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The 4 Subspace Problem

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Problem Description

Let V be a finite dimensional k-space for a field k, and let V_1, V_2 and V_3 be subspaces of V. The system $(V; V_1, V_2, V_3)$ is said to be decomposable if there exists non-trivial subspaces U and W of V such that $V = U \amalg W$, and $V_i = (V_i \cap U) \amalg (V_i \cap W)$ for $i = 1, 2, 3$. It is not difficult to show that if the system $(V; V_1, V_2, V_3)$ is indecomposable, then $\dim_k V \leq 2$ and that there are essentially only 9 such systems which are indecomposable.

The 4 subspace problem is the well known problem of classifying all indecomposable systems when one increases the number of subspaces from 3 to 4 in the description above.

The aim of this project is to give a complete solution of the 4 subspace problem.

Abstract

We present a complete solution to the 4 subspace problem in the generality of an algebraically closed field. We do this by means of Auslander-Reiten theory and give the Auslander-Reiten quiver of the extended D_4 Dynkin diagram. We also give a geometric interpretation when two configurations of four lines through the origin in the plane are equivalent.

To My Dad.

Contents

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Finally, I will like to thank Linn for her uttermost patience and caring support.

Introduction

The objective of this thesis is to solve the four subspace problem for an algebraically closed field. We give a solution to this by use of Auslander-Reiten theory.

Given a field k and a finite dimensional vector space V_0 over k, and let V_1, V_2, V_3 and V_4 be subspaces of V_0. We say that this configuration is decomposable if there exists non-trivial $U_0, W_0 \subset V_0$ such that $U_0 \amalg W_0 = V_0$ and $V_i = (U_0 \cap V_i) \amalg (W_0 \cap V_i)$ for $1 \leq i \leq 4$. If there are no such decomposition, then the configuration is said to be indecomposable. The classification of all the indecomposable configurations is the four subspace problem.

In [MZ] the four subspace problem is solved by the use of representations of partially ordered sets, however it was already solved in the 1970's by joint work of Gelfand and Ponomarev for an algebraically closed field. Later Nazarova gave a solution for an arbitrary field.

Chapter 1 deals with basic representation theory of quivers and the Coxeter functors.

In chapter 2 we go through the Auslander-Reiten theory. Introducing important notions and language

Chapter 3 deals with what can be said as the focal point of this thesis, namely the four subspace problem. Section 3.1 gives a brief introduction to representations of partially ordered sets. Section 3.2 deals with the four subspace problem in the language of Auslander-Reiten theory. We give a complete classification of the indecomposable representations of the extended D_4 Dynkin diagram.

The examples that are given will be framed in the following way

Example

clearly showing the start and the end of the example. At the end of proofs there will appear a □ to indicate that the proof is finished. Statements without proof may also append this symbol, indicating the end of the statement.

Chapter 1

Representations and Coxeter Functors

The aim of this chapter is to give an introduction to the representations theory of path algebras. While the first section goes through the rudimentary definitions and notation in connection with path algebras, the last section focuses on the Coxeter functors and their properties given in [BGP].

1.1 Preliminaries

Let $Q = (Q_0, Q_1)$ be a oriented multi graph, with Q_0 the set of vertices and Q_1 the set of edges. We call Q a **quiver**, furthermore, if Q_0 and Q_1 are both finite, then we say that Q is a finite quiver. We call an edge in Q_1 an arrow. Let $s, e\colon Q_1 \longrightarrow Q_0$ be functions defined by $s(\alpha) = i$ if $\alpha \in Q_1$ is an arrow that starts in vertex $i \in Q_0$, and $e(\alpha) = j$ if $\alpha \in Q_1$ is an arrow that ends in vertex j. A **path** in Q is a composition of arrows that make sense, i.e. p is an path if $p = \alpha_r \cdots \alpha_2\alpha_1$, with $s(\alpha_{i+1}) = e(\alpha_i)$ for $1 \leq i < r$, moreover let $s(p) = s(\alpha_1)$ and $e(p) = e(\alpha_r)$. Also, for each $i \in Q_0$ define e_i as the **trivial path** from vertex i to i, and $s(e_i) = e(e_i) = i$. A nontrivial path p is said to be an **oriented cycle** if $s(p) = e(p)$, and an arrow α is called a **loop** if $s(\alpha) = e(\alpha)$. A quiver is called **acyclic** if it contains no oriented cycles. Now given a path p, we denote by $l(p)$ the length of the path p, defined the following way, if $p = \alpha_r \cdots \alpha_2\alpha_1$, with $\alpha_i \neq e_j$ for i, j, then $l(p) = r$, and let $l(e_i) = 0$ for $i \in Q_0$.

Let k be a field. Given a finite quiver Q, we denote by kQ the vector space of all k-linear combinations of paths in Q, i.e. taking the paths of Q as basis. Furthermore, we may make this into a k-algebra by defining multiplication of two paths α and β as follows

$$\beta\alpha = \begin{cases} \beta\alpha & \text{, if } \alpha \text{ and } \beta \text{ are non-trivial and } e(\alpha) = s(\beta) \\ \beta & \text{, if } \alpha = e_{s(\beta)} \\ \alpha & \text{, if } \beta = e_{e(\alpha)} \\ 0 & \text{, otherwise} \end{cases}$$

extending this bi-linearly to the whole kQ gives the desired multiplication. We call the given algebra the **path algebra** of Q over k. Note that kQ is finite dimensional if, and only if, Q contains no oriented cycles. Also notice that $1_\Lambda = e_1 + \cdots + e_n$ where $Q_0 = \{1, ldots, n\}$.

An **admissible relation** ρ on the quiver Q is a k-linear combination of paths $\rho = a_1p_1 + \cdots + a_np_n$ with $a_i \in k$ and $s(p_1) = \cdots = s(p_n)$ and $e(p_1) = \cdots = e(p_n)$ and with $l(p_i) \geq 2$ for $i = 1, \ldots, n$. If we include paths of length 1 as a relation, then we might as well have removed the arrow corresponding to that relation from our quiver. If ρ is a set of relations on Q over k then the pair (Q, ρ) is the **quiver with relations**, and the associated path algebra is then $k(Q, \rho) = kQ/\langle\rho\rangle$, and $\langle\rho\rangle$ is the ideal in kQ generated by the set of relations ρ. If we denote by $J \subset kQ$ the ideal generated by the arrows, i.e. the paths of length 1, we have that $\langle\rho\rangle \subseteq J^2$.

A **representation** of a quiver $Q = (Q_0, Q_1)$ over a field k is a pair (V, f) such that V is a set of vector spaces $V = \{V_i\}$ for $i \in Q_0$ and f is a collection of k-linear transformations f_α for $\alpha \in Q_1$ such that $f_\alpha\colon V_i \longrightarrow V_j$ for $i = s(\alpha)$ and $j = e(\alpha)$. A morphism of two representations $h\colon (V, f) \longrightarrow (V', f')$ is a Q_0-tuple of linear maps $h_i\colon V_i \longrightarrow V'_i$, $i \in Q_0$, which makes the following diagram commute

$$\begin{array}{ccc} V_{s(\alpha)} & \xrightarrow{f_\alpha} & V_{e(\alpha)} \\ \downarrow{\scriptstyle h_i} & & \downarrow{\scriptstyle h_j} \\ V'_{s(\alpha)} & \xrightarrow{f'_\alpha} & V'_{e(\alpha)} \end{array}$$

for each $\alpha \in Q_1$. This will give a structure of a category on these representations. We will denote by Rep Q **the category of** (finite dimensional) **representations** of Q over k. If we have a quiver with relations (Q, ρ), then a representation of (Q, ρ) is an representation of Q with the extra condition that for every relation $\sigma \in \langle\rho\rangle$ we have that $f_\sigma = 0$, where f_σ is the k-linear combination of compositions of f_α corresponding to the α's in σ. That is, if $\sigma = a_1p_1 + \cdots + a_np_n$, then $f_\sigma = a_1f_{p_1} + \cdots + a_nf_{p_n}$ where $f_{p_i} = f_{\alpha_{r_i}} \cdots f_{\alpha_{i_1}}$ where $p_i = \alpha_{r_i} \cdots \alpha_{i_1}$, for $1 \leq i \leq n$.

Example 1

Let Q be the quiver $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xrightarrow{\delta} 4$, $\rho = \{\delta\beta\alpha\}$ and let k be a field. Let

$U = (U, f)$, $V = (V, g)$ and $W = (W, h)$ be the following representations

$$\begin{array}{ccc} 0 & k & k \\ \downarrow & \downarrow{\scriptstyle 0} & \downarrow \\ k & k & 0 \\ \downarrow{\scriptstyle 1} & \downarrow{\scriptstyle 1} & \downarrow \\ k & k & 0 \\ \downarrow{\scriptstyle 1} & \downarrow{\scriptstyle 1} & \downarrow \\ k & k & 0 \end{array}$$

respectively, where 1 is the identity on k, i.e. $1 = 1_k$, and the maps that are omitted are the zero map, because that is the only map there is between these spaces. Now let $T\colon U \longrightarrow V$ be given by $T_1 = 0$ and $T_i = 1$, $i = 2, 3, 4$, and let $S\colon V \longrightarrow W$ be given by $S_1 = 1$ and $S_2 = S_3 = S_4 = 0$. The reader is left to check that S and T are in fact morphisms in Rep Q.

We will say that a morphism is a **monomorphism/epimorphism** in Rep Q if each $h_i\colon V_i \longrightarrow V_i'$ is a monomorphism/epimorphism. An **isomorphism** in Rep Q is a morphism that is a monomorphism and epimorphism. Furthermore, we say that $U = (U, g)$ is a **sub representation** of $V = (V, f)$ in Rep Q, and write $U \subseteq V$, if for all $i \in Q_0$ we have $U_i \subseteq V_i$ as vector spaces, and for all $\alpha \in Q_1$, and we have that $g_\alpha = f_\alpha|_{U_{s(\alpha)}}$. Given a sub representation (U, g) of (V, f) we can construct the **factor representation** of (V, f) by taking the factor vector space at each vertex and taking the maps induced by f, that is, (W, h) is the factor representation with $W_i = V_i/U_i$ for $i \in Q_0$ and h_α is the linear map such that the following diagram commutes for all $\alpha \in Q_1$

$$\begin{array}{ccccc} U_{s(\alpha)} & \longrightarrow & V_{s(\alpha)} & \xrightarrow{p_{s(\alpha)}} & W_{s(\alpha)} \\ {\scriptstyle g_\alpha}\downarrow & & {\scriptstyle f_\alpha}\downarrow & & \downarrow{\scriptstyle h_\alpha} \\ U_{e(\alpha)} & \longrightarrow & V_{e(\alpha)} & \xrightarrow{p_{e(\alpha)}} & W_{e(\alpha)} \end{array}$$

where $p_i\colon V_i \longrightarrow W_i$ is the canonical projection. The sum of two representations (U, g) and (U', g') is the representation (V, f) where $V_i = U_i \coprod U_i'$ and $f_\alpha = \begin{pmatrix} g & 0 \\ 0 & g' \end{pmatrix}$, and we write $(V, f) = (U, g) \coprod (U', g')$. We say that a representation is **indecomposable** if $(V, f) = (U, g) \coprod (U', g')$ implies that $(U, g) = 0$ or $(U', g') = 0$, where $(U, g) = 0$ means that $U(i) = 0$ for all $i \in Q_0$. A **simple** representation is a representation different from the zero representation with no proper sub representations. Obviously, the simple representations are indecomposable. Given a quiver $Q = (Q_0, Q_1)$ and k a field, then for each $i \in Q_0$ we have that the representations $S_i = (V, f)$ are simple, where $V_t = 0$ for $t \neq i$ and $V_i = k$, and $f = 0$, meaning that all linear maps are the zero maps. We give some examples.

Example 2

If Q and k are as in the above example, and U,V and W as before, then we have that U is a sub representation of V, W is a factor representation of V. Furthermore, W is the simple corresponding to vertex 1. Even more so, U and W are indecomposable, while $V = U \amalg W$.

Another way of viewing representations of a quiver Q is in a functorial way. Let Q be the category whose objects are the vertices of Q, in other words $obj(Q) = Q_0$ and for $i, j \in Q_0$ the set $\mathrm{Hom}_Q(i, j)$ is the set of paths in Q starting at i and ending at j. Suppose $\alpha \in \mathrm{Hom}_Q(i, j)$ and $\beta \in \mathrm{Hom}_Q(j, m)$ then $\beta\alpha$ is a path from i to m by concatenation. Notice that $e_i \in \mathrm{Hom}_Q(i, i)$ is the identity on i. By $\mathrm{vec}\,(k)$ we mean the category of finite dimensional k-vector spaces and with $\mathrm{Hom}_{\mathrm{vec}\,(k)}(V, W)$ the set of linear transformations from V to W. Then Rep Q is naturally isomorphic with the category of functors from Q to $\mathrm{vec}\,(k)$, denoted by $\mathrm{vec}\,(k)^Q$. Note that in this category the objects are functors from Q to $\mathrm{vec}\,(k)$ and the morphisms are the natural transformations between the functors.

We give another and probably a bit more interesting example.

Example 3

Let k be an algebraically closed field and let $Q\colon\ 1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$. Let R_μ be the representation given by $k \underset{1}{\overset{\mu}{\rightrightarrows}} k$ and let T_λ be the representation given by $k^2 \underset{I}{\overset{J_\lambda}{\rightrightarrows}} k^2$ with I the 2×2 identity matrix over k and $J_\lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, the Jordan block of size 2 corresponding to λ, here $\mu, \lambda \in k$. We claim that $\mathrm{Hom}_k(R_\mu, T_\lambda) = (0)$ if $\mu \neq \lambda$, and if $\mu = \lambda$ then $\mathrm{Hom}_k(R_\mu, T_\lambda) \simeq k$. We have to have the following commutative diagram

$$\begin{array}{ccc} k & \xrightarrow{(x\ y)^t} & k^2 \\ {\scriptstyle 1}\downarrow\downarrow{\scriptstyle \mu} & & {\scriptstyle I}\downarrow\downarrow{\scriptstyle J_\lambda} \\ k & \xrightarrow[(u\ v)^t]{} & k^2 \end{array}$$

This yields the following equations

$$\begin{aligned} I(x\ y)^t &= (u\ v)^t 1 \\ J_\lambda(x\ y)^t &= (u\ v)^t \mu \end{aligned}$$

These two equations give that $(x\ y)^t$ is an eigenvector for J_λ and μ is the corresponding eigenvalue for J_λ. Now J_λ has only λ as eigenvalue with algebraic multiplicity 2 and geometric multiplicity 1. The eigenspace corresponding to λ has as basis $(1\ 0)^t$. Thus $\mathrm{Hom}_k(R_\lambda, T_\lambda) \simeq k$ and $\mathrm{Hom}_k(R_\mu, T_\lambda) = (0)$ if $\mu \neq \lambda$.

This algebra is called the **Kronecker** algebra and is very important in representation theory. This algebra is of infinite representation type, meaning that there is infinitely many isomorphism classes of indecomposable modules, and in some sense the smallest algebra of infinite representation type.

Above we said that we are always able to find simple representations of a given quiver, and for some quiver these are all the simple representations. However, there are some quivers for which we are able to find infinitely many simple representations. We illustrate with an example.

Example 4

Let k be a field and let Q be the quiver $\alpha \circlearrowleft \bullet \circlearrowright \beta$. Let V_n be the following representation $A \circlearrowleft k^n \circlearrowright A^t$, where

$$A = \left(\begin{array}{cccc|c} 0 & 0 & \dots & 0 & 0 \\ \hline 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{array}\right)$$

Now choose any nonzero $x \in k^n$, i.e. for some $1 \leq i \leq n$, $x_i \neq 0$ where $x = (x_1, \dots, x_n)^t$. Then $e_1 = (A^t)^{i-1}x = (x_i, \dots, x_n, 0, \dots, 0)^t$. Now let $S = \{e_1, \dots, e_n\}$, where $e_j = A^{j-1}e_1$. We easily see that S is k-linearly independent and thus is a basis for k^n. This shows that any nonzero vector x generates the whole representation. Or equivalently, there is no subspace of k^n that is invariant under A and A^t. This proves that V_n is a simple representation for any positive n.

Throughout this thesis we will mainly be concerned with quivers without oriented cycles, nevertheless, quivers with cycles give many interesting examples. The dimension vector of a representation V over a quiver $Q = (Q_0, Q_1)$ is a $|Q_0|$-tuple over $\mathbb{Z}$, with i'th entry $\dim_k V_i$, that is $(\dim_k V_1, \dots, \dim_k V_{|Q_0|})$ and we write $\underline{\dim}\, V$ for this element in $\mathbb{Z}^{|Q_0|}$.

We have already shown how to get an algebra structure over a field k given a quiver $Q = (Q_0, Q_1)$, namely take as basis the paths in the quiver. If we now consider the finitely dimensional modules over this path algebra we get a category, and it turns out that this category is equivalent to the category of finite dimensional representations. We summarize in the following Theorem.

Theorem 1
Let k be a field and let Q be a finite quiver. Then there is an equivalence of categories between Rep Q *and* mod kQ*, where* mod kQ *is the category of finite dimensional left kQ-modules.* □

What this result tells us is that we can think of our representations as modules and vice versa. This result is quite essential throughout this text, and the reader should be aware of that it will be used without any effort or concern to mentioning it. We will here give an algorithm that takes a representation and gives a module and vice versa, however, for a proof of this the reader is referred to [ARS, sec. III.1]. Given a finite quiver $Q = (Q_0, Q_1)$ and a field k. If (V, f) is a representation of Q over k, then let $M = \coprod_{i \in Q_0} V_i$. Let $m \in M$, that is $m(i) = v_i \in V_i$ for $i \in Q_0$, and for each $\alpha \in Q_1$ we have $f_\alpha \colon V_{s(\alpha)} \longrightarrow V_{e(\alpha)}$, we then define $\alpha \cdot m$ as

$$(\alpha m)(i) = \begin{cases} f_\alpha(v_{s(\alpha)}) & \text{, if } i = e(\alpha) \\ 0 & \text{, otherwise} \end{cases}$$

Thus, if $\alpha \colon i \longrightarrow j$, then αm is zero for all indices except for index j where it takes the value $f_\alpha(v_i)$. The reader may check that this gives a kQ-module structure on M. On the other hand, if M is in mod kQ, and since $1_{kQ} = e_1 + \cdots + e_n$, where e_i is a trivial path at vertex i, we have that $M = \coprod e_i M$ as vector spaces over k. Let $\alpha \colon i \longrightarrow j$ be an arrow, then we have that $\alpha(e_i M) = e_j(\alpha M)$ which is a subspace of $e_j M$, this is really coming from the fact that $\alpha = e_j \alpha e_i$. Let $f_\alpha \colon e_i M \longrightarrow e_j M$ be given by $f_\alpha(e_i m) = e_j \alpha e_i (e_i m) = e_j \alpha e_i m \in e_j M$. Thus, if $V = \{e_i M\}_{i \in Q_0}$ and $f = \{f_\alpha\}_{\alpha \in Q_1}$, then (V, f) is in Rep Q. Since we may view a Λ-module as an representation, we see that $\underline{\dim}\, M = (\dim_k e_1 M, \ldots, \dim_k e_n M)$ is in accordance with the dimension vector of a representation, where $Q_0 = \{1, \ldots, n\}$. Moreover $\underline{\dim}\, M = (\dim_k \operatorname{Hom}_\Lambda(\Lambda e_1, M), \ldots, \dim_k \operatorname{Hom}_\Lambda(\Lambda e_n, M))$, since $\operatorname{Hom}_\Lambda(\Lambda e, M) \simeq eM$ for any idempotent $e \in \Lambda$ through the identification $f \longmapsto f(e) = ef(e)$ with $f \in \operatorname{Hom}_\Lambda(\Lambda e, M)$.

Let R be a commutative ring. Recall that Λ is an R-algebra if there is a ring homomorphism $\varphi \colon R \longrightarrow \Lambda$ such that $\operatorname{Im} \varphi \subseteq Z(\Lambda)$, where $Z(\Lambda)$ denotes the center of Λ. If R is a commutative artinian ring, then we say that Λ is an **artin** (R-)**algebra** if Λ is finitely generated as a R-module.

We say that a ring Λ is left hereditary if the left ideals in Λ are all projective. This is in fact equivalent to that submodules of projective Λ-modules are projective.

1.2 The Coxeter Functors

Let $Q = (Q_0, Q_1)$ be a quiver and let k be a field. We now want to introduce the Coxeter functor and the partial Coxeter functors, but before that we have to introduce some notation. For a vertex $i \in Q_0$ we denote by $\xi(i) = \{\alpha \in$

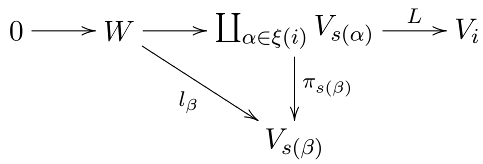
$Q_1 \,|\, s(\alpha) = i$ or $e(\alpha) = i\}$, i.e. all arrows starting or ending in the vertex i. We call a vertex $i \in Q_0$ a **sink**, or $(+)$-accessible, if for each $\alpha \in \xi(i)$ we have that $s(\alpha) \neq i$, in other words, there is no arrow going out of vertex i. Similarly, we define a **source** as a vertex $i \in Q_0$ such that for all $\alpha \in \xi(i)$, $e(\alpha) \neq i$, meaning that there is no arrow in Q ending in the vertex i, we often also call a source for $(-)$-accessible. For i in Q we get another quiver c_iQ which has the same vertices as Q. However, we reverse all the arrows either starting or ending in vertex i. That is, for each $\alpha\colon i \longrightarrow j$ and $\beta\colon h \longrightarrow i$ in Q we get $\alpha'\colon j \longrightarrow i$ and $\beta'\colon i \longrightarrow h$ in c_iQ, respectively.

sink source

Let Q be a quiver and let $V = (V, f)$ be in Rep Q. For a sink i in Q and a representation (V, f) of Q we define the **left partial Coxeter functor** from Rep Q to Rep c_iQ by $S_i^+(V, f) = (U, g)$ where

$$U_j = \begin{cases} V_j & ; \quad j \neq i \\ W & ; \quad \text{otherwise} \end{cases}$$

where W is the kernel of the map $L\colon \coprod_{\alpha\in\xi(i)} V_{s(\alpha)} \longrightarrow V_i$ with $L(v) = \sum f_j(v_j)$. We then have the following diagram



For each $\beta \in \xi(i)$ we get a map $l_\beta\colon W \longrightarrow V_{s(\beta)}$, where l_β is the composition of the inclusion of W into $\coprod_{\alpha\in\xi(i)} V_{s(\alpha)}$ and the canonical projection onto $V_{s(\beta)}$. We then set

$$g_\alpha = \begin{cases} f_\alpha & ; \quad \alpha \notin \xi(i) \\ l_\alpha & ; \quad \text{otherwise} \end{cases}$$

Notice that the arrows going into vertex i have now been reversed, i.e. we have a representation of c_i^+Q and that vertex i is now a source with respect to c_i^+Q. If we have a morphism $h\colon (V, f) \longrightarrow (V', f')$ we get the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & W & \longrightarrow & \coprod_{\alpha\in\xi(i)} V_{s(\alpha)} & \longrightarrow & V_i \\ & & \downarrow{\scriptstyle \tilde{h}} & & \downarrow{\scriptstyle \coprod_{\alpha\in\xi(i)} h_{s(\alpha)}} & & \downarrow{\scriptstyle h_i} \\ 0 & \longrightarrow & W' & \longrightarrow & \coprod_{\alpha\in\xi(i)} V'_{s(\alpha)} & \longrightarrow & V'_i \end{array}$$

where $\tilde{h}$ is the restriction of $(h_{s(\alpha)})_{\alpha\in\xi(i)}$ to W. So $C_i^+(h)_i = \tilde{h}$ and $C_i^+(h)_j = h_j$ otherwise is a map in Rep c_iQ. We have that $C_i^+\colon$ Rep $Q \longrightarrow$ Rep c_iQ is a functor. That is, we get a representation $U = (U, g)$ over the quiver c_iQ and for each map $h\colon (V, f) \longrightarrow (V', f')$ we get a map $C_i^+(h)\colon C_i^+(V, f) \longrightarrow C_i^+(V', f')$, which satisfy the requirement to be a functor.

Example 5

Let $Q\colon 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3 \xleftarrow{\gamma} 4$, k a field, and let $C_3^+\colon$ Rep $Q \longrightarrow$ Rep c_3Q be the left partial Coxeter functor for the sink at 3. Let V is the representation $k \xrightarrow{1} k \xrightarrow{1} k \xleftarrow{1} k$, U the representation $k \longrightarrow 0 \longrightarrow k \xleftarrow{1} k$ and T the representation $0 \longrightarrow 0 \longrightarrow k \longleftarrow 0$, then $C_3^+(V), C_3^+(U)$ and $C_3^+(T)$ are the following representations $k \xrightarrow{1} k \xleftarrow{-1} k \xrightarrow{1} k$, $k \longrightarrow 0 \longleftarrow 0 \longrightarrow k$ and $0 \longrightarrow 0 \longleftarrow 0 \longrightarrow 0$ respectively.

Let us spice up the quiver some and revisit the Kronecker.

Example 6

Given a field k, let Q be the quiver $1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2$ and let V be the following representation $k \underset{1}{\overset{\lambda}{\rightrightarrows}} k$. Vertex 2 is a sink so we may apply C_2^+ to V, which gives the following representation: $k \underset{-\lambda}{\overset{1}{\leftleftarrows}} k$. Now vertex 1 is a sink with respect to c_2Q. We then may apply C_1^+ to $C_2^+(V)$ which yields the representation $k \underset{1}{\overset{\lambda}{\rightrightarrows}} k$. That is, $V = C_1^+C_2^+(V)$. This is no coincidence as we will see later.

Almost identically, for a source i in Q and a representation (V, f) we define the **right partial Coxeter functor** from $C_i^-\colon$ Rep $Q \longrightarrow$ Rep c_iQ by $C_i^-(V, f) = (U, g)$

$$U_j = \begin{cases} V_j & ; \quad j \neq i \\ W & ; \quad \text{otherwise} \end{cases}$$

where $W = \text{Coker } M$ and $M\colon V_i \longrightarrow \coprod_{\alpha\in\xi(i)} V_{e(\alpha)}$ with $M(v) = (f_\alpha(v))_{\alpha\in\xi(i)}$, and

$$g_\alpha = \begin{cases} f_\alpha & ; \quad \alpha \notin \xi(i) \\ m_\alpha & ; \quad \text{otherwise} \end{cases}$$

where $m_\alpha\colon V_{e(\alpha)} \longrightarrow W$ is obtained by first taking the natural inclusion from $V_{e(\alpha)}$ to $\coprod_{\alpha\in\xi(i)} V_{e(\alpha)}$ and then passing to W through the natural projection

$\coprod_{\alpha\in\xi(i)} V_{e(\alpha)} \longrightarrow W$. In other words, we have the following commuting diagram

$$\begin{array}{ccccccc} & & V_{e(\beta)} & & & & \\ & & \downarrow{\scriptstyle \iota_{e(\beta)}} & \searrow{\scriptstyle m_\beta} & & & \\ V_i & \xrightarrow{M} & \coprod_{\alpha\in\xi(i)} V_{e(\alpha)} & \longrightarrow & W & \longrightarrow & 0 \end{array}$$

Now given a morphism $h\colon (V,f)\longrightarrow(V',f')$ we have the following commutative diagram

$$\begin{array}{ccccccc} V_i & \xrightarrow{M} & \coprod_{\alpha\in\xi(i)} V_{e(\alpha)} & \longrightarrow & W & \longrightarrow & 0 \\ \downarrow{\scriptstyle h_i} & & \downarrow{\scriptstyle \coprod_{\alpha\in\xi(i)} h_{s(\alpha)}} & & \vdots{\scriptstyle \bar{h}} & & \\ V'_i & \xrightarrow{M'} & \coprod_{\alpha\in\xi(i)} V'_{e(\alpha)} & \longrightarrow & W' & \longrightarrow & 0 \end{array}$$

Thus we get that $C_i^-(h)_j = h_j$ for $j \neq i$ and $C_i^-(h)_i = \bar{h}$, with $\bar{h}$ uniquely determined by the property of co-kernels. This shows that C_i^- is indeed a functor.

Notice that if vertex i is a sink in Q then it is a source with respect to the orientation in c_i^+Q. Hence, we may look at the composition $C_i^-C_i^+\colon\ \text{Rep } Q\longrightarrow \text{Rep } Q$. We now want to compare the representation V to the representation $C_i^-C_i^+(V)$. We construct a morphism[1] $\tau^i\colon\ C_i^-C_i^+(V)\longrightarrow V$ by $\tau^i_j = \text{Id}_{V_j}$ for $i \neq j$ and for τ^i_i we note that we have the exact sequence $0\longrightarrow C_i^+(V)_i \xrightarrow{M} \coprod_{\alpha\in\xi(i)} V_{s(\alpha)} \xrightarrow{L} V_i$, that is Ker $L = $ Im M. Let $\tau^i_i\colon\ C_i^-C_i^+(V)_i\longrightarrow V_i$ be the unique map that makes the following diagram commute.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_i^+(V)_i & \xrightarrow{M} & \coprod_{\alpha\in\xi(i)} V_{s(\alpha)} & \xrightarrow{\pi} & C_i^-C_i^+(V)_i & \longrightarrow & 0 \\ & & \| & & \| & & \downarrow{\scriptstyle \tau_i^i} & & \\ 0 & \longrightarrow & C_i^+(V)_i & \longrightarrow & \coprod_{\alpha\in\xi(i)} V_{s(\alpha)} & \xrightarrow{L} & V_i & & \end{array}$$

The uniqueness is guaranteed by the property of co-kernels. By the Snake lemma we instantaneously achieve that τ^i_i is mono which in turn yields that τ^i is mono. Thus we are able to construct the factor representation $V/\text{Im } \tau^i$. It is easy to see that $\dim_k(V/\text{Im } \tau^i)_j = 0$ when $j \neq i$, i.e. $V/\text{Im } \tau^i$ is concentrated at vertex i. We also note that we have the following split exact sequence

$$0 \longrightarrow C_i^-C_i^+(V)_i \longrightarrow V_i \longrightarrow V_i/\text{Im } \tau^i \longrightarrow 0$$

due to the fact that $V/\text{Im } \tau^i$ is concentrated at vertex i we are always able to find a morphism $V/\text{Im } \tau^i\longrightarrow V_i$ such that the above sequence splits. In other words $V_i \simeq C_i^-C_i^+(V)_i \coprod V_i/\text{Im } \tau^i$. Now if $V = C_i^-(W)$ for some W in Rep c_iQ, we see that the map L in the above diagram is onto making τ^i an isomorphism.

[1] We really should write τ^i_V, since this morphism depends on the representation V. However, we will omit the V purposefully so the notation does not get too complicated.

This is seen by observing that $\mathrm{Im}\ \tau_i^i = \mathrm{Im}\ L$ since π is onto by definition and $L = \tau_i^i \pi$.

Similarly we define[2] $\sigma^j \colon V \longrightarrow C_j^+ C_j^-(V)$ by $\sigma_i^j = \mathrm{Id}_{C_j^+ C_j^-(V)_i}$ for $i \neq j$ and let σ_j^j be the unique map which makes the following diagram commute

$$\begin{array}{ccccccccc}
 & & V_j & \xrightarrow{M} & \coprod_{\alpha \in \xi(j)} V_{s(\alpha)} & \longrightarrow & C_j^-(V)_j & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \sigma_j^j} & & \| & & \| & & \\
0 & \longrightarrow & C_j^+ C_j^-(V)_j & \xrightarrow{\iota} & \coprod_{\alpha \in \xi(j)} V_{s(\alpha)} & \xrightarrow{L} & C_j^-(V)_j & \longrightarrow & 0
\end{array}$$

Notice that the maps M and L in general are not the same as the maps in the previous diagram, however for convenience and readability we stick to this notation. Analogously, we have the exact sequence $V_j \xrightarrow{M} \coprod_{\alpha \in \xi(j)} V_{s(\alpha)} \xrightarrow{L} C_j^+(V)_j \longrightarrow 0$ and the property of kernels that ensures the unique map in the above diagram. We see that if M is one-to-one, it follows that σ_j^j is an isomorphism. This follows from Snake Lemma. Here, too, we have immediate consequences as above, however we leave the proof to the reader as an exercise and summarize our finding in a proposition.

Proposition 2
Let the notation be as above.

(a) *For a sink i we have $C_i^+(V \coprod V') = C_i^+(V) \coprod C_i^+(V')$. Equally, we have that $C_i^-(V \coprod V') = C_i^-(V) \coprod C_i^-(V')$ for a source i.*

(b) *τ^i and σ^j are mono and epi respectively.*

(c) *If τ^i is an isomorphism then the dimension of the vector spaces $C_i^+(V)_p$ are given by*

$$\begin{aligned}
\dim_k C_i^+(V)_p &= \dim_k V_p \ \text{ for } p \neq i \\
\dim_k C_i^+(V)_i &= -\dim_k V_i + \sum_{\alpha \in \xi(i)} \dim_k V_{s(\alpha)}.
\end{aligned} \tag{1.1}$$

If σ^j is an isomorphism then the dimension of the vector spaces $C_j^-(V)_q$ are given by

$$\begin{aligned}
\dim_k C_j^-(V)_q &= \dim_k V_q \ \text{ for } q \neq j \\
\dim_k C_j^-(V)_j &= -\dim_k V_j + \sum_{\beta \in \xi(j)} \dim_k V_{e(\beta)}.
\end{aligned} \tag{1.2}$$

(d) *The sub representation* Ker σ^j *is such that* $(\mathrm{Ker}\ \sigma^j)_q = 0$ *for* $q \neq j$. *The factor representation* $V/\mathrm{Im}\ \tau^i$ *is such that* $(V/\mathrm{Im}\ \tau^i)_p = 0$ *for* $p \neq j$.

[2]Here, too, we are sloppy with the notation, and omit the index V.

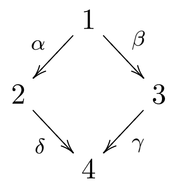
(e) *If the map L is epi then τ^i is an isomorphism. Similarly, if the map M is mono then σ^j is an isomorphism.*

(f) $V \simeq C_i^- C_i^+(V) \coprod V/\operatorname{Im} \tau^i$. *Analogously,* $V \simeq C_j^+ C_j^-(V) \coprod \operatorname{Ker} \sigma^j$. □

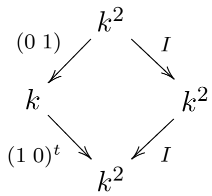
Note that if the representation $V = C_j^+(W)$ for some representation $W \in \operatorname{Rep} Qc_j$, then the map L is onto and thus $V \simeq C_j^+ C_j^-(V)$. Analogously, if $V = C_i^-(W)$, where W is a representation of $c_i Q$, then the map M is mono, and hence we have $V \simeq C_i^- C_i^+(V)$. We give an example.

Example 7

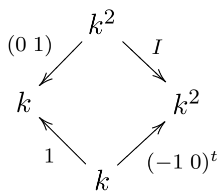
Let k be a field and let Q be the following quiver



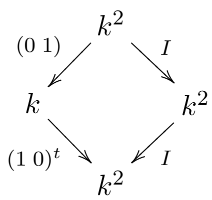
Let now V be the representation



where I is the two by two identity matrix. Then $C_4^+(V)$ is the representation



We can now apply C_4^- to the above representation, which in turn yields



In other words, $V = C_4^- C_4^+(V)$.

One of the nicest properties of the partial Coxeter functors are the ones we give in the next theorem.

Theorem 3
Let $Q = (Q_0, Q_1)$ be a quiver, and let V in Rep Q *be an indecomposable representation.*

(a) If $i \in Q_0$ is a sink, then two scenarios are possible, either $V \simeq S_i$, the simple corresponding to vertex i, and $C_i^+(V) = 0$ or $C_i^+(V)$ is indecomposable with $C_i^- C_i^+(V) = V$ and the dimension of the spaces $C_i^+(V)_p$ are given by (1.1).

(b) If $j \in Q_0$ is a source, then either $V \simeq S_j$, the simple corresponding to vertex j, and $C_j^-(V) = 0$ or $C_j^-(V)$ is indecomposable with $C_j^+ C_j^-(V) = V$ and the dimension of the spaces $C_j^-(V)_q$ are given by (1.2).

Proof. We only prove (a), (b) is shown in a similar fashion. From Proposition 2 part (f) we have that $V \simeq C_i^- C_i^+(V) \coprod V/\operatorname{Im} \tau^i$. Now if V is indecomposable then V must coincide with one of them. If $V = V/\operatorname{Im} \tau^i$ we get that $V_p = 0$ for $p \neq i$, and since V is indecomposable we must have $V \simeq S_i$. On the other hand, if $V = C_i^- C_i^+(V)$, then τ^i is an isomorphism and we have Proposition 2 part (c). We now show that $W = C_i^+(V)$ is indecomposable. Assume that $W = W_1 \coprod W_2$. We the apply C_i^- and get $V = C_i^-(W_1) \coprod C_i^-(W_2)$. V still being indecomposable, we arrive at, say, $C_i^-(W_2) = 0$. Then by Proposition 2 part (e) that $\sigma^i \colon W \longrightarrow C_i^+ C_i^-(W)$ is an isomorphism, however $\sigma^i(W_2) \subseteq C_i^+ C_i^-(W_2) = 0$. Thus we see that $W_2 = 0$. □

We now generalize the notion of a sink and a source. We say that a sequence of vertices $i_1, i_2, \ldots, i_n$ is $(+)$-accessible with respect to Q if vertex i_1 is $(+)$-accessible with respect to Q, and vertex i_2 is $(+)$-accessible with respect to $c_{i_1}Q$, and i_3 is $(+)$-accessible with respect to $c_{i_2}c_{i_1}Q$, and so on. In a similar way we define a $(-)$-accessible sequence. Now, inductively, we get a generalization of Theorem 3.

Corollary 4
For a $(+)$-accessible sequence $i_1, i_2, \ldots, i_n$ in $Q = (Q_0, Q_1)$.

(a) $C_{i_1}^- C_{i_2}^- \ldots C_{i_{r-1}}^-(S_{i_r})$ is either 0 or indecomposable, for $1 \leq r \leq n$ (here S_{i_r} is in Rep *$c_{i_{r-1}} \ldots c_{i_1}Q$).*

(b) If V in Rep Q *is indecomposable, and $C_{i_n}^+ \ldots C_{i_2}^+ C_{i_1}^+(V) = 0$. Then $V \simeq C_{i_1}^- C_{i_2}^- \ldots C_{i_{r-1}}^-(S_{i_r})$, for some $1 \leq r \leq n$.* □

We now come to the meat and bones of these functors and why they are so important to us. The next result shows that knowing the indecomposable objects of an acyclic quiver you know the indecomposable objects of any other given orientation of that quiver. But before we indulge into this we introduce a slightly different notation for a quiver. Here it will be advantageous to think of our quivers as the underlying graph Γ, that is the set of vertices and non-oriented edges, and the orientation of the quiver, denoted by Ω. We write $Q = (\Gamma, \Omega)$.

Theorem 5
Let Q be a quiver with acyclic underlying graph, and let Ω and Ω' be two orientations of Q.

(a) There exists a sequence of vertices $i_1, i_2, \ldots, i_n$ which is $(+)$-accessible with respect to Ω such that $\Omega' = c_{i_n} \ldots c_{i_1} \Omega$

(b) Let $\mathscr{M}$, $\mathscr{M}'$ be the set of isomorphism classes of indecomposable representations of (Γ, Ω) and (Γ, Ω') respectively, and let $\bar{\mathscr{M}} \subseteq \mathscr{M}$ be of the form $C^-_{i_1} C^-_{i_2} \ldots C^-_{i_{r-1}}(S_{i_r})$ for $1 \leq r \leq n$, and $\bar{\mathscr{M}}' \subseteq \mathscr{M}'$ be of the form $C^+_{i_{r-1}} \ldots C^+_{i_2} C^+_{i_1}(S_{i_r})$ for $1 \leq r \leq n$. Then the functor $C^+_{i_n} \ldots C^+_{i_2} C^+_{i_1}$ gives a one-to-one correspondence between $\mathscr{M} - \bar{\mathscr{M}}$ and $\mathscr{M}' - \bar{\mathscr{M}}'$

Proof. (a). It is enough to consider the case where Ω and Ω' differ only in one arrow, say α. If we then remove the edge α we get two disjoint connected components. Let Q' be the component containing the vertex $e(\alpha)$ with respect to the orientation Ω. In Q' we are able to find an ordering of vertices $i_1 < \ldots < i_n$ such that $e(\beta) < s(\beta)$ for all β in Q'. Such an ordering always exists since Q' is acyclic. Note now that i_1 is $(+)$-accessible with respect to Ω, i_2 is $(+)$-accessible with respect to $c_{i_1}\Omega$, etc. Thus the sequence $i_1, \ldots, i_n$ is $(+)$-accessible. Also every arrow in Q' is reversed twice, and the arrow α in Q is reversed once, that is $\Omega' = c_{i_n} \ldots c_{i_1} \Omega$.

(b). This is shown using (a) and corollary 4. □

A numbering $i_1, \ldots, i_n$ of the vertices of a quiver Q is called **suitable**, if $e(\alpha) < s(\alpha)$ for all $\alpha \in Q_1$. If Q is acyclic, then such an numbering always exists. We are usually interested in different combinations of the partial Coxeter functors which is a endofunctor, so we introduce some notation. For a acyclic quiver Q, we let $\Phi^+ = C^+_{i_n} \ldots C^+_{i_1}$ and $\Phi^- = C^-_{i_1} \ldots C^-_{i_n}$, where $i_1, \ldots, i_n$ is a suitable numbering of the vertices of Q. These functors we call the **Coxeter functors**. We state some related consequences.

Theorem 6
In the above setting, we have that the sequence $i_1, \ldots, i_n$ is $(+)$-accessible and $i_n, \ldots, i_1$ is $(-)$-accessible. Furthermore, $\Phi^+, \Phi^- \colon \operatorname{Rep} Q \longrightarrow \operatorname{Rep} Q$. Also, the Φ^+ and Φ^- are independent of the choice in a suitable numbering of the vertices of Q. □

We come to the last definition in this section. For an acyclic quiver Q we say that a representation V is **regular** if $V \simeq (\Phi^-)^m(\Phi^+)^m V \simeq (\Phi^+)^m(\Phi^-)^m V$ for all $m \geq 0$. If $(\Phi^+)^m V = 0$ ($(\Phi^-)^m V = 0$) for some $m \in \mathbb{N}$, we say that V is **$(+)$-irregular** (**$(-)$-irregular**, respectively).

Theorem 7
Let Q be an acyclic quiver. Then each indecomposable representation of Q is either regular or irregular. □

Chapter 2

Auslander-Reiten Theory

This chapter is dedicated to, as the chapter caption suggests, Auslander-Reiten theory. We shall go through some basic concepts of AR-theory, such as almost split and irreducible maps, Coxeter transformation and almost split sequences. Topping off with the AR-quiver of an artin algebra. This chapter is based upon the work of [ARS] and [ASS]. The biggest hurdle in this chapter was to give a classification of minimal almost split maps in terms of irreducible maps without going by the existence of minimal maps.

2.1 The Dual and the Transpose.

This section is devoted to two important functors in representation theory, the transpose and the dual of the transpose. We will here introduce these notions, and go through some elementary and some non-trivial properties of them. In this section, Λ will be an artinian algebra over a commutative artin ring k, that is we have a ring homomorphism $\varphi\colon k \longrightarrow \Lambda$ with $\operatorname{Im}\varphi \subseteq Z(\Lambda)$ and Λ is finitely generated as a R-module, where $Z(\Lambda)$ is the center of Λ. Recall Krull-Schmidt Theorem for finitely generated Λ-modules. It says that given two decomposition of a Λ-module M into indecomposables, then these decomposition differ only by permutation of the summands up to isomorphism. Hence for M in $\operatorname{mod}\Lambda$ we have an unique (up to isomorphism) decomposition $M = M_{\mathscr{P}} \amalg M'$, where $M_{\mathscr{P}}$ has no nonzero projective summands and M' is projective. Denote by $\mathscr{P}(\Lambda)$ the full sub category of $\operatorname{mod}\Lambda$ consisting of projective modules from $\operatorname{mod}\Lambda$. Recall that $(-)^* = \operatorname{Hom}_\Lambda(-,\Lambda)\colon \mathscr{P}(\Lambda) \longrightarrow \mathscr{P}(\Lambda^{\mathrm{op}})$ is a duality.

Given a minimal projective presentation $P_1 \xrightarrow{f} P_0 \xrightarrow{p} X \longrightarrow 0$ of X in $\operatorname{mod}\Lambda$, i.e. $p\colon P_0 \longrightarrow X$ and $f\colon P_1 \longrightarrow \operatorname{Ker} p$ are projective covers, we define the transpose of X, $\operatorname{Tr} X = \operatorname{Coker} f^*$. That is, $P_0^* \xrightarrow{f^*} P_1^* \longrightarrow \operatorname{Tr} X \longrightarrow 0$ is exact. We state some immediate consequences of this definition.

Proposition 8
All modules are in mod Λ

(a) $\operatorname{Tr} X = 0$ *if and only if* X *is projective.*

(b) $P_0^* \xrightarrow{f^*} P_1^* \xrightarrow{q} \operatorname{Tr} X \longrightarrow 0$ *is a minimal projective presentation of* $\operatorname{Tr} X$ *whenever* $P_1 \xrightarrow{f} P_0 \xrightarrow{p} X \longrightarrow 0$ *is a minimal projective presentation of a nonprojective module* X.

(c) $\operatorname{Tr}(\coprod_{i=1}^n X_i) \simeq \coprod_{i=1}^n \operatorname{Tr} X_i$ *, when* n *is finite.*

(d) $\operatorname{Tr} \operatorname{Tr} X \simeq X_{\mathscr{P}}$ *for all* X.

(e) If X *and* Y *have no nonzero projective direct summands, then* $\operatorname{Tr} X \simeq \operatorname{Tr} Y$ *if and only if* $X \simeq Y$.

Proof. (d) and (e) will follow from the other parts of the proposition.

(a) If X is projective, then $0 \xrightarrow{0} X \longrightarrow X \longrightarrow 0$ is the minimal projective presentation of X, which yields $X^* \xrightarrow{0^*} 0 \longrightarrow \operatorname{Tr} X \longrightarrow 0$, so $\operatorname{Tr} X = 0$. If $\operatorname{Tr} X = 0$ then $P_0^* \xrightarrow{f^*} P_1^* \longrightarrow 0$ is a split epimorphism, it then follows that $0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$ is split exact, and the claim follows.

(b) If X is non-projective, then $\operatorname{Tr} X \neq 0$. Clearly $P_0^* \longrightarrow P_1^* \longrightarrow \operatorname{Tr} X \longrightarrow 0$ is a projective presentation of $\operatorname{Tr} X$ in mod Λ^{op}. Assume that this is not minimal. This means that we have a non-trivial decomposition $P_0^* = Q_0 \amalg Q_0'$ and $P_1^* = Q_1 \amalg Q_1'$ with $q' \colon Q_0' \longrightarrow Q_1'$ an isomorphism. And the above sequence is isomorphic to $Q_0 \amalg Q_0' \xrightarrow{q \amalg q'} Q_1 \amalg Q_1' \longrightarrow \operatorname{Tr} X \longrightarrow 0$. This then yields $Q_1^* \xrightarrow{q*} Q_0^* \longrightarrow X \longrightarrow 0$ when we apply $(-)^*$. We arrive at a contradiction, since this then violates the minimality of the projective presentation of X.

(c) This is a direct consequence of $(-)^*$ being a duality and the universal property of Coker. □

Tr will not usually define a functor between module categories in general, in order for it to be a functor we need to move to **stable categories** modulo projectives. We will denote by $\mathscr{P}(A, B)$ the R-submodule of $\operatorname{Hom}_\Lambda(A, B)$ which consist of all morphisms $f \colon A \longrightarrow B$ which factor through a projective, i.e. there is a projective in mod Λ, say P, such that $f = hg$ for some morphisms $g \colon A \longrightarrow P$ and $h \colon P \longrightarrow B$. We then define $\underline{\operatorname{Hom}}_\Lambda(A, B) = \operatorname{Hom}_\Lambda(A, B)/\mathscr{P}(A, B)$. Furthermore, we will denote the category of finitely generated Λ-modules modulo projectives by $\underline{\operatorname{mod}}\,\Lambda$, which objects are exactly the objects of mod Λ and morphisms are the factors $\underline{\operatorname{Hom}}_\Lambda(A, B)$. For a proof of the next statement the reader is referred to [ARS, IV.1].

Proposition 9
The functor $\operatorname{Tr}\colon \underline{\operatorname{mod}}\,\Lambda \longrightarrow \underline{\operatorname{mod}}\,\Lambda^{\mathrm{op}}$ *is an equivalence of categories.* □

Since R is artin, we only have finitely many isomorphism classes of simple R-modules, say $S_1, \ldots, S_n$. Let $I = \coprod I(S_i)$, where $S_i \longrightarrow I(S_i)$ is the injective envelope. Then the contravariant functor $\operatorname{Hom}_R(-, I)\colon \operatorname{mod} R \longrightarrow \operatorname{mod} R$ is a duality, which then induces a duality $D = \operatorname{Hom}_R(-, I)\colon \operatorname{mod} \Lambda \longrightarrow \operatorname{mod} \Lambda^{\mathrm{op}}$. Suppose $\Lambda = kQ$, where k is a field and Q is an acyclic quiver. Then this duality reduces to $D = \operatorname{Hom}_k(-, k)$. If (V, T) is a representation of a quiver Γ, then $D(V, T)$ is the representation of the opposite quiver, Γ^{op}, with $(D(V))_i = D(V_i) = V_i^*$ the usual dual space of a vector space, and for $T_\alpha\colon V_i \longrightarrow V_j$ in (V, T) we have $D(T_\alpha)\colon D(V_j) \longrightarrow D(V_i)$ given by $D(T_\alpha)(g)(v) = g(T_\alpha v)$ for $g \in D(V_j)$ and $v \in V_i$. Especially, this means that if P_i and I_i are the projective and injective representations corresponding to vertex i, we have $D\left(P_i^*\right) \simeq I_i$.

Example 8

If we are in the case of the Kronecker algebra, and given the representation $k \overset{1}{\underset{t}{\rightrightarrows}} k$, then $D(V_1) \simeq D(V_2) \simeq k$ and $D(1) = 1$ and $D(t) = t$ by an appropriate choice of basis. Thus the dual representation is then $k \overset{1}{\underset{t}{\leftleftarrows}} k$.

We might now be interested in knowing what happens on $\underline{\operatorname{mod}}\,\Lambda$ under the action of D. If $f \in \mathscr{P}(A, B)$, that is

$$\begin{array}{ccc} & P & \\ {}^{h}\nearrow & & \searrow^{g} \\ A & \xrightarrow{\quad f \quad} & B \end{array}$$

is commutative for some projective P. Since D is a duality we get the following commutative diagram

$$\begin{array}{ccc} & D(P) & \\ {}^{D(g)}\nearrow & & \searrow^{D(h)} \\ D(B) & \xrightarrow{\quad D(f) \quad} & D(A) \end{array}$$

with $D(P)$ injective in mod Λ^{op}. Thus if $f\colon A \longrightarrow B$ factors through a projective, $D(f)\colon D(B) \longrightarrow D(A)$ factors through an injective. We are then tempted to introduce the stable category modulo injectives. Let A and B be in mod Λ and let $\mathscr{I}(A, B) \subseteq \operatorname{Hom}_\Lambda(A, B)$ be the R-submodule consisting of all morphism which factor through an injective, that is all morphisms $f\colon A \longrightarrow B$ which for some $g\colon A \longrightarrow I$ and $h\colon I \longrightarrow B$ and I injective in mod Λ are such that $f = hg$. We will usually denote the factor module $\operatorname{Hom}_\Lambda(A, B)/\mathscr{I}(A, B)$ by $\overline{\operatorname{Hom}}_\Lambda(A, B)$. We will then write $\overline{\operatorname{mod}}\,\Lambda$ when referring to the stable category modulo injectives, that is the category consisting of the same objects as mod Λ but the hom-sets are the factor modules $\overline{\mathrm{Hom}}_\Lambda(A,B)$ for Λ-modules A and B. From the observation above we see that the duality $D\colon \mathrm{mod}\,\Lambda \longrightarrow \mathrm{mod}\,\Lambda$ induces a duality $D\colon \underline{\mathrm{mod}}\,\Lambda \longrightarrow \overline{\mathrm{mod}}\,\Lambda$. Combining this fact with Proposition 9 results in the following proposition.

Proposition 10
The compositions $D\,\mathrm{Tr} : \underline{\mathrm{mod}}\,\Lambda \longrightarrow \overline{\mathrm{mod}}\,\Lambda$ *and* $\mathrm{Tr}\,D : \overline{\mathrm{mod}}\,\Lambda \longrightarrow \underline{\mathrm{mod}}\,\Lambda$ *are inverse equivalences of categories.* □

We now give some basic properties of $D\,\mathrm{Tr}$ following Proposition 8.

Proposition 11

(a) *If* $P_1 \xrightarrow{f} P_0 \longrightarrow X \longrightarrow 0$ *is a minimal projective presentation of an indecomposable non-projective* Λ*-module* X*, then* $0 \longrightarrow D\,\mathrm{Tr}\,X \xrightarrow{g} D\,(P_0^*) \xrightarrow{D\,(f^*)}$ *is a minimal injective copresentation, in other words* $g\colon D\,\mathrm{Tr}\,X \longrightarrow D(P_0^*)$ *and the induced morphism* $h\colon \mathrm{Coker}\ g \longrightarrow D(P_1^*)$ *are injective envelopes.*

(b) $(D\,I_0)^* \longrightarrow (D\,I_1)^* \longrightarrow \mathrm{Tr}\,D\,X \longrightarrow 0$ *is a minimal projective presentation of* $\mathrm{Tr}\,D\,X$ *whenever* $0 \longrightarrow X \longrightarrow I_0 \longrightarrow I_1$ *is a minimal injective copresentation of a noninjective module* X*.*

(c) $D\,\mathrm{Tr}\,(\coprod_{i\in I} X_i) \simeq \coprod_{i\in I} D\,\mathrm{Tr}\,X_i$ *where* I *is finite and all* X_i*'s are in* $\mathrm{mod}\,\Lambda$.

(d) $D\,\mathrm{Tr}\,X = 0$ *if and only if* X *is projective.*

(e) $D\,\mathrm{Tr}\,X$ *has no nonzero injective direct summands for all* X *in* $\mathrm{mod}\,\Lambda$.

(f) *For all* X *in* $\mathrm{mod}\,\Lambda$, $(\mathrm{Tr}\,D\,)D\,\mathrm{Tr}\,X \simeq X_{\mathscr{P}}$.

(g) *If* X *and* Y *have no nonzero projective direct summands, then* $D\,\mathrm{Tr}\,X \simeq D\,\mathrm{Tr}\,Y$ *if and only if* $X \simeq Y$*.* □

Recall that a ring R is left **hereditary** if all left ideals of R are projective. There is a homological characterization of a left hereditary ring R, namely $\mathrm{l.\,gl.\,dim}\,R \leq 1$. In fact this is equivalent to submodules of projective R-modules again being projective. Since we say modules for left modules, we only say hereditary when we mean left hereditary.

Another important connection to note is that if Λ is in addition hereditary and for X and Y in $\mathrm{mod}\,\Lambda$ with X, Y with no nonzero projective direct summands, then $\mathscr{P}(X,Y) = 0$. This can be seen the following way. Let $f \in \mathscr{P}(X,Y)$, then there is some projective P in $\mathrm{mod}\,\Lambda$ and $h\colon X \longrightarrow P$ and $g\colon P \longrightarrow Y$ such that $f = gh$. Now this implies $h\colon X \longrightarrow \mathrm{Im}\ f \subseteq P$ is a split epimorphism, in other words $X \simeq \mathrm{Im}\ h \amalg Z$, with $\mathrm{Im}\ h$ projective. This yields $\mathrm{Im}\ h = 0$ and $f = 0$. In view of this we denote by $\mathrm{mod}_{\mathscr{P}}\,\Lambda$ the full subcategory of $\mathrm{mod}\,\Lambda$ in which the objects are the objects X in $\mathrm{mod}\,\Lambda$ with $X \simeq X_{\mathscr{P}}$, that is all objects with no nonzero projective direct summands. The assertion above then yields a useful property calculation wise for $D\,\mathrm{Tr}$.

Proposition 12
Let Λ be a hereditary artin algebra. Then there is an equivalence of categories between $\mathrm{mod}_{\mathscr{P}}\,\Lambda$ and $\underline{\mathrm{mod}}\,\Lambda$. □

Notice that when Λ is hereditary, then the minimal projective presentation of $M \in \mathrm{mod}_{\mathscr{P}}\,\Lambda$ is of the following form $0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$. We then apply $\mathrm{Hom}_\Lambda(-,\Lambda)$ on the sequence above, this then yields the following exact sequence $0 \longrightarrow M^* \longrightarrow P_0^* \longrightarrow P_1^* \longrightarrow \mathrm{Ext}^1_\Lambda(M,\Lambda) \longrightarrow 0$. By the universal property of Coker we see that $\mathrm{Ext}_\Lambda(-,\Lambda)\colon \underline{\mathrm{mod}}\,\Lambda \longrightarrow \underline{\mathrm{mod}}\,\Lambda^{\mathrm{op}}$ and $\mathrm{Tr}\,(-)\colon \underline{\mathrm{mod}}\,\Lambda \longrightarrow \underline{\mathrm{mod}}\,\Lambda^{\mathrm{op}}$ are isomorphic as functors, and therefore $D\,\mathrm{Ext}^1_\Lambda(-,\Lambda) \simeq D\,\mathrm{Tr}\,(-)$ from $\underline{\mathrm{mod}}\,\Lambda$ to $\overline{\mathrm{mod}}\,\Lambda$. Hence we have proved the following Proposition.

Proposition 13
Let Λ be a hereditary artin algebra. There is a functorial isomorphism between $\mathrm{Ext}_\Lambda(-,\Lambda)\colon \underline{\mathrm{mod}}\,\Lambda \longrightarrow \underline{\mathrm{mod}}\,\Lambda^{\mathrm{op}}$ and $\mathrm{Tr}\,(-)\colon \underline{\mathrm{mod}}\,\Lambda \longrightarrow \underline{\mathrm{mod}}\,\Lambda^{\mathrm{op}}$, and followingly $D\,\mathrm{Ext}^1_\Lambda(-,\Lambda)\colon \underline{\mathrm{mod}}\,\Lambda \longrightarrow \overline{\mathrm{mod}}\,\Lambda$ and $D\,\mathrm{Tr}\,(-)\colon \underline{\mathrm{mod}}\,\Lambda \longrightarrow \overline{\mathrm{mod}}\,\Lambda$. □

We now turn our attention to studying some important modules that show up related to the functors $D\,\mathrm{Tr}$ and $\mathrm{Tr}\,D$. Let Λ be a hereditary artin algebra, we say that a Λ-module Q is **preprojective** if there is a nonnegative integer such that $(D\,\mathrm{Tr}\,)^n Q$ is projective. Furthermore, Q is **indecomposable preprojective** if Q is indecomposable. Dually, we define a Λ-module J to be **preinjective** if $(\mathrm{Tr}\,D\,)^m J$ is injective for some nonnegative integer m. We say that J is **indecomposable preinjective** if it is indecomposable and preinjective. If an indecomposable module is neither preprojective nor preinjective, then we call it **regular**. An easy observation to make is that Q is preprojective if and only if $(D\,\mathrm{Tr}\,)^n Q = 0$ for some n. Dually, J is preinjective if and only if $(\mathrm{Tr}\,D\,)^m J = 0$, where $m \geq 0$

We now give a complete characterization of finitely generated indecomposable preprojectives (preinjectives) for a hereditary artin algebra.

Proposition 14
Let Λ be an hereditary artin algebra. Q is an indecomposable preprojective Λ-module if and only if there is an indecomposable projective Λ-module, P, such that $Q \simeq (\mathrm{Tr}\,D\,)^m P$ for some $m \geq 0$.

Proof. If $Q \simeq (\mathrm{Tr}\,D\,)^m P$ then it is preprojective by definition, and indecomposable since the functor $D\,\mathrm{Tr}$ is additive. Conversely, if Q is indecomposable projective this is trivial. Assume Q non-projective indecomposable preprojective. Then there exists an indecomposable projective $P \neq 0$ and some $m \in \mathbb{N}$ such that $P \simeq (D\,\mathrm{Tr}\,)^m Q$. Applying the $\mathrm{Tr}\,D$ functor m times gives us $(\mathrm{Tr}\,D\,)^m P \simeq (\mathrm{Tr}\,D\,)^m (D\,\mathrm{Tr}\,)^m Q \simeq Q$. As desired. □

The dual of this proposition follows by a small observation, namely

Lemma 15
If Λ is an artin algebra, then $(\operatorname{Tr} D)^n DX \simeq D(D\operatorname{Tr})^n X$ for all X in $\operatorname{mod}\Lambda$.

Proof. We prove this by induction on n. For $n = 1$ this is trivially true since $D^2 \simeq 1$ as functors. Assume that it holds for $n = k$, $k \geq 1$.

$$\begin{aligned}(\operatorname{Tr} D)^{k+1} DX = \operatorname{Tr} D\,(\operatorname{Tr} D)^k DX &\simeq (\operatorname{Tr} D) D(D\operatorname{Tr})^k X\\ &\simeq D(D\operatorname{Tr})(D\operatorname{Tr})^k X = D(D\operatorname{Tr})^{k+1} X\end{aligned}$$

which concludes the proof. □

Hence we get the dual result of Proposition 14.

Proposition 16
Let Λ be as above. J is a indecomposable preinjective Λ-module if and only if there is a indecomposable injective Λ-module, I, such that $J \simeq (D\operatorname{Tr})^n I$ for some $n \geq 0$. □

The functors $\Phi^+ \colon \operatorname{Rep} Q \longrightarrow \operatorname{Rep} Q$ and $D\operatorname{Tr} \colon \operatorname{Rep} Q \longrightarrow \operatorname{Rep} Q$ have many similar features, and one may ask oneself when does these functors coincide, if they do at all? The next proposition gives an answer to this.

Proposition 17
Let Q be a tree and k a field. The functors $\Phi^+, D\operatorname{Tr} \colon \operatorname{Rep} Q \longrightarrow \operatorname{Rep} Q$ coincide. Similarly, $\Phi^- \colon \operatorname{Rep} Q \longrightarrow \operatorname{Rep} Q$ and $\operatorname{Tr} D \colon \operatorname{Rep} Q \longrightarrow \operatorname{Rep} Q$ coincide. □

This of course means that the terms (+)-irregular and preprojective also coincide for trees. Similarly, we have that (−)-irregular and preinjective coincide.

In closing we are going to go through Auslander's defect formula which is an important result in the representation theory of algebras, especially in connection with almost split sequences which we will go through in Section 2.2. First we need to define what we mean by the defect of an exact sequence. Let $\delta \colon 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be an exact sequence, we then define the **covariant defect** of $\delta_* \colon \operatorname{mod}\Lambda \longrightarrow \mathrm{Ab}$ through the exactness of the following sequence

$$0 \longrightarrow \operatorname{Hom}_\Lambda(C,-) \longrightarrow \operatorname{Hom}_\Lambda(B,-) \longrightarrow \operatorname{Hom}_\Lambda(A,-) \longrightarrow \delta_*(-) \longrightarrow 0$$

and similarly the **contravariant defect** δ^*:

$$0 \longrightarrow \operatorname{Hom}_\Lambda(-,A) \longrightarrow \operatorname{Hom}_\Lambda(-,B) \longrightarrow \operatorname{Hom}_\Lambda(-,C) \longrightarrow \delta^*(-) \longrightarrow 0$$

Theorem 18 (Auslander's defect formula)
Let $\delta \colon 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be exact in $\operatorname{mod}\Lambda$, and let $X \in \operatorname{mod}\Lambda$. Then there is an isomorphism $D\,\delta^*(X) \simeq \delta_*(D\operatorname{Tr} X)$ which is functorial in δ and X.

Proof. Let $P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$ be a minimal projective presentation of X. We then get $P_0^* \longrightarrow P_1^* \longrightarrow \operatorname{Tr} X \longrightarrow 0$. If we now apply $- \otimes_\Lambda Y$ for any $Y \in \operatorname{mod} \Lambda$.

$$\begin{array}{ccccccccc}
 & & & & P_0^* \otimes_\Lambda Y & \longrightarrow & P_1^* \otimes_\Lambda Y & \longrightarrow & \operatorname{Tr} X \otimes_\Lambda Y \longrightarrow 0 \\
 & & & & \alpha_0 \downarrow \simeq & & \alpha_1 \downarrow \simeq & & \| \\
0 & \longrightarrow & \operatorname{Hom}_\Lambda(P_0, Y) & \longrightarrow & \operatorname{Hom}_\Lambda(P_0, Y) & \longrightarrow & \operatorname{Hom}_\Lambda(P_0, Y) & \longrightarrow & \operatorname{Tr} X \otimes_\Lambda Y \longrightarrow 0
\end{array}$$

Where the isomorphisms $\alpha_i \colon \operatorname{Hom}_\Lambda(P_i, \Lambda) \otimes_\Lambda Y \longrightarrow \operatorname{Hom}_\Lambda(P_i, Y)$ are given on a generator $h \otimes y$ by $\alpha_i(h \otimes y)(p) = h(p) \cdot y$, for all $y \in Y$, $h \in \operatorname{Hom}_\Lambda(P_i, \Lambda)$ and $p \in P_i$ for $i = 0, 1$. A routine calculation shows that this isomorphisms is natural. Now consider the following diagram

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \operatorname{Hom}_\Lambda(X, A) & \longrightarrow & \operatorname{Hom}_\Lambda(P_0, A) & \longrightarrow & \operatorname{Hom}_\Lambda(P_1, A) & \longrightarrow & \operatorname{Tr} X \otimes_\Lambda A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow {\scriptstyle 1 \otimes f} & & \\
0 & \longrightarrow & \operatorname{Hom}_\Lambda(X, B) & \longrightarrow & \operatorname{Hom}_\Lambda(P_0, B) & \longrightarrow & \operatorname{Hom}_\Lambda(P_1, B) & \longrightarrow & \operatorname{Tr} X \otimes_\Lambda B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow {\scriptstyle 1 \otimes g} & & \\
0 & \longrightarrow & \operatorname{Hom}_\Lambda(X, C) & \longrightarrow & \operatorname{Hom}_\Lambda(P_0, C) & \longrightarrow & \operatorname{Hom}_\Lambda(P_1, C) & \longrightarrow & \operatorname{Tr} X \otimes_\Lambda C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \delta^*(X) & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & & & & & & & \\
 & & 0 & & & & & & & &
\end{array}$$

By the Snake lemma we see that $\operatorname{Ker}(1 \otimes f) \simeq \delta^*(X)$. Applying $D(-)$ to the right most column in the above diagram yields the following exact sequence

$$0 \longrightarrow D(\operatorname{Tr} X \otimes_\Lambda C) \longrightarrow D(\operatorname{Tr} X \otimes_\Lambda B) \longrightarrow D(\operatorname{Tr} X \otimes_\Lambda A) \longrightarrow D\,\delta^*(X) \longrightarrow 0$$

Recall that $D(\operatorname{Tr} X \otimes_\Lambda Y) = \operatorname{Hom}_R(\operatorname{Tr} X \otimes_\Lambda Y, I) \simeq \operatorname{Hom}_\Lambda(Y, \operatorname{Hom}_R(\operatorname{Tr} X, I)) = \operatorname{Hom}_\Lambda(Y, D \operatorname{Tr} X)$. Hence we have that the above sequence is isomorphic to the sequence

$$0 \to \operatorname{Hom}_\Lambda(C, D \operatorname{Tr} X) \to \operatorname{Hom}_\Lambda(B, D \operatorname{Tr} X) \to \operatorname{Hom}_\Lambda(A, D \operatorname{Tr} X) \to \delta_*(D \operatorname{Tr} X) \to 0$$

and the claim follows. □

This result has a powerful corollary

Corollary 19
Given the exact sequence $\delta \colon 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ *and* X *in* $\operatorname{mod} \Lambda$*. The following are equivalent*

(a) *For all* $s \colon X \longrightarrow C$ *there exists a* $t \colon X \longrightarrow B$ *such that* $s = gt$.

(b) *For all* $s' \colon A \longrightarrow D \operatorname{Tr} X$ *there exists a* $t' \colon B \longrightarrow D \operatorname{Tr} X$ *such that* $s' = t'f$.

Proof. This follows from the fact that (a) is equivalent to $\delta^*(X) = 0$, and (b) is the same as $\delta_*(D \operatorname{Tr} X) = 0$. □

2.2 Almost Split Sequences

Closely related to the functors $D\,\mathrm{Tr}$ and $\mathrm{Tr}\,D$ are the almost split sequences, which on the literature is often abbreviated to a.s.s. . Let

$$\eta\colon\; 0 \longrightarrow A \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0$$

be an exact sequence, which is not split. We say that η is almost split if the end terms A and C are indecomposable and for every non-isomorphism $h\colon X \longrightarrow C$ with X indecomposable there exists a $s\colon X \longrightarrow B$ such that $h = fs$. The connection with $D\,\mathrm{Tr}$ and the almost split sequences is through the end terms, namely if η is an almost split sequence then $A \simeq D\,\mathrm{Tr}\,C$, meaning that the end terms determine each other.

For a ring Λ, recall that a Λ-homomorphism $f\colon M \longrightarrow N$ is a **split epimorphism** if there exists a Λ-homomorphism $g\colon N \longrightarrow M$ such that $1_N = fg$.

$$\begin{array}{ccc} & & N \\ & \overset{g}{\swarrow} & \Big\| 1_N \\ M & \xrightarrow{f} & N \end{array}$$

Dually we call a Λ-homomorphism $f\colon M \longrightarrow N$ for a **split monomorphism** if there exists a Λ-homomorphism $g\colon N \longrightarrow M$ such that $1_M = gf$.

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ 1_M \Big\| & \underset{g}{\swarrow} & \\ M & & \end{array}$$

One can easily see that if $f\colon M \longrightarrow N$ is a split epimorphism then N is isomorphic to a direct summand in M. If f is a split monomorphism then M is isomorphic to a direct summand in N. Also note that if f is an isomorphisms then it is both a split epi- and monomorphism.

Let Λ be an artin algebra, and let A, B, C, M and N be in $\mathrm{mod}\,\Lambda$. We call a Λ-homomorphism $f\colon B \longrightarrow C$ **right minimal** if every $h \in \mathrm{End}_\Lambda B$ such that

$$\begin{array}{ccc} B & & \\ h \Big\downarrow & \overset{f}{\searrow} & \\ B & \xrightarrow{f} & C \end{array}$$

commutes, is an automorphism. A Λ-homomorphism $f\colon B \longrightarrow C$ is called **right almost split** if f is not a split epimorphism and if for every $h\colon M \longrightarrow C$ which is not a split epimorphism there exist a $g\colon M \longrightarrow B$ such that the following triangle commutes

$$\begin{array}{ccc} & & M \\ & \overset{g}{\swarrow} & \Big\downarrow h \\ B & \xrightarrow{f} & C \end{array}$$

If $f\colon B \longrightarrow C$ is right almost split and right minimal we say that f is **right minimal almost split**. We also have the dual of these statements, namely we call a Λ-homomorphism $g\colon A \longrightarrow B$ for **left minimal** if every $h \in \operatorname{End}_\Lambda B$ such that

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ & \searrow^{g} & \downarrow h \\ & & B \end{array}$$

commutes, is an automorphism. A Λ-homomorphism $g\colon A \longrightarrow B$ is called **left almost split** if g is not a split monomorphism and if for every $h\colon A \longrightarrow N$ which is not a split monomorphism there exist a $g\colon B \longrightarrow N$ such that the following triangle commutes

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ h \downarrow & \swarrow_{h} & \\ N & & \end{array}$$

If $g\colon A \longrightarrow B$ is left almost split and left minimal we say that g is **left minimal almost split**. We immediately see that left minimal is the dual of right minimal, and right almost split is the dual of left almost split.

Let us give an example of a right almost split map.

Example 9

Let Λ be an artin algebra, $\mathfrak{r} \subset \Lambda$ the Jacobson radical and let P be non-simple indecomposable projective. By Nakayama's lemma we have that $\mathfrak{r}P \subset P$ is a proper submodule, thus the inclusion map $i\colon \mathfrak{r}P \longrightarrow P$ is not a split epimorphism. Given $h\colon X \longrightarrow P$ which is not a split epimorphism we must have $\operatorname{Im} h \subseteq \mathfrak{r}P$. This follows from the fact that $\mathfrak{r}P$ is the unique maximal submodule of P. This then shows that h factors through $\mathfrak{r}P$. If P is simple, then $\mathfrak{r}P = 0$. Let $h\colon X \longrightarrow P$. Since P is simple we have that $\operatorname{Im} h = 0$ or $\operatorname{Im} h = P$, if h is not a split epimorphism, then $h = 0$ and it factors trivially through $\mathfrak{r}P$. In any case we have that $\mathfrak{r}P \longrightarrow P$ is a right almost split map. Since $\mathfrak{r}P \longrightarrow P$ is a monomorphism it is right minimal. That is, $\mathfrak{r}P \longrightarrow P$ is minimal right almost split when P is an indecomposable projective.

Dually, we see that the canonical projection $I \longrightarrow I/\operatorname{soc} I$ is minimal left almost split.

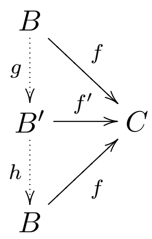
Let $f\colon B \longrightarrow C$ be a right almost split map for some $B, C \in \operatorname{mod} \Lambda$. Then $B \amalg X \xrightarrow{(f\ 0)} C$ is also right almost split for any $X \in \operatorname{mod} \Lambda$. This is rather unfortunate, however by means of minimal maps we are able to get rid of superfluous direct summands and in some sense making the maps unique up to isomorphism.

Proposition 20
Let Λ be an artin algebra.

(a) *Given C in* mod Λ*, if $f\colon B \longrightarrow C$ is minimal right almost split, then f is uniquely determined up to isomorphism.*

(b) *Given A in* mod Λ*, if $g\colon A \longrightarrow B$ is minimal left almost split, then g is unique up to isomorphism.*

Proof. We only prove (a), (b) is proven similarly. Let $f\colon B \longrightarrow C$ and $f'\colon B' \longrightarrow C$ be minimal right almost split maps.



Since f is not a split epimorphism and f' is right almost split we get $g\colon B \longrightarrow B'$ such that $f = f'g$. Similarly, since f' is not a split epimorphism and f is right almost split we get $h\colon B' \longrightarrow B$ such that $f' = fh$. This then yields $f = f(hg)$ and $f' = f'(gh)$, and hence hg and gh are isomorphisms, since f and f' are both right minimal. This yields that g is an isomorphism, because gh isomorphism implies that g is an epimorphism, and hg isomorphism implies that g is a monomorphism. □

An intrinsic property of the almost split maps is that for a left almost split map the domain is indecomposable and for a right almost split map the codomain is indecomposable.

Lemma 21
Let A, B and C be in mod Λ

(a) *If $f\colon B \longrightarrow C$ is right almost split, then C is indecomposable. Moreover, if in addition C is non-projective then f is an epimorphism.*

(b) *If $g\colon A \longrightarrow B$ is left almost split, then A is indecomposable. Moreover, if in addition A is non-injective then g is a monomorphism.*

Proof. Here too, we only prove (a), (b) follows by duality. Suppose $C = C_1 \amalg C_2$ with $C_i \neq 0$ for $i = 1, 2$. Let $u_i\colon C \longrightarrow C_i$ denote the corresponding inclusion maps for $i = 1, 2$. Since C has a non-trivial decomposition, u_i is not a split epimorphism for $i = 1, 2$. Thus there exists $h_i\colon C_i \longrightarrow B$ such that $u_i = fh_i$ for $i = 1, 2$. This yields a contradiction since we get the following commutative triangle

$$\begin{array}{ccc} & & C \\ & \overset{(h_1\ h_2)}{\swarrow} & \| \\ B & \xrightarrow{f} & C \end{array}$$

Thus C must be indecomposable. Assume that C is indecomposable non-projective, and let $p\colon P \longrightarrow C$ be the projective cover. Since C is non-projective, p is not a split epimorphism, thus p factors through f.

$$\begin{array}{ccc} & & P \\ & \swarrow & \downarrow p \\ B & \xrightarrow{f} & C \end{array}$$

Since f is last in a composition which is an epimorphism, f itself must be an epimorphism. □

There is an important characterization of the almost split maps, which makes it somewhat easier to determine whether a map is almost split or not.

Proposition 22

(a) *Let $g\colon A \longrightarrow B$ be a homomorphism in* mod Λ. *The following are equivalent:*

- *(i) g is left almost split.*
- *(ii) g is not a split monomorphism, A is indecomposable and if Y is indecomposable and $Y \not\simeq A$, then for every $s\colon A \longrightarrow Y$ there exists a $h\colon B \longrightarrow Y$ such that the following triangle commutes*

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ s \downarrow & \swarrow h & \\ Y & & \end{array}$$

(b) *Let $f\colon B \longrightarrow C$ be a homomorphism in* mod Λ. *The following are equivalent:*

- *(i) f is right almost split.*
- *(ii) f is not a split epimorphism, C is indecomposable and if X is indecomposable and $X \not\simeq C$, then for every $t\colon X \longrightarrow C$ there exists a $h\colon X \longrightarrow B$ such that the following triangle commutes*

$$\begin{array}{ccc} & & X \\ & \overset{h}{\swarrow} & \downarrow t \\ B & \xrightarrow{f} & C \end{array}$$

Proof. Notice that we only need to prove one of the to equivalence, since the other one follows by duality.

(b): If $f\colon B \longrightarrow C$ is right almost split, then by definition f is not a split epimorphism, and by Lemma 21 we have that C is indecomposable. Let $X \not\simeq C$ be indecomposable. If $t\colon X \longrightarrow C$ is a split epimorphism, then C is a direct summand in X, thus $X \simeq C$ which is impossible. Thus every such t factors through f.

Now we prove the converse. Let $t\colon Y \longrightarrow C$ be a homomorphism which is not a split epimorphism. Suppose that $Y = \coprod_{i=1}^{n} Y_i$ with Y_i indecomposable for $1 \leq i \leq n$. Then non of the compositions $Y_i \xrightarrow{q_i} Y \xrightarrow{t} C$ are split epimorphisms, otherwise g would be a split epimorphism. So if we can lift all these compositions to B, t will factor through f. Hence we can restrict to the case where Y is indecomposable. If $Y \not\simeq C$, then by assumption t factors through f. Suppose that $Y \simeq C$. Since $t\colon Y \longrightarrow C$ is not a split epimorphism, t cannot be an epimorphism, otherwise t would be an isomorphism by a length argument. Hence $\operatorname{Im} t \subsetneq C$. Let $\operatorname{Im} t = \coprod_{i=1}^{m} X_i$, where X_i is indecomposable. Then the inclusion $X_i \longrightarrow \operatorname{Im} t \longrightarrow C$ is not a split epimorphism, since $l(X_i) < l(C)$. So $X_i \longrightarrow C$ factors through f, and therefore $\operatorname{Im} t \longrightarrow C$ factors through f, which ultimately means that t factors through f, this finishes the proof. □

We now come to a powerful proposition which ties almost split maps to the functors $D\operatorname{Tr}$ and $\operatorname{Tr} D$.

Proposition 23

(a) *Let $g\colon A \longrightarrow B$ be a minimal left almost split morphism and A non-injective. Then the exact sequence $0 \longrightarrow A \xrightarrow{g} B \xrightarrow{f} \operatorname{Coker} g \longrightarrow 0$ has the following properties*

(i) $\operatorname{Coker} g \simeq \operatorname{Tr} D\, A$.

(ii) *f is minimal right almost split.*

(b) *Let $f\colon B \longrightarrow C$ be a minimal right almost split morphism with C non-projective. Then the exact sequence $0 \longrightarrow \operatorname{Ker} f \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0$ has the following properties*

(i) $\operatorname{Ker} f \simeq D\operatorname{Tr} C$.

(ii) *g is minimal left almost split.* □

In order to prove this we will need the following lemma.

Lemma 24
Let

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{g} & B & \xrightarrow{f} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} & & \\ 0 & \longrightarrow & A & \xrightarrow{g} & B & \xrightarrow{f} & C & \longrightarrow & 0 \end{array}$$

be a commutative diagram with exact, non split rows.

(a) If A is indecomposable and t is an isomorphism, then r and s are isomorphisms. In particular, if $t = 1_C$ then f is right minimal.

(b) If C is indecomposable and r is an isomorphism, then s and t are isomorphisms. In particular, if $r = 1_A$ then g is left minimal.

Proof. (a) Without loss of generality we may suppose that $t = 1_C$. If r is not an isomorphism, it has to be nilpotent since $\operatorname{End}_\Lambda A$ is local. Therefore there exists $n \in \mathbb{N}$ so that $r^n = 0$. Thus $s^n g = g r^n = 0$, and hence s^n factors through the cokernel of g, C, that is there exists a unique $h\colon C \longrightarrow B$ such that $hf = s^n$. Since $f = t^n f = f s^n$, we get $fhf = f$. This means that $hf = 1_C$ since f is an epimorphism which shows that $0 \longrightarrow A \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0$ is a split exact sequence. This is impossible, and thus r is an isomorphism.

(a) follows by duality. ☐

With this lemma in place we move on to the proposition.

Proof of Proposition 23. Let $f\colon B \longrightarrow C$ be a minimal right almost split map with C not projective. We then have the following exact sequence

$$\eta\colon\ 0 \longrightarrow \operatorname{Ker} f \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0$$

Suppose that $\operatorname{Ker} f$ has a non-trivial decomposition into indecomposable Λ-modules, i.e. $\operatorname{Ker} f = K_1 \amalg \cdots \amalg K_n$ with $K_1, \ldots, K_n \neq 0$. The sequence above is not split exact, since this will mean that f is a split epimorphism and this will violate the fact that f is right almost split. Hence $g\colon \operatorname{Ker} f \longrightarrow B$ is not a split monomorphism. This means that there is some $1 \leq i \leq n$ such that the projection map $p = p_i\colon \operatorname{Ker} f \longrightarrow K_i$ does not factor through g. Hence we get the following commutative pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ker} f & \xrightarrow{g} & B & \xrightarrow{f} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle t} & & \| & & \\ 0 & \longrightarrow & K_i & \longrightarrow & E & \xrightarrow{h} & C & \longrightarrow & 0 \end{array}$$

We must have that h is not a split epimorphism, otherwise the lower sequence would be split exact and p_i would factor through B. Thus we get the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ker} f & \xrightarrow{g} & B & \xrightarrow{f} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle t} & & \| & & \\
0 & \longrightarrow & K_i & \longrightarrow & E & \xrightarrow{h} & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle s} & & \| & & \\
0 & \longrightarrow & \operatorname{Ker} f & \xrightarrow{g} & B & \xrightarrow{f} & C & \longrightarrow & 0
\end{array}$$

Where s comes from the fact that f is right almost split and h is not a split epimorphism. The map q can be found such that every square commutes. Since f is right minimal we get that st is an automorphism and consequently qp, too, is an automorphism. This in turn implies that p is a monomorphism, since p was the canonical projection it also is an epimorphism, thus it is an isomorphism. Thus $\operatorname{Ker} f$ must be indecomposable.

We now want to show that g is left almost split. Note that η is not split exact, hence $\operatorname{Ker} f$ cannot be injective. Let Y be an indecomposable Λ-module. If Y is injective then $Y \not\simeq \operatorname{Ker} f$ and all maps $t\colon \operatorname{Ker} f \longrightarrow Y$ extend to B. On the other hand if Y is not injective and not isomorphic to $D \operatorname{Tr} C$ we have that $\operatorname{Tr} D Y$ is not isomorphic to C. Since f is right almost split every $t'\colon \operatorname{Tr} D Y \longrightarrow C$ factors through f. By Corollary 19 we get that every map $t\colon \operatorname{Ker} f \longrightarrow Y$ extends to B. Thus we see that if $\operatorname{Ker} f \not\simeq D \operatorname{Tr} C$ then η is split exact, this is impossible. Therefore we must have that $\operatorname{Ker} f \simeq D \operatorname{Tr} C$ and by Proposition 22 we have that g is left almost split. That g is left minimal follows from the fact that C is indecomposable, and f not being a split epimorphism. □

A Λ-homomorphism $f\colon A \longrightarrow B$ is said to be irreducible if f is not a split monomorphism nor a split epimorphism, and if the following triangle commutes

$$\begin{array}{ccc}
A & \xrightarrow{\quad f \quad} & B \\
{\scriptstyle t}\searrow & & \nearrow{\scriptstyle s} \\
 & M &
\end{array}$$

for some $t\colon A \longrightarrow M$ and $s\colon M \longrightarrow B$, then t is a split monomorphism or s is a split epimorphism. It is easily seen that this notion is self dual.

Let $f\colon B \longrightarrow C$ be irreducible in mod Λ. We then have the following commutative triangle

$$\begin{array}{ccc}
B & \xrightarrow{\quad f \quad} & C \\
{\scriptstyle t}\searrow & & \nearrow{\scriptstyle s} \\
 & B/\operatorname{Ker} f &
\end{array}$$

Since f is irreducible we must have that t is a split monomorphism or that s is a split epimorphism. However, if t is a split monomorphism then f is a monomorphism. On the other hand, if s is a split epimorphism then f is a epimorphism. Suppose that there is some $h \in \operatorname{End}_\Lambda C$ such that $f = hf$. Since f is not a split monomorphism, we get that h is a split epimorphism, and thus h is an automorphism since C has finite length. In a similar fashion one can show that f is right minimal. We collect our findings in the following Lemma.

Lemma 25
Let $f\colon B \longrightarrow C$ be irreducible in mod Λ.

(a) f is either a monomorphism or an epimorphism.

(b) f is both left minimal and right minimal. □

Lemma 26
Let $\eta\colon 0 \longrightarrow A \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0$ be an exact sequence in mod Λ

(a) The map $g\colon A \longrightarrow B$ is irreducible if and only if, η is nonsplit, and for every homomorphism $h\colon M \longrightarrow C$ there exists $s\colon M \longrightarrow B$ such that $h = fs$ or $t\colon B \longrightarrow M$ such that $f = ht$.

(b) The map $f\colon B \longrightarrow C$ is irreducible if and only if, for every homomorphism $h\colon M \longrightarrow C$ there exists $s\colon M \longrightarrow B$ such that $h = fs$ or $t\colon B \longrightarrow M$ such that $f = ht$.

Proof. (a): Suppose g is irreducible. Then η is not split exact since g is not a split monomorphism. Let $h\colon M \longrightarrow C$, and consider the following pullback diagram

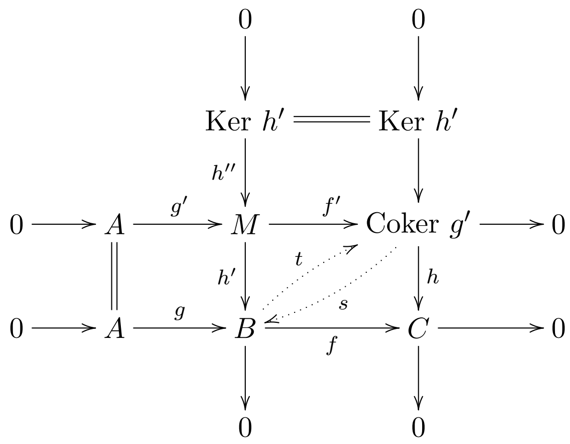
$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{g'} & E & \xrightarrow{f'} & M & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A & \xrightarrow{g} & B & \xrightarrow{f} & C & \longrightarrow & 0 \end{array}$$

Since g is irreducible we must have either that g' is a split monomorphism or that h' is a split epimorphism. If g' is a split monomorphism then there exists a homomorphism $f''\colon M \longrightarrow E$ such that $f'f'' = 1_M$. We then get that $h = h(f'f'') = (hf')f'' = f(h'f'')$, that is $s = h'f''$. On the other hand if h' is a split epimorphism, then there exists $h''\colon B \longrightarrow E$ such that $h'h'' = 1_B$, which yields $f = f(h'h'') = (fh')h'' = h(f'h'')$, and thus we have $t = f'h''$.

Now suppose that η is non-split and that for every homomorphism $h\colon M \longrightarrow C$ there exists $s\colon M \longrightarrow B$ such that $h = fs$ or $t\colon B \longrightarrow M$ such that $f = ht$. Since η is not split, we get that g is not a split epimorphism nor a split monomorphism. Suppose that we have the following factorization of g

$$\begin{array}{ccc} A & \xrightarrow{\quad g \quad} & B \\ & {\scriptstyle g'}\searrow \quad \nearrow{\scriptstyle h'} & \\ & M & \end{array}$$

Since g is a monomorphism we must have that g' also is a monomorphism. We then get the following commutative diagram



If there exists a $t\colon B\longrightarrow \text{Coker } g'$ as above, then $f(h'-tf')=0$, and hence there exists a $t'\colon M\longrightarrow A$ such that $h'=tf'+gt'$. This then yields that $t'g'=1_A$ since $g=h'g'=tf'g'+gt'g'=gt'g'$. In other words g' is a split monomorphism. Conversely, if there exists a $s\colon \text{Coker } g'\longrightarrow B$ as above, then h is an epimorphism, then so is h' by Snake lemma. Similarly as above h'' is a split monomorphism, and followingly h' is a split epimorphism. Thus we have shown that g is irreducible.

(b) follows by similar arguments as in (a). □

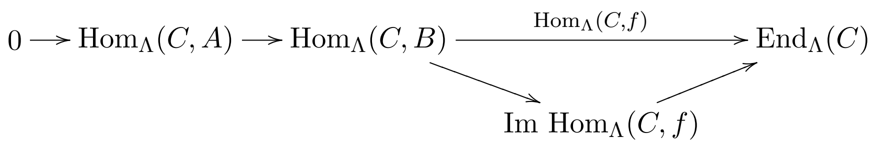
This Lemma has an important corollary.

Corollary 27

(a) If $g\colon A\longrightarrow B$ is an irreducible monomorphism, then $\text{End}_\Lambda \text{Coker } g$ *is local, in particular* $\text{Coker } g$ *is indecomposable.*

(b) If $f\colon B\longrightarrow C$ is an irreducible epimorphism, then $\text{End}_\Lambda \text{Ker } f$ *is local, in particular* $\text{Ker } f$ *is indecomposable.*

Proof. The proof of (b) is done in a similar fashion as the proof of (a), so we only give a proof for (a). Let $C=\text{Coker } g$ with $f\colon B\longrightarrow C$. We then have the exact non-split sequence $\eta\colon 0\longrightarrow A\xrightarrow{g} B\xrightarrow{f} C\longrightarrow 0$. Applying the functor $\text{Hom}_\Lambda(C,-)$ to the above sequence we get the following exact sequence of $\text{End}_\Lambda(C)^{\text{op}}$-modules



Therefor we get that $\text{Im Hom}_\Lambda(C,f)\subseteq \text{End}_\Lambda(C)$ is a right ideal. If $h\in \text{End}_\Lambda(C)$ is a non-isomorphism, then h is not an epimorphism. By Lemma 26 we must have that there is some $s\colon C\longrightarrow B$ such that $h=fs$ or some map $t\colon B\longrightarrow C$ such that

$f = ht$. If we are in the latter case, then we arrive at h being an epimorphism, this then cannot be the case. Therefor there is some $s\colon C \longrightarrow B$ such that $h = fs$, that is $h \in \operatorname{Im} \operatorname{Hom}_\Lambda(C, f)$. This means that the non-isomorphism in $\operatorname{End}_\Lambda(C)$ are a sub set of $\operatorname{Im} \operatorname{Hom}_\Lambda(C, f)$. If $h \in \operatorname{End}_\Lambda(C)$ is an isomorphism such that $h \in \operatorname{Im} \operatorname{Hom}_\Lambda(C, f)$, then η is split exact. Therefore $\operatorname{Im} \operatorname{Hom}_\Lambda(C, f)$ is precisely the non-isomorphisms of $\operatorname{End}_\Lambda(C)$. We now want to show that $\operatorname{Im} \operatorname{Hom}_\Lambda(C, f) \subseteq \operatorname{End}_\Lambda(C)$ is an ideal. Let $h \in \operatorname{Im} \operatorname{Hom}_\Lambda(C, f)$ and $h' \in \operatorname{End}_\Lambda(C)$. We have that $l(\operatorname{Im} h'h) \leq l(\operatorname{Im} h) < l(C)$, since h is not an isomorphism and $\operatorname{Im} h'h$ is a factor of $\operatorname{Im} h$. Thus $h'h$ is not an isomorphism and therefore $h'h \in \operatorname{Im} \operatorname{Hom}_\Lambda(C, f)$. Since the non-units of $\operatorname{End}_\Lambda(C)$ form an ideal, $\operatorname{End}_\Lambda(C)$ is a local ring, this in turn yields that C is indecomposable. □

We shall see that there is an characterization of the irreducible maps through the minimal left and right almost split maps.

Theorem 28

(a) Let $g\colon A \longrightarrow B$ be minimal left almost split.

(i) Then g is irreducible.

(ii) Furthermore, a homomorphism $g'\colon A \longrightarrow B'$ is irreducible if and only if $B' \neq 0$ and there exists a homomorphism $g''\colon A \longrightarrow B'$ such that $\begin{pmatrix} g' \\ g'' \end{pmatrix}\colon A \longrightarrow B' \amalg B''$ is a minimal left almost split morphism with $B \simeq B' \amalg B''$.

(b) Let $f\colon B \longrightarrow C$ be minimal right almost split.

(i) Then f is irreducible.

(ii) Furthermore, a homomorphism $f'\colon B' \longrightarrow C$ is irreducible if and only if $B' \neq 0$ and there exists a homomorphism $f''\colon B'' \longrightarrow C$ such that $\begin{pmatrix} f' & f'' \end{pmatrix}\colon B' \amalg B'' \longrightarrow C$ is a minimal right almost split morphism with $B \simeq B' \amalg B''$.

Proof. (a): Suppose that $g\colon A \longrightarrow B$ is a minimal left almost split map. By definition, g is not a split monomorphism, and by Lemma 21 we have that A is indecomposable, and therefore g is not an isomorphism, thus g cannot be a split epimorphism either. Assume that $g = st$ for some $t\colon A \longrightarrow M$ and $s\colon M \longrightarrow B$. If t is a split monomorphism then we are done, therefore suppose that t is not a split monomorphism. Since g is left almost split there exists a $h\colon B \longrightarrow M$ such that $t = hg$. Thus $g = st = shg$, since g is left minimal sh is an isomorphism, and so s is a split epimorphism. This then proves that g is irreducible.

Let $g'\colon A \longrightarrow B'$ be an irreducible morphism. We must have $B' \neq 0$. Moreover, g' is not a split monomorphism, so there exists a $h\colon B \longrightarrow B'$ such that the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ & \searrow^{g'} & \downarrow{\scriptstyle h} \\ & & B' \end{array}$$

Since g' is irreducible and g is left almost split, we get that h is a split epimorphism. Thus B' is a direct summand in B, that is $B \simeq B' \amalg B''$ where $B'' = \text{Ker } h$. Hence we have that there is a map $h'\colon B \longrightarrow B''$ such that $\left(\begin{smallmatrix} h \\ h' \end{smallmatrix}\right) : B \longrightarrow B' \amalg B''$ is an isomorphism. Consequently, we get that $\left(\begin{smallmatrix} h \\ h' \end{smallmatrix}\right) g = \left(\begin{smallmatrix} g' \\ h'g \end{smallmatrix}\right) : A \longrightarrow B' \amalg B''$ is minimal left almost split. Next assume that $\left(\begin{smallmatrix} g \\ g' \end{smallmatrix}\right) : A \longrightarrow B' \amalg B''$ is minimal left almost split with $B \simeq B' \amalg B''$. Suppose that $g = st$ for some $t\colon A \longrightarrow N$ and $s\colon N \longrightarrow B'$ with t not a split monomorphism. We now get that there exists $\left(\begin{smallmatrix} p & q \end{smallmatrix}\right) : B' \amalg B'' \longrightarrow N$ such that $s = \left(\begin{smallmatrix} p & q \end{smallmatrix}\right)\left(\begin{smallmatrix} g \\ g' \end{smallmatrix}\right)$, because $\left(\begin{smallmatrix} g \\ g' \end{smallmatrix}\right)$ is left almost split. Whence we have the following commutative diagram

$$\begin{array}{ccccc} & & A & & \\ & \swarrow^{\left(\begin{smallmatrix} g \\ g' \end{smallmatrix}\right)} & \downarrow{\scriptstyle \left(\begin{smallmatrix} t \\ g' \end{smallmatrix}\right)} & \searrow^{\left(\begin{smallmatrix} g \\ g' \end{smallmatrix}\right)} & \\ B' \amalg B'' & \xrightarrow[\left(\begin{smallmatrix} p & q \\ 0 & 1_{B''} \end{smallmatrix}\right)]{} & N \amalg B'' & \xrightarrow[\left(\begin{smallmatrix} s & 0 \\ 0 & 1_{B''} \end{smallmatrix}\right)]{} & B' \amalg B'' \end{array}$$

Thus we see that $\left(\begin{smallmatrix} sp & sq \\ 0 & 1_{B''} \end{smallmatrix}\right)$ is an automorphism by the minimality of $\left(\begin{smallmatrix} g \\ g' \end{smallmatrix}\right)$. So we see that s is a split epimorphism since sp is an automorphism. This proves that g is irreducible. The proof of (b) follows from (a) by duality. □

We are now finally able to define almost split sequences. Let η be the following short exact sequence $0 \longrightarrow A \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0$. If g is minimal left almost split and g minimal right almost split then η is called an **almost split sequence**. From our preceding observations we must have that the end terms are indecomposable. Moreover, η is non-split since g is not a split monomorphism, or f is a split epimorphism. We therefore see that A cannot be injective, and also B not projective.

Now we come to the *pièce de résistance*, the characterization of almost split sequences.

Theorem 29
Let $\eta\colon 0 \longrightarrow A \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0$ be a short exact sequence. The following are equivalent:

(a) η is almost split.

(b) g is minimal left almost split.

(c) f is minimal right almost split.

(d) *C is indecomposable and g is left almost split.*

(e) *A is indecomposable and f is right almost split.*

(f) *$C \simeq \operatorname{Tr} D\, A$ and g is left almost split.*

(g) *$A \simeq D \operatorname{Tr} C$ and f is right almost split.*

(h) *A and C are indecomposable, and f and g are irreducible.*

Proof. By definition we have that (a) implies (b) and (c). From Proposition 23 we see that we also have the converse. Lemma 21 and Proposition 23 proves (b) implies (d), and (c) implies (e). By Lemma 24 we get that (d) implies (b), and (d) implies (c). From Proposition 23 and Lemma 24 we see that (d) implies (f), and (e) implies (g). The converse follows from Lemma 21. That (a) implies (h) comes from Lemma 21 and Theorem 28. We are now left with only one implication to finish the proof, namely that (h) implies (e). Suppose that $h\colon X \longrightarrow C$ with X indecomposable and $X \not\simeq C$. Then by Lemma 26 we have that there is $s\colon X \longrightarrow B$ such that $h = fs$, and then we are done, or there is a $t\colon B \longrightarrow X$ so that $f = ht$. In the latter case we must have that t is a split monomorphism, since f is irreducible and $X \not\simeq C$. Since X is indecomposable, we must have that t is an isomorphism, and then $h = ft^{-1}$, and hence f is right almost split by Proposition 22. This completes the proof. □

Having established some basic properties of the almost split sequences, we now need to show their existence.

Theorem 30
Let Λ be an artin algebra, and let $A, C \in \operatorname{mod} \Lambda$

(a) *If C is an indecomposable non-projective Λ-module, then there is an almost split sequence $0 \longrightarrow A \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0$.*

(b) *If A is an indecomposable non-injective Λ-module, then there is an almost split sequence $0 \longrightarrow A \xrightarrow{g} B \xrightarrow{f} C \longrightarrow 0$.*

Proof. (a) We will show that there is an exact sequence $0 \longrightarrow D \operatorname{Tr} C \longrightarrow B \xrightarrow{f} C \longrightarrow 0$ where f is right almost split. Since C is indecomposable and not projective, we have $\operatorname{Tr} D\,(D \operatorname{Tr} C) \simeq \operatorname{Tr} D^2 \operatorname{Tr} C \simeq \operatorname{Tr}^2 C \simeq C$. If $D \operatorname{Tr} C$ is injective, then $D\,(D \operatorname{Tr} C) \simeq \operatorname{Tr} C$ is projective, and $\operatorname{Tr}^2 C = 0$, which is impossible. Thus we know that $\operatorname{Ext}^1_\Lambda(-, D \operatorname{Tr} C) \neq 0$. That is, there is a $V \in \operatorname{mod} \Lambda$ so that $\eta'\colon 0 \longrightarrow D \operatorname{Tr} C \xrightarrow{g'} E' \xrightarrow{f'} V \longrightarrow 0$ is not split exact. Notice that if every morphism $C \longrightarrow V$ factors through $f'\colon B \longrightarrow V$, then every morphism $D \operatorname{Tr} C \longrightarrow D \operatorname{Tr} C$ factors through $g'\colon D \operatorname{Tr} C \longrightarrow B$ by Corollary 19. This would mean that g' is a split monomorphism, contradicting the assumption that η' is non-split.

Now let $\Gamma = \operatorname{End}_\Lambda(C)^{\mathrm{op}}$, then we have the following exact sequence of Γ-modules

$$\operatorname{Hom}_\Lambda(C,B) \xrightarrow{\operatorname{Hom}_\Lambda(C,f')} \operatorname{Hom}_\Lambda(C,V) \longrightarrow \operatorname{Coker} \operatorname{Hom}_\Lambda(C,f') \longrightarrow 0$$

where $\operatorname{Coker} \operatorname{Hom}_\Lambda(C,f') \neq 0$. Therefore we may find a morphism $h\colon C \longrightarrow V$ whose image in $\operatorname{Coker} \operatorname{Hom}_\Lambda(C,f')$ generates a simple Γ-module. Now consider the pullback diagram

$$\begin{array}{lccccccccc}
\eta = p'(h)\colon & 0 & \longrightarrow & D\operatorname{Tr}C & \xrightarrow{g} & B & \xrightarrow{f} & C & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow{\scriptstyle h} & & \\
\eta'\colon & 0 & \longrightarrow & D\operatorname{Tr}C & \xrightarrow{g'} & E & \xrightarrow{f'} & V & \longrightarrow & 0
\end{array}$$

We shall now show that η is an almost split sequence. Clearly η is not split exact, otherwise h would factor through f', which would mean that $h = 0$ in $\operatorname{Coker} \operatorname{Hom}_\Lambda(C,f')$. We now show that f is right almost split. Let $s\colon X \longrightarrow C$ be a map which is not a split epimorphism. By taking the pullback of η along s we have the following diagram

$$\begin{array}{lccccccccc}
\eta \cdot s\colon & 0 & \longrightarrow & D\operatorname{Tr}C & \xrightarrow{g''} & Y & \xrightarrow{f''} & X & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow{\scriptstyle s} & & \\
\eta\colon & 0 & \longrightarrow & D\operatorname{Tr}C & \xrightarrow{g} & B & \xrightarrow{f} & C & \longrightarrow & 0
\end{array}$$

We see that s factors through $f \Leftrightarrow \eta{\cdot}s$ is split exact $\Leftrightarrow$ every $t'\colon D\operatorname{Tr}C \longrightarrow D\operatorname{Tr}C$ factors through g''. By Corollary 19 this is the same as every $t\colon C \longrightarrow X$ factors through f'' in $\eta \cdot s$. Let $t\colon C \longrightarrow X$, we shall now show that t factors through f''. Consider the following commutative diagram:

$$\begin{array}{lccccccccc}
\eta'' = \eta(st)\colon & 0 & \longrightarrow & D\operatorname{Tr}C & \longrightarrow & Y' & \longrightarrow & C & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow{\scriptstyle t} & & \\
\eta \cdot s\colon & 0 & \longrightarrow & D\operatorname{Tr}C & \xrightarrow{g''} & Y & \xrightarrow{f''} & X & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow{\scriptstyle s} & & \\
\eta = \eta' \cdot h\colon & 0 & \longrightarrow & D\operatorname{Tr}C & \xrightarrow{g} & B & \xrightarrow{f} & C & \longrightarrow & 0 \\
 & & & \| & & \downarrow & & \downarrow{\scriptstyle h} & & \\
\eta'\colon & 0 & \longrightarrow & D\operatorname{Tr}C & \xrightarrow{g'} & E & \xrightarrow{f'} & C & \longrightarrow & 0
\end{array}$$

Since $\operatorname{End}_\Lambda(C)$ is a local ring and $st \in \operatorname{End}_\Lambda(C)^{\mathrm{op}}$ is not an isomorphism, we must have that $st \in \operatorname{rad}\operatorname{End}_\Lambda(C)^{\mathrm{op}} = \operatorname{rad}\Gamma$. Then $\eta'' = \eta(st) = p(h)(st) = (st)\cdot p(h) \in (st)\cdot S = 0$, where S is the simple Γ-module generated by $p(h)$. This means that η'' is split exact, which means that t factors through f'' (for all t), and thus s factors through f, and f is right almost split. Hence η is an almost split sequence by Theorem 29 part (g).

(b) follows from the fact that $0 \to A \to B \to C \to 0$ is almost split if and only if $0 \to D\,C \to D\,B \to D\,A \to 0$ is almost split. □

The next proposition asserts that these sequences are determined by their end terms.

Proposition 31
Let $0 \to A \xrightarrow{g} B \xrightarrow{f} C \to 0$ *and* $0 \to A' \xrightarrow{g'} B' \xrightarrow{f'} C' \to 0$ *be almost split sequences. The following are equivalent:*

(a) The two sequences are isomorphic.

(b) $A \simeq A'$.

(c) $C \simeq C'$

Proof. (b) $\Leftrightarrow$ (c) follows from the fact that $D\,\mathrm{Tr}\,C \simeq A$ and $D\,\mathrm{Tr}\,C' \simeq A'$.

(c) $\Rightarrow$ (a): Suppose that $h\colon C \to C'$ is an isomorphism. It then follows that $hf\colon B \to C'$ is minimal right almost split, since f is minimal right almost split. By Proposition 20 we get that there is an isomorphism $h'\colon B \to B'$ such that $hf = f'h'$ since $f'\colon B' \to C'$ is minimal right almost split. The opposite implication is obvious. □

As we shall see, the almost split sequences tie the irreducible maps going out from a module to the irreducible maps going into it.

Proposition 32

(a) Let C *be an indecomposable non-projective module in* mod Λ. *There exists an irreducible morphism* $f\colon M \to C$ *if and only if there exists an irreducible morphism* $g\colon D\,\mathrm{Tr}\,C \to M$.

(b) Let A *be an indecomposable non-injective module in* mod Λ. *There exists an irreducible morphism* $g\colon A \to N$ *if and only if there exists an irreducible morphism* $f\colon N \to \mathrm{Tr}\,D\,A$.

Proof. (a): Suppose $f\colon M \to C$ is irreducible. Theorem 28 says that there is a $f'\colon M' \to C$ such that $(\,f\ f'\,)\colon M \amalg M' \to C$ is minimal right almost split. Since C is not projective, thus $(\,f\ f'\,)$ is an epimorphism. Let $A = \mathrm{Ker}\,(\,f\ f'\,)$, then the following sequence is almost split by Theorem 29

$$0 \longrightarrow A \xrightarrow{\binom{g}{g'}} M \amalg M' \xrightarrow{(f\ f')} C \longrightarrow 0$$

Here $g\colon A \to M$ is irreducible, and $A \simeq D\,\mathrm{Tr}\,C$. Note that an irreducible map composed with an isomorphism is still irreducible. The converse statement is proved similarly.

(b) follows by duality. □

We end this section with a proposition which, in some cases, makes it easy to verify whether a short exact sequence is almost split or not.

Proposition 33
Let C be an indecomposable module with $\underline{\mathrm{End}}_\Lambda(C)$ a division ring. Then the following are equivalent for a short exact sequence $\eta\colon\ 0 \longrightarrow D\,\mathrm{Tr}\,C \longrightarrow B \longrightarrow C \longrightarrow 0$.

(a) η is almost split.

(b) η does not split.

(c) $B \not\simeq D\,\mathrm{Tr}\,C \amalg C$. □

The dual statement also holds, namely:

Proposition 34
Let C be an indecomposable module with $\overline{\mathrm{End}}_\Lambda(A)$ a division ring. Then the following are equivalent for a short exact sequence $\eta\colon\ 0 \longrightarrow A \longrightarrow B \longrightarrow \mathrm{Tr}\,D\,A \longrightarrow 0$.

(a) η is almost split.

(b) η does not split.

(c) $B \not\simeq A \amalg \mathrm{Tr}\,D\,A$. □

2.3 The Coxeter Transformation

Let us start with introducing the **Grothendieck group** of an artin algebra Λ. Let $\mathcal{F}$ be the free abelian group with basis the isomorphism classes of finitely generated Λ-modules. We denote the isomorphism class of a module $M \in \mathrm{mod}\,\Lambda$ by $[M]$. Furthermore, let $\mathcal{K}$ be the subgroup of $\mathcal{F}$ generated by elements of the form $[M]-[L]-[N]$ whenever $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is exact in mod Λ. Let $K_0(\mathrm{mod}\,\Lambda) = \mathcal{F}/\mathcal{K}$ be the Grothendieck group of Λ. We first show that $K_0 = K_0(\mathrm{mod}\,\Lambda)$ is free. Let $[S_1], \dots, [S_n]$ be a complete list of isomorphism classes of simple Λ-modules, and let $G = \langle [S_1], \dots, [S_n] \rangle$, that is the subgroup generated by the isomorphism classes of the simple Λ-modules. We define $\varphi\colon G \longrightarrow K_0$ by $\varphi([S_i]) = [S_i]$. We now want to give a homomorphism going the other direction, however, first note that given a module M in mod Λ and a composition series $0 = M_0 \subset M_1 \subset \cdots \subset M_{r-1} \subset M_r = M$ of M, meaning that M_{i+1}/M_i is a simple Λ-module, we have the exact sequence $0 \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow M_i/M_{i-1} \longrightarrow 0$, and consequently we get $[M] = [M_r/M_{r-1}] + [M_{r-1}] = \cdots = \sum_{i=1}^{r} [M_i/M_{i-1}] = \sum_{i=1}^{n} m_{S_i}[S_i]$, where m_{S_i} is the number of times S_i occurs in the composition series of M. Hence define $\psi\colon K_0 \longrightarrow G$ by $\psi([M]) = \sum_{i=1}^{n} m_{S_i}[S_i]$. Obviously,

$\psi\varphi = 1_G$ and $\varphi\psi = 1_{K_0}$, so $G = K_0$. We have now shown that K_0 is a free abelian group of rank n, and hence we are essentially dealing with $\mathbb{Z}^n$.

If we require that Λ is of finite global dimension we get that the isomorphism classes of indecomposable projective Λ-modules generate K_0. Let P_i be the projective cover of S_i in $\{S_1, \ldots, S_n\}$ as above. Then $P_1, \ldots, P_n$ are non-isomorphic indecomposable projective Λ-modules. We want to show that $\{[P_1], \ldots, [P_n]\}$ is a basis for K_0. Since K_0 has rank m, then we only need to show that K_0 is generated by $\{[P_1], \ldots, [P_n]\}$. Since Λ has finite global dimension, there is a finite projective resolution of a simple Λ-module S, $0 \longrightarrow Q_t \longrightarrow \ldots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow S \longrightarrow 0$. Thus, $[S] = \sum_{i=0}^{t} (-1)^i [Q_i]$, where each $Q_i \simeq \coprod_j P_j$, and the claim follows. The dual is also true, i.e. the injective envelopes of the isomorphism classes of simples generate K_0, this realization is achieved by similar arguments as above. Inspired by the above, we define the **Coxeter transformation** $\Phi_\Lambda \colon K_0 \longrightarrow K_0$ by $\Phi_\Lambda([P_i]) = -[I_i]$. This is obviously an isomorphism, since it takes a basis to another basis. Moreover, since $I_i \simeq D\,P_i^*$, we see that $\Phi_\Lambda[P_i] = -[D\,P_i^*]$.

Suppose $P_1 \xrightarrow{p_1} P_0 \longrightarrow M \longrightarrow 0$ is a minimal projective presentation of M, a non projective module in mod Λ. This means that $0 \longrightarrow \operatorname{Ker} p_1 \longrightarrow P_1 \xrightarrow{p_1} P_0 \longrightarrow M \longrightarrow 0$ is exact. Thus $[M] - [\operatorname{Ker} p_1] = [P_0] - [P_1]$ in K_0. We now use the Coxeter transformation and get $\Phi_\Lambda[M] - \Phi_\Lambda[\operatorname{Ker} p_1] = [D\,P_1^*] - [D\,P_0^*]$. We also have the exact sequence $0 \longrightarrow D\operatorname{Tr} M \longrightarrow D\,P_1^* \longrightarrow D\,P_0^* \longrightarrow D\,M^* \longrightarrow 0$, and thus we get $[D\,M^*] - [D\operatorname{Tr} M] = [D\,P_0^*] - [D\,P_1^*] = \Phi_\Lambda[\operatorname{Ker} p_1] - \Phi_\Lambda[M]$. By duality we immediately see that if $0 \longrightarrow N \longrightarrow E_0 \xrightarrow{i_1} E_1$ is a minimal injective copresentation of a non injective Λ-module N, then $\Phi_\Lambda^{-1}[\operatorname{Coker} i_1] - \Phi_\Lambda^{-1}[N] = [(D\,N)^*] - [\operatorname{Tr} D\,N]$. We have just proven the following Lemma.

Lemma 35

(a) *Let M be a non-projective module and let $P_1 \xrightarrow{p_1} P_0 \longrightarrow M \longrightarrow 0$ be a minimal projective presentation of M. Then*

$$[D\operatorname{Tr} M] = \Phi_\Lambda[M] + [D\,M^*] - \Phi_\Lambda[\operatorname{Ker} p_1]$$

(b) *Let N be a non-injective module in* mod Λ *and $0 \longrightarrow N \longrightarrow E_0 \xrightarrow{i_1} E_1$ a minimal injective copresentation of N. Then*

$$[\operatorname{Tr} D\,N] = \Phi_\Lambda^{-1}[N] + [(D\,N)^*] - \Phi_\Lambda^{-1}[\operatorname{Coker} i_1]$$

□

Recall that a ring R is left hereditary if all left ideals of R are projective. There is a homological characterization of a left hereditary ring R, namely $\operatorname{l.gl.dim} R \leq 1$. In fact this is equivalent to submodules of projective R-modules again being projective. In some sense the Coxeter transformation simplifies for hereditary algebras. We fix the basis $\mathscr{B} = \{[S_1], \ldots, [S_n]\}$ for $K = K_0(\operatorname{mod} \Lambda)$. We call a nonzero element x in K_0 **positive** if all its coordinates with respect to $\mathscr{B}$ are nonnegative. Similarly, we say that a nonzero $x \in K_0$ is **negative** if all its coordinates with respect to $\mathscr{B}$ are non-positive. The two above notions are equivalent to $x = [M]$ or $x = -[M]$ for a module $M \in \text{mod}\,\Lambda$.

Proposition 36

Let Λ be a hereditary artin algebra and let Φ_Λ be the Coxeter transformation.

(a) $\Phi_\Lambda[M] = [D\ \text{Ext}^1_\Lambda(M,\Lambda)] - [D\,M^*]$ *for all X in* mod Λ.

(b) If M is an indecomposable non-projective, then $\Phi_\Lambda[M] = [D\,\text{Tr}\,M]$.

(c) Suppose M is indecomposable. Then M is projective if and only if $\Phi_\Lambda[M]$ is negative.

(d) $\Phi_\Lambda[M]$ is either positive or negative, where M is an indecomposable Λ-module.

(e) $\Phi^{-1}_\Lambda[M] = [\text{Ext}^1_{\Lambda^{\text{op}}}(D\,M,\Lambda)] - [(D\,M)^*]$ *for all M in* mod Λ.

(f) If M is non injective indecomposable Λ-module, then $\Phi^{-1}_\Lambda[M] = [\text{Tr}\,D\,M]$.

(g) Let M be indecomposable. Then M is injective if and only if $\Phi^{-1}_\Lambda[M]$ is negative.

(h) If $M \in$ mod Λ *indecomposable. Then $\Phi^{-1}_\Lambda[M]$ is either positive or negative.*

Proof. Clearly the parts (e), (f), (g) and (h) will follow by duality once we have proved the first part.

(a) Let $P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$ be a minimal projective presentation of M. We have seen in Proposition 35 that $\Phi_\Lambda[M] = [D\,\text{Tr}\,M] - [D\,M^*] + \Phi_\Lambda[\text{Ker}\ p_1]$. Since Λ is hereditary, Ker $P_1 \longrightarrow P_0 = 0$. If $M \in \underline{\text{mod}}\,\Lambda$ then the claim follows by Proposition 13. We only need to show the claim for a projective module, since everything commutes with respect to finite direct sums. If M is projective, then the claim is trivially true since $D\,\text{Tr}\,M = 0$ and $\Phi_\Lambda[M] = -[D\,M^*]$.

(b) Let M be an indecomposable nonprojective Λ-module, and let $f\colon M \longrightarrow \Lambda$. Since Im $f \subseteq \Lambda$ is a submodule we have that Im f is a projective summand of M, however M is indecomposable. Then, by Lemma 35 we get $[D\,\text{Tr}\,M] = \Phi_\Lambda[M] - \Phi_\Lambda[\text{Ker}\ p_1]$, however since Λ hereditary, Ker $p_1 = 0$.

(c) If M is projective, then $\Phi_\Lambda[M]$ is trivially negative by definition. If $\Phi_\Lambda[M]$ is negative then M is trivially projective by (b).

(d) This follows by (c). □

Some easy made observations follow from this proposition, we give them in the following corollary.

Corollary 37
Let M and N be indecomposable Λ-modules for a hereditary artin algebra. If Φ_Λ is the Coxeter transformation and $[M] = [N]$ in K_0, then the following is true:

(a) *M is projective if and only if N is projective.*

(b) *If M is projective, then $M \simeq N$.*

(c) *M is preprojective if and only if $\Phi_\Lambda^n[M]$ is negative for some $n \in \mathbb{N}$.*

(d) *If M is preprojective, then $M \simeq N$.*

(e) *M is injective if and only if N is injective.*

(f) *If M is injective, then $M \simeq N$.*

(g) *M is preinjective if and only if $\Phi_\Lambda^{-m}[M]$ is negative for some $m \in \mathbb{N}$.*

(h) *If M is preinjective, then $M \simeq N$.*

Proof. (a) If M and N are indecomposable, then by (c) in the previous proposition we get that M is an indecomposable projective module if and only if $\Phi_\Lambda[M]$ is negative, since $\Phi_\Lambda[M] = \Phi_\Lambda[N]$ we get that M is projective if and only if N is projective.

(b) Let M be projective and let $[N] = [M]$, then by (a) we get that N is projective. Since $[N] = [M]$ there is a nonzero map from M to N, say $f\colon M \longrightarrow N$. Since Λ is hereditary we get that $M \simeq \operatorname{Im} f \amalg \operatorname{Ker} f$, now since M is indecomposable and f nonzero, we get that $\operatorname{Ker} f = 0$. Hence $M \simeq N$, since $0 \longrightarrow M \longrightarrow N \longrightarrow N/\operatorname{Im} f \longrightarrow 0$ and $[N/\operatorname{Im} f] = 0$.

(c) If M is indecomposable, we know that M is preprojective if and only if $(D\operatorname{Tr})^m M$ is projective for some nonnegative m, then by the previous proposition we get that this is equivalent to $\Phi_\Lambda^n[M]$ being negative for some natural number n.

(d) For M and N indecomposable with $[M] = [N]$ and let M be preprojective. By (c) we know that $\Phi_\Lambda^n[M] = \Phi_\Lambda^n[N]$ is negative with $n \in \mathbb{N}$. Let n be the smallest such number, then $[(D\operatorname{Tr})^{n-1}M] = \Phi_\Lambda^{n-1}[M] = \Phi_\Lambda^{n-1}[N] = [(D\operatorname{Tr})^{n-1}N]$ is positive. Thus $(D\operatorname{Tr})^{n-1}M$ and $(D\operatorname{Tr})^{n-1}N$ are indecomposable projective and by (b) are isomorphic. Thus $M \simeq N$. The rest are just dual statements of the ones proven and follow trivially. □

In the case of a path algebra there is another important matrix which is closely related to the Coxeter transformation. Let $\lambda = kQ$ be the path algebra over the field k given the finite quiver $Q = (Q_0, Q_1)$. Let $e_1, \ldots, e_n$ be the trivial paths in Λ. Denote by C_Λ the **Cartan** matrix of Λ, where

$$C_\Lambda = \begin{pmatrix} \underline{\dim}\,\Lambda e_1 & \ldots & \underline{\dim}\,\Lambda e_n \end{pmatrix}.$$

It is well known that $\det C_\Lambda = \pm 1$. Before we give the relationship between the Cartan matrix and the Coxeter transformation, we define a bilinear form. For a hereditary artin R-algebra Λ, denote by $\langle -, - \rangle_\Lambda \colon K_0(\text{mod}\ \Lambda) \times K_0(\text{mod}\ \Lambda) \longrightarrow \mathbb{Z}$ the **homological bilinear form** given by

$$\langle M, N \rangle_\Lambda = l_R(\text{Hom}_\Lambda(M, N)) - l_R(\text{Ext}^1_\Lambda(M, N)).$$

Proposition 38
Let Q be a finite quiver and let k be a field. Let $\Lambda = kQ$. By identifying $K_0(\text{mod}\ \Lambda)$ with $\mathbb{Z}^n$, the following holds

(a) $\Phi_\Lambda = -C^t_\Lambda C^{-1}_\Lambda$.

(b) $\langle M, N \rangle_\Lambda = [M]^t (C^{-1}_\Lambda)^t [N]$.

(c) $\langle M, N \rangle_\Lambda = -\langle M, \Phi_\Lambda N \rangle_\Lambda = \langle \Phi_\Lambda M, \Phi_\Lambda N \rangle_\Lambda$. □

2.4 Auslander-Reiten Quiver

Here we define the Auslander-Reiten quiver of an artin algebra, and go through some properties and results concerning the AR-quiver (short for Auslander-Reiten quiver).

Let Λ be an artin algebra. We define the **Auslander-Reiten quiver** of mod Λ, Γ_Λ, in the following way. The vertices of the AR-quiver are the isomorphism classes of indecomposable, $[M]$, in mod Λ. If $[M]$ and $[N]$ are vertices, then there is an arrow $[M] \longrightarrow [N]$ if and only if there is an irreducible morphism $M \longrightarrow N$ in mod Λ. Moreover, the arrows in Γ_Λ have valuation $[M] \xrightarrow{(a,b)} [N]$ if $M^a \amalg X \longrightarrow N$ is minimal right almost split with M not a direct summand in X, and $M \longrightarrow N^b \amalg Y$ is minimal left almost split with N not a direct summand in Y. We call $[P]$ a projective vertex if P is projective, similarly we call $[I]$ an injective vertex if I is injective.

Note that Γ_Λ cannot have loops, if it did then there would be an irreducible morphism from, say N to N. We know that the irreducible maps are either monomorphisms or epimorphisms. Because $l(N) < \infty$, it must be an isomorphism. This is not possible since this would imply that an irreducible map is a split monomorphism.

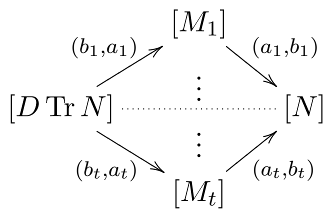
Let $[M] \xrightarrow{(a,b)} [N]$ be an arrow in Γ_Λ, and let N be non-projective. We then have that $0 \longrightarrow D\,\text{Tr}\,N \longrightarrow M^a \amalg X \longrightarrow N \longrightarrow 0$ is an almost split sequence. In particular $D\,\text{Tr}\,N \longrightarrow M^a \amalg X$ is minimal left almost split. Thus we see that we have $[D\,\text{Tr}\,N] \xrightarrow{(\ ,a)} [M]$. One can show that the missing valuation is in fact b.

This means that if N is non-projective, and $[M]\xrightarrow{(a,b)}[N]$ is an arrow in Γ_Λ, then Γ_Λ contains the arrow $[D\operatorname{Tr} N]\xrightarrow{(b,a)}[M]$.

The Auslander-Reiten quiver has an intrinsic structure, given by $D\operatorname{Tr}$. For $[N] \in (\Gamma_\Lambda)_0$ not projective, define the **Auslander-Reiten translate**, $\tau[N]$, as $[D\operatorname{Tr} N]$. This gives a bijective correspondence between the non-projective vertices with the non-injective vertices. The inverse bijection τ^{-1} is given by $\operatorname{Tr} D$. Suppose that

$$0 \longrightarrow D\operatorname{Tr} N \longrightarrow \coprod_{i=1}^{t} M_i^{a_i} \longrightarrow N \longrightarrow 0$$

is an almost split sequence, with N non-projective and M_i indecomposable and pairwise nonisomorphic. Thus in Γ_Λ we get the following picture:



where the horizontal dotted line indicates that this is an almost split sequence. In fact the Auslander-Reiten quiver is in fact a translation quiver. We like to mention here that if Λ is an algebra over an algebraically closed field, then the valuations in the Auslander-Reiten quiver are always of the form (n, n) for some $n \in \mathbb{N}$. This means that they valuation is really not necessary in these situations.

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a valued quiver, which is locally finite, that is each vertex has finitely many arrows coming in and going out. For a vertex $x \in \Gamma_0$ denote by x^- the set of immediate predecessors of x, that is $\{y \in \Gamma_0 \mid \text{there exists } y\longrightarrow x \text{ in } \Gamma_1\}$. Similarly we define the set of immediate successors of x, $x^+ = \{y \in \Gamma_0 \mid \text{there exists } x\longrightarrow y \text{ in } \Gamma_1\}$. Let $\tau\colon U\longrightarrow\Gamma_0$ be an injective map, for some $U \subseteq \Gamma_0$, τ is called the translation of Γ. We call the pair (Γ, τ) for a **translation quiver** if:

(i) Γ has no multiple arrows or loops.

(ii) For each $x \in \Gamma_0$ where $\tau(x)$ is defined, then $x^- = \tau(x)^+$.

(iii) If $x\xrightarrow{(a,b)}y$ is in Γ and $\tau(y)$ is defined, then $\tau(y)\xrightarrow{(b,a)}x$ is in Γ.

If in addition

(iv) If for each $x \in \Gamma_0$ such that $\tau(x)$ is defined, x^- is non-empty.

Then we say that (Γ, τ) is a proper translation quiver. For a translation quiver (Γ, τ) we say that a vertex $x \in \Gamma_0$ is projective if $\tau(x)$ is not defined. If $x \in \Gamma_0$ is such that $x \neq \tau(z)$ for any $z \in U \subseteq \Gamma_0$, then we say that x is injective.

Two indecomposable Λ-modules M and N are said to be related by an irreducible morphisms if there is an irreducible morphism $f\colon A \longrightarrow B$. This relation will generate an equivalence relation on the indecomposable Λ-modules. We call an equivalence class under the equivalence relation a **component**. Then M and N are in the same component if and only if there is an $n \in \mathbb{N}$ so that either there is an irreducible map $f_i\colon X_i \longrightarrow X_{i+1}$ or there is an irreducible map $g_i\colon X_{i+1} \longrightarrow X_i$ for each $0 \leq i < n$, with $X_0 = A$ and $X_n = B$. It is well known that the algebra is of finite type if and only if there is a finite component in Γ_Λ. Furthermore, components that only contain preprojective, preinjective or regular indecomposable modules are called preprojective, preinjective or regular components respectively. Usually there are many components of the same type, that is either preprojective, preinjective or regular. Nonetheless, it can be shown that there is only one preprojective component when the algebra is an indecomposable hereditary artin algebra.

Let $\Delta = (\Delta_0, \Delta_1)$ be a valued quiver without loops, define $\mathbb{Z}\Delta$ as the following translation quiver, let the vertices of $\mathbb{Z}\Delta$ be the tuple $(n, d) \in \mathbb{Z} \times \Delta$. If $\alpha\colon x \xrightarrow{(a,b)} y$ is an arrow in Δ, then $\alpha_n\colon (n, x) \xrightarrow{(a,b)} (n, y)$ and $\sigma(\alpha_n)\colon (n-1, y) \xrightarrow{(b,a)} (n, x)$ are arrows in $\mathbb{Z}\Delta$ for all $n \in \mathbb{Z}$. Moreover, for every $(n, x) \in (\mathbb{Z}\Delta)_0$ we define the translation $\tau(n, x) = (n-1, x)$. We denote by $\mathbb{Z}\Delta$ the translation quiver of Δ. Let $\mathbb{N}\,\Delta$ denote the subtranslation quiver of $\mathbb{Z}\Delta$ with vertices (n, x) where $n \in \mathbb{N}$ and $x \in \Delta_0$.

The next proposition states that for some quivers we are able to determine the shape of the preprojective component of the path algebra over some field.

Proposition 39
Let Q be a connected quiver, without cycles and multiple arrows and let k be a field. Then there is an injective translation quiver morphism of the preprojective component of Λ to $\mathbb{N}\,Q^{\text{op}}$. □

Let M be an indecomposable non-projective module, then denote by αM the number of indecomposable summands in the middle term of the almost split sequence ending in M. We now specialize to the case of Λ being a hereditary artin algebra. The next result states that the middle term in an almost split sequence ending in a regular module has at most two summands.

Proposition 40
Let Λ be a hereditary artin algebra and M an indecomposable regular module in $\operatorname{mod} \Lambda$. *Then $\alpha(M) \leq 2$.* □

We now come to the structure Theorem of the regular components for hereditary artin algebras of infinite representation type.

Theorem 41
Let Λ be a hereditary artin algebra of infinite representation type and let $\mathscr{C}$ be a regular component of Γ_Λ. Then the following is true.

(a) *In $\mathscr{C}$ there exists an infinite chain of irreducible monomorphisms $C_0 \xrightarrow{f_0} C_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} C_n \xrightarrow{f_n} \cdots$ such that $\alpha(C_0) = 1$ and $\alpha(C_i) = 2$ for $i \geq 1$.*

(b) *For each $n \in \mathbb{Z}$ and $i \in \mathbb{N}$, there is an almost split sequence $0 \longrightarrow D\operatorname{Tr}^{n+1} C_i \longrightarrow D\operatorname{Tr}^{n+1} C_{i+1} \amalg D\operatorname{Tr}^{n} C_{i-1} \longrightarrow D\operatorname{Tr}^{n} C_i \longrightarrow 0$, where $C_{-1} = 0$.*

(c) *The set $\{D\operatorname{Tr}^n C_i | n \in \mathbb{Z}, i \in \mathbb{N}\}$ constitutes a complete set of indecomposable modules in $\mathscr{C}$ up to isomorphism.*

(d) *If $h\colon D\operatorname{Tr}^n C_{i+1} \longrightarrow D\operatorname{Tr}^{n-1} C_i$ is any irreducible morphism, then* Ker $h \simeq D\operatorname{Tr}^n C_0$.

(e) *If $D\operatorname{Tr}^n C_i \simeq C_i$ for some $n \in \mathbb{Z}$ and $i \in \mathbb{N}$, then $D\operatorname{Tr}^n C_j \simeq C_j$ for all $j \in \mathbb{N}$.*

(f) *The translation quiver $\mathscr{C}$ is isomorphic to $\mathbb{Z}A_\infty/\langle\tau^n\rangle$ where n is the smallest positive integer with $D\operatorname{Tr}^n C_0 \simeq C_0$.* □

If we were to interpret this geometrically we see that the shape of the regular component(s) of the Auslander-Reiten quiver looks like figure 2.1.

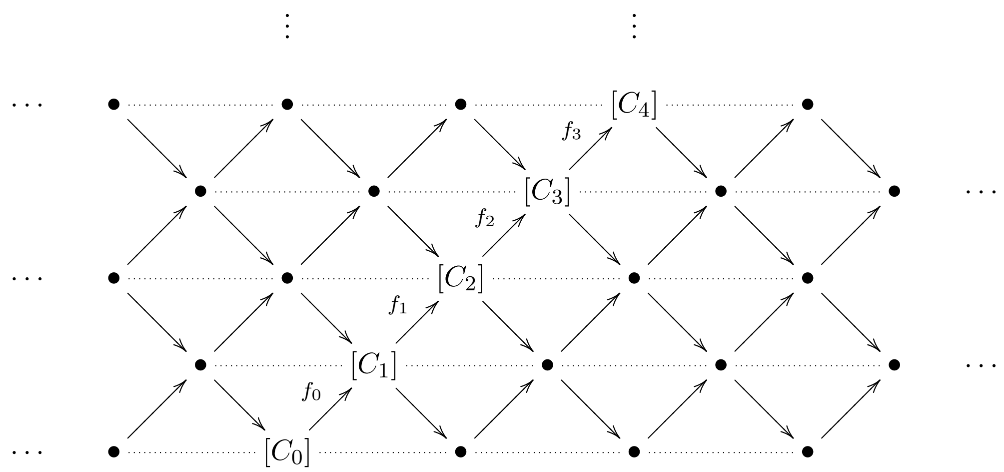


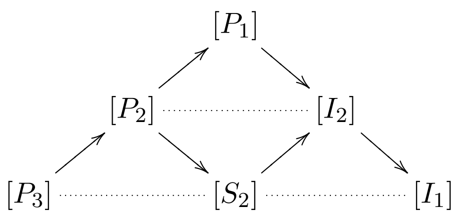
Figure 2.1: The regular components of the Auslander-Reiten quiver of an hereditary artin algebra.

If $D\,\mathrm{Tr}^n C_0 \simeq C_0$ for some $n \in \mathbb{Z}$, then $D\,\mathrm{Tr}^n C_i \simeq C_i$ for all $i \geq 0$ and we obtain what is called a stable tube. If n is the smallest positive integer for which $D\,\mathrm{Tr}^n C_i \simeq C_i$, then we say that the tube has rank n.

We end this chapter with an example.

Example 10

Let $\Lambda = k(\ 1 \longrightarrow 2 \longrightarrow 3\ )$. The Auslander-Reiten quiver of mod Λ is then



Where P_i are the indecomposable projective corresponding to vertex i, and I_j the indecomposable injective corresponding to vertex j, and S_2 is the simple at vertex 2.

Chapter 3

The Four Subspace Problem

The four subspace problem is easily formulated and a student taking a basic course in linear algebra is fully equipped to understand the problem. However, after some initial consideration one begins to see the complexity of it.

The four subspace problem can be formulated in the following fashion: Given a field k and a finite dimensional vector space V_0 over k, and let V_1, V_2, V_3 and V_4 be subspaces of V_0. We call the quintuplet $V = (V_0; V_1, V_2, V_3, V_4)$ for a **quadruple**. We say that a quadruple is **decomposable** if there exists non-trivial $U_0, W_0 \subset V_0$ such that $U_0 \amalg W_0 = V_0$ and $V_i = (U_0 \cap V_i) \amalg (W_0 \cap V_i)$ for $1 \leq i \leq 4$, and we write $V = U \amalg W$. If no such decomposition exists, then we say that the quadruple is **indecomposable**. Moreover, two quadruples, $(V_0; V_1, V_2, V_3, V_4)$ and $(V_0'; V_1', V_2', V_3', V_4')$, are said to be isomorphic if there is a vector space isomorphism $\varphi\colon V_0 \longrightarrow V_0'$, such that the subspace structure is invariant under φ, that is $\varphi(V_i) = V_i'$ for $1 \leq i \leq 4$.

The problem was first solved in the case of an algebraically closed field k by Gelfand and Ponomarev. Later Nazarova closed the gap when she considered an arbitrary field k. Most recent contributions can be found in [MZ].

3.1 Representations of Partially Ordered Sets

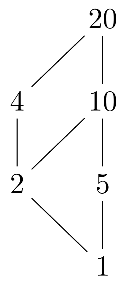
Representations of partially ordered sets come in as the first attempt to fully cope with the four subspace problem. We are here going to go briefly through the concepts presented in this theory.

Recall that a **Hasse diagram** of a finite partially ordered set $\mathfrak{M}$, is a representations of $\mathfrak{M}$ as a directed graph in which elements of $\mathfrak{M}$ is represented as vertices and there is a directed edge from x to y if $y \leq x$ in $\mathfrak{M}$. However, usually we drop the direction of an edge, and place the x above y when $y \leq x$ in $\mathfrak{M}$.

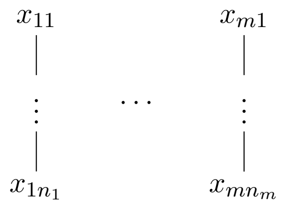
Furthermore, we may omit any edges that is covered by the transitivity of the relation since they carry redundant information, i.e. if $z \leq y \leq x$ in $\mathfrak{M}$, then we identify the edge corresponding to $z \leq x$ with the composition $z \leq y$ and $y \leq x$.

Example 11

Let $\mathfrak{M}$ be the positive divisors of 20 and let the relation be $a \leq b$ if a divides b. This yields the following Hasse diagram:



Let $\mathfrak{M} = (n_1, \dots, n_m)$ be the partially ordered set given by the following Hasse diagram



We say that a finite partially ordered set $\mathfrak{M}$ has **width** d if there are d elements in $\mathfrak{M}$ which are incomparable. In Example 11, we clearly see that the width is 2. The partially ordered set $(n_1, \dots, n_d)$ has width d.

Let $\mathfrak{M}$ be a finite partially ordered set. A representation of $\mathfrak{M}$ over a field k is a finite dimensional vector space V_0 and a collection of subsets of V_0, $V = (V_0; V_i \mid i \in \mathfrak{M})$ such that for each $i, j \in \mathfrak{M}$ with $i \leq j$ then $V_i \subseteq V_j$. Associated with a representation V of a finite partially ordered set $\mathfrak{M}$ is its **dimension vector**, $\underline{\dim}\, V = (d_0, d_1, \dots, d_m)$, where $m = |\mathfrak{M}|$ and $d_0 = \dim_k V_0$ and $d_i = \dim V_i/\overline{V}_i$, where $\overline{V}_i = \sum_{j<i} V_j$ for all $i \in \mathfrak{M}$.

Furthermore, we define the matrix presentation of V as the matrix

$$M = \left(\boxed{M_1} \ \dots \ \boxed{M_m} \right)$$

where M_i is a $d_0 \times d_i$-matrix with columns consisting of coordinate vectors of a basis in $V_i/\overline{V}_i$ with respect to a chosen basis of V_0. Moreover, we define some operations on the matrix M. In addition to elementary row operations, we allow elementary column operations inside each block M_i, and also we allow to add multiple of columns in block M_i to block M_j if $i < j$ in $\mathfrak{M}$. If one matrix presentation can be transformed into another matrix presentation with the above mentioned operations, then we say that they are equivalent. Naturally, two representations U and V of $\mathfrak{m}$ are isomorphic if and only if the corresponding matrices M_U and M_V are equivalent.

We say that a finite partially ordered set is of finite (infinite) type if there is a finite (infinite) number of indecomposable representations.

The four subspace problem comes in as in a natural way in this connection. Let $\mathfrak{m} = (1,1,1,1)$, that is the set of four non-comparable points, then a representation of $\mathfrak{m}$ is, given a finite dimensional k-space V_0, the tuple $V = (V_0; V_1, V_2, V_3, V_4)$, which of course is a quadruple. Thus one sees that the four subspace problem is in fact the classification of the indecomposable representations of $\mathfrak{m}$. One quickly sees that this gets quite cumbersome as one tries to approach the problem in this manner, nonetheless a complete analysis of the four subspace problem as representations of partially ordered sets can be found in [MZ].

It was shown already in [AV] that every finite partially ordered set of width greater then or equal to 4 is of infinite type, this of course means that $(1,1,1,1)$ is of infinite type, thus it was early known that the four subspace problem was of infinite type. A more general analysis of the representation type of partially ordered sets can be found in [Kle2, Kle1].

3.2 As Representations of Quivers

In this section we will consider the four subspace problem as a classification problem with respect to indecomposable modules over a path algebra. As we saw in Theorem 1, this is the same as classifying the indecomposable representation of the corresponding quiver.

Throughout this section k is an algebraically closed field, i.e. every algebraic equation has a solution in the field. Furthermore, Q will denote the quiver

$$\begin{array}{ccccc} & & 2 & & \\ & & \downarrow{\scriptstyle\beta} & & \\ 1 & \xrightarrow{\alpha} & 5 & \xleftarrow{\delta} & 3 \\ & & \uparrow{\scriptstyle\gamma} & & \\ & & 4 & & \end{array}$$

and let $\Lambda = kQ$. Now given a quadruple $U = (U_0; U_1, U_2, U_3, U_4)$ with matrix form as follows

$$M = \left(\begin{array}{|c|c|c|c|} \hline M_1 & M_2 & M_3 & M_4 \\ \hline \end{array} \right)$$

where $\dim_k U_i = n_i$ for $1 \leq i \leq 4$ and M has n rows, is represented by the following representation

$$\begin{array}{ccccc} & & k^{n_2} & & \\ & & \downarrow{\scriptstyle M_2} & & \\ k^{n_1} & \xrightarrow{M_1} & k^n & \xleftarrow{M_3} & k^{n_3} \\ & & \uparrow{\scriptstyle M_4} & & \\ & & k^{n_4} & & \end{array}$$

Thus we see that every quadruple is contained in Rep Q. Notice, however, that not every representation of Q is a quadruple. For instance the following representation

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ k^3 & \xrightarrow{\left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)} & k^2 & \longleftarrow & 0 \\ & & \uparrow & & \\ & & 0 & & \end{array}$$

is not, as it is presented here, a quadruple. Nonetheless, any representation can be transformed such that it corresponds to a quadruple. Given a representation

$$\begin{array}{ccccc} & & k^{n_2} & & \\ & & \downarrow{\scriptstyle f_2} & & \\ k^{n_1} & \xrightarrow{f_1} & k^n & \xleftarrow{f_3} & k^{n_3} \\ & & \uparrow{\scriptstyle f_4} & & \\ & & k^{n_4} & & \end{array}$$

by considering the subspaces $\mathrm{Im}\ f_i \subseteq k^n$ yields the following representation

$$\begin{array}{ccccc} & & \mathrm{Im}\ f_2 & & \\ & & \downarrow & & \\ \mathrm{Im}\ f_1 & \longrightarrow & k^n & \longleftarrow & \mathrm{Im}\ f_3 \\ & & \uparrow & & \\ & & \mathrm{Im}\ f_2 & & \end{array}$$

which is an quadruple.

We will fix some auxiliary notation here. Let $P_i = \Lambda e_i$ be the indecomposable projective representation corresponding to vertex i, dually we denote by $I_j = e_j\Lambda$ the indecomposable injective at vertex j, and lastly let S_i be the simple concentrated at vertex i, see Table 3.1.

We clearly see that the Coxeter and inverse Coxeter transformation are given by the following matrices

$$\Phi_\Lambda = \begin{pmatrix} 0 & 1 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 & -1 \\ 1 & 1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 1 & -1 \end{pmatrix}, \quad \Phi_\Lambda^{-1} = \begin{pmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 1 \\ -1 & -1 & -1 & -1 & 3 \end{pmatrix}$$

Projective indecomposable				
$\begin{array}{ccc} & 0 & \\ & \downarrow & \\ k \xrightarrow{1} & k & \leftarrow 0 \\ & \uparrow & \\ & 0 & \end{array}$	$\begin{array}{ccc} & k & \\ & \downarrow{\scriptstyle 1} & \\ 0 \rightarrow & k & \leftarrow 0 \\ & \uparrow & \\ & 0 & \end{array}$	$\begin{array}{ccc} & 0 & \\ & \downarrow & \\ 0 \rightarrow & k & \xleftarrow{1} k \\ & \uparrow & \\ & 0 & \end{array}$	$\begin{array}{ccc} & 0 & \\ & \downarrow & \\ 0 \rightarrow & k & \leftarrow 0 \\ & \uparrow{\scriptstyle 1} & \\ & k & \end{array}$	$\begin{array}{ccc} & 0 & \\ & \downarrow & \\ 0 \rightarrow & k & \leftarrow 0 \\ & \uparrow & \\ & 0 & \end{array}$
P_1	P_2	P_3	P_4	$P_5 = S_5$

Injective indecomposable				
$\begin{array}{ccc} & 0 & \\ & \downarrow & \\ k \rightarrow & 0 & \leftarrow 0 \\ & \uparrow & \\ & 0 & \end{array}$	$\begin{array}{ccc} & k & \\ & \downarrow & \\ 0 \rightarrow & 0 & \leftarrow 0 \\ & \uparrow & \\ & 0 & \end{array}$	$\begin{array}{ccc} & 0 & \\ & \downarrow & \\ 0 \rightarrow & 0 & \leftarrow k \\ & \uparrow & \\ & 0 & \end{array}$	$\begin{array}{ccc} & 0 & \\ & \downarrow & \\ 0 \rightarrow & 0 & \leftarrow 0 \\ & \uparrow & \\ & k & \end{array}$	$\begin{array}{ccc} & k & \\ & \downarrow{\scriptstyle 1} & \\ k \xrightarrow{1} & k & \xleftarrow{1} k \\ & \uparrow{\scriptstyle 1} & \\ & k & \end{array}$
$I_1 = S_1$	$I_2 = S_2$	$I_3 = S_3$	$I_4 = S_4$	I_5

Table 3.1: The indecomposable projective and injective representations of Q.

Also C_Λ, the Cartan matrix of Λ, is

$$C_\Lambda = \begin{pmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 1&1&1&1&1 \end{pmatrix}.$$

When we now move on to the indecomposable preprojective and preinjective representations we will have to study powers of the Coxeter transformation. Here we split the problem in even and odd powers of the matrices.

Lemma 42
The powers of the Coxeter and inverse Coxeter transformation are given by the following matrices

$$\Phi_\Lambda^{2n+1} = \begin{pmatrix} n & n+1 & n+1 & n+1 & -(2n+1) \\ n+1 & n & n+1 & n+1 & -(2n+1) \\ n+1 & n+1 & n & n+1 & -(2n+1) \\ n+1 & n+1 & n+1 & n & -(2n+1) \\ 2n+1 & 2n+1 & 2n+1 & 2n+1 & -(4n+1) \end{pmatrix} \quad , \quad \Phi_\Lambda^{2n} = \begin{pmatrix} n+1 & n & n & n & -2n \\ n & n+1 & n & n & -2n \\ n & n & n+1 & n & -2n \\ n & n & n & n+1 & -2n \\ 2n & 2n & 2n & 2n & -(4n-1) \end{pmatrix}$$

for $n \in \mathbb{Z}$ in the odd case, and $n \in \mathbb{Z} \setminus \{0\}$ in the even case.

Proof. This clearly follows from the fact that $\Phi_\Lambda \Phi_\Lambda^{2n} = \Phi_\Lambda^{2n+1}$, $\Phi_\Lambda \Phi_\Lambda^{2n+1} = \Phi_\Lambda^{2(n+1)}$, $\Phi_\Lambda^{-1} \Phi_\Lambda^{2n} = \Phi_\Lambda^{2n-1}$ and $\Phi_\Lambda^{-1} \Phi_\Lambda^{2n+1} = \Phi_\Lambda^{2n}$ for any $n \in \mathbb{Z}$. □

As an immidiate consequence we get the following.

Corollary 43
For $1 \leq i, j \leq 4$

(a) $$[\mathrm{Tr}\, D^{\,2n} P_i] = (n+1)[S_i] + \sum_{j \neq i,5} n[S_j] + (2n+1)[S_5]$$

$$[\mathrm{Tr}\, D^{\,2n+1} P_i] = n[S_i] + \sum_{j \neq i,5} (n+1)[S_j] + (2n+2)[S_5], \text{ for } n \geq 0.$$

(b) $[D\,\mathrm{Tr}^{2m} I_j] = (m+1)[S_j] + \sum_{i\neq j,5} m[S_i] + 2m[S_5]$

$[D\,\mathrm{Tr}^{2m+1} I_j] = m[S_j] + \sum_{i\neq j,5}(m+1)[S_i] + (2m+1)[S_5]$, *for* $m \geq 0$.

Moreover, for the projective and injective at vertex 5 we have

(c) $[\mathrm{Tr}\,D^{\,n} P_5] = \sum_{j\neq 5} n[S_j] + (2n+1)[S_5]$, *where* $n \geq 0$,

(d) $[D\,\mathrm{Tr}^{\,m} I_5] = \sum_{i\neq 5}(m+1)[S_i] + (2m+1)[S_5]$, *where* $m \geq 0$, □

We are now able to describe the preprojective and preinjective component of the AR-quiver of Rep Q. From Corollary 43, we clearly see that the preprojective component continues indefinitely to the right, and vice versa, the preinjective component continues backwards to the left indefinitely. This is illustrated in figure 3.1 and 3.2.

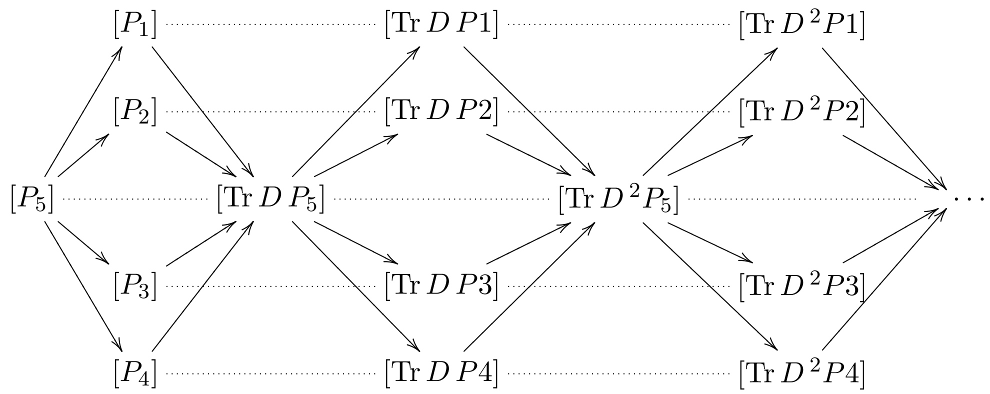


Figure 3.1: The preprojective component of the AR-quiver of Λ.

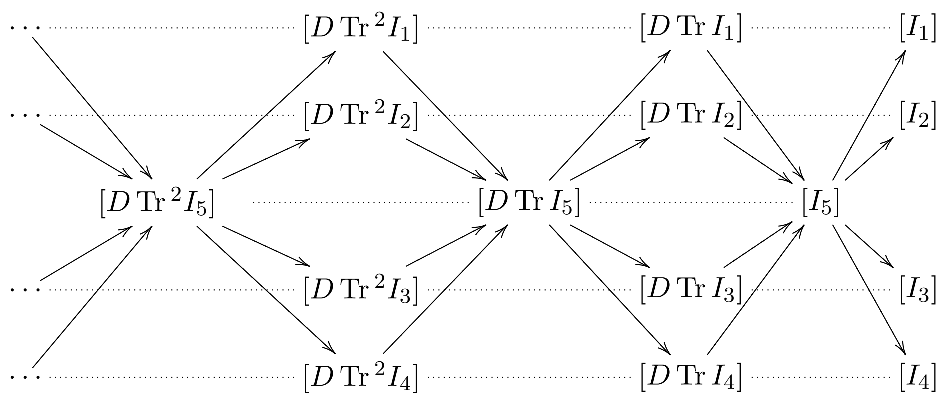


Figure 3.2: The preinjective component of the AR-quiver of Λ.

Already here we can say that Λ is not of finite representation type, since that means that every module is preprojective and preinjective.

Throughout the remainder of this section let M be the module corresponding to the representation

$$\begin{array}{ccccc} & & k & & \\ & & \downarrow{\scriptstyle f_2} & & \\ k & \xrightarrow{f_1} & k^2 & \xleftarrow{f_3} & k \\ & & \uparrow{\scriptstyle f_4} & & \\ & & k & & \end{array}$$

where $f_i = (a_i\ b_i)^t$ with $a_i, b_i \in k$, and $(a_i, b_i) \neq (0,0)$, $1 \leq i \leq 4$. Furthermore, denote by α_{ij} the determinant of $\left(\begin{smallmatrix} a_i & a_j \\ b_i & b_j \end{smallmatrix}\right)$ where $1 \leq i,j \leq 4$. Note that $\alpha_{ij} = 0$ is equivalent to Im $f_i =$ Im f_j. Hence, $\alpha_{ij} = 0$ is transitive, i.e. if $\alpha_{ij} = 0$ and $\alpha_{jm} = 0$, then $\alpha_{im} = 0$, hence we may only define α_{ij} for $1 \leq i < j \leq 4$. Thus we see that if $\alpha_{ij} = 0$ for four pairs of indices, then $\alpha_{ij} = 0$ for all indices. We therefore get the decomposition

$$\begin{array}{ccccc} & & k & & \\ & & \downarrow{\scriptstyle f_2} & & \\ k & \xrightarrow{f_1} & k^2 & \xleftarrow{f_3} & k \\ & & \uparrow{\scriptstyle f_4} & & \\ & & k & & \end{array} \quad \simeq \quad \begin{array}{ccccc} & & k & & \\ & & \downarrow{\scriptstyle 1} & & \\ k & \xrightarrow{1} & \mathrm{Im}\ f_1 & \xleftarrow{1} & k \\ & & \uparrow{\scriptstyle 1} & & \\ & & k & & \end{array} \quad \amalg \quad \begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ 0 & \longrightarrow & k & \longleftarrow & 0 \\ & & \uparrow & & \\ & & 0 & & \end{array}$$

That is $M \simeq I_5 \amalg S_5$. Now if $\alpha_{ij} = 0$ for exactly three pairs of indices, then we have that three of the spaces Im f_i coincide, say, for $i = r,s,t$, and Im $f_q \cap$ Im $f_i = 0$ for $i = r,s,t$ and q is the last index. Without loss of generality we may assume that $(r,s,t) = (2,3,4)$ and $q = 1$. We then have the split exact sequence $0 \longrightarrow P_1 \longrightarrow M \longrightarrow D\,\mathrm{Tr}\,S_1 \longrightarrow 0$.

We now come to the interesting part, namely, when $\alpha_{ij} = 0$ for exactly two pairs (i,j) and (p,q). By transitivity we see that $\{i,j\} \cap \{p,q\} = \emptyset$. Let N_{ij} be the representation given by one dimensional spaces at vertex i, j and 5, with identity as morphisms. This then gives a decomposition of $M \simeq N_{ij} \amalg N_{pq}$, with $\mathrm{End}_\Lambda(N_{ij}) \simeq k$. Now a minimal projective presentation of N_{ij} is given by $0 \longrightarrow P_5 \xrightarrow{-\cdot(\beta_i\ \beta_j)} P_i \amalg P_j \longrightarrow N_{ij} \longrightarrow 0$, where $\beta_i \colon i \longrightarrow 5$ in Q. If we now apply $\mathrm{Hom}_\Lambda(-,\Lambda)$ to the sequence above we get

$$P_i^* \amalg P_j^* \xrightarrow{(\beta_i\ \beta_j)\cdot -} P_5^* \longrightarrow \mathrm{Tr}\,N_{ij} \longrightarrow 0$$

Dualizing we arrive at $D\,\mathrm{Tr}\,N_{ij} \simeq N_{pq}$, where $\{i,j\}$ and$\{p,q\}$ are disjoint. This we can see if we for illustration purpose set $(i,j) = (1,3)$. Then we are in the situation $P_1^* \amalg P_3^* \xrightarrow{(\alpha\ \delta)\cdot -} P_5^*$, and $\mathrm{Tr}\,N_{13} = P_5^*/\langle \alpha, \delta \rangle \simeq D\,N_{24}$. Hence,

$D\,\mathrm{Tr}\,N_{13} \simeq N_{24}$. Moreover, $\Phi^+(N_{ij}) = \Phi^-(N_{ij}) = N_{pq}$. We demonstrate by showing this for N_{13}.

$$N_{13} = \begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ k & \xrightarrow{1} & k & \xleftarrow{1} & k \\ & & \uparrow & & \\ & & 0 & & \end{array} \overset{C_5^+}{\longmapsto} \begin{array}{ccccc} & & 0 & & \\ & & \uparrow & & \\ k & \xleftarrow{1} & k & \xrightarrow{-1} & k \\ & & \downarrow & & \\ & & 0 & & \end{array} \overset{C_1^+C_2^+C_3^+C_4^+}{\longmapsto} \begin{array}{ccccc} & & k & & \\ & & \downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & k & \longleftarrow & 0 \\ & & \uparrow{\scriptstyle 1} & & \\ & & k & & \end{array} = N_{24}$$

So we have that $\Phi^+(N_{ij}) = N_{pq}$. Similarly one sees that $\Phi^-(N_{ij}) = N_{pq}$, and thus N_{ij} is regular.

Let $\alpha_{ij} = 0$ for exactly one pair. Then we are in fact looking at the representation

$$\begin{array}{ccccc} & & k & & \\ & & \downarrow{\scriptstyle f_2} & & \\ k & \xrightarrow{f_1} & k^2 & \xleftarrow{f_3} & k \end{array}$$

since the missing linear map coincides with one of the remaining ones, and the three spaces Im f_1, Im f_2 and Im f_3 are different, meaning that Im $f_i \cap$Im $f_j = 0$ for $i \neq j$. Let us have a closer inspection of the endomorphisms of M.

$$\begin{array}{ccccc} & & \lambda_2 \circlearrowleft & & \\ & & k & & \\ & & \downarrow{\scriptstyle f_2} & & \\ \lambda_1 \circlearrowleft k & \xrightarrow{f_1} & k^2 & \xleftarrow{f_3} & k \circlearrowright \lambda_3 \\ & & \circlearrowleft \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) & & \end{array}$$

The requirement of commutativity yields the following equations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} f_1 = \lambda_1 f_1 \tag{3.1a}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} f_2 = \lambda_2 f_2 \tag{3.1b}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} f_3 = \lambda_3 f_3 \tag{3.1c}$$

Since the images of the maps f_i are all different we have that $\{f_i, f_j\}$ are linearly independent when $i \neq j$. Thus, $\lambda_1 = \lambda_2$, $\lambda_1 = \lambda_3$ or $\lambda_2 = \lambda_3$, since a $n \times n$-matrix has at most n distinct eigenvalues. If $\lambda_1 = \lambda_2$, then $\{f_1, f_2\}$ is a basis for the eigenspace corresponding to λ_1. Hence, f_3 lies in the eigenspace corresponding to λ_1. In other words, $\lambda_1 = \lambda_2 = \lambda_3$. A similar argument in the cases $\lambda_1 = \lambda_3$ and $\lambda_2 = \lambda_3$ yields $\lambda_1 = \lambda_2 = \lambda_3$. We then have that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right)$. Meaning that by a change of basis we can reduce the matrix A to the matrix I_λ. Hence the equations (3.1) reduce to

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} f_i = \lambda f_i$$

for $1 \leq i \leq 3$. Hence, the endomorphism is given by a parameter $\lambda \in k$, that is $\text{End}_\lambda(M) \simeq k$. We have now shown that M is indecomposable if $\alpha_{ij} = 0$ for just one pair (i, j). We may also note that if $f_i = (a_i\ b_i)^t$, a change in basis reduces the above representation to

$$\begin{array}{ccccc} & & k & & \\ & & \downarrow{\scriptstyle (0\ 1)^t} & & \\ k & \xrightarrow{(1\ 0)^t} & k^2 & \xleftarrow{(1\ 1)^t} & k \end{array}$$

This can be achieved in the following manner. Let $B = \left(\begin{smallmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{smallmatrix}\right)$, then a change in basis at each of the one dimensional spaces is equivalent to multiplying the corresponding column with a non-zero scalar. A change in basis in the two dimensional space preserving the linear maps is equivalent to performing elementary row operations on the matrix B. Since $(a_i, b_i) \neq (0, 0)$, we may without loss of generality, assume $a_1 \neq 0$, then we may choose a_1 as generator for k, and B reduces then to $\left(\begin{smallmatrix} 1 & a_2 & a_3 \\ b'_1 & b_2 & b_3 \end{smallmatrix}\right)$ and $b'_1 = a_1^{-1} b_1$. Getting rid of b'_1 yields $\left(\begin{smallmatrix} 1 & a_2 & a_3 \\ 0 & b'_2 & b'_3 \end{smallmatrix}\right)$, where $b'_i = b_i - b'_1 a_i$, $i = 1, 2$. Because the f_i's are pairwise linearly independent for $i = 1, 2, 3$ it follows that B has full rank, meaning that $(b'_2, b'_3) \neq (0, 0)$. Since $\det \left(\begin{smallmatrix} a_1 & a_2 \\ b_1 & b_2 \end{smallmatrix}\right) \neq 0$ we know that $\det \left(\begin{smallmatrix} 1 & a_2 \\ 0 & b'_2 \end{smallmatrix}\right) \neq 0$, that is $b'_2 \neq 0$. Thus choose b'_2 as basis for k, this corresponds to multiplying the second column with $(b'_2)^{-1}$. That is, the matrix reduces to $\left(\begin{smallmatrix} 1 & 0 & a'_3 \\ 0 & 1 & b'_3 \end{smallmatrix}\right)$, where $a'_3 = a_3 - a'_2 b'_3$ and $a'_2 = (b'_2)^{-1} a_2$. Now neither $a'_3 = 0$ nor $b'_3 = 0$, if it were so, then Im f_1 = Im f_3 or Im f_3 = Im f_1, since if $b'_3 = 0$ gives that $b_3 = a_1^{-1} b_1 a_3$, then $f_1(a_1^{-1} a_3) = f_3$. And if $a'_3 = 0$, i.e. $a_3 = a'_2 b'_3$, this the then leads to Im $(b_3 f_2 - a_2 b'_1 f_3)$ = Im f_3, which results in $\{f_2, f_3\}$ being linearly dependent. Thus we can reduce further by multiplying with $(a'_3)^{-1}$ in the last column, and we get $\left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & \mu \end{smallmatrix}\right)$. Multiplying the first row with μ and then multiplying with μ^{-1} in first and third column yields $\left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{smallmatrix}\right)$. We started out by assuming $a_1 \neq 0$, this may not be the case. If this indeed is not the case, then b_1 need to be different from zero, and hence we may swap the rows in the matrix B. This will lead to the same reduced matrix as we got above. We have seen that $b'_3 \neq 0$, and a similar argument forces $b'_2 \neq 0$. This means that there is just one representation, up to isomorphism, with $\alpha_{ij} = 0$. Let M_{ij} denote this representation up to isomorphism. Clearly, N_{ij} is a sub representation of M_{ij}. Since $\alpha_{ij} = 0$, there are $x, y \neq 0$ in k, such that $f_i x = f_j y$, hence, let $h\colon N_{ij} \longrightarrow M_{ij}$ by

$$h_n = \begin{cases} x & , n = i \\ y & , n = j \\ f_i & , n = 5 \\ 0 & , \text{otherwise} \end{cases}$$

Let N_{pq} be the representation such that $\{i, j\} \cap \{p, q\} = \emptyset$, then we have the exact sequence $0 \to N_{ij} \to M_{ij} \to N_{pq} \to 0$, which is non-split. Since $\operatorname{End}_\Lambda(N_{pq}) \simeq k$ we see that the sequence is in fact almost split. Let us see what the Coxeter functors give in this case. Let us say we have that $\alpha_{12} = 0$. As we saw above we may reduce the representation such that the representation looks like

$$\begin{array}{ccccc} & & k & & \\ & & \downarrow{\scriptstyle (1\ 0)^t} & & \\ k & \xrightarrow{(1\ 0)^t} & k^2 & \xleftarrow{(0\ 1)^t} & k \\ & & \uparrow{\scriptstyle (1\ 1)^t} & & \\ & & k & & \end{array}$$

If we now apply C_5^+ we see that we need to look at the kernel of the map $L = \left(\begin{smallmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{smallmatrix}\right) : k^4 \to k^2$. In other words find the nullspace of L. Let $A = \left(\begin{smallmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & -1 & 1 \end{smallmatrix}\right)^t$, and one easily sees that $LA = 0$. Hence, $C_5^+(M_{12})$ is the following representation

$$\begin{array}{ccccc} & & k & & \\ & & \uparrow{\scriptstyle (-1\ 0)} & & \\ k & \xleftarrow{(1\ -1)} & k^2 & \xrightarrow{(0\ -1)} & k \\ & & \downarrow{\scriptstyle (0\ 1)} & & \\ & & k & & \end{array}$$

Now continuing with the rest of the left partial Coxeter functors we get

$$\begin{array}{ccccc} & & k & & \\ & & \downarrow{\scriptstyle (0\ 1)^t} & & \\ k & \xrightarrow{(1\ 1)^t} & k^2 & \xleftarrow{(1\ 0)^t} & k \\ & & \uparrow{\scriptstyle (1\ 0)^t} & & \\ & & k & & \end{array}$$

that is $\Phi^+(M_{12}) = M_{34}$. One can show that $\Phi^+(M_{ij}) = \Phi^-(M_{ij}) = M_{pq}$, where i, j, p, q are as above, that is disjoint subsets of cardinality 2 of the set $\{1, 2, 3, 4\}$.

Lastly, the case $\alpha_{ij} \neq 0$ for all $1 \leq i, j \leq 4$. First, note that the representation M is isomorphic to a representation of the form

$$\begin{array}{ccccc} & & k & & \\ & & \downarrow{\scriptstyle (0\,1)^t} & & \\ k & \xrightarrow{(1\,0)^t} & k^2 & \xleftarrow{(1\,1)^t} & k \\ & & \uparrow{\scriptstyle (1\,\lambda)^t} & & \\ & & k & & \end{array}$$

Meaning that the vector spaces at vertices 1 to 4 are identified up to permutation. This is obtained by a change in basis analogously to what we did in the case $\alpha_{ij} = 0$ for one pair (i, j), that is the matrix $\left(\begin{smallmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{smallmatrix}\right)$ reduces to a matrix of the form $\left(\begin{smallmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & \lambda \end{smallmatrix}\right)$, with $\lambda \neq 0, 1$. We baptize this representation as M_λ, for $\lambda \in k \setminus \{0, 1\}$. If we now examine the endomorphisms of M_λ we see that we get the equations

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} x \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} y \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} z \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ \lambda \end{pmatrix} &= \begin{pmatrix} 1 \\ \lambda \end{pmatrix} w \end{aligned}$$

One can easily see that $a = d = x = y = z = w$ and $b = c = 0$. Hence, $\operatorname{End}_\Lambda(M_\lambda) \simeq k$ and M_λ is indecomposable. Clearly $\operatorname{Hom}_\Lambda(M_\lambda, M_\mu) = 0$ when $\lambda \neq \mu$.

We have the minimal projective presentation of M is

$$0 \longrightarrow P_5 \amalg P_5 \xrightarrow{-\cdot\left(\begin{smallmatrix} \alpha & \beta & -\gamma & 0 \\ \alpha & \lambda\beta & 0 & -\delta \end{smallmatrix}\right)} P_1 \amalg P_2 \amalg P_3 \amalg P_4 \longrightarrow M_\lambda \longrightarrow 0$$

where $n \xmapsto{-\cdot x} n \cdot x$. Passing to the transpose we get

$$P_1^* \amalg P_2^* \amalg P_3^* \amalg P_4^* \xrightarrow{\left(\begin{smallmatrix} \alpha & \beta & -\gamma & 0 \\ \alpha & \lambda\beta & 0 & -\delta \end{smallmatrix}\right)\cdot -} P_5^* \amalg P_5^* \longrightarrow \operatorname{Tr} M_\lambda \longrightarrow 0$$

That is $\operatorname{Tr} M_\lambda = P_5^* \amalg P_5^* / \langle \left(\begin{smallmatrix} \alpha \\ \alpha \end{smallmatrix}\right), \begin{pmatrix} \beta \\ \lambda\beta \end{pmatrix}, \left(\begin{smallmatrix} \gamma \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ \delta \end{smallmatrix}\right) \rangle$. A routine calculation shows that this corresponds to the representation

$$\begin{array}{ccccc} & & k & & \\ & & \uparrow{\scriptstyle (0\ 1)} & & \\ k & \xleftarrow{(1\ 0)} & k^2 & \xrightarrow{(1\ 1)} & k \\ & & \downarrow{\scriptstyle (1\ \lambda)} & & \\ & & k & & \end{array}$$

Hence we have that $D\operatorname{Tr} M_\lambda \simeq M_\lambda$. Here, too, the Coxeter functors yield no surprise, namely $\Phi^+(M_\lambda) \simeq \Phi^-(M_\lambda) \simeq M_\lambda$. We gather our findings and state some new ones in the following proposition.

Proposition 44
Notation as above. For $\{i,j\}, \{p,q\} \subset \{1,2,3,4\}$, *and* $\lambda, \mu \in k \setminus \{0,1\}$.

(a) $\operatorname{Hom}_\Lambda(N_{ij}, N_{pq}) = \begin{cases} 0 & ; \text{ if } |\{i,j\} \cap \{p,q\}| \leq 1. \\ k & ; \text{ otherwise.} \end{cases}$

(b) $\operatorname{Hom}_\Lambda(M_{ij}, M_{pq}) = \begin{cases} 0 & ; \text{ if } |\{i,j\} \cap \{p,q\}| = 1. \\ k & ; \text{ otherwise.} \end{cases}$

(c) $\operatorname{Hom}_\Lambda(M_\lambda, M_\mu) = \begin{cases} k & ; \text{ if } \lambda = \mu. \\ 0 & ; \text{ otherwise.} \end{cases}$

(d) $\operatorname{Hom}_\Lambda(N_{ij}, M_{pq}) = \begin{cases} k & ; \text{ if } (i,j) = (p,q). \\ 0 & ; \text{ otherwise.} \end{cases}$

(e) $\operatorname{Hom}_\Lambda(M_{ij}, N_{pq}) = \begin{cases} k & ; \text{ if } \{i,j\} \cap \{p,q\} = \emptyset. \\ 0 & ; \text{ otherwise.} \end{cases}$

(f) $\operatorname{Hom}_\Lambda(N_{ij}, M_\lambda) = \operatorname{Hom}_\Lambda(M_\lambda, N_{ij}) = \operatorname{Hom}_\Lambda(M_{ij}, M_\lambda) = \operatorname{Hom}_\Lambda(M_\lambda, M_{ij}) = 0.$

(g) $\dim_k \operatorname{Ext}^1_\Lambda(N_{ij}, N_{pq}) = \begin{cases} 1 & ; \text{ if } \{i,j\} \cap \{p,q\} = \emptyset. \\ 0 & ; \text{ otherwise.} \end{cases}$

(h) $\dim_k \operatorname{Ext}^1_\Lambda(M_{ij}, M_{pq}) = \begin{cases} 0 & ; \text{ if } |\{i,j\} \cap \{p,q\}| = 1. \\ 1 & ; \text{ otherwise.} \end{cases}$

(i) $\dim_k \operatorname{Ext}^1_\Lambda(M_\lambda, M_\mu) = \begin{cases} 1 & ; \text{ if } \lambda = \mu. \\ 0 & ; \text{ otherwise.} \end{cases}$

(j) $\dim_k \operatorname{Ext}^1_\Lambda(N_{ij}, M_{pq}) = \begin{cases} 1 & ; \text{ if } (i,j) = (p,q). \\ 0 & ; \text{ otherwise.} \end{cases}$

(k) $\dim_k \operatorname{Ext}^1_\Lambda(M_{ij}, N_{pq}) = \begin{cases} 1 & ; \text{ if } \{i,j\} \cap \{p,q\} = \emptyset. \\ 0 & ; \text{ otherwise.} \end{cases}$

(l) $\operatorname{Ext}^1_\Lambda(N_{ij}, M_\lambda) = \operatorname{Ext}^1_\Lambda(M_\lambda, N_{ij}) = \operatorname{Ext}^1_\Lambda(M_{ij}, M_\lambda) = \operatorname{Ext}^1_\Lambda(M_\lambda, M_{ij}) = 0.$

(m) $D\operatorname{Tr}^2 N_{ij} \simeq N_{ij}$ *and* $D\operatorname{Tr}^2 M_{ij} \simeq M_{ij}$*. Moreover* $D\operatorname{Tr} M_\lambda \simeq M_\lambda$*.*

Proof. (a), (b), (c),(d),(e) and (f) are simple calculations. What remains to be proven is (g), (h), (i), (j), (k) and (l), however we only give a proof of (g), since the other ones are done in a similar fashion. We start with a projective resolution of N_{ij}, since Λ is hereditary we get the short exact sequence $0 \to P_5 \to P_i \amalg P_j \to N_{ij} \to 0$. Applying $\operatorname{Hom}_\Lambda(-, N_{pq})$ yields the following long exact sequence

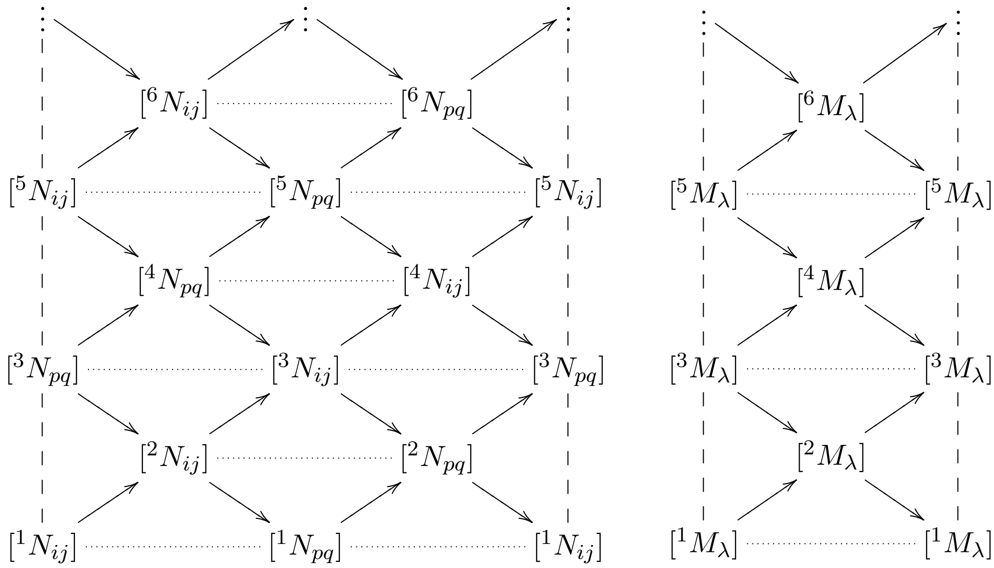
$$0 \to \operatorname{Hom}_\Lambda(N_{ij}, N_{pq}) \to \operatorname{Hom}_\Lambda(P_i \amalg P_j, N_{pq}) \longrightarrow \operatorname{Hom}_\Lambda(P_5, N_{pq}) \longrightarrow$$

$$\longrightarrow \operatorname{Ext}^1_\Lambda(N_{ij}, N_{pq}) \longrightarrow \operatorname{Ext}^1_\Lambda(P_i \amalg P_j, N_{pq}) \to \cdots$$

Here the fifth term vanishes. Since $\dim_k \operatorname{Hom}_\Lambda(P_5, N_{pq}) = 1$ for any p and q, and since $\dim_k \operatorname{Hom}_\Lambda(P_i, N_{pq}) = 1$ if $i = p$ or $i = q$ and zero otherwise, the claim follows. □

This reveals very much about the shape of the regular component. Since $D\operatorname{Tr}^2 N_{ij} \simeq N_{ij}$, we know that the regular component containing $N_{i,j}$ will be a tube of rank 2, that is we identify some vertices. From above we have that there is an almost split sequence $0 \to N_{ij} \to M_{ij} \to N_{pq} \to 0$, where $\{i,j\} \cap \{pq\} = \emptyset$. This means that the component containing N_{ij} will also contain N_{pq}. Since M_{ij} is indecomposable, the component will also contain M_{ij}. If we identify the vertices that get translated to them selves, we see that the regular component of N_{ij} looks like the picture we have drawn in figure 3.4(a). Notice that there is no indecomposable regular module with length less than 3, because the indecomposable of length 1 are the simple Λ-modules, which are either projective or injective, and the indecomposable projective modules corresponding to vertex $1 \le i \le 4$ cover the indecomposable modules of length 2. Accordingly, it is easily seen that ${}^1N_{ij} \simeq N_{ij}$ and ${}^2N_{ij} \simeq M_{ij}$ in figure 3.4(a). What we see is that there will be three tubes of this kind, one for each partition of $\{1,2,3,4\}$ into two disjoint sets.

Furthermore, since for each $\lambda \in k \setminus \{0,1\}$, we have $D\operatorname{Tr} M_\lambda \simeq M_\lambda$ and $\operatorname{Ext}^1_\Lambda(M_\lambda, M_\lambda) \neq 0$, the regular component containing M_λ is a tube of rank 1. This is what we have drawn in figure 3.4(b) we have an indecomposable Λ-module M_λ which gives a tube of rank 1. It can be shown that ${}^1M_\lambda \simeq M_\lambda$.



(a) Three exceptional tubes of rank 2, one for each partition of $\{1,2,3,4\}$ into two disjoint sets.

(b) One stable tube of rank 1 for each $\lambda \in k \setminus \{0,1\}$.

Figure 3.4: The regular components of the AR-quiver of Λ.

We can immediately read off the dimension vectors of the modules in the components in figure 3.4. Namely, for $m \geq 1$, in figure 3.4(a) when $n = 2m-1$ we have

$$\underline{\dim}\, {}^nN_{ij}(t) = \begin{cases} m & , \quad t = i, j. \\ 2m-1 & , \quad t = 5. \\ m-1 & , \quad \text{otherwise.} \end{cases}$$

If $n = 2m$, then we have

$$\underline{\dim}\, {}^nN_{ij}(t) = \begin{cases} 2m & , \quad t = 5. \\ m & , \quad \text{otherwise.} \end{cases}$$

In figure 3.4(b) we have

$$\underline{\dim}\, {}^nM_\lambda(t) = \begin{cases} 2n & , \quad t = 5. \\ n & , \quad \text{otherwise.} \end{cases}$$

We now want to show that every indecomposable regular module is in one of these components, nonetheless we need some further observations about the hom-sets before we are able to do so.

Proposition 45
Notation as above. For $n, m \in \mathbb{N}$, $\{i,j\}, \{p,q\} \subset \{1,2,3,4\}$ and $\lambda, \mu \in k \setminus \{0,1\}$, we have the following:

(a) *If* $|\{i,j\} \cap \{p,q\}| = 1$, *then* $\mathrm{Hom}_\Lambda({}^nN_{ij}, {}^mN_{pq}) = \mathrm{Ext}^1_\Lambda({}^nN_{ij}, {}^mN_{pq}) = 0$.

(b) $\mathrm{Hom}_\Lambda({}^nN_{ij}, {}^mM_\lambda) = \mathrm{Hom}_\Lambda({}^nM_\lambda, {}^mN_{pq}) = \mathrm{Ext}^1_\Lambda({}^nN_{ij}, {}^mM_\lambda) = \mathrm{Ext}^1_\Lambda({}^nM_\lambda, {}^mN_{pq}) = 0$.

(c) $\dim_k \mathrm{Hom}_\Lambda(N_{ij}, {}^nN_{pq}) = \begin{cases} 1 & , \quad (p,q) = (i,j). \\ 0 & , \quad \text{otherwise.} \end{cases}$

(d) $\dim_k \mathrm{Ext}_\Lambda({}^nN_{ij}, N_{pq}) = \begin{cases} 1 & , \quad \{p,q\} \cap \{i,j\} = \emptyset. \\ 0 & , \quad \text{otherwise.} \end{cases}$

(e) $\mathrm{Hom}_\Lambda({}^nM_\lambda, {}^mM_\mu) = \mathrm{Ext}^1_\Lambda({}^nM_\lambda, {}^mM_\mu) = 0$ *when* $\lambda \neq \mu$.

(f) $\dim_k \mathrm{Hom}_\Lambda({}^nM_\lambda, N_\lambda) = \mathrm{Hom}_\Lambda(M_\lambda, {}^nN_\lambda) = 1$.

(g) $\dim_k \mathrm{Ext}_\Lambda({}^nM_\lambda, N_\lambda) = \mathrm{Ext}_\Lambda(M_\lambda, {}^nN_\lambda) = 1$.

Proof. We only give a proof for (a)From figure 3.4 we see that we have the following almost split sequences

$$0 \longrightarrow {}^nN_{ij} \longrightarrow {}^{n-1}N_{pq} \amalg {}^{n+1}N_{ij} \longrightarrow {}^nN_{pq} \longrightarrow 0 \tag{3.2}$$

$$0 \longrightarrow {}^nM_\lambda \longrightarrow {}^{n-1}M_\lambda \amalg {}^{n+1}M_\lambda \longrightarrow {}^nM_\lambda \longrightarrow 0 \tag{3.3}$$

for $n \geq 1$ with ${}^0N_{ij} = {}^0M_\lambda = 0$. These sequences yield the following long exact sequences

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_\Lambda(-, {}^nN_{ij}) \longrightarrow \mathrm{Hom}_\Lambda(-, {}^{n-1}N_{pq} \amalg {}^{n+1}N_{ij}) \longrightarrow \mathrm{Hom}_\Lambda(-, {}^nN_{pq}) \longrightarrow \\ \longrightarrow \mathrm{Ext}^1_\Lambda(-, {}^nN_{ij}) \longrightarrow \mathrm{Ext}^1_\Lambda(-, {}^{n-1}N_{pq} \amalg {}^{n+1}N_{ij}) \longrightarrow \mathrm{Ext}^1_\Lambda(-, {}^nN_{pq}) \longrightarrow 0 \end{aligned} \tag{3.4}$$

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_\Lambda(-, {}^nM_\lambda) \longrightarrow \mathrm{Hom}_\Lambda(-, {}^{n-1}M_\lambda \amalg {}^{n+1}M_\lambda) \longrightarrow \mathrm{Hom}_\Lambda(-, {}^nM_\lambda) \longrightarrow \\ \longrightarrow \mathrm{Ext}^1_\Lambda(-, {}^nM_\lambda) \longrightarrow \mathrm{Ext}^1_\Lambda(-, {}^{n-1}M_\lambda \amalg {}^{n+1}M_\lambda) \longrightarrow \mathrm{Ext}^1_\Lambda(-, {}^nM_\lambda) \longrightarrow 0 \end{aligned} \tag{3.5}$$

for any $M \in \mathrm{mod}\,\Lambda$.

(a) We prove this by induction on n and m. Proposition 44 covers the case $n = m = 1$. Suppose it holds for $n = 1$ and $m \leq r$. By inserting[1] N_{ip} and M_{ip} into equation (3.4) we see that it holds for $m = s + 1$. The proof in the other variable is done in a similar fashion.

(c) Here too we proceed by induction. We have from Proposition 44 that the statement holds true for $n = 1$. From figure 3.4(a) we have the exact sequence $0 \longrightarrow N_{ij} \longrightarrow {}^nN_{ij} \longrightarrow {}^{n-1}N_{pq} \longrightarrow 0$. Applying $\mathrm{Hom}_\Lambda(N_{ij}, -)$ to this sequence yields

$$0 \longrightarrow \mathrm{Hom}_\Lambda(N_{ij}, N_{ij}) \longrightarrow \mathrm{Hom}_\Lambda(N_{ij}, {}^nN_{ij}) \longrightarrow \mathrm{Hom}_\Lambda(N_{ij}, {}^{n-1}N_{pq}) \longrightarrow 0$$

[1] Without loss of generality we may use N_{ip} and $M_{ip} \simeq {}^2N_{ip}$, since $|\{i,j\} \cap \{p,q\}| = 1$ gives that $|\{i,j\} \cap \{u,v\}| = |\{u,v\} \cap \{p,q\}| = 1$ for any $\{u,v\} \in \{1,2,3,4\}$ with $(i,j) \neq (u,v) \neq (p,q)$.

By the induction hypothesis we have that the last term is zero and from Proposition 44 the first term is one dimensional and thus the claim follows.

(d) As above we have the exact sequence $\delta\colon 0 \longrightarrow N_{ij} \longrightarrow {}^{n}N_{ij} \longrightarrow {}^{n-1}N_{pq} \longrightarrow 0$. If we now apply $\mathrm{Hom}_\Lambda(-, N_{ij})$ we get

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_\Lambda({}^{n-1}N_{pq}, N_{ij}) \longrightarrow \mathrm{Hom}_\Lambda({}^{n}N_{ij}, N_{ij}) \longrightarrow \mathrm{Hom}_\Lambda(N_{ij}, N_{ij}) \longrightarrow \\ \mathrm{Ext}^1_\Lambda({}^{n-1}N_{pq}, N_{ij}) \longrightarrow \mathrm{Ext}^1_\Lambda({}^{n}N_{ij}, N_{ij}) \longrightarrow \mathrm{Ext}^1_\Lambda(N_{ij}, N_{ij}) \longrightarrow 0 \end{aligned}$$

The sixth term vanishes and the third term is one dimension due to Proposition 44. If we can show that the dimension of the two first terms is the same, then the claim follows since $\dim_k \mathrm{Ext}^1_\Lambda({}^{n}N_{ij}, N_{ij})$ by the induction hypothesis. If we apply $\mathrm{Hom}_\Lambda(-, N{-}pq)$ to δ we see that $\mathrm{Hom}_\Lambda({}^{n-1}N_{pq}, N_{ij}) \xrightarrow{\simeq} \mathrm{Hom}_\Lambda({}^{n}N_{ij}, N_{ij})$ since $\mathrm{Hom}_\Lambda(N_{ij}, N_{pq}) = 0$ by Proposition 44.

The rest of the statements follow by similar arguments. □

We now fix the basis $\mathscr{B} = \{v_i\}$ for $K_0(\mathrm{mod}\,\Lambda) \simeq \mathbb{Z}^5$ (over $\mathbb{Q}$) where

$$\mathscr{B} = \{(\,1\,1\,1\,1\,2\,)^t, (\,1\,1\,0\,0\,1\,)^t, (\,1\,0\,1\,0\,1\,)^t, (\,1\,0\,0\,1\,1\,)^t, (\,0\,0\,0\,0\,1\,)^t\}.$$

Notice that $(\,1\,1\,1\,1\,2\,)^t$ is a basis for the eigenspace of Φ_Λ and Φ_Λ^{-1} with respect to the eigenvalue 1, moreover the vectors $(\,1\,1\,0\,0\,1\,)^t, (\,1\,0\,1\,0\,1\,)^t$ and $(\,1\,0\,0\,1\,1\,)^t$ together with $(\,1\,1\,1\,1\,2\,)^t$ constitute a basis for the eigenspace of Φ_Λ^2 and Φ_Λ^{-2} corresponding to the eigenvalue 1. Furthermore, for $i = 2, 3, 4$ we have that $\Phi_\Lambda(v_i) = \Phi_\Lambda^{-1}(v_i) = v_1 - v_i$. We shall use this basis to describe the dimension vectors of the indecomposable modules in mod Λ. Notice that in $\mathscr{B}$, $v_i = [N_{1i}]$ for $2 \leq i \leq 4$.

Proposition 46
Let X be an indecomposable Λ-module and let $[X] = \sum_{i=1}^{5} s_i v_i$ in $K_0(\mathrm{mod}\,\Lambda)$, with $v_i \in \mathscr{B}$ where $\mathscr{B}$ is as above.

- *(a) If $s_5 > 0$ then X is preprojective. Moreover, if $s_i \neq 0$ for some $2 \leq i \leq 4$ then X is isomorphic to one of the representations in Corollary 43 (a). Otherwise, i.e. if $s_i = 0$ for all $2 \leq i \leq 4$ then X is isomorphic to one of the representations in Corollary 43 (c).*

- *(b) If $s_5 < 0$ then X is preinjective. Moreover, if $s_i \neq 0$ for some $2 \leq i \leq 4$ then X is isomorphic to one of the representations in Corollary 43 (b). Otherwise, i.e. if $s_i = 0$ for all $2 \leq i \leq 4$ then X is isomorphic to one of the representations in Corollary 43 (d).*

- *(c) If X is regular then $s_5 = 0$. Moreover, either N_{ij} or M_λ is a submodule of X, for some $\{i, j\} \subset \{1, 2, 3, 4\}$ and $\lambda \in k \setminus \{0, 1\}$.*

Proof. (a) Corollary 37 says that X is preprojective if $\Phi_\Lambda^n([X])$ is negative. This is only possible if $s_5 > 0$. Hence X is isomorphic to one of the representations given in Corollary 43 (a) or (c).

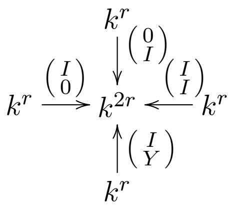
(b) We proceed with a similar argument as above, however in this case the we examine when $\Phi_\Lambda^{-m}([X])$ is negative. To obtain this we need that $s_5 < 0$. Thus X is isomorphic to one of the representations given in Corollary 43 (b) or (d).

(c) Suppose X is regular, then clearly $s_5 = 0$. Now suppose that $s_i \neq 0$ for $2 \leq i \leq 4$, then we have that $\langle v_i, X\rangle_\Lambda = s_i$. If $s_i > 0$ this means that $\mathrm{Hom}_\Lambda(N_{1i}, X) \neq 0$, which results in N_{1i} being a submodule of X. On the other hand if $s_i < 0$, then $\langle \Phi_\Lambda(v_i), X\rangle_\Lambda = \langle \Phi_\Lambda(v_i), \Phi_\Lambda^2(X)\rangle_\Lambda = \langle v_i, \Phi_\Lambda(X)\rangle_\Lambda = -\langle v_i, X\rangle_\Lambda = -s_i$. This then shows that $\mathrm{Hom}_\Lambda(D\,\mathrm{Tr}\,N_{1i}, X) \neq 0$. Hence $D\,\mathrm{Tr}\,N_{1i}$ is a submodule of X. In any case we have that N_{pq} is a submodule of X if $s_i \neq 0$ for $2 \leq i \leq 4$.

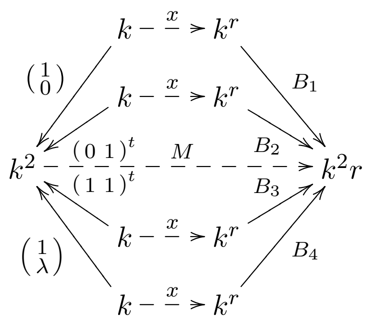
In the other case, that is $s_i = 0$ for all $2 \leq i \leq 4$ we have that X is given by the following representation

$$\begin{array}{ccccc} & & k^r & & \\ & & \downarrow{\scriptstyle A_2} & & \\ k^r & \xrightarrow{A_1} & k^{2r} & \xleftarrow{A_3} & k^r \\ & & \uparrow{\scriptstyle A_4} & & \\ & & k^r & & \end{array}$$

Here we get two possible scenarios: either $\dim_k(\mathrm{Im}\ A_i \cap \mathrm{Im}\ A_j) = 0$ for all $1 \leq i < j \leq 4$ or $\dim_k(\mathrm{Im}\ A_i \cap \mathrm{Im}\ A_j) \geq 1$ for at least one pair (i, j). Let us consider the first case. Since X is indecomposable we must have that all the matrices A_i have maximal rank, i.e. $\mathrm{rank}\, A_i = r$. Hence the block matrix $B = \left(\begin{array}{|c|c|c|c|}\hline A_1 & A_2 & A_3 & A_4 \\ \hline\end{array}\right)$ reduces to the block matrix $\left(\begin{array}{|c|c|c|c|}\hline I & 0 & X_1 & Y_1 \\ \hline 0 & I & X_2 & Y_2 \\ \hline\end{array}\right)$. Since the rank of A_3 is maximal, we are able to arrange it so that by elementary column operations the third vertical stripe from the left is reduced to $\begin{array}{|c|}\hline I \\ \hline X \\ \hline\end{array}$. What we can say here is that X also has maximal rank, this is because we have assumed that $\mathrm{Im}\ A_1 \cap \mathrm{Im}\ A_3 = 0$. Hence by simultaneously executing elementary row operations on the lower horizontal stripe and elementary column operations on the second vertical stripe from the left we are able to reduce B to the following matrix $\left(\begin{array}{|c|c|c|c|}\hline I & 0 & I & Z_1 \\ \hline 0 & I & I & Z_2 \\ \hline\end{array}\right)$. Once again we argue that Z_1 and Z_2 have maximal rank, thus by a similar argument as above we are able to reduce B to $\left(\begin{array}{|c|c|c|c|}\hline I & 0 & I & I \\ \hline 0 & I & I & Z_2 \\ \hline\end{array}\right)$. Since $\dim_k(\mathrm{Im}\ A_i \cap \mathrm{Im}\ A_j) = 0$ we must have that Y has an eigenvalue $\lambda \in k \setminus \{0, 1\}$, here we use the assumption of k being algebraically closed. What we have done here is to show that the above representation is isomorphic to the following representation

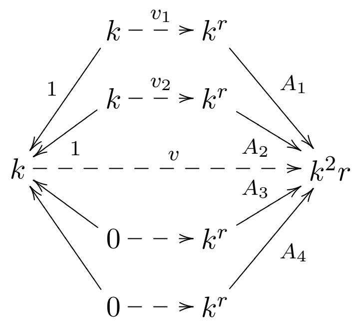


Hence let $0 \neq x \in k^r$ be an eigenvector for Y corresponding to the eigenvalue λ. Thus there exist $u_i \in \text{Im } A_i$ such that $\dim_k \text{Span}\{u_i\} = 2$ and $u_3 = u_1 + u_2$ and $u_4 = u_1 + \lambda u_2$, obviously $\lambda \neq 0, 1$. This means that we can embed M_λ into X with the following map



where B_i is the i'th block of the reduced matrix B and $M = \left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right)$.

On the other hand if $\dim_k(\text{Im } A_i \cap \text{Im } A_j) \geq 1$ for some (i, j). Then there is a nonzero $v \in (\text{Im } A_i \cap \text{Im } A_j)$ and let $v_1, v_2 \in k^r$ so that $v = A_i v_1 = A_j v_2$. Then we see that N_{ij} is a submodule of X through the following map



This concludes the proof. □

We are now closing in on the classification problem.

Proposition 47

Every indecomposable regular $X \in \text{mod } \Lambda$ is isomorphic to one of the modules given in figure 3.4.

Proof. We prove this by induction on the length on X. We have already seen that there are no regular modules with length less than 3. Therefore, suppose that $l(X) \geq 3$ and that any indecomposable regular module of length less then n is one of the ones given in figure 3.4. By Proposition 46 there is an inclusion $f\colon Y \longrightarrow X$ where $Y \simeq N_{ij}$ or $Y \simeq M_\lambda$. Suppose that we are in the first case. We therefore have the short exact sequence $\eta\colon 0 \longrightarrow N_{ij} \xrightarrow{f} X \longrightarrow Z \longrightarrow 0$, where $Z = \text{Coker } f$. Let $Z = \coprod_{i=1}^r Z_i$, with Z_i indecomposable for all i. We now argue that Z_i is regular. Z_i cannot be preprojective, since there are no nonzero maps from regular modules to preprojective modules. Furthermore, none of the Z_i's can be preinjective either, since Proposition 46 (c) says that $S_5 = 0$ where $[X] = \sum_{i=1}^{5} s_i v_i$, with v_i as in Proposition 46.

We now show that Z_i lies in the same tube as $Y \simeq N_{ij}$. Since $\text{Ext}^1_\Lambda({}^nN_{uv}, N_{ij}) = \text{Ext}^1_\Lambda({}^nM_\lambda, N_{ij}) = 0$ for $|\{u,v\} \cap \{i,j\}| \leq 1$ and for all $\lambda \in k \setminus \{0,1\}$ we have the short exact sequence

$$0 \longrightarrow \text{Hom}_\Lambda(M, N_{ij}) \longrightarrow \text{Hom}_\Lambda(M, X) \longrightarrow \text{Hom}_\Lambda(M, \textstyle\coprod_{i=1}^r Z_i) \longrightarrow 0$$

for $M \simeq {}^nN_{uv}$ or ${}^nM_\lambda$. Thus if M is a summand of $\coprod_i Z_i$ then M is also a summand of X. This is impossible since we have assumed that X is indecomposable. Thus we are left with $Z_i \simeq {}^nN_{pq}$, where $\{i,j\} \cap \{p,q\} = \emptyset$.

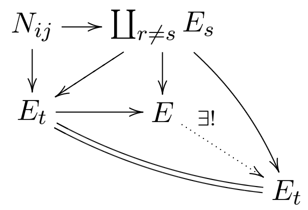
Next we show that Z is indecomposable. Suppose that $r \geq 2$. By taking the pullback of $0 \longrightarrow N_{ij} \longrightarrow X \longrightarrow \coprod_{i=1}^r Z_i \longrightarrow 0$ along the inclusion $Z_i \longrightarrow \coprod_{i=1}^r Z_i$ we get the following non-spilt exact sequences

$$\begin{array}{c} 0 \longrightarrow N_{ij} \longrightarrow E_1 \longrightarrow Z_1 \longrightarrow 0 \\ \vdots \\ 0 \longrightarrow N_{ij} \longrightarrow E_r \longrightarrow Z_r \longrightarrow 0 \end{array}$$

Since $\dim_k \text{Ext}^1_\Lambda(Z_i, N_{ij}) = 1$, we have that the E_i's are one of the modules in figure 3.4 because $l(E_i) < l(X)$. We then have the following pushout diagram

$$\begin{array}{cccccccccc} & 0 & \longrightarrow & \coprod_{i=1}^r N_{ij} & \longrightarrow & \coprod_{i=1}^r E_i & \longrightarrow & \coprod_{i=1}^r Z_i & \longrightarrow & 0 \\ & & & \sigma\downarrow & & \downarrow & & \| & & \\ \eta'\colon & 0 & \longrightarrow & N_{ij} & \longrightarrow & E & \longrightarrow & \coprod_{i=1}^r Z_i & \longrightarrow & 0 \end{array}$$

where $\sigma(y_1, \ldots, y_r) = \sum_{i=1}^r y_i$. Notice that η' is isomorphic to η. Let E_t be such that $l(E_t) \geq l(E_s)$, for $1 \leq s,t \leq r$. We make use of the following fact, for any nonzero $f\colon N_{ij} \longrightarrow E_t$ and $g\colon N_{ij} \longrightarrow E_s$ there is an inclusion $h\colon E_s \longrightarrow E_t$ such that $f = hg$, where E_t is as above and $1 \leq s \leq r$. This yields the short exact sequence $0 \longrightarrow N_{ij} \longrightarrow \coprod_{i=1}^r E_i \longrightarrow E \longrightarrow 0$. Here we must have that E_t is a summand in E since we have the following commutative diagram



However $E \simeq X$ and X is indecomposable, thus $r = 1$. Let $Z = Z_1$. Since $\dim_k \operatorname{Ext}_\Lambda(N_{ij}, Z) = 1$, we must have that X is in figure 3.4 because there is a non-split sequence $\delta\colon 0 \longrightarrow N_{ij} \longrightarrow X' \longrightarrow Z \longrightarrow 0$ with X' from figure 3.4, and thus η is a scalar multiple of δ meaning that $X \simeq X'$.

A similar argument in the case of $Y \simeq M_\lambda$ with $\lambda \in k \setminus \{0, 1\}$ will suffice. □

This show that the list $\{{}^nQ_i, {}^nJ_i, {}^nN_{pq}, {}^nM_{pq}, {}^nN_\lambda | n \in \mathbb{N}, 1 \leq i \leq 5, \{p, q\} \in \{1, 2, 3, 4\}\}$ is a complete list of finitely generated indecomposable Λ-modules. In particular this means that the figures 3.1, 3.2 and 3.4 constitute the Auslander-Reiten quiver of mod Λ. Nevertheless, one needs to mention that not every indecomposable representation given in this section is valid as an indecomposable quadruple, the exceptions are the simple injective Λ-modules. The benefits of this approach over representations of partially ordered sets is that the Auslander-Reiten quiver gives a detail geometric picture over the module category. Off the Auslander-Reiten quiver one can see how the indecomposable modules interact with each other with respect to morphisms between them.

3.3 Four Lines in the Real Plane

One way of explaining the four subspace problem is to ask how many ways may one draw four lines which intersect each other at a point on a rubber sheet, under the assumption that two such configurations are equal if one can stretch and pull the rubber sheet so that one gets from one configuration to the other. By introducing a basis and normalizing, this means, given four lines through the origin, two such configurations are equal if one can get from one to the other by means of change of basis. Or in terms of representations of quivers, when are two representations of Q over $\mathbb{R}$ (where Q is as in the previous section) with dimension vector $(1, 1, 1, 1, 2)$ isomorphic. In this section we are going to give a criterion for when two such configurations are equal in terms of projective geometry.

We have not been precise enough here. We are not interested in all representations with dimension vector $(1, 1, 1, 1, 2)$, our concern lies mainly with representations of the following type

$$\begin{array}{ccccc} & & \mathbb{R} & & \\ & & \big\downarrow f_2 & & \\ \mathbb{R} & \xrightarrow{f_2} & \mathbb{R}^2 & \xleftarrow[f_3]{} & \mathbb{R} \\ & & f_4 \big\uparrow & & \\ & & \mathbb{R} & & \end{array}$$

with $f_i \in \mathbb{R}^2 - \{(0,0)\}$, for $1 \leq i \leq 4$.

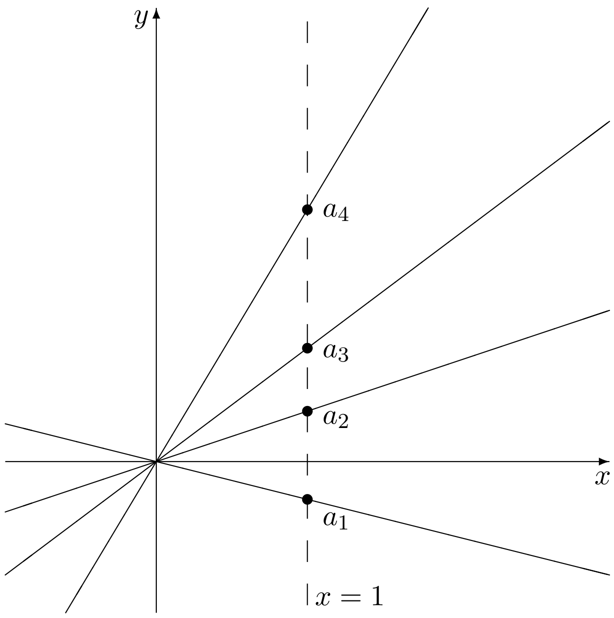


Figure 3.5: Four non-overlapping subspaces of $\mathbb{R}^2$

Given a representation of Q, this then corresponds to a picture similar to figure 3.5. Here we have drawn in the vertical line $x = 1$. We are then able to identify each non-vertical line with its gradient, that is its intersection with this line, and by identifying the line $x = 0$ with, ω, the point at infinity we get a complete identification. Thus such a representation can be represented as a 4-tuple, (a_1, a_2, a_3, a_4), where $a_i \in \mathbb{R} \cup \{\omega\} \approxeq \mathbb{S}^1$. This set is often referred to as the one point compactification of the real line. Notice that this representation is not unique, for instance, $(0, 1, 2, 3)$ and $(1, 2, 3, 0)$ are in fact the same representation.

Given four distinct collinear points A, B, C and D, that is they all lie on a single line. The **cross-ratio** of A, B with respect to C, D is given by[2]

$$\frac{DA}{DB}\Big/\frac{CB}{CA}$$

and we write $(A, B; C, D)$. Here we understand AB to be the oriented length, meaning that the length of the line segment is signed depending on the orientation. With the orientation given in figure 3.6, AB is positive. Obviously, $AB = -BA$ for any pair of points and any orientation. One easily verifies some permutation which leaves the cross-ratio symmetry

Figure 3.6: The oriented length.

Example 12

We are going to calculate the cross-ratio of points A, B, C and D where the lines $x = 0$, $y = 0$ $y = x$ and $y = \lambda x$ intersect $x = 1$. The line $x = 1$, can be parameterized in the following way:

$$l = \theta v_1 + v_2 \equiv v_1 + \tfrac{1}{\theta} v_2$$

where $v_1 = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ and $v_2 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$.Then $\theta = 0, \omega, 1, \lambda$ give the respective intersections with the line $x = 1$ and the above mentioned lines.

$$(A, B; C, D) = (0, \omega; 1, \lambda) = \frac{\lambda - 0}{1 - 0}\Big/\frac{\lambda - \omega}{1 - \omega} = \lambda \frac{(\lambda/\omega) - 1}{(1/\omega) - 1} = \lambda$$

Let O be a point and let a and b be rays starting in O. By (ab) we mean the angle between the rays in the interval $[-\pi, \pi]$ with positive or negative sign whenever a has to rotate anti-clockwise or clockwise to reach b. Since we are here only interested in the sine of the angle, there will be no ambiguity at the points $-\pi$ and π. If c is a ray starting in O, then the ratio $\sin(ca)/\sin(cb)$ will denote the ratio which c divides (ab). This, of course, presumes that $c \neq b$ and that c and b are not opposite rays. Suppose that A, B and C are collinear points and

[2]There is 24 permutations of the tuple $(A, B; C, D)$, however not every definition is equal. That is the group S_4 does not act trivially on the cross-ratio, in fact there is six different cross-ratios. These will correspond to the exceptional modules N_{ij}.

O is a point that does not lie on the line through A, B and C. Furthermore, let a be the ray starting in O and going through A, b the ray from O through B and c the ray through C starting in O. Then CA/CB and $\sin(ca)/\sin(cb)$ have the same sign. This remark shall come in handy in what now follows. Let a, b, c and d be four rays coming out O such that it intersects a line, as depicted in figure 3.7. Define

$$R(a, b; c, d) = \frac{\sin(da)}{\sin(db)} \Big/ \frac{\sin(ca)}{\sin(cb)}$$

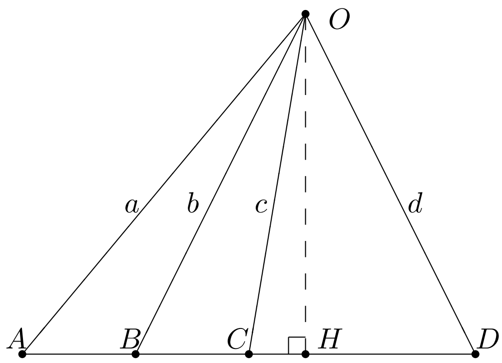


Figure 3.7: The cross-ratio.

Lemma 48
Let A, B, C, D, a, b, c and d be as above. Then $(A, B; C, D) = R(a, b; c, d)$.

Proof. Let OH be the normal from O down to the line through the four points.

$$(A, B; C, D) = \frac{DA}{DB} \Big/ \frac{CA}{CB} = \frac{\frac{1}{2}OH \cdot DA}{\frac{1}{2}OH \cdot DB} \Big/ \frac{\frac{1}{2}OH \cdot CA}{\frac{1}{2}OH \cdot CB} =$$

$$\pm \frac{\text{Area}(\triangle DOA)}{\text{Area}(\triangle DOB)} \Big/ \frac{\text{Area}(\triangle COA)}{\text{Area}(\triangle COB)} = \pm \frac{\frac{1}{2}OA \cdot OD \sin(da)}{\frac{1}{2}OB \cdot OD \sin(db)} \Big/ \frac{\frac{1}{2}OA \cdot OC \sin(ca)}{\frac{1}{2}OB \cdot OC \sin(cb)} =$$

$$\pm \frac{\sin(da)}{\sin(db)} \Big/ \frac{\sin(ca)}{\sin(cb)} = \pm R(a, b; c, d)$$

Let us examine the sign in more detail. Our earlier remark says that CA/CB and $\sin(ca)/\sin(cb)$ have same sign, and of course the same is valid for DA/DB and $\sin(da)/\sin(db)$. Thus $(A, B; C, D) = R(a, b; c, d)$. □

Let P be a point not on the lines l and l'. The map that sends a point on l to a point on l' as shown in figure 3.8 is called a **perspective projection**. P is said to be the center of the perspective projection. The composition of finitely many perspective projections is called a **projective transformation**, or sometimes a projectivity. We write $(A, B, C, \ldots) \overline{\overline{\wedge}} (A', B', C', \ldots)$ whenever A' comes from A, B' from B, and so on... under some projective transformation.

An important fact of projective geometry is that the cross-ratio is invariant under projective transformations.

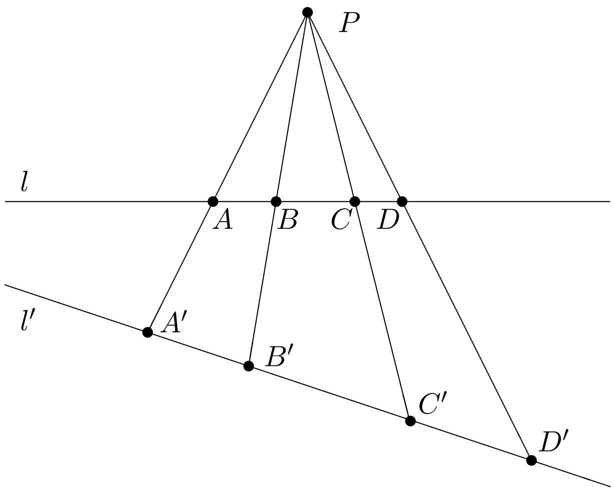


Figure 3.8: Perspective projection with center at P.

Theorem 49
$(A, B, C, D) \;\overline{\overline{\wedge}}\; (A', B', C', D')$ *if and only if* $(A, B; C, D) = (A', B'; C', D')$

Proof. $\Rightarrow$: It is clear that we only need to prove the statement for a perspective projection. Thus we are in the situation of figure 3.8, and Lemma 48 yields

$$(A, B; C, D) = R(a, b; c, d) = (A', B'; C', D')$$

where a is the ray starting in P and going through A etc.

$\Leftarrow$: We will here make use of the following result: There is a projectivity that sends the three collinear points A, B, C to the three collinear points A', B', C'. For a proof of this statement the reader is referred to [HW, p. 23] or in any other book on elementary projective geometry. Let us call this transformation for T, and suppose the image of D under T is X. We need to show that $D' = X$. From the above implication it follows that $(A, B; C, D) = (A', B'; C', X)$, combined with the presumption of the Theorem we get $(A', B'; C', D') = (A', B'; C', X)$, i.e.

$$\frac{XA'}{XB'}\bigg/\frac{C'A'}{C'B'} = \frac{D'A'}{D'B'}\frac{C'A'}{C'B'}$$

since $C'A'/C'B' \neq 0$, we end up with

$$\frac{XA'}{XB'} = \frac{D'A'}{D'B'}$$

A commonly known result of projective geometry says that there is exactly one point X such that $XA'/XB' = r$ for all $r \in \mathbb{R} \setminus \{1\}$. Hence we see that $X = D'$, and the claim follows. $\square$

The interpretation of the above results is that two configurations are equivalent if and only if they have the same cross-ratio.

Example 13

Let the four lines be given by the following equations $y_1 = -x$, $y_2 = 3x$, $y_3 = 7x$ and $y_4 = 11x$. The matrix $\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 3 & 5 & 11 \end{pmatrix}$ is reduced to the matrix $\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 3/4 \end{pmatrix}$ by elementary row operations and scalar column multiplication, that is we have changed basis. Then by Example 12, we have that the cross-ratio $(-1, 3; 5, 11) = \frac{3}{4}$.

Now let $y'_1 = 0$, $y'_2 = x$, $y'_3 = 2x$ and $y'_4 = 3x$, and let A, B, C and D be the respective intersection with $x = 0$. The $A = (1, 0)$, $B = (1, 1)$, $A = (1, 2)$ and $A = (1, 3)$. The cross-ratio of A, B with respect to C, D is

$$(A, B; C, D) = \frac{3-0}{2-0} \Big/ \frac{3-1}{2-1} = 3/4$$

Thus the two configurations are equivalent.

3.4 Closing Remarks

A special case of the four subspace problem is when we restrict the base field to a finite field. For instance, let $k = \mathbb{Z}_2$. If we consider four one dimensional subspaces of a two dimensional space, we see that we cannot have four distinct inclusion, simply because the base field is too small. This means that the indecomposable representations with dimension vector $(1, 1, 1, 1, 2)$ are exhausted by the representations N_{ij} and M_{ij} where $\{i, j\} \subset \{1, 2, 3, 4\}$. The approach described in this thesis is fully adaptable to this case, however the reasoning gets more complicated. Generally the Auslander-Reiten quiver gets more complicated when one assumes that the base field is not algebraically closed. It can be mentioned that over the real numbers the tubes of rank 1 are parameterized by the upper half plane of the complex numbers, actually they are given by the irreducible polynomials over the reals. In fact this is the case when one considers an arbitrary field. There will be an indecomposable representation with dimension vector $(n, n, n, n, 2n)$ for each irreducible polynomial $p(x) \in k[x]$ such that $\deg p(x)$ divides n, except x and $x - 1$. In the case of algebraically closed field every irreducible polynomial is linear, hence we only have the representations ${}^nM_\lambda$ with $\lambda \in k \setminus \{0, 1\}$.

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