\partial\xi}\right)^2 = rq^2 - \frac{1}{2},$$

where $q = 1/\Psi$. This can be solved (also see the soliton solution under remarks in the KdV section) to obtain[14]

$$\frac{1}{\sqrt{r}}\cosh^{-1}\sqrt{2r}q = \xi$$

or equally

$$q = \frac{1}{\sqrt{2r}}\cosh\sqrt{r}\xi$$

[14] The negative case from the square root does not yield acceptable solutions (cf. Sulem & Sulem [63]) and is therefore not pursued here.

which in terms of Ψ becomes

$$\Psi = \sqrt{2r}\frac{1}{\cosh\sqrt{r}\xi}.$$

Thus, substituting back for A (and also a, b and ξ) we have the soliton solution

$$A(x,t) = \sqrt{2r}\frac{1}{\cosh\sqrt{r}(x-ct)}e^{i\left(\frac{c}{2}x+(r-\frac{c^2}{4})t\right)}.$$

There exists a two-dimensional version of the NLS equation, similar to how the KP equation is related to the KdV equation. This will be the topic of the next section.

5.3.4 Davey-Stewartson equations

In this section we include a second horizontal dimension in the NLS equation and so obtain two equations describing the evolution of a three-dimensional wave-packet of wave number k on water of finite depth. These two equations are called the Davey-Stewartson (DS) equations, which were "originally" derived in 1974 by Davey & Stewartson [19] and with the assumption of irrotationality. We write "originally" since the same equations were actually derived by Benney and Newell seven years earlier [5], as remarked by Dingemans in [24, p. 890].

We assume the amplitude modulation to occur in both horizontal directions, while the oscillations only to occur along the x-direction. That is, as initial data we have

$$\eta(x,y,0) = A(x,y)\,e^{ikx} + \text{c.c},$$

where c.c denotes the complex conjugate.

The procedure now adopted is similar to what Davey and Stewartson did in [19] and follows previous section closely. We transform the variables by (now also including the y-dependence)

$$\zeta = \epsilon\,(x - c_{\text{g}}t)\,, \quad \xi = x - c_{\text{p}}t, \quad Y = \epsilon y, \quad \tau = \epsilon^2 t,$$

where ζ describes the modulated wave and ξ describes the carrier wave. The variables $c_{\text{p}} = c_{\text{p}}(k)$ and $c_{\text{g}} = c_{\text{g}}(k)$ are to be determined (but will actually become as in the previous section).

This leads to the following set of governing equations

$$\frac{\partial^2\phi}{\partial z^2} + \delta^2\left(\frac{\partial^2\phi}{\partial\xi^2} + 2\epsilon\frac{\partial}{\partial\zeta}\frac{\partial\phi}{\partial\xi} + \epsilon^2\frac{\partial^2\phi}{\partial\zeta^2} + \epsilon^2\frac{\partial^2\phi}{\partial Y^2}\right) = 0,$$

with on free-surface $z = \epsilon\eta$

$$\epsilon^2\frac{\partial\phi}{\partial\tau} + \eta - \left(\epsilon c_{\mathrm{g}}\frac{\partial\phi}{\partial\zeta} + c_{\mathrm{p}}\frac{\partial\phi}{\partial\xi}\right) + \frac{\epsilon}{2}\left[\frac{1}{\delta^2}\left(\frac{\partial^2\phi}{\partial z^2}\right)^2 + \left(\frac{\partial\phi}{\partial\xi} + \epsilon\frac{\partial\phi}{\partial\zeta}\right)^2 + \epsilon\left(\frac{\partial\phi}{\partial Y}\right)^2\right] = 0$$

$$\frac{\partial\phi}{\partial z} - \delta^2\left[\epsilon^2\frac{\partial\eta}{\partial\tau} - \left(\epsilon c_{\mathrm{g}}\frac{\partial\eta}{\partial\zeta} + c_{\mathrm{p}}\frac{\partial\eta}{\partial\xi}\right) + \epsilon\left(\frac{\partial\phi}{\partial\xi} + \epsilon\frac{\partial\phi}{\partial\zeta}\right)\left(\frac{\partial\eta}{\partial\xi} + \epsilon\frac{\partial\eta}{\partial\zeta}\right) + \epsilon^3\frac{\partial\phi}{\partial Y}\frac{\partial\phi}{\partial Y}\right] = 0$$

and at the bottom $z = -1$

$$\frac{\partial\phi}{\partial z} = 0$$

We note all the Y-dependencies which occur are of order $O\left(\epsilon^2\right)$ and the only contribution from the Y-dependency will, due to nonlinearities, be through the term ϕ_{YY} in the Laplace equation.

Following the procedure as in the case with the NLS equation, we search for asymptotic solutions as

$$\phi(\xi,\zeta,Y,z,\tau;\epsilon) \sim f^{(0)}(\zeta,Y,\tau) + \sum_{n=0}^{\infty}\epsilon^n\phi^{(n)}(\xi,\zeta,Y,z,\tau)$$

$$\eta(\xi,\zeta,Y,\tau;\epsilon) \sim \sum_{n=0}^{\infty}\epsilon^n\eta^{(n)}(\xi,\zeta,Y,\tau).$$

Taking into account the higher harmonics we also write

$$\phi^{(m)} = \sum_{n=0}^{m+1}\Phi^{(m,n)}E^n + \text{c.c}, \quad \eta^{(m)} = \sum_{n=0}^{m+1}A^{(m,n)}E^n + \text{c.c},$$

where $m = 1, 2, 3, ...$ and the functions $\Phi^{(m,n)} = \Phi^{(m,n)}(\zeta, Y, z, \tau)$ and $A^{(m,n)} = A^{(m,n)}(\zeta, Y, \tau)$ are to be determined.

Since the Y dependence first occurs at order $O\left(\epsilon^2\right)$ the derivation at leading and first order are very closely related to the NLS derivation. In fact, they are so closely related that we can use the results from the previous section, rather than doing the same calculations over again. As mentioned, the first difference from the earlier derivation occurs at order $O\left(\epsilon^2\right)$ and then for the equation which emerges from the boundary conditions at E^0 :

$$(1 - c_{\mathrm{g}}^2)\frac{\partial^2 f^{(0)}}{\partial\zeta^2} + \frac{\partial^2 f^{(0)}}{\partial Y^2} = \frac{\gamma}{c_{\mathrm{p}}^2}\frac{\partial}{\partial\zeta}|A^{(0)}|^2, \tag{5.134}$$

where

$$\gamma = 2c_\mathrm{p} + c_\mathrm{g}\,\mathrm{sech}^2\,\delta k.$$

This is an equation for $f^{(0)}$ and can only be solved for $A^{(0)}$ given. We can obtain $A^{(0)}$ from an equation which emerges at E^1 (again from the boundary conditions)

$$\begin{aligned} &-2ikc_\mathrm{p}\frac{\partial A^{(0}}{\partial \tau} + \alpha\frac{\partial^2 A^{(0)}}{\partial \zeta^2} - c_\mathrm{p}c_\mathrm{g}\frac{\partial^2 A^{(0)}}{\partial Y^2} \\ &\quad + \left[\beta + \frac{\gamma^2 k^2}{c_\mathrm{p}^2(1-c_\mathrm{g}^2)}\right] A^{(0)}|A^{(0)}|^2 + \gamma k^2 A^{(0)}\frac{\partial f^{(0)}}{\partial \zeta} = 0, \end{aligned} \tag{5.135}$$

with α and β are as given in the previous section (equations (5.131) and (5.132)) and γ as above. These two equations are called the Davey-Stewartson equations and describe in three dimensions the amplitude modulation of harmonic waves in water of finite depth.

Remarks

First we note how the NLS equation can be obtained from the DS equation by ignoring the second horizontal variable Y. This yields for (5.134)

$$(1-c_\mathrm{g}^2)\frac{\partial^2 f^{(0)}}{\partial \zeta^2} = \frac{\gamma}{c_\mathrm{p}^2}\frac{\partial}{\partial \zeta}|A^{(0)}|^2,$$

which can be integrated once to obtain

$$\frac{\partial f^{(0)}}{\partial \zeta} = \frac{\gamma}{(1-c_\mathrm{g}^2)c_\mathrm{p}^2}|A^{(0)}|^2.$$

The equation for $A^{(0)}$ (5.135) then reduces to, after substituting for f_ζ,

$$-2ikc_\mathrm{p}\frac{\partial A^{(0}}{\partial \tau} + \alpha\frac{\partial^2 A^{(0)}}{\partial \zeta^2} + \left[\beta + \frac{2\gamma^2 k^2}{c_\mathrm{p}^2(1-c_\mathrm{g}^2)}\right] A^{(0)}|A^{(0)}|^2 = 0,$$

which is the NLS equation.

As with the NLS equation, it is possible to make additional approximations by assuming shallow water by letting $\delta \to 0$ and similarly infinity depth by letting $\delta \to \infty$. The corresponding equations become for $\delta \to 0$

$$\begin{aligned} &-2ik\delta^2\frac{\partial A^{(0)}}{\partial \tau} + \delta^4 k^2\frac{\partial^2 A^{(0)}}{\partial \zeta^2} - \delta^2\frac{\partial^2 A^{(0)}}{\partial Y^2} \\ &\qquad - \frac{9}{2}A^{(0)}\left|A^{(0)}\right|^2 + 3\delta^2 k^2 A^{(0)}\frac{\partial f^{(0)}}{\partial \zeta} = 0 \end{aligned}$$

and

$$\delta^2 k^2 \frac{\partial^2 f^{(0)}}{\partial \zeta^2} + \frac{\partial^2 f^{(0)}}{\partial Y^2} = -3 \frac{\partial}{\partial \zeta} |A^{(0)}|^2.$$

On the other hand, for infinite depth, i.e. $\delta \to \infty$ we have (which also yields $c_{\mathrm{p}} \sim 1/\sqrt{\delta k}$)

$$\begin{aligned} - 2i\sqrt{\frac{k}{\delta}}\frac{\partial A^{(0)}}{\partial \tau} + \frac{1}{4\delta k}\frac{\partial^2 A^{(0)}}{\partial \zeta^2} \\ - \frac{1}{2\delta k}\frac{\partial^2 A^{(0)}}{\partial Y^2} + 4k^3\delta \left|A^{(0)}\right|^2 + 2k\sqrt{\frac{k}{\delta}}A^{(0)}\frac{\partial f^{(0)}}{\partial \zeta} = 0 \end{aligned}$$

and

$$\frac{\partial^2 f^{(0)}}{\partial \zeta^2} + \frac{\partial^2 f^{(0)}}{\partial Y^2} = -2\sqrt{\delta k} \frac{\partial}{\partial \zeta} |A^{(0)}|^2.$$

For a handling of the soliton solutions and the Lax pairs of the DS equation we refer to the paper of Boiti *et al.* [7].

Chapter 6

Summary and discussion

All knowledge has given me is
more questions.

Unknown

6.1 Summary

The subject of water waves is indeed immense and what we have presented in this thesis is actually just a small part of the whole topic; the derivation of equations belonging to a specific class, the class of completely integrable equations. These equations are far from all the equations describing wave motion in water. Other examples are the Green-Naghdi (GN) equations, the Benjamin-Bona-Mahoney (BBM) equation use of Airy theory or Stokes theory, to mention but a few. These do, however, not necessarily belong to the same class as the equations we have derived in this thesis.

We started out deriving the general equations for wave motion in a fluid, both with and without the condition of irrotationality. More precisely, Euler's equation with the condition of an incompressible fluid with no viscosity was derived. These assumptions gave a fairly accurate description in the case of water. The former assumption could be applied without great loss of generality, as long as the temperature is held constant. The latter assumption could be made as long as the interaction between the boundary and the water was neglected, say the wind (or another fluid) and water, which was beyond the scope of this thesis.

We continued by imposing certain conditions at the boundaries; A kinematic and dynamic condition at the free surface and a kinematic condition at the bottom. At the free surface, the boundary between air and water, we assumed the water to be able to move freely and the pressure to be constant, i.e. no wind. We included in the general equations the surface tension, which led to a pressure difference due to the curvature along the free surface. The bottom was assumed to be solid and no water could thus pass through this boundary. Later on we also assumed the bottom to be non-moving, as a non-moving bottom only excludes the modeling of underwater landslides and/or earthquakes which also was beyond the scope of this thesis.

The resulting system then became non-trivial, as one of the unknown variables, the variable η, appeared in the equation describing the free surface boundary. On several occasions we solved this by Taylor expanding the boundary conditions on $z = \epsilon\eta$ around $z = 0$, which imposed additional requirements on the respective functions (the functions $\mathbf{u}$ and p); We are only assured the Taylor expansion exists if the functions are analytical. However, as Johnson points out, any convergence requirements are unnecessary as the functions need only to satisfy the conditions laid down for asymptotic validity as $\epsilon \to 0$.

After having derived the full set of governing equations we rewrote them on a non-dimensional form, which resulted in discovering an amplitude parameter ϵ and a shallowness parameter δ. These two parameters were then used to describe the regime of water waves in a rather precise way. For $\epsilon \to 0$, the amplitude was said to be small (compared to the water depth) and for $\delta \to 0$ the water was said to be shallow (the wave length was large compared to the water depth). Conversely, for $\epsilon \to \infty$ the amplitude was large (again compared to the water depth) and for $\delta \to \infty$ the water was deep (short wave length compared to the water depth). We mainly looked at the former situation for ϵ, the small amplitude regime for fixed δ. We also derived two equations where we in addition assumed shallow water, $\delta \to 0$.

In the following sections we applied the theory of asymptotic analysis on the governing equations, after some appropriate rescaling. The asymptotic expansions were based on taking the limit (to zero) in one or both of the parameters ϵ and δ^2. Depending on how the governing equations were rescaled, if any a priori assumptions were made (e.g. one-way travelling waves) and how far the asymptotic expansions were taken, we derived different completely integrable equations describing the wave propagation. For simplicity we assumed a horizontal bottom and zero surface tension in all our derivations, except for the KdV equation, where we included the surface tension for the sake of illustration, and showed that this only led to a minor change in the final result.

At leading order of the asymptotic approximations (for $\epsilon \to 0$, δ fixed), without any rescaling or transformations, the linearised water wave equations were obtained. At the same order with the additional assumption of shallow water, by also letting $\delta \to 0$, we obtained the linearised shallow water wave equations and showed these could be combined to obtain the classical wave equation.

We then continued by expanding the asymptotic series up to the first order, still for $\epsilon \to 0$ and δ fixed. Assuming propagation only in one dimension and rescaling of the spatial variables by $\xi = (\delta/\sqrt{\epsilon})x$ and $\tau = (\delta/\sqrt{\epsilon})t$, the KdV equation was obtained. This scaling then replaced any δ^2 with ϵ in the governing equations. The KdV equation was furthermore shown to have soliton solutions and we also stated its Lax pairs.

At the same order, with the same assumption of one-directional waves but also including the second horizontal dimension (appropriately scaled by $Y = (\delta/\sqrt{\epsilon})y$), the KP equation, a two-dimensional sibling of the KdV equation, was obtained. The soliton solutions of this equation were also stated and it was shown how the KdV equation easily could be obtained when neglecting the second horizontal dependency.

Still at the same order and with the same scaling as the previous two cases, we relaxed the conditions to include waves propagating in both directions, and found that the Boussinesq equation then could be obtained. We did, however, again restrict the derivation to one horizontal dimension. A soliton solution of the Boussinesq equation was derived and it was mentioned that the KdV equation can be obtained from this equation.

All the equations at first order describe the wave propagation for $\epsilon \to 0$ and δ fixed and in the region of space and time determined by ξ, Y and τ at order $O(1)$. That is, in the far field of space, large x and y, and at long times, i.e. large t. The scaling imposed that these equations describe wave propagation in shallow water.

At second order asymptotic approximation, with the assumption one-dimensional one-way travelling waves in the spatial region determined by the scaling $\xi = \sqrt{\epsilon}(x - t)$ and $\tau = \epsilon\sqrt{\epsilon}t$, we obtained the CH and the DP equations. These equations did not, however, describe the free surface elevation η as in case with the equations at first order. These did instead describe the horizontal velocity at at certain depth and also included the possibility of the wave breaking.

For the final equations we derived at second order, the assumption of a modulated amplitude was made. The spatial region under consideration was again altered, determined by the transformations $\zeta = \epsilon\,(x - c_{\mathrm{g}}t)$, $\xi = x - c_{\mathrm{p}}t$ and $\tau = \epsilon^2 t$, where ζ described the modulated wave and ξ described the carrier wave. The variables c_{g} and c_{p} were the group and phase speed, respectively. This led us to obtain the NLS equation and its two-dimensional sibling, the DS equation (where the second horizontal variable was scaled as $Y = \epsilon y$). We showed how additional approximations could be made by either letting $\delta \to 0$ or $\delta \to \infty$ and also derived a soliton solution of the NLS equation.

All of the equations we derived, except the linear case, are completely integrable and have solutions of soliton type. We have stated some of these properties and derived some of the solutions, but have far from dwelled deep into this part of the subject.

6.2 Further work

As we have mentioned several times, the topic concerning water waves is extremely large and there are any number of areas inside the subject worth studying in more detail, which might make it difficult to choose in which direction to take the work. However, from the point we are now, the first few steps of the path ahead lies clear.

We have for each equation made the fairly strict assumptions of imposing a horizontal bottom, neglecting the surface tension (except for the KdV equation), neglecting any viscous effects, looking at one-directional waves (except for the Boussinesq equation), constant surface pressure, incompressibility and assuming a stationary underlying flow. We have also several times assumed wave propagation in only one horizontal direction, which can partially be justified by saying the waves are uniform in the second horizontal dimension (infinite plane waves) or that they describe uniform cylindrical wave propagation (then the wave is the same, no matter the direction on the horizontal plane). Every one of the simplifications introduced leads the mathematical model further away from the real physical problem and it is therefore clear that relaxing one or more of these restrictions is desirable.

Some changes lead only to minor modifications in some of the equations, e.g. inclusion/exclusion of surface tension in the KdV equation led to a minor modification in the final result, but they can equally lead to major modifications in other derivations. Some relaxations might even make the derivations in this thesis impossible. We have for example relied heavily on the assumption of an inviscid incompressible fluid and should one or both of these assumptions be changed, the basic governing equations (which we derive all the equations from) will become significantly modified. Another example is seen when avoiding the assumption of one-directional waves for the KdV equation (or equally the other with this assumption) will void the derivation from the beginning. Or, should one assume an uneven bottom the assumptions of one-directional waves cannot be held, as an uneven bottom can lead to reflected waves. Note, however, that one could then impose conditions of the bottom “varying slowly” which might make the derivation still possible, where slowly can be determined from, say, the parameter ϵ.

Other types of borders might also be of interest. For example a solid vertical boundary somewhere along the x-direction can be imposed in the simplest case. Again this might result in for example reflected waves and one must be careful which assumptions to make when deriving the corresponding equations. A solid boundary instead of air, looking at two-fluid problems (where one fluid lies on top of the other) or solid boundaries of arbitrary geometry somewhere in the domain (i.e. ships or underwater constructions) are all areas of interest and only results in relatively minor changes in the governing equations.

Other directions to be pursued are studying the properties and solutions of each equation obtained in more detail. We have already mentioned the completely integrability property and the soliton solutions but have not proven nor studied the stability and uniqueness (well-posedness) of the solutions, the interaction of different (soliton) solutions with each other or if they for example are globally valid. This is an area which certainly should obtain more attention. Furthermore we have neither studied the energy of the system and how this behaves nor studied what happens when the wave breaks.

Another question one might ask is if there exists any other equation with similar properties as derived here, but with different scaling and transformations (i.e. other regions of space and time) and perhaps with different asymptotic approaches (e.g. taking the limit on other parameters) or taking the asymptotic expansions further than second order. The CH equation was only discovered some fifteen years ago and the following Degasperis-Procesi even more recently and there is no reason to believe that there does not exists more equations yet to be discovered.

As a final note it is clear that the theory of water waves has given much insight in not only to the physical problem but also to the mathematics hiding behind the problem. Due to the remarkable properties of the KdV equation (and later it was shown that other equations also had similar properties) many powerful tools have been developed; The Inverse Scattering Transform, Hirota's method, the Miura transformation and the Lax equation to mention but a few. Continuing discovering similar equations and developing tools to analyse these will lead to a more comprehensive understanding of mathematics itself and to the world around us.

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Appendix A

Derivation of the unsteady Bernoulli equation for irrotational flow

In this section we derive the unsteady Bernoulli equation for irrotational flow, which means that the velocity potential ϕ exists and is defined by $\mathbf{u} = \nabla\phi$. The free surface is at $z = \eta(x, y, t)$ and the solid (non-moving) bottom at $z = -h(x, y)$. We repeat Euler's equation on compent form, which reads

$$\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \quad \frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \quad \frac{\mathrm{D}w}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g.$$

where

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right).$$

By using the definition of ϕ, these equations can be written as

$$\frac{\partial}{\partial x}\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}\left(u^2 + v^2 + w^2\right) + \frac{p}{\rho}\right] = 0,$$
$$\frac{\partial}{\partial y}\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}\left(u^2 + v^2 + w^2\right) + \frac{p}{\rho}\right] = 0,$$
$$\frac{\partial}{\partial z}\left[\frac{\partial\phi}{\partial t} + \frac{1}{2}\left(u^2 + v^2 + w^2\right) + \frac{p}{\rho} + gz\right] = 0,$$

or by integrating once

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\left(u^2 + v^2 + w^2\right) + \frac{p}{\rho} = f_1(y, z, t),$$
$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\left(u^2 + v^2 + w^2\right) + \frac{p}{\rho} = f_2(x, z, t),$$
$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\left(u^2 + v^2 + w^2\right) + \frac{p}{\rho} = -gz + f_3(x, y, t),$$

where f_i are the integration constants. Subtracting the first equation from the second gives us $f_1 = f_2$, saying these two equations actually are the same and that both f_1 and f_2 only depends on z and t. Due to this, we will only consider the first of the two equations from this point on. Subtracting the third equation from the first, on the other hand, gives us a relation between the integration constants

$$f_3(x, y, t) = f_1(z, t) + gz.$$

We notice that the right hand-side is independent of both x and y which means that also f_3 is independent of x and y and $f_3(x, y, t) = f(t)$. Furthermore this yields $f_1(z, t) = f(t) - gz$. Substituting this into the first equation then gives

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(u^2 + v^2 + w^2\right) + \frac{p}{\rho} = f(t) - gz,$$

and we have obtained the unsteady Bernoulli equation, valid in the region $-h(x, y) \leq z \leq \eta(x, y, t)$.

Note that the integration constant $f(t)$ can be eliminated by defining

$$\frac{\partial \Phi}{\partial t} = \frac{\partial \phi}{\partial t} - f(t)$$

and the unsteady Bernoulli equation then becomes

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}\left(u^2 + v^2 + w^2\right) + \frac{p}{\rho} + gz = 0$$

or

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}\nabla\Phi^2 + \frac{p}{\rho} + gz = 0.$$

Appendix B

Calculations of dimensionless variables

In this section we show the calculations that lead to the governing equations on non-dimensional form. The choice of scales and justification of the choices can be found in chapter 4. We repeat the non-dimensional variables, denoted by primes,

$$
\begin{aligned}
&x' = \frac{x}{\lambda}, \quad y' = \frac{y}{\lambda}, \quad z = \frac{z}{h_0}, \quad t' = \frac{U}{\lambda}t, \quad h' = \frac{h}{h_0} \\
&u' = \frac{u}{U}, \quad v' = \frac{v}{U}, \quad w' = \frac{\lambda}{h_0 U}w, \quad \eta' = \frac{\eta}{a}.
\end{aligned}
$$

and the pressure rewritten as

$$p = p_{\mathrm{a}} + \rho g z + \rho g h_0 p'.$$

The two terms of the first component in Euler's equation are

$$
\begin{aligned}
\frac{\mathrm{D}u}{\mathrm{D}t} &= \frac{U}{\frac{\lambda}{U}}\frac{\mathrm{D}u'}{\mathrm{D}t'} = \frac{U^2}{\lambda}\frac{\mathrm{D}u'}{\mathrm{D}t'} \\
\frac{\partial p}{\partial x} &= \frac{1}{\lambda}\frac{\partial}{\partial x'}(p_a + \rho g h_0 z' + \rho g h_0 p') = \frac{\rho U^2}{\lambda}\frac{\partial p'}{\partial x'},
\end{aligned}
$$

which gives the first component on non-dimensional form

$$
\begin{aligned}
&\frac{U^2}{\lambda}\frac{\mathrm{D}u'}{\mathrm{D}t'} = -\frac{1}{\rho}\frac{\rho U^2}{\lambda}\frac{\partial p'}{\partial x'} \\
&\Rightarrow \frac{\mathrm{D}u'}{\mathrm{D}t'} = -\frac{\partial p'}{\partial x'}.
\end{aligned}
$$

Similar calculations yields for the second component. The vertical component in Euler's equation is slightly different due to use of a different scale:

$$\begin{aligned}
\frac{\mathrm{D}w}{\mathrm{D}t} &= \frac{h_0 U}{\lambda \frac{\lambda}{U}} \frac{\mathrm{D}w'}{\mathrm{D}t'} = \frac{h_0 U^2}{\lambda^2} \frac{\mathrm{D}w'}{\mathrm{D}t'} \\
\frac{\partial p}{\partial z} &= \frac{1}{h_0} \frac{\partial}{\partial z'} (p_a + \rho g h_0 z' + \rho g h_0 p') = \rho g \frac{\partial p'}{\partial z'} + \rho g,
\end{aligned}$$

which gives

$$\begin{aligned}
\frac{h_0 U^2}{\lambda^2} \frac{\mathrm{D}w'}{\mathrm{D}t'} &= \frac{1}{\rho} \left(\rho g \frac{\partial p'}{\partial z'} + \rho g \right) - g \\
\Rightarrow \delta^2 \frac{\mathrm{D}w'}{\mathrm{D}t'} &= -\frac{\partial p'}{\partial z'},
\end{aligned}$$

where $\delta = h_0/\lambda$ is the ratio between the depth and wavelength and is the first of two parameters we will obtain. This parameter is called the "long wavelength" or "shallowness" parameter and tells us how "deep" the water is. We also note that one term has been eliminated due to how we redefined p.

The conservation of mass yields

$$\begin{aligned}
0 &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \\
&= \frac{U}{\lambda} \frac{\partial u'}{\partial x'} + \frac{U}{\lambda} \frac{\partial v'}{\partial y'} + \frac{h_0 U}{\lambda h_0} \frac{\partial w'}{\partial z'} \\
&= \frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} + \frac{\partial w'}{\partial z'},
\end{aligned}$$

where we now see that the vertical velocity scale is correct.

The kinematic free-surface boundary condition (3.7) on non-dimensional form becomes

$$\begin{aligned}
0 &= w - \frac{\partial \eta}{\partial t} - \left(u \frac{\partial \eta}{\partial x} + v \frac{\partial \eta}{\partial y} \right) \\
&= \frac{U h_0}{\lambda} w' - \frac{a}{\frac{\lambda}{U}} \frac{\partial \eta'}{\partial t'} - \frac{aU}{\lambda} \left(u' \frac{\partial \eta'}{\partial x'} + v' \frac{\partial \eta'}{\partial y'} \right),
\end{aligned}$$

where we can divide throughout by λ/U, rearrange the terms and define the second parameter $\epsilon \equiv a/h_0$

$$w' = \epsilon \left[\frac{\partial \eta'}{\partial t'} + \left(u' \frac{\partial \eta'}{\partial x'} + v' \frac{\partial \eta'}{\partial y'} \right) \right] \qquad z' = \epsilon \eta'(x, y, t).$$

The dynamic free surface condition reads (on $z = \eta$)

$$\begin{aligned} 0 &= p - p_{\mathrm{a}} \\ &= \left(p_{\mathrm{a}} + \rho g \eta + \rho g h_0 p'\right) - p_{\mathrm{a}} \\ &= a\eta' - h_0 p' \end{aligned}$$

and thus $p' = \epsilon\eta'$. If surface tension should be included we also need to non-dimensionalize $\mathcal{T}$

$$\begin{aligned} \mathcal{T} &= \Gamma \frac{\left(1+\eta_y^2\right)\eta_{xx} + \left(1+\eta_x^2\right)\eta_{yy} - 2\eta_{xy}\eta_x\eta_y}{\left(1+\eta_x^2+\eta_y^2\right)^{\frac{3}{2}}} \\ &= \Gamma \frac{\left(1+\frac{a^2}{\lambda^2}(\eta'_{y'})^2\right)\frac{a}{\lambda^2}\eta'_{x'x'} + \left(1+\frac{a^2}{\lambda^2}(\eta'_{x'})^2\right)\frac{a}{\lambda^2}\eta'_{y'y'} - 2\frac{a^3}{\lambda^4}\eta'_{x'y'}\eta'_{x'}\eta'_{y'}}{\left(1+\frac{a^2}{\lambda^2}(\eta'_{x'})^2+\frac{a^2}{\lambda^2}(\eta'_{y'})^2\right)^{\frac{3}{2}}} \\ &= \Gamma \frac{\epsilon\delta}{\lambda} \frac{\left(1+\epsilon^2\delta^2(\eta'_{y'})^2\right)\eta'_{x'x'} + \left(1+\epsilon^2\delta^2(\eta'_{x'})^2\right)\eta'_{y'y'} - 2\epsilon^2\delta^2\eta'_{x'y'}\eta'_{x'}\eta'_{y'}}{\left(1+\epsilon^2\delta^2(\eta'_{x'})^2+\epsilon^2\delta^2(\eta'_{y'})^2\right)^{\frac{3}{2}}}, \end{aligned}$$

where we have used that $a/\lambda = \epsilon\delta$. Thus

$$p' = \epsilon\left(\eta' - \frac{1}{\rho g \lambda^2}\mathcal{T}'\right).$$

The last boundary condition is rewritten in the same manner and reads

$$\begin{aligned} 0 &= w - \frac{\partial h}{\partial t} - \left(u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}\right) \\ &= \frac{h_0 U}{\lambda}w' - \frac{h_0 U}{\lambda}\frac{\partial h'}{\partial t'} - \frac{U h_0}{\lambda}\left(u'\frac{\partial h'}{\partial x'} + v\frac{\partial h'}{\partial y'}\right) \\ &= w' - \frac{\partial h'}{\partial t'} - \left(u'\frac{\partial h'}{\partial x'} + v\frac{\partial h'}{\partial y'}\right) \end{aligned}$$

on $z' = -h(x,y,t)/h_0$.

We can also use these non-dimensional variables to find the scaling for the velocity potential, e.g. the first velocity component in terms of the velocity potential gives

$$u' = \frac{u}{U} = \frac{1}{U}\frac{\partial\phi}{\partial x} = \frac{1}{\lambda U}\frac{\partial\phi}{\partial x'} = \frac{\partial\phi'}{\partial x'}$$

and we find

$$\phi' = \frac{1}{\lambda U}\phi.$$

If the flow is irrotational, the zero-vorticity condition reads

$$\delta^2\frac{\partial w'}{\partial y'} - \frac{\partial v'}{\partial z'} = 0, \quad \frac{\partial u'}{\partial z'} - \delta^2\frac{\partial w'}{\partial x'} = 0, \quad \frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'} = 0$$

and the dynamic boundary condition instead becomes

$$\epsilon\eta' + \frac{\partial \phi'}{\partial t'} + \frac{1}{2}\mathbf{u}' \cdot \mathbf{u}' = 0, \qquad \text{on} \quad z' = \epsilon\eta'(x, y, t).$$

Appendix C

Original derivation of the Boussinesq equation

We will here derive the equations that Boussinesq himself obtained in 1872 (Boussinesq [8]) as described by Dingemans [24]. The assumption of an uniform depth $h(x, y) = h_0$ is made and only wave propagation in one dimension is considered. Furthermore we assume irrotationality, such that the velocity potential ϕ exists.

In chapter 3 we derived the following governing equations (and ignoring the second horisontal coordinate)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0, \qquad -h \leq z \leq \eta(x, t)$$

with on the free surface $z = \eta(x, t)$

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} = \frac{\partial \phi}{\partial z}\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \phi^2}{\partial x} + \frac{\partial \phi^2}{\partial z}\right) + gz = 0$$

and on the bottom $z = -h$

$$\frac{\partial \phi}{\partial z} = 0.$$

The first equation is the Laplace equation in two dimensions and can be integrated twice with respect to z to obtain

$$\phi(x, z, t) = \phi(x, -h, t) - \int_{-h}^{z} \int_{-h}^{z'} \frac{\partial^2 \phi}{\partial x^2} \,\mathrm{d}z \,\mathrm{d}z', \tag{C.1}$$

where we have used the kinematic condition $\phi_z = 0$ at the bottom $z = -h$.

The main approximation now made is assuming the velocity potential ϕ is almost the same both at the top and bottom of the fluid, which can be expected in shallow water. We denote the velocity potential at the bottom by $\phi_b(x,t) = \phi(x,-h,t)$ and the approximation is made by setting $\phi \approx \phi_b$. Now the integral in (C.1) becomes

$$
\begin{aligned}
\Phi(x,z,t) &\approx \phi_b(x,t) - \int_{-h}^{z}\int_{-h}^{z'} \frac{\partial^2\phi_b}{\partial x^2}\,\mathrm{d}z\,\mathrm{d}z' \\
&= \phi_b(x,t) - \frac{\partial^2\phi_b}{\partial x^2}\int_{-h}^{z}\int_{-h}^{z'} \mathrm{d}z\,\mathrm{d}z' \\
&= \phi_b(x,t) - \frac{\partial^2\phi_b}{\partial x^2}\int_{-h}^{z} z' + h\,\mathrm{d}z' \\
&= \phi_b(x,t) - \frac{\partial^2\phi_b}{\partial x^2}\left(\frac{1}{2}z^2 + hz + \frac{1}{2}h^2\right) \\
&= \phi_b(x,t) - \frac{1}{2}\left(z^2 + 2hz + h^2\right)\frac{\partial^2\phi_b}{\partial x^2} \\
&= \phi_b(x,t) - \frac{(z+h)^2}{2}\frac{\partial^2\phi_b}{\partial x^2}.
\end{aligned}
$$

If we subsitute this once more back into (C.1) it is easy to verify that we obtain

$$
\phi(x,z,t) \approx \phi_b(x,t) - \frac{(z+h)^2}{2}\frac{\partial^2\phi_b}{\partial x^2} + \frac{(z+h)^4}{4!}\frac{\partial^4\phi_b}{\partial x^4}.
$$

In this way higher order approximations can be obtained by repeated back-subsitutions. We now use the last approximation in the kinematic and dynamic free surface conditions to obtain

$$
\begin{aligned}
\frac{\partial u_b}{\partial t} + u_b\frac{\partial u_b}{\partial x} + g\frac{\partial \eta}{\partial t} &= \frac{1}{2}h^2\frac{\partial^2 u_b}{\partial x^2\partial t} \\
\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left[(h+\eta)u_b\right] &= \frac{1}{6}\frac{\partial^3 u_b}{\partial x^3},
\end{aligned}
$$

which are the equations Boussinesq derived in 1872.

Appendix D

Additional scaling of the Boussinesq equation

In this section we do some additional scaling on the Boussinesq equation derived in section 5.2.3 to transform it onto a more common form. In the derivation we obtained at order $O(\epsilon)$, for $\eta \sim \eta^0 + \epsilon\eta^{(1)}$,

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 \eta}{\partial x^2} - \epsilon \frac{\partial^2}{\partial x^2}\left[\frac{1}{2}\eta^2 + \left(\int_{-\infty}^{x} \frac{\partial \eta}{\partial t}\,\mathrm{d}x\right)^2\right] - \frac{\epsilon}{3}\frac{\partial^4 \eta}{\partial x^4} = O\left(\epsilon^2\right), \tag{D.1}$$

where

$$u^{(0)} = -\int_{-\infty}^{x} \frac{\partial \eta^{(0)}}{\partial t}\,\mathrm{d}x,$$

with the condition of $u^{(0)} \to 0$ as $x \to \infty$. This is then a Boussinesq equation. To obtain the equation on its more common form we first need to set

$$H = \eta - \epsilon\eta^2$$

and define

$$X = x + \epsilon \int_{-\infty}^{x} \eta(x, t; \epsilon)\,\mathrm{d}x,$$

which yields after inserting into (D.1)

$$\frac{\partial^2 H}{\partial t^2} - \frac{\partial^2 H}{\partial X^2} - \frac{3\epsilon}{2}\frac{\partial^2 H^2}{\partial X^2} - \frac{\epsilon}{3}\frac{\partial^4 H}{\partial X^4} = O\left(\epsilon^2\right).$$

A last scaling of the variables via

$$H' = -\frac{\epsilon}{2}H, \quad X' = \sqrt{\frac{3}{\epsilon}}, \quad t' = \sqrt{\frac{3}{\epsilon}}t$$

and then neglecting the $O\left(\epsilon^2\right)$ will give us the traditional Boussinesq equation

$$\frac{\partial^2 H'}{\partial t'^2} - \frac{\partial^2 H'}{\partial X'^2} - 3\frac{\partial^2}{\partial X'^2}(H')^2 - \frac{\partial^4 H'}{\partial X'^4} = 0.$$