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Partial Orders in Representation Theory of Algebras

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Problem description

Let k be a field and Λ a finitely generated k-algebra, say generated by n elements $x_1, \ldots, x_n$. Consider the space of d-dimensional Λ-modules. This space can be identified with a subspace of the space $\mathcal{M}_d(k)^n$, where $\mathcal{M}_d(k)$ denotes the space of $d \times d$-matrices with entries from k. Determine when the ranks of matrices in $\mathcal{M}_m(\Lambda)$ applied to M^m will determine the isomorphism type of the Λ-module M. Especially, look at this for the path algebras of Dynkin quivers.

Abstract

In this paper we investigate some partial orders used in representation theory of algebras. Let K be a commutative ring, Λ a finitely generated K-algebra and d a natural number. We then study partial orders on the set of isomorphism classes of Λ-modules of length d. The orders degeneration, virtual degeneration and hom-order are discussed.

The main purpose of the paper is to study the relation $\leq_n$ constructed by considering the ranks of $n \times n$-matrices over Λ as K-endomorphisms on M^n for a Λ-module M. We write $M \leq_n N$ when for any $n \times n$-matrix the rank with respect to M is greater than or equal to the rank with respect to N. We study these relations for various algebras and determine when $\leq_n$ is a partial order.

Preface

This paper was written as the final part of my Master of Science degree. In January 2007, after stumbling around in the Bachelor-program for many years, I finally started in the Master-program at the Department of Mathematics at NTNU. The bulk of the thesis was written in the spring of 2008.

I would like to thank my family for the support during this period. Also, a special thanks to Professor Sverre Smalø and Post.doc. Bernt Tore Jensen for guiding me through the mathematics.

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Chapter 1

Introduction and Notation

1.1 Introduction

In this paper we will look at some partial orders used in the representation theory of algebras. Specifically, for a finitely generated algebra Λ over a commutative ring K and a natural number d, we are interested in partial orders on the set of isomorphism classes of Λ-modules of length d.

In chapter 2 the three most important such orders are described. These are called degeneration, virtual degeneration and the hom-order. The notion of degeneration originally comes from algebraic geometry, and there it only applies to finite-dimensional algebras over algebraically closed fields. Thanks to a theorem by Grzegorz Zwara we can also define degeneration in purely algebraic terms. After giving the geometric definition of degeneration, we state Zwara's theorem without proof in section 2.1. The new definition that this theorem gives us is easier to work with, and it also allows us to expand the notion of degeneration to finitely generated algebras over commutative rings. In section 2.2 we give the definitions of virtual degeneration and the hom-order, and briefly discuss the connections between these three orders.

Recently a new order was discovered, or rather a set of relations, some of which are partial orders. These relations come from considering $n \times n$-matrices over Λ as K-endomorphisms on M^n for a module M, and looking at the ranks. When for all $n \times n$-matrices the rank with respect to M is greater than or equal to the rank with respect to N we write $M \leq_n N$. Chapter 3 is devoted to studying these relations, which is the main purpose of this paper. A precise definition of $\leq_n$ is given in section 3.1. In the following sections we try to determine for which n $\leq_n$ is a partial order for various algebras. The central problem in this is to find out when the ranks completely determine the isomorphism class of a module M.

1.2 Notation

Throughout this paper, all rings have unity, and all modules are unitary. All modules are left modules unless otherwise noted.

For a ring R, $\operatorname{mod} R$ denotes the category of finitely generated R-modules. The subcategory $\operatorname{ind} R \subseteq \operatorname{mod} R$ consists of exactly one representative of each isomorphism class of indecomposable modules in $\operatorname{mod} R$. If $\operatorname{ind} R$ is finite, R is said to be of finite representation type. $\mathcal{M}_n(R)$ denotes the ring of $n \times n$-matrices with entries from R.

An R-module M is called *artin* if every descending chain of proper submodules $M \supsetneq M_1 \supsetneq M_2 \supsetneq \dots$ is finite. A ring is called artin if it is artin as a module over itself.

Let K be a commutative ring. A K*-algebra* Λ is a K-module which is also a ring such that

$$a(xy) = (ax)y = x(ay)$$

for all $a \in K$ and $x, y \in \Lambda$.

A subset $X \subseteq \Lambda$ is said to *generate* Λ if any element in Λ can be written as a sum of products of elements from X and elements from K (the products may contain several copies of each element). The elements of X are called *generators* of Λ. If there exists a finite set that generates Λ, Λ is said to be *finitely generated.*

Similarly, for a Λ-module M, a subset $Y \subseteq M$ is said to *generate* M if every element in M can be written as a sum $\sum_{y \in Y} a_y y$ with $a_y \in \Lambda$. If there exists a finite set that generates M, M is called *finitely generated.*

For a commutative artin ring K, a K-algebra Λ is called an *artin algebra* if it is finitely generated as a K-module.

Let Λ be an artin algebra and let $\Lambda \simeq \bigoplus_{i=1}^{n} P_i$ be a decomposition of Λ as a Λ-module into indecomposable projective modules. If $P_i \not\simeq P_j$ whenever $i \neq j$, Λ is called *basic.*

For a module X, $D(X)$ and $\mathrm{Tr}X$ denote the dual of X and the transpose of X respectively (see chapter IV in [1] for details).

Examples: Let K be a commutative ring.

1. $\Lambda = K$ is a K-algebra. It is generated by $\{1_K\}$ both as a K-algebra and as a K-module, so it is a finitely generated algebra, and if K is artin, Λ is also an artin algebra.

2. $\Lambda = K[X]$, the ring of polynomials in one variable over K, is a K-algebra. It is generated as an algebra by $\{1_K, X\}$, so it is a finitely generated algebra. However, it is not finitely generated as a K-module, and hence it is not an artin algebra even if K is artin.

3. $\Lambda = K\langle X, Y\rangle$, the free algebra in two non-commuting variables over K, is a K-algebra. Again, this is finitely generated as an algebra, but not as a module.

Another important example is the path algebra over a quiver. A *quiver* Γ is an oriented graph, i.e. it consists of a set of vertices, denoted Γ_0, and a set of arrows between the vertices, denoted Γ_1.

A vertex i is called a *sink* if there are no arrows starting in i. A vertex j is called a *source* if there are no arrows ending in j.

Example:

$$Q : 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$$

In Q we have $Q_0 = \{1, 2, 3\}$ and $Q_1 = \{\alpha, \beta\}$. The only sink is 3, and the only source is 1.

A *path* in the quiver is a concatenation of arrows that obeys the orientation. There is also for each vertex i a *trivial path* e_i, which is the path of length zero in the vertex i. Q has six paths: e_1, e_2, e_3, α, β and $\beta\alpha$.

Given a quiver Γ and a field k we construct the *path algebra* $k\Gamma$ in the following way: Let $k\Gamma$ be a k-vector space with the paths in Γ as basis. For two paths x and y let the product $y \cdot x$ be x concatenated with y when x ends in the vertex y starts in, and zero otherwise. The multiplication is then expanded linearly to the rest of $k\Gamma$.

Using the quiver Q from above, we then see that kQ is a six-dimensional k-algebra. We have

$$\begin{gathered} e_1 \cdot e_1 = e_1 \\ e_2 \cdot e_2 = e_2 \\ e_3 \cdot e_3 = e_3 \\ \alpha \cdot e_1 = \alpha \\ \beta\alpha \cdot e_1 = \beta\alpha \\ \beta \cdot e_2 = \beta \\ \beta \cdot \alpha = \beta\alpha \\ e_2 \cdot \alpha = \alpha \\ e_3 \cdot \beta = \beta \\ e_3 \cdot \beta\alpha = \beta\alpha. \end{gathered}$$

Any other product of two paths is zero.

For a field k and a finite quiver Γ, it is easy to see that $k\Gamma$ is a finitely generated k-algebra. If furthermore Γ has no oriented cycles, $k\Gamma$ is also finitely generated as a k-module, and since all fields are artin $k\Gamma$ is then an artin algebra. All quivers considered in this paper will be finite.

In this paper we will focus in particular on the path algebras of *Dynkin quivers*, i.e. quivers where the underlying graph is one of the following:

$$A_n: \qquad 1 \text{ —— } \cdots \text{ —— } n \qquad n \geq 2$$

$$D_n: \quad \begin{tikzcd} 1 \arrow[dr, dash] & & & \\ & 3 \arrow[r, dash] & \cdots \arrow[r, dash] & n \\ 2 \arrow[ur, dash] & & & \end{tikzcd} \qquad n \geq 4$$

$$E_6: \quad \begin{tikzcd} & & 3 \arrow[d, dash] & & \\ 1 \arrow[r, dash] & 2 \arrow[r, dash] & 4 \arrow[r, dash] & 5 \arrow[r, dash] & 6 \end{tikzcd}$$

$$E_7: \quad \begin{tikzcd} & & 3 \arrow[d, dash] & & & \\ 1 \arrow[r, dash] & 2 \arrow[r, dash] & 4 \arrow[r, dash] & 5 \arrow[r, dash] & 6 \arrow[r, dash] & 7 \end{tikzcd}$$

$$E_8: \quad \begin{tikzcd} & & 3 \arrow[d, dash] & & & & \\ 1 \arrow[r, dash] & 2 \arrow[r, dash] & 4 \arrow[r, dash] & 5 \arrow[r, dash] & 6 \arrow[r, dash] & 7 \arrow[r, dash] & 8 \end{tikzcd}$$

For a field k and a quiver Γ, a *representation* (V, f) of Γ over k consists of a k-vector space V_i for each vertex i in Γ_0 and a linear map $f_\phi : V_i \to V_j$ for each arrow ϕ from vertex i to vertex j. For example

$$0 \to k \xrightarrow{1} k$$

is a representation of Q.

A representation (V,f) of Γ gives rise to a $k\Gamma$-module M in the following way: Let $M = \bigoplus_{i \in \Gamma_0} V_i$ as a k-vectorspace. For each trivial path e_i and $x = (x_1, \ldots, x_n) \in M$ let $e_i x = x'$ where $x'_i = x_i$ and $x'_h = 0$ for $h \neq i$. For each arrow $\phi : i \to j$ and $y = (y_1, \ldots, y_n) \in M$ let $\phi y = y'$ where $y'_j = f_\phi(y_i)$ and $y'_h = 0$ for $h \neq j$. This completely determines the $k\Gamma$-multiplication on M.

Conversely, from a $k\Gamma$-module M we can construct a representation (V, f). Let $V_i = e_i M$ for all $i \in \Gamma_0$. For each arrow $\phi : i \to j$ let f_ϕ be given by $f_\phi(x) = \phi x$ for all $x \in e_i M$. The maps given in this way are k-linear, so (V, f) is a representation.

In fact, the above constructions are inverse equivalences between the category of k-representations of Γ and the category of finite dimensional $k\Gamma$-modules (see section III.1 in [1] for details). From now on we will identify modules over a path algebra with the correponding representations through this equivalence.

Chapter 2

Partial Orders

2.1 Degeneration

Definition 2.1.1. Let k be an algebraically closed field and let Λ be an artin k-algebra. Then $\text{rep}_d\Lambda$ is the set of k-algebra-homomorphisms from Λ to $\mathcal{M}_d(k)$.

To every $f \in \text{rep}_d\Lambda$ we can associate a d-dimensional module $M_f \in \text{mod}\,\Lambda$ in the following way:

Let $M_f = k^d$ as k-vector spaces, and define Λ-multiplication by $\lambda \cdot x = f(\lambda)x$ for all $\lambda \in \Lambda$ and $x \in M_f$.

Conversely, from a d-dimensional Λ-module M we can obtain a function $f_M \in \text{rep}_d\Lambda$ by fixing a k-basis for M and identifying M with k^d through this basis, and letting $f_M(\lambda)$ be the matrix where the ith column is λ times the ith basis vector. It is easily verified that f_M becomes a k-algebra homomorphism.

Example: Let $\Lambda = k(1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3)$ and let M be the Λ-module $(0 \to k \xrightarrow{1} k)$. M is 2-dimensional and we identify its elements with column vectors in k^2 through $\phi : M \to k^2$ where $(0, a, b) \mapsto (a, b)^{tr}$. Then we have

$$e_1 \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) = \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), e_1 \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) = \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right) \Rightarrow f_M(e_1) = \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$$

$$e_2 \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), e_2 \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) = \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right) \Rightarrow f_M(e_2) = \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$$

$$e_3 \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) = \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), e_3 \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \Rightarrow f_M(e_3) = \left(\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right)$$

$$\alpha \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) = \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right), \alpha \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) = \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right) \Rightarrow f_M(\alpha) = \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$$

$$\beta \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), \beta \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) = \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right) \Rightarrow f_M(\beta) = \left(\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right).$$

Then f_M is expanded linearly to all other elements in Λ.

Any $f \in \mathrm{rep}_d\Lambda$ is completely determined by its values on the generators of Λ. Since Λ is finitely generated, we can then identify f with an element in $\mathcal{M}_d(k)^n$, where n is the number of generators of Λ. The set $\mathrm{rep}_d\Lambda$ then becomes a subset of $\mathcal{M}_d(k)^n$.

Let $Gl_d(k)$ be the group of invertible $d \times d$-matrices over k. This group acts on $\mathcal{M}_d(k)^n$ by conjugation, i.e. for $G \in Gl_d(k)$ and $(A_1, A_2, \ldots, A_n) \in \mathcal{M}_d(k)^n$ we have $G*(A_1, A_2, \ldots, A_n) = (GA_1G^{-1}, GA_2G^{-1}, \ldots, GA_nG^{-1})$. For $f \in \mathrm{rep}_d\Lambda$ with $f = A \in \mathcal{M}_d(k)^n$, the map $G * f = G * A$ again is a k-algebra homomorphism, so $\mathrm{rep}_d\Lambda$ is closed under this action. In fact, for each $f \in \mathrm{rep}_d\Lambda$ and $G \in Gl_d(k)$ the module corresponding to f is isomorphic to the module corresponding to $G*f$. Hence we have a 1-1 correspondence between the isomorphism classes of d-dimensional modules and the $Gl_d(k)$-orbits in $\mathrm{rep}_d\Lambda$.

A polynomial p in nd^2 variables over k can be interpreted as a function $p : \mathcal{M}_d(k) \to k$ in the following way: For each $A = ((x^1_{ij}), (x^2_{ij}), \ldots, (x^n_{ij})) \in \mathcal{M}_d(k)$ let $p(A) = p(x^1_{11}, x^1_{12}, \ldots, x^1_{1d}, x^1_{21}, \ldots, x^1_{dd}, x^2_{11}, \ldots, x^n_{dd})$.

Definition 2.1.2. Let $f \in \mathrm{rep}_d\Lambda$ and let $Gl_d(k)f$ be its $Gl_d(k)$-orbit. The *Zariski closure* of $Gl_d(k)f$ is

$$\overline{Gl_d(k)f} = \{g \in \mathrm{rep}_d\Lambda | p(g) = 0 \text{ for all polynomials } p \text{ such that } p(Gl_d(k)f) = 0\}$$

Definition 2.1.3. Let M and N be d-dimensional Λ-modules, and let f_M and f_N be the corresponding elements in $\mathrm{rep}_d\Lambda$. *M degenerates to N*, written $M \leq_{deg} N$, if $Gl_d(k)f_N \subseteq \overline{Gl_d(k)f_M}$.

As a relation on the set of isomorphism classes of d-dimensional Λ-modules, $\leq_{deg}$ is obviously reflexive. If $M \leq_{deg} M'$ and $M' \leq_{deg} N$ then $Gl_d(k)f_{M'} \subseteq \overline{Gl_d(k)f_M}$ and hence $Gl_d(k)f_N \subseteq \overline{Gl_d(k)f_{M'}} \subseteq \overline{Gl_d(k)f_M}$, so $M \leq_{deg} N$ and hence $\leq_{deg}$ is transitive.

That the relation is also antisymmetric is easier to see using an alternative charachterization of degeneration given by the following theorem by Grzegorz Zwara:

Theorem 2.1.4. *Let k be an algebraically closed field, Λ an artin k-algebra and M and N finite-dimensional Λ-modules. $M \leq_{deg} N$ if and only if there exists a module $X \in \mathrm{mod}\,\Lambda$ and an exact sequence*

$$0 \to X \to X \oplus M \to N \to 0.$$

A proof of this theorem can be found in [5].

If $M \leq_{deg} N$ and $N \leq_{deg} M$ we have the exact sequences

$$0 \to X \to X \oplus M \to N \to 0$$

$$0 \to Y \to Y \oplus N \to M \to 0.$$

For any $A \in \mathrm{mod}\,\Lambda$ we then have the exact sequences

$$0 \to \mathrm{Hom}_\Lambda(N, A) \to \mathrm{Hom}_\Lambda(X \oplus M, A) \to \mathrm{Hom}_\Lambda(X, A)$$

$$0 \to \mathrm{Hom}_\Lambda(M,A) \to \mathrm{Hom}_\Lambda(Y \oplus N, A) \to \mathrm{Hom}_\Lambda(Y,A)$$

from which we get

$$\dim_k(\mathrm{Hom}_\Lambda(M,A)) + \dim_k(\mathrm{Hom}_\Lambda(X,A)) \leq \dim_k(\mathrm{Hom}_\Lambda(X,A)) + \dim_k(\mathrm{Hom}_\Lambda(N,A))$$

$$\Rightarrow \dim_k(\mathrm{Hom}_\Lambda(M,A)) \leq \dim_k(\mathrm{Hom}_\Lambda(N,A))$$

$$\dim_k(\mathrm{Hom}_\Lambda(N,A)) + \dim_k(\mathrm{Hom}_\Lambda(Y,A)) \leq \dim_k(\mathrm{Hom}_\Lambda(Y,A)) + \dim_k(\mathrm{Hom}_\Lambda(M,A))$$

$$\Rightarrow \dim_k(\mathrm{Hom}_\Lambda(N,A)) \leq \dim_k(\mathrm{Hom}_\Lambda(M,A))$$

and hence $\dim_k(\mathrm{Hom}_\Lambda(M,A)) = \dim_k(\mathrm{Hom}_\Lambda(N,A))$ for any $A \in \mathrm{mod}\,\Lambda$. But if M and N are nonisomorphic there exists a module B with $\dim_k(\mathrm{Hom}_\Lambda(M,B)) \neq \dim_k(\mathrm{Hom}_\Lambda(N,B))$, as will be shown in Corollary 3.4.3.

Thanks to Theorem 2.1.4 we can expand the notion of degeneration to algebras over commutative rings. Let K be a commutative ring and let Λ be a finitely generated K-algebra. We can not use the old definition of $\mathrm{rep}_d\Lambda$ for such an algebra, so we simply let $\mathrm{rep}_d\Lambda$ be the set of isomorphism classes of Λ-modules which have lenght d as K-modules. Then we use Theorem 2.1.4 as the new definition of degeneration: $M \leq_{deg} N$ if there exists an $X \in \mathrm{mod}\,\Lambda$ and an exact sequence

$$0 \to X \to X \oplus M \to N \to 0.$$

2.2 Virtual Degeneration and Hom-order

Virtual degeneration and the hom-order are other important partial orders on $\mathrm{rep}_d\Lambda$.

Definition 2.2.1. For two Λ-modules M and N, M *virtually degenerates to* N if $M \oplus X \leq_{deg} N \oplus X$ for some $X \in \mathrm{mod}\,\Lambda$. We write this as $M \leq_{vdeg} N$.

Definition 2.2.2. For two Λ-modules M and N we write $M \leq_{hom} N$ if $\ell_K(\mathrm{Hom}_\Lambda(X,M)) \leq \ell_K(\mathrm{Hom}_\Lambda(X,M))$ for all $X \in \mathrm{mod}\,\Lambda$.

It's easy to see that $M \leq_{deg} N$ implies $M \leq_{vdeg} N$, but the reverse implication does not hold in general. A counterexample was constructed by Jon Carlson, and this can be found in [3].

Proposition 2.2.3. *Let Λ be an artin algebra and let M and N be Λ-modules such that $M \leq_{vdeg} N$. Then $M \leq_{hom} N$.*

Proof. M virtually degenerates to N, so there exist Λ-modules A and B and an exact sequence

$$0 \to A \to A \oplus B \oplus M \to B \oplus N \to 0$$

Then for any Λ-module X we have an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(A,X) \to \mathrm{Hom}_\Lambda(A \oplus B \oplus M, X) \to \mathrm{Hom}_\Lambda(B \oplus N, X)$$

From this we get

$$\ell(\mathrm{Hom}_\Lambda(A \oplus B \oplus M, X)) \leq \ell(\mathrm{Hom}_\Lambda(A, X)) + \ell(\mathrm{Hom}_\Lambda(B \oplus N, X))$$

Subtracting $(\ell(\mathrm{Hom}_\Lambda(A, X)) + \ell(\mathrm{Hom}_\Lambda(B, X)))$ from each side we see that $\ell(\mathrm{Hom}_\Lambda(M, X)) \leq \ell(\mathrm{Hom}_\Lambda(N, X))$ for any Λ-module X, hence $M \leq_{hom} N$. $\square$

Here it is not known if the reverse implication holds, but it can be shown that when Λ is of finite representation type, $M \leq_{hom} N$ implies $M \leq_{deg} N$, so in that case $\leq_{deg}$, $\leq_{vdeg}$ and $\leq_{hom}$ are all equivalent. This also holds for the algebra $k[X]$, where k is a field.

Chapter 3

A New Order

3.1 The Order $\leq_n$

Let Λ be a finitely generated algebra over a commutative ring K. Throughout this section, lenght of a module always refers to its length as a K-module.

Definition 3.1.1. For a Λ-module M of finite length and an $n \times n$-matrix (λ_{ij}) with entries from Λ, let $\phi_M((\lambda_{ij}))$ be the length of the K-module $(\lambda_{ij})M^n$.

Definition 3.1.2. For two Λ-modules M and N with $\ell(M) = \ell(N)$ we write $M \leq_n N$ if $\phi_M((\lambda_{ij})) \geq \phi_N((\lambda_{ij}))$ for all $(\lambda_{ij}) \in \mathcal{M}_n(\Lambda)$.

Clearly $\leq_n$ is a quasiordering on $\text{rep}_d\Lambda$, but it is not necessarily antisymmetric. However, if n is large enough, $\leq_n$ is a partial order.

When $M \leq_n N$ we also have $M \leq_m N$ for all $m \leq n$, since any $m \times m$-matrix (λ_{ij})can be expanded to an $n \times n$-matrix (λ'_{ij}) simply by letting $\lambda'_{ij} = \lambda_{ij}$ for $i \leq m, j \leq m$ and $\lambda'_{ij} = 0$ otherwise. Consequently, if $\leq_n$ is not antisymmetric, then neither is $\leq_m$ for all $m \leq n$. Conversely, if $\leq_m$ is a partial order, then so is $\leq_n$ for all $n \geq m$.

Definition 3.1.3. Let Λ be an artin algebra.

1. For a finitely generated projective Λ-module P, $m^{\Lambda}_{Pr}(P)$ is the maximum of the multiplicities of the indecomposable Λ-modules in a decomposition of P into a direct sum of indecomposable modules.

2. For a nonprojective module $X \in \operatorname{mod}\Lambda$ with minimal projective presentation
$$P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0$$
let
$$m^{\Lambda}_{Pr}(X) = \max(m^{\Lambda}_{Pr}(P_0), m^{\Lambda}_{Pr}(P_1))$$

3. When Λ has finite representation type let
$$m_{Pr}(\Lambda) = \max_{X \in \operatorname{ind}\Lambda} m^{\Lambda}_{Pr}(X)$$

Example: Let k be a field and let Γ be the quiver

$$\begin{tikzcd}
1 & & \\
& 3 \arrow[ul, "\alpha"'] \arrow[dl, "\beta"] \arrow[r, "\gamma"] & 4 \\
2 & &
\end{tikzcd}$$

Let $\Lambda = k\Gamma$ and let M be the module with representation

$$\begin{tikzcd}
k & & \\
& k^2 \arrow[ul, "(1\ 0)"'] \arrow[dl, "(0\ 1)"] \arrow[r, "(1\ 1)"] & k \\
k & &
\end{tikzcd}.$$

M has minimal projective presentation

$$P_1 \oplus P_2 \oplus P_4 \to P_3^2 \to M \to 0$$

where $P_i = \Lambda e_i$. We have

$$m^{\Lambda}_{Pr}(P_1 \oplus P_2 \oplus P_4) = 1$$
$$m^{\Lambda}_{Pr}(P_3^2) = 2$$

and hence

$$m^{\Lambda}_{Pr}(M) = 2$$

Proposition 3.1.4. *Let Λ be an artin algebra over a commutative artin ring K, and let M, N and X be Λ-modules of finite length, with $\ell(\mathrm{Hom}_\Lambda(X, M)) \neq \ell(\mathrm{Hom}_\Lambda(X, N))$ and let $m^{\Lambda}_{Pr}(X) = n$. Then there exists an $n \times n$-matrix $(\lambda_{ij}) \in \mathcal{M}_n(\Lambda)$ such that $\phi_M((\lambda_{ij})) \neq \phi_N((\lambda_{ij}))$*

Proof. First assume that there exists an indecomposable projective module P with $\ell(\mathrm{Hom}_\Lambda(P, M)) \neq \ell(\mathrm{Hom}_\Lambda(P, N))$. We have that $P \simeq \Lambda e$ for some primitive idempotent $e \in \Lambda$. Furthermore $\mathrm{Hom}_\Lambda(\Lambda e, Y) \simeq eY$ for any $Y \in \mathrm{mod}\,\Lambda$ so we have $\phi_M(e) \neq \phi_N(e)$.

Then we look at the case where no such P exists, and consequently $\ell(\mathrm{Hom}_\Lambda(Q, M)) = \ell(\mathrm{Hom}_\Lambda(Q, N))$ for any finitely generated projective module Q.

Let

$$\eta : P_1 \to P_0 \to X \to 0$$

be a minimal projective presentation of X. For $i \in \{0, 1\}$, we then have $P_i \oplus Q_i \simeq \Lambda^n$ for some projective module Q_i. Adding the exact sequence $Q_1 \xrightarrow{0} Q_0 \xrightarrow{id} Q_0 \to 0$ to η we get

$$\mu : \Lambda^n \xrightarrow{f} \Lambda^n \to X \oplus Q_0 \to 0,$$

an exact sequence where f can be expressed by a matrix $A \in \mathcal{M}_n(\Lambda)$. Applying $\mathrm{Hom}_\Lambda(-, M)$ and $\mathrm{Hom}_\Lambda(-, N)$ to μ we get that

$$0 \to \mathrm{Hom}_\Lambda(X, M) \oplus \mathrm{Hom}_\Lambda(Q_0, M) \to M^n \overset{\mathrm{Hom}_\Lambda(f,M)}{\to} M^n$$

and

$$0 \to \mathrm{Hom}_\Lambda(X, N) \oplus \mathrm{Hom}_\Lambda(Q_0, N) \to N^n \overset{\mathrm{Hom}_\Lambda(f,N)}{\to} N^n$$

are exact sequences of K-modules. This gives us

$$\ell(M^n) = \ell(\mathrm{Hom}_\Lambda(X, M)) + \ell(\mathrm{Hom}_\Lambda(Q_0, M)) + \ell(\mathrm{im}\,(\mathrm{Hom}_\Lambda(f, M)))$$

$$\ell(N^n) = \ell(\mathrm{Hom}_\Lambda(X, N)) + \ell(\mathrm{Hom}_\Lambda(Q_0, N)) + \ell(\mathrm{im}\,(\mathrm{Hom}_\Lambda(f, N)))$$

M and N have the same length, and Q_0 is projective so by assumption $\ell(\mathrm{Hom}_\Lambda(Q_0, M)) = \ell(\mathrm{Hom}_\Lambda(Q_0, N))$. Hence we get

$$\ell(\mathrm{Hom}_\Lambda(X, M)) + \phi_M(A) = \ell(\mathrm{Hom}_\Lambda(X, N)) + \phi_N(A)$$

and since $\ell(\mathrm{Hom}_\Lambda(X, M)) \neq \ell(\mathrm{Hom}_\Lambda(X, N))$ we get $\phi_M(A) \neq \phi_N(A)$ □

This proposition gives us a nice way to decide if $\leq_n$ is a partial order on $\mathrm{rep}_d\Lambda$ when Λ has finite representation type.

Proposition 3.1.5. *Let Λ be a basic artin algebra of finite representation type. Then $\leq_n$ is a partial order on $\mathrm{rep}_d\Lambda$ for all d if and only if $n \geq m_{Pr}(\Lambda)$.*

Proof. It follows from Proposition 3.1.4 that $\leq_n$ is a partial order when $n \geq m_{Pr}(\Lambda)$. Now assume that $m_{Pr}(\Lambda) = n + 1$ and $\leq_n$ is a partial order. Then there exists an indecomposable Λ-module X with $m^\Lambda_{Pr}(X) = n + 1$. Since X is indecomposable with $m^\Lambda_{Pr}(X) \geq 2$ it can't be projective, and so there exists an almost split sequence

$$\nu : 0 \to D\mathrm{Tr}X \xrightarrow{f} E \xrightarrow{g} X \to 0.$$

Let $Z = D\mathrm{Tr}X \oplus X$. Since $\leq_n$ is a partial order there exists an $n \times n$-matrix A with $\phi_Z(A) \neq \phi_E(A)$. This matrix gives us the Λ-module $Y = \Lambda^n/(\Lambda^n A)$ which has projective presentation

$$\Lambda^n \xrightarrow{\cdot A} \Lambda^n \to Y \to 0$$

and since Λ is basic we have $m^\Lambda_{Pr}(Y) \leq n$ Applying $\mathrm{Hom}_\Lambda(-, E)$ and $\mathrm{Hom}_\Lambda(-, Z)$ to the presentation of Y we get

$$0 \to \mathrm{Hom}_\Lambda(Y, E) \to \mathrm{Hom}_\Lambda(\Lambda^n, E) \to \mathrm{Hom}_\Lambda(\Lambda^n, E)$$

$$0 \to \mathrm{Hom}_\Lambda(Y, Z) \to \mathrm{Hom}_\Lambda(\Lambda^n, Z) \to \mathrm{Hom}_\Lambda(\Lambda^n, Z)$$

This gives us $\ell(\mathrm{Hom}_\Lambda(Y, E)) = \ell(E^n) - \phi_E(A)$ and $\ell(\mathrm{Hom}_\Lambda(Y, Z)) = \ell(Z^n) - \phi_Z(A)$. Since $\ell(E) = \ell(Z)$ and $\phi_E(A) \neq \phi_Z(A)$ we then have $\ell(\mathrm{Hom}_\Lambda(Y, E)) \neq \ell(\mathrm{Hom}_\Lambda(Y, Z))$.

If X is not a direct summand in Y, any homomorphism from Y to X will factor through g, since ν is almost split. This means that $\mathrm{Hom}_\Lambda(Y, g)$ is an epimorphism, and thus

$$0 \to \mathrm{Hom}_\Lambda(Y, D\mathrm{Tr}X) \to \mathrm{Hom}_\Lambda(Y, E) \to \mathrm{Hom}_\Lambda(Y, X) \to 0$$

is exact. But then $\ell(\mathrm{Hom}_\Lambda(Y, E)) = \ell(\mathrm{Hom}_\Lambda(Y, D\mathrm{Tr}X)) + \ell(\mathrm{Hom}_\Lambda(Y, X)) = \ell(\mathrm{Hom}_\Lambda(Y, Z))$. Hence X must be a direct summand in Y. On the other hand we have $m^\Lambda_{Pr}(X) = n + 1 > n = m^\Lambda_{Pr}(Y)$, so X can't be a direct summand in Y. Hence $\leq_n$ is not a partial order. □

3.2 Hereditary Algebras of Finite Type

We will now apply Proposition 3.1.5 to some path algebras. First we look at the algebra kQ where k is a field and Q is the quiver

$$Q : 1 \xrightarrow{\alpha} 2.$$

As in section 1.2, we identify kQ-modules with representations. The only non-projective indecomposable kQ-module up to isomorphism is $(k \to 0)$ which has projective presentation

$$(0 \to k) \longrightarrow (k \xrightarrow{1} k) \longrightarrow (k \to 0) \longrightarrow 0$$

Thus we have $m_{Pr}(kQ) = 1$ and $\geq_n$ is a partial order for any $n \in \mathbb{N}$. In fact this holds for any quiver where the underlying graph is A_n.

Lemma 3.2.1. *Let k be a field, Q a quiver without oriented cycles and X an indecomposable kQ-module. Then*

$$m^{kQ}_{Pr}(X) = \max_{1\leq i\leq n}(\max\{\dim_k e_i(X/\mathrm{rad}X), \dim_k e_i(\mathrm{Tr}X/\mathrm{rad}\mathrm{Tr}X)\})$$

where n is the number of vertices in Q and e_i is the trivial path in the ith vertex.

Proof. Let $P' \to P \to X \to 0$ be a minimal projective presentation of X. Then $P \to X$ is a projective cover of X, and we have $P/\mathrm{rad}P \simeq X/\mathrm{rad}X$. The multiplicity of an indecomposable summand P_i in P is equal to the multiplicity of the corresponding simple summand S_i in $P/\mathrm{rad}P$, and thus to its multiplicity in $X/\mathrm{rad}X$. Since $X/\mathrm{rad}X$ is semisimple, this is again equal to $\dim_k e_i(X/\mathrm{rad}X)$. Hence we have $m^{kQ}_{Pr}(P) = \max_{1\leq i\leq n}(\dim_k e_i(X/\mathrm{rad}X))$.

By the definition of the transpose, $P'^* \to \mathrm{Tr}X$ is a projective cover of $\mathrm{Tr}X$, and as above we get $m^{kQ^{op}}_{Pr}(P'^*) = \max_{1\leq i\leq n}(\dim_k e_i(\mathrm{Tr}X/\mathrm{rad}\mathrm{Tr}X))$.

$(-)^* : \mathcal{P}(kQ) \to \mathcal{P}(kQ^{op})$ is a duality, so we have $m^{kQ}_{Pr}(P') = m^{kQ^{op}}_{Pr}(P'^*)$. This means that

$$\begin{aligned} m^{kQ}_{Pr}(X) &= \max\{m^{kQ}_{Pr}(P), m^{kQ^{op}}_{Pr}(P'^*)\} \\ &= \max_{1\leq i\leq n}(\max\{\dim_k e_i(X/\mathrm{rad}X), \dim_k e_i(\mathrm{Tr}X/\mathrm{rad}\mathrm{Tr}X)\}) \end{aligned}$$

□

Proposition 3.2.2. *Let Q be a quiver with underlying graph A_n, $n \in \mathbb{N}$, and let k be a field. Then $\leq_1$ is a partial order on $\mathrm{rep}_d kQ$ for any d.*

Proof. For any indecomposable kQ-module X we have that $\dim_k e_i X \leq 1$ for any vertex i in Q (2.2 in [2]). This obviously implies $\dim_k e_i(X/\mathrm{rad}X) \leq 1$. kQ^{op} also has underlying graph A_n, and thus we get $m_{Pr}(kQ) = 1$ from Lemma 3.2.1. The proposition then follows from Proposition 3.1.5. □

Similarly we can show that $\leq_2$ is a partial order for $\mathrm{rep}_d kQ$ when D_n is the underlying graph of Q. But for some orientations even $\leq_1$ is a partial order, for example the quiver Q:

$$\begin{tikzcd} 1 \arrow[dr, "\alpha"'] & & \\ & 3 \arrow[r, "\gamma"] & 4 \\ 2 \arrow[ur, "\beta"'] & & \end{tikzcd}$$

Computing the minimal projective presentations for the 12 indecomposable modules we find that $m_{Pr}(kQ) = 1$ On the other hand the quiver Q':

$$\begin{tikzcd} 1 \arrow[dr, "\alpha"'] & & \\ & 3 & 4 \arrow[l, "\gamma"] \\ 2 \arrow[ur, "\beta"'] & & \end{tikzcd}$$

has the indecomposable representation X:

$$\begin{tikzcd} k \arrow[dr, "1"'] & & \\ & k & k \arrow[l, "1"'] \\ k \arrow[ur, "1"'] & & \end{tikzcd} .$$

Denoting the projective module corresponding to the ith vertex by P_i, X has minimal projective presentation

$$P_3^2 \to P_1 \oplus P_2 \oplus P_4 \to X \to 0$$

so $m_{Pr}(kQ') = 2$.

Now consider the representations M:

$$
\begin{tikzcd}
k^2 \arrow[dr, "{\left(\begin{smallmatrix}1&0\\0&1\\0&1\end{smallmatrix}\right)}"'] & & & & k \arrow[dr, "1"'] & & & & k \arrow[dr, "{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}"'] & & \\
& k^3 & k^2 \arrow[l, "{\left(\begin{smallmatrix}1&0\\0&1\\0&0\end{smallmatrix}\right)}"'] & \simeq & & k & k \arrow[l, "1"'] & \oplus & & k^2 & k \arrow[l, "{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)}"'] \\
k^2 \arrow[ur, "{\left(\begin{smallmatrix}1&0\\0&0\\0&1\end{smallmatrix}\right)}"'] & & & & k \arrow[ur, "1"'] & & & & k \arrow[ur, "{\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)}"'] & &
\end{tikzcd}
$$

and N:

$$
\begin{tikzcd}
k^2 \arrow[dr, "{\left(\begin{smallmatrix}0&0\\1&0\\0&1\end{smallmatrix}\right)}"'] & & & & k \arrow[dr, "1"'] & & & & 0 \arrow[dr] & & & & k \arrow[dr, "1"'] & & \\
& k^3 & k^2 \arrow[l, "{\left(\begin{smallmatrix}1&0\\0&1\\0&0\end{smallmatrix}\right)}"'] & \simeq & & k & k \arrow[l, "1"'] & \oplus & & k & k \arrow[l, "1"'] & \oplus & & k & 0 \arrow[l] \\
k^2 \arrow[ur, "{\left(\begin{smallmatrix}1&0\\0&0\\0&1\end{smallmatrix}\right)}"'] & & & & 0 \arrow[ur] & & & & k \arrow[ur, "1"'] & & & & k \arrow[ur, "1"'] & &
\end{tikzcd}
$$

Here we have

$$
\begin{gathered}
\phi_M(0) = \phi_N(0) = 0 \\
\phi_M(e_1) = \phi_N(e_1) = 2 \\
\phi_M(e_2) = \phi_N(e_2) = 2 \\
\phi_M(e_3) = \phi_N(e_3) = 3 \\
\phi_M(e_4) = \phi_N(e_4) = 2 \\
\phi_M(\alpha) = \phi_N(\alpha) = 2 \\
\phi_M(\beta) = \phi_N(\beta) = 2 \\
\phi_M(\gamma) = \phi_N(\gamma) = 2
\end{gathered}
$$

For any nonzero $a \in k$ and any $x \in kQ'$ we have $\phi_M(ax) = \phi_M(x)$ and $\phi_N(ax) = \phi_N(x)$. For a general $x \in kQ'$ given by

$$x = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5\alpha + a_6\beta + a_7\gamma$$

we then have

$$\phi_M(x) = \sum_{i=1}^{4} \phi_M(a_ie_i) + \max\{3, \phi_M(a_3e_3) + \phi_M(a_5\alpha) + \phi_M(a_6\beta) + \phi_M(a_7\gamma)\}$$

$$\phi_N(x) = \sum_{i=1}^{4} \phi_N(a_ie_i) + \max\{3, \phi_N(a_3e_3) + \phi_N(a_5\alpha) + \phi_N(a_6\beta) + \phi_N(a_7\gamma)\}$$

Thus we get that $\phi_M(x) = \phi_N(x)$ for any $x \in kQ'$, but since their decompositions into direct sums of indecomposable modules are different, they are clearly nonisomorphic. Hence $\leq_1$ is not a partial order on $\mathrm{rep}_9 kQ'$.

In general we have the following:

Proposition 3.2.3. *Let Q be a quiver with underlying graph D_n :*

$$\begin{tikzcd} 1 \arrow[dr, no head] & & & \\ & 3 \arrow[r, no head] & \cdots \arrow[r, no head] & n \\ 2 \arrow[ur, no head] & & & \end{tikzcd}$$

$n \geq 4$, and let k be a field. Then $\leq_2$ is a partial order on $\mathrm{rep}_d kQ$ for any d. Furthermore, $\leq_1$ is a partial order if and only if when $3 \leq i \leq n-1$ the ith vertex is neither a sink nor a source.

Proof. Similarly to the case with A_n we have that for any kQ-module X, $\dim_k(e_i X) \leq 2$ for all i (3.2 in [2]). The first part of the proposition then follows from Lemma 3.2.1 and Proposition 3.1.5.

Now we look at the special case where sinks and sources only occur in the endpoints. Let X be an indecomposable kQ-module and let i be a vertex in Q. Assume that $\dim_k(e_i X) = 2$. Since X is indecomposable, the vector spaces in the endpoints have dimension at most one. Therefore i is not an endpoint and consequently not a source. This means that in the representation of X there is a linear map ending in i. Since X is indecomposable this map must be non-zero, and since $\mathrm{rad}X$ is generated by the linear maps in the representation, we have that $\dim_k e_i(\mathrm{rad}X) \geq 1$, and thus $\dim_k e_i(X/\mathrm{rad}X) \leq 1$.

i is not a sink in Q, and thus not a source in Q^{op}. Similar to the above we then get $\dim_k e_i(\mathrm{Tr}X/\mathrm{rad}\mathrm{Tr}X) \leq 1$.

The proposition then follows from Lemma 3.2.1 and Proposition 3.1.5. □

For quivers with underlying graphs E_6, E_7 and E_8 the minimal i that makes $\leq_i$ a partial order depends on the orientation of the quiver, just like it does for D_n. For a given orientation one can find the minimal i by computing $m_{Pr}(kQ)$. We give a summary in the following proposition:

Proposition 3.2.4. *Let k be a field.*

1. *For a quiver Q with underlying graph E_6, $\leq_3$ is a partial order on $\mathrm{rep}_d kQ$ for any d. For some orientations of Q, $\leq_2$ is also a partial order, but $\leq_1$ is not an order for any orientation.*

2. *For a quiver Q' with underlying graph E_7, $\leq_4$ is a partial order on $\mathrm{rep}_d kQ'$ for any d. For some orientations of Q', $\leq_3$ and even $\leq_2$ are also partial orders, but $\leq_1$ is not an order for any orientation.*

3. *For a quiver Q'' with underlying graph E_8, $\leq_6$ is a partial order on $\mathrm{rep}_d kQ''$ for any d. For some orientations of Q'', lower i, down to $i = 3$, make $\leq_i$ a partial order. $\leq_1$ and $\leq_2$ are not partial orders for any orientations.*

Proof. That $\leq_3$, $\leq_4$ and $\leq_6$ are always partial orders for the respective quivers follows from 4.2, 4.3 and 4.4 in [2], Lemma 3.2.1 and Proposition 3.1.5. Computing the projective presentations we see that for the quivers
Q:

$$\begin{tikzcd} & & 3 \arrow[d] & & \\ 1 \arrow[r] & 2 \arrow[r] & 4 \arrow[r] & 5 \arrow[r] & 6 \end{tikzcd}$$

Q':

$$\begin{tikzcd} & & 3 \arrow[d] & & & \\ 1 \arrow[r] & 2 \arrow[r] & 4 \arrow[r] & 5 \arrow[r] & 6 \arrow[r] & 7 \end{tikzcd}$$

Q'':

$$\begin{tikzcd} & & 3 \arrow[d] & & & & \\ 1 \arrow[r] & 2 \arrow[r] & 4 \arrow[r] & 5 \arrow[r] & 6 \arrow[r] & 7 \arrow[r] & 8 \end{tikzcd}$$

we have $m_{Pr}(kQ) = 2$, $m_{Pr}(kQ') = 2$ and $m_{Pr}(kQ'') = 3$, and hence $\leq_2$, $\leq_2$ and $\leq_3$ respectively are partial orders.

To see that $\leq_1$ is not a partial order for any path algebra over E_6 consider the indecomposable module X

$$\begin{tikzcd} & & k^2 \arrow[d, "\gamma"] & & \\ k \arrow[r, dash] & k^2 \arrow[r, dash] & k^3 \arrow[r, dash] & k^2 \arrow[r, dash] & k \end{tikzcd}$$

Regardless of the rest of the orientation, the projective cover of X must contain two copies of P_3, the indecomposable projective module corresponding to the third vertex.

If γ has the opposite direction, the indecomposable module Y:

$$\begin{tikzcd} & & k & & \\ k \arrow[r, dash] & k^2 \arrow[r, dash] & k^3 \arrow[r, dash] \arrow[u, "\gamma"'] & k^2 \arrow[r, dash] & k \end{tikzcd}$$

must have two copies of P_3 in the syzygy.

Similarly for E_7 and E_8 consider the indecomposable modules

$$\begin{tikzcd}[arrows=-] & & k^2 \arrow[d] & & & \\ k^2 \arrow[r] & k^3 \arrow[r] & k^4 \arrow[r] & k^3 \arrow[r] & k^2 \arrow[r] & k \end{tikzcd}$$

and

$$\begin{tikzcd}[arrows=-] & & k^3 \arrow[d] & & & & \\ k^2 \arrow[r] & k^4 \arrow[r] & k^6 \arrow[r] & k^5 \arrow[r] & k^3 \arrow[r] & k^2 \arrow[r] & k \end{tikzcd}$$

respectively. □

Assuming that k is algebraically closed, we have now investigated all the representation-finite, hereditary artin k-algebras, and conclude this section with

Corollary 3.2.5. *Let k be an algebraically closed field and let Λ be a basic hereditary k-algebra of finite representation type. Then $\leq_6$ is a partial order on* $\mathrm{rep}_d\Lambda$ *for any d.*

Proof. Follows from Propositions 3.2.2, 3.2.3 and 3.2.4. □

3.3 Trivial Extensions

Let R be a basic hereditary artin algebra of finite representation type, let $Q = D(R)$ as an R-R-bimodule, and let $\Lambda = R \ltimes Q$ be the trivial extension of R by Q. In this section we show that when $\leq_n$ is a partial ordering on $\mathrm{rep}_d R$, it is also a partial ordering of $\mathrm{rep}_d\Lambda$.

The additive structure of Λ is just the direct sum of R and Q, and the multiplication is defined as follows: For $r, r' \in R$ and $f, f' \in Q$ let $(r, f)(r', f') = (rr', fr' + rf')$.

Proposition 3.3.1. Λ *is self-injective.*

Proof. First we need to find the top of Λ. Let $\mathfrak{r}$ be the radical of R. By Proposition I.3.3 in [1] an ideal J in a left artin ring R is the radical if and only if it is nilpotent and R/J is semisimple. Let n be the smallest number such that $\mathfrak{r}^n = 0$. Clearly $(\mathfrak{r}, Q) = \{(r, q) \in \Lambda | r \in \mathfrak{r}, q \in Q\}$ is an ideal in Λ. We have that $(\mathfrak{r}, Q)^n \subseteq (0, Q)$, and thus $(\mathfrak{r}, Q)^{2n} = 0$ so $(\mathfrak{r}, Q)$ is nilpotent. Further, $\Lambda/(\mathfrak{r}, Q) \simeq R/\mathfrak{r}$ is semisimple, so $(\mathfrak{r}, Q)$ is the radical of Λ. Then

$$\Lambda/\mathrm{rad}\Lambda = (R, Q)/(\mathfrak{r}, Q) \simeq R/\mathfrak{r}$$

Next we find the socle of Λ. Assume that (U, V) is a semisimple submodule of Λ. $(0, V)$ is a submodule of (U, V), and since (U, V) is semisimple, $(0, V)$ is a direct summand. Then $(U, 0)$ is also a submodule. Since $(r, f)(u, 0) = (ru, fu)$ for $(r, f) \in \Lambda, (u, 0) \in (U, 0)$ we must have $fu = 0$ for all $f \in Q, u \in U$, hence

$U = 0$. It follows that $\operatorname{soc}\Lambda = (0, \operatorname{soc} Q) \simeq \operatorname{soc} Q$. Since $Q = D(R)$ we have $\operatorname{soc} Q \simeq R/\mathfrak{r}$.

Thus we have $\Lambda/\mathrm{rad}\Lambda \simeq \operatorname{soc}\Lambda$, which means that Λ is self-injective. $\square$

Let X be a Λ-module. We identify Q with the ideal $(0, Q)$ and let $U = X/QX$ and $V = QX$. X can be described by an R-homomorphism $Q \otimes_R U \xrightarrow{\psi} V$. This is called the canonical expression of X. We have that $X \simeq U \oplus V$ as R-modules. The Λ-multiplication in X is then defined by

$$(r, f)(u, v) = (ru, rv + \psi(f \otimes u))$$

where $r \in R$, $f \in D(R)$, $u \in U$ and $v \in V$.

A Λ-homomorphism can be described by a 2×2-matrix of R-homomorphisms in the following way:

$$(Q \otimes_R U \xrightarrow{\psi} V) \xrightarrow{\left(\begin{smallmatrix} f & 0 \\ g & h \end{smallmatrix}\right)} (Q \otimes_R U' \xrightarrow{\psi'} V')$$
$$(u, v) \mapsto (f(u), g(u) + h(v))$$

where $f : U \to U'$, $g : U \to V'$, $h : V \to V'$ and the diagram

$$\begin{tikzcd} Q \otimes_R U \arrow[r, "\psi"] \arrow[d, "1_Q \otimes f"'] & V \arrow[d, "h"] \\ Q \otimes_R U' \arrow[r, "\psi'"] & V' \end{tikzcd}$$

commutes.

We will need the following propositions from [4]:

Proposition 3.3.2. *Let $(\psi : Q \otimes_R U \to V)$ be the canonical expression of a Λ-module X. When $\psi \neq 0$, X is indecomposable if and only if one of the following conditions hold:*

1. *ψ is an isomorphism and U is indecomposable and projective. In this case, X is a projective and injective Λ-module.*

2. *ψ is an epimorphism, U is projective and $\ker\psi$ is indecomposable and is an essential submodule of $Q \otimes_R U$.*

Proof. First we show that when X is indecomposable, U is projective and ψ is an epimorphism.

The R-homomorphism

$$\phi : \bigoplus_{x \in X} Q \to QX$$

$$(q_x)_{x \in X} \mapsto \sum_{x \in X} q_x x$$

is an epimorphism and Q is injective. Thus $V = QX$ is a factor of an injective R-module, and since R is hereditary, V is also injective. Applying $\mathrm{Hom}_R(Q, -)$ to $Q \otimes_R U \xrightarrow{\psi} V$ we get

$$\mathrm{Hom}_R(Q, Q \otimes_R U) \xrightarrow{\mathrm{Hom}_R(Q,\psi)} \mathrm{Hom}_R(Q, V).$$

By the adjoint isomorphism we have

$$\mathrm{Hom}_R(Q, Q \otimes_R U) \simeq \mathrm{Hom}_R(\mathrm{Hom}_R(Q, Q), U) \simeq \mathrm{Hom}_R(R, U) \simeq U$$

so $U' = \mathrm{im}\,\mathrm{Hom}_R(Q, \psi)$ is a factor of U. It is also a submodule of $\mathrm{Hom}_R(Q, V)$, which is projective since V is injective. R is hereditary so U' is also projective. Therefore U' is a direct summand of U. Since $\psi \neq 0$ we have $U' \neq 0$. Let $V' = \psi(Q \otimes_R U')$. Since U' is projective, $Q \otimes_R U'$ is injective, and so is its factor module V'. Therefore V' is a direct summand in V, and hence $(Q \otimes_R U' \xrightarrow{\psi|_{Q \otimes_R U'}} V')$ is a direct summand of X. If X is indecomposable we then have $U = U'$ and $V = V'$ and the claim follows.

Now assume that ψ is an isomorphism. If $U \simeq U_1 \oplus U_2$ is a decomposition of U then $(Q \otimes_R U_1 \xrightarrow{1} Q \otimes_R U_1) \oplus (Q \otimes_R U_2 \xrightarrow{1} Q \otimes_R U_2)$ is a decomposition of X.

Conversely assume that $X \simeq X_1 \oplus X_2$ is a decomposition of X. Let $(Q \otimes_R U_i \xrightarrow{\psi|_{Q \otimes_R U_i}} V_i)$ be the canonical expression of X_i. If U is indecomposable either U_1 or U_2 must be zero, but if $U_i = 0$ then $V_i = \psi(Q \otimes_R U_i) = 0$ and consequently $X_i = 0$. Hence, when ψ is an isomorphism, X is indecomposable as a Λ-module if and only if U is indecomposable as an R-module.

Since U is an indecomposable and projective R-module, it is a direct summand of R. Thus $X = (Q \otimes_R U \xrightarrow{1} Q \otimes_R U)$ is a direct summand of $\Lambda = (Q \otimes_R R \to Q)$, so X is a projective Λ-module. Since Λ is self-injective by Proposition 3.3.1, X is then also injective.

Now we look at the case where ψ is not an isomorphism. Assume first that $\ker\psi$ is not essential. Then there exists a submodule $Y \subseteq Q \otimes_R U$ with $\ker\psi \cap Y = 0$. Let $U' = \mathrm{Hom}_R(Q, Y)$ and $V' = \psi(Y)$. Since Y does not intersect $\ker\psi$, ψ restricted to Y is a monomorphism, and by the definition of V' it is also an epimorphism. Hence it is an isomorphism and $(Q \otimes_R U' \to V')$ is an injective submodule of X, and therefore a direct summand. If X is indecomposable this contradicts the assumption that ψ is not an isomorphism.

Now assume that $\ker\psi$ is essential and let $X \simeq (Q \otimes_R U_1 \xrightarrow{\psi_1} V_1) \oplus (Q \otimes_R U_2 \xrightarrow{\psi_2} V_2)$ be a proper decomposition of X. Then $\ker\psi \simeq \ker\psi_1 \oplus \ker\psi_2$. If $\ker\psi_i = 0$ then $\ker\psi \cap (Q \otimes_R U_i) = 0$ but this is impossible since $\ker\psi$ is essential. Hence $\ker\psi$ is decomposable.

Conversely, let $\ker\psi \simeq Y_1 \oplus Y_2$ be a proper decomposition. Since $Q \otimes_R U$ is injective and $\ker\psi$ is an esential submodule, $Q \otimes_R U$ is the injective envelope of $\ker\psi$. Hence $Q \otimes_R U \simeq I_1 \oplus I_2$ where I_i is the injective envelope of Y_i. Let $U_i = \mathrm{Hom}_R(Q, I_i)$ and $V_i = \psi(I)$. Then $X \simeq (Q \otimes_R U_1 \xrightarrow{\psi_1} V_1) \oplus (Q \otimes_R U_2 \xrightarrow{\psi_2} V_2)$ is a proper decomposition of X.

$\square$

Proposition 3.3.3. *1. For an indecomposable R-module X let $\rho : P \to X$ be the projective cover. Then*

$$(Q \otimes_R P \xrightarrow{1} Q \otimes_R P) \xrightarrow{\left(\begin{smallmatrix} \rho & 0 \\ 0 & 0 \end{smallmatrix}\right)} (Q \otimes_R X \to 0) \to 0$$

is the projective cover of X as a Λ-module.

2. For an indecomposable Λ-module $(Q \otimes_R U \xrightarrow{\psi} V)$ with $\psi \neq 0$

$$(Q \otimes_R U \xrightarrow{1} Q \otimes_R U) \xrightarrow{\left(\begin{smallmatrix} 1_U & 0 \\ 0 & \psi \end{smallmatrix}\right)} (Q \otimes_R U \xrightarrow{\psi} V) \to 0$$

is the projective cover.

Proof. Let $Y = (Q \otimes_R P \xrightarrow{1} Q \otimes_R P)$. Then we have $Y/(\mathfrak{r}, Q)Y \simeq P/\mathfrak{r}P \simeq X/\mathfrak{r}X$. $\left(\begin{smallmatrix} \rho & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is clearly an epimorphism, so it is the projective cover of X as a Λ-module.

In case 2., let $Y' = (Q \otimes_R U \xrightarrow{1} Q \otimes_R U)$. Then we have $Y'/(\mathfrak{r}, Q)Y' \simeq U/\mathfrak{r}U \simeq X/(\mathfrak{r}, Q)X$. Again, $\left(\begin{smallmatrix} 1_U & 0 \\ 0 & \psi \end{smallmatrix}\right)$ is an epimorphism, so it is the projective cover of X.

$\square$

Any module over Λ is also a module over R. Thus the indecomposable Λ-modules can be divided into two cases, those that decompose over R and those that don't. First we look at the indecomposable Λ-modules that are also indecomposable over R. For such a module X we have that $QX = 0$ and hence the canonical expression is $(Q \otimes_R X \to 0)$.

Proposition 3.3.4. *Let X be an indecomposable non-projective R-module. Then*

$$m^R_{Pr}(X) = m^\Lambda_{Pr}(X)$$

Proof. Let

$$0 \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} X \to 0$$

be a minimal projective resolution of X as an R-module. Then the sequence

$$0 \to (Q \otimes_R P_1 \xrightarrow{1_Q \otimes f_1} Q \otimes_R P_0) \xrightarrow{\left(\begin{smallmatrix} f_1 & 0 \\ 0 & 1_{P_0} \end{smallmatrix}\right)} (Q \otimes_R P_0 \xrightarrow{1} Q \otimes_R P_0) \xrightarrow{\left(\begin{smallmatrix} f_0 & 0 \\ 0 & 0 \end{smallmatrix}\right)} (Q \otimes_R X \to 0) \to 0$$

is exact, and by Proposition 3.3.3 $\left(\begin{smallmatrix} f_0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is a projective cover of X as a Λ-module. Since $(Q \otimes_R P_1 \xrightarrow{1_Q \otimes f_1} Q \otimes_R P_0)$ is a canonical expression, we know from Proposition 3.3.2 that $1_Q \otimes f_1$ is an epimorphism. Thus

$$(Q \otimes_R P_1 \xrightarrow{1} Q \otimes_R P_1) \xrightarrow{\left(\begin{smallmatrix} 1_{P_1} & 0 \\ 0 & 1_Q \otimes f_1 \end{smallmatrix}\right)} (Q \otimes_R P_1 \xrightarrow{1_Q \otimes f_1} Q \otimes_R P_0)$$

is an epimorphism, and then

$$(Q \otimes_R P_1 \xrightarrow{1} Q \otimes_R P_1) \xrightarrow{\begin{pmatrix} f_1 & 0 \\ 0 & 1_Q \otimes f_1 \end{pmatrix}} (Q \otimes_R P_0 \xrightarrow{1} Q \otimes_R P_0) \xrightarrow{\begin{pmatrix} f_0 & 0 \\ 0 & 0 \end{pmatrix}} (Q \otimes_R X \to 0) \to 0$$

is a projective presentation of X as a Λ-module.

For any direct sum of projective R-modules $P_1 \oplus P_2$ we have

$$(Q \otimes_R (P_1 \oplus P_2) \xrightarrow{1} Q \otimes_R (P_1 \oplus P_2)) \simeq (Q \otimes_R P_1 \xrightarrow{1} Q \otimes_R P_1) \oplus (Q \otimes_R P_2 \xrightarrow{1} Q \otimes_R P_2)$$

Together with Proposition 3.3.2 this shows that for any projective R-module P, $m^R_{Pr}(P) = m^\Lambda_{Pr}(Q \otimes_R P \xrightarrow{1} Q \otimes_R P)$. Hence we have

$$m^R_{Pr}(X) = m^\Lambda_{Pr}(X)$$

□

Proposition 3.3.5. *Let P be an indecomposable projective R-module. Then*

$$m^\Lambda_{Pr}(P) \leq m_{Pr}(R)$$

Proof.

$$0 \to (Q \otimes_R Q \otimes_R P \to 0) \xrightarrow{\begin{pmatrix} 0 & 0 \\ 1_{Q \otimes P} & 0 \end{pmatrix}} (Q \otimes_R P \xrightarrow{1} Q \otimes_R P) \xrightarrow{\begin{pmatrix} 1_P & 0 \\ 0 & 0 \end{pmatrix}} (Q \otimes_R P \to 0) \to 0$$

is an exact sequence of Λ-modules, and by Proposition 3.3.3 $\begin{pmatrix} 1_P & 0 \\ 0 & 0 \end{pmatrix}$ is a projective cover of P.

Let

$$f : P' \to Q \otimes_R P$$

be a projective cover of $Q \otimes_R P$ as an R-module. Again by Proposition 3.3.3

$$(Q \otimes_R P' \xrightarrow{1} Q \otimes_R P') \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}} (Q \otimes_R Q \otimes_R P \to 0)$$

is a projective cover of $(Q \otimes_R Q \otimes_R P \to 0)$, and hence

$$(Q \otimes_R P' \xrightarrow{1} Q \otimes_R P') \xrightarrow{\begin{pmatrix} 0 & 0 \\ f & 0 \end{pmatrix}} (Q \otimes_R P \xrightarrow{1} Q \otimes_R P) \xrightarrow{\begin{pmatrix} 1_P & 0 \\ 0 & 0 \end{pmatrix}} (Q \otimes_R P \to 0) \to 0$$

is a projective presentation of P as a Λ-module. $(Q \otimes_R P \xrightarrow{1} Q \otimes_R P)$ is indecomposable, so we have

$$m^\Lambda_{Pr}(P) = m^\Lambda_{Pr}((Q \otimes_R P' \xrightarrow{1} Q \otimes_R P')) = m^R_{Pr}(P') \leq m_{Pr}(R)$$

since $Q \otimes_R P$ is an indecomposable R-module.

□

Now we turn to the indecomposable Λ-modules that decompose over R. For such a module $(Q \otimes_R U \xrightarrow{\psi} V)$ we have that ψ is non-zero.

Proposition 3.3.6. *Let $X = (Q \otimes_R U \xrightarrow{1} V)$ be an indecomposable Λ-module with $\psi \neq 0$. Then*

$$m^{\Lambda}_{Pr}(X) \leq m_{Pr}(R)$$

Proof. Let $i : \ker\psi \to Q \otimes_R U$ be the inclusion. Then

$$0 \to (Q \otimes_R \ker\psi \to 0) \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ i & 0 \end{smallmatrix}\right)} (Q \otimes_R U \xrightarrow{1} Q \otimes_R U) \xrightarrow{\left(\begin{smallmatrix} 1_U & 0 \\ 0 & \psi \end{smallmatrix}\right)} (Q \otimes_R U \xrightarrow{\psi} V) \to 0$$

is an exact sequence of Λ-modules where $\left(\begin{smallmatrix} 1_U & 0 \\ 0 & \psi \end{smallmatrix}\right)$ is a projective cover by Proposition 3.3.3. Let $(Q \otimes_R P \xrightarrow{1} Q \otimes_R P) \xrightarrow{\left(\begin{smallmatrix} f & 0 \\ 0 & 0 \end{smallmatrix}\right)} (Q \otimes_R \ker\psi \to 0)$ be a projective cover of $\ker\psi$ as a Λ-module. Then

$$(Q \otimes_R P \xrightarrow{1} Q \otimes_R P) \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ if & 0 \end{smallmatrix}\right)} (Q \otimes_R U \xrightarrow{1} Q \otimes_R U) \xrightarrow{\left(\begin{smallmatrix} 1_U & 0 \\ 0 & \psi \end{smallmatrix}\right)} (Q \otimes_R U \xrightarrow{\psi} V) \to 0$$

is a projective presentation of X as a Λ-module. By Proposition 3.3.2 $\ker\psi$ is an indecomposable R-module, and so by Proposition 3.3.5 (if it is projective) or Proposition 3.3.4 (if it is not) we have $m^{\Lambda}_{Pr}((Q \otimes_R P \xrightarrow{1} Q \otimes_R P)) \leq m_{Pr}(R)$.

$Q \otimes_R U$ is an injective R-module, and by Proposition 3.3.2 $\ker\psi$ is an essential submodule of $Q \otimes_R U$, hence $i : \ker\psi \to Q \otimes_R U$ is an injective envelope. Then by Proposition IV.1.12 in [1] $(D(Q \otimes_R U))^* \to \mathrm{Tr}D\ker\psi \to 0$ is a projective cover. Thus we get

$$m^{\Lambda}_{Pr}(Q \otimes_R U \xrightarrow{1} Q \otimes_R U) = m^{R}_{Pr}(D(Q \otimes_R U)^*) \leq m^{R}_{Pr}(\mathrm{Tr}D\ker\psi) \leq m_{Pr}(R).$$

□

Now we have checked all non-projective indecomposable Λ-modules and can conclude with

Corollary 3.3.7. *Let R be a basic hereditary artin algebra of finite representation type with $m_{Pr}(R) = n$ and let $\Lambda = R \ltimes D(R)$ be a trivial extension. Then $\leq_n$ is a partial order for $\mathrm{rep}_d\Lambda$ for any d.*

Proof. From Propositions 3.3.4, 3.3.5 and 3.3.6 it follows that $m_{Pr}(R) = m_{Pr}(\Lambda)$. The statement then follows from Proposition 3.1.5. □

3.4 Algebras of Infinite Representation Type

For an algebra Λ of infinite representation type there is of course no n such that $\leq_n$ is a partial order on $\mathrm{rep}_d\Lambda$ for all d. However, for a given d one can still find an n that makes $\leq_n$ a partial order. In [3] it was stated without a proof that $\leq_{d^5}$ always is a partial order on $\mathrm{rep}_d\Lambda$. In this section we will prove this result.

Let K be a commutative ring and let Λ be a finitely generated K-algebra. Throughout this section, lenght of a module always refers to its length as a K-module. We need the following lemma:

Lemma 3.4.1. *Let K be a commutative ring, and let M and N be K-modules of length m and n respectively, $m, n \in \mathbb{N}$. Then*

$$\ell(\mathrm{Hom}_K(M, N)) \leq mn$$

Proof. We prove this by induction on the lengths of M and N. First we show that $\ell(\mathrm{Hom}_K(S', S)) \leq 1$ where S and S' are simple K-modules. Since S and S' are simple any morphism between them is either 0 or an isomorphism. Thus if $\mathrm{Hom}_K(S', S) \neq (0)$ we have $S \simeq S' \simeq K/I$ where I is a maximal ideal in K. Then we have

$$\mathrm{Hom}_K(S', S) \simeq \mathrm{Hom}_K(K/I, K/I) \simeq \mathrm{Hom}_{K/I}(K/I, K/I) \simeq K/I$$

so it has length 1.

Then we use induction on the length of M to show that $\ell(\mathrm{Hom}_K(M, S)) \leq m$ for any simple module S. We have that $M/M' \simeq S'$ for some M' of length $m-1$ and a simple module S'. This gives us the exact sequence

$$0 \to M' \to M \to S' \to 0$$

Applying $\mathrm{Hom}_K(-, S)$ we get the exact sequence

$$0 \to \mathrm{Hom}_K(S', S) \to \mathrm{Hom}_K(M, S) \to \mathrm{Hom}_K(M', S)$$

We assume by induction that $\ell(\mathrm{Hom}_K(M', S)) \leq m - 1$, so

$$\ell(\mathrm{Hom}_K(M, S)) \leq \ell(\mathrm{Hom}_K(S', S)) + \ell(\mathrm{Hom}_K(M', S)) \leq m$$

Similarly we have $N/N' \simeq S$ for a submodule N' of length $n-1$ and a simple module S. The exact sequence

$$0 \to N' \to N \to S \to 0$$

gives us that

$$0 \to \mathrm{Hom}_K(M, N') \to \mathrm{Hom}_K(M, N) \to \mathrm{Hom}_K(M, S)$$

is also exact. $\ell(\mathrm{Hom}_K(M, N')) \leq m(n-1)$ by the induction hypothesis, so

$$\ell(Hom_K(M, N)) \leq \ell(\mathrm{Hom}_K(M, N')) + \ell(\mathrm{Hom}_K(M, S)) \leq m(n-1) + m = mn$$

□

Proposition 3.4.2. *Let Λ be a finitely generated algebra over a commutative ring K. For $n \geq d^5$ the relation $\leq_n$ is a partial order on* $\mathrm{rep}_d\Lambda$.

Proof. We need to show that $\leq_n$ is antisymmetric when $n \geq d^5$. Let M and N be non-isomorphic Λ-modules of length d. First we want to show that there exists a Λ-module X with $\ell(\mathrm{Hom}_\Lambda(X, M)) \neq \ell(\mathrm{Hom}_\Lambda(X, N))$ and $\ell(X) \leq d^3$.

Let $\{f_1, f_2, \ldots, f_m\}$ be a generating set for $\mathrm{Hom}_\Lambda(M, N)$ as a K-module. Then we have an exact sequence

$$M \stackrel{(f_1,f_2,\ldots,f_m)^{tr}}{\longrightarrow} N^m \to C \to 0$$

where C is the cokernel of $(f_1, f_2, \ldots, f_m)^{tr}$. We have $\mathrm{Hom}_\Lambda(M, N) \subseteq \mathrm{Hom}_K(M, N)$ as a K-module, so $m \leq d^2$ by Lemma 3.4.1. Hence $\ell(C) \leq \ell(N^m) \leq d^3$. Applying $\mathrm{Hom}_\Lambda(-, M)$ and $\mathrm{Hom}_\Lambda(-, N)$ to the above sequence we get

$$\mu : 0 \to \mathrm{Hom}_\Lambda(C, M) \to \mathrm{Hom}_\Lambda(N^m, M) \stackrel{\mathrm{Hom}_\Lambda((f_1,f_2,\ldots,f_m)^{tr},M)}{\longrightarrow} \mathrm{Hom}_\Lambda(M, M)$$

and

$$\nu : 0 \to \mathrm{Hom}_\Lambda(C, N) \to \mathrm{Hom}_\Lambda(N^m, N) \stackrel{\mathrm{Hom}_\Lambda((f_1,f_2,\ldots,f_m)^{tr},N)}{\longrightarrow} \mathrm{Hom}_\Lambda(M, N)$$

$\mathrm{Hom}_\Lambda((f_1, f_2, \ldots, f_m)^{tr}, N)$ is an epimorphism by construction. Now we assume

$$\ell(\mathrm{Hom}_\Lambda(C, M)) = \ell(\mathrm{Hom}_\Lambda(C, N)) \tag{3.1}$$

$$\ell(\mathrm{Hom}_\Lambda(M, M)) = \ell(\mathrm{Hom}_\Lambda(M, N)) \tag{3.2}$$

$$\ell(\mathrm{Hom}_\Lambda(N, M)) = \ell(\mathrm{Hom}_\Lambda(N, N)) \tag{3.3}$$

From the sequences μ and ν we then get

$$\ell(\mathrm{im}\,\mathrm{Hom}_\Lambda((f_1, f_2, \ldots, f_m)^{tr}, M)) = \ell(\mathrm{Hom}_\Lambda(N^m, M)) - \ell(\mathrm{Hom}_\Lambda(C, M))$$

$$= \ell(\mathrm{Hom}_\Lambda(N^m, N)) - \ell(\mathrm{Hom}_\Lambda(C, N)) = \ell(\mathrm{Hom}_\Lambda(M, N)) = \ell(\mathrm{Hom}_\Lambda(M, M))$$

and hence $\mathrm{Hom}_\Lambda((f_1, f_2, \ldots, f_m)^{tr}, M)$ is an epimorphism. In particular the identity on M factors through $(f_1, f_2, \ldots, f_m)^{tr}$, so $(f_1, f_2, \ldots, f_m)^{tr}$ must be a split monomorphism. Then M and N must have common nonzero direct summand. Consequently, if M and N have no common nonzero summands, one of the assumptions (3.1), (3.2) and (3.3) must fail and we have found the desired X. If M and N have a common summand then for some Y we have $M \simeq M' \oplus Y$ and $N \simeq N' \oplus Y$ where M' and N' have no common nonzero direct summands, and we can use the above argument on M' and N' to obtain X.

We now show that there exists a $d^5 \times d^5$-matrix (λ_{ij}) with $\phi_M((\lambda_{ij})) \neq \phi_N((\lambda_{ij}))$

If $\mathrm{ann}\, M \neq \mathrm{ann}\, N$ there is a $\lambda \in \Lambda$ with $\phi_M(\lambda) \neq \phi_N(\lambda)$, so we assume $\mathrm{ann}\, M = \mathrm{ann}\, N$. We have $\mathrm{Hom}_\Lambda(X, M) \simeq \mathrm{Hom}_\Lambda(X/(\mathrm{ann}\, M)X, M)$, so we may assume that X is annihilated by $\mathrm{ann}\, M$. Then we have $\mathrm{Hom}_\Lambda(X, M) \simeq \mathrm{Hom}_{\Lambda/\mathrm{ann}\, M}(X, M)$.

We have $\Lambda/\operatorname{ann} M \subseteq \operatorname{End}_K M$ and thus $\ell(\Lambda/\operatorname{ann} M) \leq d^2$. We can now make a free resolution of X as a $\Lambda/\operatorname{ann} M$-module:

$$(\Lambda/\operatorname{ann} M)^{d^2x} \to (\Lambda/\operatorname{ann} M)^x \to X \to 0$$

where $x = \ell(X)$, and from this we get the exact sequence

$$(\Lambda/\operatorname{ann} M)^{d^2x} \stackrel{(\lambda_{ij})}{\to} (\Lambda/\operatorname{ann} M)^{d^2x} \stackrel{g}{\to} X \oplus (\Lambda/\operatorname{ann} M)^{(d^2x-x)} \to 0$$

Applying the functors $\operatorname{Hom}_{\Lambda/\operatorname{ann} M}(-, M)$ and $\operatorname{Hom}_{\Lambda/\operatorname{ann} M}(-, N)$ to this sequence we get

$$0 \longrightarrow \operatorname{Hom}_{\Lambda/\operatorname{ann} M}(X, M) \oplus M^{d^2x-x} \longrightarrow M^{d^2x} \xrightarrow{(\lambda_{ij})} M^{d^2x}$$

and

$$0 \longrightarrow \operatorname{Hom}_{\Lambda/\operatorname{ann} M}(X, N) \oplus N^{d^2x-x} \longrightarrow N^{d^2x} \xrightarrow{(\lambda_{ij})} N^{d^2x}$$

and this gives

$$\phi_M(\lambda_{ij}) = \ell(M^{d^2x}) - (\ell(M^{d^2x-x}) + \ell(\operatorname{Hom}_\Lambda(X, M))) = dx - l(\operatorname{Hom}_\Lambda(X, M))$$

$$\phi_N(\lambda_{ij}) = \ell(N^{d^2x}) - (l(N^{d^2x-x}) + \ell(\operatorname{Hom}_\Lambda(X, N))) = dx - \ell(\operatorname{Hom}_\Lambda(X, N)).$$

Since $\ell(\operatorname{Hom}_\Lambda(X, M)) \neq \ell(\operatorname{Hom}_\Lambda(X, N))$ we have $\phi_M(\lambda_{ij}) \neq \phi_N(\lambda_{ij})$. □

Corollary 3.4.3. *Let M and N be non-isomorphic Λ-modules. Then there exists a Λ-module X such that $\ell(\operatorname{Hom}_\Lambda(X, M)) \neq \ell(\operatorname{Hom}_\Lambda(X, N))$.*

Proof. Follows from the first part of the preceding proof. □

The result in Corollary 3.4.3 was first proved by Maurice Auslander, and later generalized by Klaus Bongartz (see comments on p. 223 in [1]).

Chapter 4

Summary

Proposition 3.4.2 ensures that for any finitely generated algebra and a number d we can find an n such that $\leq_n$ is a partial order on $\text{rep}_d\Lambda$, namely $n = d^5$. However, in all the examples we have computed, the smallest n that makes $\leq_n$ a partial order is much smaller than d^5, except of course when $d = 1$. For the hereditary algebras of finite representation type over an algebraically closed field, we have that $\leq_6$ is always a partial order regardless of d, and for many of these algebras even lower values of n do the trick. The same goes for the trivial extensions of these algebras by their dual. This suggests it might be possible to find a better bound for when $\leq_n$ is a partial order.

Other questions that have yet to be answered:

- Does Proposition 3.1.5 hold also for non-basic algebras?
- Hilbert's basis theorem ensures that when n is large enough, $\leq_n$ is equivalent to $\leq_{hom}$. How large must n be for this to happen?
- Does there exist a Λ, $\text{rep}_d\Lambda$ and n such that $\leq_n$ is a partial order but not equivalent to $\leq_{hom}$?

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