

Homotopy coherent centers versus centers of homotopy categories

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ABSTRACT. Centers of categories capture the natural operations defined on their objects. Homotopy coherent centers are an extension of this notion to categories with an associated homotopy theory. These centers can also be interpreted as Hochschild cohomology type invariants in contexts that are not necessarily linear or stable, and we argue that they are more appropriate to higher categorical contexts than the centers of their homotopy or derived categories. We present an obstruction theory for realizing elements in the centers of homotopy categories, and a Bousfield–Kan type spectral sequence that computes the homotopy groups. Several non-trivial classes of examples are given as illustrations of the general theory throughout.

Introduction

Whenever we are given a small category \mathbf{C} , we may ask for a description of all natural operations on its objects. These are the families $\Phi = (\Phi_x : x \rightarrow x \mid x \in \text{Obj}(\mathbf{C}))$ of endomorphisms that are natural in the objects x , and they form a monoid under composition. In fact, a more conceptual description of this monoid presents it as the endomorphism monoid of the identity functor $\mathbf{C} \rightarrow \mathbf{C}$ in the category of all such functors and natural transformations. Naturality implies immediately that this monoid is always abelian. This observation is usually attributed to Eckmann and Hilton. As a classical example, for the category of commutative rings of characteristic p , where p is a prime number, this monoid is the free (abelian) monoid on one generator: Frobenius. For a more recent computation with a similar outcome see [Szy]. On the other hand, if the category in question is a monoid itself—a category with one object, then we have just given a long-winded description of the center of this monoid, the subset of elements that commute with all of its elements. In general, the endomorphism monoid of the identity functor is often referred to as the *center* $Z(\mathbf{C})$ of \mathbf{C} , for instance by Bass [Bas68, II, §2] and Mac Lane [Mac71, II.5, Exercise 8], and we will follow this terminology. Bernstein, in [Ber84, 1.9], defined the center of abelian categories, but his main object of interest was the category of smooth representations of a p -adic group.

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In the present paper, we study a refined notion of center, the homotopy coherent center $\mathcal{Z}(\mathbf{C})$, for categories \mathbf{C} that admit a homotopy theory. This center is defined directly within \mathbf{C} rather than on the level of the homotopy category. Briefly, its elements will determine families $\Phi = (\Phi_x: x \rightarrow x \mid x \in \text{Obj}(\mathbf{C}))$ of endomorphisms in \mathbf{C} , but it is no longer required that these are natural in the strict sense. Instead, these elements will also come with continuously chosen homotopies $\Phi_y f \simeq f \Phi_x$ for all arrows $f: x \rightarrow y$ in \mathbf{C} and additional higher homotopies that, for instance, show that for any other arrow $g: y \rightarrow z$ the two evident homotopies $\Phi_z g f \simeq g f \Phi_x$ that can now be obtained from Φ_f , Φ_g , and Φ_{gf} , are also homotopic. The actual formulas bear very close resemblance to those used in the definition of Hochschild(–Mitchell) cohomology [Mit72]. This suggests that the idea of deriving the notion of the center of a category is not new, in particular in the additive context of homological algebra and differential graded categories. There, it already appeared in an unpublished preprint by Keller [Kel]. The ideas presented in the present paper lead to new results in the differential graded context as well. This has been realized in joint work with Neumann [NS], with applications to the characteristic homomorphism from the Hochschild cohomology of a differential graded algebra, or differential graded category, to the graded center of its derived category.

Here we will work in a non-linear and unstable context: categories enriched in spaces. Spaces will be (fibrant) simplicial sets, so that categories enriched in spaces will be categories enriched in (fibrant) simplicial sets or (fibrant) simplicial categories for short. The reader unfamiliar with simplicial categories will be able to replace them by topological categories, but even in that setting, (co)simplicial methods are indispensable for our approach. Simplicial categories are by now a well-established context in which to do homotopy theory, and there is even a homotopy theory of simplicial categories themselves [Ber07]. This is relevant here because homotopy coherent centers are invariant under the corresponding notion of weak equivalence for simplicial categories, Dwyer–Kan equivalence (Theorem 4.1). We also note that the present definition automatically extends to other contexts that have an associated homotopy theory, such as Quillen model categories or quasi-categories, by passing to the associated simplicial categories that they define.

The homotopy coherent center of any category enriched in spaces has a canonical E_2 multiplication (Theorem 3.1). This means, in particular, that these centers are A_∞ monoids (coherently associative, i.e. have an action of an $A_\infty = E_1$ operad), and that the multiplication is homotopy commutative (because it extends to an action of an E_2 operad). This is the analogue of Eckmann–Hilton in the present setting, and it can be proved using the methods introduced by McClure and Smith in [MS02], [MS04a], and [MS04b] for their solution of the Deligne conjecture.

In Section 5, we discuss the examples given by simplicial monoids and, in particular, simplicial groups. In the latter case, the homotopy coherent centers are related to other notions of centers studied before in homotopical group theory, see the ICM surveys by Dwyer [Dwy98] and Grodal [Gro10]. For instance, if G is a simplicial group, then its center consists of the fixed points under the conjugation action, and the homotopy fixed points are equivalent to the homotopy coherent center as defined here (Theorem 5.1). On the other hand, the homotopy center of a p -compact group is defined as the loop space of the space of self-maps of the classifying space based at the identity, see [MN94] and [DW95], and this is also

equivalent to the homotopy coherent center in the case where both of them are defined (Corollary 5.3).

Back in our general context, there is also a very naive and rigid notion $Z(\mathbf{C})$ of a center for simplicial categories \mathbf{C} based on (strict) equalities. The relation of the homotopy coherent center to this simplicial center takes the form of a morphism

$$Z(\mathbf{C}) \longrightarrow \mathcal{Z}(\mathbf{C}).$$

The question whether such a map from a strict limit to a homotopy limit is an equivalence (or at least some sort of completion) is called a homotopy limit problem, following Thomason [Tho83] and Carlsson [Car87]. We will discuss this question for the map displayed above in Section 6 where we also give an example that shows that the situation in the present context is more complicated.

There is a canonical map $\mathcal{Z}(\mathbf{C}) \rightarrow Z(\mathrm{Ho} \mathbf{C})$ from the homotopy coherent center to the center (in the ordinary sense) of its homotopy category $\mathrm{Ho} \mathbf{C}$. Because the target is discrete, this map factors through the components:

$$\pi_0 \mathcal{Z}(\mathbf{C}) \longrightarrow Z(\mathrm{Ho} \mathbf{C}).$$

In general, this latter map will be neither surjective nor injective. Here we will explain the difference between the homotopy coherent centers of simplicial categories \mathbf{C} and the centers of their homotopy categories $\mathrm{Ho} \mathbf{C}$. In Section 7, using the methods initiated by Bousfield and Kan, see [BK72] and [Bou89], we provide means to study the failure of surjectivity by developing an obstruction theory (Theorem 7.5) for the realization of an element in the target by an element in the homotopy coherent center. As for injectivity, we show that there is a spectral sequence (Theorem 7.1) with $E_2^{0,0}$ related to $Z(\mathrm{Ho} \mathbf{C})$ that not only targets the kernel of that map but also the higher homotopy groups of the homotopy coherent center, information that is entirely invisible from the perspective of the homotopy category. The passage to the center of the homotopy category reappears from this viewpoint as an edge homomorphism.

Fringed Bousfield–Kan type spectral sequences in unstable general contexts may have a less pleasant algebraic behavior than the spectral sequences of abelian groups that we usually meet in homological algebra. In our situation, the homotopy commutative multiplication of the coherent centers leads to some simplification. In an earlier version of this paper, this was illustrated by spelling out the details for the category of groups, where the homotopy theory is induced by conjugation of homomorphisms, so that the homotopy category is the category of representations, and which is related to the theory of bands in the sense of Giraud [Gir71]. This has now been developed with Meir [MS15] in the general setting of Drinfeld centers of bicategories.

The final Section 8 deals with another class of examples of simplicial categories: simplicial groupoids. We show that the theory presented here is related to very classical and difficult questions in (unstable) homotopy theory such as spaces of homotopy self-equivalences. That section also includes examples of simplicial categories where the homotopy coherent centers have higher homotopy groups, and where the map $\pi_0 \mathcal{Z}(\mathbf{C}) \rightarrow Z(\mathrm{Ho} \mathbf{C})$ is not injective or not surjective.

All categories considered in this paper are small. The definitions, theorems, and examples exposed here are the basis for the treatment of homotopy coherent centers of large categories such as categories of simplicial diagrams, Bousfield localizations, and categories of universal algebras with Dwyer [DS].

1. Categories enriched in spaces and their centers

We will use combinatorial models (based on (co)simplicial techniques) throughout this text. If X and Y are simplicial sets, then $\text{Mor}(X, Y)$ will denote the set of simplicial maps $X \rightarrow Y$. These are the vertices in the mapping space $\text{Map}(X, Y)$. The general formula for the n -simplices is $\text{Map}(X, Y)_n = \text{Mor}(\Delta^n \times X, Y)$.

A *space* will be a Kan complex, that is a fibrant simplicial set with respect to the Kan model structure. If Y is a space in this sense, so is $\text{Map}(X, Y)$.

We will consider small categories \mathbf{C} that are enriched in spaces, and we will write

$$\mathbf{C}(x, y)$$

for the space of maps from the object x to the object y . Of course, if we are considering the category of spaces, we will continue to write $\text{Map}(X, Y)$. Categories enriched in spaces are the fibrant simplicial categories of [Ber07].

Let \mathbf{C} be a category enriched in spaces. For any integer n such that $n \geq 0$, we will use \mathbf{C}_n as our notation for the (ordinary) category of n -simplices in \mathbf{C} . In particular, we will call \mathbf{C}_0 the *underlying category* of \mathbf{C} . There is another way to pass from a category \mathbf{C} enriched in spaces to an ordinary category: the *homotopy category* $\text{Ho } \mathbf{C}$. It has the same set of objects, but the set of morphisms $x \rightarrow y$ in $\text{Ho } \mathbf{C}$ is the set $\pi_0 \mathbf{C}(x, y)$ of components of the mapping space $\mathbf{C}(x, y)$.

Simplicial centers. We can now define a strict notion of center for categories enriched in spaces.

DEFINITION 1.1. Let \mathbf{C} be a category enriched in spaces. The *simplicial center* $Z(\mathbf{C})$ of \mathbf{C} is the equalizer (in the category of spaces) of the two maps

$$(1.1) \quad \prod_x \mathbf{C}(x, x) \longrightarrow \prod_{y, z} \text{Map}(\mathbf{C}(y, z), \mathbf{C}(y, z))$$

that are given by sending a given family $\Phi = (\Phi_x)$ to the map $f \mapsto f\Phi_y$ and the map $f \mapsto \Phi_z f$, respectively, in the (y, z) component.

For discrete categories \mathbf{C} , we may replace the target with

$$\prod_{f: y \rightarrow z} \mathbf{C}(y, z),$$

but the alternative above is adapted to work in the simplicial context as well.

We can determine the set of n -simplices of the simplicial center by direct inspection.

PROPOSITION 1.2. *For every category \mathbf{C} enriched in spaces, there are isomorphisms*

$$Z(\mathbf{C})_n \cong Z(\mathbf{C}_n)$$

that are natural in n .

In particular, the elements of the center $Z(\mathbf{C}_0)$ of the underlying category \mathbf{C}_0 appear as the vertices of the simplicial center $Z(\mathbf{C})$ of \mathbf{C} .

The simplicial center of a category enriched in spaces is a simplicial monoid: a simplicial submonoid of the product of the endomorphism monoids $\mathbf{C}(x, x)$, by Definition 1.1.

COROLLARY 1.3. *For every category \mathbf{C} enriched in spaces, the simplicial center $Z(\mathbf{C})$ is a simplicial abelian monoid.*

PROOF. As has been mentioned in the introduction, by Eckmann and Hilton, the centers $Z(\mathbf{C}_n)$ of the discrete categories \mathbf{C}_n are abelian. (One way to see it is in [Kas95, XI.2.4].) Now the corollary follows from the preceding proposition. \square

The following result states that the homotopy types of simplicial centers are as simple as possible.

COROLLARY 1.4. *For every category \mathbf{C} enriched in spaces, the simplicial center $Z(\mathbf{C})$ is a product of Eilenberg–Mac Lane spaces.*

PROOF. This follows immediately from the preceding corollary and Moore’s theorem about the homotopy types of simplicial abelian monoids, see [Moo55]. \square

We remark that centers in enriched contexts have been studied before, see Lindner’s paper [Lin80], for instance.

2. Homotopy coherent centers

The simplicial center of a category enriched in spaces has been defined, in the preceding Section 1, as a certain limit, see also Remark 2.1 below. Now we will define the homotopy coherent center of a category enriched in spaces as the corresponding homotopy limit. We will start with a brief review of the cosimplicial replacement of the homotopy end construction in this particular case, referring to [CP97] for more general information. See also [Wei01] for the case when the source category is discrete.

Cosimplicial replacements. Let \mathbf{C} be a category enriched in spaces. For any integer $n \geq 0$ we can consider the space

$$\Pi^n \mathbf{C} = \prod_{x_0, \dots, x_n} \text{Map}(\mathbf{C}(x_1, x_0) \times \dots \times \mathbf{C}(x_n, x_{n-1}), \mathbf{C}(x_n, x_0))$$

where the product runs over the $(n + 1)$ -tuples of objects of \mathbf{C} . If Φ is a vertex in $\Pi^n \mathbf{C}$, then it can be evaluated on n -tuples (f_1, \dots, f_n) of composable arrows

$$x_0 \xleftarrow{f_1} x_1 \xleftarrow{f_2} x_2 \xleftarrow{\dots} x_{n-1} \xleftarrow{f_n} x_n$$

to give $\Phi(f_1, \dots, f_n) \in \mathbf{C}(x_n, x_0)$. Together with the evident structure maps, this defines a cosimplicial space $\Pi^\bullet \mathbf{C}$: The coface maps $\Pi^{n-1} \mathbf{C} \rightarrow \Pi^n \mathbf{C}$ are given by

$$(d^k \Phi)(f_1, \dots, f_n) = \begin{cases} f_1 \Phi(f_2, \dots, f_n) & k = 0 \\ \Phi(f_1, \dots, f_k f_{k+1}, \dots, f_n) & 0 < k < n \\ \Phi(f_1, \dots, f_{n-1}) f_n & k = n \end{cases}$$

for $k = 0, \dots, n$, and the codegeneracy maps $\Pi^{n+1} \mathbf{C} \rightarrow \Pi^n \mathbf{C}$ are given by

$$(s^k \Phi)(f_1, \dots, f_n) = \Phi(f_1, \dots, f_k, \text{id}, f_{k+1}, \dots, f_n)$$

for $k = 0, \dots, n$.

REMARK 2.1. For $n = 0$ and $n = 1$, we obtain the source and target of (1.1), and we recover the definition of the simplicial center $Z(\mathbf{C})$ as the equalizer of the two coboundaries $\Pi^0 \mathbf{C} \rightarrow \Pi^1 \mathbf{C}$. This equalizer is the limit $\lim \Pi^\bullet \mathbf{C}$ of the cosimplicial space $\Pi^\bullet \mathbf{C}$.

We note that the cosimplicial space $\Pi^\bullet \mathbf{C}$ is canonically pointed by the families of composition maps (interpreted as identities in low dimensions).

The category of cosimplicial simplicial sets comes with a least three different Quillen model category structures with level-wise equivalences: The most useful one is the Reedy structure that is already implicit in [BK72]. The other two are the projective and injective model structure on the diagram category of functors from Δ to simplicial sets. In the projective model structure, the fibrant objects are the cosimplicial spaces, that is the cosimplicial simplicial sets that are level-wise Kan complexes. So $\Pi^\bullet \mathbf{C}$ is a cosimplicial space, as has already been implicit in the terminology used above. It is actually better:

LEMMA 2.2. *If \mathbf{C} is a category enriched in spaces, the cosimplicial space $\Pi^\bullet \mathbf{C}$ is Reedy fibrant.*

PROOF. This can be checked by hand: It has to be verified that certain maps

$$(2.1) \quad \Pi^n \mathbf{C} \longrightarrow M^{n-1} \Pi^\bullet \mathbf{C},$$

for all $n \geq 0$, are fibrations. Here, the target $M^{n-1} \Pi^\bullet \mathbf{C} \subseteq (\Pi^{n-1} \mathbf{C})^n$ is the matching space, the subspace defined by the n -tuples with consistent codegeneracies, and the map (2.1) is also given by codegeneracies. This already implies the result, because the codegeneracies are given by (projections and) evaluations on identities. The latter, in turn, are induced by insertions of identity factors, which are injective, hence cofibrations. \square

Fibrancy in the injective model structure is the strongest condition. It implies the equivalence of the homotopy limit with the actual limit, see [GJ99, VIII.2.11]. It will follow from our discussion in Section 6 that $\Pi^\bullet \mathbf{C}$ is not injectively fibrant in general.

Homotopy coherent centers. We now come to the definition that is basic to the rest of this text.

DEFINITION 2.3. Let \mathbf{C} be a category enriched in spaces. The *homotopy coherent center* $\mathcal{Z}(\mathbf{C})$ of \mathbf{C} is defined as the totalization

$$\mathcal{Z}(\mathbf{C}) = \text{Tot}(\Pi^\bullet \mathbf{C}) \simeq \text{holim } \Pi^\bullet \mathbf{C}$$

of the cosimplicial space $\Pi^\bullet \mathbf{C}$.

We note that the indicated equivalence of the totalization with the homotopy limit follows from Lemma 2.2 above: This is a general property of Reedy fibrant cosimplicial spaces, see [GJ99, VIII.2.12]. It also follows that $\mathcal{Z}(\mathbf{C})$ is a space (Kan complex) if the category \mathbf{C} is enriched in spaces (locally Kan). See also [CP97, Proposition 2.1] for a direct proof of a more general statement.

A vertex in the homotopy coherent center $\mathcal{Z}(\mathbf{C})$ is given by (the adjoint to) a family $\Phi = (\Phi^n \mid n \geq 0)$ of maps

$$\Phi^n: \Delta^n \times \mathbf{C}(x_1, x_0) \times \cdots \times \mathbf{C}(x_n, x_{n-1}) \longrightarrow \mathbf{C}(x_n, x_0)$$

indexed by all choices of n and $(n+1)$ -tuples x_0, \dots, x_n of objects that are consistent with the cosimplicial structure maps. If we choose $n = 0$, then we obtain a family of morphisms $\Phi_x^0: x \rightarrow x$, and if we choose $n = 1$, then we obtain homotopies $\Delta^1 \times \mathbf{C}(y, z) \rightarrow \mathbf{C}(y, z)$ between the maps $f \mapsto \Phi_z^0 f$ and $f \mapsto f \Phi_y^0$. In particular, these give homotopies $\Phi_z^0 f \simeq f \Phi_y^0$ that show that the homotopy classes of

the Φ_x^0 define an element in the center of the homotopy category. For $n \geq 2$, the Φ^n contain higher coherence information.

REMARK 2.4. The center of an ordinary category is the monoid of all natural transformations from the identity to itself, and we have here presented a homotopy coherent generalization of this idea that is adapted to the context of categories enriched in spaces. For later purposes, it will be useful to know that there are similar models for spaces of coherent natural transformations between simplicial functors $F, G: \mathbf{C} \rightarrow \mathbf{D}$ as well: We build a cosimplicial space $\Pi^\bullet(F, G)$ with

$$\Pi^n(F, G) = \prod_{x_0, \dots, x_n} \text{Map}(\mathbf{C}(x_1, x_0) \times \cdots \times \mathbf{C}(x_n, x_{n-1}), \mathbf{D}(Fx_n, Gx_0))$$

in dimension n . Then $\mathcal{N}(F, G) = \text{Tot } \Pi^n(F, G)$ may be considered as the space of all coherent natural transformations $F \rightarrow G$. Taking $F = \text{id}_{\mathbf{C}} = G$, we recover the homotopy coherent center as $\mathcal{Z}(\mathbf{C}) = \mathcal{N}(\text{id}_{\mathbf{C}}, \text{id}_{\mathbf{C}})$. See [CP96] and [CP97] for this and generalizations.

3. Multiplicative structure

We will now address the algebraic structure on the homotopy coherent centers that reflects the ‘composition’ of coherent natural transformations.

An A_∞ multiplication. We have seen in Corollary 1.3 that the simplicial center $\mathcal{Z}(\mathbf{C})$ of a category \mathbf{C} enriched in spaces is a simplicial abelian monoid. I know of no reason why the homotopy coherent center $\mathcal{Z}(\mathbf{C})$ should have a canonical simplicial monoid structure as well. However it does come with a homotopy commutative A_∞ structure, and that is just as good for the purposes of homotopy theory. To see this, we would like to use the criteria for operad actions on totalizations of cosimplicial objects as presented in [MS02], [MS04a], and [MS04b].

These cited papers are written topologically, so as to be able to concatenate paths, for instance. Still their results can be used here: Both geometric realization and singular complex commute with finite products, and the homotopy coherent center will turn out to be invariant under equivalences (Theorem 4.1). Therefore, given a simplicial category \mathbf{C} we can first check that the homotopy coherent center of the topological category $|\mathbf{C}|$ has an action of a topological A_∞ operad using the results from McClure and Smith. Then we infer that the homotopy coherent center of the simplicial category $\text{Sing}(|\mathbf{C}|)$ has an action of a simplicial A_∞ operad. Since $\text{Sing}(|\mathbf{C}|)$ is equivalent to \mathbf{C} , this proves our claim.

With these technicalities out of the way, we need to find strictly associative and unital pairings

$$m = m_{p,q}: \Pi^p \mathbf{C} \times \Pi^q \mathbf{C} \longrightarrow \Pi^{p+q} \mathbf{C}$$

that satisfy

$$\begin{aligned} d^k m(\Phi, \Psi) &= \begin{cases} m(d^k \Phi, \Psi) & k \leq p \\ m(\Phi, d^{k-p} \Psi) & k > p \end{cases} \\ m(d^{p+1} \Phi, \Psi) &= m(\Phi, d^0 \Psi) \\ s^k m(\Phi, \Psi) &= \begin{cases} m(s^k \Phi, \Psi) & k < p \\ m(\Phi, s^{k-p} \Psi) & k \geq p \end{cases} \end{aligned}$$

in order to deduce that $\mathcal{Z}(\mathbf{C}) = \text{Tot}(\Pi^\bullet \mathbf{C})$ is an A_∞ monoid [MS04b, Section 3]. The reader may check that

$$m_{p,q}(\Phi, \Psi)(f_1, \dots, f_p, f_{p+1}, \dots, f_{p+q}) = \Phi(f_1, \dots, f_p)\Psi(f_{p+1}, \dots, f_{p+q})$$

satisfies these conditions in our case.

Homotopy commutativity. As suggested by the observation of Eckmann and Hilton, we will now proceed to see that the homotopy coherent center not only has an $A_\infty = E_1$ structure, but in fact is automatically commutative up to homotopy in the sense that this structure can be enhanced to an E_2 structure.

THEOREM 3.1. *For every category \mathbf{C} enriched in spaces, the homotopy coherent center $\mathcal{Z}(\mathbf{C})$ comes with an E_2 -multiplication; it is a homotopy commutative A_∞ monoid.*

PROOF. Again, we use the work of McClure and Smith cited above, with similar provisos concerning topological versus simplicial categories. Accordingly, we need to turn $\Pi^\bullet \mathbf{C}$ into a (non-symmetric) operad with an associative multiplication with unit [MS04b, Section 10].

For the operad structure, it suffices to give the insertion maps

$$\circ_i: \Pi^n \mathbf{C} \times \Pi^j \mathbf{C} \longrightarrow \Pi^{n+j-1} \mathbf{C},$$

and we do so by defining

$$(\Phi \circ_i \Psi)(f_1, \dots, f_{n+j-1}) = \Phi(f_1, \dots, f_{i-1}, \Psi(f_i, \dots, f_{i+j-1}), f_{i+j}, \dots, f_{n+j-1}).$$

The additional structure of an associative multiplication with unit boils down to a pair of vertices $\mu \in \Pi^2 \mathbf{C}$ and $\epsilon \in \Pi^0 \mathbf{C}$. These have to be chosen such that the two conditions $\mu(\mu, \text{id}) = \mu(\text{id}, \mu)$ and $\mu(\epsilon, \text{id}) = \text{id} = \mu(\text{id}, \epsilon)$ are satisfied. In our case, we can define ϵ to be the family (id_x) of identity morphisms $\text{id}_x: x \rightarrow x$, and μ to be the family of composition maps $\mathbf{C}(x_1, x_0) \times \mathbf{C}(x_2, x_1) \rightarrow \mathbf{C}(x_2, x_0)$ that send (f_1, f_2) to $f_1 f_2$.

Again, it is straightforward to verify that these maps satisfy the required conditions. \square

COROLLARY 3.2. *For every category \mathbf{C} enriched in spaces, the set $\pi_0 \mathcal{Z}(\mathbf{C})$ of components of the homotopy coherent center $\mathcal{Z}(\mathbf{C})$ is a commutative monoid.*

REMARK 3.3. In the additive context of homological algebra and differential graded categories, the idea of deriving the notion of the center of a category has already appeared in an unpublished preprint by Keller [Kel]. In the same context, Tamarkin [Tam07] considers a subspace of a more general many objects Hochschild object. Batanin and Markl [BM12] obtain the classical Hochschild complex of a differential graded category as a special case of a more general construction. See [KT05] for another non-linear version of Deligne's conjecture in a different setting. In the linear context, a version for differential graded categories was proven by Tamarkin [Tam07]. Recently, Batanin and Markl [BM15] gave a proof of a more general statement which is applicable, in particular, to topologically enriched categories.

4. Functoriality and equivalences

In general, a simplicial functor $F: \mathbf{C} \rightarrow \mathbf{D}$ between categories \mathbf{C} and \mathbf{D} enriched in spaces does not induce a morphism between the centers, in neither direction. This is already clear for discrete monoids. However, centers do allow for some functoriality, and this will be discussed in the present section. The main result will be the invariance of the homotopy coherent center under weak equivalences of simplicial categories as defined by Dwyer and Kan in [DK80, 2.4]. Before we do so, let us discuss a different situation that also arises sufficiently often.

Quotients. There is always a morphism

$$\mathcal{Z}(\mathbf{C}) \rightarrow Z(\mathrm{Ho} \mathbf{C}),$$

defined by sending a coherent family $\Phi = (\Phi^n)$ to the homotopy class $[\Phi^0]$. (We use the homotopies Φ^1 to show that this is well-defined.) The target of the morphism is discrete, and this map factors to give the second map

$$\pi_0 \mathcal{Z}(\mathbf{C}) \longrightarrow Z(\mathrm{Ho} \mathbf{C})$$

displayed in the introduction.

More generally, using functorial Postnikov approximations for spaces, which are simplicial functors $P_n: \mathbf{S} \rightarrow \mathbf{S}$ that preserve products, we may also consider the Postnikov approximations $\mathbf{C} \rightarrow P_n \mathbf{C}$ of the categories enriched in spaces \mathbf{C} . The homotopy category is the special case $n = 0$, i.e. $P_0 \mathbf{C} = \mathrm{Ho} \mathbf{C}$.

We obtain maps $\mathcal{Z}(\mathbf{C}) \rightarrow \mathcal{Z}(P_n \mathbf{C})$ and more generally $\mathcal{Z}(P_m \mathbf{C}) \rightarrow \mathcal{Z}(P_n \mathbf{C})$ for all $m \geq n$. We will later see that $\mathcal{Z}(P_n \mathbf{C})$ is n -truncated, but it will not be the n -truncation of $\mathcal{Z}(\mathbf{C})$ in general. For instance, in the case $n = 0$, the purpose of the obstruction theory that will be developed in Section 7 is to explain the (potential) failure of the map $\pi_0 \mathcal{Z}(\mathbf{C}) \rightarrow Z(\mathrm{Ho} \mathbf{C})$ to be a bijection.

Equivalences. Recall from [DK80, 2.4] that a simplicial functor $F: \mathbf{C} \rightarrow \mathbf{D}$ between simplicial categories is called a *weak equivalence* if the following two conditions are satisfied: First, the induced functor $\mathrm{Ho} F: \mathrm{Ho} \mathbf{C} \rightarrow \mathrm{Ho} \mathbf{D}$ between the homotopy categories has to be an equivalence of (ordinary) categories, and second, for each pair of objects x, y of \mathbf{C} , the induced map $\mathbf{C}(x, y) \rightarrow \mathbf{D}(Fx, Fy)$ has to be a weak equivalence of spaces. Two simplicial categories are *weakly equivalent* if there exists a (finite) zig-zag of weak equivalences between them.

THEOREM 4.1. *Weakly equivalent categories enriched in spaces have weakly equivalent homotopy coherent centers.*

PROOF. Let \mathbf{C} and \mathbf{D} be weakly equivalent categories enriched in spaces. It will be sufficient to deal with the case in which we have a weak equivalence $F: \mathbf{C} \rightarrow \mathbf{D}$ between them, and to produce a zig-zag of weak equivalences between their homotopy coherent centers.

As an intermediate step, we can use the space $\mathcal{N}(F, F)$ of homotopy coherent natural transformations from the functor F to itself as explained in Remark 2.4. For any simplicial functor $F: \mathbf{C} \rightarrow \mathbf{D}$, pre- and post-composition with F induces maps

$$\mathcal{Z}(\mathbf{C}) \xrightarrow{F_*} \mathcal{N}(F, F) \xleftarrow{F^*} \mathcal{Z}(\mathbf{D})$$

that are compatible with the multiplications.

If $F: \mathbf{C} \rightarrow \mathbf{D}$ is a weak equivalence, then so are F_* and F^* , and the result follows. \square

5. Monoids and groups

To illustrate the two different notions of centers that we have defined for categories enriched in spaces so far, simplicial centers and homotopy coherent centers, let us study these in the case where the category enriched in spaces has just one object, so that it is a simplicial monoid M whose underlying simplicial set is a Kan complex. This happens, for instance, if M is a simplicial group, or if $M = \text{Map}(X, X)$ for a Kan complex X .

From Section 1, we recall that the simplicial center $Z(M)$ is a simplicial abelian monoid with n -simplices $Z(M)_n = Z(M_n)$, and its homotopy type is determined by its homotopy groups since it is a product of Eilenberg–Mac Lane spaces.

Let us now inspect the homotopy coherent center $\mathcal{Z}(M)$. The definition from Section 2 gives

$$\mathcal{Z}(M) = \text{Tot}(\Pi^\bullet M).$$

By inspection, the cosimplicial space $\Pi^\bullet M = \text{Map}(M^\bullet, M)$ is obtained by mapping the bar construction $M^\bullet = B_\bullet(M)$ into M . The bar construction $B_\bullet(M)$ is the simplicial space with $B_n(M) = M^n$, faces are given by multiplication in M , except for the first and last one, which omit the corresponding entries, and degeneracies are given by inserting the identity.

The vertices of $\mathcal{Z}(M)$ are the coherent families of maps

$$\Delta^n \longrightarrow \text{Map}(M^n, M),$$

or equivalently

$$z_n: M^n \longrightarrow \text{Map}(\Delta^n, M)$$

by adjunction. (Note the different meanings of the superscript: M^n is the n -th cartesian power, whereas in Δ^n the n indicates the dimension.) A vertex in the homotopy coherent center $\mathcal{Z}(M)$ therefore gives for $n = 0$ an element $z = z_0$ in M , and for $n = 1$ and each element $m \in M$ a path

$$z_1(m): mz \longrightarrow zm$$

in M , and for $n = 2$ it gives for each pair of elements m, n in M a 2-simplex $z_2(m, n)$ in M that gives a homotopy between the two potentially different paths $mnz \rightarrow zmn$ which are $z_1(m \cdot n)$ and $(z_1(m) \cdot n) \circ (m \cdot z_1(n))$.

$$\begin{array}{ccc} & mzn & \\ m \cdot z_1(n) \nearrow & & \searrow z_1(m) \cdot n \\ mnz & \xrightarrow{z_1(m \cdot n)} & zwn \end{array}$$

This should give an idea of the data encoded in the homotopy coherent center in the case of simplicial monoids.

We will now specialize a bit more in order to relate our definitions to another branch of contemporary group theory: homotopical group theory.

Simplicial groups. In contrast to the general monoid, the elements in a group G all have inverses, so that we have a conjugation action of G on itself. The fixed points of this action form the center $Z(G)$ of G in the traditional sense.

This suggests that the homotopy fixed point space of the conjugation action of G on itself is another natural candidate for the notion of a homotopy coherent center. It can be defined as the space

$$\mathrm{Map}_G(EG, \mathrm{Ad}(G))$$

of equivariant maps from a free resolution $EG \rightarrow \star$ of the universal G -fixed point \star to $\mathrm{Ad}(G)$. The following result shows that the homotopy coherent center in our sense agrees with this concept, when the latter is defined.

THEOREM 5.1. *For every simplicial group G , there is an equivalence*

$$\mathcal{Z}(G) \simeq \mathrm{Map}_G(EG, \mathrm{Ad}(G))$$

of its homotopy coherent center with the homotopy fixed point space of the conjugation action of G on itself.

PROOF. This follows from the induction-restriction-adjunction associated to the diagonal

$$G \longrightarrow G^{\mathrm{op}} \times G, g \longmapsto (g^{-1}, g).$$

Namely, we have

$$G \cong \mathrm{ind}(\star),$$

where G is the G -biset with the action of G by multiplication from the right and from the left, as well as

$$\mathrm{Ad}(G) \cong \mathrm{res}(G).$$

Using this, we can write

$$\begin{aligned} \mathrm{Map}_G(EG, \mathrm{Ad}(G)) &\cong \mathrm{Hom}(\mathrm{B}(\star, G, G), \mathrm{res}(G)) \\ &\cong \mathrm{Hom}(\mathrm{ind} \mathrm{B}(\star, G, G), G) \\ &\cong \mathrm{Hom}(\mathrm{B}(\mathrm{ind}(\star), G, G), G) \\ &\cong \mathrm{Hom}(\mathrm{B}(G, G, G), G). \end{aligned}$$

Since the G -space G , thought of as a functor from G to G -spaces, is full and faithful, this is isomorphic to

$$\mathrm{Hom}(\mathrm{B}(G, G, \mathrm{Id}_G), \mathrm{Id}_G) = \mathcal{Z}(G),$$

as was to be shown. □

We note that the equivalence in the theorem allows us to model the naturally given map $Z(G) \rightarrow \mathcal{Z}(G)$ as the inclusion of the fixed points into the homotopy fixed points.

The following (folklore) result gives another presentation of the homotopy fixed point space of the conjugation action of G on itself, and hence also of the homotopy coherent center of G .

PROPOSITION 5.2. *For every simplicial group G , there is an equivalence*

$$\mathrm{Map}_G(EG, \mathrm{Ad}(G)) \simeq \Omega(\mathrm{Map}(\mathrm{BG}, \mathrm{BG}), \mathrm{id})$$

with the space of loops in the space of self-maps of BG which are based at the identity.

PROOF. The space $\text{Map}_G(EG, \text{Ad}(G))$ can be identified with the space of sections of the Borel construction bundle

$$EG \times_G \text{Ad}(G) \longrightarrow BG,$$

and the space $\Omega(\text{Map}(BG, BG), \text{id})$ can be identified with the space of sections of the free loop space evaluation bundle

$$\Lambda BG \longrightarrow BG.$$

It remains to note that there is an equivalence

$$EG \times_G \text{Ad}(G) \simeq \Lambda BG$$

of spaces over BG . See the original sources [Goo85], [BF86], or [BHM93]. \square

COROLLARY 5.3. *For every simplicial group G , there is an equivalence*

$$\mathcal{Z}(G) \simeq \Omega(\text{Map}(BG, BG), \text{id})$$

of its homotopy coherent center with the space of loops in the space of self-maps of BG which are based at the identity.

The right hand side of this equivalence is used as a definition for the homotopy center of p -compact groups by Dwyer and Wilkerson [DW95].

A still more general formula for the homotopy coherent center of a simplicial groupoid will be given later, see Proposition 8.2.

6. The homotopy limit problem

It is clear from our description of the simplicial center and the homotopy coherent center as a limit and as a homotopy limit, respectively, that there is a natural map

$$(6.1) \quad \mathcal{Z}(\mathbf{C}) \longrightarrow \mathcal{Z}(\mathbf{C}).$$

The question whether this map is an equivalence, or how far off it is, can be compared with the difficult homotopy limit problem, see Thomason [Tho83] and Carlsson [Car87].

The two notions of center agree for discrete categories.

PROPOSITION 6.1. *For every discrete category \mathbf{C} , the arrow (6.1) is an equivalence; the domain is discrete.*

Of course, this result applies, in particular, to discrete monoids, discrete groups, and discrete groupoids.

We are now in a position to use the different descriptions of the homotopy coherent center of a simplicial group G given in Section 5 in order to discuss the homotopy limit map $\mathcal{Z}(G) \rightarrow \mathcal{Z}(G)$ in various other special cases. Here is another case that can be dealt with.

PROPOSITION 6.2. *Let A be a simplicial abelian group. If A is an Eilenberg–Mac Lane space or if A has the homotopy type of an abelian compact Lie group, then the arrow $\mathcal{Z}(A) \rightarrow \mathcal{Z}(A)$ is an equivalence.*

PROOF. We have to show that the natural map $A = \text{Map}(\star, A) \rightarrow \text{Map}(\text{BA}, A)$ is an equivalence. This follows in the first case from Thom’s determination [Tho56] of the homotopy type of the mapping space into an Eilenberg–Mac Lane space. The second case follows from the same result and the fact that A is a product of a finite group with a torus. \square

We will revisit the first case of the preceding result in Example 8.6 from a different perspective.

The consequence of Proposition 6.2 does not hold for all simplicial abelian groups in general, even though they are known to be products of Eilenberg–Mac Lane spaces by Moore’s theorem [Moo55].

EXAMPLE 6.3. Let us consider the (abelian) group A of all complex line bundles over the circle. Up to homotopy (or isomorphism), there is only one. This immediately gives

$$\pi_0 Z(A) = \pi_0 A = 0.$$

But as a space (and as a group), this can be modeled as

$$A = \text{Map}(S^1, \mathbb{C}P^\infty) \simeq S^1 \times \mathbb{C}P^\infty.$$

Therefore

$$\pi_0 \mathcal{Z}(A) \cong \pi_1(\text{Map}(\text{BA}, \text{BA}), \text{id}) \cong \mathbb{Z}.$$

This shows that the behavior of the map $Z(A) \rightarrow \mathcal{Z}(A)$ can be rather unpredictable on components in general.

This example shows that the homotopy limit map (6.1) need not be an equivalence, not even after passage to classifying spaces and on homology with finite coefficients. This may be compared to the following result.

THEOREM 6.4. ([DW95, Theorem 1.4]) *Let G be connected compact Lie group. Then the natural map*

$$\text{BZ}(G) \longrightarrow \text{B}\mathcal{Z}(G)$$

induces an isomorphism on homology with finite coefficients.

Earlier results in this direction had been obtained in [JMO92, Theorem 3] in the case when G is simple, and in [DM87] when $G = \text{SU}(2)$.

7. Spectral sequences and obstructions

In this section we answer the question about the kernel and image of the map

$$\pi_0 \mathcal{Z}(\mathbf{C}) \longrightarrow Z(\text{Ho } \mathbf{C}),$$

by means of spectral sequences and obstruction theory, respectively.

A ubiquitous problem when discussing realization questions is the potential emptiness of realization spaces. This is irrelevant here, since the (homotopy coherent) center is always nonempty. Moreover, its set of components has the structure of an abelian monoid. This leads to a simplification of another difficulty that we often encounter when computing homotopy groups of spaces by means of such spectral sequences: the lack of (abelian) group structures on π_0 and π_1 .

Spectral sequences. If \mathbf{C} is a category enriched in spaces, then its homotopy coherent center $\mathcal{Z}(\mathbf{C})$ has been defined as the totalization of the cosimplicial space $\Pi^\bullet \mathbf{C}$. This is pointed and fibrant, see Lemma 2.2. Bousfield and Kan, see [BK72] and [Bou89] as well as the textbook [GJ99, VIII.1], have shown that there is always a spectral sequence for the totalization of such a cosimplicial space.

THEOREM 7.1. *For every category \mathbf{C} enriched in spaces, there exists a fringed spectral sequence with target $\pi_{t-s}\mathcal{Z}(\mathbf{C})$ and*

$$E_2^{s,t} = \pi^s \pi_t \Pi^\bullet \mathbf{C}$$

for $t \geq s \geq 0$. In particular,

$$E_2^{0,0} \leq Z(\mathrm{Ho} \mathbf{C})$$

is a submonoid of the center of its homotopy category.

PROOF. Only the last statement requires proof. The E_1 term is given as

$$E_1^{s,t} = N^s \pi_t \Pi^\bullet \mathbf{C}.$$

In particular, the 0-line is

$$E_1^{0,t} = \pi_t \Pi^0 \mathbf{C}$$

with differential given by the difference between the two coface operators d_0 and d_1 . This gives

$$E_2^{0,t} = \pi^0 \pi_t \Pi^\bullet \mathbf{C}.$$

For $t = 0$, we have

$$E_1^{0,0} = \prod_x \pi_0 \mathbf{C}(x, x),$$

and this consists of the families Φ of homotopy classes $\Phi_x: x \rightarrow x$. These lie in the equalizer $\pi^0 \pi_0 \Pi^\bullet \mathbf{C}$ if $f\Phi_x$ and $\Phi_y f$ are homotopic for all $f: x \rightarrow y$ in \mathbf{C} , and these homotopies can be chosen continuously in f . In particular, the family Φ is in the center $Z(\mathrm{Ho} \mathbf{C})$ of the homotopy category $\mathrm{Ho} \mathbf{C}$, where continuity in f is not required. \square

REMARK 7.2. As the proof has shown, the difference between $E_2^{0,0}$ and the center of the homotopy category is owed by the fact that the maps

$$\pi_0 \mathrm{Map}(\mathbf{C}(y, z), \mathbf{C}(y, z)) \longrightarrow \mathrm{Map}(\pi_0 \mathbf{C}(y, z), \pi_0 \mathbf{C}(y, z))$$

need not be injective.

Compared with the general spectral sequence of Bousfield–Kan type, the spectral sequences in Theorem 7.1 have a relatively well-behaved left lower corner. The monoidal structure on $\Pi^\bullet \mathbf{C}$ leads to abelian group structures from the E_1 term on, with the possible exception of $E_r^{0,0}$. In that spot, our spectral sequence starts with

$$E_1^{0,0} = \prod_x \pi_0 \mathbf{C}(x, x),$$

which—in the most general case that we consider—is just a monoid. But, the structure $E_2^{0,0} \cong Z(\mathrm{Ho} \mathbf{C})$ on the next term is already an abelian monoid. This is as good as we may hope for in the context of centers.

The manner of convergence of Bousfield–Kan type spectral sequences is often a delicate issue, and we do not have to add anything to the original results here. See again [BK72], [Bou89, §4], and [GJ99, VI.2]. We would only like to point out the following result for truncated situations.

PROPOSITION 7.3. *If \mathbf{C} is a category enriched in spaces such that all of its mapping spaces $\mathbf{C}(x, y)$ are n -truncated for an n that is independent of the objects x and y , then the spectral sequence of Theorem 7.1 converges completely to its target.*

This proposition applies, in particular, to categories enriched in spaces that arise from categories enriched in groupoids ($n = 1$).

Before we turn towards the obstruction theory that always comes with a fringed Bousfield–Kan type spectral sequence, let us briefly collect the different types of truncations for homotopy coherent centers that we have considered so far. For all $s \geq 0$ and $n = t - s \geq 0$ there are maps

$$\begin{array}{ccccc} \mathcal{Z}(\mathbf{C}) & \longrightarrow & P_n \mathcal{Z}(\mathbf{C}) & \longrightarrow & \mathcal{Z}(P_n \mathbf{C}) \\ \parallel & & & & \\ \mathrm{Tot}(\Pi^\bullet \mathbf{C}) & \longrightarrow & & \longrightarrow & \mathrm{Tot}_s(\Pi^\bullet \mathbf{C}) \end{array}$$

and from the position of the corresponding regions in our spectral sequence it is clear that we cannot expect these to be equivalences in general.

Obstructions. The fringed spectral sequence of Theorem 7.1 comes with an obstruction theory that allows the investigation of the edge homomorphism

$$\pi_0 \mathcal{Z}(\mathbf{C}) \longrightarrow Z(\mathrm{Ho} \mathbf{C}),$$

in particular of its image. Note that the range of that map contains the $E_2^{0,0}$ term of the spectral sequence, whereas the domain belongs to the target of that spectral sequence. Therefore, we can start our discussion with an element of the abelian monoid $\pi_0 \mathrm{Tot}_1(\Pi^\bullet \mathbf{C}) \leq Z(\mathrm{Ho} \mathbf{C})$ and ask whether or not we can lift it to $\pi_0 \mathrm{Tot}_s(\Pi^\bullet \mathbf{C})$ for a given s . We will be able to do this for every s if and only if the given element lifts to $\pi_0 \mathrm{Tot}(\Pi^\bullet \mathbf{C}) = \pi_0 \mathcal{Z}(\mathbf{C})$.

Since this is by now standard, we will only sketch how the general obstruction theory applies to our specific context and refer to [Bou89, §5 and §10] and [GJ99, VIII.4] for details.

First of all the representatives of the classes in $\pi_0 \mathrm{Tot}_s(\Pi^\bullet \mathbf{C})$ can be fairly explicitly described. They are given by $(s + 1)$ -tuples (Φ_0, \dots, Φ_s) of simplices

$$\Phi_p: \Delta^p \longrightarrow \Pi^p \mathbf{C}$$

that are compatible with the coface and codegeneracy maps whenever these are defined. The maps $(\Phi_0, \dots, \Phi_s) \mapsto (\Phi_0, \dots, \Phi_{s-1})$ and $(\Phi_0, \dots, \Phi_s) \mapsto \Phi_s$ induce the upper left of the following two pullback squares.

$$\begin{array}{ccccc} \mathrm{Tot}_s(\Pi^\bullet \mathbf{C}) & \longrightarrow & \mathrm{Map}(\Delta^s, \Pi^s \mathbf{C}) & & \\ \downarrow & & \downarrow f & & \\ \mathrm{Tot}_{s-1}(\Pi^\bullet \mathbf{C}) & \longrightarrow & P & \longrightarrow & \mathrm{Map}(\Delta^s, M^{s-1} \Pi^\bullet \mathbf{C}) \\ & & \downarrow & & \downarrow \\ & & \mathrm{Map}(\partial \Delta^s, \Pi^s \mathbf{C}) & \longrightarrow & \mathrm{Map}(\partial \Delta^s, M^{s-1} \Pi^\bullet \mathbf{C}) \end{array}$$

The map f is the fibration determined by the fibrant cosimplicial space $\Pi^\bullet \mathbf{C}$. Thus, in order to produce a lift of an element in $\pi_0 \mathrm{Tot}_s(\Pi^\bullet \mathbf{C})$, we have to find a lift of

its image in the pullback

$$P = \text{Map}(\partial\Delta^s, \Pi^s \mathbf{C}) \times_{\text{Map}(\partial\Delta^s, M^{s-1}\Pi^s \mathbf{C})} \text{Map}(\Delta^s, M^{s-1}\Pi^s \mathbf{C})$$

as illustrated in the following diagram.

$$\begin{array}{ccc} \partial\Delta^s & \longrightarrow & \Pi^s \mathbf{C} \\ \downarrow & \nearrow \text{dashed} & \downarrow \\ \Delta^s & \longrightarrow & M^{s-1}\Pi^s \mathbf{C} \end{array}$$

The element in $\pi_{s-1}(\Pi^s \mathbf{C})$ represented by $\partial\Delta^s \rightarrow \Pi^s \mathbf{C}$ is an obstruction to the existence of such a lift. In fact, this obstruction lives on the E_1 term, more precisely in $E_1^{s,s-1}$. It turns out that we can pass to the E_2 term to obtain a well-defined obstruction in $E_2^{s,s-1}$ to the liftability of the restriction in $\pi_0 \text{Tot}_{s-1}(\Pi^s \mathbf{C})$ of our element in $\pi_0 \text{Tot}_s(\Pi^s \mathbf{C})$ to $\pi_0 \text{Tot}_{s+1}(\Pi^s \mathbf{C})$.

REMARK 7.4. A little extra care has to be taken when there are fundamental groups involved. It is common to assume that Whitehead products vanish in the spaces involved in $\Pi^s \mathbf{C}$, or at least that the fundamental groupoids act trivially on all fundamental groups. Both hypotheses are satisfied in the linear case, when the mapping spaces are simplicial abelian groups, or in the stable case, when the mapping spaces are infinite loop spaces.

The following result summarizes the situation.

THEOREM 7.5. *Let \mathbf{C} be a category enriched in spaces that satisfies a hypothesis on the fundamental groups as in Remark 7.4 above. Then any element in the center*

$$Z(\text{Ho } \mathbf{C}) = \text{Tot}_1(\Pi^s \mathbf{C})$$

of the homotopy category can be lifted to an element in the homotopy coherent center

$$\mathcal{Z}(\mathbf{C}) = \text{Tot}(\Pi^s \mathbf{C})$$

if and only if it lies in $E_2^{0,0} \leq Z(\text{Ho } \mathbf{C})$ and the obstruction classes in

$$E_2^{s,s-1} = \pi^s \pi_{s-1} \Pi^s \mathbf{C}$$

vanish for all $s \geq 2$.

We note that the obstruction classes certainly vanish if the obstruction groups are all trivial. This happens, of course, in the case when \mathbf{C} is homotopically discrete.

8. Groupoids and spaces

In this last section, we will discuss another class of examples of categories enriched in spaces: simplicial groupoids. By definition, a simplicial groupoid \mathbf{G} is a simplicial category such that the categories \mathbf{G}_n of n -simplices are groupoids. For instance, simplicial groups, as discussed already in Section 5, are simplicial groupoids. Here, we are interested mainly in the case where there are many objects. Let us start by reviewing the discrete case first.

Fundamental groupoids. If X is a space, its fundamental groupoid $\Pi_1 X$ is a (discrete) groupoid, and up to equivalence, every discrete groupoid has this form: Take X to be its classifying space.

PROPOSITION 8.1. *For every space X , the center of the fundamental groupoid splits as a product*

$$Z(\Pi_1 X) \cong \prod_{[x] \in \pi_0 X} Z(\pi_1(X, x))$$

of centers of its fundamental groups over a set of representatives of its path components.

PROOF. This is certainly true for path connected spaces X , since in that case the fundamental groupoid is equivalent to the fundamental group of any of its points. The general case follows from the compatibility of centers with disjoint unions. \square

After this review of the discrete case, let us now move on to simplicial groupoids.

Path groupoids. While the fundamental groupoid $\Pi_1 X$ of a space X is useful for many purposes, the passage to homotopy classes that attends it is a rather drastic simplification of the situation. It is preferable to work with the path groupoid $\mathbf{G}X$ of X that has been introduced by Dwyer and Kan in [DK84]. It is a simplicial groupoid that can be thought of as a simplicially enriched refinement of the fundamental groupoid of the space X . In fact, in the cited paper it is shown that the homotopy theory of simplicial groupoids is equivalent to the homotopy theory of spaces. More precisely, inverse equivalences are given by the pair of adjoint functors $\mathbf{G} \mapsto \mathbf{B}\mathbf{G}$, the classifying space, and $X \mapsto \mathbf{G}X$. Consequently, we may expect the centers of simplicial groupoids to be related to the classical homotopy theory of spaces, and as we will see in the examples below, this turns out to be true.

The homotopy category of $\mathbf{G}X$ is the fundamental groupoid $\Pi_1 X$ of X , so that the formula

$$\Pi_1 X = \mathbf{Ho}(\mathbf{G}X),$$

relates the two incarnations of the fundamental groupoid idea.

The center of the fundamental groupoid of X has been described in Proposition 8.1, and we can now ask for the homotopy coherent center $\mathcal{Z}(\mathbf{G}X)$ of the Dwyer–Kan path groupoid of X , and the relation between the two.

We have seen that the homotopy coherent center in general has an E_2 structure by Theorem 3.1. *A fortiori*, it is an $E_1 = A_\infty$ monoid. For simplicial groupoids, it will turn out to be group-like by Corollary 8.3 below. Therefore, by Stasheff’s recognition theorem [Sta63] (as improved by May [May74]), it must be equivalent to the loop space of some (classifying) space. In fact, using the E_2 structure, that space will have its own delooping as well. The reader may wonder what these spaces are, and the following results will answer this question.

PROPOSITION 8.2. *For every simplicial groupoid \mathbf{G} , we have*

$$\mathcal{Z}(\mathbf{G}) \simeq \Omega(\mathbf{Map}(\mathbf{B}\mathbf{G}, \mathbf{B}\mathbf{G}), \text{id}).$$

PROOF. Similarly to the preceding proposition, this follows from Corollary 5.3 and the compatibility of centers with disjoint unions. \square

The center of any groupoid is always an abelian group rather than just an abelian monoid. This is far from obvious in the homotopy coherent setting, but the following result affirms that it still holds true.

COROLLARY 8.3. *For every simplicial groupoid \mathbf{G} , the abelian monoid $\pi_0\mathcal{Z}(\mathbf{G})$ is an abelian group.*

PROOF. Proposition 8.2 allows us to identify $\pi_0\mathcal{Z}(\mathbf{G})$ with a fundamental group of a space. \square

If X is a Kan complex, then it is known that the mapping space $\text{Map}(X, X)$ already models the derived mapping space, and it is also a Kan complex. Since there is an equivalence $X \simeq \mathbf{B}\mathbf{G}X$, the proposition above has also the following corollary.

COROLLARY 8.4. *For every Kan complex X , we have*

$$\mathcal{Z}(\mathbf{G}X) \simeq \Omega(\text{Map}(X, X), \text{id}).$$

The description of spaces such as $\Omega(\text{Map}(X, X), \text{id})$ is a classical subject of (unstable) homotopy theory. We will now discuss more specific classes of examples in order to demonstrate the complexity of the matter.

Specific classes of spaces. Given a space X , we have the canonical homomorphism

$$(8.1) \quad \pi_0\mathcal{Z}(\mathbf{G}X) \longrightarrow \mathcal{Z}(\Pi_1 X).$$

This is the edge homomorphism of our obstruction theory spectral sequence in Section 7, and we can discuss the problem whether or not this map (8.1) is injective or surjective here. We will see that neither has to be the case.

EXAMPLE 8.5. As the simplest case, let us consider a space that comes to us as a classifying space $X = \mathbf{B}\Gamma = \mathbf{K}(\Gamma, 1)$ for a discrete group Γ . In this case, we should not expect any higher homotopy structure, and indeed, the space $\Omega(\text{Map}(X, X), \text{id})$ is homotopically discrete, with

$$\mathcal{Z}(\mathbf{G}X) \simeq \pi_0\mathcal{Z}(\mathbf{G}X) \cong \mathcal{Z}(\Gamma),$$

the center of the fundamental group.

In order to encounter higher homotopy structure, we may generalize this example in at least two ways: Replace $X = \mathbf{B}\Gamma = \mathbf{K}(\Gamma, 1)$ with an Eilenberg–Mac Lane space $\mathbf{K}(A, n)$ for any abelian group A and any $n \geq 2$. Or, replace the aspherical circle $\mathbf{B}\mathbb{Z} = \mathbf{K}(\mathbb{Z}, 1) = \mathbb{S}^1$ by a sphere \mathbb{S}^n of dimension $n \geq 2$. These will be our next two classes of examples.

EXAMPLE 8.6. Let $X = \mathbf{K}(A, n)$ be an Eilenberg–Mac Lane space of type (A, n) for an abelian group A and an integer $n \geq 2$. Then the homotopy type of the space of self-maps is

$$\text{Map}(\mathbf{K}(A, n), \mathbf{K}(A, n)) \simeq \text{Hom}(A, A) \times \mathbf{K}(A, n),$$

see [Tho56] for the original argument and [May67, 25.2] for the simplicial version. Since the space $\Omega(\text{Map}(X, X), \text{id})$ of loops based at the identity depends only on the component of the identity, we therefore obtain an equivalence

$$\mathcal{Z}(\mathbf{G}\mathbf{K}(A, n)) \simeq \Omega\mathbf{K}(A, n) \simeq \mathbf{K}(A, n-1).$$

This means that the abelian group $\pi_0\mathcal{Z}(\mathbf{GK}(A, n))$ is trivial. (In particular, it is isomorphic to $\mathcal{Z}(\Pi_1\mathbf{K}(A, n))$ which is also trivial.) But, we see that the center can have arbitrary higher homotopy groups. This information will already be lost when we pass from the center $\mathcal{Z}(\mathbf{GX})$ to its group $\pi_0\mathcal{Z}(\mathbf{GX})$ of components, let alone $\mathcal{Z}(\Pi_1 X)$.

EXAMPLE 8.7. Let $X = S^n$ be an n -dimensional sphere, $n \geq 2$. The homotopy groups of the homotopy coherent center $\mathcal{Z}(\mathbf{GS}^n)$ can be computed from the fibration sequence

$$\Omega^n S^n \longrightarrow \text{Map}(S^n, S^n) \longrightarrow S^n.$$

Let us first consider the stable case $n \geq 3$. In that case, in order to compute π_0 , we note that the boundary operator $0 = \pi_2 S^n \rightarrow \pi_{n+1} S^n$ has to be zero, so that $\pi_0\mathcal{Z}(\mathbf{GS}^n)$ is the stable 1-stem $\mathbb{Z}/2$. The computation of the higher homotopy groups involves the boundary operator

$$\pi_k S^n \longrightarrow \pi_{k+(n-1)} S^n.$$

This is the Whitehead product with the identity ι_n of S^n . In particular, the first potentially nonzero contribution is given by the Whitehead square of the identity, and that element figures prominently in the Hopf invariant one problem: the Whitehead square $[\iota_n, \iota_n]$ is zero if and only if there is an element of Hopf invariant one in $\pi_{2n+1}(S^{n+1})$, and this is very rarely the case (by [Ada60]: only if $n = 3, 7$ in our range). And, the divisibility of the class $[\iota_n, \iota_n]$ is related to the (strong) Kervaire invariant one problem, see [BJM83]. In the meta-stable case $n = 2$, in order to compute $\pi_0\mathcal{Z}(\mathbf{GS}^2)$, the boundary operator $\mathbb{Z} \cong \pi_2(S^2) \rightarrow \pi_3(S^2) \cong \mathbb{Z}$ hits $[\iota_2, \iota_2]$ which is divisible by 2 but not by 4, so that $\pi_0\mathcal{Z}(\mathbf{GS}^2) \cong \mathbb{Z}/2$ also in this case. Indeed, for S^2 , the entire homotopy type of the identity component of the mapping space is known, see [Han83] and [Han90]. The result is

$$\text{Map}(S^2, S^2)_{\text{id}} \simeq \text{SO}(3) \times \tilde{\Omega}_0^2 S^2,$$

where $\tilde{\Omega}_0^2 S^2$ is the universal cover of a component of the double loop space of the 2-sphere; it does not matter which component. The existence of such a homotopy equivalence also implies that the abelian group $\pi_0\mathcal{Z}(\mathbf{GS}^2)$ has order 2, of course.

Example 8.7 shows that the map (8.1) need not be injective, and surjectivity of (8.1) can also fail in this context. In fact, the image has been studied from a different point of view in [Got65], and is commonly called the *Gottlieb subgroup* of the fundamental group. It can be as complicated as algebraically possible: If Γ is a discrete group, and G is a subgroup of its center, then there is a connected space X and an isomorphism $\Gamma \cong \pi_1(X)$ such that this isomorphism sends G isomorphically to the Gottlieb subgroup of X , see [Var74].

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