NTNU
Norwegian University of
Science and Technology

Market Risk in Turbulent Markets

Martin Børter

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Supervisor: Jacob Laading, MATH

Norwegian University of Science and Technology
Department of Mathematical Sciences

Problem Description

This thesis is a study of market risk under different market conditions. As the world has witnessed large price movements in the stock market during the current financial crisis, followed by frequent interest rate cuts, we analyze to what extend these observations are extreme according to broadly adopted market risk models.

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Preface

This thesis was carried out at the Department of Mathematical Sciences at the Norwegian University of Science and Technology (NTNU), Trondheim, during the period January 2009 to June 2009. It represents a semester of work load, and completes the five year master program in Industrial Mathematics.

I thank my fellow student Øystein S. Koren for rewarding discussions, particularly during my project thesis "Pricing Equity and Interest Rate Derivatives" the previous semester. The experiences from the project were necessary for writing this thesis. I would also thank Sindre Hilden for reviewing.

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Finally, I would like to express my gratefulness to my supervisor Associate Professor Jacob K. Laading for providing excellent guidance and feedback. Despite the fact that we have been communicating over the North Sea during the process, the cooperation has worked out fine.

Martin Børter June 11, 2009

Abstract

In this thesis we study market risk in turbulent markets over different risk horizons. We construct portfolios which represent possible investments for a life assurance fund. The portfolios consist of equities, fixed income instruments, cash positions and interest rate derivatives. The most commonly used metrics for measuring market risk are Value-at-Risk (VaR) and Expected Shortfall (ES), and they will be central. For the completeness of the thesis, we introduce necessary theory from quantitative finance related to asset price dynamics and security pricing. Interest rate related instruments are handled by the LIBOR Market Model (LMM), while equity prices are modeled as geometric Brownian motions. By combinding the two, we implement a risk model and make daily and quarterly market risk estimates between 2000-2008 for the portfolios. We choose some central events from the last quarter of 2008, a critical phase of the ongoing financial crisis, and analyze how the portfolios and the corresponding risk estimates are affected. Comparison of the portfolio losses against the risk estimates allows us to evaluate the reliability of the broadly adopted model.

Our findings show that large losses occure more frequently than expected from the model for all market conditions, but in particular during the turbulent 2008. The high frequency is most evident from violations of risk estimates far into the tails of the loss distribution. This is a result of our assumption of normal distributed logreturns for assets, which provides a better fit closer to the center of the loss distribution. For a daily risk horizon we find strong tendencies of clustering between extreme losses, especially around October 2008.

In the light of the extreme volatility under the current market conditions, combined with the weaknesses of our model assumptions, we expected the model to be flawed. However, it performed better than expected, and it is clear that in many cases it gives valuable information about the level of market risk for the tested portfolios, even during the financial crisis of 2008. To exploit this information it is important to be aware of the weaknesses of the model.

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Chapter 1

Introduction

This thesis is a study of market risk over different time horizons during different market regimes. We focus on the volatile markets observed recently. As there have been extreme price movements in the stock market followed by frequent interest rate cuts, we analyze to what extend these observations are extreme according to common market risk models.

1.1 Background

Whether you are a major bank with billions of dollars worth of assets, a professional investor, or a pensioner with only a few hundreds, you would like to have a perception of how large losses may occur as a result of market deviations. For risk managers, asset managers, or anyone exposed to market risk, it is preferable to have standardized techniques to measure risk quantitatively. After the market crash in the 90's it has been developed metrics such as Value at Risk (VaR) and Expected Shortfall (ES). VaR has become the benchmark for both measuring risk, and for controlling and manage risk actively. Increased pressure from regulators such as Basel Committee on Banking Supervision[1], globalization of the market which has introduced more sources of risk, as well as technological advances improving computational speed have all been factors contributing to the development and focus on quantitative risk management. There is however a fundamental problem with the models. According to the paper written by Y. Yamai and T. Yoshiba, Bank of Japan [21] available at BIS[2], it is a well known fact that VaR models do not work under market stress. They are usually based on normal asset returns and do not work under extreme price fluctuations. Despite this, we will not modify well established models to try to cope with the problems of fat tails and asset returns. This thesis is merely a discussion of the already existing models seen in light of the new extreme market data. We implement a VaR and ES model and analyze

[1] The Basel Committee on Banking Supervision provides a forum for regular cooperation on banking supervisory matters. Its objective is to enhance understanding of key supervisory issues and improve the quality of banking supervision worldwide.

[2] The Bank for International Settlements (BIS) is an international organisation which fosters international monetary and financial cooperation and serves as a bank for central banks.

its performance during the latest decade, focusing on the current financial crisis.

The global financial crisis of 2008-2009 is an ongoing major crisis. It became prominently visible in September 2008 with the failure and merger of several large United States-based financial firms. The underlying causes leading to the crisis had been reported for many months before September 2008. Several financial papers have written comments about the financial stability of leading U.S. and European investment banks, insurance firms and mortgage banks consequent to the subprime mortgage crisis. Beginning with failures of large financial institutions in the United States, it rapidly evolved into a global credit crisis, deflation and sharp reductions in shipping, resulting in a number of European bank failures. There has been large reductions in the market value of equities, commodities and risky bonds world wide. The crisis led to a liquidity problem and the de-leveraging of financial institutions especially in the United States and Europe, which further accelerated the liquidity crisis, and a decrease in international shipping and commerce. World political leaders, national ministers of finance, and central bank directors have coordinated their efforts to reduce fear, but the crisis is ongoing and continues to change. In October 2008 it evolved into a currency crisis with investors transferring vast capital resources into stronger currencies such as the yen, the dollar and the Swiss franc. This forced many emergent economies to seek aid from the International Monetary Fund (IMF) [6].

1.2 Thesis Outline

To evaluate the accuracy of market risk models on empirical market data we need to define some portfolios to work with. We are concerned with both interest rate risk and risk from exposure to the stock market. Since the latter tend to dominate between the two, we will consider portfolios with little or nothing invested in the stock market for analysing interest rate risk. In addition, we want to analyze portfolios which might represent real market participants or professional investors. With this in mind, an obvious candidate is a life assurance fund. The largest investments will be in fixed income instruments. These investments are typical since the funds, due to regulations, are required to guarantee a certain rate of return to its investors. Smaller amounts will be placed in equities, cash positions and interest rate derivatives.

In chapter 2, we introduce the theory from financial mathematics describing models for asset dynamics. In chapter 3, we present different approaches to quantify market risk and clarify our choice of approach. In chapter 4, we describe some of the central events during the second half of 2008, and discuss broadely adopted model assumptions in the light of the extreme price fluctuations. In chapter 5, we describe the data material and portfolio structures. In chapter 6, the most relevant details from the implementation are described. In chapter 7, we presents our results and in chapter 8, we present a conclusion and possible extensions of this thesis.

Chapter 2

Equity and Interest Rate Dynamics

2.1 Asset Classes and Price Dynamics

Asset classes are groups of securities which exhibit similar characteristics and are subject to the same laws and regulations. We will focus on the three main classes, namely equities, fixed income and money market instruments. The two last classes are closely related and we model their value based on the evolution of interest rates.

It should be noted that in addition to the three main asset classes, some investment professionals would add real estate and commodities, and possibly other types of investments. Mathematically, these classes are more challenging to model, and the fields are less developed compared to equity and interest rate modelling.

2.1.1 Equity

Equity captures shares or any other securities representing ownership interest in companies. The main objective for companies is typically seen as maximizing the wealth of its share-holders. This is done through a balance between paying dividends to share-holders and reinvestment to create future dividend payments or capital gain. The share-price is based on investors expectations of the future cash flow for the companies which generate dividends and capital gain for the share holders. Share prices and indices are frequently quoted in the media. The latter is often used as a benchmark for portfolios since they reflect how the general stock market is performing. Since they are calculated based on a set of shares they should have similar nature as individual stocks when it comes to price evolution.

As emphasized in Wilmott (1995) [14] we will not try to predict tomorrows stock prices. Even though we have a time series of historical prices we cannot use it to forecast future prices, but this does not mean that the history tells us nothing. Looking at the history we can get information of the likely range or even distribution of future prices. It is common to argue that the stock prices must move randomly because of the *efficient market hypothesis*. The hypothesis says two tings [14]

- The past history is fully reflected in the present price, which does not hold any further information about an asset.
- Markets respond immediately to any new information about an asset.

This means that the evolution of asset prices is based on the stream of new information. The efficient market hypothesis suggest the prices to follow a Markov process. We denote the asset price at time t as S. Over a small time interval dt the asset price have changed to $S + dS$. We can model the return of the asset dS/S by dividing it into two parts. Analogue to the risk-free return obtained from having money in the bank we express the deterministic contribution to the return of the asset as

$$\mu dt,$$

where μ is called the drift of the asset and corresponds to the risk-free rate r if investing in a bank account. Investing in stocks or portfolios of stocks need not give a positive return. This is captured by a random term

$$\sigma dW,$$

where $dW \sim N(0, dt)$. σ is called the volatility and measures the standard deviation of the returns. Combining the two terms gives

$$\frac{dS}{S} = \mu dt + \sigma dW. \tag{2.1}$$

By letting dt get infinitesimally small, the dW term becomes Brownian motion and we have a stochastic differential equation. This is a common mathematical representation for the price evolution of assets related to share prices.

2.1.2 Interest Rates

Interest rates can be viewed as the price of borrowing or lending money. Since a lender lose the possibility of investing or spending money he receives interest as compensation. If this was not the case, lenders would not be willing to lend money. Interest rates are partly determined by the government and the market. In order to control the level of inflation the central banks set interest rates. An example is the U.S. Fed Funds rate. It is the amount the U.S. government charge other banks to borrow money. It affects the entire supply of money.

Swap rates are the borrowing rates between financial institutions with solid credit ratings. Swap rates are calculated using the fixed rate leg of interest rate swaps[1]. Swap

[1] See chapter 2.4.1 for an explanation of interest rate swaps.

rates form the basis of the swap curve (also known as LIBOR curve). In most emerging markets with underdeveloped government bond markets, the swap curve is more complete than the treasury yield curve, and is thus used as the benchmark curve. In this thesis we will use both the yield on government bonds and the swap rates. Due to the credit crisis the two rates behave more differently than earlier and will thus be treated separately.

There are some fundamental differences between the behavior of equities and interest rates. Mathematically, it is easier to model the evolution of equities. We list some reasons described in Hull (2000) [8].

- Interest rates have term structure. There are many interest rates quoted every day corresponding to the length of their contract (time to maturity).
- The behavior of an individual interest rate is more complicated than that of a stock price. For instance, interest rates cannot grow unbounded and they are mean reverting, i.e. large interest rates tend to move towards a historical mean and similar for small rates.
- The volatilities for the rates corresponding to the different maturities are different. Short-term rates typically have greater volatility than long-term rates. In addition the different rates are strongly correlated.

2.2 Derivatives and Hedging

2.2.1 Derivatives

A derivative is a financial instrument whose value is derived from one or more variables, known as the underlying. It can be described as a side bet on the value of the underlying. The underlying can be everything from asset prices to weather conditions. In this study we restrict ourselves to derivatives on interest rates. As there are different types of assets, there are numerous forms of contracts. They are classified as forwards/futures, options and swaps. The **forward** contract is an agreement between to parties. At a set date in the future, the holder will receive a unit of the underlying asset for paying the delivery price, or the forward price. The forward price is determined so that the value of the contract is initially equal to zero. The contract is an obligation for both parts and there is counter party risk associated with the agreement. That is, there is a risk that one of the parties will not be able to pay due to liquidity problems. To avoid this risk, **futures** were invented. They can be thought of as a set of short term forwards, which yield the same payoff in the end. They are exchange-traded and often liquid contracts. An **option** is a contract written by a seller that gives the buyer the right, but not the obligation, to buy (in the case of a call option) or to sell (in the case of a put option) a stated number of units of a particular security at a specified time at a specified price. In return for granting the option, the seller collects a payment from the buyer. The payment is known as the option price. The position of the seller is known as "going short" while the buyer "goes long". A **swap** is an agreement between two parties to exchange, or swap, future cash flows.

2.2.2 Hedging

Hedging is the process of reducing the financial risk that either arise in the course of business operations or investments. Hedging is and important aid in the financial market. One common form of hedging is insurance where, by paying a fixed amount, you can protect yourself against certain specified losses. These losses might be due to fire, theft, or even adverse price movements. Hedging against potential losses from investments can be done by derivatives. Since the the value of the derivative depend on the underlying, a position in both might offset the net risk. Derivatives might also be used to the opposite of hedging, i.e. increase a portfolios exposure against market fluctuations. If an investor posseses a particular market view, he or she might use derivatives to profit from price movements. But the major use of financial derivatives, by far, is for hedging and not speculation.

2.3 Interest Rate Modelling

A rich theory of interest rates is explored in this section. It allows us to connect a whole family of interest rates, one for each maturity, and provides a clearer understanding of the interest rate market. Interest rates are closely related to bonds and their yield. A bonds yield is the interest rate implied by the payment structure. Specifically, it is the internal rate of return (IRR) of investing in the bond. Thus, the price of government bonds indirectly reflect the market's belief of future risk free interest rates. By plotting the IRRs against the maturity dates of the bonds we get the yield curve $Y(t;T)$. In addition, we introduce the forward rate curve $F(t;T)$. The forward rate reflects the market's belief of the future instantaneous rate at T today at t. The zero-coupon bond price $Z(t;T)$ at time t for receiving 1 unit of a currency at time T is given by

$$Z(t;T) = e^{-\int_t^T F(t,s)ds},$$

or

$$F(t;T) = -\frac{\partial \log Z(t;T)}{\partial T}.$$

This representation of the interest rate relies on bond prices being differentiable with respect to the maturity date. Consider the value of zero-coupon bonds $Z(t;T)$ taken from real data. Define $Y(t;T)$ by

$$Y(t;T) = -\frac{\log(Z(t:T))}{T-t}.$$

A more precise definition of the yield curve is the plot of $Y(t;T)$ against $(T-t)$.

The curve is used to predict changes in the economy and future growth. There are three main types of yield curve shapes; normal, inverted and flat (humped). A normal yield curve is one where longer-term bonds have higher yield than shorter-term bonds. Investors should be given a larger compensation to tie up money for a long time period, however this is not always the case. If the short term rate is high and expected to fall the curve is inverted. It has higher yield on shorter term bonds, and can be a sign of an upcoming recession. The flat curve is one where the shorter and longer term yields are close to each other. In Figure 2.1, the yield on US government bonds for 3, 5 and 10 years are plotted. As we can see, the 10 year yield is less volatile than the shorter ones. This yield is larger than the shorter most of the time (normal shape). There are two periods where the three rates are fairly similar (flat curve) or even that the short rates are higher (inverted). These periods are around year 2000 and 2007. As we know today, both were followed by recessions.

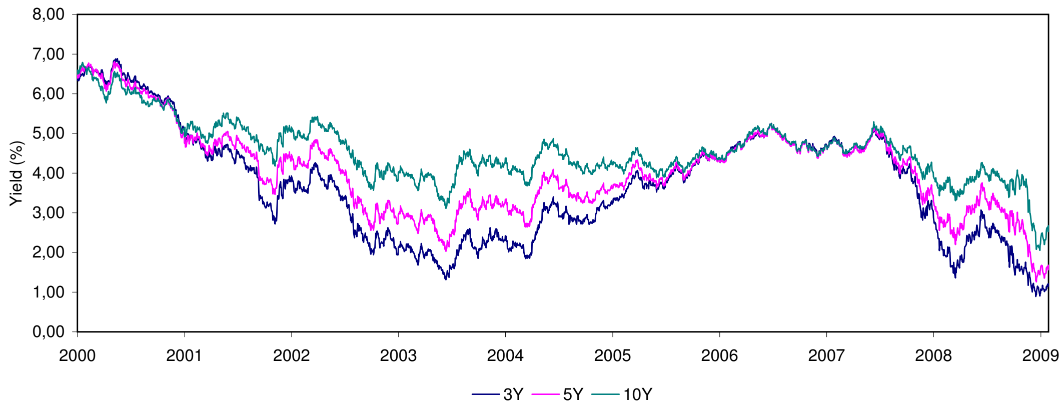

Figure 2.1: Yield on US government bonds.

2.3.1 LIBOR Market Model

The LIBOR Market Model (LMM) describe the arbitrage-free dynamics of the term structure of interest rates through the evolution of forward rates. The model is the discrete counterpart to the Heat-Jarrow-Mortan (HJM) model, but LMM is based on simple rather than continuously compounded forward rates. This shift has far-reaching practical and theoretical implications since most market interest rates are based on simple compounding over intervals. The model has gained rapid acceptance in the financial industry. An important benchmark in the money market and the standard for the model is the London Interbank Offered Rate (LIBOR). This is the rate at which banks can borrow funds, in marketable size, from other banks in the London interbank market. It is also the rate at which the important largest commercial borrowers are able to borrow money. Typically, a multinational corporation with a very good credit rating may be able to borrow money for one year at LIBOR plus four or five basis points (1 bp = 0.01%). The LIBOR is fixed on a daily basis by the British Bankers' Association. The

LIBOR is calculated by a weighted average of the world's most credit worthy banks' interbank deposit rates for larger loans with maturities between overnight and one full year. Rates longer than a year are usually found from the liquid swap market.

We denote the forward LIBOR rate $L(0,T)$ as the rate set at time 0 for the interval $[T, T+\delta]$. If we enter a contract at time 0 to borrow 1 at time T and repay it with interest at time $T+\delta$, the interest due will be $\delta L(0,T)$. The identity between forward LIBOR rates and bond prices is given by

$$L(0,T) = \frac{B(0,T) - B(0,T+\delta)}{\delta B(0,T+\delta)}. \tag{2.2}$$

It should be noted that equation (2.2) may not hold if the bonds on the one side and the forward rates on the other reflects different levels of credit-worthiness.

Many derivatives tied to LIBOR and swap rates are sensitive only to a finite set of maturities, and it is not necessary to introduce a continuum of maturity dates to price and hedge these securities. We consider a class of models in which a finite set of maturities

$$0 = T_0 < T_1 < \cdots < T_M < T_{M+1}$$

are fixed in advance. Let

$$\delta_i = T_{i+1} - T_i, \quad i = 0, ..., M$$

denote the length of the intervals between tenor dates. Furthermore, let $B_n(t)$ denote the price at time t of a bond maturing at T_n, and write $L_n(t)$ for the forward rate as of time t for the accrual period $[T_n, T_{n+1}]$. Thus, we have

$$L_n(t) = \frac{B_n(t) - B_{n+1}(t)}{\delta_n B_{n+1}(t)}, \qquad 0 \le t \le T_n, \quad n = 0, 1, ..., M. \tag{2.3}$$

This relation between bond prices and forward rates can be inverted to produce

$$B_n(T_i) = \prod_{j=i}^{n-1} \frac{1}{1 + \delta_j L_j(T_i)}, \quad n = i+1, ..., M+1.$$

If we want to find the price $B_n(t)$ for some $n > i+1$ the factor

$$\prod_{j=i+1}^{n-1} \frac{1}{1 + \delta_j L_j(t)}$$

discounts the bond's payment at T_n back to time T_{i+1}. But in addition we need a discount factor from T_{i+1} to t. We introduce a function $\eta : [0, T_{M+1}) \to \{1, ..., M+1\}$ by taking $\eta(t)$ to be the integer satisfying

$$T_{\eta(t)-1} \le t < T_{\eta(t)},$$

thus $\eta(t)$ gives the next tenor date at time t. Now, we can express the bond prices at times between tenor dates as

$$B_n(t) = B_{\eta(t)}(t) \prod_{j=\eta(t)}^{n-1} \frac{1}{1+\delta_j L_j(t)}.$$

Spot Measure

The evolution of the forward LIBOR rates can be described by a system of SDEs of the form

$$\frac{dL_n(t)}{L_n(t)} = \mu_n(t)dt + \sigma_n(t)^T dW(t), \quad 0 \leq t \leq T_n, \quad n = 1, ..., M, \tag{2.4}$$

where dW is a d-dimensional standard Brownian motion. In the HJM setting, the numeraire associated with the risk neutral measure is

$$\beta(t) = \exp\left(\int_0^t r(u)du\right).$$

The simply compounded counter party of $\beta(t)$ is

$$B^*(t) = B_{\eta(t)}(t) \prod_{j=0}^{\eta(t)-1} [1 + \delta_j L_j(T_j)].$$

The bond price divided by this numeraire is called the deflated bond price and is given by

$$D_n(t) = \frac{B_n(t)}{B^*(t)} = \left(\prod_{j=0}^{\eta(t)-1} \frac{1}{1+\delta_j L_j(T_j)}\right) \prod_{j=\eta(t)}^{n-1} \frac{1}{1+\delta_j L_j(t)}.$$

The $B_{\eta(t)}(t)$ term is canceled out and $D_n(t)$ is expressed by the LIBOR rates only. The deflated bond prices should be positive martingales which imposes restrictions on the dynamics in equation (2.4). The drift of the forward LIBOR rates must be

$$\mu_n(t) = \sum_{j=\eta(t)}^{n} \frac{\delta_j L_j(t) \sigma_n^T \sigma_j(t)}{1+\delta_j L_j(t)}$$

in order for this to be the case. A sketch of a proof is given in Glassermann [7].

Forward measure

The LIBOR market model may alternatively be formulated under the forward measure. That is, the numeraire asset is replaced with the bond B_{M+1}. The new deflated bond price is given by

$$D_n(t) = \frac{B_n(t)}{B_{M+1}(t)} = \prod_{j=n+1}^{M} (1 + \delta_j L_j(t)).$$

Again the factor $B_{\eta(t)}(t)$ has been canceled out. This causes the arbitrage-free dynamics of the forward LIBOR rates to have drift

$$\mu_n(t) = -\sum_{j=n+1}^{M} \frac{\delta_j L_j(t)\sigma_n(t)^T \sigma_j(t)}{1+\delta_j L_j(t)},$$

as shown in Glassermann (2000) [7]. The system of SDEs can now be written as

$$\frac{dL_n(t)}{L_n(t)} = -\sum_{j=n+1}^{M} \frac{\delta_j L_j(t)\sigma_n(t)^T \sigma_j(t)}{1+\delta_j L_j(t)} dt + \sigma_n(t)^T dW(t), \quad 0 \le t \le T_n, \quad n = 1, ..., M. \tag{2.5}$$

2.4 Interest Rate Instruments

2.4.1 Swaps, Caps and Floors

A swap is an agreement between two parties to exchange, or swap, future cash flows. For an interest rate swap, each party agrees to pay either a fixed or floating rate multiplied by a notional principal amount over a certain time period. The most common agreement is where one party pays a fixed rate and receives floating rates. The swap rate is the fixed rate which makes the initial value of the contract equal to zero. Thus, the interest rate swap is initially an agreement with no upfront payment. Interest rate swaps can be used by hedgers to manage their risk against changes in rates or by speculators who possess a particular market view. They are very popular and highly liquid instruments. The swap which pays fixed and receives floating rates is called a *payer swap* and its counter party is called a *receiver swap*.

An *interest rate cap* is a derivative in which the buyer receives payments at the end of each period if the interest rate exceeds the agreed strike price. The interest rate cap can be viewed as a series of European call options or caplets which exist for each period during the agreement. An investor which is obligated to pay the floating rate may purchase the cap so that the net payment each period will not exceed the strike. Similarly, an *interest rate floor* is a series of European put options or floorlets on a specified reference rate. The parity between, swaps, caps and floors on the same rate with the same strike is given by

$$\text{Swap} = \text{Cap} - \text{Floor}$$

2.4.2 Bonds

Bonds represent deterministic cash flows, meaning that the coupons and the face value are known in advance. However, the price for two bonds with identical payment structure may differ. This is due to the creditworthiness of the bond issuer. For government bonds, the default risk is usually considered to be zero, but for bonds issued by leveraged companies the default risk can not be neglected. While it is very rare that a state defaults (even though this happened in Russia in the late 90's), exchange traded firms with a higher leverage such as commercial banks and other financial institutions, have over and over again showed their sensitivity to sudden market collapses, and are treated thereby. When companies face problems with their liquidity they might not be able to meet their obligations to the bond holders. For the market to have a perception of the economic robustness of the companies, they are rated by independent credit rating agencies (such as Fitch, Moody's, or Standard & Poors). Risky bonds (speculative bonds) trade at lower prices than safe (investment-grade bonds), which means that investors demand higher yield for higher risk.

Many pension funds and other investors (banks, insurance companies), however, are prohibited in their by-laws from investing in bonds which have ratings below a particular level. As a result, the lower-rated securities have a different investor base than investment-grade bonds. The value of speculative bonds is affected to a higher degree than investment grade bonds by the possibility of default. For example, in a recession interest rates may drop, and the drop in interest rates tends to increase the value of investment grade bonds. However, a recession tends to increase the possibility of default in speculative-grade bonds.

To price a bond, the creditworthiness of the issuer must be reflected in the discount rate for its cash flow. Therefore, if we want to model the evolution of bond prices, we need to calibrate the Market Model[2] to data reflecting the corresponding creditworthiness. That is, for government bonds we use their historical yield, while for money market instruments we use LIBOR rates.

[2]We write Market Model since the model introduced as LMM can be used on other rates than the LIBOR.

Chapter 3

Market Risk

Generally, financial risk is classified into the broad categories; market risk, credit risk, liquidity risk, operational risk and legal risk. There are strong dependencies between the categories. E.g. for speculative bonds the credit risk is reflected in the prices. Since market risk is concerned with price fluctuation it will be affected by the change in credit risk. We focus on market risk.

3.1 Forms of Market Risk

Market risk arises from the movements in market prices (volatility). Market risk can take two forms, *absolute risk* measured in dollar terms and *relative risk* measured relative to a benchmark index. While the former focuses on the volatility of the total return, the latter measures risk in terms of deviation from the index. The latter is typical of interest for funds which are benchmarked against a certain index, such as The Norwegian Government Pension Fund. For our purposes it is natural to evaluate the volatility of the total return since we implement fictitious portfolios. Market risk is also divided into directional and non directional risk. Directional risk involve exposure to the direction of movements in financial variables. These exposures are measured by linear approximations such as beta for stocks and delta for options. Non directional risk then involves the remaining risk which consist of nonlinear exposures and exposures to hedged positions or to volatilities. An example is quadratic exposure when dealing with options which is measured by gamma. Market risk is controlled by limits on notionals, maximum exposures, VaR limits and supervision by risk managers.

3.2 Risk Measures

There are four different approaches to measuring risk according to McNeil, Frey and Embrechts (2005) [1]. They are listed as the notional amount approach, factor-sensitivity measures, risk measures based on loss distributions and risk measures based on scenarios.

The simplest of the four is **the notional amount approach**. This approach estimates risk by a weighted sum of the notional values, where the weights assigned to the underlying securities reflect how risky they are. Its advantage is the simplicity, but there are several disadvantages. Netting effects from positions canceling each other are ignored as well as diversification effects of the portfolio. Also, there are problems in assigning weights to derivatives since their value can differ widely from their notional amount.

Factor sensitivity measures give the change in the portfolio value due to a predetermined change in one of the underlying factors. The "Greeks" from option pricing are examples of this. Factor sensitivity measures give information about how robust the value of the portfolio is with respect to the underlying variables, but it does not give the overall riskiness of the position.

Risk measures based on loss distributions are today the most used methods. They estimate the loss distribution corresponding to a predetermined time horizon. Two central approaches are VaR and ES. Using the loss distribution is justified since the worrying part of risk is possible losses and not so much the possible profits. It also overcomes the problems of the two former methods since both netting effects and diversification are captured, as well as it gives a quantification of the overall risk for the portfolio. Last, it does not lay restrictions on the shape of the portfolio distributions.

Scenario based risk measures are concerned with future scenarios with insulated or simultaneous change in the risk factors. The worst case scenario can be measured as the maximum loss under all tested scenarios. If appropriate, the most extreme ones can be weighted down. Scenario based risk measures are widely used during turbulent markets. We will, however, focus on risk measures based on loss distributions.

3.2.1 The Loss Distribution

Consider a portfolio with value $V(s)$ at time s. It may consist of bonds, collections of equities, interest rate derivatives or more general the overall position of a financial institution. The loss of the portfolio over a period Δt is denoted by

$$L_{[s,s+\Delta t]} = -(V(s + \Delta t) - V(s)). \tag{3.1}$$

It is an estimated probability density distribution for the portfolio value at time Δt into the future at time s. The most critical part is the probability of large losses and hence the upper tail of the loss distribution. The concept makes sense on all levels of aggregation from a portfolio consisting of a single instrument to the overall position of a financial institution, and can be compared across different portfolios.

An important accounting detail that will affect the results is whether the fund manager have decided to keep the securities until they expire or not. We assume the latter and thus need to make a *mark-to-market*[1] valuation of the portfolio. Mark-to-market or fair value accounting refers to the accounting standards of assigning a value to a position held in a financial instrument based on the current fair market price for the instrument or similar instruments. It is the benchmark for the risk metrics we will consider.

3.3 Value-at-Risk (VaR)

Let $F_L(l)$ denote the cumulative probability function corresponding to the loss distribution of a portfolio, i.e. $F_L(l) = P(L < l)$. Given some confidence level $\alpha \in (0, 1)$, the VaR of a portfolio is given by the smallest number l so that the probability of loss L exceeding l is no larger than $(1 - \alpha)$. Formally

$$VaR_\alpha = \inf\{l \in \mathbb{R} : P(L > l) \leq 1 - \alpha\} = \inf\{l \in \mathbb{R} : F_L(l) \geq \alpha\}. \tag{3.2}$$

VaR is thus a quantile of the loss distribution corresponding to a certain confidence level. The time horizon of interest may be one day for trading activities and months or years for portfolio management. It is supposed to be the timescale associated with the orderly liquidation of the portfolio, meaning the sale of assets at a sufficiently low rate for the sale to have little impact on the market. The VaR estimate is thus an estimate of loss that can be realized, not just a paper loss. For a small investor investing in liquid assets, the liquidation horizon may be much shorter than months, but longer horizon VaR can be of interest for other reasons. For portfolio managers releasing quarterly reports, the three months VaR might be appropriate to disclose. The confidence level α is typically set between 95% and 99%. Naturally, we are most concerned with the tails of the loss distribution and hence the relative high α values. On the other hand, it is very challenging to estimate the probability of rare events, due to statistical significance of past data, so the α should not be set unreasonably high.

VaR is usually estimated assuming normal market circumstances, meaning extreme market conditions such as crashes are not considered, or they are examined separately. In addition, no trading is assumed over the VaR horizon. The calculation of VaR requires the following data: the current price of all assets in the portfolio, their volatility and the correlation between them. For traded assets we take prices quoted in the market (mark-to-market). For OTC[2] contracts we must use some "approved model". This is called mark-to-model.

[1] Fair value accounting has been a part of US Generally Accepted Accounting Principles (GAAP) since the early 1990s. The use of fair value measurements has increased steadily over the past decade, primarily in response to investor demand for relevant and timely financial statements that will aid in making better informed decisions.

[2] Over the counter (OTC) contracts are not exchange traded, but are agreements between specific buyers and sellers. There is counter party risk associated with these contracts.

3.3.1 Parametric Methods

The most common parametric method used is the Variance-Covariance method which assumes that the risk factor changes are multivariate normally distributed, $X_{t+1} \sim N_d(\mu, \Sigma)$, where μ is the mean vector and Σ is the covariance matrix. Further it assumes that the portfolio distribution can be expressed sufficiently accurate through a linear relation to the distribution of the risk factors. That is

$$L^{\Delta}_{t+1} = -(c_t + b_t' X_{t+1}), \tag{3.3}$$

for some constant scalar c_t and vector b_t assumed known at time t. The VaR can thus be found analytically.

For complex portfolios with interest rate instruments, and especially interest rate derivatives, the linear assumption is poor, and Monte Carlo based methods are preferable.

3.3.2 Sampling from Empirical Distribution

Sampling from historical data does not assume any particular distribution for the risk factor changes. Instead, it uses the empirical distribution of the historical data, $x_{t-n+1}, ..., x_t$. Based on these observations one can simulate losses by drawing risk factor changes from the past data. Let $\hat{L}_m$ be the resulting loss if the risk factor changes from period m would reoccur, then

$$\hat{L}_m = -(f(t+1, z_t + x_m) - f(t, z_t)). \tag{3.4}$$

This can be repeated n times and we have a simulated distribution of the portfolio change. VaR_α and ES_α can be estimated by the use of empirical quantile estimation. That is, if we have $n = 1000$ we estimate $\hat{L}_{t+1,1}, ..., \hat{L}_{t+1,1000}$, sort them, and use the tenth largest value as an estimate of $\text{VaR}_{\alpha=0.99}$. $\text{ES}_{0.99}$ is found from averaging over the ten largest. Historical sampling is easy to implement and we need not make parametric assumptions of the risk factor changes. However, it needs a large amount of data. There is no more information added by increasing the number of simulations when the sample is limited. In addition, this method does not capture the current volatility at time t. It would make no difference if the dataset contained a volatile period in the beginning of the data set or in the end, which is relevant since volatility vary over time.

3.3.3 Monte Carlo Simulation

In the Monte Carlo approach we fit the historical data to a parametric model. By simulating possible paths according to stochastic models we get an empirical distribution for the losses for the next period. The VaR quantile can be estimated as in the historical simulations, by sorting the M simulated losses in increasing order and pick element $\alpha \cdot M$. The procedure is described superficially by algorithm 3.1. Because of its flexibility, Monte Carlo simulation is by far the most powerful approach to VaR and ES. The accuracy of the method will naturally depend on the model assumptions and specifications. The simulation may be computationally expensive which is the drawback of the method.

Algorithm 3.1 Monte Carlo Simulation

Fit data $\mathbf{x}_{t-n+1}, ..., \mathbf{x}_t$ to a parametric distribution $D(\boldsymbol{\theta})$
for $i = 1$ to M **do**
  Draw $\mathbf{X}_{t+1}^i \sim D(\boldsymbol{\theta})$
  Calculate $\hat{L}_{t+1}^i = -(f(t+1, z_t + x_{t+1}^i) - f(t, z_t))$
end for
Return $\hat{\mathbf{L}}_{t+1}$

3.4 Expected Shortfall (ES)

By definition, the VaR at confidence level α does not give any information about the severity of losses which occur with probability $(1-\alpha)$. Two portfolios with equal VaR estimates for a certain confidence level might not be equally risky. If one portfolio have much heavier tails than the other for the distribution beyond the VaR, it is clearly more risky. A complementary measure for VaR is the expected shortfall (ES). It measures the expected loss given that it exceeds the VaR limit. The ES is thus concerned with the distribution of the tails and it is not just a quantile as VaR. ES is also referred to as Conditional Value at Risk (CVaR) and Expected Tail Loss (ETL).

For a loss L with $\mathbb{E}(|L|) < \infty$ and cumulative distribution function F_L the expected shortfall at confidence level $\alpha \in (0,1)$ is defined as

$$ES_\alpha = \frac{1}{1-\alpha} \int_\alpha^1 q_u(F_L) du, \tag{3.5}$$

where $q_u(F_L)$ is the quantile function of the loss distribution F_L. ES is closely related to VaR. This becomes evident when the quantile function is written as $q_u(F_L) = VaR_u(L)$ and we can write ES as

$$ES_\alpha = \frac{1}{1-\alpha} \int_\alpha^1 VaR_u(L) du, \tag{3.6}$$

Expected shortfall can also be written more intuitively as the conditional expectation

$$ES_\alpha = \mathbb{E}(L | L \geq VaR_\alpha) = \frac{\mathbb{E}(L; L \geq VaR_\alpha)}{1-\alpha}, \tag{3.7}$$

where $E(X;A) := E(XI_A)$ and $I_A = 1$ if $X \in A$ and $I_A = 0$ else.

Chapter 4

Financial Crisis and Model Assumptions

4.1 Central Events During the Crisis

We analyze the effect of some of the major events during the critical phase of the latest financial crisis. We focus on the following four events.

4.1.1 March 16 2008, JP Morgan Chase Acquires Bear Stearns

Bear Stearns, with base in New York City, was one of the largest global investment banks and securities trading and brokerage firms prior to its sudden collapse and distress sale to JP Morgan Chase Sunday March 16 2008. Bear Stearns pioneered the securitization and asset-backed securities markets, and as investor losses mounted in those markets in 2006 and 2007, the company actually increased its exposure, especially against the mortgage-backed assets that were central to the subprime mortgage crisis. On March 14 2008, the Federal Reserve Bank of New York provided an emergency loan through JP Morgan to try to avert a sudden collapse of the company. They feared the potential market crash that would result from Bear Stearns becoming insolvent because of its commitments to the other large financial institutions. The Fed's decision to guarantee a temporary credit line from JP Morgan was the first time the central bank has bailed out a brokerage firm since the Great Depression of the 1920s. However the company could not be saved, and was sold to JP Morgan Chase for as low as ten dollars per share, far below the traded $93 a share as late as February 2008 [17, 3].

4.1.2 September 15 2008, Lehman Brothers Collapses

Lehman Brothers filed for bankruptcy protection on September 15, 2008. The bankruptcy of Lehman Brothers is the largest bankruptcy filing in U.S. history with Lehman holding over $600 billion in assets. In 2008, Lehman faced an unprecedented loss due to the continuing subprime mortgage crisis. Lehman's loss was a result of having held on to large positions in subprime and other lower-rated mortgage tranches when securitizing the underlying mortgages. Whether Lehman did this because it was simply unable to sell the lower-rated bonds, or made a conscious decision to hold them, is unclear. In any event, huge losses accrued in lower-rated mortgage-backed securities throughout 2008. In the second fiscal quarter, Lehman reported losses of $2.8 billion and was forced to sell off $6 billion in assets. In the first half of 2008 alone, Lehman stock lost 73% of its value as the credit market continued to tighten. In August 2008, Lehman reported that it intended to release 6% of its work force, 1,500 people, just ahead of its third-quarter-reporting deadline in September. The collapse of Lehman deepend the fear in the credit market. The large financial institutions were tied to each other through complicated OTC agreements. The lending between banks dried up, and the de-leveraging of financial institutions accelerated [13].

4.1.3 October 3 2008, $700 Billion US Bailout Package Gets Approved

The Emergency Economic Stabilization Act of 2008, commonly referred to as a bailout package to the U.S. financial system, is a law enacted in response to the global financial crisis of 2008 authorizing the United States Secretary of the Treasury to spend up to US $700 billion to purchase distressed assets, especially mortgage-backed securities, and make capital injections into banks. Both foreign and domestic banks are included in the bailout. The purpose of the plan is to purchase bad assets, reduce uncertainty regarding the worth of the remaining assets, and restore confidence in the credit markets. On October 3, the Senate voted 263-171 to enact the bill into law. President Bush signed the bill within hours of its enactment, creating a $700 billion Troubled Assets Relief Program to purchase failing bank assets.

Supporters of the bailout plan argued that the market intervention called for by the plan was vital to prevent further erosion of confidence in the U.S. credit markets, and that failure to act could lead to an economic depression. Opponents of the rescue plan emphasized that the problems of the American economy were created by excess credit and debt, and that a massive infusion of credit and debt into the economy only would exacerbate the problems with the economy [4].

4.1.4 October 9 2008, Central Banks Execute Coordinated Interest Rate Cuts

The world's major central banks lowered their benchmark interest rates Wednesday October 9 2008. It was a coordinated effort to halt a collapse of share prices and a freeze in credit markets that threatened to set off the first global recession since the early 1970s. The Federal Reserve, the European Central Bank, the Bank of England and the central banks of Canada and Sweden all reduced primary lending rates by a half percentage point. Switzerland also cut its benchmark rate, while the Bank of Japan endorsed the moves without changing its rates. In addition, the Chinese central bank joined the effort without explicitly saying it was doing so by reducing its key interest rate and lowering bank reserve requirements to free up cash for lending. Together with other moves in the United States, Britain and Continental Europe the previous days, the rate cuts was part of a broader, global strategy that embraces aggressive use of monetary policy and taxpayer recapitalization of ailing banks, generating cautious optimism among crisis-weary analysts [5].

4.2 Standard & Poor 500 and The VIX Index

The Standard & Poor 500 (S&P 500) is a value weighted index published since 1957 of the prices of 500 common stocks actively traded in the United States. The stocks included in the S&P 500 are those of large publicly held companies that trade on either of the two largest American stock markets, the New York Stock Exchange and NASDAQ. Almost all of the stocks included in the index are among the 500 American stocks with the largest market capitalizations. Figure 4.1 shows the historical level for the index from June 2007 to January 2009. The key events during the beginning of the financial crisis are shown. The short term reaction of JP Morgans acquisition of Bear Stearns seem to result in a rally, since the government shows willingness to rescue a troubled financial institution. After the fall of Lehman Brothers the market collapses. The US Bailout package does not seem to calm the market either. However, the dramatic fall seems to slow down when the coordinated interest rate cut is announced.

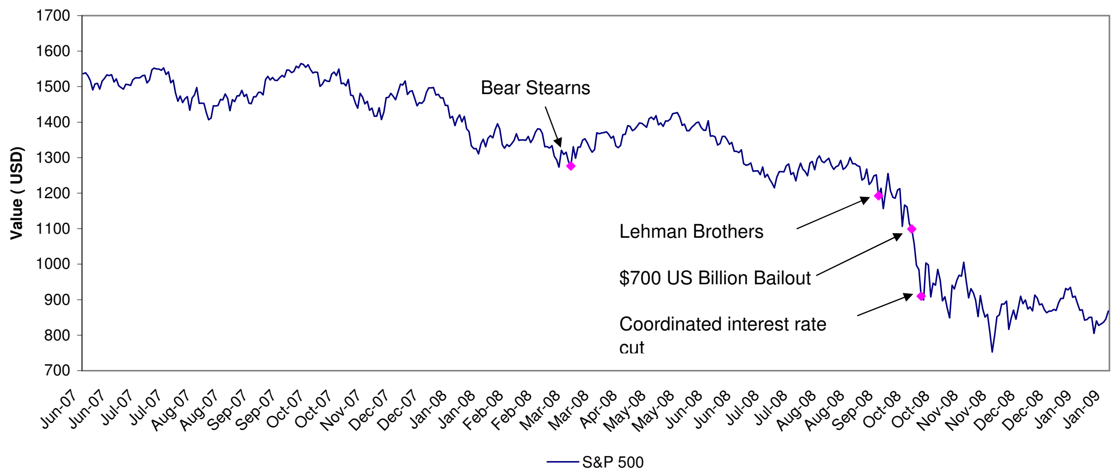

Figure 4.1: S&P 500 and central events during the financial crisis.

VIX is the ticker symbol for the Chicago Board Options Exchange Volatility Index, a popular measure of the implied volatility of S&P 500 index options. A high value corresponds to a more volatile market and therefore more costly options, which can be used to defray risk from volatility. If investors see high risks of a change in prices, they require a greater premium when issuing options. Often referred to as the fear index, it represents a measure of the market's expectation of volatility over the next 30 day period. Since the S&P index represents a broad part of the US stock market, the volatility index reflects the level of market stress in the US. In Figure 4.2 the four selected events are shown in the plot of the historical levels of the VIX. After Lehman collapses, the market volatility picks up dramatically and does not seem to be affected by the announcement of the US bailout package. The increase in volatility slows down after the coordinated interest rate cuts. Since wall street and other financial centers are known to be well informed and influenced by rumors, market reactions may start in advance of major events. It is thus challenging to determine how the events are actually affecting the market.

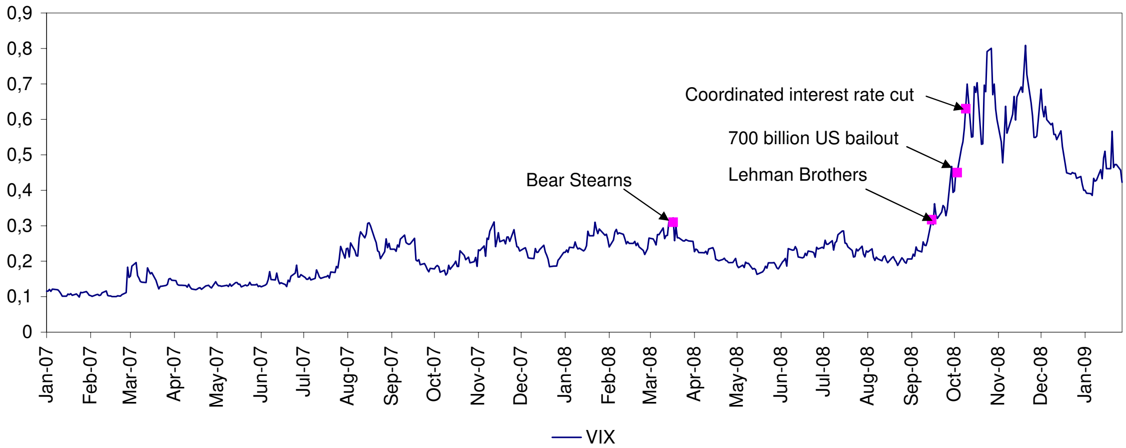

Figure 4.2: The market volatility index for S&P 500, VIX.

As stock- and commodity prices have dropped worldwide during the global financial crisis, there have been spectacular price movements. It has been fundamental uncertainty about the direction of the future global economy, and there still is. The crisis is more global than earlier and it is hard to see how the world is going to break the negative spiral of the bearish markets. On the other hand, the central banks, and in particular the US government, have shown willingness to take dramatic actions to prevent the world from going into a depression. Since this is done in a much larger scale than earlier, economists are uncertain about the outcome. Some mean that the markets should be left to their own destiny, while others argue that stimuli from the government is inevitable. In addition to the uncertainty about the companies future earnings, the small trading volumes have contributed to even greater volatility, since small trades have moved prices.

4.3 Model Assumptions and Financial Turbulence

We will base our model on Monte Carlo simulation, and we will need to make numerous assumptions for the behavior of the relevant financial instruments. The VaR and ES estimates are forward looking in the sense that they measure future loss quantiles and expectations. Ideally we would base the model on forward looking market estimates, like implied volatilities. However, a major part of the calibration will be based on historical data, such as volatility for government bonds and correlations between the risk factors. This means that we are predicting the future based on the past, and there are no guarantees that future scenarios are reflected in the past. Large and unexpected losses can not be foreseen. Nevertheless, there is information in historical data and there are different approaches for how to utilize it. Where it is necessary to estimate parameters based on historical market data we will follow the guide lines of the technical document of RiskMetrics[1].

4.3.1 Multivariate Normal Distributed Risk Factors

In order to get a VaR estimate from a Monte Carlo simulation, we need to assume a probability distributions for the relevant risk factors. It is common to base the marginal distributions of the risk factors on the normal distribution. We will additionally assume that the risk factors are multivariate normal. Historical logreturns for equity and interest rates are more sharp-peaked and fat-tailed than suggested by the normal distribution. This means that models based on the normal assumption should underestimate the likelihood of extreme events. This might be a poor assumption, especially during financial turbulence.

4.3.2 Correlation Between the Risk Factors

One of the main challenges when calibrating the risk model is estimating the correlations between the risk factors. The correlation structure has great impact on the total volatility for the portfolio and the risk estimates. As the correlations between assets might change rapidly, this can be a source to misleading VaR estimates. Typically, markets which are independent or very little correlated during normal market conditions might become strongly correlated during a recession. There are common factor for all markets, supply of capital (liquidity) and investors willingness to take risk. During recessions with huge price drops, large investors may be forced to liquidate their positions in risky assets due to margin calls, loss limits or risk regulations. This leads to further price drops, and further liquidations of positions. The market for risky assets become illiquid, and the prices are affected by the more overall market view. For "well diversified" portfolios of assets representing a certain level of risk, it seems reasonable that the volatility for the total portfolio would have potential to increase due to increased correlation. We will

[1]RiskMetrics started as a free service offered by JP Morgan in 1994 to promote VaR as a risk management tool. Later the bank spun off its risk-management group as the RiskMetrics Group. They developed the benchmark for modelling market risk.

make simulations based on the estimated correlation from the simulation point in time. From there we assume a constant correlation matrix. For daily risk estimates this might be a reasonable assumption, but for long horizons, e.g. quarterly, this is not as realistic. A plot of how estimated correlations change over time is shown in Figure 6.2.

Chapter 5

Data Material and Portfolios

The data set contains market quotes for S&P 500 and Euro Stoxx 50, and USD swap rates and yield on US government bonds. In addition, we have market volatilities for S&P 500 and USD swap rates. The market quotes are from January 3 2000 to January 28 2009. The data is provided by DnB NOR Markets through their access to Reuters and Bloomberg.

Dow Jones Euro Stoxx 50 is a stock index designed by Stoxx Ltd, a joint venture of Deutsche Boerse AG, Dow Jones & Company and SIX Swiss Exchange. According to Stoxx, its goal is "to provide a blue-chip[1] representation of Supersector leaders in the Eurozone."

For all interest rate related instruments, we consider products with maturity of 3, 5, and 10 year. These rates are liquid and the quality of the market data is therefore higher.

5.1 Portfolio Structures

5.1.1 Markowitz Portfolio Theory

The Markowitz Portfolio Theory is a famous theory about how to allocate capital in possible investments. It gives answers to how a portfolio should be diversified to obtain the lowest risk subject to a predetermined expected rate of return. By diversifying our investments, we reduce specific risk. Opposite to systematic risk, specific risk does not contribute to a higher expected rate of return. Risk is in this context defined as the standard deviation of the portfolio returns. We would expect a life insurance fund to have a dynamic investment strategy, at least for the equity positions. Therefore, we implement a Markowitz model for asset allocation. Let $S_{i,t}$ denote the price (adjusted

[1] A blue chip stock is the stock of a well-established company having stable earnings and no extensive liabilities. The term derives from casinos, where blue chips stand for counters of the highest value. Most blue chip stocks pay regular dividends, even when business is faring worse than usual.

for dividend payments and stock splits) for stock i at time t. The relative price change over one period (e.g. one day) is

$$R_t = \frac{S_t - S_{t-1}}{S_{t-1}} \tag{5.1}$$

Assuming we have data for $t \in \{1, 2, ..., T\}$ we estimate the expected return $\bar{r}_i$ for the stock i by

$$\bar{r}_i = \frac{1}{T} \sum_{t=0}^{T} R_{i,t}. \tag{5.2}$$

As suggested in RiskMetrics we use an exponential weighted moving average to estimate the covariance matrix at time T, $\Sigma(T)$. The covariance between asset i and j corresponding to price changes over one period is estimated by

$$\hat{\Sigma}_{i,j}(T) = \sqrt{\frac{1}{\sum_{t=1}^{T} \lambda^{T-t}} \sum_{t=1}^{T} \lambda^{T-t}(R_{t,i} - \bar{r}_i)(R_{t,j} - \bar{r}_j)},$$

where $\lambda \in (0, 1)$. RiskMetrics suggest $\lambda = 0.94$. Having estimated the expected rate of return and the covariance matrix, and determined the risk free rate for the relevant investment horizon we turn to the modern portfolio theory to determine the weights for the risky assets. To develop a solution, suppose that there are n risky assets in the market. We assign weights $w_1, w_2, ..., w_n$ to the assets such that $\sum_{i=1}^{n} w_i = 1$. This will result in a expected return for the portfolio $r_p = \sum_{i=1}^{n} w_i \bar{r}_i$. The standard deviation for the portfolio is given by $\sigma_p = \sqrt{\sum_{i,j} w_i w_j \Sigma_{i,j}}$. Figure 5.1 shows possible combinations of risk/return. By investing in the risky assets alone, the efficient frontier is the optimal portfolio given a certain expected rate of return $\bar{r}$. Having the opportunity to borrow and lend at the risk free rate r_f, the optimal portfolio will occur at the line which tangent the efficient frontier. However, we will use the model to determine only the allocations in equity, and we are thus interested in the market portfolio.

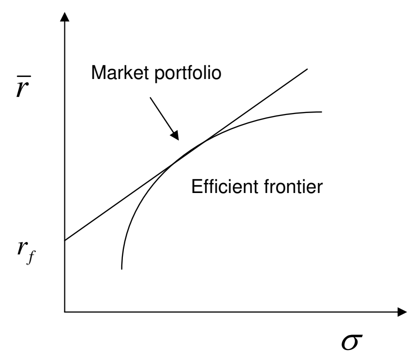

Figure 5.1: Efficient frontier and the market portfolio.

The problem of determining the market portfolio corresponds to maximizing the sharp ratio $(r_p - r_f)/\sigma_p$. This leads to solving a system of linear equations [12]. Let $v = (v_1, ..., v_n)^T$ and the excess return vector $\bar{r}_e = (\bar{r}_1 - r_f, ..., \bar{r}_n - r_f)^T$. Solve

$$\hat{\Sigma} v = \bar{r}_e \tag{5.3}$$

for v. Normalizing the v's gives the desired weights

$$w_i = \frac{v_i}{\sum_{i=1}^{n} v_i}. \tag{5.4}$$

This solution does not guarantee the weights to be positive. Since we are only allocating the investments in two indices, we will prevent shorting by substituting negative weights to zero and place all equity investments in the other index.

5.1.2 Portfolios

Portfolio 1

We start our risk analysis on a simple portfolio consisting of US government bonds. This allows us to analyze interest rate risk isolated. There will be equal amounts invested in the 3Y, 5Y and 10Y zero coupon bonds.

Portfolio 2

We continue by implementing portfolio 2 which reflect the level of market risk suitable for a life assurance company. The portfolio consist of 20% equities and 80% US government bonds. The bond portfolio have the same structure as in portfolio 1. The indices are weighted according to the Markowitz Portfolio Theory (market portfolio).

Portfolio 3

In portfolio 3, we add cash positions in the money market. They receive LIBOR rates and are hedged with interest rate floors. The interest rate derivatives increase the complexity of the portfolio. There will be 5% invested in stock indices, 30% in US government bonds and 65% invested in cash and floors. The floors will initially be out-of-the money. Capital invested in equities is allocated dynamically according to the Markowitz theory.

We do not allocate investments according to Markowitz across different asset classes. Since they are different of nature, the risk in form of standard deviation is not directly comparable. Moreover, if we have a static portfolio (between the asset classes) it is easier to make comparisons of results over time and interpret the results. That is, we do not want to add unnecessary complexity to the portfolio. The only dynamic elements come from the market portfolio for equities and the hedged cash positions. Since the floor prices change over time, the positions are adjusted so that the value of the floors and the cash add up to 65% of the portfolio value. In addition, the notional amount for the floors have to equal the notional on the cash position.

Since bonds change characteristics as they move towards maturity, we want to update the portfolio along the way. That is, at the beginning of all time periods evaluated, the bonds have equal time to maturity, i.e. 3Y, 5Y and 10Y. In practice, we would have to re-balance the portfolio after each time period. The position receiving LIBOR rates with corresponding floors are updated similarly. We will try to keep the portfolio as unchanged as possible as time evolves. We do not focus on how well the portfolios are performing in form of ROI or similar. We neglect transaction costs, and whether the bonds are really traded. The weights are shown in the table in Figure 5.2.

Investment	Portfolio 1	Portfolio 2	Portfolio 3
Stock Indecies	**0 %**	**20 %**	**5 %**
Standard & Poor 500	0 %	Markowitz	Markowitz
Euro Stoxx 50	0 %	Markowitz	Markowitz
Government bonds	**100 %**	**80 %**	**30 %**
US 3Y	33 %	27 %	10 %
US 5Y	33 %	27 %	10 %
US 10Y	33 %	27 %	10 %
Cash and floors	**0 %**	**0 %**	**65 %**

Figure 5.2: Portfolio weights.

The proportions of the total equity position invested in S&P 500 and Euro Stoxx 50 are shown in Figure 5.3. As we see, the estimated optimal portfolio change rapidly over time.

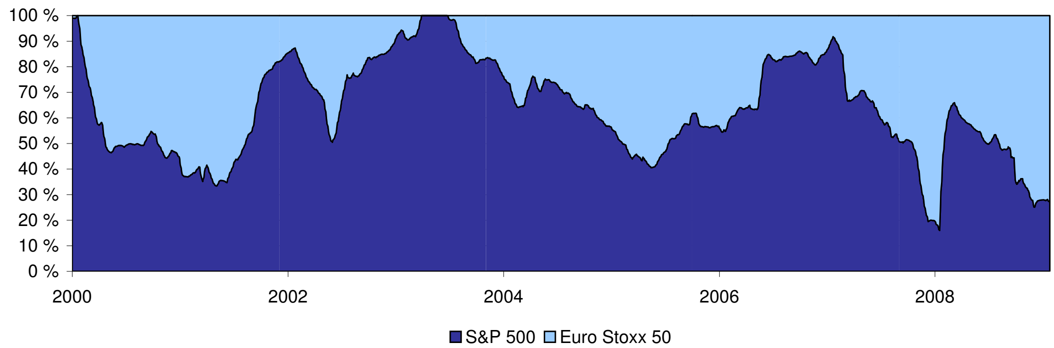

Figure 5.3: Proportion of equity invested in S&P 500 and Euro Stoxx 50.

Chapter 6

Implementation

6.1 Volatilities

The technical document of RiskMetrics [10] suggests using an exponential weighted moving average (EWMA) to estimate the volatility for the risk factors, with $\lambda = 0.94$. The alternative is to use implied volatilities. Figure 6.1 shows the volatility estimated from historical data versus the VIX for S&P500. The two measures follow each other relatively tightly, but the historical lays systematically below the implied. A more qualitatively argument for using implied volatilities is that they are forward looking and based on expectations of market participants. The historical estimates based on the past do not necessarily determine future risk. We will use implied volatilities for assets where these are available. For S&P500 we use the VIX index, and for LIBOR rates we use cap volatilities. For Euro Stoxx 50 and the yield on government bonds we use historical. Since the implied volatility systematically lays above the historical, we approximate implied volatilities by scaling up the historical. We will estimate the scaling factors based on comparable instruments. Even if we prefer to use implied volatilities, it should be stressed that their quality rely on liquid markets for the options and derivatives they are based on.

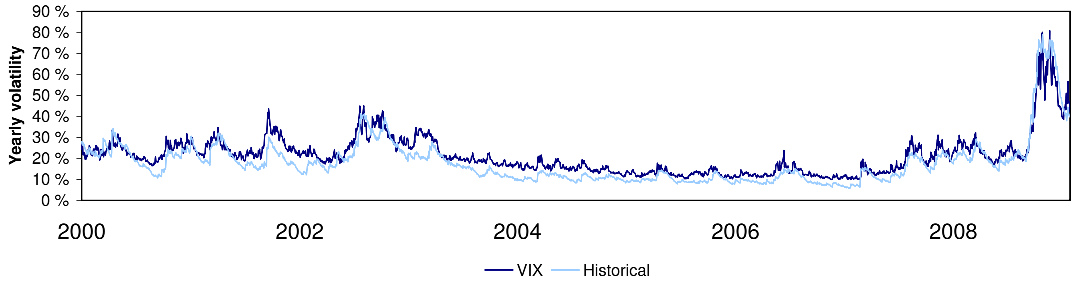

Figure 6.1: The yearly volatility for S&P 500 estimated by EWMA from historical data and the VIX index.

6.2 Correlations

Similarly to volatilities, historical correlations are not constant over time. They change rapidly. In addition, correlations depend on the size of the time horizon considered. Two assets might have almost independent daily fluctuations, but be stronger correlated when we consider a longer time scale. Ideally, if we consider a quarterly VaR we would like to estimate the correlations based on quarterly data. Thus, if we want to estimate the VaR for a long time horizon we should have a large data set with prices going far into the history.

We estimate the correlations by EWMA. The risk factors are defined as the logchanges of the stock prices (indices) adjusted for dividends and stock splits, and for fixed income the logchange of the LIBOR rates and yield on government bonds. Consider the historical prices $S_0, S_1, ..., S_N$, we estimate the covariance between asset i and j over a time horizon from t_N to t_{N+1} by

$$\hat{\sigma}_{i,j,t_{N+1}|t_N} = (1-\lambda)(r_{i,t_N} - \bar{r}_{i,t_N})(r_{j,t_N} - \bar{r}_{j,t_N}) + \lambda\ \hat{\sigma}_{i,j,t_N|t_{N-1}} \tag{6.1}$$

where

$$r_{k,t_N} = \ln(S_{k,t_N}/S_{k,t_{N-1}}) \quad \text{and} \quad \bar{r}_{k,t_N} = \frac{1}{N}\ln(S_{k,t_N}/S_{k,t_0}). \tag{6.2}$$

The correlation between asset i and j is estimated from

$$\hat{\rho}_{i,j} = \frac{\hat{\sigma}_{i,j}}{\hat{\sigma}_i \hat{\sigma}_j}, \tag{6.3}$$

where $\hat{\sigma}_i^2 = \hat{\sigma}_{i,i}$. Figure 6.2 shows how some of the estimated correlations vary over time. The correlation between the yield on US government bonds with maturity 3 and 5 years varies between 0.35 and 0.9, while the correlation between the yield on US government bonds with maturity 3 years and S&P500 fluctuates between -0.2 and 0.6. This shows that correlations are far from constant. An observation worth mentioning is that the latter correlation seems to increase during bearish markets. The explanation might be that during normal market conditions both markets are liquid and there are no reasons for the daily changes of the two to have an effect on each other. However, when the stock markets have large drops, large funds can be forced to turn to government bonds as safe harbors for their assets. Large transactions from a falling stock market to government bonds would force up the bond prices, and thus yields on these bonds drops in line with the stock market.

The main conclusion from the figure is that the way to estimate correlation, either by EWMA with different levels of λ or equally weighted history will have a huge impact on the estimated volatility for the total portfolio and the VaR estimate. Since the correlations seem to be time dependent we believe the EWMA is the better choice.

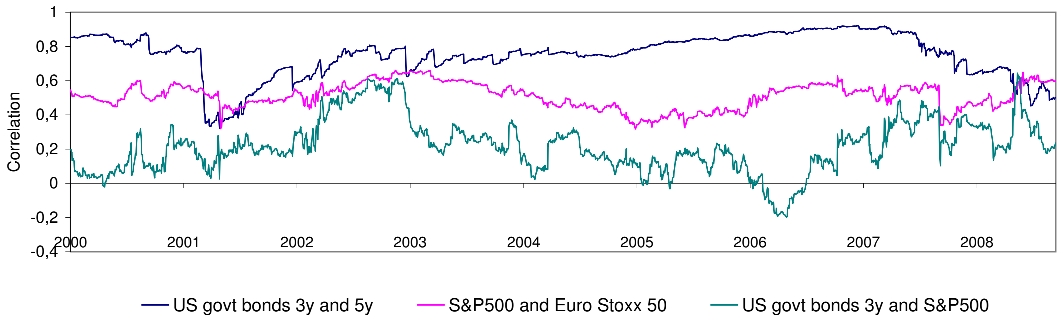

Figure 6.2: Estimated correlation between the yield on US government bonds with maturity 3 and 5 years, S&P 500 and Euro Stoxx 50, and between the yield on US government bonds with maturity 3 years and S&P 500.

6.3 Pricing and Modeling of Portfolio Instruments

When we model the dynamics of the portfolio instruments during the VaR horizon, we assume the same behavior as when pricing them. That is, geometric Brownain motion (GBM) for equity and LMM for forward rates. There is however one important difference. While the derivatives are priced under a risk neutral measure, the real dynamics should be modeled under the real measure. That is, instead of using risk neutral drift we use the historical drift.

The correlation matrix contains the correlation between all portfolio risk factors. We write

$$\hat{\rho} = BB^T,$$

where B is found by an eigenvalue-decomposition. If we let $Z \sim N(0, I)$, then $\text{Cov}(BZ) = \mathbb{E}\{BZ(BZ)^T\} = BB^T$. We can thus generate multivariate normal variables according to the historical correlation structure by setting $\tilde{Z} = BZ$, where $Z \sim N(0, I)$ [9].

6.3.1 Equities

The indices have similar nature as liquid stocks and are modeled as geometric Brownian motion according to equation (2.1). For this we need the current stock price S_t and estimates for the volatility σ as well as the real drift μ. The result is lognormal distributions for the future prices given by

$$\hat{S}_{T,i} = S_{t,i} \exp\left((\hat{\mu}_i - \frac{1}{2}\hat{\sigma}_i^2)(T-t) + \sqrt{T-t}\,\hat{\sigma}_i \tilde{Z}_i \right). \tag{6.4}$$

6.3.2 Interest Rate Related Instruments

To capture the price dynamics of interest rate instrument, we first simulate the underlying rates by Market Models using the real drift parameters. The Market Models are especially appropriate for pricing interest rate derivatives for which we have no closed form formula, and is thus suitable for all three portfolios.

Simulation of forward rates is a special case of simulating a system of SDEs. There are many choices of how to discretize and under which measure to simulate. In addition we need to choose a numerical scheme. In order to avoid negative rates, which is highly relevant today, we choose the Euler Scheme to $\log L_n$. That is, we simulate the forward rates as geometric Brownian motions over $[t_i, t_{i+1}]$ with drift and volatility parameters fixed at t_i. We will use the spot measure.

When pricing interest rate derivatives, we are first of all interested in the simulated forward rates when they coincide with the simply compounded spot rates, i.e. $L_i(T_i)$. They are used to both calculate the payoffs and discounting. Thus, it is sensible to let the maturity dates be grid points in the time discretization, i.e. $t_i = T_i$. Restricting ourself to the Euler scheme to $\log L_n$ under the spot measure leads to the discretized version of the system of SDEs in equation (2.4)

$$\hat{L}_n(t_{i+1}) = \hat{L}_n(t_i) \times \exp\left(\left[\mu_n(\hat{L}(t_i), t_i) - \frac{1}{2}\sigma_n(t_i)^2\right][t_{i+1} - t_i]\sqrt{t_{i+1} - t_i}\sigma_n(t_i)^T \tilde{Z}_{i,n}\right),$$

where

$$\mu_n(\hat{L}_n(t_i), t_i) = \sum_{j=\eta(t_i)}^{n} \frac{\delta_j \hat{L}_n(t_i)\sigma_n(t_i)\sigma_j}{1 + \delta_j \hat{L}_j(t_i)},$$

and $\tilde{Z}_i$ is generated according to the correlation structure. We assume the volatilities are functions of time to maturity (stationary) and constant between tenor dates, thus for each rate we have

$$\int_0^{T_n} \sigma_n^2(t)dt = \sigma_n^2(T_0)\delta_0 + \sigma_n^2(T_1)\delta_1 + \cdots + \sigma_n^2(T_{n-1})\delta_{n-1}.$$

The volatilities are calibrated to the market from cap volatilities at-the-money. To find the $\sigma_n^2(T_i)$'s we solve

$$\sigma_c^2(0, T_n)T_n = \sigma_n^2(T_0)\delta_0 + \sigma_n^2(T_1)\delta_1 + \cdots + \sigma_n^2(T_{n-1})\delta_{n-1}, \qquad n = 1, ..., M$$

where $\sigma_c^2(0, T_n)$ denotes the cap volatility for the cap maturing at T_n.

Government Bonds

A zero cupon bond with face value 1 and maturity at T_m have a marking-to-market value at $t = 0$ given by

$$\hat{B}(0, T_m) = \prod_{j=0}^{m-1} \frac{1}{(1 + \delta_j \hat{L}_j(0))}, \tag{6.5}$$

where $L_j(t)$ represents the simply compounded forward rate between t_j and t_{j+1} at time t. Note that this rate is not the LIBOR rate, but the forward rate corresponding to the yield on government bonds. To capture the dynamics of the bond prices we simulate the forward rates for the desired time horizon, Δt. Assume $\Delta t < \delta_0$. The simulated bond value is approximated by

$$\hat{B}(\Delta t, T_m) = \frac{1}{1 + (\delta_0 - \Delta t)\hat{L}_0(0)} \prod_{j=1}^{m-1} \frac{1}{(1 + \delta_j \hat{L}_j(\Delta t))}. \tag{6.6}$$

This will be one realization of a Monte Carlo simulation.

Interest rate derivatives

If we want to price a derivative with payoff $g(\hat{L}_n(T_n))$ at time T_n where $\hat{L}_n(T_n)$ is simulated under the spot measure, Glassermann [7] states that we can average over independent replications of

$$g(\hat{L}_n(T_n)) \cdot \prod_{j=0}^{n-1} \frac{1}{1 + \delta_j \hat{L}_j(T_j)} \tag{6.7}$$

For interest floors, the payment at T_i is determined by the spot rate at T_{i-1}. The value of an interest rate floor with strike K and with payments at T_i, $i = 2, ..., M$, and with a notional amount of 1, is thus found by averaging over independent replications of

$$\sum_{n=2}^{M} \left(\delta_{n-1}(K - \hat{L}_{n-1}(T_{n-1}))^+ \cdot \prod_{j=0}^{n-1} \frac{1}{(1 + \delta_j \hat{L}_j(T_j))} \right).$$

As mentioned in the description of portfolio 3, the floor contract will initially be out-of-the money. More precisely we set $K = L_1(0) - 0.25\%$.

For a payer swap contract, with the same payment legs as above, we average over independent replications of

$$\sum_{n=2}^{M} \left(\delta_{n-1}(\hat{L}_{n-1}(T_{n-1}) - K) \cdot \prod_{j=0}^{n-1} \frac{1}{(1 + \delta_j \hat{L}_j(T_j))} \right).$$

We implement a cash position that receive the floating LIBOR rates, like the floating legs of the payer interest rate swap. A notional amount of 1 is payed back at the maturity of the contract. Since

$$\big(K - L_i(T_i)\big)^+ + L_i(T_i) = \max\big(K, L_i(T_i)\big), \tag{6.8}$$

the value of the net cash flow from both contracts is found by averaging over independent replications of

$$\sum_{n=2}^{M} \left(\delta_{n-1} \max(K, \hat{L}_{n-1}(T_{n-1})) \cdot \prod_{j=0}^{n-1} \frac{1}{(1 + \delta_j \hat{L}_j(T_j))} \right) + \prod_{j=0}^{M-1} \frac{1}{(1 + \delta_j \hat{L}_j(T_j))}. \tag{6.9}$$

In addition, we are concerned with how the value of the contract might change over the VaR horizon and we want to simulate prices at Δt into the future. This is done by taking the time step Δt under the real probability measure. The resulting rates $L_j(\Delta t)$ are then used as the initial forward rates, and the valuation at Δt is based on these. The time to maturity for all payment legs are reduced by Δt. Figure 6.3 shows a simplified visualization of the simulation.

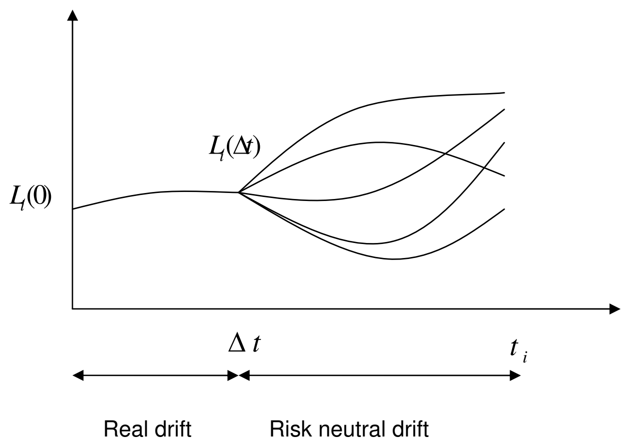

Figure 6.3: Simulation of future values for interest rate derivatives. Paths are simulated up to time Δt using the real drift parameters. The resulting rates are used as the initial rates for the derivative pricing under the risk neutral measure.

This is much more computationally expensive than the modeling of the other portfolio instruments. If we need to make B realizations to price a floor, we would need $M \cdot B$ runs to get the VaR contribution from the derivative. And if we want to estimate daily VaR for the entire history (10 years) the simulation becomes time consuming.

For the interest rate derivatives we assume payment legs at the tenors dates for which we have data. An alternative would be to have e.g. yearly payments, and approximate the missing rates based on the spot rate, the 3Y, 5Y and 10Y rates. This would describe more realistic contracts, but make minor differences to the price dynamics and the market risk estimates. Since we focus on the computational efficiency, we make this simplification.

A more comprehensive description of the implementation is given in Appendix A. Pseudocode for estimating drift, and making time steps and pricing interest rate derivatives by LMM, is enclosed. The routine for estimating risk is also described through an algorithm.

6.4 Number of Replications

In general, Monte Carlo simulation has a slow convergence. The error due to the simulation decrease as $\mathcal{O}(1/M)$. By making many VaR estimates and measuring the standard deviation we can get an idea of how many replications are necessary. With 10000 replications the standard deviation is less than 0.5% of the mean VaR, which seems acceptable. This result is based on the 97.5% quantile. Quantiles further into the tails are more sensitive to the number of replications. If there are significant large errors due to a low number of replications, this will be visible from the VaR plots, with jagged estimates from period to period, and unstable distances between the quantiles.

6.5 Time Horizons and Confidence Levels for Risk Estimates

We make VaR estimates for both daily and quarterly changes. Daily VaR is typically relevant for traiders and investors with highly liquid positions, i.e. the positions are small compared to the total market and it is possible to sell out quickly. Quarterly VaR can be of interest for larger funds, with longer liquidation horizon or simply for accounting purposes. The simplest VaR models based on normal distributed asset returns suggest that the VaR for a longer time horizon can be found by multiplying the VaR for a short period by a the square root of a time factor. Since we have instruments with non-linear dependence of the underlying risk factors, both bonds and interest rate derivatives, and the correlation structure depends on the time horizon we avoid this assumption. Naturally, daily VaR is estimated based on daily historical data. For the quarterly VaR we would ideally use a long time series of quarterly data. But since we only have data from 2000 to the beginning of 2009, and since we want to estimate VaR from 2001, we use weekly data as a compromise. We will estimate VaR for the 95%, 97.5% and 99% confidence levels and expected shortfall for the 99% level. Evaluating a set of quantiles allows us to determine how far into the tails extreme events occur and we can get a deeper insight to the behavior of the portfolio compared to the model.

The process to obtain VaR and ES estimates from the data material is illustrated through a flowchart in the next section.

6.6 Flowchart

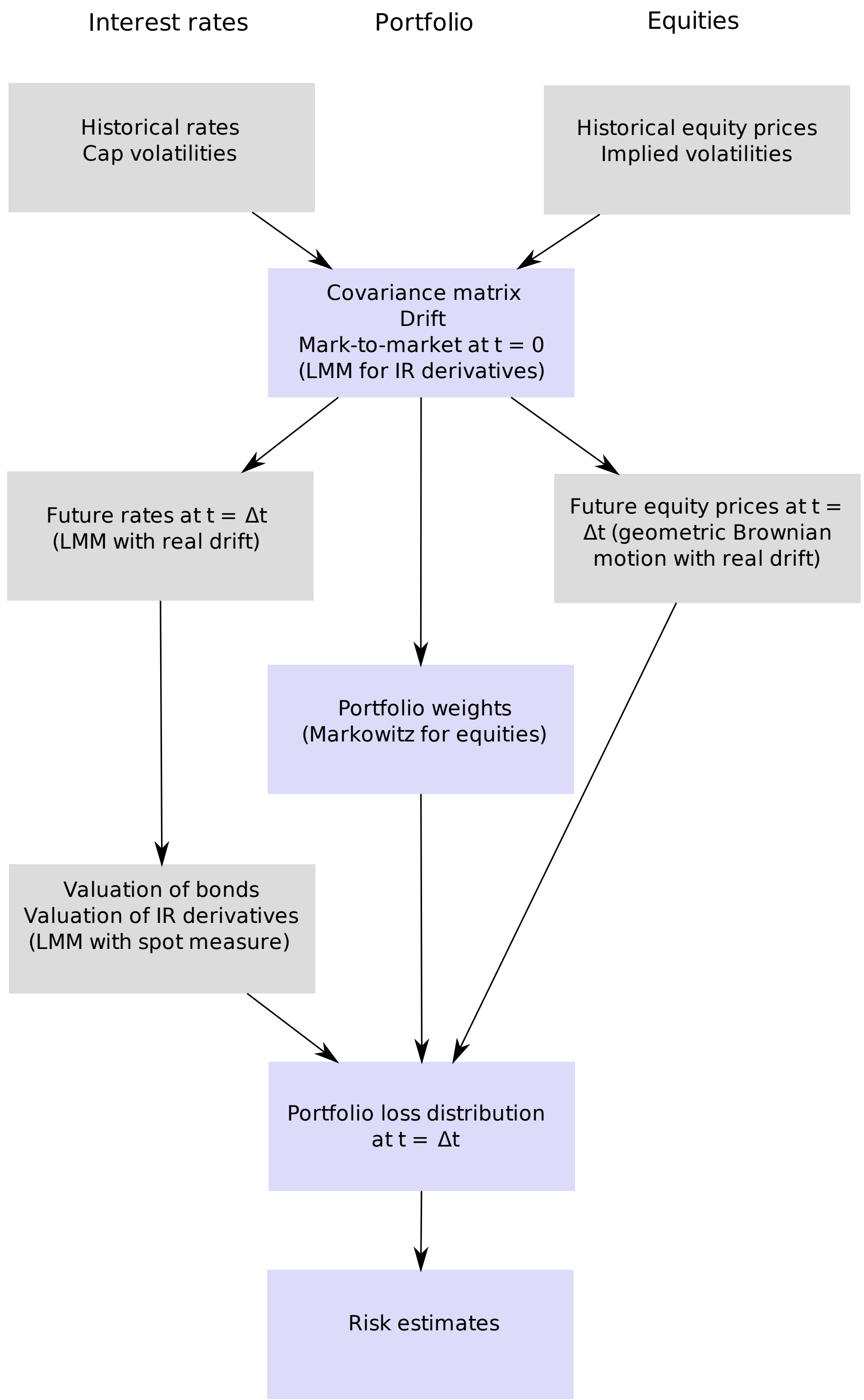

Figure 6.4: Flowchart for risk estimation routine.

Chapter 7

Results

Obviously, the VaR metric is a widely general term, and the properties will depend on the models we choose and how we construct them. Parametric or non parametric models, the distribution of asset returns, how to estimate volatilities and correlation, the portfolio structure are all factors with impact on the estimates. Naturally, we will focus on the features of our model with the corresponding assumptions, and the results will be valid only under these conditions, not for VaR and ES in general.

Having implemented a model describing the total portfolio dynamics, we are interested in how this model fits the real market observations. This can be checked by making back testing. Given the historical data up to time t we estimate VaR for the next period $t + \Delta t$. This estimate is compared with the actual portfolio loss over the same period. This is done for the whole data history. When the portfolio loss exceeds the VaR limit, this is referred to as a *VaR break*. If the model is a good representation of the portfolio dynamics, the VaR breaks should be randomly distributed over the data history. In addition the losses should exceed the VaR limits in line with the confidence level, i.e. the losses should be greater than the 95% limit in 5 of 100 observations over time. We examine if there exist any pattern for when the losses exceed the VaR and ES thresholds, with focus on the level of market stress.

Do the model make reasonable forecasts for the VaR and ES? How much larger are the VaR estimates during volatile markets? Do VaR breaks occur more often during market stress? Are the extreme observations further into the tails during turbulent markets? What happened to the risk estimates and portfolio losses during the most central events of the financial crisis of 2007- 2009? These are questions we address and will try to answer.

7.1 Daily Risk Estimates

7.1.1 Comparison of Risk Estimates and Real Losses

In this section we evaluate the VaR and ES estimates versus the actual portfolio losses. This allows us to evaluate the nature of the portfolio fluctuations, and the reliability of the models on a daily basis. We will analyze how the models perform at different levels of market stress, by measuring the frequency of VaR breaks during predefined periods. We divide the dataset into periods of either normal or turbulent marked conditions. Based on the VIX index, the periods from September 2001 to December 2003, and July 2007 to January 2009 are defined to be turbulent, while the period between January 2004 and June 2007 which is characterized by low volatility, is defined as normal.

Portfolio 1

Figure 7.1 shows the daily VaR estimates and the losses for portfolio 1. The estimates seem to follow the size of the large losses, which clearly is a good sign. There are no observations unreasonable far into the tails. The VaR estimates are largest during the turbulent periods in the beginning and at the end of the data set. For a portfolio consisting of pure investment grade bonds, it seems like the interest rate risk is not much larger during the late credit crunch than during the previous recession.

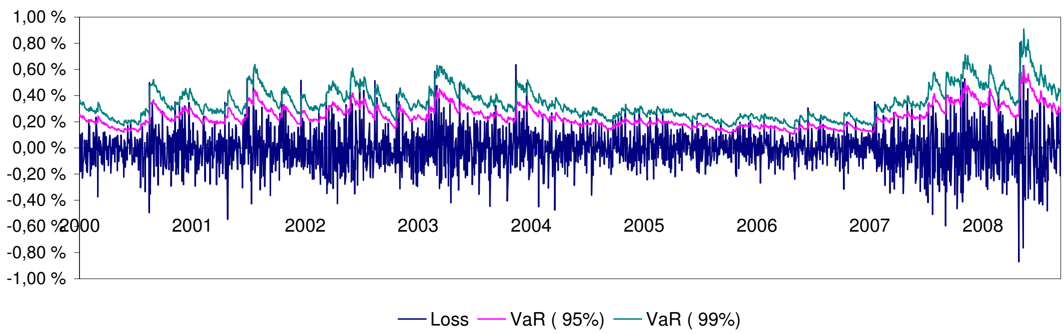

Figure 7.1: Daily VaR estimates and losses for portfolio 1.

The table in Figure 7.2 shows the number and frequency of VaR violations during the predefined periods. The VaR(95%) estimates seem to match the actual losses well. This limit is violated only slightly more than expected from the confidence level. As we move further into the tails the breaks deviate more from what we would expect. The VaR(99%) is violated at a frequency of almost 2%. This indicate that the loss distribution for the bond portfolio has fatter tails than our model suggests, which is most likely a consequence of our assumption of normal distributed logreturns for the interest rates. An interesting observation is that the risk thresholds are violated less during the financial crisis, and especially for the VaR(99%).

For the first period the ES(99%) is violated 15 times over 867 days. Since the tail distributions for losses are right skewed, the ES should be further out than the median of the tail distribution. For observations breaking VaR(99%) we would expect 50% to exceed a tail median by definition, and thus less than 50% to violate the ES(99%). That means that less than 0.5% of the losses should be larger than the ES(99%). The later periods have less ES breaks and seem to match the theoretical expectations better.

Portfolio 1	From	To	Trading days	VaR(95%)	VaR(97.5%)	VaR(99%)	ES(99%)
Number of breaks							
Turbulent	Sep-01	Dec-03	867	45	28	17	15
Normal	Jan-04	Jun-07	911	54	29	17	7
Turbulent	Jul-07	Jan-09	412	21	11	4	1
Break frequency							
Turbulent	Sep-01	Dec-03	867	5,19 %	3,23 %	1,96 %	1,73 %
Normal	Jan-04	Jun-07	911	5,93 %	3,18 %	1,87 %	0,77 %
Turbulent	Jul-07	Jan-09	412	5,10 %	2,67 %	0,97 %	0,24 %

Figure 7.2: VaR violations versus market conditions for portfolio 1.

Portfolio 2

Figure 7.3 shows the portfolio losses and the daily VaR estimates for portfolio 2. From the second half of 2008 the VaR(95%) estimates are almost four times larger than it was at its lowest during the bull market from 2004 to mid-2007. The largest portfolio loss in one day was 1.3% at January 21 2008. This day was a "Black Monday" in worldwide stock markets, especially for the European market. The Guardian wrote on January 22 2008: "Since the start of the year share prices have dropped by 14%, with the near 900-point fall in the FTSE 100 wiping out all the gains of the last 18 months and putting renewed pressure on pension funds. Yesterday's 5.48% fall was the biggest in percentage terms since the immediate aftermath of the 9/11 terrorist attacks but less than half as big as the record 12.2% drop in October 1987."

The Euro stoxx 50 fell 7.3% that day. In addition, at this time 80% of the equity was invested in this index. This explains the size of the portfolio loss.

Many of the largest portfolio losses occurred during October 2008, e.g. a 1.12% drop at the 9th. But for comparison, the S&P500 index fell more than 9% three of the days during the same period. Not surprisingly, the portfolio is much less risky than portfolios consisiting purely of stocks. On average, the positions in indices contribute to approximately 60% of the portfolio fluctuations. This means that the relatively small fraction invested in equities dominate the portfolio losses.

It is clear from Figure 7.3 that the size of the relative profits are often larger than the losses. This has a natural explanation. If a stock price is first halved and later reach its original price, it has dropped 50% and then had a positive gain of 100%.

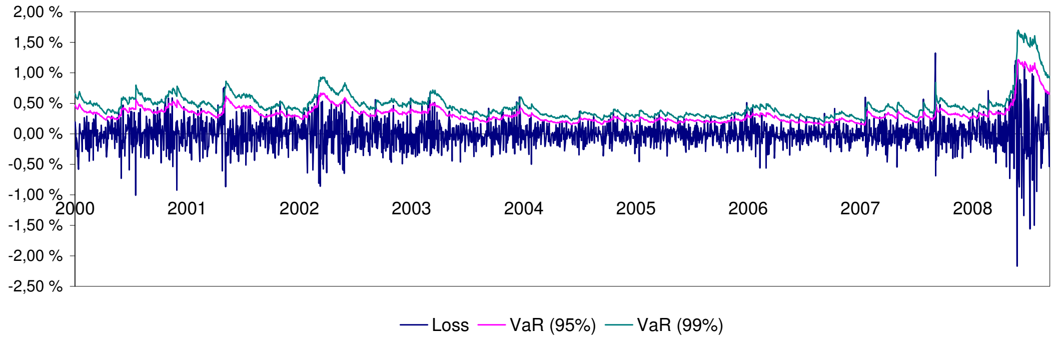

Figure 7.3: Daily VaR estimates and losses for portfolio 2.

The VaR violations for the different periods are shown in figure 7.4. As before the limits are violated at a higher frequency than what would be natural from the confidence level. The deviations are larger further into the tails for this portfolio as well. The VaR(99%) is violated almost twice as often as desired, and surprisingly most during the normal market conditions.

For the ES breaks, close to half of the losses exceeding VaR(99%) also violate the ES(99%). The latest period has the highest frequency of breaks. Again, the ES violations support a more heavy tailed distribution for the actual losses.

Portfolio 2	From	To	Trading days	VaR(95%)	VaR(97.5%)	VaR(99%)	ES(99%)
Number of breaks							
Turbulent	Sep-01	Dec-03	867	52	27	15	7
Normal	Jan-04	Jun-07	911	55	35	20	10
Turbulent	Jul-07	Jan-09	412	31	16	8	5
Break frequency							
Turbulent	Sep-01	Dec-03	867	6,00 %	3,11 %	1,73 %	0,81 %
Normal	Jan-04	Jun-07	911	6,04 %	3,84 %	2,20 %	1,10 %
Turbulent	Jul-07	Jan-09	412	7,52 %	3,88 %	1,94 %	1,21 %

Figure 7.4: VaR violations versus market conditions for portfolio 2.

Portfolio 3

This portfolio has more complex dynamics than the previous. The VaR estimates in Figure 7.5 indicate that it has a sligtly smaller risk level than portfolio 1, and a significantly smaller level than portfolio 2. The latter is primarily due to the lower fraction invested in equities, the first is probably the result of diversification and hedging. In addition, the estimates are more stable over the data history, meaning that the portfolio according to the model is less sensitive to the changes in market regimes.

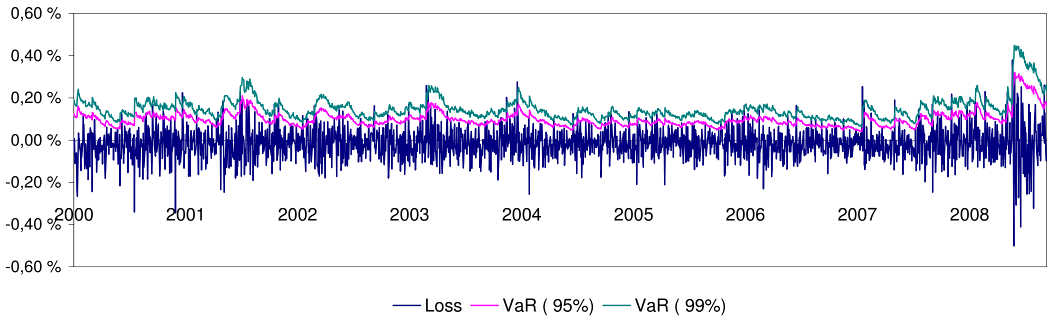

Figure 7.5: Daily VaR estimates and losses for portfolio 3.

In line with what we have seen earlier, figure 7.6 shows that also for portfolio 3, the VaR limits are violated more frequent than expected from the confidence level, especially for the latest period. Here, almost three percent of the losses are larger than VaR(99%).

The number of ES(99%) breaks are high for all three periods, and particularly the 1.94% for the latest period. A frequency of 2% corresponds to more than four times the number of violation, than what would be in line with the 99% confidence level.

Portfolio 3	From	To	Trading days	VaR(95%)	VaR(97.5%)	VaR(99%)	ES(99%)
Number of breaks							
Turbulent	Sep-01	Dec-03	867	45	26	15	9
Normal	Jan-04	Jun-07	911	49	29	16	9
Turbulent	Jul-07	Jan-09	412	25	17	12	8
Break frequency							
Turbulent	Sep-01	Dec-03	867	5,19 %	3,00 %	1,73 %	1,04 %
Normal	Jan-04	Jun-07	911	5,38 %	3,18 %	1,76 %	0,99 %
Turbulent	Jul-07	Jan-09	412	6,07 %	4,13 %	2,91 %	1,94 %

Figure 7.6: VaR violations versus market conditions for portfolio 3.

7.1.2 Central Events and Portfolio Losses

In this subsection we evaluate how the portfolios were affected by some of the most critical announcements during the most rough period of the financial crisis. October 2008 has later been referred to as "Black October". Due to the market collapse, the market volatility increase dramatically. The risk estimates more than triples during this month.

Portfolio 1

The VaR estimates versus the losses for the government bond portfolio are shown in Figure 7.7. The announcement of the bankruptcy of Lehman Brothers causes the large profit of 0.87% the very same day. This is the largest one day profit during the whole data set. It shows how the risk willingness of investors was reduced by this event. The two other events do not seem to have caused large market reactions, at least not on the same days as the news were announced. This might be due to the flow of information.

The VaR limits are not violated more than expected even though it is the most volatile period of the dataset. This agrees with the results in Figure 7.2. That is, for portfolio 1, the VaR breaks are less frequent and close to what is expected from the confidence level during the financial crisis.

Figure 7.7 also shows a one day lag for the size of the risk estimates compared to the size of the profits and losses. As the volatilities are based on historical data for the government bonds, large fluctuation will cause larger VaR estimates for the next trading day. This can be seen from the sharp increase in the VaRs for the day after the huge loss at September 19, followed by a decrease the next trading day September 22, which agree with the small loss that day. This confirmes that the indecies in the implementation are correct.

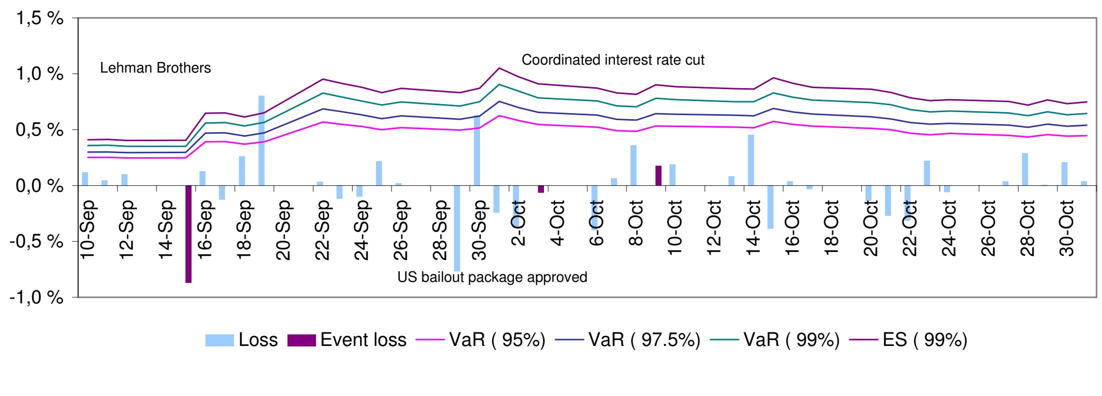

Figure 7.7: Losses and risk estimates during the market crash of September and October 2008 for portfolio 1.

Portfolio 2

From figure 7.8 we can get a closer look at the losses and risk estimates for portfolio 2, during the same period. As we see, the portfolio did not have the largest fluctuations at any of the event dates. This might be due to the flow of information to investors, or because the diversification between equities and government bonds cancel large portfolio fluctuations. As shown earlier in figure 6.2, the correlation between the logreturns on the 3 year government bond yield and the Standard & Poor 500, is high during the last part of 2008. This means that the bond and equity prices tend to move in the opposite direction, especially during financial turbulence.

By analyzing the portfolio closer, it seems like the direction of the losses occur as we would expect for an equity portfolio. That is, the collapse of Lehman Brothers causes a portfolio loss. For the three events, we can calculate how much of the fluctuations were caused by the equity positions versus the bond positions. At the day Lehman collapses, 71% of the portfolio fluctuation are caused by the stocks. The day of the bailout package, 80% of the fluctuation were from stocks. And last, for the coordinated interest rate cuts 70% of the fluctuations were from stocks. Since "only" 60% of the portfolio fluctuations are caused by the equity positions on average, it seem like the equity market is affected more than usual by the critical announcements, compared to the bond market.

October 6-10 was the worst week for the stock market in 75 years. The Dow Jones loses 22.1 percent, its worst week on record, down 40.3 percent since reaching a record high a year earlier. The S&P 500 loses 18.2 percent, its worst week since 1933, down 42.5 percent since its own high October 9, 2007. That this week was a disaster for the portfolio is evident from Figure 7.8, with four of five days breaking the VaR(95%), and three days breaking VaR(99%).

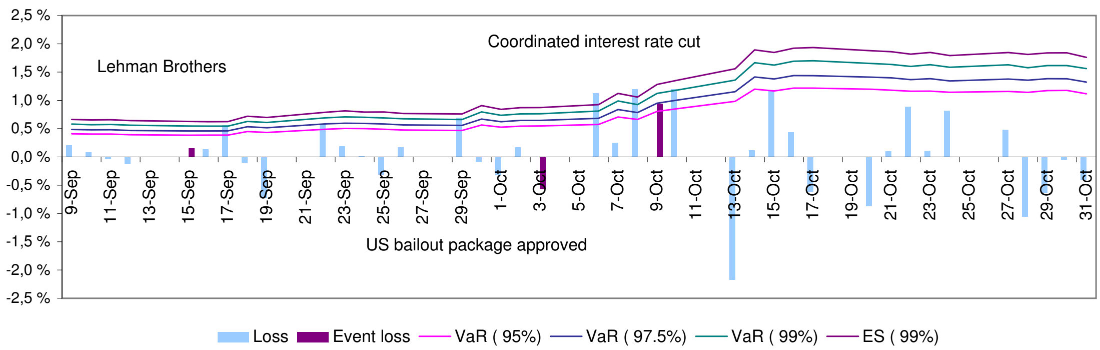

Figure 7.8: Losses and risk estimates during the market crash of September and October 2008 for portfolio 2.

Portfolio 3

Figure 7.9 shows the losses around October 2008 for portfolio 3. Since the portfolio structure is more complex, it is more difficult to have a perception of how the underlying risk factors contribute to the losses. However, this portfolio have the largest number of breaks during the rough period. The largest loss occur at the day of the coordinated interest rate cuts. This might look awkward since the floors will raise in value with sinking rates. But as we saw earlier, portfolio 1 also had a loss that day despite it consists purely of bonds. According to our data, the yield on government bonds did not sink that day. It was again a signal of the severity of the crisis, and might have increased investors fear of a state default.

For the previous two portfolios, the distance between the VaR thresholds seem to increase roughly proportional to the VaRs. This indicate that the shapes of the loss distributions are preserved over time. For portfolio 3, the distances are much larger during turbulent periods. The shape of the estimated loss distribution change becomes more heavy tailed during turbulence.

As for portfolio 2, the number of VaR breaks are high. The VaR(95%) and VaR(97.5%) thresholds are violated seven times, while the VaR(99%) four times.

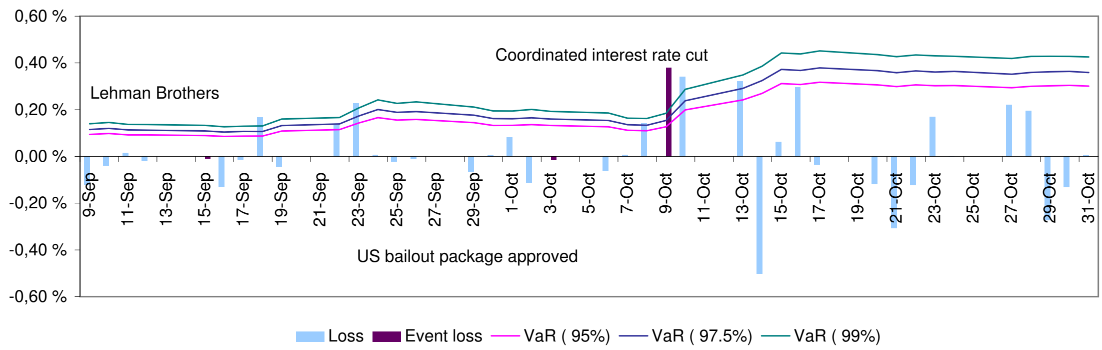

Figure 7.9: Losses and risk estimates during the market crash of September and October 2008 for portfolio 3.

7.1.3 Cumulative Number of VaR Violations

Figure 7.10 shows the cumulative number of VaR breaks versus time for portfolio 2. If the VaR breaks were evenly distributed, this would be a perfectly straight line. But since the observations should be randomly spread out, we expect a jagged line. To have a perception of how this line should look, we have generated 2263 independent realizations from a Bernoulli distribution, the number of historical observations. By assigning a probability of success to be 5%, this line represents how the cumulative number of breaks for the VaR(95%) might look if the model was a perfect fit. The most significant difference between the two is that the cumulative break for VaR(95%) have a steeper trend. Further, the cumulative breaks show two periods with extra high frequencies of breaks. The end of the periods are marked with vertical lines. They are both during financial turbulence with increasing volatility, which support that the models might perform slightly worse during market stress. However, as an overall picture, the cumulative number of violations seem to behave fairly similar to the randomly generated line. And it does not appear to be any obvious changes in regimes from evaluating the cumulative number of VaR violations. Before the analysis we expected the model to fail during the market crash of 2008. Interestingly, there is not a clear change in the accuracy of the estimates compared to previous periods, which are under completely different market conditions.

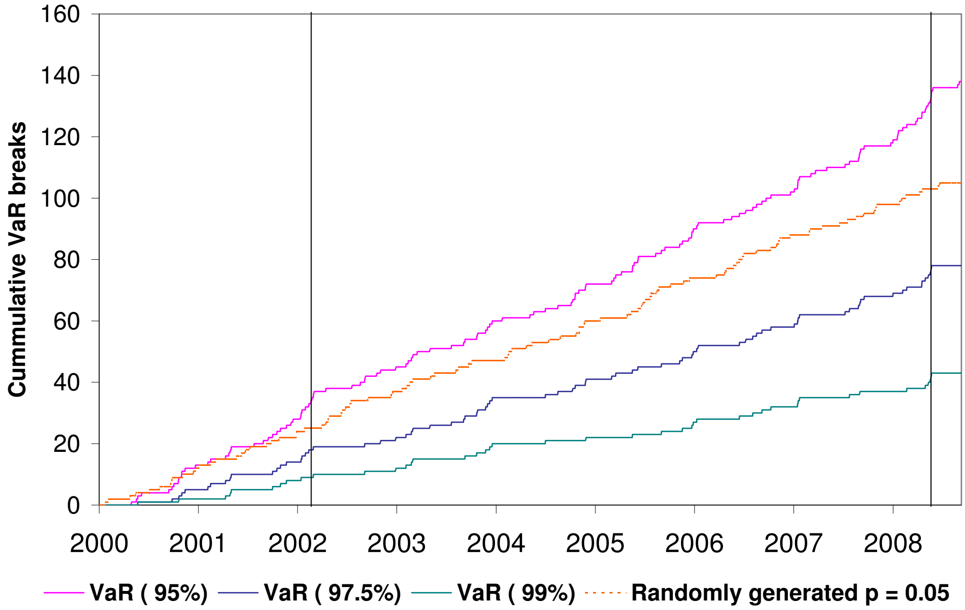

Figure 7.10: Cumulative number of losses exceeding VaR(95%), VaR(97.5%) and VaR(99%) from 2000 to 2008 for portfolio 2. For comparison we have generated independent realizations from a Bernoulli distribution.

7.1.4 Likelihood of Extreme Observations

October 2008 was the most volatile period of the data set, and the number of VaR breaks are high. For portfolio 2, the VaR(95%) is violated 7 of 39 days, and both the VaR(97.5%) and the VaR(99%) are violated 5 times. This corresponds to a frequency of 17.9% and 12.8%, respectively. All quantiles are therefore violated way more than desired. It is of interest to find the probability of these extreme observations given that the model fit the data. Let B_α be the number of VaR breaks for the VaR corresponding to the confidence level α. Based on the model assumptions, the probability of a VaR violation p should be constant with $p = 1-\alpha$, and the violations should be independent. Over this subset with $n = 39$ observations, the B_α should be binomial distributed, $\text{bin}(n, p)$. That is, the probability for B_α to take the value b is given by

$$p(B_\alpha = b) = \begin{pmatrix} n \\ b \end{pmatrix} (1-\alpha)^b \alpha^{n-b}. \tag{7.1}$$

The probability of observing b or more violations is thus

$$p(B_\alpha \geq b) = 1 - p(B_\alpha < b) = 1 - \sum_{k=0}^{b-1} p(B_\alpha = k). \tag{7.2}$$

This value is known as the p-value. Table 7.1 shows the p-values for the VaR breaks. The probability of observing the same number of VaR violations or more is 0.29%, 0.28% and 0.0043% for the α's in increasing order. Five violations or more for the 99% VaR in 39 days is expected to happen once out of 23,256 such periods. This is thus almost unlikely to happen, given that our model represents the data[1]. This is a strong indication of clustering between tail events.

α for VaR	breaks	frequency	p-value
95 %	7	17.9%	0.29%
97.5%	5	12.8%	0.28%
99 %	5	12.8%	0.0043%

Table 7.1: Test for the probability of the large number of VaR breaks.

[1]The VaR breaks should theoretically be independent of the level of market stress. Since we choose to evaluate this period beforehand, we can evaluate these extreme data isolated. It would in contrast be wrong to take a view at the entire dataset, find the periods with the largest number of VaR breaks, and then do this test isolated for that particular subset.

7.2 Quarterly Risk Estimates

There is even more uncertainty when making risk estimates for a quarter of a year ahead. Our model assumes that the correlations and volatilities are constant during the time horizon for the simulation. In addition, it is challenging to estimate both volatility and correlation since they should be based on quarterly data and we have a limited length on the data set. As emphasized in the previous chapter, we make a compromise and base our estimates on weekly data. Figure 7.11 shows the VaR(95%) from models calibrated based on data with increments of one day, three days and one week for portfolio 2. As confirmed by the figure, the choice of increment length have a significant effect on the estimates. And when comparing the relative differences between the three, it seems like this assumption will dominate the historical versus implied volatility choice. This would however not have been an issue if we had had historical prices for decades back in time, and only were interested in making future risk estimates from the last data point, but in this thesis we make risk estimates for the whole data set. Since there are only 35 quarterly observations in the dataset, there is little significance in comparing risk estimates with actual losses.

Another observation worth mentioning appears when comparing Figure 7.3 and Figure 7.11. While the daily risk estimates for portfolio 2 are largest during the late financial crisis, the quarterly VaRs in 7.11 are largest during the less volatile period in the beginning of the dataset. If the daily market risk is largest during the latest period, then we would believe this to be the case for the quarterly market risk estimates as well. This is explained by a time lag for quarterly estimates. The largest price fluctuations for the portfolio were during October 2008. For the daily volatility estimates, these large price fluctuations will be picked up fast and increase the estimates. The quarterly volatility estimates for Q4 of 2008 are in contrast estimated based on data up to September 2008. The size of the quarterly VaR estimates will therefore have a time lag compared to the daily.

Figure 7.11 shows 35 observations of losses and VaR estimates. The expected number of VaR breaks for $\alpha = 95\%$ is $35 \cdot 5\% = 1.75$, and the actual breaks are one for all the three VaR estimates. Because of the limited number of observations, it makes no sense to draw any conclusions. The estimates do at least look reasonable compared to the sizes of the actual losses.

The period from 2003 to 2007 is characterized by increasing interest rates in the U.S. and a bull stock market. This results in a profitable stock portfolio and losses from the government bond portfolio. As we have concluded earlier, the equity positions have the largest impact on the portfolio returns, and this is supported by Figure 7.11. During this bull period, 14 of the quarterly returns are profits in contrast to only four losses. This period is followed by four quarterly losses on a row when the financial crisis becomes evident.

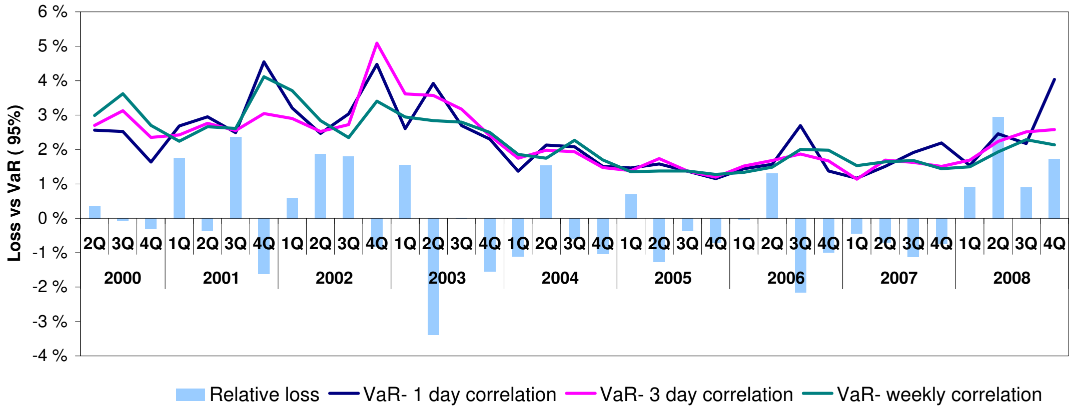

Figure 7.11: Quarterly VaR(95%) estimates and losses for portfolio 2 . The correlations are estimated based on daily, three days and weekly history.

Chapter 8

Conclusion and Further Work

8.1 Conclusion

In order to get VaR and ES estimates from a Monte Carlo simulation, we need to assume probability distributions for the relevant risk factors. It is common to assume that the logreturns of assets are normal distributed, and this is the case for our model. The empirical logreturns for equity and interest rates are however more sharp-peaked and fat-tailed than suggested by the normal distribution. This means that the model should underestimate the likelihood of extreme losses, and may be an especially poor assumption during financial turbulence. Additionally, we assume that the logreturns are multivariate normal, and that the covariance matrix is constant over the risk horizon. A commonly observed property of asset returns is that during periods with large price moves, there is a greater degree of co-movements across different assets. This is often referred to as tail dependence [15]. These two last assumptions introduce additional weaknesses to the model since the multivariate normal distribution do not exhibit tail dependence and volatilities are not constant over time. However, broadely adopted models are constructed this way.

We analyze the performance of the model by comparing the daily risk estimates with the portfolio losses between 2000 and 2008. The results show that the losses exceed the VaR quantiles more frequent than what would be natural from the corresponding confidence level α, i.e. the VaR(α) should be violated $(1-\alpha)$ of the days, by definition. Further, the larger we set α, the more misguiding are the estimates. While the 95% quantiles are violated 5-6% of the days, the 99% quantiles are violated about 2% of the days. The highest frequencies of violations registered, are from the current financial crisis.

In contrast to VaR(α), we do not know exactly how often ES(α) should be violated directly from its definition. However, we know that it is less than $(1-\alpha)/2$. Since the tail distributions for losses are right skewed, the ES should be further out than the median of the tail distribution. For observations violating VaR(99%) we would expect 50% to exceed a tail median, and thus less than 50% to violate the ES(99%). That means that less than 0.5% of the losses should be larger than the ES(99%). On average, the losses exceed the ES more than twice as often. And for the last portfolio we tested, the ES(99%) is violated about 2% of the days during the latest financial crisis, which is more than four times more than expected from the confidence level.

The high frequency of VaR and ES violations indicate that the normal assumption for logreturns result in a model that under-estimate the probability of extreme losses. Large parts of the theory of mathematical finance is however based on normal distributed asset returns, such as the Black Scholes framework. Practitioners have found ways to work around this for many problems, e.g. working with different volatilities for options with different strikes. In the risk context we are not only estimating one parameter like an option price, but we are interested in the shape of the whole loss distribution. With the normal assumption, it seems impossible to get resonable estimates for all VaR quantiles, using only one loss distribution for all confidence levels.

Investigation of the losses during the rough period around October 2008 shows strong indications of clustering between large losses. E.g. in one single week, the Var(95%) is violated four times for one of the portfolios.

In addition to daily horizons, we have analyzed quarterly risk estimates. From the results, the estimates seem to be reasonable compared to the size of the portfolio losses. There are however little significance to such analysis, since we only have 39 observations for quarterly portfolios losses over the data set.

Even if the daily losses exceed the risk estimates too often compared to the corresponding confidence levels, and even though we see strong tendencies of clustering, it is evident from the results that the risk estimates do give valuable information about the market risk of the portfolios. For instance, the frequencies of losses exceeding the risk estimates are relatively stable over time. A priori, we expected the model to be flawed during the extreme volatile market observed recently, but our analysis indicate that the model in many cases performes only slightly worse. The risk estimates from the model can thus be informative even under turbulent market conditions. However, it is important to be aware of the weaknesses of the model.

8.2 Further Work

To deal with the clustering of VaR violations, plausible solutions point towards time correlated frameworks for volatilities, such as GARCH models. Among academics and practitioners, GARCH-type models have gained attention. This is due to the evidence that time series realizations of returns often exhibit time dependent volatility. Numerous tests of GARCH-type models to foreign exchange and stock markets have demonstrated that these approaches can provide somewhat better estimates of volatility than simple moving averages, particularly over short time horizons such as a day or a week [10].

A more radical model modification would be to consider alternative distributions for the risk factors, such as the Normal Inverse Gaussian distribution (NIG). It is more flexible and provides a better fit to the logreturns for asset.

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Appendix A

Pseudocode

The model is implemented in R. Algorithm A.1 shows how the drift is calculated at a time step under the spot measure, and algorithm A.2 shows how the forward rates are simulated from a time step to the next. Algorithm A.3 describes the pricing routine, and the procedure for estimating VaR is found in algorithm A.4. To make the pseudocode easy to interpret matrix operations are described as loops.

Algorithm A.1 Drift - Calculate drift at t_i under the spot measure

Input:
Volatilities $\sigma_n(t_i)$, $n = i+1, ..., M$
Forward rates $\hat{L}_n(t_i)$, $n = i+1, ..., M$
Time between tenors δ_j, $j = i, ..., M$
for n = i+1 to M **do**
    for j = i to n **do**
        $\mu_n(\hat{L}_n(t_i), t_i) = \mu_n(\hat{L}_n(t_i), t_i) + \frac{\delta_j \hat{L}_n(t_i)\sigma_n(t_i)\sigma_j}{1+\delta_j \hat{L}_j(t_i)}$
    end for
end for
Return: $\mu_n(\hat{L}(t_i), t_i), \quad n = i+1, ..., M$

Algorithm A.2 LMM - Simulate forward rates for t_{i+1} based on rates at t_i

Inputs:
$\sigma_n(t_i), \quad n = i+1, ..., M$
$\hat{L}_n(t_i), \quad n = i+1, ..., M$
Do:
Draw $\tilde{Z}_i \sim N(0, \hat{\rho}_{rates})$
if spot measure **then**
    Drift by algorithm A.1
else
    if real drift **then**
        $\hat{\mu}_n = \hat{\mu}^n_{real}$
    end if
end if
for $n =$ i+1 to M **do**
    $\hat{L}_n(t_{i+1}) = \hat{L}_n(t_i) \times \exp\left(\left[\mu_n(\hat{L}(t_i), t_i) - \frac{1}{2}\sigma_n(t_i)^2\right][t_{i+1} - t_i] + \sqrt{t_{i+1} - t_i}\sigma_n(t_i)\tilde{Z}_{i,n}\right)$
end for
Return: $\hat{L}_n(t_{i+1}), \quad n = i+1, ..., M$

Algorithm A.3 LMM - Price interest rate derivative by LMM

Inputs:
Vol $\sigma_n(t_i)$, $n = 1, ..., M-1$, $t_i = T_i$, $i = 0, ..., n-1$
Initial rates $\hat{L}_n(0), \quad n = 0, ..., M$
Time between tenors δ_j, $j = 0, ..., M-1$
for b in $1..B$ **do**
    Get $\hat{L}_n(T_n)^b$, $n = 1, ..., M$ from algorithm A.2 ($t_i = T_0, ..., T_{M-2}$)
    $PV_b = \sum_{n=1}^{M} g(\hat{L}_n(T_n)^b) \cdot \prod_{j=0}^{n-1} \frac{1}{1+\delta_j \hat{L}_j(T_j)^b}$
end for
Price $P = \frac{1}{B}\sum_{b=1}^{B} PV_b$
Return P

Algorithm A.4 Estimate VaR at $t = 0$

Input:
History for risk factors $x_0, ..., x_K$, K = history length, $x_d \in \mathbb{R}^D$ # D # instruments
Forward rates $L_n(0)$, $n = 0, ..., M$
Share prices $S_{0,}$
Cap vol $\sigma_c(0, T_i)$, i = 1,...,M, VIX
Δt, α
Portfolio weights w_i, $i = 1, ..., d$
Estimate:
Covariance between all risk factors $\hat{\Sigma}$ by EWMA
Correlation $\hat{\rho}$ from $\hat{\Sigma}$
Real drift $\hat{\mu}_{real} \in \mathbb{R}^d$
Mark-to-market:
Bonds $\hat{B}(0, T_m) = \prod_{j=0}^{m-1} \frac{1}{(1+\delta_j \hat{L}_{govt,j}(0))}$, $m = 1, ..., M$
IR derivatives by algorithm A.3$(L_{LIBOR,n}(0), n = 0, ..., M)$
for $s = 1$ to S **do**
    Draw $Z^s \sim N_d(0, \hat{\rho})$
    $\hat{S}^s_{\Delta t,d} = S_{0,d} \exp\left((\hat{\mu} - \frac{1}{2}\hat{\sigma}^2)\Delta t + \sqrt{\Delta t}\hat{\sigma} Z^s_d\right)$ $d = 1, ..., d_{stocks}$
    Simulate $\hat{L}_n(\Delta t)$ by algoritm A.2$(\hat{\mu}_{real}$, $L_n(0)$, $n = 1, ..., M)$
    $\hat{B}(\Delta t, T_m)^s = \frac{1}{1+(\delta_0 - \Delta t)L_{govt,0}(0)} \prod_{j=1}^{m-1} \frac{1}{(1+\delta_j \hat{L}_{govt,j}(\Delta t))}$, $m = 1, ..., M$
    Price IR derivative $P(\Delta t)^s$ by algorithm A.3$(\hat{L}_n(\Delta t)$, $n = 1, ..., M)$
    Calculate return $r_{d,s}$, $d = 1, ..., D$
    $r_s = \sum_{d=1}^{D} w_d r_{d,s}$
end for
$r = (r_1, ..., r_S)'$
$r_{sorted} = sort(r)$
$\text{VaR}_{\alpha,\Delta t} = r_{sorted}[(1-\alpha) \cdot S]$
Return: $\text{VaR}_{\alpha,\Delta t}$