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On the Convergence of Limit-Periodic Continued Fractions

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Master of Science in Mathematics
Submission date: May 2008
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Problem description

The application of continued fractions to various parts of mathematics has been viewed with renewed interest in recent years. These applications are often rooted in the classical theory. We analyze properties of the continued fractions that are helpful in various settings.

Abstract

We give a brief account of the general analytic theory of continued fractions and state and prove the Lorentzen bestness theorem. We investigate the possibility of a new proof of the Lorentzen bestness theorem and we give a related convergence theorem together with a conjecture. We explore some connections between the limit periodic continued fractions and other parts of mathematics and we give a suggestion of a topic suitable for further research.

Preface

This thesis is the consequence of the single burning desire to actually *do* mathematics after all the years of observing and following, more or less, the work of others. In spite of the endless list of faliures along the way, the last year has seen some of the most satisfying moments of my life. To suspect, conjecture, prove and finally publish a result is all an aspiring mathematician could ever hope for and I somehow managed it all. But alas, I did not manage to reach the goal I set for myself; to supply a new proof of the Lorentzen bestness theorem. All effort was not in vain, however, since the process revealed results and demonstrates connections that would otherwise remain unknown.

Acknowledgements

I would first and foremost thank professor Lisa Lorentzen for her limitless enthusiasm, patience and careful reflection upon any more or less intelligent question I might have during the work on this thesis. I would also like to thank professor Alexei Rudakov for introducing me to the subject. The girls at the reading room, Benedikte, Kari-Lise, Trude and Yvonne, also deserves mention for their efforts to make the lunch-breaks long and entertaining. Finally, I would like to thank my family and in particular my wife-to-be Camilla for her invaluable support and kindness during the last year. Without you I would not have succeeded.

Trondheim, May 2008

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Chapter 1

Introduction

To do mathematics can be compared with tourism. When we study for exams we travel through an unknown land by giudebook and we are shown the proper landmarks of the region in turn, perhaps with the time for a picture or two. More advanced study is usually done under the guidance of an advisor, not unlike the local guide that shows us that really spectacular view or hidden museum. But to really know a place, one has to go alone and explore. The thesis here presented is an attempt to leave the beaten path and explore the world of continued fractions. While this exploration is an extremely interesting activity, it is also rather dangerous since one runs the risk of finding little of note, or spending a lot of time chasing a proof that skillfully eludes capture. Both problems are illustrated in this thesis; we did find some new results, but not as many as we hoped for, and the really nice result that we glimpsed skulking around in the bushes avoided capture.

The main goal of this thesis was to do research and consequently we wanted to minimize the time spent on writing. Hence, the thesis differs from the norm in that it is rather short and that the presentation of the basic material contains very few proofs. However, a few proofs of the more important theorems are included for completeness and to give an illustration of the theory at work.

As for the contents of this thesis, the natural center gradually became the Lorentzen bestness theorem originally given in [Lor92] and reproduced with a slightly more polished proof in chapter 3. Consequently, chapter 2 contains all the material we need to state and prove the Lorentzen bestness theorem and also the few additional facts from the theory that we needed to prove our few original results that we stated in chapter 4. The article containing our most interesting result is given in appendix A. The pictures that was generated to get a grip on the subject and the MATLAB code made to produce them are given in appendix B. As for chapter 4 we present the results in the order that they were found. This is done to illustrate our chain of thought that lead us from the Lorentzen bestness theorem to the Jacobsthal polynomials. We also give an idea for further work and a concluding remark in chapter 5.

Chapter 2

Definitions and basic notions

The aim of this chapter is to give a short presentation of the basics that we use to study infinite continued fractions with complex coefficients. The material presented is taken more or less directly from [LW08], [LW92] and [JT80] and we will in general skip the proofs when we state results that are given here. However, results that are used but not stated in the sources will be proven.

2.1 The class $\mathcal{M}$

We define the class $\mathcal{M}$ of *Möbius transformations* or *linear fractional transformations* by the set of functions

$$\tau(w) = \frac{aw+b}{cw+d} \tag{2.1}$$

where $ad - bc \neq 0$ and $a, b, c, d \in \mathbb{C}$. As we know from complex analysis, functions τ from this class $\mathcal{M}$ has several nice properties. In particular we have that $\tau(\widehat{\mathbb{C}}) = \widehat{\mathbb{C}}$, that τ is one to one and that any τ is analytic in $\widehat{\mathbb{C}}$ except for one pole. Here we let $\widehat{\mathbb{C}}$ denote the *extended complex plane*, given by $\mathbb{C} \cup \{\infty\}$. For easy reference we make a list of some basic properties of $\mathcal{M}$ as found in [LW08], [Ahl79] and [Gam00]:

- The *cross ratio*

$$\frac{u-z}{u-w} \cdot \frac{v-w}{v-z}$$

  where $u, v, w, z \in \widehat{\mathbb{C}}$ and $u \neq v \neq w \neq z$ is invariant under linear fractional transformations, that is

$$\frac{\tau(u)-\tau(z)}{\tau(u)-\tau(w)} \cdot \frac{\tau(v)-\tau(w)}{\tau(v)-\tau(z)} = \frac{u-z}{u-w} \cdot \frac{v-w}{v-z}. \tag{2.2}$$

- It follows that if w_1, w_2, w_3 are distinct in $\widehat{\mathbb{C}}$ and u_1, u_2, u_3 are distinct in $\widehat{\mathbb{C}}$, then there exists a unique $\tau \in \mathcal{M}$ such that

$$\tau(w_1) = u_1, \quad \tau(w_2) = u_2 \quad \text{and} \quad \tau(w_3) = u_3$$

- Hence if a sequence $\{\tau_n\}$ from $\mathcal{M}$ converges pointwise at three distinct points to distinct values, then it converges to some $\tau \in \mathcal{M}$ in all of $\widehat{\mathbb{C}}$.

- If $\{\tau_n\}$ converges at three or more distinct points but not to a $\tau \in \mathcal{M}$, then the limit is the same at all these points, except possibly at one.
- The functions τ maps circles to circles in $\widehat{\mathbb{C}}$.
- Every map τ is a composition of translations, dilations and inversions.

Since infinity is not a special point in our setting, we choose the Riemann-sphere as our representation of the extended complex plane $\widehat{\mathbb{C}}$. We also need a suitable metric on the Riemann-sphere in order to define convergence. The metric we use is the *chordal metric* inroduced by L. Ahlfors, see for instance [Ahl79]. The metric is given by

$$d(w_1, w_2) = \begin{cases} \frac{2|w_1 - w_2|}{\sqrt{1+|w_1|^2}\sqrt{1+|w_2|^2}} & \text{for } w_1, w_2 \in \mathbb{C}. \\ \frac{2}{\sqrt{1+|w_1|^2}} & \text{if } w_1 \in \mathbb{C},\ w_2 = \infty. \\ 0 & \text{for } w_1 = w_2 = \infty. \end{cases}$$

This metric has the following nice properties: it is compact, bounded by 2 and $w_n \to \widehat{w} \in \widehat{\mathbb{C}}$ if and only if $d(w_n, \widehat{w}) \to 0$. With the chordal metric we can define an equivalence relation on the sequences from $\widehat{\mathbb{C}}$:

Definition 2.1. *Let two sequences $\{w_n\}$ and $\{v_n\}$ from $\widehat{\mathbb{C}}$ be given where $\widehat{\mathbb{C}}$ is equipped with the chordal metric. We say that the sequences are* equivalent *if and only if*

$$\lim_{n\to\infty} d(w_n, v_n) = 0.$$

If two sequences $\{w_n\}$ and $\{v_n\}$ are equivalent we write $\{w_n\} \sim \{v_n\}$.

We now define convergence of a sequence of transformations from $\mathcal{M}$.

Definition 2.2. *A sequence $\{\tau_n\}$ from $\mathcal{M}$ converges to some $\tau \in \mathcal{M}$ if and only if*

$$\lim_{n\to\infty} \sigma(\tau_n, \tau) = 0$$

where $\sigma(\tau_i, \tau_j) = \sup_{w\in\widehat{\mathbb{C}}} d(\tau_i(w), \tau_j(w))$.

In other words, we demand that the sequence of linear fractional transformations $\{\tau_n\}$ converges uniformly on $\widehat{\mathbb{C}}$ with respect to the chordal metric. However, a different notion of convergence as introduced by L. Lorentzen in [Jac86] will be of more use to us:

Definition 2.3. *A sequence $\{\tau_n\}$ from $\mathcal{M}$ converges generally to a constant $\gamma \in \widehat{\mathbb{C}}$ if and only if there exists a sequence $\{w_n^\dagger\}$ from $\widehat{\mathbb{C}}$ such that*

$$\lim_{n\to\infty} \tau_n(w_n) = \gamma \quad \textit{whenever} \quad \liminf_{n\to\infty} d(w_n, w_n^\dagger) > 0 \tag{2.3}$$

Informally we may say that we have general convergence if we stay away from a certain sequence of points $\{w_n^\dagger\}$ as n goes to infinity. We write $\tau_n \to \gamma$ to denote that $\{\tau_n\}$ converges generally to the constant $\gamma \in \widehat{\mathbb{C}}$. The sequence $\{w_n^\dagger\}$ is called an *exceptional sequence* for the sequence $\{\tau_n\}$ in this case. Following [LW08] we make the following observations:

- The exceptional sequence is not unique. To see this, let $\{w_n^\dagger\}$ be a exceptional sequence. Then any sequence $\{w_n^\star\}$ that satisfies $\lim d(w_n^\dagger, w_n^\star) = 0$ is exceptional.

- All exceptional sequences are equivalent by Definition 2.1.
- We cannot have uniform convergence to a constant γ of a sequence $\{\tau_n\}$ in $\widehat{\mathbb{C}}$. One always has to accept the existence of exceptional sequences. To see this, assume that $\tau_n(w_n) \to \gamma$ for all w_n. Let then $w_n = \tau_n^{-1}(\mu)$ for all n and observe the contradiction.

The definition of general convergence given in Definition 2.2 is unfortunately rather cumbersome to work with since it requires that we know the exceptional sequence in advance. Luckily we have a definition that is equivalent and easier to use:

Definition 2.4. *A sequence $\{\tau_n\}$ from $\mathcal{M}$ converges generally to a constant $\gamma \in \widehat{\mathbb{C}}$ if and only if there exists two sequences $\{v_n\}$ and $\{w_n\}$ from $\widehat{\mathbb{C}}$ such that*

$$\liminf_{n\to\infty} d(v_n, w_n) > 0$$

and

$$\lim_{n\to\infty} \tau_n(v_n) = \lim_{n\to\infty} \tau_n(w_n) = \gamma$$

For the proof of equivalence of Definition 2.3 and 2.4 we refer to [LW08]. Informally, this definition simply requires that the sequence $\{\tau_n\}$ converges to the same value γ for sufficiently different sequences $\{v_n\}$ and $\{w_n\}$. Related to general convergence is the notion of *restrained sequences*:

Definition 2.5. *A sequence $\{\tau_n\}$ from $\mathcal{M}$ is restrained if and only if there exists a sequence $\{w_n^\dagger\}$ from $\widehat{\mathbb{C}}$ such that whenever*

$$\liminf d(v_n, w_n^\dagger) > 0 \quad \textit{and} \quad \liminf d(w_n, w_n^\dagger) > 0$$

$$\lim_{n\to\infty} d(\tau_n(v_n), \tau_n(w_n)) = 0. \tag{2.4}$$

Definition 2.6. *A sequence $\{\tau_n\}$ from $\mathcal{M}$ is restrained if and only if there exists two sequences $\{v_n\}$ and $\{w_n\}$ from $\widehat{\mathbb{C}}$ with $\liminf d(v_n, w_n) > 0$ such that (2.4) holds.*

Definition 2.7. *A sequence $\{\tau_n\}$ from $\mathcal{M}$ is restrained if and only if no subsequence of $\{\tau_n\}$ converges to some $\tau \in \mathcal{M}$.*

The Definitions 2.5, 2.6 and 2.7 are all equivalent. A restrained sequence $\{\tau_n\}$ can informally be understood as a sequence whose asymptotic behaviour is independent of the sequence $\{w_n\}$ as long as the sequence $\{w_n\}$ stay far enough away from $\{w_n^\dagger\}$. We observe that if $\{\tau_n\}$ is restrained, all sequences $\{\tau_n(w_n)\}$ with $\liminf d(w_n, w_n^\dagger) > 0$ are equivalent, and we say that a sequence from this equivalence class is a *generic sequence* for $\{\tau_n\}$. For later reference have

Theorem 2.1. *If $\{\tau_n\}$ converges generally, then $\{\tau_n\}$ is restrained.*

Proof. Assume that $\{\tau_n\}$ converges generally. We know then by Definition 2.4 that there exists two sequences $\{v_n\}$ and $\{w_n\}$ from $\widehat{\mathbb{C}}$ such that $\liminf d(v_n, w_n) > 0$ and $\lim \tau_n(v_n) = \lim \tau_n(w_n) = \gamma$. To show that $\{\tau_n\}$ is restrained we must show that $\lim d(\tau_n(v_n), \tau_n(w_n)) = 0$. But this is obvious by property of the metric d since we have that $\lim \tau_n(v_n) = \lim \tau_n(w_n) = \gamma$. □

A sequence $\{\tau_n\}$ from $\mathcal{M}$ is either restrained or has a subsequence $\{\tau_{n_k}\}$ which converges to a non-singular transformation, and these are the only possibilities. If $\{\tau_n\}$ has no restrained subsequence, i.e every subequence $\{\tau_{n_k}\}$ has a subsequence $\{\tau_{n_{k_m}}\}$ that converges to a non-singular transformation, we say that $\{\tau_n\}$ is *totally non-restrained*.

2.1.1 Mapping properties of linear fractional transformations

The mappings τ from the class $\mathcal{M}$ has some very nice geometrical properties. A circle C on the Riemann sphere $\widehat{\mathbb{C}}$ is also a circle in the complex plane $\mathbb{C}$ if $\infty \notin C$. If $\infty \in \mathbb{C}$ then $C \setminus \infty$ is a straight line in $\mathbb{C}$ and therefore circles and lines in $\mathbb{C}$ are refered to as *generalized circles*. Before we proceed, we need some aditional notation. We shall consider the family $\mathfrak{V}$ of closed sets V on $\widehat{\mathbb{C}}$ where the boundary ∂V of V is a circle on $\widehat{\mathbb{C}}$. If now $\infty \notin V$, we say that V is a *closed (circular) disk*, and we write $V = \mathfrak{B}(\Gamma, \rho) = \{w \in \mathbb{C} : |w - \Gamma| \leq \rho\}$ where $\Gamma \in \mathbb{C}$ is the center and $\rho > 0$ is the (euclidian) radius of V. If $\infty \in V$ we say that V is a *closed half plane*, and we write $V = \xi + e^{i\alpha}\overline{\mathbb{H}}$ where $\xi \in \mathbb{C}, \alpha \in \mathbb{R}, \mathbb{H} = \{w \in \mathbb{C} : \Re(w) > 0\}$ and where $\overline{\mathbb{H}}$ is the closure of $\mathbb{H}$ in $\widehat{\mathbb{C}}$. Let now $\tau \in \mathcal{M}$ be given by

$$\tau = \frac{aw + b}{cw + d} \quad \text{with } \Delta = ad - bc \neq 0. \tag{2.5}$$

We then obtain

Lemma 2.1. *If $V \in \mathfrak{V}$ and τ is given by (2.5) and $c = 0$, we have that*

$$\tau(\mathfrak{B}(\Gamma, \rho)) = \mathfrak{B}\left(\frac{a}{d}\Gamma + \frac{b}{d}, \left|\frac{a}{d}\right| \rho\right)$$
$$\tau(\xi + e^{i\alpha}\overline{\mathbb{H}}) = \tau(\xi) + e^{i(\alpha+\beta)}\overline{\mathbb{H}};\ \beta = \arg\frac{a}{d}.$$

If $c \neq 0$, $-\frac{d}{c} \notin \partial V$, $V = \mathfrak{B}(\Gamma, \rho)$ with $\Gamma \in \mathbb{C}$ and $\rho \in \mathbb{R} \setminus \{0\}$, then $\tau(V) = \mathfrak{B}(\Gamma_1, \rho_1)$ where

$$\Gamma_1 = \frac{a}{c} - \frac{\frac{1}{c}\Delta(\bar{c}\overline{\Gamma} + \bar{d})}{|c\Gamma + d|^2 - |c\rho|^2} \quad \textit{and} \quad \rho_1 = \frac{\rho|\Delta|}{|c\Gamma + d|^2 - |c\rho|^2}$$

If $c \neq 0$, $-\frac{d}{c} \notin \partial V$, $V = \xi + e^{i\alpha}\overline{\mathbb{H}}$ with $\xi \in \mathbb{C}$ and $\alpha \in \mathbb{R}$, then $\tau(V) = \mathfrak{B}(\Gamma_1, \rho_1)$ where

$$\Gamma_1 = \frac{a}{c} - \frac{\frac{1}{2c^2}\Delta e^{-i\alpha}}{\Re[(\xi + \frac{d}{c})e^{-i\alpha}]} \quad \textit{and} \quad \rho_1 = \frac{\frac{1}{2|c|^2}|\Delta|}{\Re[(\xi + \frac{d}{c})e^{-i\alpha}]}.$$

For a proof see [LW08]. The next result is taken from [Lor07].

Lemma 2.2. *Let $\mathcal{T}_n = \tau_1 \circ \tau_2 \circ \cdots \circ \tau_n$ for all $n \in \mathbb{N}$ where all $\tau_n \in \mathcal{M}$ map the unit disk $\mathbb{D}$ into itself with radii such that*

$$\limsup_{n\to\infty} \operatorname{rad}\ \tau_n(\mathbb{D}) < 1$$

and assume that there exists a sequence $\{w_n\} \subseteq \widehat{\mathbb{C}}$ such that

$$\liminf \big||w_n| - 1\big| > 0 \quad \textit{and} \quad \liminf \big||\tau_n(w_n)| - 1\big| > 0.$$

Then $\{\mathcal{T}_n\}$ converges generally to some constant $\gamma \in \overline{\mathbb{D}}$ with an exceptional sequence $\{w_n^\dagger\}$ with $w_n^\dagger \in \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. If also $\lim \operatorname{rad} \mathcal{T}_n(\overline{\mathbb{D}}) > 0$, then

$$\sum_{n=n_0}^{\infty} |\mathcal{T}_{n+1}(w_{n+1}) - \mathcal{T}_n(w_n)| < \infty$$

i.e. $\{\mathcal{T}_n(w_n)\}_{n_0}^{\infty}$ converges absolutely to γ for some $n_0 \in \mathbb{N}$.

2.2 Continued fractions

2.2.1 Initial definitions

Definition 2.8. *A* continued fraction $b_0 + \mathbf{K}(a_n/b_n)$ *is an ordered pair* $\big((\{a_n\}, \{b_n\}), \{S_n\}\big)$ *where* $\{a_n\}$ *and* $\{b_n\}$ *are sequences of complex numbers with all* $a_n \neq 0$ *and* $\{S_n\}$ *is a sequence from* $\mathcal{M}$ *given by*

$$s_0 = b_0 + w \quad , \quad s_n = \frac{a_n}{b_n + w}$$

and

$$S_n(w) = s_0 \circ s_1 \circ \cdots \circ s_n(w) = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_2}{\ddots + \cfrac{a_n}{b_n}}}} \tag{2.6}$$

where

- a_n *and* b_n *are called the* elements *of* $b_0 + \mathbf{K}(a_n/b_n)$
- $\frac{a_n}{b_n}$ *is called a* fraction term *for* $b_0 + \mathbf{K}(a_n/b_n)$
- *evaluations* $S_n(w)$ *of* $\{S_n\}$ *are called the* n-th approximants
- *evaluations of the form* $f_n = S_n(0)$ *of* $\{S_n\}$ *are called the* classical approximants.

We shall also make use of an alternative notation for $S_n(w)$ in (2.6), namely

$$S_n(w) = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n + w}.$$

We now have

Lemma 2.3. *Let* S_n *be given as in* (2.6)*. Then*

$$S_n(w) = \frac{A_{n-1}w + A_n}{B_{n-1}w + B_n} \quad \textit{for} \quad n = 1, 2, 3, \ldots \tag{2.7}$$

where

$$A_n = b_n A_{n-1} + a_n A_{n-2}, \quad B_n = b_n B_{n-1} + a_n B_{n-2} \tag{2.8}$$

with initial values $A_{-1} = 1,\ A_0 = b_0,\ B_{-1} = 0$ *and* $B_0 = 1$.

The proof is by simple induction and will be omitted. We say that A_n and B_n are the *n-th canonical numerator and denominator* of $b_0 + \mathbf{K}(a_n/b_n)$. By induction we also have the *determinant formula* which is given by

$$\Delta_n = A_{n-1}B_n - A_n B_{n-1} = \prod_{k=1}^{n}(-a_k). \tag{2.9}$$

The classical approximant f_n is made by truncating the continued fraction after n terms. The part cut off is also a continued fraction,

$$\frac{a_{n+1}}{b_{n+1}} + \frac{a_{n+2}}{b_{n+2}} + \frac{a_{n+3}}{b_{n+3}} + \cdots. \tag{2.10}$$

The continued fraction in (2.10) is called the *n-th tail* of the continued fraction $b_0 + \mathbf{K}(a_n/b_n)$ and its m-th approximant is denoted by $S_m^{(n)}(w)$. As in Lemma 2.3 we have that

$$S_m^{(n)}(w) = \frac{A_{m-1}^{(n)} w + A_m^{(n)}}{B_{m-1}^{(n)} w + B_m^{(n)}} \quad \text{for} \quad m = 1, 2, 3, \ldots \tag{2.11}$$

where

$$A_m^{(n)} = b_{m+n} A_{m-1}^{(n)} + a_{m+n} A_{m-2}^{(n)}, \quad B_m^{(n)} = b_{m+n} B_{m-1}^{(n)} + a_{m+n} B_{m-2}^{(n)} \tag{2.12}$$

with initial values $A_{-1}^{(n)} = 1,\ A_0^{(n)} = 0,\ B_{-1}^{(n)} = 0$ and $B_0^{(n)} = 1$. With the notation we have now introduced we have the following lemma

Lemma 2.4. *The following equalities hold:*

$$A_m^{(n)} = a_{n+1} B_{m-1}^{(n+1)} \qquad \textit{for} \quad n \geq 0, \quad m \geq 0, \tag{2.13}$$

$$B_m^{(n)} = b_{n+1} B_{m-1}^{(n+1)} + a_{n+2} B_{m-2}^{(n+2)} \qquad \textit{for} \quad n \geq 0, \quad m \geq 0, \tag{2.14}$$

$$A_{m-1}^{(n)} B_{m+k}^{(n)} - A_{m+k}^{(n)} B_{m-1}^{(n)} = B_k^{(m+n)} \prod_{j=n+1}^{m+n} (-a_j). \tag{2.15}$$

The proof is by induction and we will omit it but the interested reader can see [LW92]. As we can see from Lemma 2.4, we have a determinant-like formula for the tail if $k = 0$.

2.2.2 Convergence

We may now define various types of convergence for continued fractions.

Definition 2.9. *A continued fraction* $\boldsymbol{K}(a_n/b_n)$ converges classically *to a value* $f \in \widehat{\mathbb{C}}$ *if and only if* $\lim f_n = f$*, that is*

$$\lim_{n\to\infty} S_n(0) = f.$$

Definition 2.10. *A continued fraction* $\boldsymbol{K}(a_n/b_n)$ converges generally *to a value* $f \in \widehat{\mathbb{C}}$ *with exceptional sequence* $\{w_n^\dagger\}$ *if and only if the sequence* $\{S_n\}$ *of linear fractional transformations converges generally to* f *with exceptional sequence* $\{w_n^\dagger\}$*.*

Definition 2.11. *A continued fraction* $\boldsymbol{K}(a_n/b_n)$ converges absolutely *if its classical approximants* f_n *satisfy*

$$\sum_{n=1}^{\infty} |f_{n+1} - f_n| < \infty \tag{2.16}$$

If $\mathbf{K}(a_n/b_n)$ converges neither in the general or classical way, we say that $\mathbf{K}(a_n/b_n)$ *diverges generally*. We note that if the sequence $\{S_n\}$ is totally non-restrained the continued fraction $\mathbf{K}(a_n/b_n)$ diverges generally. If the limit in Definition 2.9 fails to exist, we say that the continued fraction $\mathbf{K}(a_n/b_n)$ diverges classically. It is here worth noting that classical convergence implies general convergence. To see this, observe that $S_{n+1}(\infty) = S_n(0)$, and hence if now $\mathbf{K}(a_n/b_n)$ converges classically this implies by Definition 2.4 that we have general convergence as well. We also see that classical divergence does not imply general divergence. This follows from the fact that the classical divergence of $\mathbf{K}(a_n/b_n)$ may be caused by a zero in the exceptional sequence of $S_n(w)$. We also point out that we allow convergence to infinity.

2.2.3 Equivalence transformations

Definition 2.12. *The continued fractions* $b_0 + \mathbf{K}(a_n/b_n)$ *and* $b_0^\star + \mathbf{K}(a_n^\star/b_n^\star)$ *with classical approximants* f_n *and* $f_n^\star$ *respectively are said to be* equivalent *if*

$$f_n = f_n^\star, \quad \textit{for all } n = 0, 1, 2, \ldots$$

We introduce the notation $b_0 + \mathbf{K}(a_n/b_n) \approx b_0^\star + \mathbf{K}(a_n^\star/b_n^\star)$ for this equivalence. We immideately obtain

Theorem 2.2. *The continued fractions* $b_0 + \mathbf{K}(a_n/b_n)$ *and* $b_0^\star + \mathbf{K}(a_n^\star/b_n^\star)$ *are equivalent if and only if there exists a sequence of non-zero constants* $\{r_n\}$ *with* $r_0 = 1$ *such that*

$$a_n^\star = r_n r_{n-1} a_n \qquad \textit{where} \quad n = 1, 2, 3, \ldots \tag{2.17}$$

$$b_n^\star = r_n b_n \qquad \textit{where} \quad n = 0, 1, 2, \ldots \tag{2.18}$$

The proof will be omitted. There are at least two different proofs of this in the literature; one in [JT80] and one in [LW08]. Thus we have that $b_0 + \mathbf{K}(a_n/b_n) \approx b_0 + \mathbf{K}(1/c_n)$ whenever $b_n \neq 0$ for all n and

$$c_n = b_n \prod_{k=1}^{n} a_k^{(-1)^{n+k-1}} \quad \text{for } n = 1, 2, 3, \ldots \tag{2.19}$$

and that $b_0 + \mathbf{K}(a_n/b_n) \approx b_0 + \mathbf{K}(d_n/1)$ whenever

$$d_1 = \frac{a_1}{b_1} \quad \text{and} \quad d_n = \frac{a_n}{b_n b_{n-1}} \quad \text{for } n = 1, 2, 3, \ldots \tag{2.20}$$

We note that equivalence transformations preserves classical and absolute convergence since the classical approximants are the same. For general convergence the picture is more complicated. In general, general convergence is not preserved by equivalence transformations and there exist examples of equivalence transformations that turn a generally convergent continued fraction into a generally divergent continued fraction. The following theorem is quite helpful when considering generally convergent continued fractions

Theorem 2.3. *Let* $\mathbf{K}(a_n/b_n)$ *converge generally to* f. *If the sequence* $\{r_n\}$ *of complex numbers is bounded and bounded away from* 0, *then also* $\mathbf{K}(r_{n-1}r_n a_n/r_n b_n)$ *converges generally to* f.

Proof. The two equivalent continued fractions are given by

$$\frac{a_1}{b_1} \underset{+}{} \frac{a_2}{b_2} \underset{+}{} \cdots \underset{+}{} \frac{a_n}{b_n} \underset{+}{} \cdots \tag{2.21}$$

and

$$\frac{r_1 a_1}{r_1 b_1} \underset{+}{} \frac{r_1 r_2 a_2}{r_2 b_2} \underset{+}{} \cdots \underset{+}{} \frac{r_{n-1} r_n a_n}{r_n b_n} \underset{+}{} \cdots. \tag{2.22}$$

By the recursion formulas we have, if A_n and B_n are canonical numerators and denominators for (2.21) and C_n and D_n are canonical numerators and denominators for (2.22), that

$$\begin{array}{llll}
C_1 & = r_1 b_1 C_0 + r_1 r_0 a_1 C_{-1} & = r_1 r_0 a_1 & = r_1 r_0 A_1 \\
D_1 & = r_1 b_1 D_0 + r_1 r_0 a_1 D_{-1} & = r_1 b_1 & = r_1 B_1 \\
C_2 & = r_2 b_2 C_1 + r_2 r_1 a_2 C_0 & = r_2 r_1 r_0 b_2 b_1 & = r_2 r_1 r_0 A_2 \\
D_2 & = r_1 b_2 D_1 + r_2 r_1 a_2 D_0 & = r_2 r_1 b_2 b_1 + r_2 r_1 a_2 & = r_2 r_1 B_2.
\end{array}$$

Since $r_0 = 1$ we have by induction that

$$C_n = A_n \prod_{k=1}^{n} r_k \quad \text{and} \quad D_n = B_n \prod_{k=1}^{n} r_k. \tag{2.23}$$

For the approximants $S_n(w)$ we by (2.23) have that

$$S_n(w) = \frac{A_{n-1}w + A_n}{B_{n-1}w + B_n} = \frac{\dfrac{C_{n-1}w}{\prod_{k=1}^{n-1} r_k} + \dfrac{C_n}{\prod_{k=1}^{n} r_k}}{\dfrac{D_{n-1}w}{\prod_{k=1}^{n-1} r_k} + \dfrac{D_n}{\prod_{k=1}^{n} r_k}} = \frac{C_{n-1}r_n w + C_n}{D_{n-1}r_n w + D_n}$$

Hence this implies that $S_n(w) = T_n(r_n w)$ where $S_n(w)$ is the approximants for $\mathbf{K}(a_n/b_n)$ and $T_n(w)$ is the approximants for $\mathbf{K}(r_{n-1}r_n a_n/r_n b_n)$. Let now S_n converge generally. We then have by Definition 2.4 that this happens if and only if there exists two sequences $\{v_n\}$ and $\{w_n\}$ such that

$$\liminf_{n\to\infty} d(v_n, w_n) > 0$$

and

$$\lim_{n\to\infty} S_n(v_n) = \lim_{n\to\infty} S_n(w_n) = \gamma$$

Since we now have that

$$\liminf_{n\to\infty} d(v_n, w_n) > 0 \Leftrightarrow \liminf_{n\to\infty} d(r_n v_n, r_n w_n) > 0$$

under our assumptions for r_n, we see that we may now substitute $T_n(r_n w)$ for $S_n(w)$ and obtain

$$\lim_{n\to\infty} T_n(r_n v_n) = \lim_{n\to\infty} T_n(r_n w_n) = \gamma.$$

Thus T_n also converges generally, and that proves our theorem. □

2.2.4 Value- and element sets

Definition 2.13. *A sequence $\{V_n\}_{n=0}^{\infty}$ of sets $V_n \subseteq \widehat{\mathbb{C}}$ is a* sequence of value sets *for $\boldsymbol{K}(a_n/b_n)$ if and only if both V_n and $\widehat{\mathbb{C}} \setminus V_n$ contain at least two points and*

$$s_n(V_n) = \frac{a_n}{b_n + V_n} \subseteq V_{n-1} \quad \textit{for} \quad n = 1, 2, 3, \ldots \tag{2.24}$$

If $\{V_n\}_{n=0}^{\infty}$ is 1-perodic, that is $V_n = V$ for all n, we say that V is a *simple value set* for $\mathbf{K}(a_n/b_n)$. A useful observation here is that

$$K_n = S_n(V_n) = S_{n-1}(s_n(V_n)) \subseteq S_{n-1}(V_{n-1}) = K_{n-1} \tag{2.25}$$

and hence we have that

$$S_n(w_n) \in K_n \subseteq K_{n-1} \subseteq V_0 \quad \text{for} \quad w_n \in V_n.$$

If we assume that the sets V_n are closed, then the sets K_n are closed and by (2.25) we have that the limit set

$$K = \lim_{n\to\infty} K_n = \bigcap_{n=1}^{\infty} K_n$$

exists and is closed and non-empty. If $\mathrm{diam}(K) = 0$ we say that we have the *limit point case.* We then have that K consists of only one point f where $f \in \widehat{\mathbb{C}}$, and that $S_n(w_n) \to f$ whenever $w_n \in V_n$ for all n. Also, if $0 \in V_n$ we have that $\mathbf{K}(a_n/b_n)$ converges classically to f. Furthermore, if $\liminf \mathrm{diam}_d(V_n) > 0$ for the chordal diameter

$$\mathrm{diam}_d(V_n) = \sup\{d(v, w) : v, w \in V_n\}$$

of V_n, then $\mathbf{K}(a_n/b_n)$ converges generally to f. A notion related to the notion of value sets is given in the following definition.

Definition 2.14. *For a given sequence $\{V_n\}_{n=0}^{\infty}$ where $V_n \in \widehat{\mathbb{C}}$ and both V_n and $\widehat{\mathbb{C}} \setminus V_n$ contain at least two points, the sequence $\{\Omega_n\}$ given by*

$$\Omega_n = \left\{(a, b) \in \widehat{\mathbb{C}} : \frac{a}{b + V_n} \subseteq V_{n-1}\right\}$$

is called the element sets *for continued fractions $\boldsymbol{K}(a_n/b_n)$ corresponding to $\{V_n\}$.*

2.3 Periodic and limit periodic continued fractions

2.3.1 Initial definitions

Definition 2.15. *A continued fraction $\boldsymbol{K}(a_n/b_n)$ is called p-*periodic *or simply* periodic *if the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are p-periodic; i.e if*

$$a_{n+p} = a_n, \quad b_{n+p} = b_n \quad \textit{for all } n \in \mathbb{N}, \tag{2.26}$$

and p is the smallest interger such that (2.26) *holds.*

Definition 2.16. *A continued fraction $\boldsymbol{K}(a_n/b_n)$ is called* limit p-periodic *or just* limit periodic *if the limits*

$$\lim_{n\to\infty} a_{np+m} = \tilde{a}_m, \quad \lim_{n\to\infty} b_{np+m} = \tilde{b}_m \quad \textit{for } m = 1, 2, \ldots, p \tag{2.27}$$

exist in $\widehat{\mathbb{C}}$, and p is the smallest positive integer for which (2.27) *holds.*

We shall in this master thesis only consider limit periodic continued fractions where the limits in (2.27) are finite and where $\tilde{a}_n \neq 0$. For a periodic continued fraction $\mathbf{K}(a_n/b_n)$ we may write

$$S_{np+m}(w) = S_p^{[n]} \circ S_m(w) = S_m \circ (S_p^{(m)})^{[n]}(w)$$

where we by $F^{[n]}$ mean the n-th iterate of F, i.e F iterated n times. Moreover,

$$S_p^{(m)}(w) = S_m^{-1} \circ S_p \circ S_m(w) = \frac{a_{m+1}}{b_{m+1}} + \frac{a_{m+2}}{b_{m+2}} + \ldots + \frac{a_{m+p}}{b_{m+p} + w}$$

and hence the convergence properties of $\mathbf{K}(a_n/b_n)$ depends only on how $S_p(w)$ behaves asymptotically. The approximant $S_p(w)$ we shall call the *period approximant* of the periodic continued fraction $\mathbf{K}(a_n/b_n)$. For simplicity we shall make use of the notation $S_m^{(0)}(w) = S_m(w)$. Since we are only considering limit periodic continued fractions whose limits are finite, we have that the np-th tails of the limit p-periodic continued fraction $\mathbf{K}(a_n/b_n)$ look more and more like the continued fraction

$$\underset{k=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{\tilde{a}_k}{\tilde{b}_k}\right) = \frac{\tilde{a}_1}{\tilde{b}_1} + \frac{\tilde{a}_2}{\tilde{b}_2} + \ldots + \frac{\tilde{a}_p}{\tilde{b}_p} + \frac{\tilde{a}_1}{\tilde{b}_1} + \frac{\tilde{a}_2}{\tilde{b}_2} + \ldots$$

as n increases. Following what we did for periodic continued fractions, we shall call

$$\tilde{S}_p(w) = \frac{\tilde{a}_1}{\tilde{b}_1} + \frac{\tilde{a}_2}{\tilde{b}_2} + \ldots + \frac{\tilde{a}_p}{\tilde{b}_p + w}$$

the *period limit approximant* of the limit periodic continued fraction $\mathbf{K}(a_n/b_n)$.

2.3.2 Classification of linear fractional transformations

A linear fractional transformation $\tau \in \mathcal{M}$, i.e

$$\tau = \frac{aw+b}{cw+d} \quad \text{where } \Delta = ad - bc \neq 0,$$

that is not equal to the identity transformation has two (possibly coinciding) fixed points x and y. The fixed points are taken care of in the following teorem.

Theorem 2.4. *The fixed points of the linear fractional transformation*

$$\tau = \frac{aw+b}{cw+d}$$

where $\Delta = ad - bc \neq 0$ *are given by*

$$x, y = \begin{cases} \frac{a-d\pm(a+d)u}{2c} & \text{if } c \neq 0,\ a+d \neq 0, \\ \frac{a}{c}\left(1 \pm \sqrt{-\frac{\Delta}{a^2}}\right) & \text{if } c \neq 0,\ a+d = 0,\ a \neq 0, \\ \pm\sqrt{\frac{b}{c}} & \text{if } c \neq 0,\ a+d = 0,\ a = 0, \\ \frac{b}{d-a}, \infty & \text{if } c = 0 \text{ and } \tau \text{ is not the identity transformation} \end{cases} \tag{2.28}$$

where $u = \sqrt{1 - \frac{4\Delta}{(a+d)^2}}$.

For the proof of this see [LW08]. Whenever applicable we will choose x and y from the formulas (2.28) above such that $|cy + d| < |cx + d|$.

Definition 2.17. *For* $\tau \in \mathcal{M}$ *with fixed points* x *and* y *(possibly coinciding) the* ratio $\mathcal{R}$ *is a complex number* $0 < |\mathcal{R}| \leq 1$ *given by*

$$\mathcal{R} = \begin{cases} \frac{cy+d}{cx+d} & \text{if } c \neq 0, \\ \frac{a}{d} & \text{if } c = 0,\ |a| \leq |d|, \\ \frac{d}{a} & \text{if } c = 0,\ |a| > |d|. \end{cases} \tag{2.29}$$

The ratio $\mathcal{R}$ is used to classify linear fractional transformations in the following way:

Definition 2.18. *Let* $\tau \in \mathcal{M}$ *be a linear fractional transformation. If* $|\mathcal{R}| < 1$, τ *is called* loxodromic. *If* $|\mathcal{R}| = 1$ *and* $\mathcal{R} \neq 1$, τ *is called* elliptic. *If* $\mathcal{R} = 1$ *and* τ *is not equal to the identity transformation,* τ *is called* parabolic.

The value $\mathcal{R}$ might seem a bit cumbersome to use since it requires us to know the fixed points of τ, but this is simplified by the observation that

$$\mathcal{R} = \begin{cases} \frac{1-u}{1+u} & \text{if } c \neq 0 \text{ and } a+d \neq 0 \\ -1 & \text{if } c \neq 0 \text{ and } a+d = 0 \end{cases} \tag{2.30}$$

where $u = \sqrt{1 - \frac{4\Delta}{(a+d)^2}}$. However, from [Bea83] we have the following equivalent classification

Definition 2.19. *For the linear fractional transformation*

$$\tau(w) = \frac{aw + b}{cw + d}$$

where $\Delta = ad - bc \neq 0$*, we let* t *be given by*

$$t = \frac{(a+d)^2}{4\Delta}. \tag{2.31}$$

If $t = 1$*,* $\tau(w)$ *is parabolic. If* $t \in [0, 1)$*,* $\tau(w)$ *is elliptic and for any other vaule of* t*,* $\tau(w)$ *is loxodromic.*

To see that Definition 2.18 and 2.19 are equivalent, we observe that we have that

$$t = \frac{1}{1 - \left(\frac{1-\mathcal{R}}{1+\mathcal{R}}\right)^2} \tag{2.32}$$

by equation (2.30). By using properties of basic mappings from complex analysis we see that the mapping in (2.32) maps the punctured disk in Definition 2.18 to the entire plane. But we need to show that the boundary of the disk in Definition 2.18 is mapped to the interval $[0, 1]$ and in the right order. Clearly, for $\mathcal{R} = 1$ we have that $t = 1$. For the other points on the boundary, we let $\mathcal{R} = e^{i\theta}$ where $\theta \in (0, 2\pi)$. We have that

$$\frac{1-\mathcal{R}}{1+\mathcal{R}} = \frac{e^{-\frac{i\theta}{2}} - e^{\frac{i\theta}{2}}}{e^{-\frac{i\theta}{2}} + e^{\frac{i\theta}{2}}} = -\frac{i \sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = -i\tan\frac{\theta}{2}$$

which gives that

$$t = \frac{1}{1 + \tan^2\frac{\theta}{2}}$$

and hence we have that $t \in [0, 1)$ for $\mathcal{R}$ in the circle, as we wanted. The classifications given in the Definitions 2.18 and 2.19 both have advantages and disadvantages. We will try to use the one most suitable to us in any given situation. Finally, we note that if $\tau \in \mathcal{M}$ is loxodromic, then τ has exactly two distinct fixed points. Furthermore, $\{\tau^{[n]}\}$ converges generally to x with exceptional sequence $\{y\}_{n=1}^{\infty}$. We call x the *attracting fixed point* of τ and y the *repelling fixed point* of τ.

2.3.3 Convergence of periodic- and limit periodic continued fractions

We classify periodic and limit-periodic continued fractions according to the classification of the period approximant or period limit approximant as done in Definitions 2.18 or 2.19. For p-periodic continued fractions we then have the following result

Theorem 2.5. *Let* $\mathbf{K}(a_n/b_n)$ *be a* p*-periodic continued fraction with period approximant* S_p*. If* S_p *is classfied as parabolic or loxodromic by Definition 2.18 or 2.19 then* $\mathbf{K}(a_n/b_n)$ *converges generally. If* S_p *is classified as elliptic or is the identity transformation then* $\{S_m\}$ *is totally non-restrained and* $\mathbf{K}(a_n/b_n)$ *diverges generally.*

The possible cases of classical convergence in Theorem 2.5 are taken care of in the following theorems

Theorem 2.6. *A periodic continued fraction of parabolic type converges classically.*

Theorem 2.7. *Let $\mathbf{K}(a_n/b_n)$ be a p-periodic continued fraction of loxodromic type. If $S_m(0) = y$ for some $m \in \{1, 2, \ldots, p\}$, then $\mathbf{K}(a_n/b_n)$ diverges in the classical sense. Otherwise $\mathbf{K}(a_n/b_n)$ converges in the classical sense.*

The type of divergence described in Theorem 2.7 is called *Thiele oscillation* after the danish mathematichian Thorvald Nicolai Thiele who was the first to point out that this might happen. For limit periodic contiuned fractions the picture is more complicated, as expected. However, for limit periodic continued fractions with finite limits most of the nice structure is preserved:

Theorem 2.8. *Let $\mathbf{K}(a_n/b_n)$ be a limit p-periodic continued fraction of loxodromic type with finite limits. Then $\mathbf{K}(a_n/b_n)$ converges generally. Also, $\mathbf{K}(a_n/b_n)$ converges classically if all $\tilde{S}_m(0) \neq y$ for $m \in \{1, 2, \ldots, p\}$.*

Unfortunately some nice structure is lost as well. In particular there exists parabolic limit periodic continued fractions that diverges and elliptic limit periodic continued fractions that converges but we will not concern ourselves with this. For our purposes, the result presented in Theorem 2.8 is sufficient.

Chapter 3

Core theorems

The goal of this chapter is to give the original proof of the Lorentzen bestness theorem and to give a proof of the Parabola theorem. Since both theorems are strongly connected to all the work done in this thesis and the results in themselves are interesting, both theorems are stated properly and proven. In addition, we state and prove a few results that are needed when proving the Parabola theorem.

Theorem 3.1 (The Stern-Stolz divergence theorem)**.** *If*

$$\sum_{n=1}^{\infty} |c_n| < \infty$$

then the continued fraction $\mathbf{K}(1/c_n)$ *diverges generally. Furthermore, the sequences* $\{A_{2n+m}\}_n$ *and* $\{B_{2n+m}\}_n$ *converge absolutely to finite values and the relation* $\mathcal{A}_1\mathcal{B}_0 - \mathcal{A}_0\mathcal{B}_1 = 1$ *holds where*

$$\begin{aligned}
\mathcal{A}_m &= \lim_{n\to\infty} A_{2n+m} \\
\mathcal{B}_m &= \lim_{n\to\infty} B_{2n+m}
\end{aligned}$$

Proof. This proof is based on the slightly different versions found in [Kho63], [JT80] and [LW08] and contains some elements from all. Although the proof is based on exactly the same idea in all the cases, they differ in terms of method and ease of understanding. We aim at giving a proof that is both easy and up to date but a bit long.

For the continued fraction $\mathbf{K}(1/c_n)$ with convergents A_n/B_n we have the recurrence relations given in Lemma 2.3 hold, and thus we estimate that

$$\begin{aligned}
|A_1| &\le |c_1||A_0| + |A_{-1}| = 1 < 1 + |c_1| \\
|A_2| &\le |c_2||A_1| + |A_0| = |c_2|(1 + |c_1|) + 1 < (1 + |c_1|)(1 + |c_2|) \\
&\vdots
\end{aligned}$$

and we may prove by induction that

$$|A_n| \le \prod_{k=1}^{n} (1 + |c_k|).$$

Since $\ln(1+|c_n|) < |c_n|$ for all $c_n \neq 0$ we have that

$$|A_n| \leq \prod_{k=1}^{n}(1+|c_k|) < e^{\sum_{k=1}^{n}|c_k|}$$

and if we let $n \to \infty$ and recall that the series $\sum_{k=1}^{\infty}|c_k|$ converges, then we find that A_n is bounded. In a similar manner the boundedness of B_n follows. Hence the sequences $\{A_n\}$ and $\{B_n\}$ are bounded. Also, the sum $\sum_{k=1}^{\infty}|c_k A_k| < \infty$ since

$$\sum_{k=1}^{\infty}|c_k A_k| < \sum_{k=1}^{\infty}|c_k| M < \infty$$

where M is the upper bound for $|A_n|$. By a similar argument one may show that $\sum_{k=1}^{\infty}|c_k B_k| < \infty$. The recursion formula $A_n = c_n A_{n-1} + A_{n-2}$ from Lemma 2.3 gives that

$$|A_{2k} - A_{2k-2}| = |c_{2k} A_{2k-1}|$$

and by summation over k we obtain

$$\sum_{k=1}^{n}|A_{2k} - A_{2k-2}| = \sum_{k=1}^{n}|c_{2k}A_{2k-1}| < \sum_{k=1}^{\infty}|c_{2k}A_{2k-1}| < \infty. \tag{3.1}$$

and hence the sequence $\{A_{2n}\}$ converges absolutely to a finite value. By similar arguments the sequences $\{A_{2n+1}\}$, $\{B_{2n}\}$ and $\{B_{2n+1}\}$ also converges absolutely to finite values. We also see that since

$$|A_{2n} - A_0| = \left|\sum_{k=1}^{n}(A_{2k} - A_{2k-2})\right| = \left|\sum_{k=1}^{n} c_{2k}A_{2k-1}\right| \leq \sum_{k=1}^{n}|c_{2k}A_{2k-1}| < \infty$$

by the recursion $A_n = c_n A_{n-1} + A_{n-2}$ and (3.1), it follows that the limit

$$\lim_{n\to\infty} A_{2n} = A_0 + \sum_{k=1}^{\infty} c_{2k}A_{2k-1}$$

exists. This can be done in a similar manner for the sequences $\{A_{2n+1}\}$, $\{B_{2n}\}$ and $\{B_{2n+1}\}$. In addition, by the determinant formula we have that $A_{2n+1}B_{2n} - A_{2n}B_{2n+1} = 1$ for all n, so $\mathcal{A}_1\mathcal{B}_0 - \mathcal{A}_0\mathcal{B}_1 = 1$ holds where

$$\mathcal{A}_m = \lim_{n\to\infty} A_{2n+m}$$
$$\mathcal{B}_m = \lim_{n\to\infty} B_{2n+m}.$$

To prove divergence of the continued fraction we observe that for the sequence $\{S_n\}$ we obtain two non-singular transformations

$$\lim_{n\to\infty} S_{2n}(w) = \frac{\mathcal{A}_1 w + \mathcal{A}_0}{\mathcal{B}_1 w + \mathcal{B}_0}, \quad \lim_{n\to\infty} S_{2n+1}(w) = \frac{\mathcal{A}_0 w + \mathcal{A}_1}{\mathcal{B}_0 w + \mathcal{B}_1}$$

as limits. Since now any subsequence of $\{S_{2n}\}$ and $\{S_{2n+1}\}$ now converges to a non-singular linear fractional transformation, the sequence is totally non-restrained and by negation of Lemma 2.1 we have general divergence. $\square$

For a continued fraction $\mathbf{K}(a_n/b_n)$ we call the series

$$\mathcal{S} = \sum_{n=1}^{\infty} \left| b_n \prod_{k=1}^{n} a_k^{(-1)^{n+k-1}} \right| < \infty$$

the *Stern-Stolz series* of the continued fraction $\mathbf{K}(a_n/b_n)$. The Stern-Stolz series $\mathcal{S}_1$ and $\mathcal{S}_2$ of two equivalent continued fractions $\mathbf{K}(a_n/b_n)$ and $\mathbf{K}(a_n^\star/b_n^\star)$ are identical. This is easy to see if we rewrite $\mathcal{S}_1$ as

$$\mathcal{S}_1 = \sum_{n=1}^{\infty} \left| b_{2n} \frac{a_1 a_3 \dots a_{2n-1}}{a_2 a_4 \dots a_{2n}} \right| + \sum_{n=1}^{\infty} \left| b_{2n+1} \frac{a_2 a_4 \dots a_{2n}}{a_1 a_3 \dots a_{2n+1}} \right|$$

and substitute $r_n r_{n-1} a_n$ for $a_n^\star$ and $r_n b_n$ for $b_n^\star$.

Corollary 3.1. *Let b_n be finite and nonzero for all n. If the Stern-Stolz series*

$$\mathcal{S} = \sum_{n=1}^{\infty} \left| b_n \prod_{k=1}^{n} a_k^{(-1)^{n+k-1}} \right| < \infty$$

then the continued fraction $\mathbf{K}(a_n/b_n)$ diverges classically. If in addition

$$\inf_n \left| \prod_{k=1}^{n} a_k^{(-1)^{n+k-1}} \right| > 0$$

then $\mathbf{K}(a_n/b_n)$ diverges generally.

Proof. Classical divergence follows immideately since equivalence transformations preserves both the expressions for $\mathcal{S}$ and the classical approximants as we saw in Theorem 2.2 and the remarks following that statement. As for the general divergence, this is by Theorem 2.3 preserved by equivalence transformations if

$$\inf_n \left| b_n \prod_{k=1}^{n} a_k^{(-1)^{n+k-1}} \right| > 0 \quad \text{and} \quad \sup_n \left| b_n \prod_{k=1}^{n} a_k^{(-1)^{n+k-1}} \right| < \infty.$$

Since b_n is assumed to be finite and nonzero we are done. □

Corollary 3.2. *Let the continued fraction $\mathbf{K}(a_n/1)$ be given. If the Stern-Stolz series $\mathcal{S}$ of $\mathbf{K}(a_n/1)$ converges, then the continued fraction $\mathbf{K}(a_n/1)$ diverges generally.*

Proof. We observe that for a continued fraction $\mathbf{K}(a_n/1)$, the approximant

$$S_n(w) = \frac{a_1}{1} + \frac{a_2}{1} + \dots + \frac{a_{n-2}}{1} + \frac{a_{n-1}}{1} + \frac{a_n}{1+w}$$

has the property that $S_n(-1) = S_{n-2}(0)$. Thus we have by Definition 2.4 that we have general convergence for the sequences of linear fractional transformations $\{S_{2n}\}$ and $\{S_{2n-1}\}$. But we know from the proof of the Stern-Stolz theorem and the convergence of $\mathcal{S}$ that $\{S_{2n}\}$ and $\{S_{2n-1}\}$ converges to distinct values. Hence $\{S_n\}$, with two subsequences that converges to distinct values, diverges generally. □

The convergence of the series $\mathcal{S}$ for a given continued fraction $\mathbf{K}(a_n/b_n)$ can sometimes be rather difficult to determine, and to ease our use of the Stern-Stolz theorem we have the following useful result:

Theorem 3.2. *The Stern-Stolz series $\mathcal{S}$ of $\mathbf{K}(a_n/b_n)$ has sum ∞ if at least one of the following conditions hold:*

$$\text{I:} \qquad \sum_{n=2}^{\infty}\sqrt{\left|\frac{b_n b_{n-1}}{a_n}\right|}=\infty$$

$$\text{II:} \qquad \liminf_{n\to\infty}\left|\frac{a_n}{b_n b_{n-1}}\right|<\infty$$

$$\text{III:} \qquad \sum_{n=2}^{\infty}\left|\frac{b_n b_{n-1}}{n a_n}\right|=\infty$$

For a proof see [LW08].

Theorem 3.3 (The Lane-Wall characterization theorem)**.** *Let the continued fraction $\mathbf{K}(a_n/b_n)$ with classical approximant $f_n = S_n(0)$ satisfy*

$$\sum_{n=m}^{\infty}|f_{n+1}-f_{n-1}|<\infty \tag{3.2}$$

for some fixed $m \in \mathbb{N}$. Then $\mathbf{K}(a_n/b_n)$ converges classically if and only if the corresponding Stern-Stolz series

$$\mathcal{S}=\sum_{n=1}^{\infty}\left|b_n\prod_{k=1}^{n}a_k^{(-1)^{n+k-1}}\right|$$

of $\mathbf{K}(a_n/b_n)$ diverges.

Proof. We need to prove both directions of the implication in the theorem. If we assume that $\mathbf{K}(a_n/b_n)$ converges classically then we have by the Stern-Stolz theorem that the series $\mathcal{S}$ of $\mathbf{K}(a_n/b_n)$ diverges and this establishes one direction of the implication. Hence we must prove that if the series $\mathcal{S}$ of $\mathbf{K}(a_n/b_n)$ diverges, then we have convergence of the continued fraction $\mathbf{K}(a_n/b_n)$. We observe that by Theorem 2.2 the continued fraction $\mathbf{K}(a_n/b_n)$ may by an equivalence transformation be transformed into a continued fraction $\mathbf{K}(1/c_n)$ where the Stern-Stolz series $\mathcal{S}$ is unchanged. Also, since it is the tails of a continued fraction that determines its convergence properties, we may without loss of generality set $m = 1$. By the assumption that $m = 1$ we have by (3.2) that $f_n < \infty$ and it follows that $B_n \neq 0$. From the Stern-Stolz theorem we have that $\mathbf{K}(1/c_n)$ diverges if $\sum|c_n| < \infty$ and consequently we assume that $\sum|c_n| = \infty$. By equation (3.2) we have absolute convergence of the sequences $\{f_{2n}\}$ and $\{f_{2n+1}\}$ and the limits L_0 of $\{f_{2n}\}$ and L_1 of $\{f_{2n+1}\}$ exist and are finite. To prove the convergence of $\mathbf{K}(a_n/b_n)$, we must show that the limits L_0 and L_1 coincide under the assumptions given. To do this we will assume that $L_0 \neq L_1$ and derive a contradiction. If $L_0 \neq L_1$ we have that $|f_{n+1} - f_n| \to |L_0 - L_1| > 0$. We now define

$$\delta_n=-\frac{f_{n+1}-f_{n-1}}{f_{n+1}-f_n}$$

and we see by Abel's test that the series $\sum|\delta_n|$ and $\sum|f_{n+1}-f_{n-1}|$ converge and diverge together and hence $\sum|\delta_n|$ converges. We have from the recurrence relations and determinant formula in Lemma (2.3) and equation (2.9) that

$$\begin{aligned}\delta_n&=-\frac{\frac{A_{n+1}}{B_{n+1}}-\frac{A_{n-1}}{B_{n-1}}}{\frac{A_{n+1}}{B_{n+1}}-\frac{A_n}{B_n}}=-\frac{A_{n+1}B_{n-1}-A_{n-1}B_{n+1}}{A_{n+1}B_n-A_nB_{n-1}}\cdot\frac{B_n}{B_{n-1}}\\&=-c_{n+1}\frac{A_nB_{n-1}-A_{n-1}B_n}{A_{n+1}B_n-A_nB_{n+1}}\cdot\frac{B_n}{B_{n-1}}=c_{n+1}\frac{B_n}{B_{n-1}}\end{aligned}$$

Hence we have that

$$\delta_n B_{n-1} = c_{n+1} B_n \tag{3.3}$$

for all $n \in \mathbb{N}$. We want to show, by using(3.3), that $\sum |\delta_n| < \infty$ implies that $\sum |c_n| < \infty$ and thus obtain a contradiction. We solve for c_{n+1} to obtain

$$\begin{aligned} c_{n+1} = \delta_n \frac{B_{n-1}}{B_n} &= \frac{\delta_n B_n - 1}{c_n B_{n-1} + B_{n-2}} = \frac{\delta_n B_n - 1}{\delta_{n-1} B_{n-2} + B_{n-2}} \\ &= \frac{\delta_n}{\delta_{n-1}+1} \cdot \frac{c_{n-1}B_{n-2} + B_{n-3}}{B_{n-2}} = \frac{\delta_n}{\delta_{n-1}+1} \cdot \frac{\delta_{n-2}B_{n-3} + B_{n-3}}{B_{n-2}} \\ &= \frac{\delta_n(\delta_{n-2}+1)}{\delta_{n-1}+1} \cdot \frac{B_{n-3}}{B_{n-2}} \end{aligned} \tag{3.4}$$

We see from (3.4) that a pattern is emerging. When we continue in this way we obtain

$$\begin{aligned} c_{2n+1} &= \frac{\delta_{2n}(\delta_{2n-2}+1)(\delta_{2n-4}+1)\cdots(\delta_2+1)}{(\delta_{2n-1}+1)(\delta_{2n-3}+1)\cdots(\delta_3+1)} \cdot \frac{B_1}{B_2} \\ c_{2n+2} &= \frac{\delta_{2n+1}(\delta_{2n-1}+1)(\delta_{2n-3}+1)\cdots(\delta_1+1)}{(\delta_{2n}+1)(\delta_{2n-2}+1)\cdots(\delta_2+1)} \cdot \frac{B_0}{B_1} \end{aligned}$$

We observe that since $f_{n+1} \neq f_n$ we have that $\delta_n \neq \infty, -1$. We also recall that $B_n \neq 0$ for all $n \in \mathbb{N}$. Hence we have that if $\sum |\delta_n| < \infty$ this implies that $\sum |c_n| < \infty$. But this contradicts our assumption that $\sum |c_n| = \infty$. Hence our assumption that the limits L_0 and L_1 are distinct is false, and we have convergence of $\mathbf{K}(a_n/b_n)$. □

Theorem 3.4 (The parabola theorem)**.** *Let V_α and E_α be given by*

$$V_\alpha = -\frac{1}{2} + e^{i\alpha}\overline{\mathbb{H}} = \left\{ w \in \mathbb{C} : \Re(we^{-i\alpha} \geq -\frac{1}{2}\cos\alpha \right\} \cup \{\infty\} \tag{3.5}$$

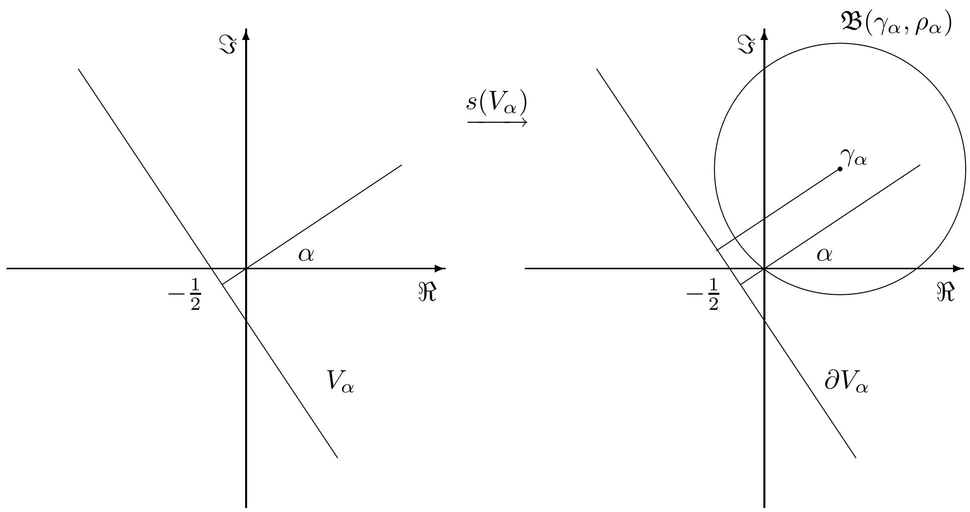
$$E_\alpha = \left\{ a \in \mathbb{C} : |a| - \Re(ae^{-2ia}) \leq \frac{1}{2}\cos^2\alpha \right\} \tag{3.6}$$

Then, for fixed $\alpha \in \mathbb{R}$ where $|\alpha| < \frac{\pi}{2}$, the set E_α is the element set for continued fractions $\mathbf{K}(a_n/1)$ corresponding to the value set V_α. Let $\mathbf{K}(a_n/1)$ be a continued fraction from E_α. If $\mathcal{S} = \infty$, then $\mathbf{K}(a_n/1)$ converges to a finite value. If $\mathcal{S} < \infty$, then $\{f_{2n}\}$ and $\{f_{2n+1}\}$ converge absoultely to distinct finite values, and $\{S_{2n}\}$ and $\{S_{2n+1}\}$ converge generally to these vaules.

Proof. The proof consists of two seperate parts. First we prove that $s(V_\alpha) = \frac{a}{1+V_\alpha} \subseteq V_\alpha$ if and only if $a \in E_\alpha$. Then we prove that if we are given a continued fraction $\mathbf{K}(a_n/1)$ with E_α as element set, then it converges if and only if $\mathcal{S} = \infty$.

We see from Definition 2.13 of value sets that if $a = 0$, then the inclusion is clear. Let us then set $a \neq 0$. We have that by Lemma 2.1 the transformation s maps the half plane V_α to a finite disk $\mathfrak{B}(\gamma_\alpha, \rho_\alpha)$ where $\gamma_\alpha = \frac{ae^{-i\alpha}}{\cos\alpha}$ and $\rho_\alpha = \frac{|a|}{\cos\alpha}$. Now $\mathfrak{B}(\gamma_\alpha, \rho_\alpha)$ is inside the half-plane V_α if and only if the distance from the center γ_α of the disk $\mathfrak{B}(\gamma_\alpha, \rho_\alpha)$ to the boundary ∂V_α of the half plane V_α is greater than or equal to the radius ρ_α of the disk. The distance from the center γ_α to the boundary ∂V_α, denoted by δ_a, is given by $\frac{\cos\alpha}{2} + \Re(\gamma_\alpha e^{-i\alpha})$. An illustration of the situation is given in figure 1

Figure 1



Hence $s(V_\alpha) \subseteq V_\alpha$ if and only if

$$\begin{array}{rcl} \dfrac{\cos\alpha}{2} + \Re(\gamma_\alpha e^{-i\alpha}) & \geq & \dfrac{|a|}{\cos\alpha} \\ & \Updownarrow & \\ \dfrac{\cos\alpha}{2} + \Re\left(\dfrac{ae^{-i\alpha}}{\cos\alpha} e^{-i\alpha}\right) & \geq & \dfrac{|a|}{\cos\alpha} \\ & \Updownarrow & \\ \dfrac{\cos^2\alpha}{2} & \geq & |a| - \Re(ae^{-2i\alpha}) \end{array}$$

Which is clearly equal to our set E_α.

We now let $\mathbf{K}(a_n/1)$ be a continued fraction from E_α. We observe that $\infty \notin S_1(V_\alpha)$ since $\infty \notin E_\alpha$. From this it follows that the nested sets $K_n = S_n(V_\alpha)$ are bounded and finite disks. If the size of these disks goes to zero, that is $\text{diam}(K_n) \to 0$ where $\text{diam}(K_n)$ is the diameter of K_n, then the convergence is clear since we end up in only one point, see (2.25). Since we have convergence, it follows by logical negation of Lemma 3.2 that the series $\mathcal{S} = \infty$.

What remains is the case where $\text{diam}(K_n) \to K$ where $\text{diam}(K) > 0$. To establish the proof, we make use of the following linear fractional transformation:

$$\varphi(w) = \frac{-1 + e^{i\alpha}\cos\alpha - w}{1 + w}$$

whose inverse is exactly the same:

$$\varphi^{-1}(z) = \frac{-1 + e^{i\alpha}\cos\alpha - z}{1 + z}.$$

The function φ has the useful property that $\varphi(\infty) = -1$ and $\varphi(-1) = \infty$. Furthermore, it maps the closed unit disk $\overline{\mathbb{D}}$ onto V_α and hence the inverse image of every point in V_α is in the closed unit disk $\overline{\mathbb{D}}$. Since we want to use Lemma 2.2, we want to find a linear fractional transformation τ_n that maps the closed unit disk into itself and also maps infinity inside the unit disk. We define

$$\tau_n = \varphi^{-1} \circ s_{2n-1} \circ s_{2n} \circ \varphi$$

for $n \in \mathbb{N}$. We now observe that

$$\begin{aligned}
\tau_n(\overline{\mathbb{D}}) &= \varphi^{-1} \circ s_{2n-1} \circ s_{2n} \circ \varphi(\overline{\mathbb{D}}) \\
&= \varphi^{-1} \circ s_{2n-1} \circ s_{2n}(V_\alpha) \\
&\subseteq \varphi^{-1} \circ s_{2n-1}(V_\alpha) \\
&\subseteq \varphi^{-1}(V_\alpha) \\
&= \overline{\mathbb{D}}
\end{aligned}$$

and hence that τ_n maps the closed unit disk into itself. Furthermore we have that

$$\begin{aligned}
\tau_n(\infty) &= \varphi^{-1} \circ s_{2n-1} \circ s_{2n} \circ \varphi(\infty) \\
&= \varphi^{-1} \circ s_{2n-1} \circ s_{2n}(-1) \\
&= \varphi^{-1} \circ s_{2n-1}(\infty) \\
&= \varphi^{-1}(0) \\
&= -1 + e^{i\alpha} \cos\alpha.
\end{aligned}$$

We observe that if we multiply $\tau_n(\infty)$ by $e^{-i\alpha}$ we get that $\tau_n(\infty)e^{-i\alpha} = -e^{-i\alpha} + \cos\alpha$ and thus, by a simple trigronometric identity for complex values we have that

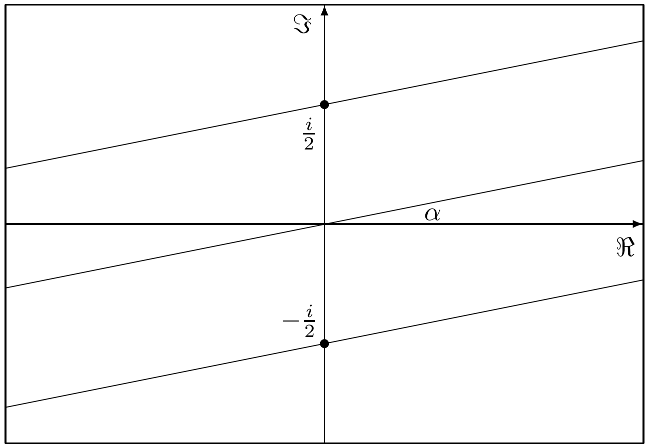
$$|\tau_n(\infty)| = |\tau_n(\infty)e^{-i\alpha}| = |i \sin\alpha| < 1$$

Hence, by setting $w_n = \infty$, it follows from Lemma 2.2 that the sequence of linear fractional transformations $\{\mathcal{T}_n\}$ given by $\{\mathcal{T}_n\} = \tau_1 \circ \tau_2 \circ \tau_3 \circ \cdots \circ \tau_n$ converges generally to some value $\gamma \in \overline{\mathbb{D}}$ and also that the sum $\sum |\mathcal{T}_n(\infty) - \mathcal{T}_{n-1}(\infty)| < \infty$. Moreover

$$\begin{aligned}
\mathcal{T}_n &= \tau_1 \circ \tau_2 \circ \tau_3 \circ \cdots \circ \tau_n \\
&= (\varphi^{-1} \circ s_1 \circ s_2 \circ \varphi) \circ (\varphi^{-1} \circ s_2 \circ s_3 \circ \varphi) \circ \cdots \circ (\varphi^{-1} \circ s_{2n-1} \circ s_{2n} \circ \varphi) \\
&= \varphi^{-1} \circ S_{2n} \circ \varphi
\end{aligned}$$

which means that $\mathcal{T}_n(\infty) = \varphi^{-1}(S_{2n}(-1)) = \varphi^{-1}(S_{2n-2}(0)) = \varphi^{-1}(f_{2n-1})$ where φ^{-1} has a pole at -1. Since all $f_n \in s_1(V_\alpha)$ and thus $\{f_n\}$ is bounded away from ∞ and -1 it follows that $\sum |f_{2n} - f_{2n-2}| \leq \infty$. Similarly it follows by using $\varphi^{-1} \circ s_{2n} \circ s_{2n+1} \circ \varphi$ that $\sum |f_{2n+1} - f_{2n-1}| < \infty$. Thus, since we have absolute convergence of the even and odd classical convergents, we have by the Lane-Wall characterization theorem that $\mathbf{K}(a_n/1)$ converges if and only if $\mathcal{S} = \infty$. $\square$

Figure 2



A useful observation here is that the set E_α can be written as

$$E_\alpha = \left\{ c^2 \in \mathbb{C} : |\Im(ce^{-i\alpha})| \leq \frac{1}{2}\cos\alpha \right\} \tag{3.7}$$

where $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. We see that the set from which c is taken from in (3.7) is a strip in the complex plane. The strip is illustrated in figure 2. We observe that the strip always has vertical width equal to 1, and that the points $\frac{i}{2}, -\frac{i}{2}$ always are points on its boundary.

Theorem 3.5 (The Lorentzen bestness-theorem)**.** *Let b be a complex number such that $b \notin E_\alpha$ for a fixed α where $|\alpha| < \frac{\pi}{2}$. Then there exists a point a on the boundary ∂E_α of E_α such that the linear fractional transformation*

$$S_2(w) = \cfrac{a}{1 + \cfrac{b}{1+w}} = \frac{a(1+w)}{1+b+w}$$

is elliptic.

Proof. The proof is split into three parts. The first part deals with the setup of the problem, where we obtain a system of two equations that we must prove has a solution in a desired range. In the second part we prove that this sytem has a solution in all but a special case, and finally in the third part we obtain a solution in the special case.

Part 1
We see from equation (3.7) that an element w lies on the boundary or inside a parabola ∂E_α if and only if we can write $w = \zeta^2$ where ζ lies on the strip $|\Im(\zeta e^{-i\alpha})| \leq \frac{1}{2}\cos\alpha$. Our goal is to prove the existence of a point $a = c^2 \in \partial E_\alpha$ such that the linear fractional transformation

$$S_2(w) = \frac{a(1+w)}{1+b+w} = \frac{c^2(1+w)}{1+d^2+w}$$

is elliptic for a given b outside the parabola. We have that b can be written as $b = d^2 \in \mathbb{C} \setminus E_\alpha$. Since c lies on the boundary of the strip it can be written as $(x+iy)e^{i\alpha}$ where $y = \frac{1}{2}\cos\alpha$. Furthermore, it is sufficient to use only this part of the boundary since both parts coincide when we square. Similarly we can write d as $(u+iv)e^{i\alpha}$ where $v > y$ or $v < -y$ since d^2 is placed outside the parabola and hence lies outside the strip. It is also worth noting that neither point is zero since zero is an interior point of the parabola.

By Definition 2.19 we have that $S_2(w)$ is elliptic if and only if $\delta \in [0, 1)$ where

$$\delta = \frac{(1+c^2+d^2)^2}{4c^2d^2}. \tag{3.8}$$

We take the square root of (3.8) and have that $S_2(w)$ is elliptic if and only if $t \in (-1, 1)$ where

$$\frac{1 + d^2 + c^2}{2cd} = t. \tag{3.9}$$

Hence we must show that a solution of equation (3.9) exists for t in the interval $(-1, 1)$. We multiply to get rid of the fraction and insert the expressions for c and d to obtain

$$(x + iy)^2 e^{2i\alpha} + 1 + (u + iv)^2 e^{2i\alpha} = 2t(x + iy)(u + iv)e^{2i\alpha}.$$

Dividing by $e^{2i\alpha}$ and using some trigonometric identities yield the expression

$$x^2 - y^2 + 2ixy + \cos 2\alpha - i \sin 2\alpha + u^2 - v^2 + 2iuv = 2t(xu - yv + ivx + iuy).$$

If we collect the real and imaginary parts of equation we obtain the two equations

$$x^2 - y^2 + \cos 2\alpha + u^2 - v^2 = 2t(xu - vy) \tag{3.10}$$

and

$$2xy - \sin 2\alpha + 2uv = 2t(vx - uy). \tag{3.11}$$

Hence the goal of our proof is to show that this system of equations (3.10), (3.11) has a solution (x, t) where $x \in \mathbb{R}$ and $t \in (-1, 1)$ for all $u \in \mathbb{R}$ and $v \in \mathbb{R}$ with $|v| > y$.

Part 2

Without further ado, it seems natural to try to solve (3.11) for x to obtain

$$x = \frac{tuy + \sin\alpha\cos\alpha - uv}{y - tv}. \tag{3.12}$$

But now we have a rather serious problem. If t is set to the value $\frac{y}{v}$ and inserted into (3.12) we divide by zero, which we usually try to avoid. This is the special case that we will deal with in part 3. For now we simply ignore the whole problem and assume that $t \neq \frac{y}{v}$. We continue to solve our system of equations, and insert the value for x obtained in (3.12) into (3.10) and rearrange everyting neatly to obtain

$$\begin{aligned}&(\sin\alpha\cos\alpha - uv + tuy)^2 + (\cos 2\alpha + u^2 - v^2 - y^2 + 2tvy)(y - tv)^2 \\ &-2tu(y - tv)(\sin\alpha\cos\alpha - uv + tuy) = 0.\end{aligned}$$

This we define as a function with t as variable:

$$\begin{aligned}F(t) \quad = \quad &(\sin\alpha\cos\alpha - uv + tuy)^2 + (\cos 2\alpha + u^2 - v^2 - y^2 + 2tvy)(y - tv)^2 \\ &-2tu(y - tv)(\sin\alpha\cos\alpha - uv + tuy) = 0.\end{aligned}$$

Hence we need to prove that a solution $F(t) = 0$ exists in our interval $(-1, 1)$. As a polynomial of t, $F(t)$ is obviously a continuous function and furthermore we observe that if we let $t = \frac{y}{v}$, most of our functional equation disappears and we are left with the function

$$F\left(\frac{y}{v}\right) = \left(\sin\alpha\cos\alpha - uv + \frac{uy^2}{v}\right)^2.$$

This is obviously positive and nonzero since it is a square of real numbers with positive and nonzero absolute values. Hence we have obtained a positive value for our function $F(t)$ in our interval. If we can create a negative one in the same interval, we are done. The first candidates for these negative values are the end-points of the interval. We simply insert to obtain

$$\begin{aligned} F(\pm 1) &= (\sin\alpha\cos\alpha - uv \pm uy)^2 + (\cos 2\alpha + u^2 - v^2 - y^2 \pm 2vy)(y \mp v)^2 \\ &\mp 2u(y \mp v)(\sin\alpha\cos\alpha - uv \pm uy). \end{aligned}$$

If we write $G = \frac{\sin\alpha\cos\alpha}{y \mp v}$ then $F(\pm 1)$ can be written as

$$F(\pm 1) = (y \mp v)^2((G \pm u)^2 + \cos 2\alpha + u^2 - v^2 - y^2 \pm 2vy \mp 2u(G \pm u)).$$

We now have two cases to consider: $v > y = \frac{1}{2}\cos\alpha$ and $v < -y = \frac{1}{2} - \cos\alpha$. We will only give details in one case, since the other is analogus. If $v > y$ we have by simplification of the above equation that

$$F(-1) = (y+v)^2(G^2 - (y+v)^2 + \cos 2\alpha). \tag{3.13}$$

If we estimate the size of this G we obtain

$$|G| = \left|\frac{\sin\alpha\cos\alpha}{y+v}\right| < \left|\frac{\sin\alpha\cos\alpha}{2y}\right| = |\sin\alpha| \quad \text{since } v > y \text{ and } y = \frac{1}{2}\cos\alpha.$$

Hence if we insert this into (3.13) we get that

$$F(-1) = (y+v)^2(G^2 - (y+v)^2 + \cos 2\alpha) < (y+v)^2(\sin^2\alpha - (y+v)^2 + \cos 2\alpha). \tag{3.14}$$

Now we observe that $\cos 2\alpha = 1 - 2\sin^2\alpha$ and we then get by equation (3.14) that

$$F(-1) < (y+v)^2(\cos^2\alpha - (y+v)^2)$$

which is clearly negative since $(y+v)^2 > (2y)^2 = \cos^2\alpha$. Thus we have proven that a negative value exists and we are done. If $v < -y$ we obtain the same conclusion by a similar argument and by using the other end-point of our interval, namely we show that $F(1) < 0$ and the conclusion follows.

Part 3
We now remedy the problem that became apparant when expressing x as in (3.12). This is done by first finding under what conditions it is possible that $t = \frac{y}{v}$ and then we examine the behaviour of x as expressed by our equations. We then solve the equations for t and prove that the solutions are on the desired form.

Under what conditions is $t = \frac{y}{v}$ possible? To find out this, we insert $t = \frac{y}{v}$ into (3.11) and solve for u to obtain

$$u = \frac{v\sin 2\alpha}{2(v^2 - y^2)}.$$

We insert this value for u into (3.10) and (3.11) to obtain

$$x^2 - y^2 + \cos 2\alpha + \frac{v^2\sin^2 2\alpha}{4(v^2 - y^2)^2} - v^2 = 2t\left(\frac{vx\sin 2\alpha}{2(v^2 - y^2)} - vy\right). \tag{3.15}$$

and

$$2xy - \sin 2\alpha + \frac{v\sin 2\alpha}{(v^2 - y^2)}v = 2t\left(vx + \frac{vy\sin 2\alpha}{2(v^2 - y^2)}\right). \tag{3.16}$$

We further simplify (3.16) and get

$$(y - tv)\left(2x + \frac{y\sin 2\alpha}{v^2 - y^2}\right) = 0. \tag{3.17}$$

Now we see that in (3.17) we managed to factor out our troublesome denominator and hence we may freely insert $t = \frac{y}{v}$ and see what happens to (3.15). We have that

$$x^2 - y^2 + \cos 2\alpha + \frac{v^2 \sin^2 2\alpha}{4(v^2 - y^2)^2} - v^2 = 2\frac{y}{v}\left(\frac{vx\sin 2\alpha}{2(v^2 - y^2)} - vy\right).$$

which we can simplify to

$$x^2 - x\frac{y\sin 2\alpha}{(v^2 - y^2)} + y^2 + \cos 2\alpha + \frac{v^2 \sin^2 2\alpha}{4(v^2 - y^2)^2} - v^2 = 0.$$

Now the square formula yields:

$$x = \frac{\frac{y\sin 2\alpha}{(v^2-y^2)} \pm \sqrt{\left(\frac{y\sin 2\alpha}{(v^2-y^2)}\right)^2 - 4\left(y^2 + \cos 2\alpha + \frac{v^2\sin^2 2\alpha}{4(v^2-y^2)^2} - v^2\right)}}{2}$$

which by simplification is equivalent to

$$x = \frac{y\sin 2\alpha}{2(v^2 - y^2)} \pm \sqrt{v^2 - y^2 - \cos 2\alpha - \frac{\sin^2 2\alpha}{4(v^2 - y^2)}}.$$

Some rearrangement of the expression under the root-sign yields

$$x = \frac{y\sin 2\alpha}{2(v^2 - y^2)} \pm \sqrt{\left(1 + \frac{\sin^2\alpha}{v^2 - y^2}\right)(v^2 - y^2 - \cos^2\alpha)}.$$

and hence x is real whenever $v^2 - y^2 - \cos^2\alpha$ is positive, that is when $v^2 > y^2 + \cos^2\alpha$. Hence we have a solution (x, t) of our system (3.10), (3.11) when $v^2 > y^2 + \cos^2\alpha$. What remains is to consider the possibility that $y^2 < v^2 < y^2 + \cos^2\alpha$. To deal with this, we let x be a real number and we note that we do nothing wrong if we express x as $-\frac{y\sin 2\alpha}{2(v^2-y^2)}$, since this can express any real number. If we insert this into (3.15) we have the expression

$$\frac{y\sin 2\alpha}{2(v^2 - y^2)}^2 - y^2 + \cos 2\alpha + \frac{v^2\sin^2 2\alpha}{4(v^2 - y^2)^2} - v^2 = 2t\left(\frac{-\frac{vy\sin 2\alpha}{2(v^2-y^2)}\sin 2\alpha}{2(v^2 - y^2)} - vy\right).$$

Expanding the parantheses and moving everyting over to one side, we obtain

$$\frac{y^2\sin^2 2\alpha}{4(v^2 - y^2)^2} - y^2 + \cos 2\alpha + \frac{v^2\sin^2 2\alpha}{4(v^2 - y^2)^2} - v^2 + \frac{2tvy\sin^2 2\alpha}{4(v^2 - y^2)^2} + 2tvy = 0. \tag{3.18}$$

Solving (3.18) with respect to t yields

$$t = \frac{(v^2 + y^2)(4(v^2 - y^2)^2 - \sin^2 2\alpha) - 4\cos 2\alpha(v^2 - y^2)^2}{2vy(\sin^2 2\alpha + 4(v^2 - y^2)^2)}.$$

We immideatly see that t is a real number, and the only thing we need to show is that $|t| < 1$. This is clearly the case if our denominator is greater in absoulte value than the numerator in the expression above and hence we have that $|t| < 1$ if and only if

$$\begin{aligned}
&-2|v|y(\sin^2 2\alpha + 4(v^2 - y^2)^2) \\
&< 4(v^2 - y^2)^2(y^2 + v^2) - (y^2 + v^2)\sin^2 2\alpha - 4(v^2 - y^2)^2 \cos 2\alpha \\
&< 2|v|y(\sin^2 2\alpha + 4(v^2 - y^2)^2).
\end{aligned} \tag{3.19}$$

From equation (3.19) we easily see that we have the inequality

$$0 > 4(v^2-y^2)^2(y^2+v^2)-(y^2+v^2)\sin^2 2\alpha-4(v^2-y^2)^2\cos 2\alpha-2|v|y(\sin^2 2\alpha+4(v^2-y^2)^2) \tag{3.20}$$

by substracting one side from the other. The equation in (3.20) can be simplified to

$$0 > 4(v + y)^2(v - y)^2(|v| - y)^2 - (y + |v|)^2 \sin^2 2\alpha - 4(v + y)^2(v - y)^2 \cos 2\alpha$$

which we can divide by $(|v| + y)^2$ to obtain the inequality

$$4(|v| - y)^4 - \sin^2 2\alpha - 4(|v| - y)^2 \cos 2\alpha < 0. \tag{3.21}$$

By the same procedure but with the other inequality in (3.19) we can obtain

$$4(y + |v|)^4 - \sin^2 2\alpha - 4(|v| + y)^2 \cos 2\alpha > 0. \tag{3.22}$$

Thus $|t| < 1$ if and only if (3.21) and (3.22) hold for $y = \frac{1}{2}\cos\alpha < |v| < \frac{3}{2}\cos\alpha = y + \frac{1}{2}\cos\alpha$. Now the inequalities (3.21) and (3.22) can be written as

$$4s^4 - \sin^2 2\alpha - 4s^2 < 0 \quad \text{where } s \in (0, \cos\alpha) \tag{3.23}$$

$$4r^4 - \sin^2 2\alpha - 4r^2 > 0 \quad \text{where } r \in (\cos\alpha, 2\cos\alpha). \tag{3.24}$$

This can be rewritten as

$$s^4 - \sin^2\alpha\cos^2\alpha - s^2 < 0 \quad \text{where } s \in (0, \cos\alpha) \tag{3.25}$$

$$r^4 - \sin^2\alpha\cos^2\alpha - r^2 > 0 \quad \text{where } r \in (\cos\alpha, 2\cos\alpha). \tag{3.26}$$

Let us take a closer look at (3.25). At $s = 0$ it is obviously negative. If we treat it as a function of s it is also obviously increasing in the interval, and hence our only concern is what happens very close to the end-point where $s = \frac{2-\epsilon}{2}\cos\alpha$. We calculate:

$$\begin{gathered}
s^4 - \sin^2\alpha\cos^2\alpha - s^2 \\
\Updownarrow \\
\left(\frac{2-\epsilon}{2}\cos\alpha\right)^4 - \sin^2\alpha\cos^2\alpha - \left(\frac{2-\epsilon}{2}\cos\alpha\right)^2 \\
\Updownarrow \\
\left(\frac{2-\epsilon}{2}\right)^4 + \left[\left(\frac{2-\epsilon}{2}\right)^4 - 1\right]\sin^2\alpha - \left(\frac{2-\epsilon}{2}\right)^2.
\end{gathered} \tag{3.27}$$

We now observe that $\left[\left(\frac{2-\epsilon}{2}\right)^4 - 1\right]\sin^2\alpha$ can be made as small as we please, and it can hence be ignored when we decide whether the equation in (3.27) is negative, zero or positive. Hence we look at

$$\left(\frac{2-\epsilon}{2}\right)^4 - \left(\frac{2-\epsilon}{2}\right)^2. \tag{3.28}$$

The equation (3.28) is clearly negative, since the lower power dominates the higher one when inside the interval $(0, 1)$. Hence inequality (3.21) hold in our interval. By a similar argument inequality (3.22) also holds. Hence we have a desired solution (x, t) and that concludes the proof. $\square$

We notice however, that the converse does not hold.

Theorem 3.6. *Let a be a complex number such that $a \in \partial E_\alpha$ where α is fixed and $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then there exists a $b \notin E_\alpha$ such that*

$$S_2(w) = \cfrac{a}{1 + \cfrac{b}{1+w}} = \frac{a(1+w)}{1+b+w}$$

is parabolic.

Proof. This is obvious since we have two distinct parabolas through any point a that also goes through $-\frac{1}{4}$. $\square$

Chapter 4

Results

The following material is a consequence of efforts made by the author to give a new and simple, i.e short, proof of the Lorentzen bestness theorem presented in chapter 3. When searching for such a proof, several nice facts was found. Among the most interesting of these facts is a nice connection to the Fibonacci polynomials in number theory and a conjecture concerning the convergence properties of periodic continued fractions with a particular type of period.

The results that we present in this chapter are presented in the order that they were found. The initial object of study is the 2-periodic continued fraction that we used in the proof of Theorem 3.5.

We will approach the Lorentzen bestness theorem in a different manner than in the proof of Theorem 3.5. Our method will be to choose an a on the parabola, and then find the set of all b-s that together with a gives that $S_2(w)$ is classified as elliptic. The Lorentzen bestness theorem is then equivalent to that every point outside the parabolic region E_α is in at least one of the sets consisting of b-s for some a. We will state this idea in a more precise manner below.

4.1 Direct generaliztion

In the proof of the Lorentzen bestness theorem we made use of the 2-periodic continued fraction

$$\frac{a}{1} + \frac{u}{1} + \frac{a}{1} + \frac{u}{1} + \frac{a}{1} + \frac{u}{1} + \cdots \tag{4.1}$$

where u was allowed to be choosen freely outside the parabolic region E_α. The Lorentzen bestness theorem established that for any $u \in \mathbb{C} \setminus E_\alpha$ there exists an $a \in \partial E_\alpha$ such that (4.1) diverges and from this one may ask the following question: If we increase the length of the period by adding u-s, for what choice of u does such an $a \in \partial E_\alpha$ that gives divergence still exist?

4.1.1 A useful definition

Our object of study is the class of continued fractions

$$\frac{a}{1} + \underbrace{\frac{u}{1} + \frac{u}{1} + \ldots + \frac{u}{1}}_{(p-1)\text{ terms}} + \frac{a}{1} + \frac{u}{1} + \ldots \tag{4.2}$$

where $p \in \mathbb{N}$ and where $a \in \partial E_\alpha$ and $u \in \mathbb{C}$. We want to study the convergence properties of the continued fractions defined in (4.2) for various fixed p, and in particular we want to see where the corresponding approximant $S_p(w)$ is classified as elliptic. To do this, we make use of the Definitions 2.18 and 2.19 and see that in this case $S_p(w) = \frac{A_{p-1}w+A_p}{B_{p-1}w+B_p}$ is elliptic if and only if

$$t = \frac{(A_{p-1} + B_p)^2}{4\Delta} \in [0, 1) \tag{4.3}$$

where $\Delta = (-1)^p u^{p-1} a$. The set of roots we obtain when we solve (4.3) for u for fixed $a \in \partial E_\alpha$ and $t \in [0, 1)$ we denote by $D_u(a, t)$ and the union

$$\mathfrak{D}(a, t) = \bigcup_{\substack{a \in \partial E_\alpha \\ t \in [0,1)}} D_u(a, t)$$

we call *the elliptic set of* $S_p(w)$ *with regard to* a *and* t, or just *the elliptic set* whenever there is no risk of confusion. Similarly, the *parabolic set of* $S_p(w)$ *with regard to* a *and* t is given by the union

$$\bigcup_{\substack{a \in \partial E_\alpha \\ t=1}} D_u(a, t)$$

and denoted by $\mathfrak{P}_u(a, t)$. The notion of the elliptic and parabolic sets is introduced simply to save time when we examine properties of convergence for a few low values for p.

4.1.2 The 2-periodic case

Initially, $\mathfrak{D}_u(a, t)$ for $p = 2$ was studied in an effort to give a new and simple proof of the Lorentzen bestness theorem. The idea was that for $p = 2$ we may rewrite equation (4.3) as a quadratic equation in u,

$$u^2 + 2(a - 2at + 1)u + a^2 + 2a + 1 = 0 \tag{4.4}$$

and hence the roots that we obtain for u in (4.4) for fixed $a \in \partial E_\alpha$ and a variable $t \in [0, 1)$ are two curves in $\mathbb{C}$ denoted by f_1 and f_2 and given by

$$\begin{aligned} f_1(t) &= 2at - a - 1 + 2\sqrt{a^2t^2 - a^2t - at} \\ f_2(t) &= 2at - a - 1 - 2\sqrt{a^2t^2 - a^2t - at}. \end{aligned} \tag{4.5}$$

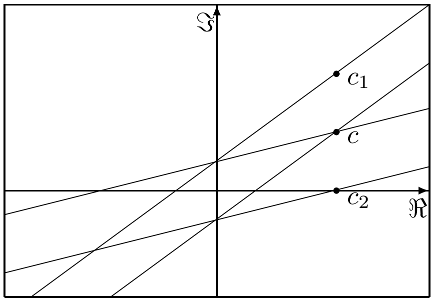
These curves are drawn for a few different a-s in the figures B.14 - B.16 in appendix B. The idea was to show that the curves f_1 and f_2 fills the set $\mathbb{C} \setminus E_\alpha$ when we take all possible values of $a \in \partial E_\alpha$ for some fixed $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. At $t = 0$ the curves f_1, f_2 coincide in the point $-a - 1$ and if we look at the limit of $f_1(t)$ and $f_2(t)$ when $t \to 1$, we get that

$$\begin{aligned} \lim_{t \to 1} f_1(t) &= a + 2i\sqrt{a} - 1 = (\sqrt{a} + i)^2 = c_1^2 \\ \lim_{t \to 1} f_2(t) &= a - 2i\sqrt{a} - 1 = (\sqrt{a} - i)^2 = c_2^2 \end{aligned}$$

We now recall that for $a \in \partial E_\alpha$ we may write $a = c^2$ where

$$|\Im(ce^{-i\alpha})| = \frac{1}{2}\cos\alpha$$

Figure 3



and we note that, depending on where we put a, we see that one of the points c_1^2, c_2^2 lie on the same parabola as a, since c_1 or c_2 lies on the same strip as c. The point that does not lie on the same parabola, lies on the other possible parabola that goes through a. The situation is illustrated in figure 3. Here we have placed c such that c_2 lies on the same strip and c_1 lies on the other possible strip. Hence we may see the curves $f_1(t)$ and $f_2(t)$ as a single cuve g that goes from one possible parabola to the other and we need to show that the collection of all these curves g for all a fills the set $\mathbb{C} \setminus E_\alpha$. A few of these curves for various $a \in E_\alpha$ are drawn in the figures B.14 - B.16 in appendix B. The union of all these curves g obviously gives us set $\mathfrak{D}_u(a,t)$ as we defined above. As we see, it would not be too surprising if the collection of all these curves fill the plane and indeed, the Lorentzen bestness theorem implies that $\mathfrak{D}_u(a,t) = \mathbb{C} \setminus E_\alpha$ and that $\mathfrak{D}_u(a,t)$ is simply connected. Unfortunately, we have not been able to supply a rigorous proof of this by using the functions $f_1(t)$ and $f_2(t)$.

4.1.3 The 3-periodic case

Following the idea in the case where $p = 2$ in (4.2), we tried to solve (4.3) for u when $p = 3$ and we came across our first surprise. We managed to conjecture and after some work prove the following nice theorem:

Theorem 4.1. *Let the continued fraction* $\mathbf{K}(a_n/1)$ *be given where* $a_{3k} = a, a_{3k+1} = u$ *and* $a_{3k+2} = u$ *for* $k \in \mathbb{N}$ *and where* $0 \neq a \in \partial E_\alpha$ *for some fixed* $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$. *Then the continued fraction* $\mathbf{K}(a_n/1)$ *converges if* $u \in -\frac{1}{4} + e^{i\alpha}\overline{\mathbb{H}} = \{w \in \mathbb{C} : \Re(we^{-i\alpha}) \geq -\frac{1}{4}\cos\alpha\}$.

Theorem 4.1 also holds for limit-periodic continued fractions:

Theorem 4.2. *Let the limit periodic continued fraction* $\mathbf{K}(a_n/1)$ *be given where* $\lim a_{3k} = a$ *and* $\lim a_{3k+1} = \lim a_{3k+2} = u \neq \infty$ *and where* $a \in E_\alpha$ *for some fixed* $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$. *Then the continued fraction* $\mathbf{K}(a_n/1)$ *converges if* $u \in -\frac{1}{4} + e^{i\alpha}\mathbb{H} = \{w \in \mathbb{C} : \Re(we^{-i\alpha}) > -\frac{1}{4}\cos\alpha\}$.

For proofs of these theorems, see section A.3.2 in appendix A where the 3-periodic case is studied in detail. For an illustration of the situation, see appendix A or figure B.3 in appendix B. Theorem 4.1 clearly implies that $\mathfrak{D}_u(a,t) \subseteq \mathbb{C} \setminus \{-\frac{1}{4} + e^{i\alpha}\mathbb{H}\}$ and hence $\mathfrak{D}_u(a,t)$ is *not* equal to the set $\mathbb{C} \setminus E_\alpha$ when $p = 3$.

4.1.4 The 4-periodic case

With our success in the case where $p = 3$ in mind, we immideatly tried to investigate similar curves for $p = 4$. The polynomial we obtain from (4.3) in u whose roots we must find is given by

$$\left(u^2 + (3+a)u + a + 1\right)^2 - 4tau^3. \tag{4.6}$$

As we see, (4.6) is a polynomial of fourth degree in u and the general expression for its roots are quite horrible. Thus, to get a grip on this at all, we instead solve (4.6) when $t = 1$ which will give us the parabolic set $\mathfrak{P}_u(a,t)$. We solve by Maple and obtain

$$
\begin{aligned}
u_1 =& -\frac{3}{2}+\frac{a}{2}+2i\sqrt{a}+\frac{1}{2}\sqrt{\frac{5i\sqrt{a}-18ia\sqrt{a}+a^2i\sqrt{a}+16a-8a^2}{i\sqrt{a}}}\\
u_2 =& -\frac{3}{2}+\frac{a}{2}+2i\sqrt{a}-\frac{1}{2}\sqrt{\frac{5i\sqrt{a}-18ia\sqrt{a}+a^2i\sqrt{a}+16a-8a^2}{i\sqrt{a}}}\\
u_3 =& -\frac{3}{2}+\frac{a}{2}-2i\sqrt{a}+\frac{1}{2}\sqrt{\frac{5i\sqrt{a}-18ia\sqrt{a}+a^2i\sqrt{a}+16a-8a^2}{i\sqrt{a}}}\\
u_4 =& -\frac{3}{2}+\frac{a}{2}-2i\sqrt{a}-\frac{1}{2}\sqrt{\frac{5i\sqrt{a}-18ia\sqrt{a}+a^2i\sqrt{a}+16a-8a^2}{i\sqrt{a}}}
\end{aligned}
$$

We recall that we may write $a=c^2$ and we simplify under the root-sign to obtain

$$
\begin{aligned}
u_1 &= \frac{1}{2}(c+i)(c+3i)+\frac{1}{2}(c+i)\sqrt{(c+5i)(c+i)}\\
u_2 &= \frac{1}{2}(c+i)(c+3i)-\frac{1}{2}(c+i)\sqrt{(c+5i)(c+i)}\\
u_3 &= \frac{1}{2}(c-i)(c-3i)+\frac{1}{2}(c+i)\sqrt{(c+5i)(c+i)}\\
u_4 &= \frac{1}{2}(c-i)(c-3i)-\frac{1}{2}(c+i)\sqrt{(c+5i)(c+i)}.
\end{aligned}
$$

A plot of u_1,u_2,u_3,u_4 for $a\in\partial E_\alpha$ are given in figure B.4 in appendix B. As we see, some of the u_i-s apparantly traces two circles but we have not been able to give an equation for them based on the equations above. The plot of $\mathfrak{P}_u(a,t)$ suggests a theorem for $p=4$ that resembles Theorem 4.1. This is however nigh impossible to prove in the same way since the roots are simply given by too complicated expressions. The reader is encouraged to have a go at the polynomial in (4.6) in Maple to be convinced of this.

4.1.5 The 5-periodic case

With the apparent faliure in the case where $p=4$, we examine the case where $p=5$. An illustration is given in figure B.5. Proceeding as we did for $p=4$, we first find $\mathfrak{P}_u(a,t)$ by setting $t=1$ in (4.3). The polynomial is given by

$$(a+2ua+1+4u+3u^2)^2+4atu^4 \tag{4.7}$$

and for $t=1$ we obtain the roots

$$
\begin{aligned}
u_1 =& -1+i\sqrt{a}\\
u_2 =& -1-i\sqrt{a}\\
u_3 =& -\frac{2a+3-i\sqrt{a}}{9+4a}\\
u_4 =& -\frac{2a+3+i\sqrt{a}}{9+4a}.
\end{aligned}
$$

We once more recall that we may set $a = c^2$ and we obtain

$$\begin{aligned}
u_1 &= -1 + ic \\
u_2 &= -1 - ic \\
u_3 &= -\frac{2(c+i)(c-\frac{3i}{2})}{4(c+\frac{3i}{2})(c-\frac{3i}{2})} = -\frac{c+i}{2c+3i} \\
u_4 &= -\frac{2(c-i)(c+\frac{3i}{2})}{4(c+\frac{3i}{2})(c-\frac{3i}{2})} = -\frac{c-i}{2c-3i}.
\end{aligned}$$

We observe that u_1 and u_2 traces lines as c traces the boundary of the strip and that u_3 and u_4 are linear fractional transformations and hence sends the boundary of the strip to generalized circles. The line where c lies can be seen as the boundary of a rotated half plane which is given by $-\frac{1}{2}\cos\alpha + e^{i\left(\frac{\pi}{2}-\alpha\right)}\mathbb{H}$. We apply Lemma 2.1 to the linear fractional transformations u_3 and u_4 and after some simplification obtain that u_3 maps the boundary of the strip to the circle $\partial\mathfrak{B}_3(\Gamma_3, \rho_3)$ where

$$\begin{aligned}
\Gamma_3 &= -\frac{1}{2} - \frac{e^{-i(\pi-\alpha)}}{8\Re\left[\left(-\frac{1}{2}\cos\alpha + \frac{3i}{2}\right)e^{-i\left(\frac{\pi}{2}-\alpha\right)}\right]} \\
\rho_3 &= \frac{1}{8\Re\left[\left(-\frac{1}{2}\cos\alpha + \frac{3i}{2}\right)e^{-i\left(\frac{\pi}{2}-\alpha\right)}\right]}.
\end{aligned} \tag{4.8}$$

Similary, u_4 maps the strip to the circle $\partial\mathfrak{B}_4(\Gamma_4, \rho_4)$ where

$$\begin{aligned}
\Gamma_4 &= -\frac{1}{2} - \frac{e^{i\alpha}}{8\Re\left[\left(-\frac{1}{2}\cos\alpha - \frac{3i}{2}\right)e^{-i\left(\frac{\pi}{2}-\alpha\right)}\right]} \\
\rho_4 &= \frac{1}{8\Re\left[\left(-\frac{1}{2}\cos\alpha - \frac{3i}{2}\right)e^{-i\left(\frac{\pi}{2}-\alpha\right)}\right]}.
\end{aligned} \tag{4.9}$$

We now try to simplify the denominator for $\Gamma_3, \Gamma_4, \rho_3$ and ρ_4. For Γ_3, ρ_3 we rewrite to obtain

$$8\Re\left[-\frac{1}{2}\left(\cos\alpha - 3i\right)\left(\cos\phi + i\sin\phi\right)\right]$$

where $\phi = \alpha - \pi/2$. We expand and seperate the real part and we get

$$-4\left(\cos\alpha\cos\phi + 3\sin\phi\right). \tag{4.10}$$

We now insert $\phi = \alpha - \pi/2$ and we see that $\cos\left(\alpha - \frac{\pi}{2}\right) = \sin\alpha$ and that $\sin\left(\alpha - \frac{\pi}{2}\right) = -\cos\alpha$ and hence we have that (4.10) can be written as

$$-4\cos\alpha(\sin\alpha - 3)$$

In a similar manner we have that the denominator for Γ_4, ρ_4 that can be written as

$$-4\cos\alpha(\sin\alpha + 3)$$

Hence we may rewrite the equations in (4.8) and (4.9) and obtain the following formulas for the center and radius for the circles $\partial\mathfrak{B}_3(\Gamma_3, \rho_3)$ and $\partial\mathfrak{B}_4(\Gamma_4, \rho_4)$:

$$\begin{aligned}
\Gamma_3 &= -\frac{1}{2} + \frac{e^{-i(\pi-\alpha)}}{4\cos\alpha(\sin\alpha - 3)} \\
\rho_3 &= -\frac{1}{4\cos\alpha(\sin\alpha - 3)} \\
\Gamma_4 &= -\frac{1}{2} + \frac{e^{i\alpha}}{4\cos\alpha(\sin\alpha + 3)} \\
\rho_4 &= -\frac{1}{4\cos\alpha(\sin\alpha + 3)}.
\end{aligned}$$

A figure illustrating the situation is given in figure B.5 in appendix B. Both the lines traced by u_1 and u_2 and the circles $\partial\mathfrak{B}_3(\Gamma_3, \rho_3), \partial\mathfrak{B}_4(\Gamma_4, \rho_4)$ traced by u_3 and u_4 are clearly visible. Unfortunately, despite that the figure suggests a nice result lurking in the background, we have not been able to supply a theorem in the 5 periodic case in a manner similar to Theorem 4.1.

4.1.6 Two conjectures

The behavior of the set $\mathfrak{P}_u(a,t)$ as demonstrated in the cases where $p = 3$, $p = 4$ and $p = 5$ suggests that as p increases, the set $\mathfrak{D}_u(a,t)$ becomes smaller and smaller. A presice statement of this is included in the following conjecture

Conjecture 1. Let $a \in \partial E_\alpha$ and let $\alpha \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be fixed. Let $\mathfrak{D}_{u,p}(a,t)$ denote the set $\mathfrak{D}_u(a,t)$ for the continued fraction in (4.2) for a given $p \in \mathbb{N}$. Then

$$\begin{aligned}
\cdots \subset \mathfrak{D}_{u,2p}(a,t) \subset \mathfrak{D}_{u,2p-2}(a,t) \subset \cdots \subset \mathfrak{D}_{u,6}(a,t) \subset \mathfrak{D}_{u,4}(a,t) \subset \mathfrak{D}_{u,2}(a,t) \\
\cdots \subset \mathfrak{D}_{u,2p+1}(a,t) \subset \mathfrak{D}_{u,2p-1}(a,t) \subset \cdots \subset \mathfrak{D}_{u,7}(a,t) \subset \mathfrak{D}_{u,5}(a,t) \subset \mathfrak{D}_{u,3}(a,t)
\end{aligned}$$

In other words: we claim that as p increases, the sets $\mathfrak{D}_u(a,t)$ form two collections of sets, one for odd p and one for even p and the elements of either collection are nested and decreases in size as p increases. The conjecture also suggests that

Conjecture 2. Let $a \in \partial E_\alpha$ and let $\alpha \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be fixed. Let $\mathfrak{D}_{u,p}(a,t)$ denote the set $\mathfrak{D}_u(a,t)$ for the continued fraction in (4.2) for a given $p \in \mathbb{N}$. Then

$$\lim_{p\to\infty} \mathfrak{D}_{u,p}(a,t) = \left(-\infty, -\tfrac{1}{4}\right]$$

Both conjectures are in a way intuitively obvious in the sense that we would expect the continued fracton in (4.2) as p grows large to depend less and less upon a and behave more and more like a 1-periodic continued fraction, and we know that the elliptic set of a 1-periodic continued fraction are given by $\left(-\infty, -\frac{1}{4}\right]$. Unfortunately, we have not managed to prove any of these conjectures.

4.1.7 Further investigations

When trying to prove conjecture 1, a calculation was performed in which we calculated several of the classical approximants of the continued fraction in (4.2) for large values of p. We obtained the following expressions for B_i for $0 \leq i < 8 < p$

$$\begin{aligned}
B_0 &= 1\\
B_1 &= 1\\
B_2 &= u+1\\
B_3 &= 2u+1\\
B_4 &= u^2+3u+1\\
B_5 &= 3u^2+4u+1\\
B_6 &= u^3+6u^2+5u+1\\
B_7 &= 4u^3+10u^2+6u+1.
\end{aligned} \tag{4.11}$$

We recognize the polynomials given in (4.11) as the *Jacobsthal polynomials*, as defined in [Kos01] by the recursion

$$\mathfrak{J}_n(x) = \mathfrak{J}_{n-1}(x) + x\mathfrak{J}_{n-2}(x) \tag{4.12}$$

where $\mathfrak{J}_1(x) = 1 = \mathfrak{J}_2(x)$. The first few Jacobsthal polynomials are given by

$$\begin{aligned}
\mathfrak{J}_3(x) &= x+1\\
\mathfrak{J}_4(x) &= 2x+1\\
\mathfrak{J}_5(x) &= x^2+3x+1\\
\mathfrak{J}_6(x) &= 3x^2+4x+1\\
\mathfrak{J}_7(x) &= x^3+6x^2+5x+1
\end{aligned} \tag{4.13}$$

We thus have that $B_n = \mathfrak{J}_{n+1}(u)$, and since $A_n = aB_{n-1}$ by Lemma 2.4 we have that $A_n = a\mathfrak{J}_n(u)$. In addition, we have an explicit formula for the Jacobsthal polynomials:

$$\mathfrak{J}_n(x) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{\lfloor \frac{n}{2} \rfloor + j}{\lfloor \frac{n-1}{2} \rfloor - j} x^{\lfloor \frac{n-1}{2} \rfloor - j}. \tag{4.14}$$

The Jacobsthal polynomials are quite useful when performing calculations that involve continued fractions of the type given in (4.2) since it allows us to skip some calculations of A_n and B_n for $n < p$ and instead choose A_n and B_n from a list of Jacobsthal polynomials. We will make use of this later when we write a computer program to draw various subsets of $\mathfrak{D}_u(a,t)$.

4.1.8 An application to recursive polynomials

The Jacobsthal polynomials that was encountered in the previous section established a link between the theory of continued fractions and the recursive polynomials that are studied in number theory. This link is not really surprising since both the recursive polynomials and the continued fractions are closely related to the study of three term recurrence relations. Hence the idea appeared that it might be possible to study these recursive polynomials through the use of continued fraction theory. A rather thorough investigation of the litterature gave no hint of any previous attempts in this direction, although a continued fraction was expressed in [Swa99] without reference to any continued fraction theory. However, this apparent lack of application may be caused by the limited sources available to the author.

In [Kos01] a lot of attention is given to various identities for Fibonacci and related numbers as well as identities for the various recursive polynomials. There are quite a lot of three-term recurrence relations given in [Kos01] and hence the potential results of an application of continued fraction theory is quite impressive, but due to various constraints on this thesis we shall limit ourselves to an example. As a proof of concept we will apply the determinant formula to a class of recursive polynomials known as the *Morgan-Voyce polynomials* and obtain a nice identity.

Following [Kos01] we define the Morgan-Voyce polynomials $b_n(x)$ and $B_n(x)$ recursively by

$$B_n(x) = (x+2)B_{n-1}(x) - B_{n-2}(x) \tag{4.15}$$
$$b_n(x) = (x+2)b_{n-1}(x) - b_{n-2}(x) \tag{4.16}$$

where $B_0 = 1, B_1 = x+2, b_0 = 1$ and $b_1 = x+1$. Clearly, the n-th canonical denominator B_n of the continued fraction

$$\frac{-1}{x+1} + \frac{-1}{x+2} + \frac{-1}{x+2} + \frac{-1}{x+2} + \dots \tag{4.17}$$

gives the Morgan-Voyce polynomial $b_n(x)$ and similarly we have that the n-th canonical denominator D_n of the continued fraction

$$\frac{-1}{x+2} + \frac{-1}{x+2} + \frac{-1}{x+2} + \frac{-1}{x+2} + \dots \tag{4.18}$$

gives the Morgan-Voyce polynomial $B_n(x)$. We also observe that the continued fraction in (4.18) is the 1-st tail of the continued fraction in (4.17). If we set $k = 0$ in Lemma 2.4 we have that

$$A_n^{(N)} B_{n-1}^{(N)} - A_{n-1}^{(N)} B_n^{(N)} = -B_0^{(N+n)} \prod_{j=N+1}^{N+n} (-a_j).$$

We now set $N = 0$ and since $A_n^{(N)} = a_{N+1} B_{n-1}^{(N+1)}$ we obtain

$$a_1 B_{n-1}^{(1)} B_{n-1} - a_1 B_{n-2}^{(1)} B_n = -\prod_{j=1}^{n} (-a_j). \tag{4.19}$$

Combining equation (4.17) with (4.19) and remembering that $a_i = -1$ for all i and that (4.18) is the first tail for (4.17), we have after some rearrangement that for the Morgan-Voyce polynomials $b_n(x)$ and $B_n(x)$ the following identity hold

$$B_{n-1}(x)b_{n-1}(x) - B_{n-2}(x)b_n(x) = 1. \tag{4.20}$$

The identity in (4.20) is given in both [Swa66b] and [Swa66a] but there it is deduced by other means. From our viewpoint the deduction of (4.20) borders to triviality and we may by Lemma 2.4 easily deduce an infinite amount of related results. It is thus the opinion of the author that the study of the various recursive polynomials and related topics as given in [Kos01] would greatly benefit from an application of continued fraction theory. This is currently a topic of further investigation.

4.1.9 Another generalization

The continued fraction in equation 4.1 was in equation 4.2 generalized by adding more variables u. We will now investigate what happens when we substitute the u-s for a-s and vice verca in equation 4.2. This leads to the problem of deciding for what u-s the continued fraction

$$\underbrace{\frac{a}{1} + \frac{a}{1} + \dots + \frac{a}{1}}_{(p-1) \text{ terms}} + \frac{u}{1} + \frac{a}{1} + \dots \tag{4.21}$$

is elliptic, where $p \in \mathbb{N}$. As before, the corresponding approximant of (4.21) is given by $S_p(w) = \frac{A_{p-1}w+A_p}{B_{p-1}w+B_p}$ and $S_p(w)$ is classified as elliptic if and only if

$$t = \frac{(A_{p-1} + B_p)^2}{4\Delta} \in [0, 1). \tag{4.22}$$

Now only $\Delta = (-1)^p a^{p-1} u$ and B_p depend on u. Hence we may write (4.22)

$$t = \frac{(A_{p-1} + B_{p-1} + uB_{p-2})^2}{4\delta u} \tag{4.23}$$

where $\delta = (-1)^p a^{p-1}$ and solve (4.23) as a second degree polynomial in u. The roots of (4.23) are given by

$$\begin{aligned} u_1 &= \frac{2t\delta - A_{p-1}B_{p-2} - B_{p-1}B_{p-2} + \sqrt{\delta t(\delta t - A_{p-1}B_{p-2} - B_{p-1}B_{p-2})}}{B_{p-2}^2} \\ u_2 &= \frac{2t\delta - A_{p-1}B_{p-2} - B_{p-1}B_{p-2} - \sqrt{\delta t(\delta t - A_{p-1}B_{p-2} - B_{p-1}B_{p-2})}}{B_{p-2}^2}. \end{aligned}$$

Since $A_m = a\mathfrak{J}_m(a)$ and $B_m = \mathfrak{J}_{m+1}(a)$, the roots u_1 and u_2 are also expressible in terms of Jacobsthal polynomials:

$$\begin{aligned} u_1 &= \frac{2t\delta - a\mathfrak{J}_{p-1}^2(a) - \mathfrak{J}_p(a)\mathfrak{J}_{p-1}(a) + \sqrt{\delta t\big(\delta t - a\mathfrak{J}_{p-1}^2(a) - \mathfrak{J}_p(a)\mathfrak{J}_{p-1}(a)\big)}}{\mathfrak{J}_{p-1}^2(a)} \\ u_2 &= \frac{2t\delta - a\mathfrak{J}_{p-1}^2(a) - \mathfrak{J}_p(a)\mathfrak{J}_{p-1}(a) - \sqrt{\delta t\big(\delta t - a\mathfrak{J}_{p-1}^2(a) - \mathfrak{J}_p(a)\mathfrak{J}_{p-1}(a)\big)}}{\mathfrak{J}_{p-1}^2(a)}. \end{aligned} \tag{4.24}$$

We see that for $p = 2$ we obtain f_1 and f_2 as expressed in 4.5, as expected. Due to constraints om time we have not been able to exploit the equations in (4.24) to find out more about the set $\mathfrak{D}_{(u,p)}(a, t)$.

Chapter 5

Ideas and conclusion

An idea for further work

During the work on this thesis, a new proof of the Lorentzen bestness theorem gradually became a goal to strive for. Even though a lot of effort was put into this, the efforts of the author were ultimately unsuccessful. As unfortunate this might seem, this faliure prompted some careful reflection on the more general geometric nature of convergence for periodic continued fractions. We will try to give an account of an idea that we obtained during our work in order to encourage others to explore this particular corner of the theory.

As we have seen in the preceeding chapter, we found a nice convergence theorem for 3-periodic continued fractions that resembles the Parabola theorem, were it not for that it works for a much smaller class of continued fractions. Both theorems are examples of what we might call *geometrical convergence theorems* for continued fractions. Such geometrical convergence theorems may be seen, as a contrast to algebraic convergence theorems such as Theorem 2.5, as theorems that gives us element sets such that any continued fraction from that element set converges. Geometrical convergence theorems are in particular connected to the periodic and limit periodic continued fractions since the classification of these continued fractions determines their convergence properties, at least in terms of general convergence and limit periodic continued fractions with finite nonzero limits. Since various practical applications of continued fraction theory often yields periodic or limit periodic continued fractions, the task of determining convergence for such continued fractions is a rather important one. Also, since Theorem 4.1 demonstrates that there exists continued fractions that converges in a much larger subset of the plane than the parabola, there is a distinct possibility that we might find some interesting convergence theorems for various special class of periodic continued fractions.

The idea is to generalize further the notion of elliptic set given in section 4.1.1. We know that a periodic or limit periodic continued fraction is classified as elliptic if and only if for its period approximant or period limit approximant $S_p(w) = \frac{A_{p-1}w+A_p}{B_{p-1}w+B_p}$ we have that $t \in [0,1)$ where

$$t = \frac{(A_{p-1}+B_p)^2}{4\Delta}.$$

The most general case then to consider is the case where we are given the p-periodic continued fraction

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots + \frac{a_p}{b_p} + \frac{a_1}{b_1} + \ldots.$$

Then a complete description of where the approximant of this continued fraction is elliptic is given by the object

$$W = \{(a_1, b_1, \ldots, a_p, b_p, t) : (A_{p-1} + B_p)^2 - 4t\Delta\}$$

which is a subset of $\mathbb{C}^{2p} \times [0, 1)$. A case less general and also more closely connected to our work occurs if we are given the p-periodic continued fraction $\mathbf{K}(a_i/1)$. Then the object

$$V = \{(a_1, \ldots, a_p, t) : (A_{p-1} + B_p)^2 - 4t\Delta\}$$

as a subset of $\mathbb{C}^p \times [0, 1)$ tells us where the approximant of $\mathbf{K}(a_i/1)$ is elliptic. The study of objects like W and V are the topic of algebraic geometry, a part of mathematics of which the author unfortunately knows nothing. It is our belief that the study of the sets W and V for various p would yield interesting results.

Conclusion

In this thesis we have tried to give an account of the periodic continued fractions and their interrelationship with the Lorentzen bestness theorem. While we failed to give a new proof of the Lorentzen bestness theorem based on what we found, we proved a nice and related theorem for a class of limit-periodic continued fractions. We have also sketched some connections to other parts of mathematics, in particular number theory and algebraic geometry.

Appendix A

Article

Presented here is the article ”Convergence properties of 3-periodic continued fractions” which is to be published in the journal *Communications in the analytic theory of continued fractions.*

Convergence properties of 3-periodic continued fractions

by N. G. Voll

Abstract

We prove classical convergence of the 3-periodic continued fraction $\mathbf{K}(a_n/1) = \frac{a}{1} + \frac{u}{1} + \frac{u}{1} + \frac{a}{1} + \frac{u}{1} \ldots$ whenever $a \in P_\alpha = \left\{a \in \mathbb{C} : |a| - \Re(ae^{-i2\alpha}) \leq \frac{1}{2}\cos^2\alpha\right\}$ and $u \in \left\{w \in \mathbb{C} : \Re(we^{-i\alpha}) \geq -\frac{1}{4}\cos\alpha\right\}$ for a fixed $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. A conjecture connected to this result is also given.

A.1 Introduction

The following article is a result of questions asked when reading the article [Lor92] by L. Lorentzen on the bestness of the parabola theorem for continued fractions. The original idea was that since Lorentzen's bestness theorem only proved the bestness of the parabola theorem with respect to 2-periodic continued fractions, there might be periodic continued fractions with longer periods for which the parabola theorem was not the best possible. The 1-periodic continued fractions is an example of a continued fraction for which the parabola theorem is not the best possible, but one might argue that the 1-periodic example can be dismissed as a special case. However, as we will see in the following investigations, the 1-periodic continued fraction is not a special case but part of a bigger picture.

A.2 Machinery

The basic machinery that we will use is the one presented by L. Lorentzen and H. Waadeland in [LW92] and more recently in [LW08]. We will for the sake of completeness clearly state every notion used, but since we assume some familiarity with the subject, our presentation of the basics will be quite short and compact. If the reader is unfamilliar with this material, a look at the sources is highly recommended.

A.2.1 Basic concepts

A *continued fraction* $\mathbf{K}(a_n/b_n)$ is the object given by

$$\mathbf{K}(a_n/b_n) = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots}} = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots \tag{A.1}$$

where $a_i \neq 0$ and b_i are complex numbers. The *n-th classical approximant* of a continued fraction is the expression we get when we stop the process in (A.1) after n terms, i.e $S_n(0)$ where

$$S_n(w) = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots + \frac{a_n}{b_n + w}. \tag{A.2}$$

We have by induction that

$$S_n(w) = \frac{A_{n-1}w + A_n}{B_{n-1}w + B_n} \tag{A.3}$$

where A_i and B_i are recursively defined by

$$A_i = b_i A_{i-1} + a_i A_{i-2}$$
$$B_i = b_i B_{i-1} + a_i B_{i-2}$$

with starting values $A_{-1} = 1$, $A_0 = 0$, $B_{-1} = 0$ and $B_0 = 1$. The *m-th tail* of a continued fraction $\mathbf{K}(a_n/b_n)$ is the continued fraction

$$\frac{a_{m+1}}{b_{m+1}} + \frac{a_{m+2}}{b_{m+2}} + \frac{a_{m+3}}{b_{m+3}} + \cdots \tag{A.4}$$

and the approximants of (A.4) are denoted by

$$S_n^{(m)}(w) = \frac{a_{m+1}}{b_{m+1}} + \frac{a_{m+2}}{b_{m+2}} + \ldots + \frac{a_n}{b_n + w}.$$

A *p-periodic continued fraction* is a continued fraction $\mathbf{K}(a_n/b_n)$ where the coefficients repeat, i.e where $a_{n+p} = a_n$ and $b_{n+p} = b_n$ for all $n \in \mathbb{N}$ and p is the smallest integer such that this holds. If we denote iteration of the mapping F n times by $F^{[n]}$, we may write $S_{np+m}(w) = S_p^{[n]} \circ S_m(w) = S_m \circ (S_p^{(m)})^{[n]}(w)$ and we call $S_p^{(m)}(w)$ the *corresponding approximant* of the p-periodic continued fraction $\mathbf{K}(a_n/b_n)$. To prove convergence we may ignore the beginning $S_m(w)$ of the p-periodic continued fraction $\mathbf{K}(a_n/b_n)$. We then have $m = 0$ and we just denote the corresponding approximant $S_p^{(0)}(w)$ by $S_p(w)$. In a similar manner we define *limit p-periodic continued fractions* to be the continued fraction $\mathbf{K}(a_n/b_n)$ where $\lim_{k\to\infty} a_{kp+n} = \tilde{a}_n$ and $\lim_{k\to\infty} b_{kp+n} = \tilde{b}_n$ for $n = 1, 2, \ldots, p$ and where $\tilde{a}_n$ and $\tilde{b}_n$ both are in $\mathbb{C}$ and $\tilde{a}_n \neq 0$. We denote by $\tilde{S}_p(w)$ the *corresponding limit approximant* of the limit p-periodic continued fraction which is given by

$$\tilde{S}_p(w) = \frac{\tilde{a}_1}{\tilde{b}_1} + \frac{\tilde{a}_2}{\tilde{b}_2} + \ldots + \frac{\tilde{a}_p}{\tilde{b}_p + w} = \frac{\tilde{A}_{p-1}w + \tilde{A}_p}{\tilde{B}_{p-1}w + \tilde{B}_p}$$

where $\tilde{A}_{-1} = 1, \tilde{A}_0 = 0, \tilde{B}_{-1} = 0$ and $\tilde{B}_0 = 1$. We define $\widehat{\mathbb{C}}$ to denote the extended complex plane, i.e. the complex plane with a point infinity added. We choose the Riemann-sphere to reperesent $\widehat{\mathbb{C}}$ and on $\widehat{\mathbb{C}}$ we will for later reference define the *chordal metric*, which was introduced by [Ahl79] and defined in [LW08] by

$$d(w_1, w_2) = \begin{cases} \frac{2|w_1 - w_2|}{\sqrt{1+|w_1|^2}\sqrt{1+|w_2|^2}} & \text{for } w_1, w_2 \in \mathbb{C}. \\ \frac{2}{\sqrt{1+|w_1|^2}} & \text{if } w_1 \in \mathbb{C},\ w_2 = \infty. \\ 0 & \text{for } w_1 = w_2 = \infty \end{cases} \tag{A.5}$$

For any approximant $S_n(w) = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots + \frac{a_n}{b_n + w}$, the *determinant formula* by $\Delta_n = A_{n-1}B_n - A_n B_{n-1} = (-1)^n \prod_{k=1}^{n} a_k$ holds by induction. Finally, we define the class $\mathcal{M}$ of linear fractional transformations, or Möbius transformations

$$\tau(w) = \frac{aw + b}{cw + d}$$

on $\widehat{\mathbb{C}}$ where $ad - bc \neq 0$ and $a, b, c, d \in \mathbb{C}$. We observe that $S_n(w)$ are elements of this class $\mathcal{M}$. We also recall that elements of the class $\mathcal{M}$ are bijective and meromorphic mappings of $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$.

A.2.2 Convergence

We say we have *classical convergence* of the continued fraction $\mathbf{K}(a_n/b_n)$ if $\lim_{n\to\infty} f_n = f$ where $f \in \widehat{\mathbb{C}}$. If the limit $\lim_{n\to\infty} f_n$ does not exist, we say we have *classical divergence*. We now define *general convergence*, as done in [LW92] and [LW08]

Definition A.1. *A sequence of linear fractional transformations $\{\tau_n\}$ converges generally to a constant $\gamma \in \widehat{\mathbb{C}}$ if and only if there exists an exceptional sequence $\{w_n^\star\}$ from $\widehat{\mathbb{C}}$ such that*

$$\lim_{n\to\infty} \tau_n(w_n) = \gamma \quad \textit{whenever} \quad \liminf_{n\to\infty} d(w_n, w_n^\star) > 0$$

where our metric is the chordal metric as defined in (A.5)*.*

Hence we say that a continued fraction $\mathbf{K}(a_n/b_n)$ converges generally to a constant $f \in \widehat{\mathbb{C}}$ with exceptional sequence $\{w_n^\star\}$ if and only if the sequence of approximants $\{S_n(w)\}$ converges generally as a sequence of linear fractional transformations to f with exceptional sequence $\{w_n^\star\}$. If $\mathbf{K}(a_n/b_n)$ does not converge generally, we say that it diverges generally.

A.2.3 Properties of periodic and limit-periodic continued fractions

A corresponding approximant $S_p(w)$ to a periodic or limit periodic continued fraction has one or two fixed points if we discard the case where $S_p(w) = I$ and I is the identity transformation. The *fixed points* will be denoted by x and y if they are distinct and by x if they coincide. For the corresponding approximant $S_p(w)$ the fixed points are given by the equations

$$x, y = \frac{A_{p-1} - B_p \pm \sqrt{(A_{p-1} + B_p)^2 - 4A_pB_{p-1}}}{2B_{p-1}} \quad \text{if } B_{p-1} \neq 0 \tag{A.6}$$

We choose x and y from this equation such that $|B_{p-1}y + B_p| \leq |B_{p-1}x + B_p|$. As done in [LW08] we may now classify the periodic continued fractions in the following way:

Definition A.2. *Let $S_p(w) = \frac{A_{p-1}w+A_p}{B_{p-1}w+B_p}$ be the corresponding approximant of the p-periodic continued fraction $\mathbf{K}(a_n/b_n)$. The ratio $\mathcal{R}$ of $S_p(w)$ is then defined to be a complex number in the closed unit disk given by*

$$\mathcal{R} = \frac{1-u}{1+u} \quad \textit{where} \quad u = \sqrt{1 - \frac{4\Delta_p}{(A_{p-1} + B_p)^2}}.$$

We set $\mathcal{R} = -1$ if $B_{p-1} \neq 0$ and $A_{p-1} + B_p = 0$. If $\mathcal{R} = 1$ the periodic continued fraction is called parabolic. If $|\mathcal{R}| < 1$ then the periodic continued fraction is called loxodromic. If $|\mathcal{R}| = 1$ with $\mathcal{R} \neq 1$ then the periodic continued fraction is called elliptic.

This classification is also used on the limit p-periodic continued fractions, we just replace $S_p(w)$ by $\tilde{S}_p(w)$ in definition A.2. We now have

Theorem A.1. *Let $\mathbf{K}(a_n/b_n)$ be a p-periodic continued fraction. If the corresponding approximant $S_p(w)$ is parabolic or loxodromic, then $\mathbf{K}(a_n/b_n)$ converges generally to the attractive fixed point x. If the corresponding approximant $S_p(w)$ is elliptic or the identity transformation then $\mathbf{K}(a_n/b_n)$ diverges generally.*

We also have that

Theorem A.2. *Let $\mathbf{K}(a_n/b_n)$ be a p-periodic continued fraction of loxodromic type. If $S_m(0) = y$ for some $m \in \{1, 2, \ldots, p\}$, then $\mathbf{K}(a_n/b_n)$ diverges classically. Otherwise $\mathbf{K}(a_n/b_n)$ converges classically.*

The type of divergence of a loxodromic continued fraction described in theorem A.2 is called *Thiele oscillation*. We note that if Thiele oscillation occurs for a p-periodic continued fraction of loxodromic type, then

$$\lim_{n\to\infty} S_{np+m}(0) = \begin{cases} x & \text{whenever } S_m(0) \neq y \\ y & \text{whenever } S_m(0) = y. \end{cases} \tag{A.7}$$

For parabolic p-periodic continued fractions $S_p(w)$ has only one fixed point $x = y$.

Theorem A.3. *A p-periodic continued fraction $\mathbf{K}(a_n/b_n)$ of parabolic type converges to the fixed point x in the classical sense and $S_n(w) \to x$ for every $w \in \widehat{\mathbb{C}}$.*

For proofs of the theorems A.1, A.2 and A.3 see [LW08].

A.2.4 Other useful results

We need a result due to L. J. Lange, as presented in [Lan92]:

Theorem A.4. *The nonsingular linear fractional transformation*

$$\tau(w) = \frac{aw + b}{cw + d}$$

is elliptic if and only if

$$t = \frac{(a + d)^2}{4(ad - bc)} \in [0, 1) \tag{A.8}$$

We observe that if this is applied to a corresponding approximant $S_p(w)$ we have that $t = \frac{(A_{p-1}+B_p)^2}{4\Delta_p}$. Another useful result is due to H. S. Wall, given in [Wal57]:

Theorem A.5. *If the continued fraction $\mathbf{K}(a_n/1)$ is such that*

$$\lim_{n\to\infty} f_{3n+r} = L_r \quad \textit{where } L_r \in \mathbb{C} \textit{ for } r = 0, 1, 2$$

and also none of the sequences $\{a_{3n}\}$, $\{a_{3n+1}\}$ and $\{a_{3n+2}\}$ has limit -1, then $\mathbf{K}(a_n/1)$ converges classically.

A.3 Results on convergence

We shall now study p-periodic continued fractions of the form $\mathbf{K}(a_n/1)$ and some of their convergence properties. In particular we are interested in for what restrictions on the choice of a two-point set $\{a, u\} \subset \widehat{\mathbb{C}}$ the p-periodic continued fraction

$$\frac{a}{1} + \underbrace{\frac{u}{1} + \frac{u}{1} + \ldots + \frac{u}{1}}_{(p-1)\text{ terms}} + \frac{a}{1} + \frac{u}{1} + \ldots \tag{A.9}$$

converges. We shall make use of the *parabolic region* P_α for a fixed angle $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, defined by

$$\begin{aligned} P_\alpha &= \left\{ a \in \mathbb{C} : |a| - \Re(ae^{-i2\alpha}) \leq \frac{1}{2}\cos^2\alpha \right\} \\ &= \left\{ c^2 \in \mathbb{C} : |\Im(ce^{-i\alpha})| \leq \frac{1}{2}\cos\alpha \right\}. \end{aligned} \tag{A.10}$$

The boundary of this parabolic region P_α is a parabola denoted by ∂P_α and given by

$$\begin{aligned} \partial P_\alpha &= \left\{ a \in \mathbb{C} : |a| - \Re(ae^{-i2\alpha}) = \frac{1}{2}\cos^2\alpha \right\} \\ &= \left\{ c^2 \in \mathbb{C} : |\Im(ce^{-i\alpha})| = \frac{1}{2}\cos\alpha \right\}. \end{aligned} \tag{A.11}$$

Then $0 \in P_\alpha$ and $-\frac{1}{4} \in \partial P_\alpha$. Furthermore, for every point a in the *cut plane* $\mathbb{C} \setminus (-\infty, -\frac{1}{4})$ there exist $\alpha_1, \alpha_2 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with $\alpha_1 \neq \alpha_2$ such that $a \in \partial P_{\alpha_1}$ and that $a \in \partial P_{\alpha_2}$. Also observe that if $a \in \left(-\infty, -\frac{1}{4}\right)$ then $a \notin P_\alpha$ for any choice of $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The *half plane* is also needed and we define it by

$$\mathbb{H} = \{w \in \mathbb{C} : \Re(w) \geq 0\}.$$

We shall also make some use of the *parabola theorem*, which can be stated as follows:

Theorem A.6 (Parabola Theorem). *Let $\mathbf{K}(a_n/1)$ be a continued fraction whose coefficients a_n is contained in P_α for some fixed $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. If*

$$\mathcal{S} = \sum_{n=1}^{\infty} \left| \frac{a_1 a_3 \dots a_{2n-1}}{a_2 a_2 \dots a_{2n}} \right| + \sum_{n=1}^{\infty} \left| \frac{a_2 a_2 \dots a_{2n}}{a_1 a_3 \dots a_{2n+1}} \right| = \infty$$

then $\mathbf{K}(a_n/1)$ converges classically to a finite value. If $\mathcal{S} < \infty$, then $\{f_{2n}\}$ and $\{f_{2n+1}\}$ converge absolutely to distinct finite values, and $\{S_{2n}\}$ and $\{S_{2n+1}\}$ converge generally to these values.

This result is assumed to be well known. A sketch of the proof can be found in [LW92] and a complete proof can be found in [LW08].

A.3.1 The two-periodic case and the Lorentzen bestness theorem.

The simplest of the periodic continued fractions defined in equation (A.9) is the 2-periodic continued fraction $\mathbf{K}(a_n/1)$ given by

$$\underset{n=1}{\overset{\infty}{\mathbf{K}}} \left(\frac{a_n}{1} \right) = \frac{a}{1} \, {}_{+} \, \frac{u}{1} \, {}_{+} \, \frac{a}{1} \, {}_{+} \, \frac{u}{1} \, {}_{+} \cdots \tag{A.12}$$

where $a \in \mathbb{C} \setminus (-\infty, -\frac{1}{4})$ and $u \in \mathbb{C}$. The continued fraction defined in (A.12) was investigated by L. Lorentzen in [Lor92] and she used it to obtain the following result:

Theorem A.7 (Lorentzen bestness theorem). *For fixed $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ let u be a complex number $u \notin P_\alpha$. Then there exists a point a on the boundary ∂P_α of P_α such that the linear fractional transformation*

$$S(w) = \frac{a}{1} \, {}_{+} \, \frac{u}{1+w} = \frac{a(1+w)}{1+u+w}$$

is elliptic.

A.3.2 The three-periodic case and a convergence theorem

A question that comes to mind when we see theorem A.7 is the following: is a similar result possible when we increase the period in equation (A.9)? We shall in this section try to partially answer this question by examining 3- periodic continued fractions $\mathbf{K}(a_n/1)$ on the form

$$\mathop{\mathbf{K}}_{n=1}^{\infty}\left(\frac{a_n}{1}\right) = \frac{a}{1} + \frac{u}{1} + \frac{u}{1} + \frac{a}{1} + \frac{u}{1} + \frac{u}{1} + \cdots \tag{A.13}$$

where $a \in \mathbb{C} \setminus (-\infty, -\frac{1}{4})$ and $0 \neq u \in \mathbb{C}$.

Theorem A.8. *Let the continued fraction $\mathbf{K}(a_n/1)$ be given where $a_{3k} = a, a_{3k+1} = u$ and $a_{3k+2} = u$ for $k \in \mathbb{N}$ and where $0 \neq a \in P_\alpha$ for some fixed $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then the continued fraction $\mathbf{K}(a_n/1)$ converges if $u \in -\frac{1}{4} + e^{i\alpha}\mathbb{H} = \{w \in \mathbb{C} : \Re(we^{-i\alpha}) \geq -\frac{1}{4}\cos\alpha\}$.*

Proof. We have two cases: either $a \in \partial P_\alpha$ or $a \in P_\alpha \setminus \partial P_\alpha$. Let first $a \in \partial P_\alpha$. We have that the corresponding approximant for $\mathbf{K}(a_n/1)$ is given by

$$S_3(w) = \frac{a(1+w+u)}{1+2u+w(u+1)}. \tag{A.14}$$

Then by theorem A.4 $S_3(w)$ is elliptic if and only if

$$t = \frac{(a+2u+1)^2}{-4au^2} \in [0,1) \tag{A.15}$$

We solve for u in the quadratic formula the expression in (A.15) and simplify to obtain that $S_3(w)$ is elliptic if and only if u lies on one of the analytic curves

$$g_1(t) = \frac{(a+1)(-1+\sqrt{-at})}{2(at+1)}$$
$$g_2(t) = \frac{(a+1)(-1-\sqrt{-at})}{2(at+1)}$$

where $t \in [0,1)$. We observe that if $t = 0$ we have that $g_1(0) = g_2(0) = -\frac{1}{2}(a+1)$. Hence the two curves $g_1(t)$ and $g_2(t)$ combines to one continuous curve with end points

$$\begin{aligned}\lim_{t\to 1} g_1(t) &= -\frac{1}{2}(1-i\sqrt{a})\\ \lim_{t\to 1} g_2(t) &= -\frac{1}{2}(1+i\sqrt{a}).\end{aligned} \tag{A.16}$$

With the notation $a = c^2$ the end points can be written $-\frac{1}{2}(1+i\sqrt{a}) = -\frac{1}{2}(1+ic)$ and $-\frac{1}{2}(1-i\sqrt{a}) = -\frac{1}{2}(1-ic)$. Hence we see that we have a curve G that goes continuously from a point given by $-\frac{1}{2}(1+ic)$ through $-\frac{1}{2}(c^2+1)$ to a point given by $-\frac{1}{2}(1-ic)$, $S_3(w)$ is elliptic if and only if u lies on this curve. We also observe that when a traces the parabola ∂P_α, then $-\frac{1}{2}(1+ic)$ and $-\frac{1}{2}(1-ic)$ traces straight lines, since c is on a straight line by (A.11). Hence, the possible choices for u which makes $S_3(w)$ elliptic is bounded by the lines

$$\begin{aligned}l_1(x) &= -\frac{1}{2}(1+ic)\\ l_2(x) &= -\frac{1}{2}(1-ic)\end{aligned} \tag{A.17}$$

where $c = (x + \frac{i}{2}\cos\alpha)e^{i\alpha}$ and $x \in \mathbb{R}$. By theorem A.6 we have that none of the lines l_1, l_2 in (A.17) can have more than one point in common with P_α, otherwise we would have a contradiction. We now see that for $x = \frac{1}{2}\sin\alpha$ we have

$$l_1\left(\frac{1}{2}\sin\alpha\right) = -\frac{1}{2} - \frac{i}{2}(\frac{1}{2}\sin\alpha + \frac{i}{2}\cos\alpha)e^{i\alpha} = -\frac{1}{2} - \frac{i^2}{4}(\cos\alpha - i\sin\alpha)e^{i\alpha} = -\frac{1}{4} \in \partial P_\alpha.$$

Hence the line l_1 is tangent to the parabola ∂P_α at $z = -\frac{1}{4}$. For l_1 we now have that

$$\begin{aligned} l_1(x)e^{-i\alpha} &= -\frac{1}{2}e^{-i\alpha} - \frac{i}{2}(x + \frac{i}{2}\cos\alpha) \\ &= -\frac{1}{2}e^{-i\alpha} - \frac{i}{2}x + \frac{1}{4}\cos\alpha \\ &= -\frac{1}{4}\cos\alpha + i\frac{\sin\alpha - x}{2} \end{aligned}$$

and thus we have that $\Re(l_1(x)e^{-i\alpha}) = -\frac{1}{4}\cos\alpha$. Since all curves G has their end points on l_1 which is the boundary of the half plane $-\frac{1}{4} + e^{i\alpha}\mathbb{H}$, we have that any choice of $u \in -\frac{1}{4} + e^{i\alpha}\mathbb{H}$ unless it is on the line l_1 gives us a loxodromic $S_3(w)$ and by theorem A.2 we have classical convergence unless we have Thiele oscillation. However, by (A.7) we have that the limits

$$\lim_{n\to\infty} f_{3n+r}$$

exists, and hence we may use theorem A.5 to conclude that we have classical convergence whenever $u \in -\frac{1}{4} + e^{i\alpha}\mathbb{H}$ since $-1 \notin -\frac{1}{4} + e^{i\alpha}\mathbb{H}$. As for the parabolic cases on the line l_1 we have that they all give classical convergence by theorem A.3. Let now $a \in P_\alpha \setminus \partial P_\alpha$. We then know that there exists two parabolas ∂P_{α_1} and ∂P_{α_2} such that $a \in \partial P_{\alpha_1}$ and $a \in \partial P_{\alpha_2}$. Obviously, we have that $\alpha < \alpha_1$ and $\alpha > \alpha_2$, since a is an interior point of P_α. By our work above it now follows that for any u such that $u \in \left\{-\frac{1}{4} + e^{i\alpha_1}\mathbb{H}\right\} \cup \left\{-\frac{1}{4} + e^{i\alpha_2}\mathbb{H}\right\}$ we have convergence for an $a \in \partial P_{\alpha_1}$ and $a \in \partial P_{\alpha_2}$. Since

$$\left\{-\frac{1}{4} + e^{i\alpha}\mathbb{H}\right\} \subset \left\{ \left\{-\frac{1}{4} + e^{i\alpha_1}\mathbb{H}\right\} \cup \left\{-\frac{1}{4} + e^{i\alpha_2}\mathbb{H}\right\} \right\}$$

for any $\alpha < \alpha_1$ and $\alpha > \alpha_2$ and this always holds since a is an interior point of P_α, we have convergence for any $u \in \left\{-\frac{1}{4} + e^{i\alpha}\mathbb{H}\right\}$ and $a \in P_\alpha \setminus \partial P_\alpha$. Hence our theorem follows. □

The figure (1) is included to give an illustration of the situation in theorem A.8.

It is possible to expand the result given in theorem A.8 to limit 3-periodic continued fractions:

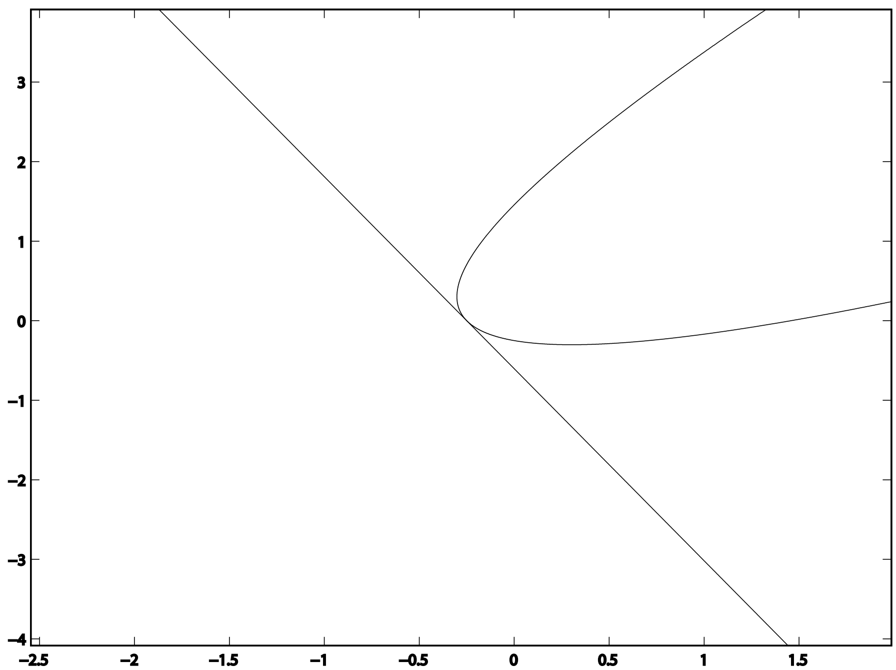
Theorem A.9. *Let the limit continued fraction* $\mathbf{K}(a_n/1)$ *be given where* $\lim a_{3k} = a$ *and* $\lim a_{3k+1} = u = \lim a_{3k+2} = u \neq \infty$ *and where* $a \in P_\alpha$ *for some fixed* $\alpha \in (-\frac{\pi}{2}, \frac{\pi}{2})$. *Then the continued fraction* $\mathbf{K}(a_n/1)$ *converges if* $u \in -\frac{1}{4} + e^{i\alpha}\mathbb{H}^o = \left\{w \in \mathbb{C} : \Re(we^{-i\alpha}) > -\frac{1}{4}\cos\alpha\right\}$

Proof. The theorem follows since what we did for corresponding approximants $S_3(w)$ in the proof of theorem A.8 also hold for corresponding limit approximants $\tilde{S}_3(w)$. The classical convergence in the half plane $-\frac{1}{4} + e^{i\alpha}\mathbb{H}^o$ is guaranteed since the limits

$$\lim_{n\to\infty} f_{3n+r}$$

still exists and we may use theorem A.5. □

Figure A.1: The situation described in theorem A.8 for $\alpha = \frac{\pi}{8}$. The curve is the boundary for the parabolic region $P_{\frac{\pi}{8}}$ and the line boundary for the half plane $-\frac{1}{4} + e^{i\frac{\pi}{8}}\mathbb{H}$.



A.3.3 The four- and five-periodic cases

For the 4- and 5-periodic continued fractions of the type given in (A.9), similar calculations as presented in the proof of theorem A.8 are possible but more complicated since we now obtain a quartic equation from the application of theorem A.4 to $S_p(w)$. Rather than presenting several pages of calculations we will just illustrate the situation with a few pictures. Figure 2 is for the case where we have a 4-periodic continued fraction of the type given in (A.9). Figure 3 is for the case where we have a 5-periodic continued fraction of the type given in (A.9).

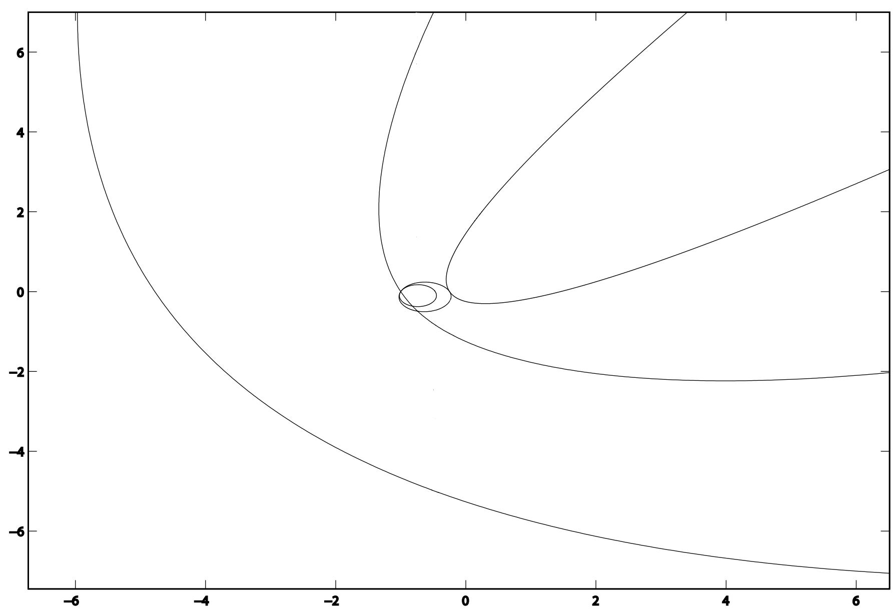
A.3.4 A Conjecture and its implications

As we see in the sections A.3.1, A.3.2 and A.3.3 there seems to be a general pattern in the size of the set where the the p-periodic continued fraction $\mathbf{K}(a_n/1)$ where $a_1 = a$ and $a_2 = a_3 = \cdots = a_p = u$ converges; The set where we have convergence apparently increases in size when p grows, but only when we compare the convergence sets for odd and even p's separately. Hence we state the following conjecture:

Conjecture 3. Let $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be fixed and let C_p denote the set of possible choices for u for which there exists an $a \in \partial P_\alpha$ such that

$$S_p(w) = \frac{a}{1} + \frac{u}{1} + \frac{u}{1} + \ldots + \frac{u}{1+w}$$

Figure A.2: The collection of end points for the curves of choices for u such that we have that $S_4(w)$ elliptic when we have a 4-periodic continued fraction as given in (A.9) and $\alpha = \frac{\pi}{8}$. The boundary for the parabolic region $P_{\frac{\pi}{8}}$ is the curve in the upper right corner of the figure. The area enclosed by the parabola $P_{\frac{\pi}{8}}$, the closest curve to the parabola on the figure and the largest disk is believed to be free of choices for u such that $S_4(w)$ is elliptic for any $a \in \partial P_{\frac{\pi}{8}}$.



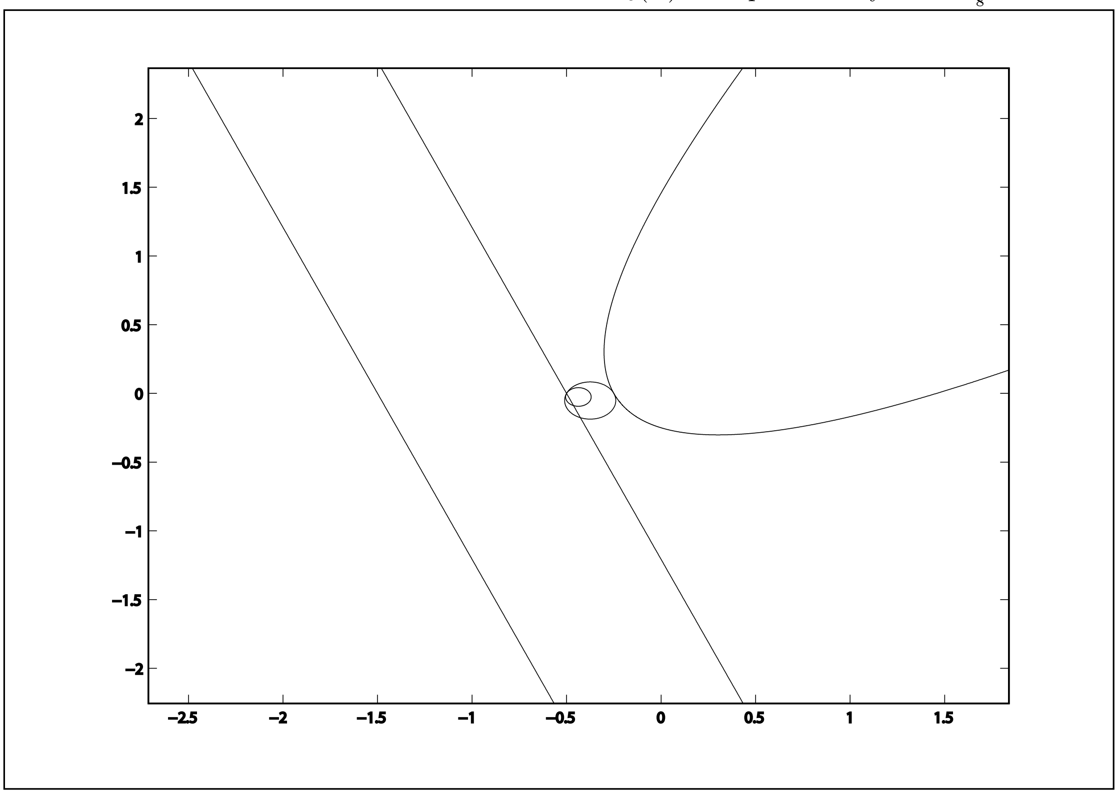
is classified as elliptic. Then

$$C_2 \supset C_4 \supset C_6 \supset \dots$$
$$C_3 \supset C_5 \supset C_7 \supset \dots$$

One reason why this conjecture seems plausible is that when p grows, the continued fraction $\mathbf{K}(a_n/1)$ looks more and more like the continued fraction $\mathbf{K}(u/1)$ which we know converges to $\frac{1}{2}(-1+\sqrt{1+4u})$ for all $u \in \mathbb{C} \setminus \left(-\infty, -\frac{1}{4}\right)$. Hence it is reasonable to assume that the set of choices for u such that the corresponding approximant of $\mathbf{K}(a_n/1)$ is classified as loxodromic must approach $\mathbb{C} \setminus \left(-\infty, -\frac{1}{4}\right)$ in some way. Our pictures seem to support this conjecture.

It is the opinion of the author that theorem A.8 can be strengthened so that for any $u \notin -\frac{1}{4}+e^{i\alpha}\mathbb{H}$ there exists an $a \in \partial P_\alpha$ such that we have divergence. Unfortunately, the author is unable to give a proof of this statement.

Figure A.3: The collection of end points for the curves of choices for u such that we have that $S_5(w)$ elliptic when we have a 5-periodic continued fraction as given in (A.9) and $\alpha = \frac{\pi}{8}$. The boundary for the parabolic region $P_{\frac{\pi}{8}}$ is the curve in the upper right corner of the figure. The area enclosed by the parabola $P_{\frac{\pi}{8}}$, the line closest to the parabola on the figure and the largest disk is believed to be free of choices for u such that $S_5(w)$ is elliptic for any $a \in \partial P_{\frac{\pi}{8}}$.



Appendix B

Images and source code

In this appendix we will present a few pictures and the source codes of the MATLAB programs that generated them. The source codes used to generate the pictures in this master thesis is a result of quite a lot of experimentation and the programs are quite far from being perfect. It is the opinion of the author that to make use of the Jacobsthal polynomials is at the time of writing the best way to proceed and this will be demonstrated by the beautiful images below. For completeness we also include the source code of the program that was written to study the curves obtained in (4.5).

B.1 Source code program 1

This MATLAB code relies on a matrix containing in essence the 19 first Jacobsthal polynomials. It is most likely quite easy to generate this matrix in MATLAB, but initial efforts in this direction failed and to avoid the possibility of spending a lot of time on this problem, the matrix was just hand-coded. As we have seen in section 4.1.7 the canonical numerators and denominators of the continued fraction in (4.2) are tied to the Jacobsthal polynomials. In particular, we have from equation (4.3) and section 4.1.7 that t is given by

$$t = \frac{(a\mathfrak{J}_{p-1}(u) + \mathfrak{J}_{p+1}(u))^2}{a(-1)^p u^{p-1}}$$

and its roots for various a-s and t-s gives us a glimpse of the elliptic set for any p. The program presented here uses this fact, and the program allows us to study the elliptic set of the continued fractions given in (4.2) up to $p = 18$. The source code is given by:

```
function a= rotter(intervall, start, stopp, n, alfa,t)
M=[ 0 0 0 0 0 0 0 0 0 0 1;
 0 0 0 0 0 0 0 0 0 0 1;
 0 0 0 0 0 0 0 0 0 1 1;
 0 0 0 0 0 0 0 0 0 2 1;
 0 0 0 0 0 0 0 0 1 3 1;
 0 0 0 0 0 0 0 0 3 4 1;
 0 0 0 0 0 0 0 1 6 5 1;
 0 0 0 0 0 0 0 4 10 6 1;
 0 0 0 0 0 0 1 10 15 7 1;
 0 0 0 0 0 0 5 20 21 8 1;
```

```
 0 0 0 0 1 15 35 28 9 1;
 0 0 0 0 6 35 56 36 10 1;
 0 0 0 1 21 70 84 45 11 1;
 0 0 0 7 56 126 120 55 12 1;
 0 0 1 28 126 210 165 66 13 1;
 0 0 8 84 252 330 220 78 14 1;
 0 1 36 210 462 495 286 91 15 1;
 0 9 120 462 792 715 364 105 16 1;
 1 45 330 924 1287 1001 455 120 17 1;];
%t=1;
K1=M(n-1,1:10);
K2=M(n+1,1:10);
J1=conv(K1,K1);
J2=conv(K1,K2);
J3=conv(K2,K2);
size(J1);
size(J2);
size(J3);
H=zeros(size(J2));
H(1,19-n+1)=4;
E=(start:intervall:stopp);
F=zeros(size(E));
G=[E;F];
j=1;
while G(1,j)<stopp;
    G(2,j)= (((G(1,j)-i*(0.5*cos(alfa)))))^2*(exp(i*2*alfa));
    j=j+1;
end;
j=1;
while G(1,j)<stopp;
    R=(-1)^(n)*H*G(2,j)*1*t - G(2,j)^2*J1 - 2*G(2,j)*J2 - J3;
    KK1(j,:)=roots(R);
    j=j+1;
end
j=1;
while G(1,j)<stopp;
    R=(-1)^(n)*H*G(2,j)*.875*t - G(2,j)^2*J1 - 2*G(2,j)*J2 - J3;
    KK2(j,:)=roots(R);
    j=j+1;
end
j=1;
while G(1,j)<stopp;
    R=(-1)^(n)*H*G(2,j)*.75*t - G(2,j)^2*J1 - 2*G(2,j)*J2 - J3;
    KK3(j,:)=roots(R);
    j=j+1;
end
j=1;
while G(1,j)<stopp;
    R=(-1)^(n)*H*G(2,j)*.625*t - G(2,j)^2*J1 - 2*G(2,j)*J2 - J3;
```

```
    KK4(j,:)=roots(R);
    j=j+1;
end
j=1;
while G(1,j)<stopp;
    R=(-1)^(n)*H*G(2,j)*0.5*t - G(2,j)^2*J1 - 2*G(2,j)*J2 - J3;
    KK5(j,:)=roots(R);
    j=j+1;
end
j=1;
while G(1,j)<stopp;
    R=(-1)^(n)*H*G(2,j)*.375*t - G(2,j)^2*J1 - 2*G(2,j)*J2 - J3;
    KK6(j,:)=roots(R);
    j=j+1;
end
j=1;
while G(1,j)<stopp;
    R=(-1)^(n)*H*G(2,j)*.25*t - G(2,j)^2*J1 - 2*G(2,j)*J2 - J3;
    KK7(j,:)=roots(R);
    j=j+1;
end
j=1;
while G(1,j)<stopp;
    R=(-1)^(n)*H*G(2,j)*.125*t - G(2,j)^2*J1 - 2*G(2,j)*J2 - J3;
    KK8(j,:)=roots(R);
    j=j+1;
end
KK=[KK1;KK2;KK3;KK4;KK5;KK6;KK7;KK8];
plot(imag(KK),real(KK),'.',imag(G(2,:)),real(G(2,:)),'.','markersize',1)
```

B.2 Source code program 2

The following program simply plots 7 versions of the curve given in (4.5) for a few values of a and a fixed value for α by use of the strip region that represents E_α. The program also draws the parabola E_α and the curves traced by the various limits of the curves in (4.5). The program is long and written in a horrible way but it gives a useful picture of the behaviour of the curves involved.

```
function a= paras(parabelp,stopp, intervall,intervall2,antall, alfa, pardist)
%regner ut punkta på parabelen
p=(((parabelp*(exp(i*alfa))+i*(0.5*cos(alfa)*(exp(i*alfa))))))^2;
p1=((((pardist +parabelp)*(exp(i*alfa))+i*(0.5*cos(alfa)*(exp(i*alfa))))))^2;
p2=((((.5*pardist +parabelp)*(exp(i*alfa))+i*(0.5*cos(alfa)*(exp(i*alfa))))))^2;
p4=((((1*pardist +parabelp)*(exp(i*alfa))+i*(0.5*cos(alfa)*(exp(i*alfa))))))^2;
p5=((((1.5*pardist +parabelp)*(exp(i*alfa))+i*(0.5*cos(alfa)*(exp(i*alfa))))))^2;
p6=((((2*pardist +parabelp)*(exp(i*alfa))+i*(0.5*cos(alfa)*(exp(i*alfa))))))^2;
p7=((((2.5*pardist +parabelp)*(exp(i*alfa))+i*(0.5*cos(alfa)*(exp(i*alfa))))))^2;
p8=((((3*pardist +parabelp)*(exp(i*alfa))+i*(0.5*cos(alfa)*(exp(i*alfa))))))^2;
```

```
%tegner parabelen
E=(-stopp:intervall2:stopp);
F=zeros(size(E));
G=[E;F];
j=1;
while G(1,j)<stopp;
    G(2,j)= (((G(1,j)*(exp(i*alfa))+i*(0.5*cos(alfa)*(exp(i*alfa))))))^2;
    j=j+1;
end;
%tegner de to parablene som endepunktene skal ligge på
j=1;
%while G(1,j)<stopp;
%    G(3,j)= ((-(G(1,j)*(exp(i*alfa))+i*(0.5*cos(alfa)*(exp(i*alfa))))))^2;
%    j=j+1;
%end;

while G(1,j)<stopp;
    G(3,j)= ((G(1,j)*(exp(i*alfa))+i*(0.5*cos(alfa)*(exp(i*alfa)))-i))^2;
    j=j+1;
end;
j=1;
while G(1,j)<stopp;
    G(4,j)= ((G(1,j)*(exp(i*alfa))+i*(0.5*cos(alfa)*(exp(i*alfa)))+i))^2;
    j=j+1;
end;
j=1;
while G(1,j)<stopp;
    G(5,j)= -G(2,j)-1;
    j=j+1;
end;
j=1;

%tegner kurvene på parabelen
P=(1:intervall:antall);
Q=zeros(size(P));
R=[P;Q];
j=1;
while R(1,j)<antall;
    R(2,j)= (2*p)/R(1,j) - p - 1
+ 2*p*(sqrt(((1/(R(1,j)^2))-(1/R(1,j))-(1/(p*R(1,j))))));
    j=j+1;
end;
j=1;
while R(1,j)<antall;
    R(3,j)= (2*p)/R(1,j) - p - 1
- 2*p*(sqrt(((1/(R(1,j)^2))-(1/R(1,j))-(1/(p*R(1,j))))));
    j=j+1;
end;
j=1
```

```
while R(1,j)<antall;
    R(4,j)= (2*p1)/R(1,j) - p1 - 1
+ 2*p1*(sqrt((1/(R(1,j)^2))-(1/R(1,j))-(1/(p1*R(1,j)))));
    j=j+1;
end;
j=1;
while R(1,j)<antall;
    R(5,j)= (2*p1)/R(1,j) - p1
- 1 - 2*p1*(sqrt((1/(R(1,j)^2))-(1/R(1,j))-(1/(p1*R(1,j)))));
    j=j+1;
end;

j=1
while R(1,j)<antall;
    R(6,j)= (2*p2)/R(1,j) - p2
- 1 + 2*p2*(sqrt((1/(R(1,j)^2))-(1/R(1,j))-(1/(p2*R(1,j)))));
    j=j+1;
end;
j=1;
while R(1,j)<antall;
    R(7,j)= (2*p2)/R(1,j) - p2 - 1
- 2*p2*(sqrt((1/(R(1,j)^2))-(1/R(1,j))-(1/(p2*R(1,j)))));
    j=j+1;
end;

j=1
while R(1,j)<antall;
    R(8,j)= (2*p4)/R(1,j) - p4 - 1
+ 2*p4*(sqrt((1/(R(1,j)^2))-(1/R(1,j))-(1/(p4*R(1,j)))));
    j=j+1;
end;
j=1;
while R(1,j)<antall;
    R(9,j)= (2*p4)/R(1,j) - p4 - 1
- 2*p4*(sqrt((1/(R(1,j)^2))-(1/R(1,j))-(1/(p4*R(1,j)))));
    j=j+1;
end;
j=1
while R(1,j)<antall;
    R(10,j)= (2*p5)/R(1,j) - p5 - 1
+ 2*p5*(sqrt((1/(R(1,j)^2))-(1/R(1,j))-(1/(p5*R(1,j)))));
    j=j+1;
end;
j=1;
while R(1,j)<antall;
    R(11,j)= (2*p5)/R(1,j) - p5 - 1
- 2*p5*(sqrt((1/(R(1,j)^2))-(1/R(1,j))-(1/(p5*R(1,j)))));
    j=j+1;
end;
```

```
j=1
while R(1,j)<antall;
    R(12,j)= (2*p6)/R(1,j) - p6 - 1
+ 2*p6*(sqrt((1/(R(1,j)^2))-(1/R(1,j))-(1/(p6*R(1,j)))));
    j=j+1;
end;
j=1;
while R(1,j)<antall;
    R(13,j)= (2*p6)/R(1,j) - p6 - 1
- 2*p6*(sqrt((1/(R(1,j)^2))-(1/R(1,j))-(1/(p6*R(1,j)))));
    j=j+1;
end;
j=1
while R(1,j)<antall;
    R(14,j)= (2*p7)/R(1,j) - p7 - 1
+ 2*p7*(sqrt((1/(R(1,j)^2))-(1/R(1,j))-(1/(p7*R(1,j)))));
    j=j+1;
end;
j=1;
while R(1,j)<antall;
    R(15,j)= (2*p7)/R(1,j) - p7 - 1
- 2*p7*(sqrt((1/(R(1,j)^2))-(1/R(1,j))-(1/(p7*R(1,j)))));
    j=j+1;
end;
j=1
while R(1,j)<antall;
    R(16,j)= (2*p8)/R(1,j) - p8 - 1
+ 2*p8*(sqrt((1/(R(1,j)^2))-(1/R(1,j))-(1/(p8*R(1,j)))));
    j=j+1;
end;
j=1;
while R(1,j)<antall;
    R(17,j)= (2*p8)/R(1,j) - p8 - 1
- 2*p8*(sqrt((1/(R(1,j)^2))-(1/R(1,j))-(1/(p8*R(1,j)))));
    j=j+1;
end;

Y=size(R)
R(2,Y(1,2)) = -p-1;
R(3,Y(1,2)) = -p-1;
R(4,Y(1,2)) = -p1-1;
R(5,Y(1,2)) = -p1-1;
R(6,Y(1,2)) = -p2-1;
R(7,Y(1,2)) = -p2-1;
R(8,Y(1,2)) = -p4-1;
R(9,Y(1,2)) = -p4-1;
R(10,Y(1,2)) = -p5-1;
R(11,Y(1,2)) = -p5-1;
R(12,Y(1,2)) = -p6-1;
```

```
R(13,Y(1,2)) = -p6-1;
R(14,Y(1,2)) = -p7-1;
R(15,Y(1,2)) = -p7-1;
R(16,Y(1,2)) = -p8-1;
R(17,Y(1,2)) = -p8-1;
U=size(G)
plot(real(R(2,:)),imag(R(2,:)),real(R(3,:)),imag(R(3,:))
,real(R(4,:)),imag(R(4,:)),real(R(5,:)),imag(R(5,:))
,real(R(6,:)),imag(R(6,:)),real(R(7,:)),imag(R(7,:))
,real(R(8,:)),imag(R(8,:)),real(R(9,:)),imag(R(9,:))
,real(R(10,:)),imag(R(10,:)),real(R(11,:)),imag(R(11,:))
,real(R(12,:)),imag(R(12,:)),real(R(13,:)),imag(R(13,:))
,real(R(14,:)),imag(R(14,:)),real(R(15,:)),imag(R(15,:))
,real(R(16,:)),imag(R(16,:)),real(R(17,:)),imag(R(17,:))
,real(G(2,1:(U(1,2)-1))),imag(G(2,1:(U(1,2)-1)))
,real(G(3,1:(U(1,2)-1))),imag(G(3,1:(U(1,2)-1)))
,real(G(4,1:(U(1,2)-1))),imag(G(4,1:(U(1,2)-1)))
,real(G(5,1:(U(1,2)-1))),imag(G(5,1:(U(1,2)-1)))
,real(p),imag(p),'MarkerSize',50)
```

B.3 Images

The images presented here are generated by the programs given in section B.1 and B.2. The images (B.1) to (B.13) are generated by the programs given in section B.1 with fixed values for t where $t \in \{\frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8}, 1\}$. Furthermore, α is chosen such that $\alpha = 0$ or $\alpha = \frac{\pi}{7}$ and we let $a \in E_\alpha$. The images (B.14) to (B.16) are generated by the programs given in section B.2 where $\alpha = 0, \alpha = \frac{\pi}{7}$ and $\alpha = \frac{\pi}{4}$. The figures has been rotated to fit the page better. Program 1 is executed by the command

```
rotter(.005,-10,10,5,0*pi/7,1)
```

here with $p = 5$ as the fourth in-value and with $\alpha = 0$ as the fifth in-value. Program 2 is executed by the command

```
paras(-1,5,.1,.0001,1000,pi/4,1)
```

where the angle $\alpha = \pi/4$ can be recognized by the sixth variable.

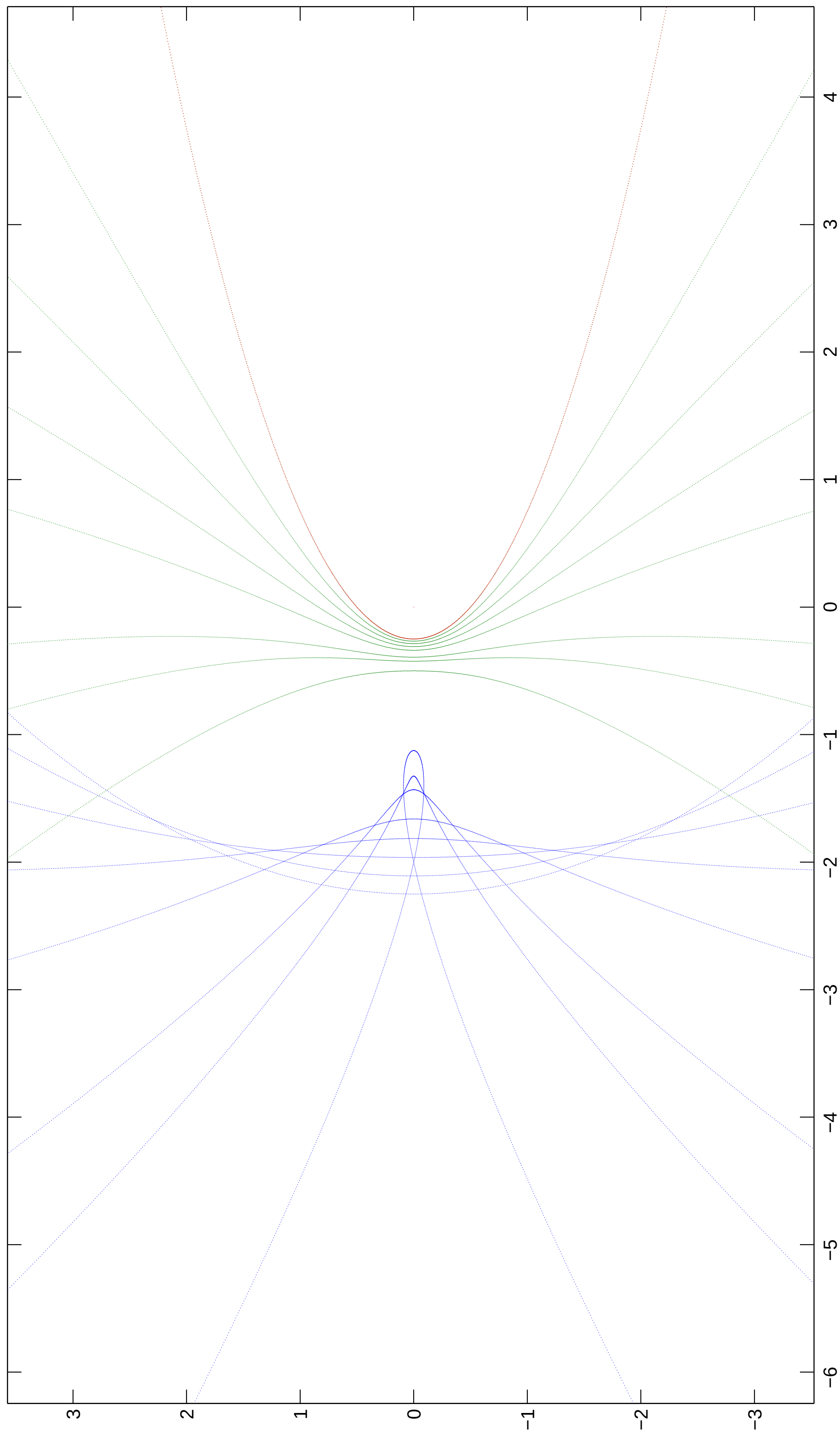


Figure B.1: A plot of some of the elliptic set for a 2-periodic continued fraction of type given in 4.2 where $\alpha = 0$.

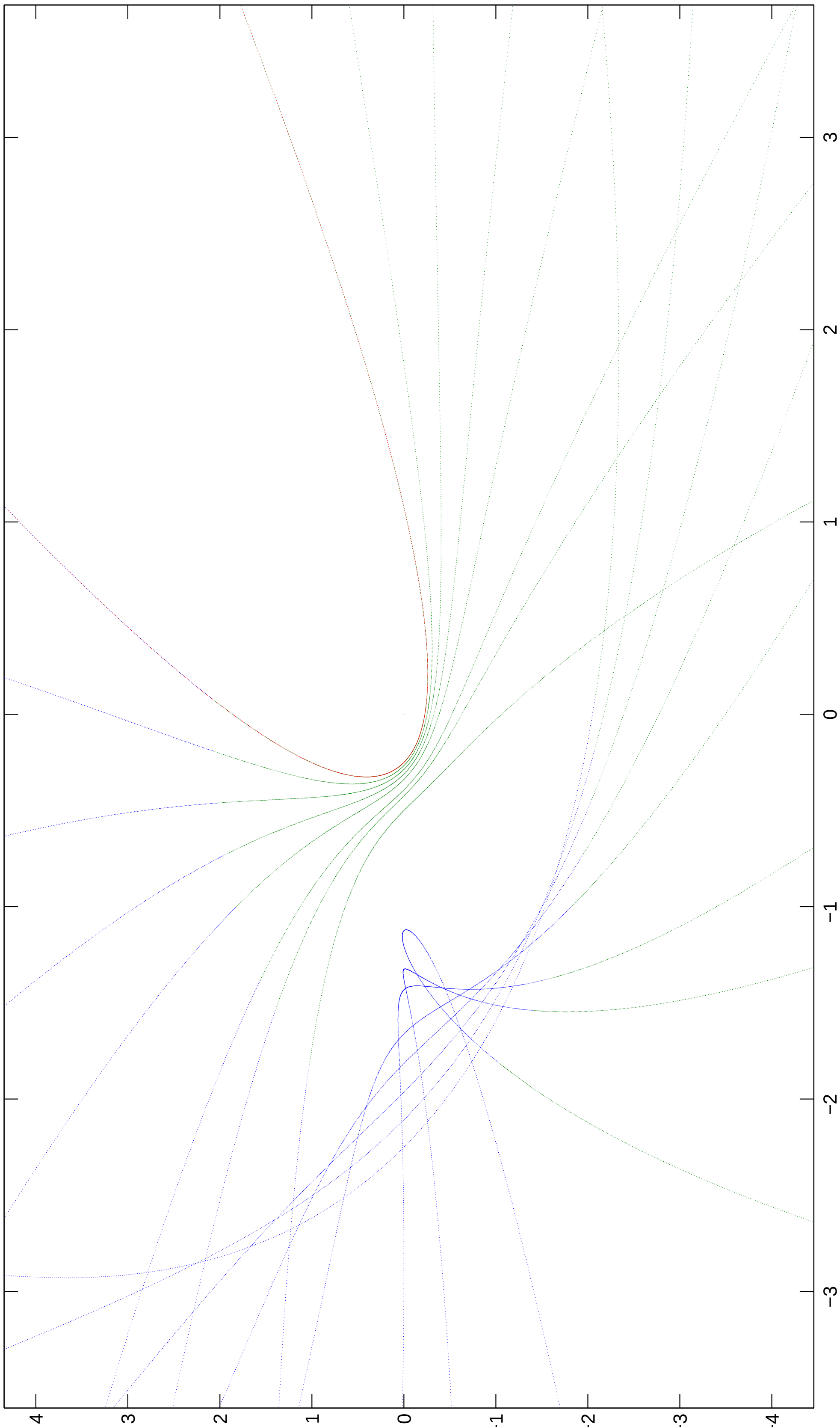


Figure B.2: A plot of some of the elliptic set for a 2-periodic continued fraction of type given in 4.2 where $\alpha = \frac{\pi}{7}$.

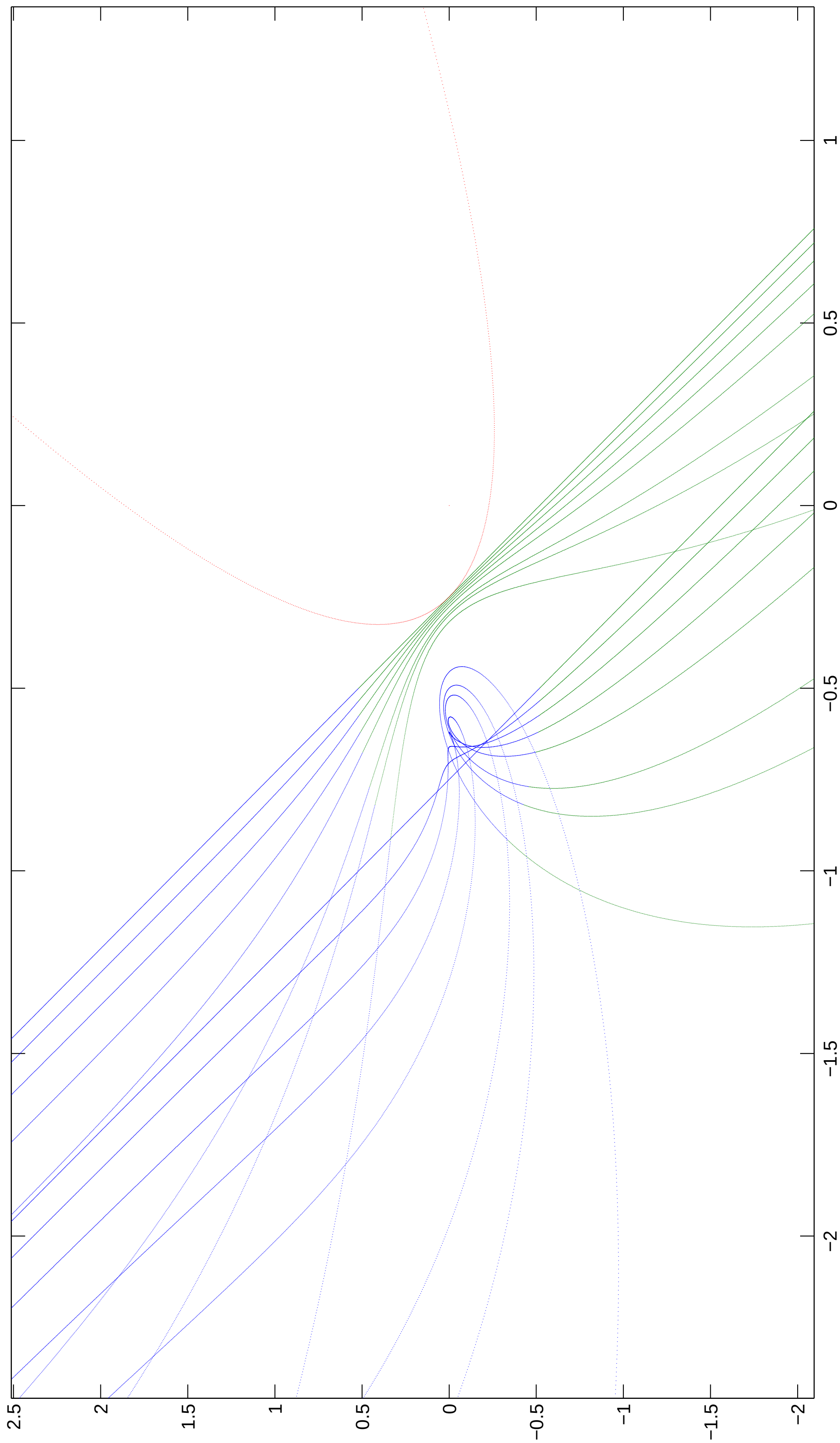


Figure B.3: A plot of some of the elliptic set for a 3-periodic continued fraction of type given in 4.2 where $\alpha = \frac{\pi}{7}$.

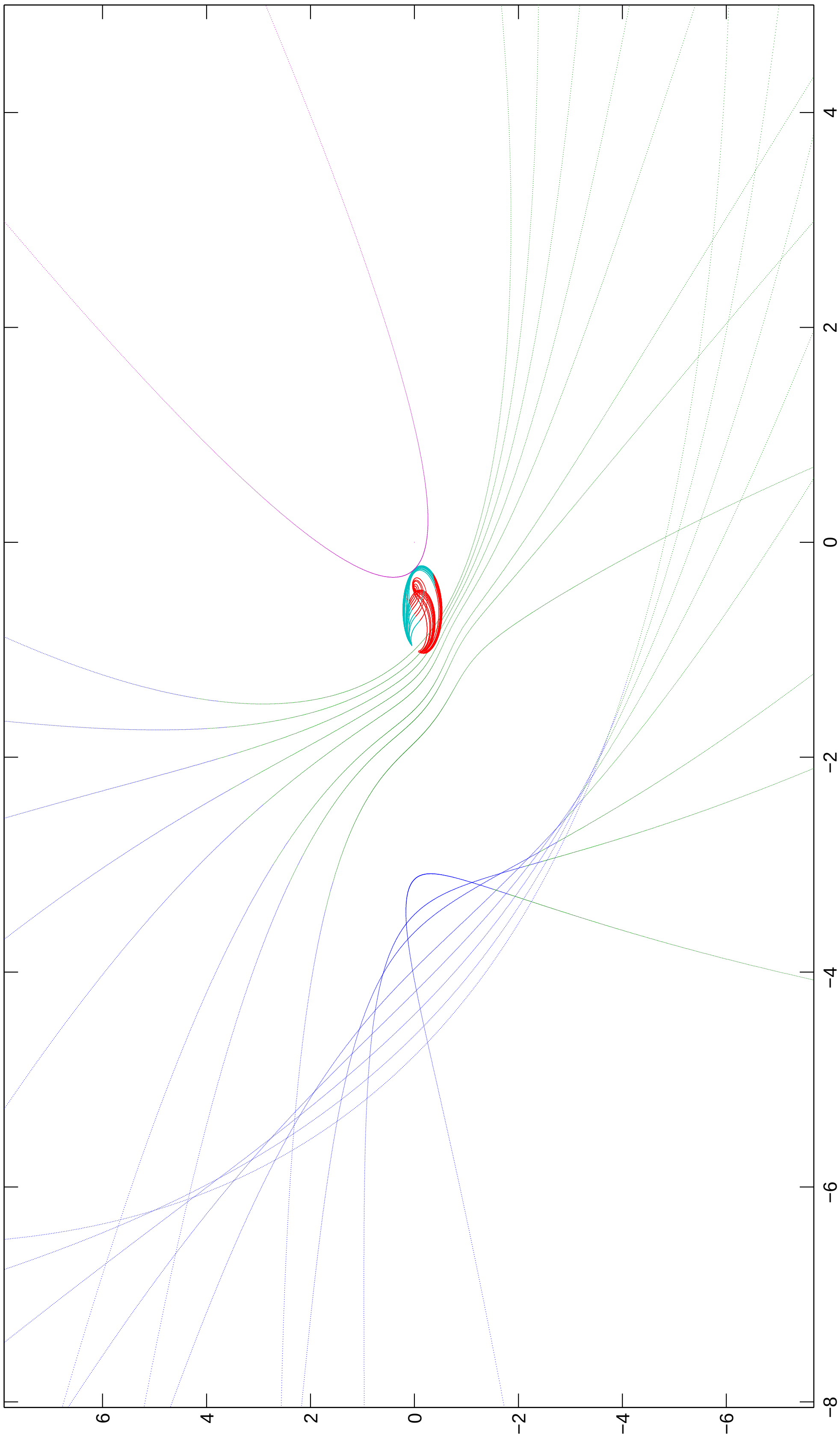


Figure B.4: A plot of some of the elliptic set for a 4-periodic continued fraction of type given in 4.2 where $\alpha = \frac{\pi}{7}$.

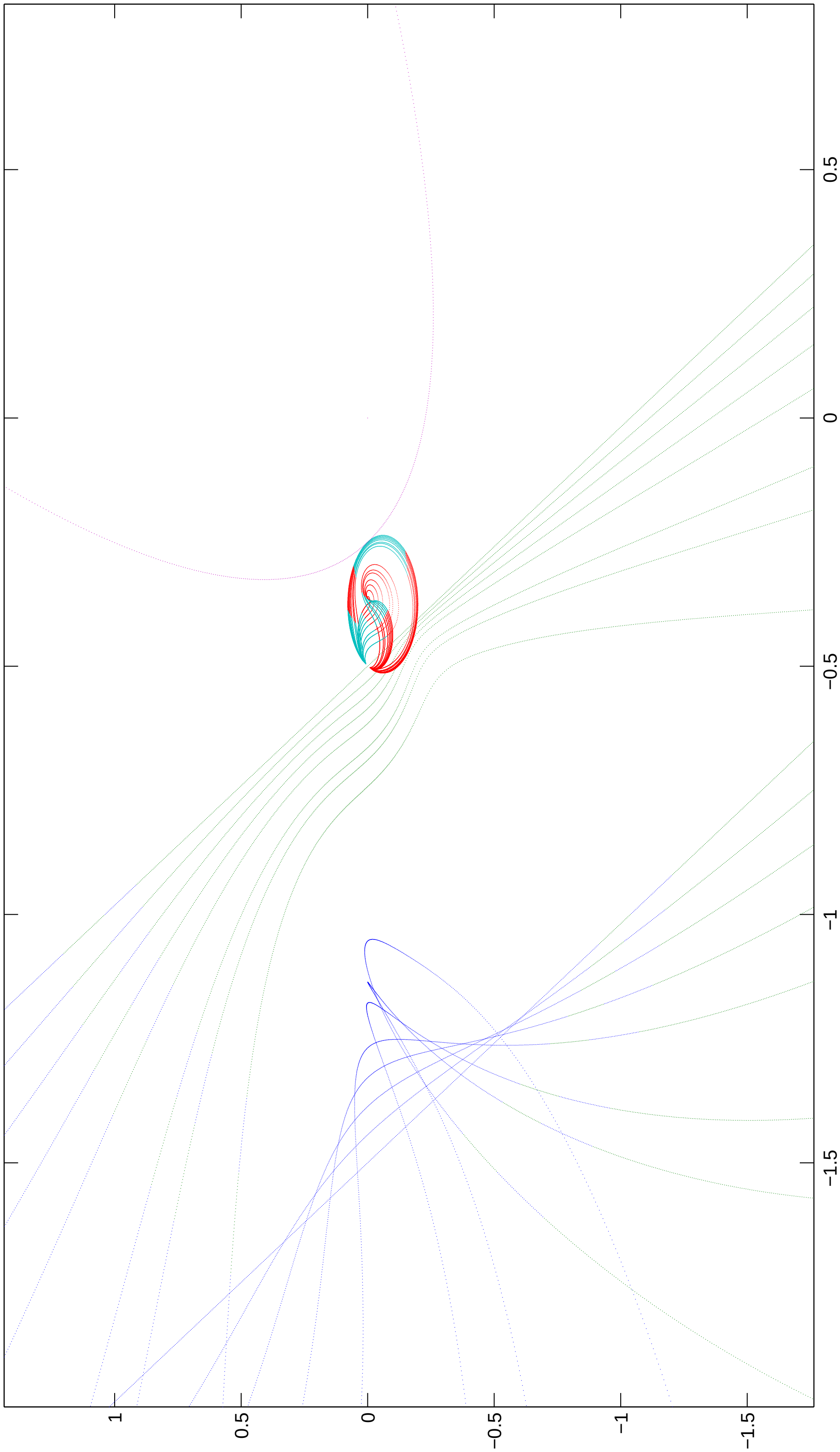


Figure B.5: A plot of some of the elliptic set for a 5-periodic continued fraction of type given in 4.2 where $\alpha = \frac{\pi}{7}$.

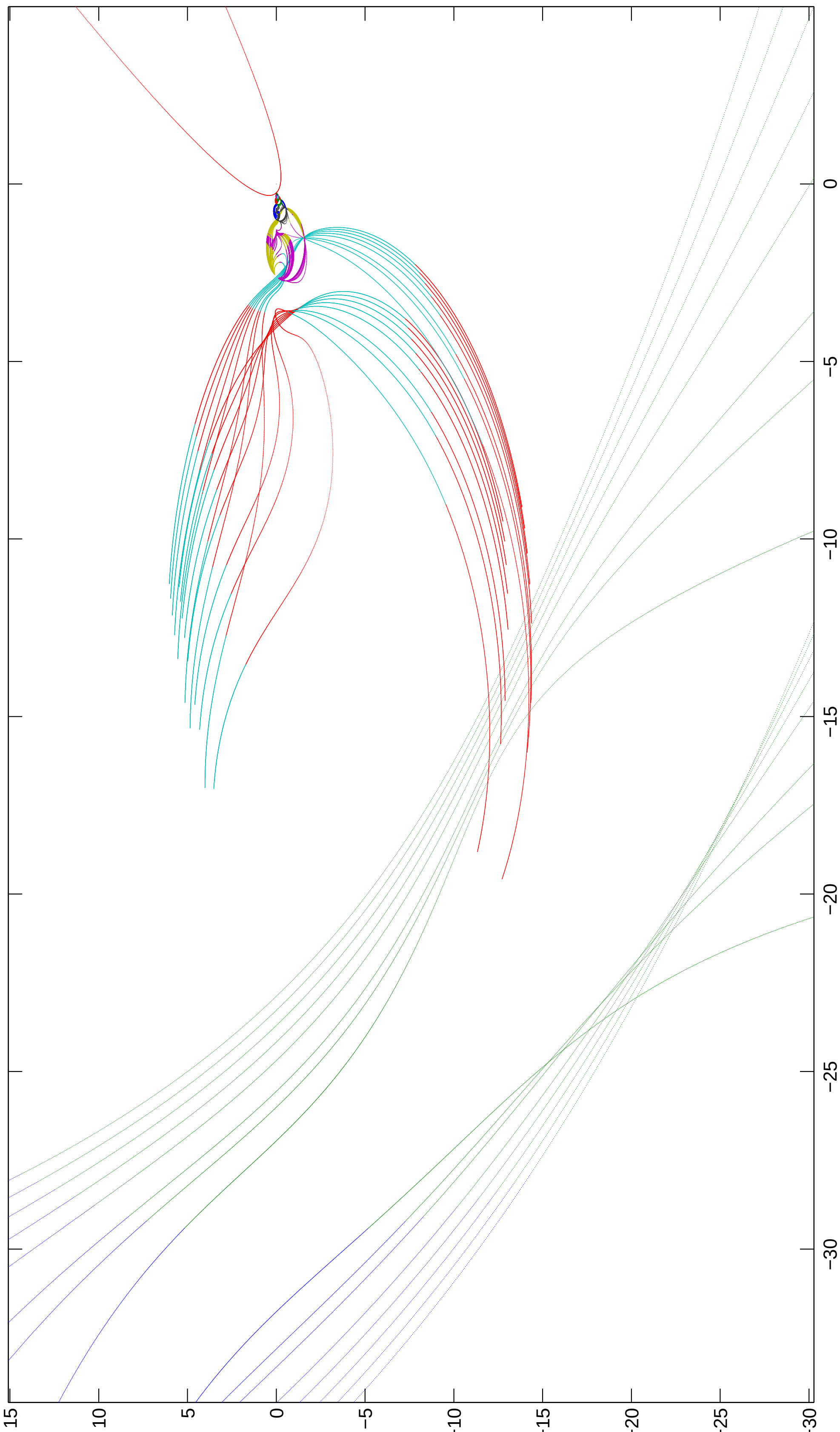


Figure B.6: A plot of some of the elliptic set for a 16-periodic continued fraction of type given in 4.2 where $\alpha = \frac{\pi}{7}$.

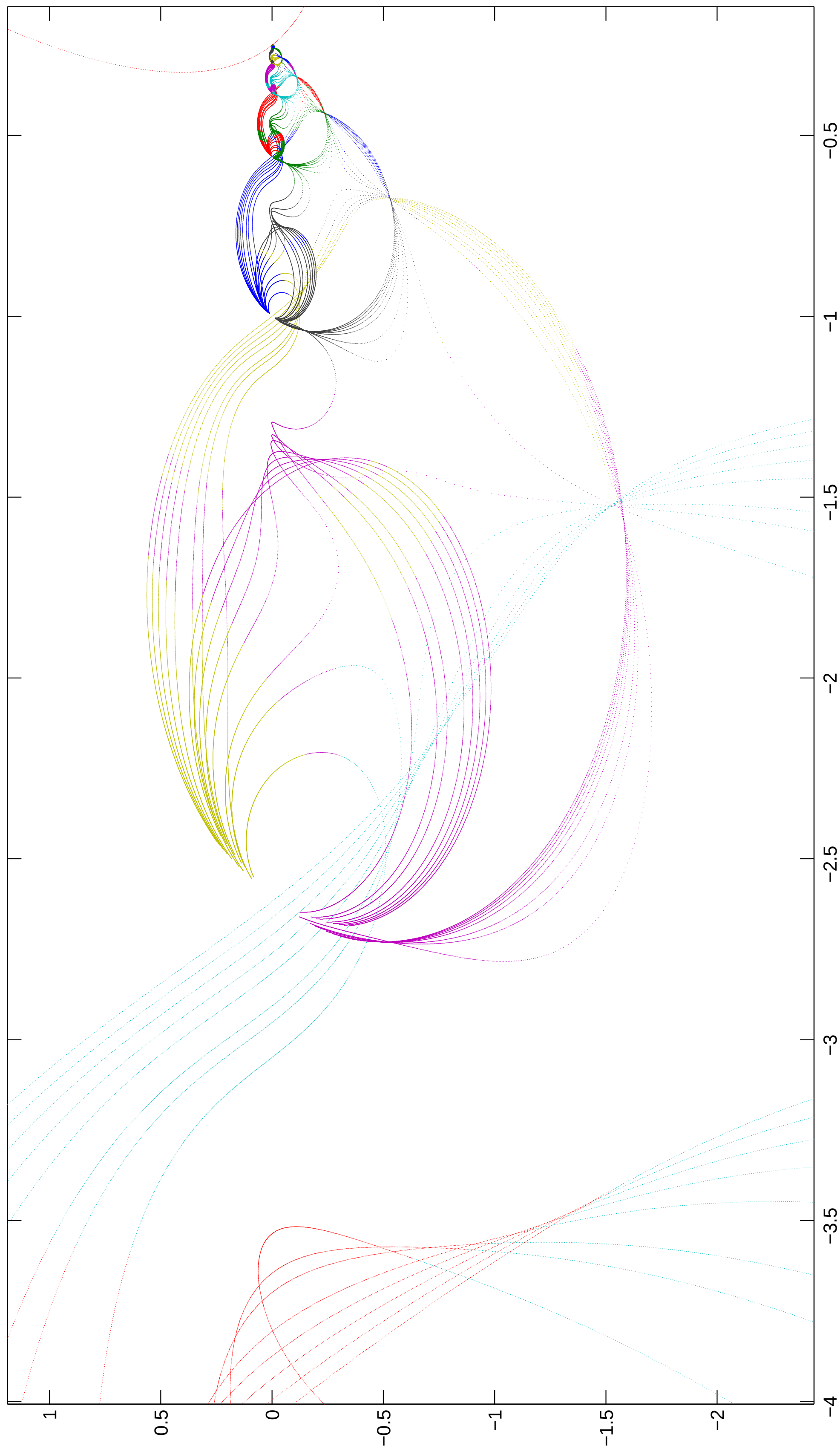


Figure B.7: A slightly zoomed plot of some of the elliptic set for a 16-periodic continued fraction of type given in 4.2 where $\alpha = \frac{\pi}{7}$.

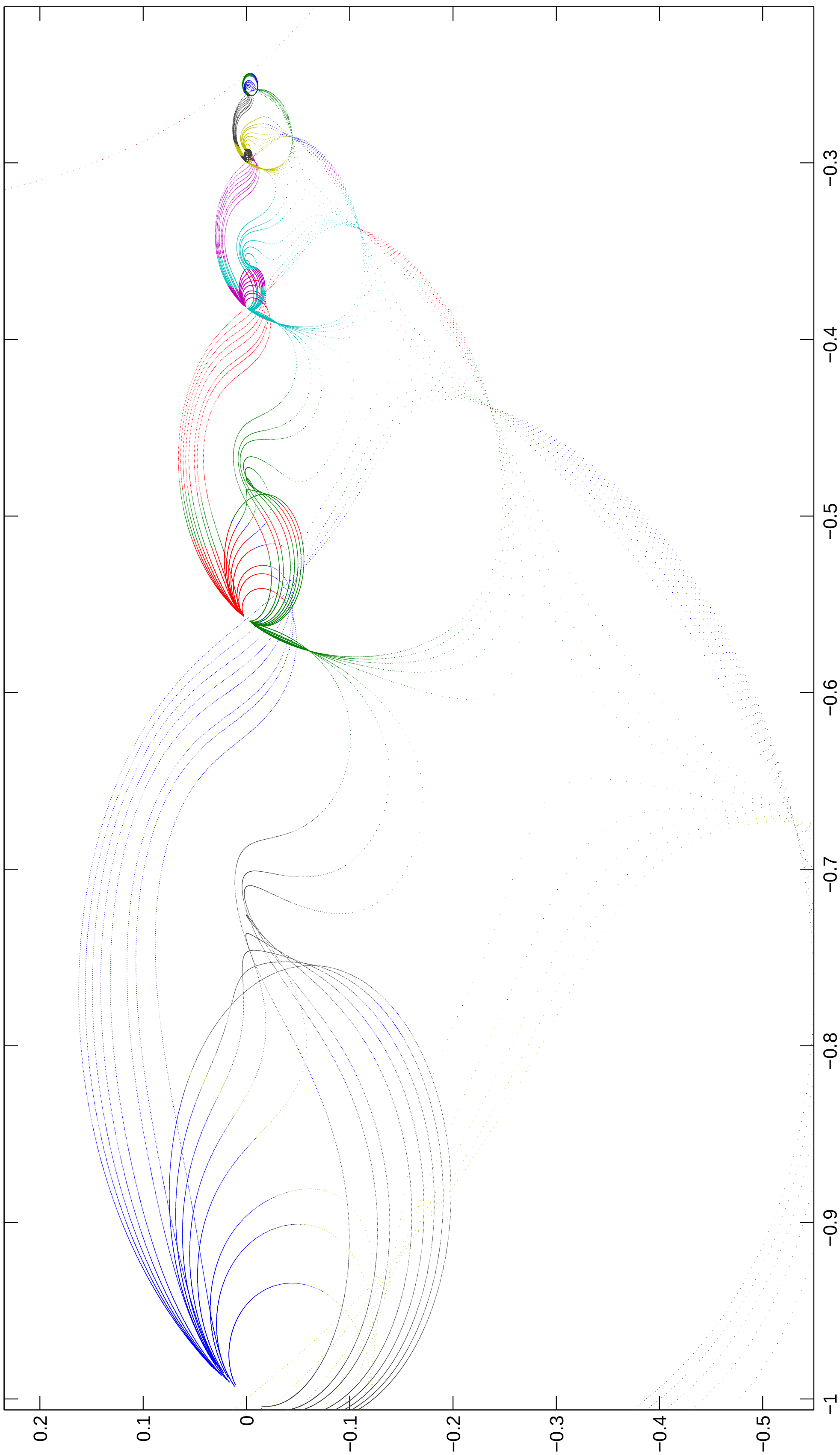


Figure B.8: A zoomed plot of some of the elliptic set for a 16-periodic continued fraction of type given in 4.2 where $\alpha = \frac{\pi}{7}$.

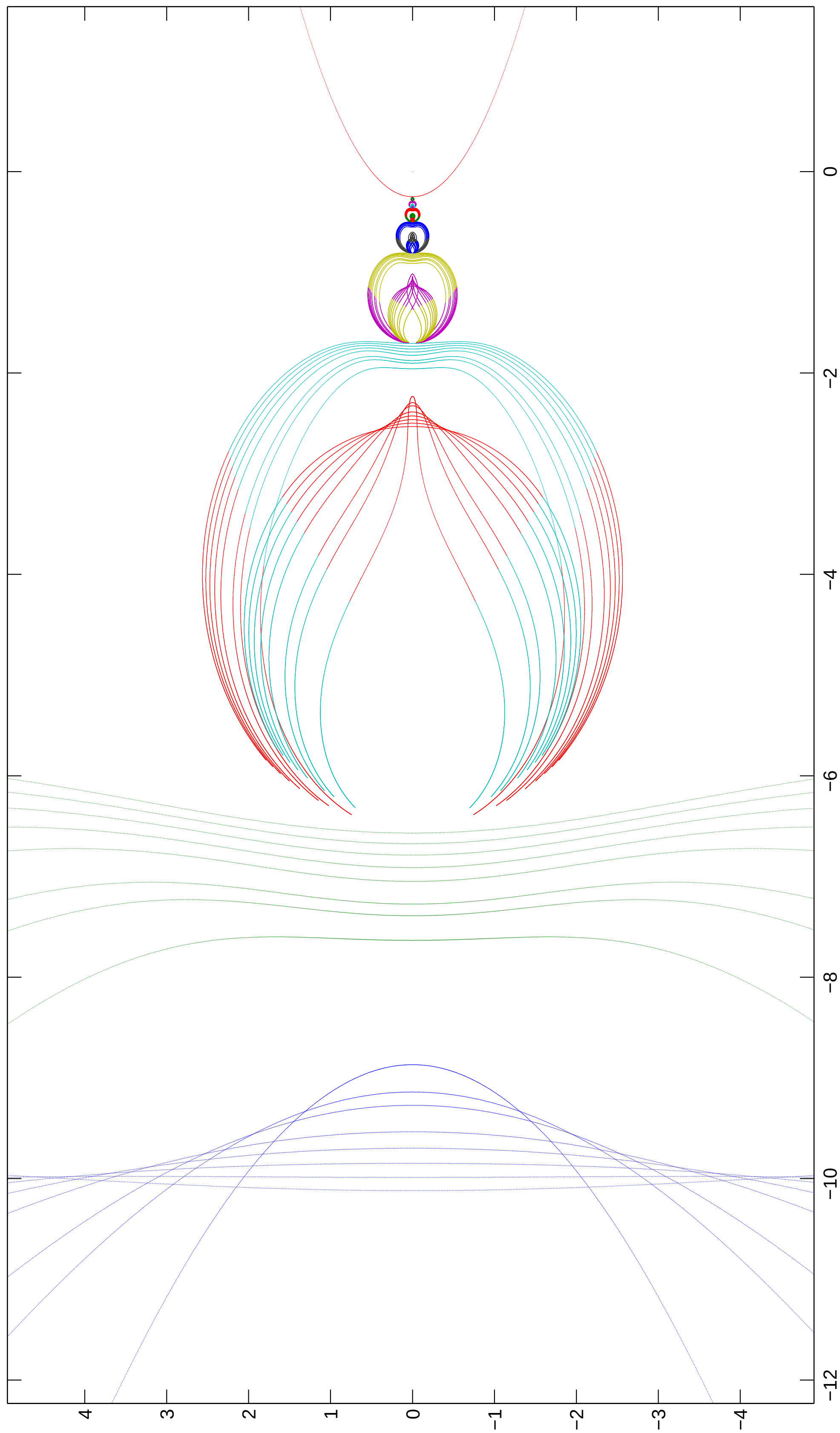


Figure B.9: A plot of some of the elliptic set for a 17-periodic continued fraction of type given in 4.2 where $\alpha = 0$.

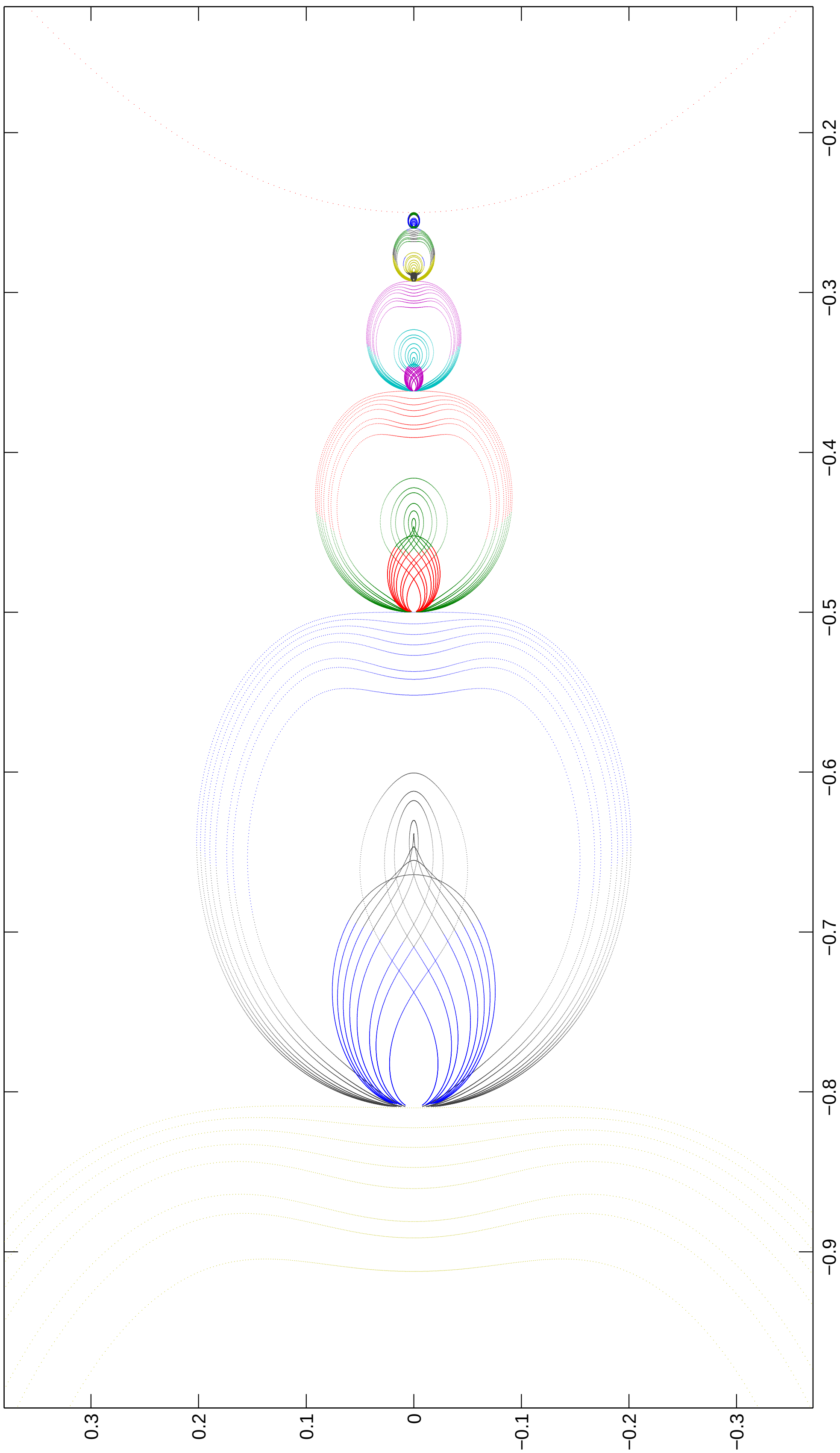


Figure B.10: A slightly zoomed plot of some of the elliptic set for a 17-periodic continued fraction of type given in 4.2 where $\alpha = 0$.

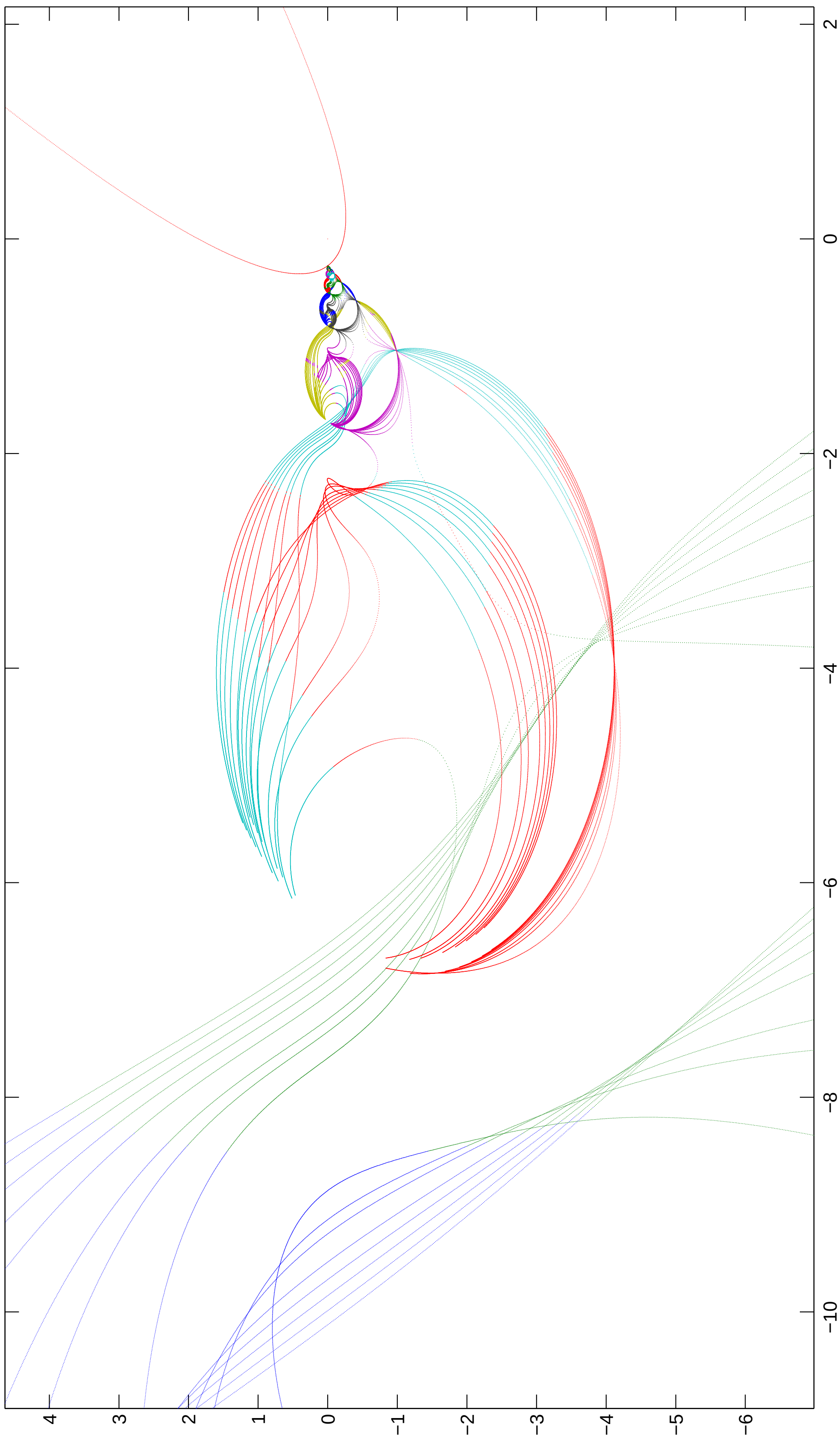


Figure B.11: A plot of some of the elliptic set for a 17-periodic continued fraction of type given in 4.2 where $\alpha = \frac{\pi}{7}$.

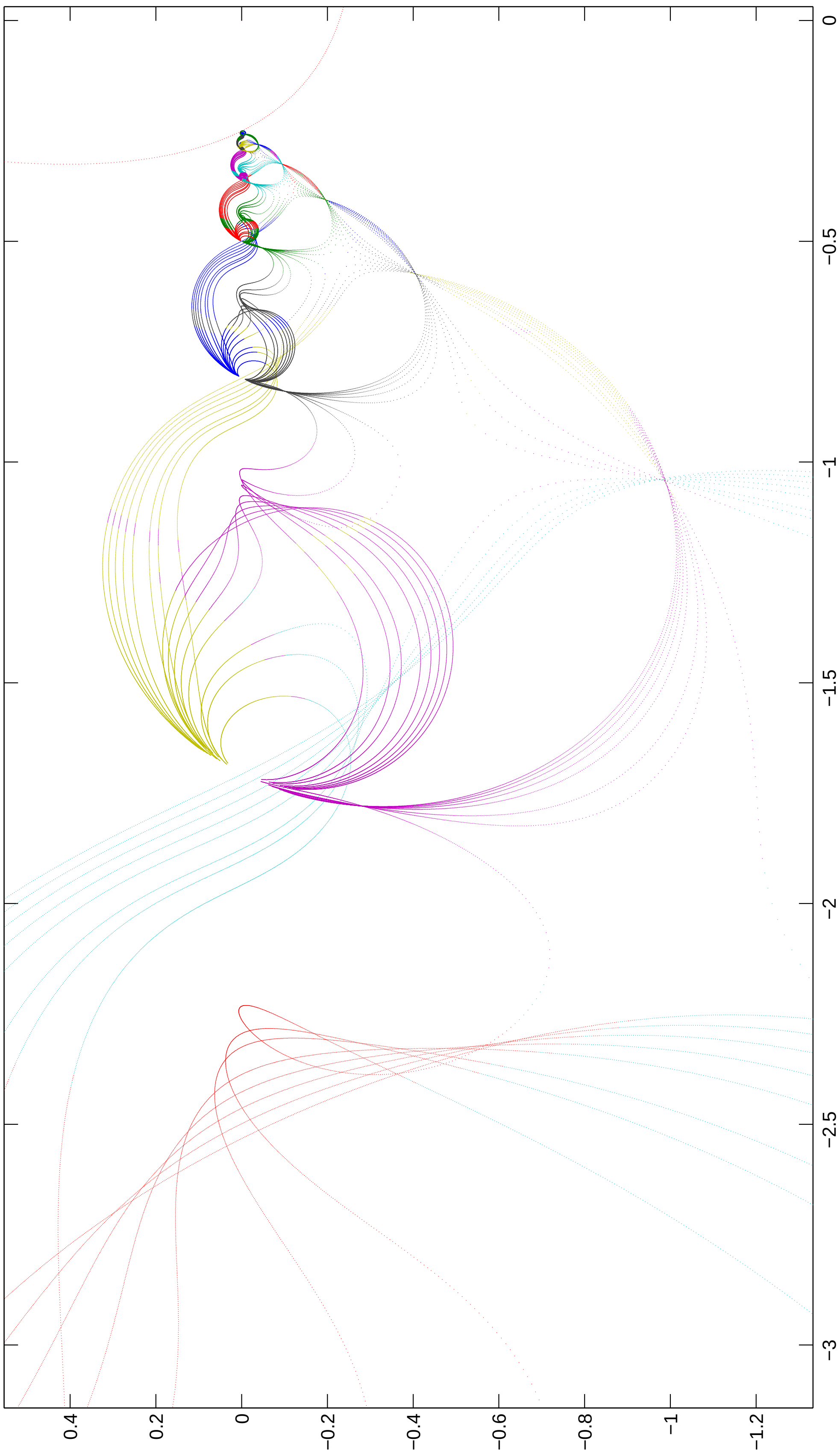


Figure B.12: A slightly zoomed plot of some of the elliptic set for a 17-periodic continued fraction of type given in 4.2 where $\alpha = \frac{\pi}{7}$.

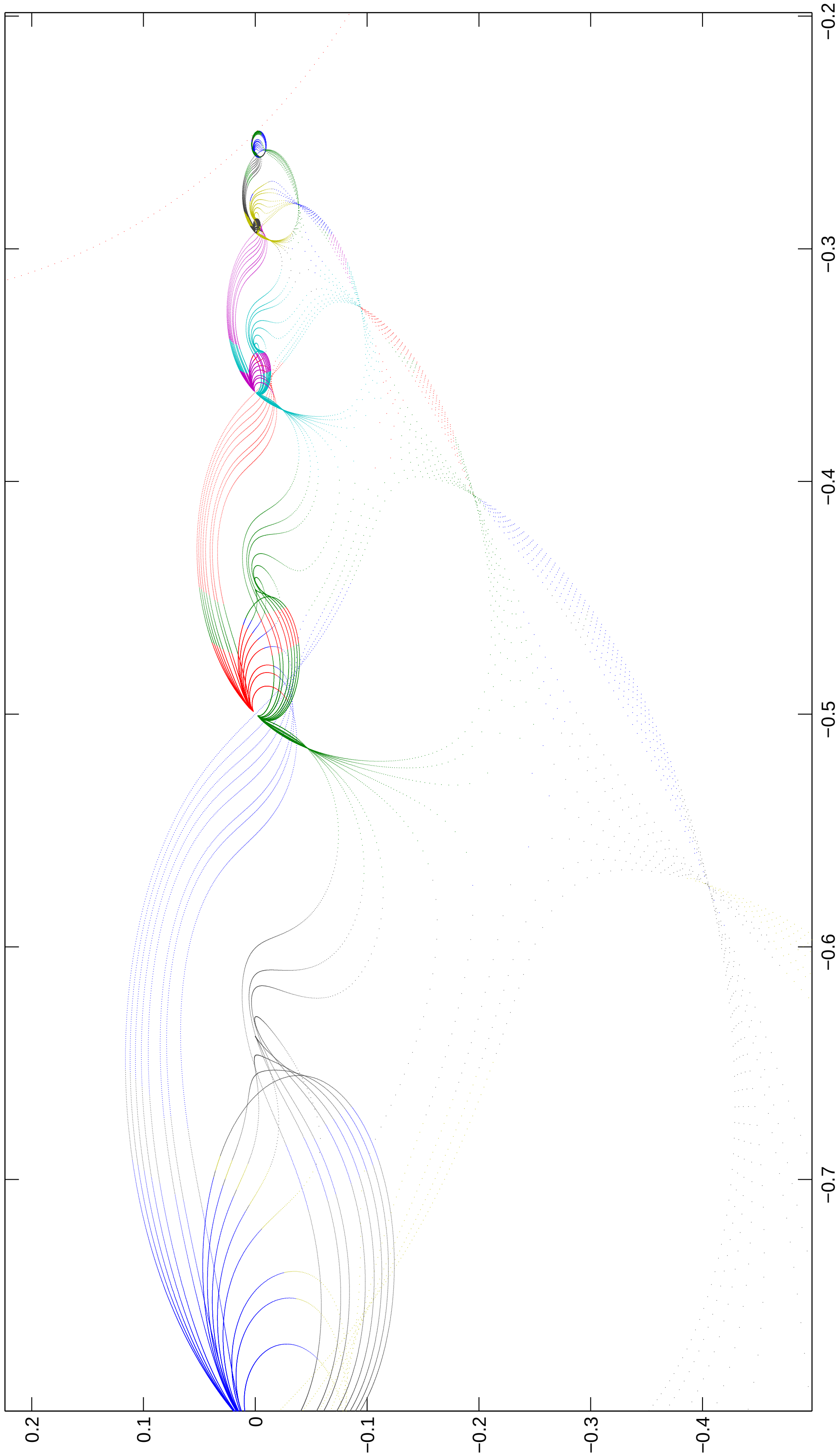


Figure B.13: A zoomed plot of some of the elliptic set for a 17-periodic continued fraction of type given in 4.2 where $\alpha = \frac{\pi}{7}$.

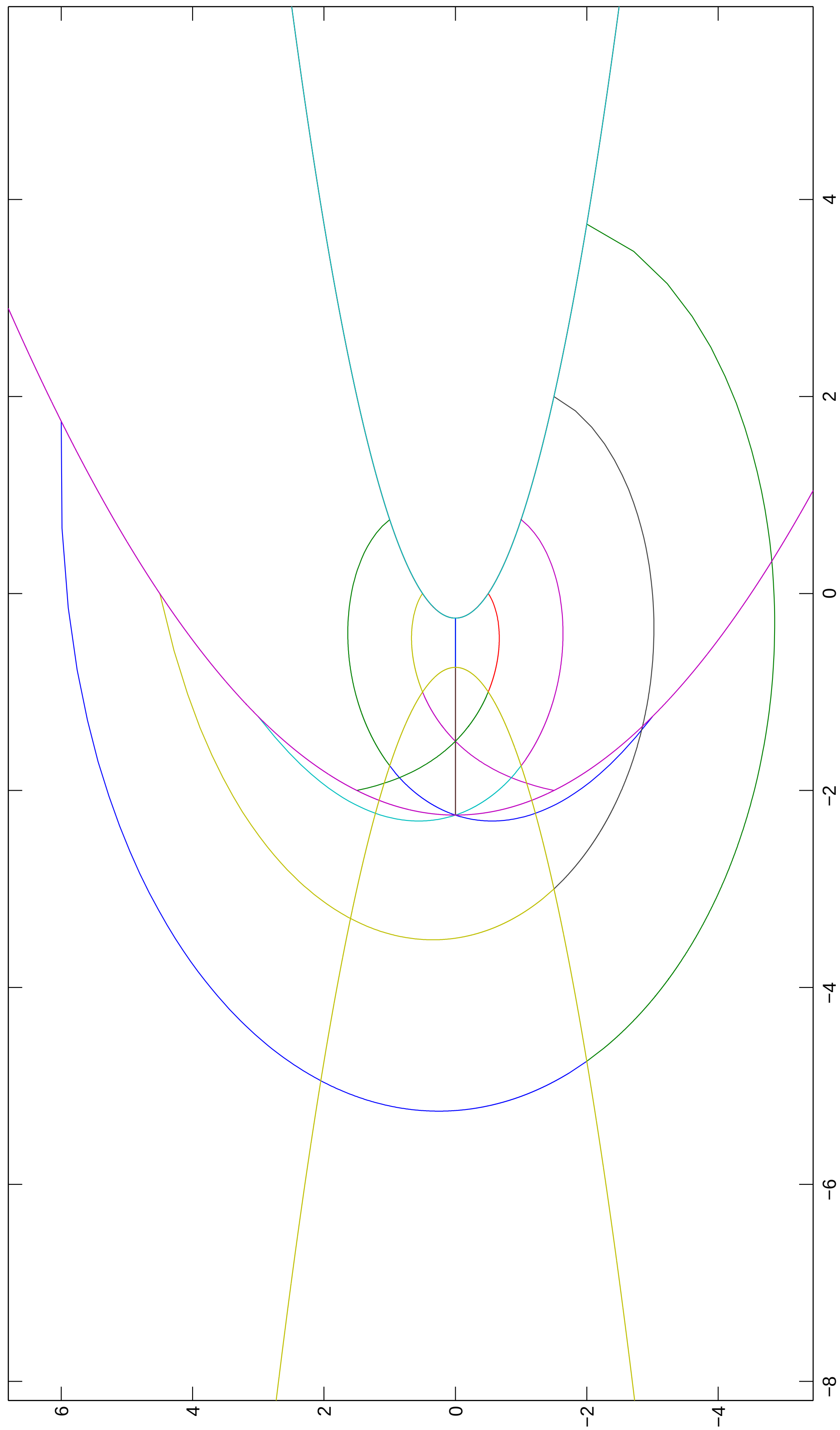


Figure B.14: A plot of some of the curves in (4.5) where $\alpha = 0$.

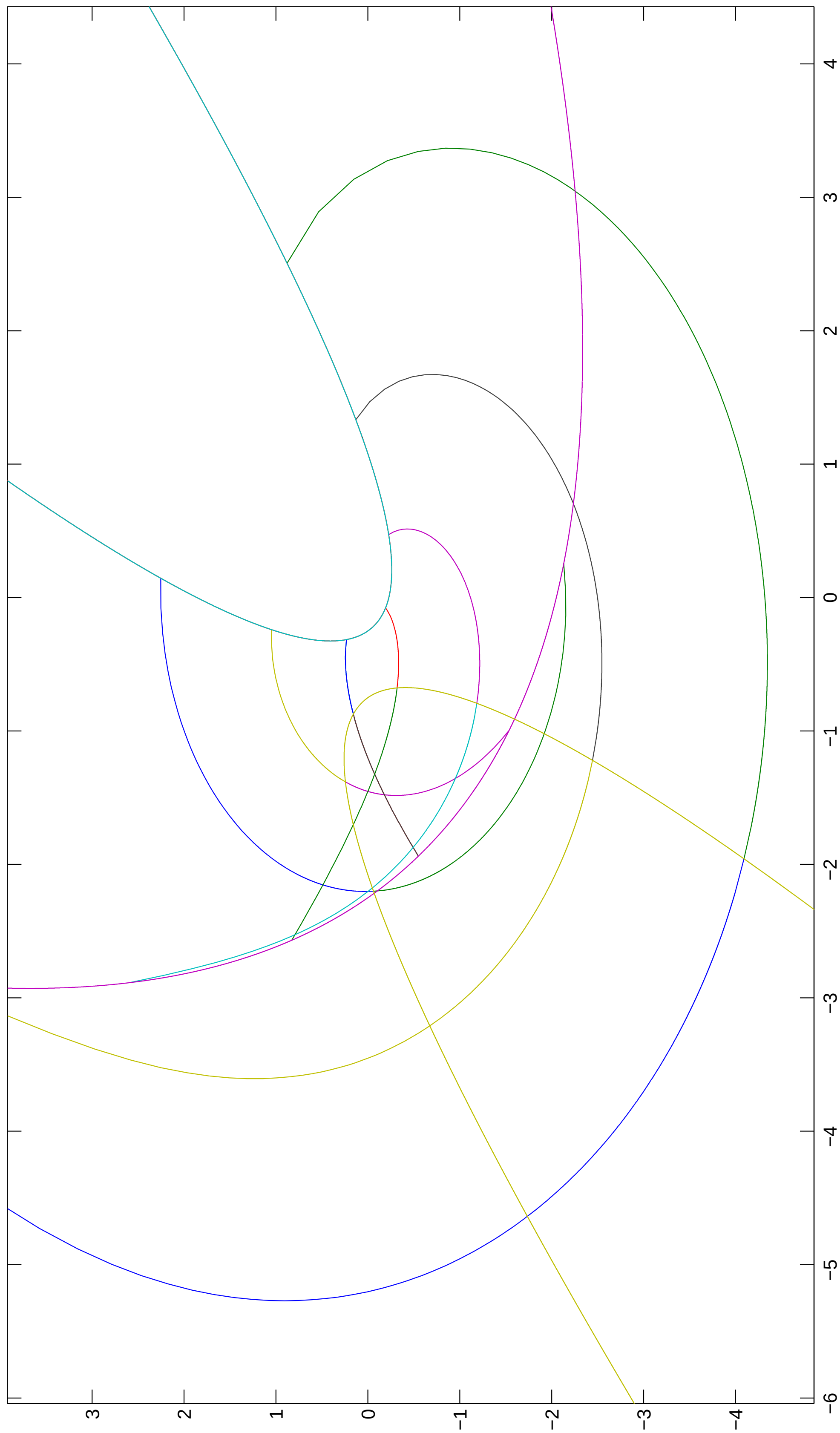


Figure B.15: A plot of some of the curves in (4.5) where $\alpha = \frac{\pi}{7}$.

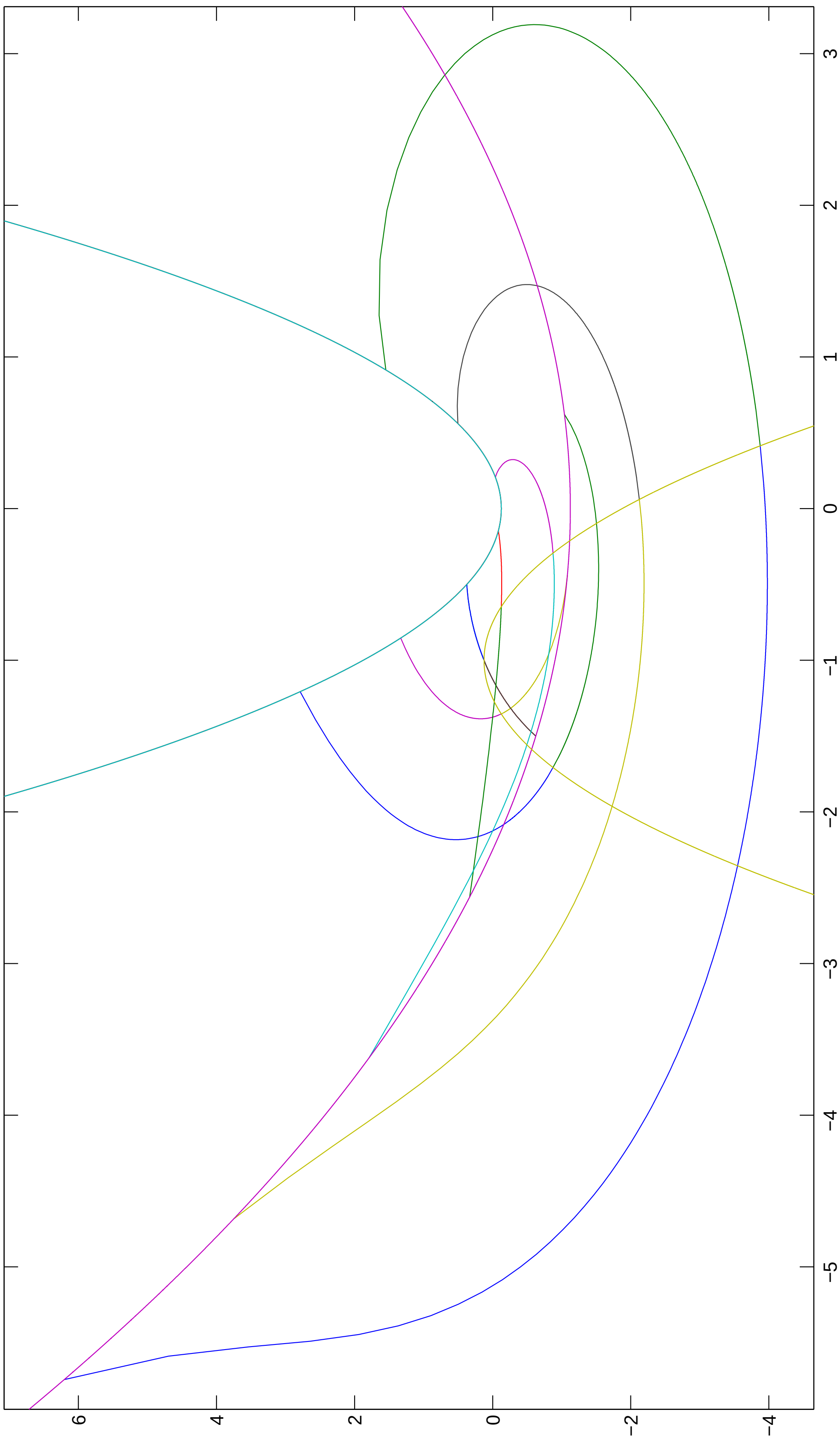


Figure B.16: A plot of some of the curves in (4.5) where $\alpha = \frac{\pi}{4}$.

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