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Degeneration as a Partial Order on Module Categories

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Introduction

The purpose of this work is to present some of the theory developed during the last 30 years on the subject of degeneration as a partial order on module categories. One of the highlights in this development was an article by Grzegorz Zwara in 1999 called "Degeneration for modules over representation-finite algebras" [16]. It had been a long standing problem whether the degeneration order coincided with the homomorphism order for representation-finite algebras. One of the main result in the article of Zwara was a corollary which stated exactly this. The corollary made it rather easy to decide degeneration for algebras of finite representation type, since once the isomorphism class of two representations is known, the dimension of the homomorphism spaces between them can be deduced from the AR-quiver of the algebra. In Chapter 2 we provide a formal method for determining degeneration for algebras of finite representation type. The method relies on Zwaras result, which is stated in Section 1.5.

It is well known (see [14]) that for a representation X over an algebra of finite representation type, there exists a set of algebraic equations S_X, by which one can determine whether a second representation Y is a degeneration of X. The set of algebraic equations is not unique. It is also known how one can find such algebraic equations (see [14]). In Chapter 3 we give an alternative procedure for finding such algebraic equations. We believe that this is an easier and more effective way to obtain such a set of algebraic equations.

In this thesis we have assumed that the reader is on a graduate level in algebra, familiar with rings, modules and some homological algebra. Some of the examples in this thesis extend over several pages. To help the reader recognize the end of an example, we use the symbol $\triangle$.

I want to thank my advisor Bernt Tore Jensen for excellent guidance and great patience. I also want to thank my formal advisor Sverre O. Smalø which have managed to give many helpful comments through e-mail. In general I am grateful to the entire Algebra group at NTNU. With no exceptions, I have always felt welcome on the 8th floor.

Chapter 1

Some Partial Orders on Module Categories

1.1 Preliminaries

1.1.1 Affine Varieties

The affine space $\mathbb{A}^n$ is K^n with a topology, where K is an algebraically closed field. The topology is the Zariski topology and is defined by the following:

$V \subseteq \mathbb{A}^n$ is closed if $\exists f_1, ..., f_m \in K[x_1, ..., x_n]$ such that $V = \{(v_1, ..., v_n) \mid f_i(v_1, ..., v_n) = 0, 1 \leq i \leq m\}$.

Definition 1.1.1 *An affine variety is a closed subset of $\mathbb{A}^n$ in the Zariski topology.*

1.1.2 Hasse Diagrams

A Hasse diagram is a graphical representation of a partially ordered set consisting of vertices and line segments. A vertex is drawn for each element of the poset, and line segments are drawn between these vertices according to the following two rules:

    1. If $x < y$ in the poset, then the vertex corresponding to x appears lower in the drawing than the vertex corresponding to y.

    2. The line segment between the vertices corresponding to any two elements x and y of the poset is included in the drawing if $x < y$ while $x < z < y$ implies that $z = x$ or $z = y$ or if $y < x$ while $y < z < x$ implies that $z = x$ or $z = y$.

So if $\{$x, y$\}$ is a partially ordered set with $x < y$ the Hasse diagram would look like this

$$y$$
$$\mid$$
$$x$$

1.2 Background

Let A be a finite dimensional associative K-algebra with an identity. If $\mathbf{a} = (1 = a_1, a_2, ..., a_t)$ is a basis of A over K, we have the structure constants a_{ijk} defined by $a_i a_j = \sum a_{ijk} a_k$. Let M be a d-dimensional left A-module with $\mathbf{b} = (b_1, b_2, ..., b_d)$ as basis. Then M is a d-dimensional vector space M with a multiplication by A from the left. By our choice of basis in M we can identify M with the vector space K^d. The elements of $\mathrm{End}_K(M)$ can be represented by $d \times d$-matrices over K. There is a correspondence between d-dimensional left A-modules and t-tuples $m = (m_1, ..., m_t)$ of

$d \times d$-matrices over K. For each d-dimensional left A-module M as above, M is determined as an A-module by the algebra homomorphism

$$\phi : A \longrightarrow \mathrm{End}_K M, \ \phi(a_i) : M \longrightarrow M,$$

where $\phi(a_i)(m) = a_i m$, $a_i \in A$. Hence M corresponds to the t-tuple $m = (m_1, ..., m_t)$, where $m_i = \phi(a_i)$. We have that m_1 is the identity matrix and $m_i m_j = \sum a_{ijk} m_k$, $1 \le i, j \le t$. For each module M the t-tuple m and all polynomials f in t non-commuting variables over K with the property that $f(a_1, ... a_t) = 0$ we have that $f(m_1, ..., m_t) = 0$ in the ring of $d \times d$-matrices.

Conversely, each t-tuple m, where m_1 is the identity matrix and $m_i m_j = \sum a_{ijk} m_k$, $1 \le i, j \le t$, corresponds to an K-algebra homomorphism $\phi_m : A \to \mathrm{End}_K(K^d)$. Then the A-module structure on K^d is defined by $a_i x = m_i x$, for $x \in K^d$.

Let us illustrate with an example.

Example 1.2.1 *Let $A = \mathbb{C}[x]/x^2$ be the two dimensional associative $\mathbb{C}$-algebra with the basis $\{1, x\}$ over $\mathbb{C}$, when one represent also the residues of 1 and x by 1 and x. The structure constant is determined by the following multiplication table*

$\times$	1	x
1	1	x
x	x	0

Let M be the 2-dimensional vector space $\mathbb{C}^2$. The elements of $\mathrm{End}_{\mathbb{C}}(M)$ can be represented by 2×2-matrices over $\mathbb{C}$. The two dimensional left A-module structure on M corresponds to the 2-tuple

$$m = (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}), \ \textit{which makes } M \textit{ isomorphic to } A \textit{ as an left } A\textit{-module.}$$

Conversely, the 2-tuple m corresponds to A as an A-module.

$\triangle$

In the following example the field $\mathbb{R}$ is not algebraically closed. The example is included because of its nice and perhaps familiar form.

Example 1.2.2 *Let A be the 4 dimensional associative $\mathbb{R}$-algebra with the basis $\{1, i, j, k\}$ over $\mathbb{R}$ where the structure constant is determined by the following multiplication table*

$\times$	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

Note that A is the quaternions $\mathbb{H}$ over $\mathbb{R}$.

Let M be the 4-dimensional vector space $\mathbb{R}^4$. The elements of $\mathrm{End}_{\mathbb{R}}(M)$ can be represented by 4×4-matrices over $\mathbb{R}$. The 4-dimensional left A-module structure on M corresponds to the 4-tuple

$$m = (\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}),$$

which makes M isomorphic to A as an left A-module.

Conversely, the 4-tuple m corresponds to M as an A-module.

$\triangle$

Definition 1.2.3 *The set of t-tuples $\{(m_1, m_2, ..., m_t)\}$ of $d \times d$-matrices over K where m_1 is the identity matrix and $m_i m_j = \sum a_{ijk} m_k$, for all $1 \le i, j \le t$, is an affine variety and is denoted by $\mathrm{mod}_A(d)$.*

The general linear group $Gl_d(K)$ is the group of $d \times d$ invertible matrices with entries in K. The elements of $Gl_d(K)$ acts on $\mathrm{mod}_A(d)$ by conjugation, $g * x = (g x_1 g^{-1}, ..., g x_t g^{-1})$, $g \in Gl_d(K)$, $x \in \mathrm{mod}_A(d)$, and the orbits $O(x)$ under this action correspond to the isomorphism classes of d-dimensional A-modules. The following lemma should make this clear:

Lemma 1.2.4 *The orbits of $O(x)$ for $x \in mod_A(d)$ corresponds to the isomorphism classes of d-dimensional left A-modules.*

Proof: Let $m = (m_1, ..., m_t)$ and $n = (n_1, ..., n_t)$ be two points in $mod_A(d)$. Let M and N be the A-modules corresponding to m and n respectively. We must show that

$$n \in O(m) \Leftrightarrow N \cong M.$$

"$\Rightarrow$": Assume there exist a $g \in Gl_d(K)$ such that $n = g * m$. We want to show that $N \cong M$. By the choice of basis we have identified M and N with K^d as K-modules. We define a map

$$\Theta : M \longrightarrow N, \text{ where } \Theta(x) = gx,$$

and prove that Θ is an isomorphism of A-modules. First one can verify that Θ is an A-homomorphism, i. e.

$$\Theta(x + y) = g(x + y) = gx + gy = \Theta(x) + \Theta(y)$$

and

$$\Theta(a_i x) = ga_i x = gm_i x,$$

by assumption $n = g * m \Leftrightarrow n_i = gm_i g^{-1}, 1 \le i \le t$. So by inserting $g^{-1}n_i g$ for m_i, we obtain

$$gm_i x = n_i gx = a_i \Theta(x),$$

$a_i \in \mathbf{a}$, x and y in M. One can define the inverse map $\Theta^{-1} : N \longrightarrow M$ given by $y \longrightarrow g^{-1}y$. As for Θ one can verify that θ^{-1} is an A-homomorphism. We see that $\Theta^{-1}\Theta$ is the identity on M and $\Theta\Theta^{-1}$ is the identity on N. Hence Θ is an isomorphism.

"$\Leftarrow$": Assume that $M \cong N$. We want to show that there exist a $g \in Gl_d(K)$ such that $n = g*m$. By assumption there exist an isomorphism $\phi : M \longrightarrow N$. We know that ϕ is an A-isomorphism, thus ϕ is invertible and $\phi(a_i x) = a_i \phi(x)$, $a_i \in \mathbf{a}$. By the choice of basis we have identified M and N with K^d. We have $R_\phi \in Gl_d(K)$ such that $\phi(x) = R_\phi x$. Thus for all $1 \le i \le t$ and $x \in mod_A(d)$ we have

$$\phi(a_i x) = a_i \phi(x), \forall x \Leftrightarrow R_\phi m_i x = n_i R_\phi x, \forall x \Leftrightarrow R_\phi m_i = n_i R_\phi \Leftrightarrow n_i = R_\phi m_i R_\phi^{-1},$$

for all $1 \le i \le t$. Hence $n = g * m$.

QED

We recall from the preliminaries that the topology on $mod_A(d)$ is the subspace topology of the vector space of all t-tuples $m = (m_1, ..., m_t)$ in $\mathbb{A}^{td^2}$.

Degeneration is defined on $mod_A(d)$ by the following:

$$M \preceq_{deg} N :\Leftrightarrow n \in \overline{O(m)}.$$

1.3 Degeneration $\preceq_{deg}$ in $\mathrm{Rep}_K(Q)$

1.3.1 The Category of Finite Dimensional Representations $Rep_K(Q)$

A quiver is a finite directed graph, possibly with multiple arrows between the vertices, and possibly with loops. Formally a quiver $Q = (Q_0, Q_1, s, t)$ consists of a finite set of vertices Q_0, a finite set of arrows Q_1 and two maps $s, t : Q_1 \to Q_0$ which sends an arrow α to its starting vertex $s(\alpha)$ and its terminating vertex $t(\alpha)$. Thus we write $\alpha : i \to j$ for an arrow starting in i and terminating in j. A path p in the quiver Q is either an ordered sequence of arrows $p = \alpha_n...\alpha_1$ with $t(\alpha_l) = s(\alpha_{l+1})$ for $1 \le l \le n - 1$, or a trivial path e_i for $i \in Q$. By a trivial path e_i, we mean a path of length zero where $s(e_i) = t(e_i) = i$.

If Q is a quiver and K a field, then the path algebra $\Lambda = KQ$ is defined as follows: it is the vector space having all the paths in the quiver as basis; multiplication is given by concatenation of paths. If two paths cannot be concatenated because the end vertex of the first is not equal to the starting vertex of the second, their product is defined to be zero. This defines an associative algebra over K. The unit element of the algebra is the sum of the trivial paths corresponding to the vertices. For a more detailed introduction to path algebras see [1].

Example 1.3.1 *Let*

$$Q: \ 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3.$$

The path algebra $\Lambda = KQ$ will be a six dimensional algebra, with basis e_1, e_2, e_3, α, β and $\beta\alpha$. Here e_1, e_2 and e_3 represent the trivial paths at the vertices 1, 2 and 3 respectively. One has to make a convention about how to represent a path and here, in accordance with the convention above, an oriented path is ordered from right to left, that is $\beta\alpha$ means first $\alpha : 1 \to 2$ then $\beta : 2 \to 3$.

The multiplication table for this algebra

$e_1 * e_1 = e_1$	$e_1 * e_2 = 0$	$e_1 * e_3 = 0$	$e_1 * \alpha = e_1$	$e_1 * \beta = 0$	$e_1 * \beta\alpha = 0$
$e_2 * e_1 = 0$	$e_2 * e_2 = e_2$	$e_2 * e_3 = 0$	$e_2 * \alpha = \alpha$	$e_2 * \beta = 0$	$e_2 * \beta\alpha = 0$
$e_3 * e_1 = 0$	$e_3 * e_2 = 0$	$e_3 * e_3 = 0$	$e_3 * \alpha = 0$	$e_3 * \beta = 0$	$e_3 * \beta\alpha = \beta\alpha$
$\alpha * e_1 = \alpha$	$\alpha * e_2 = 0$	$\alpha * e_3 = 0$	$\alpha * \alpha = 0$	$\alpha * \beta = 0$	$\alpha * \beta\alpha = 0$
$\beta * e_1 = 0$	$\beta * e_2 = \beta$	$\beta * e_3 = 0$	$\beta * \alpha = \beta\alpha$	$\beta * \beta = 0$	$\beta\alpha * \beta\alpha = 0$
$\beta\alpha * e_1 = \beta\alpha$	$\beta\alpha * e_2 = 0$	$\beta\alpha * e_3 = 0$	$\beta\alpha * \alpha = 0$	$\beta\alpha * \beta = 0$	$\beta\alpha * \beta\alpha = 0$

The unit element of Λ is $e_1 + e_2 + e_3$.

$\triangle$

A representation (V, f) of a quiver Q over a field K is a realization of its diagram of vertices in the category of vector spaces, where each vertex $i \in Q_0$ is replaced by a vector space $V(i)$ and each arrow $\alpha : i \to j$ in Q_1 is replaced by a K-linear map f_α from $V(i)$ to $V(j)$. Here we assume that the representations of Q are finite dimensional, i. e. $dim_K V(i) < \infty$, $i \in Q_0$. The dimension vector of a representation (V, f) is the vector $\mathbf{d} \in \mathbb{Z}^{Q_0}$, given by $\mathbf{d}(i) := dim_K V(i)$.

A morphism $\phi : (V, f) \to (V', f')$ between two representations of Q is a collection of K-linear maps $\phi_i : V(i) \to V'(i)$, $i \in Q_0$, such that the diagram

$$
\begin{array}{ccc}
V(i) & \xrightarrow{\phi_i} & V'(i) \\
\downarrow{\scriptstyle f_\alpha} & & \downarrow{\scriptstyle f'_\alpha} \\
V(j) & \xrightarrow{\phi_j} & V'(j),
\end{array}
$$

commutes $\forall \alpha \in Q_1$. Let (V, f), (V', f') and (V'', f'') be three representations and let

$$\phi = \{\phi_i : V(i) \to V'(i)\} : \ (V, f) \to (V', f') \text{ and } \psi = \{\psi_i : V'(i) \to V''(i) : \ (V', f') \to (V'', f''),$$

be morphisms. Then

$$\psi\phi = \{\psi_i\phi_i : V(i) \to V''(i)\} : \ (V, f) \to (V'', f'').$$

We see that we have associativity of composition of morphisms, i.e. if $\phi : (V, f) \to (V', f')$, $\psi : (V', f') \to (V'', f'')$ and $\zeta : (V'', f'') \to (V''', f''')$ then $\phi(\psi\zeta) = (\phi\psi)\zeta$. Also for every representation (V, f), there exists an identity morphism $1_{(V,f)} : (V, f) \to (V, f)$, such that for every morphism $\phi : (V, f) \to (V', f')$, we have $1_{(V',f')}\phi = \phi = \phi 1_{(V,f)}$. By these properties we see that we get a category consisting of finite dimensional representations of Q over K, which we denote by $Rep_K(Q)$.

A morphism $\phi : (V, f) \to (V', f')$ is an isomorphism if ϕ_i is an isomorphism for each $i \in Q_0$.

We can also take direct sums of representations. Let (V, f) and (V', f') be two representations in $Rep_K(Q)$. We let the direct sum $(V, f) \oplus (V', f')$ be defined by:

$$(V \oplus V')(i) = V(i) \oplus V'(i), \ \forall i \in Q_0,$$

and

$$(f \oplus f')_\alpha = \begin{pmatrix} f_\alpha & 0 \\ 0 & f'_\alpha \end{pmatrix} : V(s(\alpha)) \oplus V'(s(\alpha)) \to V(t(\alpha)) \oplus V'(t(\alpha)), \ \forall \alpha \in Q_1.$$

By a trivial representation (V, f) we mean a representation where the dimension vector $\mathbf{d}$ is the zero vector. If $(V, f) \cong (V', f') \oplus (V'', f'')$, where both (V', f') and (V'', f'') are nontrivial representations, we say that (V, f) is decomposable, if not, (V, f) is said to be indecomposable.

The Krull-Remak-Scmidt theorem holds in $Rep_K(Q)$ (see [1]), that is every representation (V, f) in $Rep_K(Q)$ has a unique decomposition into a direct sum of indecomposable summands (unique up to isomorphism and ordering of summands). Let $\Lambda = KQ$ be the path algebra of Q over K, we have that $Rep_K(Q)$ and mod Λ are equivalent as categories, where mod Λ denotes the category of finite dimensional Λ-modules (see [1]).

1.3.2 Degeneration $\preceq_{deg}$ in $Rep_K(Q)$

Now we define what is meant by degeneration $\preceq_{deg}$ in $Rep_K(Q)$. We start by choosing a basis for $V(i)$, $i \in Q_0$. We define

$$Rep_K(Q, \mathbf{d}) = \prod_{\alpha \in Q_1} \mathrm{Hom}(K^{d_{s(\alpha)}}, K^{d_{t(\alpha)}}),$$

where $\mathbf{d} = (d_i)_{i \in Q_0}$ is a dimension vector. We see that $Rep_K(Q, \mathbf{d})$ is a vector space. We choose a basis $\mathbf{b}$ for the vector space $Rep_K(Q, \mathbf{d})$, where the basis element $b^\alpha_{(u,v)}$ is the $dim(s(\alpha)) \times dim(t(\alpha))$-matrix with one in entry (u, v) and zero elsewhere. To each basis element $b^\alpha_{(u,v)}$ we associate a variable $x^\alpha_{(u,v)}$. One realize that $Rep_K(Q, \mathbf{d})$ is an affine space. The polynomials determining the closed subsets in $Rep_K(Q, \mathbf{d})$ are polynomials in the polynomial ring

$$K[x^\alpha_{(u,v)}], \text{ where } \alpha \in Q_1, \ 1 \le u \le s(\alpha) \text{ and } 1 \le v \le t(\alpha).$$

We also define a group action of

$$Gl_{\mathbf{d}}(K) = \prod_{i \in Q_0} Gl_{d_i}(K) \text{ on } Rep_K(Q, \mathbf{d}),$$

given by

$$g * x = (g_{t(\alpha)} x_\alpha g_{s(\alpha)}^{-1})_{\alpha \in Q_1}, \ g = (g_i)_{i \in Q_0} \in Gl_{\mathbf{d}}(K), \ x = (x_\alpha)_{\alpha \in Q_1} \in Rep_K(Q, \mathbf{d}).$$

The orbit of x under the group action above is denoted by $O(x)$.

There is a correspondence between representations and points in $Rep_K(Q, \mathbf{d})$. By choosing basis the correspondence follows naturally.

As for the t-tuples corresponding to the finite dimensional A-modules in $mod_A(d)$, the orbits $O(x)$ for $x \in Rep_K(Q, \mathbf{d})$ corresponds to the isomorphism classes of representations with fixed dimension vector $\mathbf{d}$.

Lemma 1.3.2 *The orbits of $O(x)$ for $x \in Rep_K(Q, \mathbf{d})$ corresponds to the isomorphism classes of representations with dimension vector $\mathbf{d}$.*

Proof: Let $x = (x_\alpha)_{\alpha \in Q_1}$ and $y = (y_\alpha)_{\alpha \in Q_1}$ be two points in $Rep_K(Q, \mathbf{d})$. Let X and Y be representations with dimension vector $\mathbf{d}$ corresponding to x and y respectively. We must show that

$$y \in O(x) \Leftrightarrow Y \cong X.$$

"$\Rightarrow$": Assume there exist a $g \in Gl_{\mathbf{d}}(K)$ such that $y = g * x$. We want to show that $X \cong Y$. We have

$$y = g * x = (g_{t(\alpha)} x_\alpha g_{s(\alpha)}^{-1}), \ g = (g_i)_{i \in Q_0} \in Gl_{\mathbf{d}}(K), \ x = (x_\alpha)_{\alpha \in Q_1} \in Rep_K(Q, \mathbf{d}).$$

So the following diagram commutes

$$
\begin{array}{ccc}
Y: & K^{d_{s(\alpha)}} \xrightarrow{\ y_\alpha\ } K^{d_{t(\alpha)}} \\
& g_{s(\alpha)} \uparrow \qquad \qquad g_{t(\alpha)} \uparrow \\
X: & K^{d_{s(\alpha)}} \xrightarrow{\ x_\alpha\ } K^{d_{t(\alpha)}}
\end{array}
$$

Since $(g_i)_{i \in Q_0}$ are invertible, they are isomorphisms of vector spaces. It follows that $Y \cong X$.

"⇐": Assume that $X \cong Y$. We want to show that there exist a $g \in Gl_d(K)$ such that $y = g * x$. By assumption there exist an isomorphism

$$\phi : X \longrightarrow Y, \ \phi \in Gl_{\mathbf{d}}(K),$$

which is determined by the choice of basis. By the commutativity property of the diagrams arising from this isomorphism

$$
\begin{array}{ccc}
X : & K^{d_{s(\alpha)}} \xrightarrow{x_\alpha} K^{d_{t(\alpha)}} \\
& \downarrow{\phi_{s(\alpha)}} \quad\quad \downarrow{\phi_{t(\alpha)}} \\
Y : & K^{d_{s(\alpha)}} \xrightarrow{y_\alpha} K^{t_{s(\alpha)}}
\end{array}
$$

where $(\phi_i)_{i \in Q_0}$ are isomorphisms, we see that

$$y_\alpha = (\phi_{t(\alpha)} x_\alpha \phi_{s(\alpha)}^{-1}).$$

It follows that $y = g * x$.

$$QED$$

Degeneration is defined on $Rep_K(Q, \mathbf{d})$ by the following:

$$X \preceq_{deg} Y :\Leftrightarrow y \in \overline{O(x)},$$

where $\overline{O(x)} = \{z \in Rep_K(Q, \mathbf{d}) \mid f(z) = 0, \ \forall f \in K[x_{(u,v)}^\alpha] \text{ such that } f(O(x)) = 0\}$.

By Bongartz (see [5]) $Rep_K(Q)$ and $mod\ A$ has the same degeneration order.

We illustrate with an example of two representations.

Example 1.3.3 *Let Q be the quiver*

$$1 \xrightarrow{\alpha} 2.$$

Consider the following two representations:

$$(V, f) : K^2 \xrightarrow{f_\alpha} K^2 \ and \ (V', f') : K^2 \xrightarrow{f'_\alpha} K^2$$

$$f_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ f'_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The orbit of f'_α, denoted by $O(f'_\alpha)$, is all invertible 2×2 matrices. We have that $O(f_\alpha)$ is all matrices of rank one. So we see that $O(f_\alpha) \subset \overline{O(f'_\alpha)}$. In more detailed terms

$$\{\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \mid t \neq 0\} \subseteq O\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right).$$

In general we have:

$$V \subseteq W \Rightarrow \overline{V} \subseteq \overline{W},$$

so in particular:

$$\{\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \mid t \neq 0\} \subseteq O\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \Rightarrow \overline{\{\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \mid t \neq 0\}} \subseteq \overline{O\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)}.$$

If $\phi : Rep_K(Q) \to K$ is a polynomial function in $K[x_{(1,1)}^\alpha, x_{(2,1)}^\alpha, x_{(1,2)}\alpha, x_{(2,2)}^\alpha]$ such that

$$\phi \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} = 0, \ \forall \ t \neq 0,$$

then

$$\phi \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

If $\psi : Rep_K(Q) \to K$ is a polynomial function such that

$$\psi \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} = 0, \ \forall \ t \neq 0,$$

then

$$\psi \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \psi(0) = 0.$$

We see that $\overline{O(f'_\alpha)} = \{O(f'_\alpha) \cup O(f_\alpha) \cup O(0)\}$, i. e. all 2×2 matrices over K. It follows that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \overline{O\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)},$$

which implies that $(V', f') \preceq_{deg} (V, f)$.

$\triangle$

1.4 Virtual Degeneration $\preceq_{vdeg}$ and the Hom-order $\preceq_{hom}$

Let A be a finite dimensional associative K-algebra with an identity over an algebraically closed field K. Let M, N and L be finite dimensional A-modules.

The following result is generalized by K. Bongartz [3]:

Proposition 1.4.1 *Let A be a finite dimensional associative K-algebra with an identity. Then two finite dimensional A-modules M and N are isomorphic if and only if*

$$dim_K(\mathrm{Hom}_A(M, X)) = dim_K(\mathrm{Hom}_A(N, X)), \ \forall \ X \in mod \ A.$$

Proof: We assume that $dim_K(\mathrm{Hom}_A(N, X)) = dim_K(\mathrm{Hom}_A(M, X))$. We claim that if M and N are different from the zero-module, they have a non-zero direct summand in common. This is obtained by taking generators $f_1, f_2, ..., f_n$ of $\mathrm{Hom}(M, X)$ as a K-module and looking at the map $f : M^n \to N$ given by

$$f(m_1, m_2, ..., m_n) = f_1 m_1 + f_2 m_2 + ... + f_n m_n.$$

By construction, the exact sequence

$$0 \to T \xrightarrow{g} M^n \xrightarrow{f} N$$

induces an exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(M, T) \xrightarrow{\mathrm{Hom}_A(M,g)} \mathrm{Hom}_A(M, M^n) \xrightarrow{\mathrm{Hom}_A(M,f)} \mathrm{Hom}_A(M, N),$$

when we apply the left exact functor $\mathrm{Hom}_A(M, -)$. Take $y \in \mathrm{Hom}_A(M, N)$, since $f_1, f_2, ..., f_n$ is generators we may write

$$y = \sum r_i f_i, \ r_i \in R, \ f_i \in \mathrm{Hom}_A(M, N).$$

We have the commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\ y\ } & N \\ & {\scriptstyle \exists \ x} \searrow & \uparrow {\scriptstyle f} \\ & & M^n, \end{array}$$

where $fx = y$. We can choose $x = (r_1 id_M, r_2 id_M, ..., r_n id_M)$, so $\mathrm{Hom}_A(M, f)$ is surjective. By comparing dimension with the previous sequence we see that the induced sequence

$$0 \longrightarrow \mathrm{Hom}_A(N, T) \xrightarrow{\mathrm{Hom}_A(N,g)} \mathrm{Hom}_A(N, M^n) \xrightarrow{\mathrm{Hom}_A(N,f)} \mathrm{Hom}_A(N, N) \longrightarrow 0,$$

has to be exact too. Since $\text{Hom}_A(N, f)$ is surjective we have that $\forall\, h \in \text{Hom}_A(N, N), \exists x \in \text{Hom}_A(N, M^n)$ such that $h = \text{Hom}_A(N, f)(x)$, i. e. $h = fx$. We have that $\exists\, x$, such that $id_N = fx$, therefore f splits and N is a direct summand of M^n.

The theorem of Krull-Remak-Schmidt (see Theorem 7.5 chapter 10 in [9]) asserts that in the case where a module M is of finite length, it decompose into indecomposable summands and the decomposition is unique up to isomorphism and ordering of summands. When the ring is a field as in our case K, finite dimension implies finite length. Since the Krull-Remak-Schmidt theorem holds in $mod\, A$, we can conclude that N and M have a non-zero indecomposable summand in common, since N is a direct summand of M^n.

The proof of the theorem proceeds by induction on the dimension of $\text{Hom}_A(N, N)$. The case when $\text{Hom}_A(N, N) = 0$ is trivial. Let us denote the common indecomposable direct summand of M and N by $U \neq (0)$, such that $M = M' \oplus U$ and $N = N' \oplus U$. We have that

$$dim_K(\text{Hom}_A(M' \oplus U, X)) = dim_K(\text{Hom}_A(N' \oplus U, X)), \forall X.$$

We can cancel $dim_K(\text{Hom}_A(U, X))$ on both sides and if N' and M' differ from the zero-module we can repeat all the argument to obtain another common non-zero indecomposable direct summand. If not, the assertion follows. Since $dim_K(\text{Hom}_A(N', N')) < dim_K(\text{Hom}_A(N, N))$, we get by induction that M' and N' are isomorphic, therefore $M \cong N$.

If $M \cong N$ then trivially $dim_K(\text{Hom}_A(N, X)) = dim_K(\text{Hom}_A(M, X)), \forall\, X$.

$$QED$$

Recall the Hom-order (see [16]):

$$N \preceq_{hom} M :\Leftrightarrow dim_K(\text{Hom}_A(M, X)) \leq dim_K(\text{Hom}_A(N, X)), \text{ for all } A\text{-modules } X.$$

Lemma 1.4.2 $\preceq_{hom}$ *is a partial order on the set of isomorphism classes of d-dimensional modules.*

Proof: For $\preceq_{hom}$ to be a partial order on the set of isomorphism classes of d-dimensional modules, it has to be reflexive, antisymmetric and transitive.

(i) Reflexive:
$$dim_K(\text{Hom}_A(M, X)) \leq dim_K(\text{Hom}_A(M, X)), \ \forall\, X,$$

$\preceq_{hom}$ is reflexive.

(ii) Antisymmetric: To be antisymmetric in this context means that if

$$dim_K(\text{Hom}_A(M, X)) \leq dim_K(\text{Hom}_A(N, X))$$

and

$$dim_K(\text{Hom}_A(N, X)) \leq dim_K(\text{Hom}_A(M, X)), \ \forall\, X,$$

then $M \cong N$. This follows by Proposition 1.4.1.

(iii) Transitive: If

$$dim_K(\text{Hom}_A(M, X)) \leq dim_K(\text{Hom}_A(N, X)), \ \forall\, X,$$

and

$$dim_K(\text{Hom}_A(N, X)) \leq dim_K(\text{Hom}_A(L, X)), \ \forall\, X,$$

then trivially

$$dim_K(\text{Hom}_A(M, X)) \leq dim_K(\text{Hom}_A(L, X)), \ \forall\, X.$$

$$QED$$

We also recall a generalization of degeneration called virtual degeneration denoted by $\preceq_{vdeg}$[10]. In general one cannot cancel common direct summands from M and N when $M \preceq_{deg} N$ (see example due to J. Carlson [10]), and obtain a degeneration of the remaining complements. But if there exist an X such that $M \oplus X \preceq_{deg} N \oplus X$, one says that $M \preceq_{vdeg} N$. One can choose X to be the zero module, so obviously $\preceq_{deg} \Rightarrow \preceq_{vdeg}$.

1.5   Zwara's Theorems

The main point of this section will be to present two very important theorems. Both are in the generality presented here, due to Zwara. We omit the proofs of both theorems because they are considered to be too long for this text.

We start with a proposition which proves one of the implications in Theorem 1.5.2 of this section. A more general version of the following proposition is due to Riedtmann [10]. In [10] the algebra is assumed to be a finitely generated K-algebra.

Proposition 1.5.1 *Let A be a finite dimensional K-algebra. If there exists a short exact sequence*

$$0 \longrightarrow Z \longrightarrow Z \oplus M \longrightarrow N \longrightarrow 0$$

of A-modules with Z, M and N finitely dimensional as A-modules, then $dim(M) = dim(N)$ and $M \preceq_{deg} N$.

Proof: We see that $dim(Z \oplus M) = dim(Z) + dim(N)$, and since dimension is additive it is clear by cancelation of $dim(Z)$ on both sides that $dim(M) = dim(N)$.

Next we consider a short exact sequence

$$0 \longrightarrow Z \xrightarrow{h = \binom{f}{g}} Z \oplus M \longrightarrow N \longrightarrow 0,$$

where $f : Z \longrightarrow Z$ and $g : Z \longrightarrow M$ are A-homomorphisms. Now consider for each λ in K the short exact sequence:

$$0 \longrightarrow Z \xrightarrow{h_\lambda = \binom{f - \lambda I_Z}{g}} Z \oplus M \longrightarrow N_\lambda \longrightarrow 0,$$

of A-modules where I_Z is the identity on Z and N_λ is the cokernel of the A-homomorphism h_λ. By change of the the basis of $Z \oplus M$ as a vector space we can obtain the short exact sequence

$$0 \to Z \xrightarrow{\binom{r}{s}} Im(h) \oplus C \xrightarrow{(u \;\; v)} N \longrightarrow 0,$$

where C is a vector space complement of $Im(h)$. Where $s = 0$. Since the sequence is exact and $\binom{r}{0}$ is injective, i.e. r is invertible, we have

$$\begin{pmatrix} u & v \end{pmatrix} \begin{pmatrix} r \\ 0 \end{pmatrix} = 0 \Rightarrow ur = 0 \Rightarrow u = 0.$$

Without loss of generality we may assume that $r = id$ and $v = id$. Also notice that $N \cong C$ as a vector space. Now consider the short exact sequence

$$0 \to Z \xrightarrow{h_\lambda = \binom{r_\lambda}{s_\lambda}} Im(h) \oplus N \xrightarrow{(u \;\; id)} N \longrightarrow 0,$$

where r_λ is invertible. We see that if $a \neq 0$ then $\begin{pmatrix} r_\lambda(a) \\ s_\lambda(a) \end{pmatrix} \neq \begin{pmatrix} 0 \\ s_\lambda(a) \end{pmatrix}$, hence

$$r_\lambda \text{ invertible} \Leftrightarrow Im(h_\lambda) \cap N = (0).$$

If r_λ is not invertible then $det\ (r_\lambda) = 0$. First we must check that there exist a λ such that $det(r_\lambda) \neq 0$. For $\lambda = 0$ we have $det\ (r) \neq 0$, so such a λ do exist. Let $K' = \{K \backslash \text{zeros of } det(r_\lambda)\}$. Since $det(r_\lambda)$ is a polynomial in λ and $det(r_\lambda)$ has a finite number of zeros, we conclude that there is a finite number of λ's for which r_λ is non invertible.

Now to see how the module structure on N vary as a function of λ consider the following commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Z & \xrightarrow{h_\lambda = \begin{pmatrix} r_\lambda \\ s_\lambda \end{pmatrix}} & Im(h) \oplus N & \xrightarrow{(u \ \ id)} & N & \longrightarrow & 0 \\
& & \downarrow & & \downarrow \begin{pmatrix} z & \delta \\ 0 & n \end{pmatrix} & & \downarrow n^\lambda & & \\
0 & \longrightarrow & Z & \longrightarrow & Im(h) \oplus N & \longrightarrow & N & \longrightarrow & 0
\end{array}
$$

The exactness of the sequence gives us

$$ur_\lambda + s_\lambda = 0 \Rightarrow u = -s_\lambda r_\lambda^{-1},$$

which gives us that u is a rational function of λ.

We know that there is a correspondence between N and a tuple $n = (n_1, ..., n_t) \in mod_A(d)$ (as with M and a tuple $m = (m_1, ..., m_t)$) and by the commutativity property of the diagram we can give an explicit formula for how the structure of N, and hence the tuple vary with λ. We have

$$n^\lambda(x) = \begin{pmatrix} u & id \end{pmatrix} \begin{pmatrix} z & \delta \\ 0 & n \end{pmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix} = \begin{pmatrix} -s_\lambda r_\lambda^{-1} & id \end{pmatrix} \begin{pmatrix} z & \delta \\ 0 & n \end{pmatrix} \begin{pmatrix} 0 \\ x \end{pmatrix} = (-s_\lambda r_\lambda^{-1} \delta x + nx),$$

so the function ζ describing the structure of N as a function of λ is defined by:

$$\zeta : K' \to mod_A(d),$$

$$\lambda \mapsto (-s_\lambda r_\lambda^{-1} \delta + n).$$

Let us return to the following sequence:

$$0 \longrightarrow Z \xrightarrow{h_\lambda = \begin{pmatrix} f - \lambda I_Z \\ g \end{pmatrix}} Z \oplus M \longrightarrow N_\lambda \longrightarrow 0.$$

For all $\lambda \neq 0$ not an eigenvalue of f, h_λ is a split monomorphism, thus

$$N_\lambda \cong M.$$

Let $n^\lambda = (n_1^\lambda, ..., n_t^\lambda)$ denote the t-tuple corresponding to N_λ and

$$V = \{n^\lambda = (n_1^\lambda, ..., n_t^\lambda) \mid r_\lambda \text{ invertible and } \lambda \text{ not an eigenvalue of } f\}.$$

Since the inverse of a matrix can be obtained by cofactor expansion the function $\zeta : K' \to mod_A(d)$, $\lambda \longrightarrow (-s_\lambda r_\lambda^{-1} \delta + n)$, gives us that the coefficients in the matrices of the t-tuple n^λ must be on the form $\frac{p(\lambda)}{det(r_\lambda)}$, where $det(r_\lambda) \neq 0$. Let

$$\phi : mod_A(d) \longrightarrow K$$

be a polynomial in the coefficient of the matrices such that $\phi(n^\lambda) = 0$, when r_λ is invertible and λ is not an eigenvalue of f. Since the denominators of all the coefficients in the matrix is either 1 or $det\ r_\lambda$, we have that there exist a $t \in \mathbb{N}$ such that $\phi(n^\lambda) = \frac{v(n^\lambda)}{(det(r_\lambda))^t} = 0 \Leftrightarrow v(n^\lambda) = 0$. Since

$v(n^\lambda) = 0$ for all except a finite number of λ's, we have that $v(n^\lambda) = 0$, $\forall\ \lambda \in K$. In particular we have that $\phi(n^0) = 0$. We also have that

$$V = \{n^\lambda \mid r_\lambda \text{ is invertible and } \lambda \text{ is not an eigenvalue of } f\} \subseteq O(m),$$

which implies

$$\overline{V} \subseteq \overline{O(m)}.$$

To complete the proof we see that

$$n = n^0 \in \overline{V} \subseteq \overline{O(m)} \Rightarrow M \preceq_{deg} N.$$

$$QED$$

We now state the first theorem of this section.

Theorem 1.5.2 (Zwara) *Let A be a finite dimensional K-algebra and let M and N be finite dimensional A-modules. Then the following conditions are equivalent:*

(1) $M \preceq_{deg} N$.
(2) There is a short exact sequence $0 \longrightarrow N \longrightarrow M \oplus Z \longrightarrow Z \longrightarrow 0$ in mod A for some module Z in mod A.
(3) There is a short exact sequence $0 \longrightarrow Z \longrightarrow Z \oplus M \longrightarrow N \longrightarrow 0$ in mod A for some module Z in mod A.

Proof: $(3) \Rightarrow (1)$: See Proposition 1.5.1.
$(2) \Rightarrow (1)$: Follows by dual arguments.
For the rest of the proof, the reader is referred to [17].

$$QED$$

For a more general version of this theorem the interested reader is referred to [15]. Let us recall a definition:

Definition 1.5.3 *An algebra A_f is of finite representation type if the category mod A_f contains only finitely many isomorphism classes of indecomposable modules.*

We now state what will be a crucial theorem for the theory developed later in this thesis.

Theorem 1.5.4 (Zwara) *If A_f is a finite dimensional K-algebra of finite representation type, and M and N are two A_f-modules of the same dimension as K-modules, then the three following statements are equivalent:*

(1) $M \preceq_{deg} N$
(2) $M \preceq_{vdeg} N$
(3) $M \preceq_{hom} N$

Proof: $(1) \Rightarrow (2)$ is obvious, one can always take an extra summand $X = 0$ and obtain virtual degeneration.

$(2) \Rightarrow (3)$ By Theorem 1.5.2 we have that to assume that $M \preceq_{vdeg} N$ is equivalent to assuming that there is an exact sequence of the form

$$0 \longrightarrow Y \longrightarrow Y \oplus Z \oplus M \longrightarrow Z \oplus N \longrightarrow 0,$$

for some module Z in *mod* A_f. Let X be a A_f-module which has finite dimension as a K-module. We then apply the functor $Hom_{A_f}(-, X)$ to this sequence and obtain the following exact sequence

$$0 \longrightarrow \mathrm{Hom}_{A_f}(Z \oplus N, X) \longrightarrow \mathrm{Hom}_{A_f}(Y \oplus Z \oplus M, X) \longrightarrow \mathrm{Hom}_{A_f}(Y, X),$$

of K-modules. Then by counting dimension as K-modules, one obtains the inequality:

$$dim_K(\mathrm{Hom}_{A_f}(Y \oplus Z \oplus M, X)) \leq dim_K(\mathrm{Hom}_{A_f}(Z \oplus N, X)) + dim_K(\mathrm{Hom}_{A_f}(Y, X)).$$

This gives:

$$dim_K(\text{Hom}_{A_f}(Y,X)) + dim_K(\text{Hom}_{A_f}(Z,X)) + dim_K(\text{Hom}_{A_f}(M,X)) \leq$$

$$dim_K(\text{Hom}_{A_f}(N,X)) + dim_K(\text{Hom}_{A_f}(Z,X)) + dim_K(\text{Hom}_{A_f}(Y,X)).$$

We can now subtract $(dim_K(\text{Hom}_{A_f}(Z,X)) + dim_K(\text{Hom}_{A_f}(Y,X)))$ from each side of this inequality and obtain the desired result $dim_K(\text{Hom}_{A_f}(M,X)) \leq dim_K(\text{Hom}_{A_f}(N,X))$ for each A_f module X which has finite dimension as a K-module. By definition $dim_K(\text{Hom}_{A_f}(M,X)) \leq dim_K(\text{Hom}_{A_f}(N,X))$ implies $M \preceq_{hom} N$.

$(3) \Rightarrow (1)$ For the rest of the proof, the reader is referred to [12]. The version of the proof in [12] is more general. One do not need K to be an algebraically closed field, it is sufficient to assume that K is a commutative artin ring.

$$QED$$

When one works with algebras of finite representation type the Hom-order is equivalent to the degeneration-order (see Theorem 1.5.4). The AR-quiver in general gives all the isomorphism classes of indecomposable modules in a category (see [1]). When the category is of finite representation type it is no problem to number the indecomposable modules. If we fix a numbering, we associate a fixed indecomposable module which we denote by idb_i to the ith isomorphism class of indecomposable modules. The t-dimensional vector H^x determined by

$$H^x(i) = dim_K\text{Hom}_{A_f}(idb_i, X), 1 \leq i \leq t,$$

where t is the number of isomorphism classes of indecomposable modules in the algebra, gives us a vector by which we can decide where X is in the Hasse diagram of equal dimensional modules in the same category. There will be given several examples and applications of this in Chapter 2.

1.6 The Tensor-Order $\preceq_{\otimes}$

The *tensor*-order is defined in the following way:

$$N \preceq_{\otimes} M :\Leftrightarrow dim_K(X \otimes_A N) \leq dim_K(X \otimes_A M), \ \forall \ X,$$

where N and M are left A-modules and X is a finite dimensional right A-module.

We shall see that the *tensor*-order is equivalent to the Hom-order:

Let R be a k-algebra over some commutative ring k and consider the category Mod R of (left) R-modules. We fix an injective k-module I and denote by $D_I = \text{Hom}_k(-, I)$ the corresponding functor Mod $k \to$ Mod k.

A short version of this proof can be found in [7].

Proposition 1.6.1 *Let R be an k-algebra over some commutative ring k. Let X be an R^{op}-module and Y be an R-module. Then there is an isomorphism*

$$D_I(X \otimes_R Y) \cong \text{Hom}_R(X, D_I Y), \tag{1.1}$$

which is functorial in X and Y.

Proof: We know from homological algebra that both $\text{Hom}_R(-, D_I Y)$ and $D_I(- \otimes_R Y)$ are both left exact contravariant functors.

It is also well known that in general $\mathrm{Hom}_R(-, B)$ converts sums into products. In other words: if λ_i is the ith injection $X_j \longrightarrow \coprod X_i$ and if B is a module, then the map

$$\Phi : \mathrm{Hom}(\coprod X_i, B) \longrightarrow \prod \mathrm{Hom}(X_i, B)$$

given by $\phi \longrightarrow (\phi\lambda_i)$ is an isomorphism.

Thus it is obvious that the functor $\mathrm{Hom}_R(-, D_I Y)$ converts sums to product, but so does $D_I(-\otimes_R Y)$, this is due to $D_I(-\otimes_R Y) = \mathrm{Hom}_k(-\otimes_R Y, k)$. We also know that in general tensor products commutes with direct sums.

Notice that $D_I(R \otimes_R Y) = D_I Y$ and $\mathrm{Hom}_R(R, D_I Y) = D_I Y$, we say that the functors coincide on R.

In the final step of the proof we then take a free presentation of X: $F' \longrightarrow F \longrightarrow X \longrightarrow 0$, and apply both functors, obtaining:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & D_I(X \otimes_R Y) & \longrightarrow & D_I(F \otimes_R Y) & \longrightarrow & D_I(F' \otimes_R Y) \\
& & \| & & \| & & \| \\
0 & \longrightarrow & \mathrm{Hom}_R(X, D_I Y) & \longrightarrow & \mathrm{Hom}_R(F, D_I Y) & \longrightarrow & \mathrm{Hom}_R(F', D_I Y)
\end{array}
$$

Which is a commutative diagram, due to the fact that we can choose $F = R^n$, $F' = R^m$, and

$$D_I(F \otimes_R Y) \cong D_I(\coprod R \otimes_R Y) \cong \prod D_I(R \otimes_R Y) \cong \prod D_I Y$$

also

$$\mathrm{Hom}_R(F, Y) \cong \mathrm{Hom}_R(\coprod R, Y) \cong \mathrm{Hom}_R(\coprod R, Y) \cong \prod D_I Y,$$

similar for F'.

The assertion follows by the Five Lemma (see [11]).

$$QED$$

The proof above is valid if we replace $D_I = \mathrm{Hom}_k(-, I)$ by the duality $D_K = \mathrm{Hom}_K(-, K)$.

If we are considering finite dimensional modules, we know that an isomorphism preserves dimension.

If we use the dual $D_K = \mathrm{Hom}_K(-, K)$ we obtain:

$$dim_K(X \otimes_R Y) = dim_K \mathrm{Hom}_R(X, D_K Y),$$

Corollary 1.6.2 *Let A_f be a finite dimensional K-algebra of finite representation type, and M and N are two A_f-modules of the same dimension as K-modules, then the following four statements are equivalent:*

(1) $M \preceq_{deg} N$
(2) $M \preceq_{vdeg} N$
(3) $M \preceq_{hom} N$
(4) $M \preceq_{\otimes} N$

Proof: The fact that (1), (2) and (3) are equivalent statements is Theorem 1.5.4. We will prove that (3) and (4) is equivalent. Let X be a finite dimensional right A_f-module. It follows from Proposition 1.6.1 that
$$dim_K(X \otimes_{A_f} M) = dim_K \mathrm{Hom}_{A_f}(X, D_K M).$$

Since $D_K = \mathrm{Hom}_K(-, K)$ is a duality, we have that

$$dim_K \mathrm{Hom}_{A_f}(X, D_K M) = dim_K \mathrm{Hom}_{A_f}(D_K^2 M, D_K X) = dim_K \mathrm{Hom}_{A_f}(M, D_K X).$$

So

$$dim_K(X \otimes_{A_f} M) = dim_K \mathrm{Hom}_{A_f}(M, D_K X).$$

As X varies over all the isomorphism classes of indecomposable finite dimensional left A_f-modules, so will $D_K X$ vary over all the isomorphism classes of indecomposable finite dimensional right A_f^{op}-modules. Thus the $\preceq_\otimes$-order will give the same partial order as the $\preceq_{hom}$-order.

QED

Chapter 2

Degeneration on Quivers of Finite Representation Type

In this chapter we first recall some classic representation theory in Section 2.1. We then develop a method for determining degeneration on quivers of finite representation type in Section 2.2. We define two types of matrices H^Q and T^Q, which will determine degeneration. In Section 2.4 we analyze the structure of these matrices and see how they are related.

2.1 Gabriel's Theorem and the Coxeter functors

Let us recall two definitions:

Definition 2.1.1 *A quiver Q is of finite representation type if the category $Rep_K(Q)$ contains only finitely many isomorphism classes of indecomposable objects.*

Definition 2.1.2 *For a quiver Q we define $\Gamma(Q)$ to be the graph we get when we forget the orientation of the arrows in Q.*

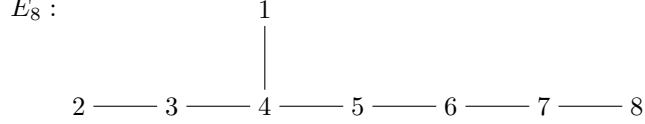We shall now state but not prove what is called Gabriel's Theorem, a classic result!

Theorem 2.1.3 (Gabriel, 1972, [6]) *A quiver Q is of finite representation type if and only if the graph $\Gamma(Q)$ is a Dynkin quiver, i.e. $\Gamma(Q)$ is one of the following graphs.*

$$A_n: \quad 1 \text{——} 2 \text{——} \cdots \text{——} n \ , 1 \leq n$$

$$D_n: \quad \begin{array}{c} 1 \\ | \\ 2 \text{——} 3 \text{——} \cdots \text{——} n \end{array} \quad , 4 \leq n$$

$$E_6: \quad \begin{array}{c} 1 \\ | \\ 2 \text{——} 3 \text{——} 4 \text{——} 5 \text{——} 6 \end{array}$$

$$E_7: \quad \begin{array}{c} 1 \\ | \\ 2 \text{——} 3 \text{——} 4 \text{——} 5 \text{——} 6 \text{——} 7 \end{array}$$

E_8 :

$$
\begin{array}{c}
1 \\
| \\
2 \text{ —— } 3 \text{ —— } 4 \text{ —— } 5 \text{ —— } 6 \text{ —— } 7 \text{ —— } 8
\end{array}
$$

In particular the property of being representation finite does not depend on the orientation of the edges.

For a more thorough and detailed elaboration on the theory in this section, the reader is referred to [13]. We have in the following introduction to the partial Coxeter functors and the Coxeter functors adopted much of the notation and structure from there.

Let Q_Ω denote the quiver Q with orientation Ω on the arrows. By $Q_{\Omega'}$ we mean the quiver with the same underlying graph, i. e. $\Gamma(Q_\Omega) = \Gamma(Q_{\Omega'})$, but with, possibly, a different orientation Ω' on the arrows.

To elucidate how the structural properties of $Rep_K(Q_\Omega)$ is connected to the properties of $Rep_K(Q_{\Omega'})$, we can recall the partial Coxeter functors and the Coxeter functors. These functors leads to an equivalence of categories, which will turn out useful in our work on determining degeneration on $Rep_K(Q)$, where Q is of finite representation type.

Given $Q_\Omega = (Q_0, Q_1, s, t)$ and a vertex $v \in Q_0$, denote by

$$
\Sigma_v = \{\alpha \in Q_1 \mid s(\alpha) = v \text{ or } t(\alpha) = v\}.
$$

We say that a vertex $v \in Q_0$ is a sink if $\{\alpha \in Q_1 \mid s(\alpha) = v\} = \varnothing$, and a source if $\{\alpha \in Q_1 \mid t(\alpha) = v\} = \varnothing$. So no arrow is ending in a source and no arrow is starting in a sink.

For a sink v of a quiver Q_Ω there exist a left partial Coxeter functor

$$
C_v^+ : \ Rep_K(Q_\Omega) \longrightarrow Rep_K(Q_{\Omega'}),
$$

where $Q_{\Omega'}$ obtained from Q_Ω by shifting the orientation on all the arrows ending in v. We define C_v^+ by the following. For $X \in Rep_K(Q_\Omega)$, let $C_v^+(X) = Y$, where $X_i = Y_i$, for all $i \neq v$, and Y_v is the kernel of the rightmost map in the following sequence

$$
0 \longrightarrow Y_v \xrightarrow{\ i\ } \coprod_{\alpha \in \Sigma_v} X_{s(\alpha)} \xrightarrow{(X_\alpha)} X_v. \tag{2.1}
$$

We have $Y_\beta = X_\beta : X_{s(\beta)} \to X_{t(\beta)}$, for all $\beta \notin \Sigma_v$ and $Y_\alpha = \pi_\alpha i : Y_{s(\alpha)} \to Y_{t(\alpha)}$, the composition of the natural embedding of Y_v into $\coprod_{\alpha \in \Sigma_v} X_{s(\alpha)}$ and the projection of this sum onto the term $X_{s(\alpha)}$, for each $\alpha \in \Sigma_v$. Note that $C_v^+(S_v) = 0$, where S_v is the simple indecomposable representation corresponding to the vertex v.

Let $h : \ X \longrightarrow X'$ be a morphism in $Rep_K(Q_\Omega)$. Then $C_v^+(h) : \ C_v^+(X) \longrightarrow C_v^+(X')$, where $C_v^+(h_i) = h_i$ for all $i \neq v$ and $C_v^+(h_v)$ is the unique morphism which makes the following diagram commute.

$$
\begin{array}{ccccccc}
0 & \longrightarrow & Y_v & \xrightarrow{\ i\ } & \coprod_{\alpha \in \Sigma_v} X_{s(\alpha)} & \xrightarrow{(X_\alpha)} & X_v \\
& & \Big\downarrow{\scriptstyle C_v^+(h_v)} & & \Big\downarrow{\scriptstyle h_{s(\alpha)}} & & \Big\downarrow{\scriptstyle h_v} \\
0 & \longrightarrow & Y_v' & \xrightarrow{\ i'\ } & \coprod_{\alpha \in \Sigma_v} X_{s(\alpha)}' & \xrightarrow{(X_\alpha')} & X_v'
\end{array}
$$

For a source v of a quiver Q_Ω there exist a right partial Coxeter functor

$$
C_v^- : \ Rep_K(Q_\Omega) \longrightarrow Rep_K(Q_{\Omega'}),
$$

where $Q_{\Omega'}$ differs from Q_Ω by that it has the shifted orientation on all the arrows starting in v. We define C_v^- by the following. For $X \in Rep_K(Q_\Omega)$, let $C_v^-(X) = Y$, where $X_i = Y_i$, for all $i \neq v$, and Y_v is the cokernel of the leftmost map in the following sequence

$$
X_v \xrightarrow{(X_\alpha)} \coprod_{\alpha \in \Sigma_v} X_{t(\alpha)} \xrightarrow{\ \pi\ } Y_v \longrightarrow 0. \tag{2.2}
$$

We have $Y_\beta = X_\beta : X_{s(\beta)} \to X_{t(\beta)}$, for all $\beta \notin \Sigma_v$ and $Y_\alpha = \pi i_\alpha : Y_{s(\alpha)} \to Y_{t(\alpha)}$, the composition of the embedding of $X_{t(\alpha)}$ into $\coprod_{\alpha \in \Sigma_v} X_{t(\alpha)}$ and π is the natural projection of $\coprod_{\alpha \in \Sigma_v} X_{t(\alpha)}$ onto the term Y_v, for each $\alpha \in \Sigma_v$. Note that $C_v^-(S_v) = 0$, where S_v is the simple indecomposable representation corresponding to the vertex v.

If $h : X \longrightarrow X'$ is a morphism in $Rep_K(Q_\Omega)$. Then $C_v^-(h) : Y_v \longrightarrow Y_v'$ is a morphism in $Rep_K(Q_{\Omega'})$, where $C_v^-(h_i) = h_i$ for all $i \neq v$ and $C_v^-(h_v)$ is the unique morphism which makes the following diagram commute.

$$
\begin{array}{ccccccc}
X_v & \xrightarrow{(X_\alpha)} & \coprod_{\alpha \in \Sigma_v} X_{s(\alpha)} & \xrightarrow{\pi} & Y_v & \longrightarrow & 0 \\
\downarrow{\scriptstyle h_v} & & \downarrow{\scriptstyle h_{s(\alpha)}} & & |\ {\scriptstyle C_v^-(h_v)} & & \\
X_v' & \xrightarrow{(X_\alpha')} & \coprod_{\alpha \in \Sigma_v} X_{s(\alpha)}' & \xrightarrow{\pi} & Y_v' & \longrightarrow & 0
\end{array}
$$

Theorem 2.1.4 *The functors*

$$C_v^+ \colon Rep_K(Q_\Omega)/S_v \to Rep_K(Q_{\Omega'})/S_v \;\; (\text{if } v \text{ is a sink})$$

and

$$C_v^- \colon Rep_K(Q_\Omega)/S_v \to Rep_K(Q_{\Omega'})/S_v \;\; (\text{if } v \text{ is a source})$$

are both equivalences of categories.

Proof: For the proof we refer the reader to [2].

$$QED$$

Theorem 2.1.5 *Let Q be a quiver and let $\Gamma(Q)$ be the underlying graph with no cycles, let Ω and Ω' be two orientations of it. Then there exists a sequence of vertices such that*

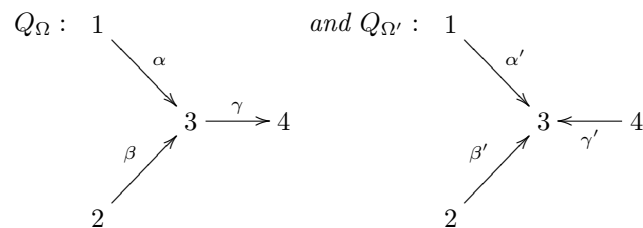$$C_{v_k}^* ... C_{v_1}^* : \; Rep_K(Q_\Omega) \to Rep_K(Q_{\Omega'}),$$

where each $C_{v_i}^*$, $1 \leq i \leq k$, is either a right partial Coxeter functor or a left partial Coxeter functor.

Proof: We refer the reader to [13].

$$QED$$

Let us illustrate how we can use the partial Coxeter functors by an example of categories which will be revisited later and work as our crown example for the theory we are developing in this chapter.

Example 2.1.6 *Let*

$$
Q_\Omega : \quad 1 \qquad\qquad and\ Q_{\Omega'} : \quad 1
$$

with quiver Q_Ω having vertex 1 mapping via α to 3, vertex 2 mapping via β to 3, and $3 \xrightarrow{\gamma} 4$; and $Q_{\Omega'}$ having vertex 1 mapping via α' to 3, vertex 2 mapping via β' to 3, and $3 \xleftarrow{\gamma'} 4$.

Note that $C_v^+(V \oplus V') \cong C_v^+(V) \oplus C_v^+(V')$, so it is enough to analyze what the functors do with the indecomposable representations of the category representing the isomorphism classes.
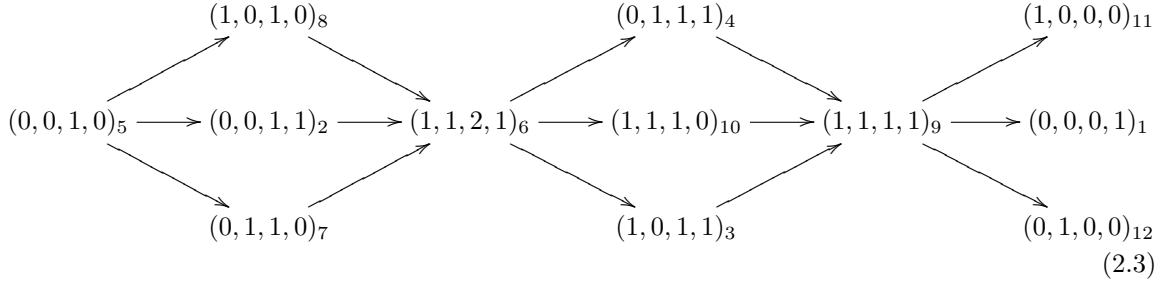
Further note that vertex 4 is a source of the quiver $Q_{\Omega'}$. We are going to apply

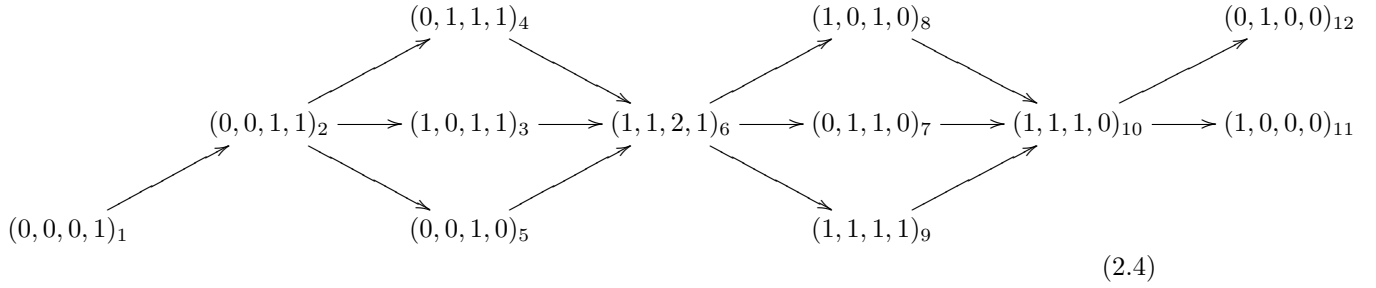$$C_4^- : \; Rep_K(Q_{\Omega'}) \longrightarrow Rep_K(Q_\Omega).$$

We start by computing the AR-quivers of the categories. The indecomposable representations are represented by their corresponding dimension vectors. Notice that the numbering is fixed in the two categories with respect to the dimension vectors of the indecomposable representations. This is done

to illustrate what happens when we apply the partial Coxeter functors, and is not something we shall do throughout this thesis.
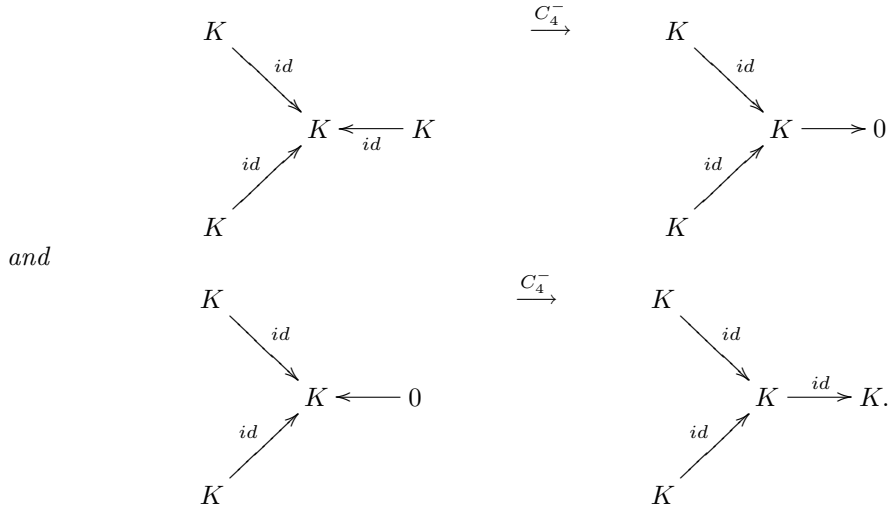
$Rep_K(Q_{\Omega'})$:

$$
\begin{array}{ccccccccccc}
& (1,0,1,0)_8 & & & (0,1,1,1)_4 & & & (1,0,0,0)_{11} & \\
& \nearrow \quad \searrow & & \nearrow & & \searrow & & \nearrow & \\
(0,0,1,0)_5 \rightarrow (0,0,1,1)_2 \rightarrow (1,1,2,1)_6 \rightarrow (1,1,1,0)_{10} \rightarrow (1,1,1,1)_9 \rightarrow (0,0,0,1)_1 & \\
& \searrow \quad \nearrow & & \searrow & & \nearrow & & \searrow & \\
& (0,1,1,0)_7 & & & (1,0,1,1)_3 & & & (0,1,0,0)_{12} &
\end{array}
$$

$$\tag{2.3}$$

$Rep_K(Q_{\Omega})$:

$$
\begin{array}{ccccccccccc}
& & (0,1,1,1)_4 & & & (1,0,1,0)_8 & & & (0,1,0,0)_{12} \\
& \nearrow & & \searrow & & \nearrow & & \searrow & \nearrow \\
& (0,0,1,1)_2 \rightarrow (1,0,1,1)_3 \rightarrow (1,1,2,1)_6 \rightarrow (0,1,1,0)_7 \rightarrow (1,1,1,0)_{10} \rightarrow (1,0,0,0)_{11} \\
\nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow \\
(0,0,0,1)_1 & & & (0,0,1,0)_5 & & & (1,1,1,1)_9 &
\end{array}
$$

$$\tag{2.4}$$

By the definition of the right partial Coxeter functor C_4^- we see that for example

$$
\begin{array}{ccc}
K & & K \\
\ \ \searrow id & \xrightarrow{\ C_4^-\ } & \ \ \searrow id \\
K \xleftarrow{id} K & & K \longrightarrow 0 \\
\ \ \nearrow id & & \ \ \nearrow id \\
K & & K
\end{array}
$$

and

$$
\begin{array}{ccc}
K & & K \\
\ \ \searrow id & \xrightarrow{\ C_4^-\ } & \ \ \searrow id \\
K \longleftarrow 0 & & K \xrightarrow{id} K. \\
\ \ \nearrow id & & \ \ \nearrow id \\
K & & K
\end{array}
$$

By the definition of the right partial Coxeter functor we see that C_4^- sends the indecomposable representation with dimension vector number 9 in $Rep_K(Q_\Omega)$ to the indecomposable representation with dimension vector number 10 in $Rep_K(Q_{\Omega'})$, and vice versa. It is straightforward to verify that all the indecomposable representations are switched, with one exception. That is for the simple representation corresponding to vertex 4, $S_4 = (0,0,0,1)_1$. We see that $C_v^-(S_4) = 0$. By the definition of the right partial Coxeter functor we see that 2 is switched by 5, 3 by 8, 4 by 7, 6 by 6, 11 by 11 and 12 by 12. By adding S_4 to the image of C_4^- we obtain the full abelian category $Rep_K(Q_\Omega)$.
$$\triangle$$

2.2 The Structure of a Representation and the Matrix H^Q

We recall from the previous section that a representation with underlying quiver of finite representation type is classified by Gabriel([6]) to be a Dynkin diagram, i. e. $A_n(n \geq 1)$, $D_n(n \geq 4)$, E_6, E_7, E_8. The number of isomorphism classes of indecomposable representation is independent of orientation on the quiver and it is well known to be $\frac{n(n-1)}{2}$, $n(n-1)$, 36, 63 and 120 respectively. We shall determine degeneration as a partial order on the representations with equal dimension vector in the category $Rep_K(Q)$ for all quivers Q of finite representation type. This we will do by first decomposing into indecomposable summands, thus we give a procedure to determine the decomposition of a given representation $X \in Rep_K(Q)$.

Assume that Q is Dynkin and let Λ denote the path algebra KQ (see Section 1.3.1). The corresponding AR-quiver of the category $Rep_K(Q)$ is directed (i. e. contains no loops). Since there is a finite number of isomorphism classes of indecomposable representations in the category, there is no problem indexing the fixed indecomposables representing the isomorphism classes of indecomposable representations in the category from the projectives to the injectives. That is for all $j < i$, $dim_K\mathrm{Hom}_\Lambda(idb_i, idb_j) = 0$. This indexing is not always unique. After indexing the indecomposables it is possible to define the entries of a matrix H^Q by:

$$(i,j) = dim_K\mathrm{Hom}_\Lambda(idb_i, idb_j), \ \forall \ 1 \leq i, \ j \leq m,$$

where m is the number of isomorphism classes of indecomposables in the category $Rep_K(Q)$. We will use that the ith row vector of H^Q, denoted r_i, is given by:

$$r_i(j) = dim_K\mathrm{Hom}_\Lambda(idb_i, idb_j)_j,$$

where $1 \leq j \leq m$.

The matrix H^Q will be upper triangular since the AR-quiver is directed (i. e. contains no loops). Also, since the AR-quiver is directed there will always be ones on the diagonal since there does not exist a path in the AR-quiver from idb_i to idb_i (see [1]). The ones on the diagonal corresponds to the identity morphisms, thus these matrices will always be invertible and the determinant will always be one. The matrix H^Q is used to determine the structure of a representation X in $\mathrm{Rep}(Q)$. This can be done by first inverting H^Q and then compute the vector

$$H^X = dim_K\mathrm{Hom}_\Lambda(idb_i, X), \ \forall \ 1 \leq i \leq m,$$

where m is the number of isomorphism classes of indecomposables in $Rep_K(Q)$. One should now multiply the vector H^X by $(H^Q)^{-1}$. The result is a vector giving the multiplicity of idb_i as a summand of X. We have that if $H^X \leq H^Y$, where X and Y are representations with equal dimension vector in $Rep_K(Q)$, then $X \preceq_{deg} Y$, this follows from Theorem 1.5.4.

Let us now have an example to illustrate how this method works:

Example 2.2.1 *Let Q be the oriented quiver:*

$$(2.4)$$

We compute the matrix H^Q, where

$$(i,j) = dim_K\mathrm{Hom}_\Lambda(idb_i, idb_j), \ \forall \ 1 \leq i, \ j \leq 12.$$

So

$$H^Q = \begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

Let X be the following representation:

$$K \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}} \quad K^2 \xrightarrow{\begin{pmatrix}1&0\\0&1\end{pmatrix}} K^2, \quad K \xrightarrow{0}$$

we get the vector

$$H^X = \begin{pmatrix} 2 & 2 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}^{tr}$$

By elementary linear algebra one obtains

$$(H^Q)^{-1} = \begin{pmatrix}
1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

So the corresponding row operations on H^X gives:

$$\overline{X} = (H^Q)^{-1}H^X = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{tr}$$

Which is equivalent to say that

$$X \cong idb_2 \oplus idb_3 \oplus idb_{12}.$$

To relate this to degeneration which is our main focus, let us compare X with two other representations with the same dimension vector with respect to degeneration. Let T be:

$$K \xrightarrow{0} \quad K^2 \xrightarrow{0} K^2. \quad K \xrightarrow{\begin{pmatrix}1\\0\end{pmatrix}}$$

One obtains

$$H^T = \begin{pmatrix} 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}^{tr}.$$

Let S be:

One obtains

$$H^S = \begin{pmatrix} 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 \end{pmatrix}^{tr}.$$

So with respect to degeneration, X and T are incomparable, but S is a degeneration of both X and T. So in the Hasse diagram (see Subsection 1.1.2) of the representations with dimension vector $\mathbf{d} = (1, 1, 2, 2)$ in $Rep_K(Q)$ X, T and S would be ordered like this:

One can also multiply H^T and H^S by $(H^Q)^{-1}$ from the left, this would give:

$$\overline{T} = (H^Q)^{-1} H^T = \begin{pmatrix} 2 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^{tr}$$

and

$$\overline{S} = (H^Q)^{-1} H^S = \begin{pmatrix} 2 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}^{tr}$$

.

Which is equivalent to say that

$$T \cong idb_1 \oplus idb_1 \oplus idb_5 \oplus idb_7 \oplus idb_{12}$$

and

$$S \cong idb_1 \oplus idb_1 \oplus idb_5 \oplus idb_5 \oplus idb_{11} \oplus idb_{12}.$$

$\triangle$

Let Q_Ω and $Q_{\Omega'}$ be as in example 2.1.6. We apply

$$C_4^+ : \ Rep_K(Q_\Omega) \longrightarrow Rep_K(Q_{\Omega'}).$$

We shall analyze the relation between H^{Q_Ω} and $H^{Q_{\Omega'}}$. First notice that by the definition of the partial Coxeter functors we see that C_4^+ makes all the same changes of indecomposables in the category as C_4^-. That is: S_4 is sent to 0, 2 is switched by 5, 3 by 8, 4 by 7, 6 by 6, 9 by 10, 11 by 11 and 12 by 12. We add S_4 to the image of C_4^+ to obtain the full abelian category $Rep_K(Q_{\Omega'})$.

We obtain the matrix $H^{Q_{\Omega'}}$, where we use the same indexing of the dimension vectors of the indecomposables in the category $Rep_K(Q_{\Omega'})$ as for the category $Rep_K(Q_\Omega)$.

$$H^{Q_{\Omega'}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that the differences between H^{Q_Ω} and $H^{Q_{\Omega'}}$ can be explained by the partial Coxeter functors. A general fact is that when applying C_v^+ to $Rep_K(Q)$ we loose a simple projective representation, just as we loose a simple injective when applying C_v^-. This is obvious from the fact that v is assumed to be either a sink or a source respectively, and S_v must therefore either be a simple projective representation or a simple injective representation. On the other hand if S_v is added to the image of either C_v^+ or C_v^-, it will have the opposite status, since all arrows ending or starting in v are turned. So note that in our case $C_4^+(S_4) = 0$, where $S_4 = idb_1$. So idb_1 is not in the image of C_4^+. When we add it to the image, it is no longer a projective representation, but an injective representation.

Let $S_v = idb_i$. Recall the definition of the partial Coxeter functors effect on the morphisms in $Rep_K(Q)$. We can see that this corresponds to interchanging row i with column i in the matrix H^Q. Notice that in general when one change the ith row with the ith column in a upper triangular invertible matrix such as H^Q, the matrix will remain invertible! In our case, row one is interchanged with column one.

The rows and columns of the remaining matrix consisting of all entries in H^{Q_Ω} except for those in the ith row or ith column are switched just as the functor C_v^+ switches the indecomposables in the categories. The result is independent of whether you first switch rows or columns. Since this actions are elementary row and column operations on a linearly independent set of vectors, one is guaranteed that $H^{Q_{\Omega'}}$ is an invertible matrix.

In our case row 2 and 5 are switched, 3 and 8, 4 and 7 and 9 and 10. The columns are switched in exactly the same way. So as one could expect the relations between the categories $Rep_K(Q_\Omega)$ and $Rep_K(Q_{\Omega'})$ are closely connected to the relations between H^{Q_Ω} and $H^{Q_{\Omega'}}$ by the partial Coxeter functors.

To compare the matrices H^{Q_Ω} and $H^{Q_{\Omega'}}$ when the indexing of the dimension vectors of the representations corresponding to the isomorphism classes of indecomposables is fixed for both categories, are really not that interesting. The structure of the AR-quiver of the two categories are identical with the exception of the one simple representation. So it should be clear that by altering the indexing of the indecomposables in the new category $Rep_K(Q_{\Omega'})$, we can obtain the same matrix with the exception of row one and column one. In the rest of this thesis, we make the convention that the representations corresponding to the isomorphism classes of indecomposables in a category $Rep_K(Q)$, where Q is Dynkin is indexed according to the following rule:
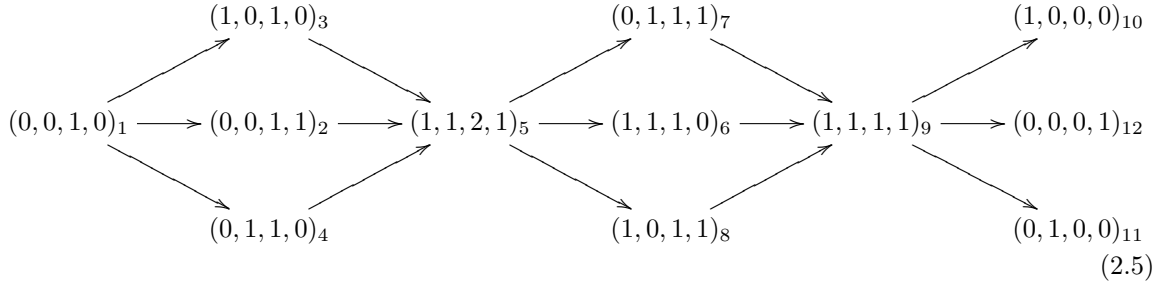
$$j < i \Leftrightarrow dim_K \text{Hom}_\Lambda(idb_i, idb_j) = 0.$$

So when we apply one of the partial Coxeter functors, f. ex.

$$C_4^+ : Rep_K(Q_\Omega) \longrightarrow Rep_K(Q_{\Omega'}),$$

we should change the indexing of the indecomposables according to this rule in $Rep_K(Q_{\Omega'})$. Then if the indexing matches the indexing in $Rep_K(Q_\Omega)$ relative to the structure of the respective AR-quivers (the index of a representation is lowered by one for each place in the AR-quiver), we should get the same matrix with the exception that row one and column one now are replaced by a new row 12 and column 12. So instead of the indexing in Section 2.1 we should order the indecomposable representations in the AR-quiver of $Rep_K(Q_{\Omega'})$ in the following way:

$Rep_K(Q_{\Omega'})$:

$$
\begin{array}{ccccc}
(1,0,1,0)_3 & & (0,1,1,1)_7 & & (1,0,0,0)_{10} \\
\end{array}
$$

$$\tag{2.5}$$

If we now compute the matrix $H^{Q_{\Omega'}}$, where $(i,j) = dim_K \mathrm{Hom}_\Lambda(idb_i, idb_j)$, $\forall\ 1 \leq i,\ j \leq 12$, we see that

$$
H^{Q_{\Omega'}} = \left(
\begin{array}{ccccccccccc|c}
1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\right) .
$$

As we can see $H^{Q_{\Omega'}}$ has a 11×11-submatrix which can be found in

$$
H^Q = H^{Q_\Omega} = \left(
\begin{array}{c|ccccccccccc}
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\right) .
$$

2.3 $(H^Q)^{-1}$

An interesting thing about the inverse matrix of H^Q is that there exist a one-to-one correspondence between the rows in $(H^Q)^{-1}$ and the minimal left almost split morphisms (see [1]) of the category $Rep_K(Q)$. We shall soon state this correspondence in a proposition, but we have to do some preparations first.

Let r'_i denote the ith row in $(H^Q)^{-1}$ and let $TrD(i)$ denote the index of the indecomposable representation $TrD(idb_i)$ corresponding to the $TrD(i)$th isomorphism class in $Rep_K(Q)$.

Due to the ordering of the indecomposables in $Rep_K(Q)$ (if $j < i$ then $dim_K(\mathrm{Hom}_\Lambda(idb_i, idb_j)) = 0$), we have that for every minimal left almost split morphism f_i from a indecomposable idb_i, where $TrD(idb_i) \neq 0$, there exist a almost split sequence in $Rep_K(Q)$ of the form:

$$
0 \longrightarrow idb_i \xrightarrow{f_i} idb_j \oplus ... \oplus idb_k \longrightarrow TrD(idb_i) \longrightarrow 0,
$$

where for all $l \in \{j, \,\ldots\,, k\}$, we have that $l > i$. All almost split sequences starting in idb_i, where $TrD(idb_i) \neq 0$, are on this form (see 1). If $TrD(idb_i) = 0$, such a sequence does not exist. If $TrD(idb_i) = 0$, then the minimal left almost split morphism f_i, starting in idb_i, is a morphism of the form:

$$idb_i \xrightarrow{f_i} idb_j \oplus \ldots \oplus idb_k \longrightarrow 0,$$

where for all $l \in \{j, \,\ldots\,, k\}$, we have that $l > i$. This is an epimorphism, since in fact, it is a morphism on the form

$$I \xrightarrow{f_i} I/\mathrm{soc}\ I, \ (\text{see } [1]).$$

The set $\{j, \,\ldots\,, k\}$ may be the empty set $\emptyset$. Of technical reasons, if $\{j, \,\ldots\,, k\} = \emptyset$, we let $idb_j \oplus \ldots \oplus idb_k = 0$, the zero module. So then f_i will trivially be an epimorphism on the form

$$I \longrightarrow I/\mathrm{soc}\ I \cong 0,$$

since $I = 0$ if and only if $\mathrm{soc}\ I = 0$ (see [1]).

Proposition 2.3.1 *Let Q be of finite representation type. There exist a one-to-one correspondence between the rows in the matrix $(H^Q)^{-1}$ and the minimal left almost split morphisms starting at the indecomposables in $Rep_K(Q)$.*

If $TrD(idb_i) \neq 0$, the minimal left almost split morphism f_i starting at idb_i gives a almost split sequence

$$0 \longrightarrow idb_i \xrightarrow{f_i} idb_j \oplus \ldots \oplus idb_k \longrightarrow TrD(idb_i) \longrightarrow 0,$$

where for all $l \in \{j, \,\ldots\,, k\}$, we have that $l > i$,

which corresponds to:

$$r_i'(t) = \begin{cases} 1, \ if\ t = i\ or\ t = TrD(i), \\ -1, \ if\ t \in \{j, \,\ldots\,, k\}, \\ 0, \ else. \end{cases}$$

If $TrD(idb_i) = 0$, the minimal left almost split morphism f_i starting at idb_i

$$idb_i \xrightarrow{f_i} idb_j \oplus \ldots \oplus idb_k \longrightarrow 0,$$

where for all $l \in \{j, \,\ldots\,, k\}$, we have that $l > i$,

corresponds to:

$$r_i'(t) = \begin{cases} 1, \ if\ t = i, \\ -1, \ if\ t \in \{j, \,\ldots\,, k\}, \\ 0, \ else. \end{cases}$$

Proof: We may assume that $Rep_K(Q)$ contains n distinct isomorphism classes of indecomposable representations.

If $TrD(idb_i) \neq 0$, the minimal left almost split morphism f_i, starting at idb_i, gives the almost split sequence

$$0 \longrightarrow idb_i \xrightarrow{f_i} idb_j \oplus \ldots \oplus idb_k \longrightarrow TrD(idb_i) \longrightarrow 0,$$

where for all $l \in \{j, \,\ldots\,, k\}$, we have that $l > i$.

Apply $\mathrm{Hom}_\Lambda(-, X)$ to the sequence, where $X \neq idb_i$, we get

$$0 \longrightarrow \mathrm{Hom}_\Lambda(TrD(idb_i), X) \longrightarrow \mathrm{Hom}_\Lambda(idb_j \oplus \ldots \oplus idb_k, X) \longrightarrow \mathrm{Hom}_\Lambda(idb_i, X) \longrightarrow 0,$$

where the right map is an epimorphism due to the minimal left almost split property of the first sequence. Let $X = idb_t$, we see that when $t \neq i$:

$$dim_K(\mathrm{Hom}_\Lambda(idb_j \oplus \ldots \oplus idb_k, idb_t)) = dim_K(\mathrm{Hom}_\Lambda(idb_i, idb_t)) + dim_K(\mathrm{Hom}_\Lambda(TrD(idb_i), idb_t))$$

and

$$0 = dim_K(\mathrm{Hom}_\Lambda(idb_i, idb_t)) + dim_K(\mathrm{Hom}_\Lambda(TrD(idb_i), idb_t)) - dim_K(\mathrm{Hom}_\Lambda(idb_j \oplus \ldots \oplus idb_k, idb_t)).$$

We know that

$$dim_K(\text{Hom}_\Lambda(idb_i, idb_i)) = 1, \ dim_K(\text{Hom}_\Lambda(TrD(idb_i), idb_i)) = 0 \text{ and}$$

$$dim_K(\text{Hom}_\Lambda(idb_j, idb_i)) + ... + dim_K(\text{Hom}_\Lambda(idb_k, idb_i)) = 0.$$

By the definition of the row vectors of H^Q we have that

$$r_i = dim_K(\text{Hom}_\Lambda(idb_i, idb_t))_t, \ r_{TrD(i)} = dim_K(\text{Hom}_\Lambda(TrD(idb_i), idb_t))_t \text{ and}$$

$$r_j = dim_K(\text{Hom}_\Lambda(idb_j, idb_t))_t, \ ..., \ r_k = dim_K(\text{Hom}_\Lambda(idb_k, idb_t))_t,$$

where $1 \le t \le n$.

Let $e_i = (0_1, ..., 0_{i-1}, 1_i, 0_{i+1}, ..., 0_n)$, where $1 \le i \le n$. By the equations above it follows that:

$$e_i = r_i - r_j - ... - r_k + r_{TrD(i)}.$$

It follows that $r'_i(t)$, is given in the following way:

$$r'_i(t) = \begin{cases} 1, \text{ if } t = i \text{ or } t = TrD(i), \\ -1, \text{ if } t \in \{j, \ ... \ , k\}, \\ 0, \text{ else.} \end{cases}$$

If $TrD(idb_i) = 0$, then the minimal left almost split morphism, starting at idb_i, f_i is on the form

$$idb_i \xrightarrow{f_i} idb_j \oplus ... \oplus idb_k \longrightarrow 0,$$

where for all $l \in \{j, \ ... \ , k\}$, we have that $l > i$. Note that if $\{j, \ ... \ , k\} = \emptyset$, then we let $idb_j \oplus ... \oplus idb_k = 0$, the zero module.

Apply $\text{Hom}_\Lambda(-, X)$ to the morphism, where $X \ne idb_i$, we get

$$0 \longrightarrow \text{Hom}_\Lambda(idb_j \oplus ... \oplus idb_k, X) \longrightarrow \text{Hom}_\Lambda(idb_i, X) \longrightarrow 0,$$

where the map is an epimorphism due to the minimal left almost split property of f_i. Let $X = idb_t$, we see that when $t \ne i$:

$$dim_K(\text{Hom}_\Lambda(idb_i, idb_t)) = dim_K(\text{Hom}_\Lambda(idb_j \oplus ... \oplus idb_k, idb_t)).$$

When $t = i$, we know that

$$dim_K(\text{Hom}_\Lambda(idb_i, idb_i)) = 1 \text{ and}$$

$$dim_K(\text{Hom}_\Lambda(idb_j, idb_i)) + ... + dim_K(\text{Hom}_\Lambda(idb_k, idb_i)) = 0.$$

By the definition of the row vectors of H^Q we have that

$$r_i = dim_K(\text{Hom}_\Lambda(idb_i, idb_t))_t \text{ and}$$

$$r_j = dim_K(\text{Hom}_\Lambda(idb_j, idb_t))_t, \ ..., \ r_k = dim_K(\text{Hom}_\Lambda(idb_k, idb_t))_t,$$

where $1 \le t \le n$.

By the equations above it follows that:

$$e_i = r_i - r_j - ... - r_k.$$

Hence $r'_i(t)$, is given in the following way:

$$r'_i(t) = \begin{cases} 1, \text{ if } t = i, \\ -1, \text{ if } t \in \{j, \ ... \ , k\}, \\ 0, \text{ else.} \end{cases}$$

Thus the matrix $(H^Q)^{-1}$ is determined by the minimal left almost split morphisms in $Rep_K(Q)$.

$$QED$$

Let us illustrate with some examples, let r'_i denote the ith row in $(H^Q)^{-1}$:

Example 2.3.2

$$0 \longrightarrow (0,0,0,1)_1 \longrightarrow (0,0,1,1)_2 \longrightarrow (0,0,1,0)_5 \longrightarrow 0$$

corresponds to

$$r_1' = \begin{pmatrix} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$\triangle$

A more interesting example is the following:

Example 2.3.3

$$0 \longrightarrow (0,0,1,1)_2 \longrightarrow (1,0,1,1)_3 \oplus (0,1,1,1)_4 \oplus (0,0,1,0)_5 \longrightarrow (1,1,2,1)_6 \longrightarrow 0$$

corresponds to

$$r_2' = \begin{pmatrix} 0 & 1 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$\triangle$

2.4 The Structure of H^Q and T^Q

A similar matrix T^Q is obtained by setting

$$(i,j) = dim_K(idb^i \otimes_\Lambda idb_j), \ \forall 1 \leq i, \ j \leq m,$$

where m is the number of isomorphism classes of indecomposables in $Rep_K(Q)$. The numbering idb^i and idb_j refers to the same numbering of dimension vectors of the indecomposables in the AR-quiver as in the previous example, but the underlying quiver of idb^i is the opposite of idb_j. Let $idb^i = D_K(idb_i)$, where $D_K = \text{Hom}_K(-, K)$. The (i,j) entry is obtained by choosing a projective resolution of idb^i and then tensoring the projective resolution with $(- \otimes_\Lambda idb_j)$. We can then easily calculate $(i,j) = dim_K(idb^i \otimes_\Lambda idb_j)$. In more precise terms:

$$Q \xrightarrow{q} P \longrightarrow idb^i \longrightarrow 0$$

tensoring by $(- \otimes_\Lambda idb_j)$ gives

$$(Q \otimes_\Lambda idb_j) \xrightarrow{(f_q \otimes_\Lambda id)} (P \otimes_\Lambda idb_j) \longrightarrow (idb_i \otimes_\Lambda idb_j) \longrightarrow 0.$$

We then see that

$$dim_K(idb^i \otimes_\Lambda idb_j) = dim_K(P \otimes_\Lambda idb_j) - rank(f_q \otimes_\Lambda id).$$

Here it is easy to calculate the right side of the equation.

An example of this calculation could be:

Example 2.4.1 *Let*

$$Q:\ \begin{array}{c} 1 \\ \ \ \searrow \alpha \\ \ \ \ \ \ 3 \xrightarrow{\gamma} 4, \\ \ \beta \nearrow \\ 2 \end{array}$$

and

$$e_2\Lambda \xrightarrow{\beta} e_3\Lambda \longrightarrow idb^8 \longrightarrow 0$$

tensoring by $(- \otimes_\Lambda idb_2)$ *gives*

$$(e_2\Lambda \otimes_{KQ} idb^2) \xrightarrow{(f_\beta \otimes_\Lambda id)} (e_3\Lambda \otimes_\Lambda idb_2) \longrightarrow (idb^8 \otimes_\Lambda idb_2) \longrightarrow 0.$$

Now

$$dim_K(idb^8 \otimes_\Lambda idb_2) = dim_K(e_3\Lambda \otimes_\Lambda idb_2) - rank(f_\beta \otimes_\Lambda id),$$

but then

$$dim_K(idb^8 \otimes_\Lambda idb_2) = dim_K(e_3idb_2) - rank(idb_2)_\beta.$$

We see that $dim_K(e_3idb_2) = 1$ and $rank(idb_2)_\beta = 0$, thus entry

$$(8,2) = dim_K(idb^8 \otimes_\Lambda idb_2) = 1 - 0 = 1.$$

$\triangle$

We end up with the following matrix T^Q:

$$T^Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

Note that

$$H^Q = (T^Q)^T.$$

There is much that could be said about the matrices H^Q and T^Q. There is a pattern in both matrices due to the fact that

$$dim_K\mathrm{Hom}_\Lambda(X,Y) = dim_K\mathrm{Hom}_\Lambda(TrD(X),TrD(Y)), \forall \, X, \, Y \text{ not injectives (see[1])},$$

where TrD denotes the transpose of the dual of a module (see[1]). We have that

$$dim_K\mathrm{Hom}_\Lambda(P_i^Q, M) = dim_K(M_i),$$

where $P_i^Q = \Lambda * e_i$ and $dim_K(M_i)$ equals the dimension of M in the ith vector space. So if we take the projectives from the AR-quiver and analyze the dimension of the Hom-spaces from them to all the others:

$$P^Q = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix},$$

the rest of H^Q is just a repetition of this pattern, where we shift all the representations by the TrD to the right in the AR-quiver, always leaving out the injectives. Thus

$$TrD(P^Q) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix},$$

and

$$TrD(TrD(P^Q)) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

So by the notation above we can write:

$$H^Q = \begin{pmatrix} P^Q \\ TrD(P^Q) \\ TrD(TrD(P^Q)) \end{pmatrix}.$$

This also explains why the matrices H^Q and T^Q only consists of the number 0, 1 and 2. If we look at the AR-quiver of the category, we see that there are no higher dimensional vector spaces in any of the indecomposable modules, they have either dimension 0, 1 or 2. This is a general fact for all quiver-algebras with D_n as the underlying graph and a similar argument can be made for all the other quiver-algebras with Dynkin diagrams as underlying graph.

The following should indicate the form of all the matrices that settles the question of the structure of representations with quivers of finite representation type, and hence settle the question of degeneration for this class of algebras. Remember that the isomorphism classes of indecomposable representations are indexed from the projectives to the injectives according to the following rule if $j < i$ then $dim_K \text{Hom}_\Lambda(idb_i, idb_j) = 0$. This does not always give a unique indexing, but all numberings having this property gives upper triangular matrices with ones on the diagonal.

$$H^{A_n} = \begin{pmatrix} P^{A_n} \\ TrD(P^{A_n}) \\ (TrD)^2(P^{A_n}) \\ . \\ . \\ . \\ (TrD)^m(P^{A_n}) \end{pmatrix}, \text{ where } m \text{ depends on } n \text{ and the orientation on } A_n .$$

where P^Q is a matrix which corresponds to the dimension of the Hom-spaces from the projectives of the AR-quiver of $Rep_K(Q)$ to all the others.

We see that in addition to depending on n, m depends on the orientation on A_n. This is because when we apply TrD to the projective representations in the AR-quiver of the category $\text{Rep}_K A_n$ we might loose projectives, since a representation can be projective and injective at the same time, TrD of an injective is zero (see [1]). That is why even if $(TrD)^t(P^{A_n})$ is a $k \times n(n-1)$ matrix, $(TrD)^{t+1}(P^{A_n})$ can be a $l \times n(n-1)$ matrix, where $l < k$ and $0 \le t \le n-1$.

To illustrate and clarify we give an example:

Example 2.4.2 *Let*
$$A_5^{eq} : \quad 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow 4 \longrightarrow 5 .$$

Then

$$H^{A_5^{eq}} = \begin{pmatrix} P^{A_5^{eq}} \\ TrD(P^{A_5^{eq}}) \\ (TrD)^2(P^{A_5^{eq}}) \\ (TrD)^3(P^{A_5^{eq}}) \\ (TrD)^4(P^{A_5^{eq}}) \end{pmatrix} =$$

$$
\left(
\begin{array}{cccccccccccccc}
H^{A_5^{eq}} & - & - & - & - & & - & - & - & & - & - & & - & - \\
| & 1 & . & . & . & & . & & & & . & & & . & . \\
| & 0 & 1 & . & . & & . & & & & . & & & . & . \\
| & 0 & 0 & 1 & . & & . & & & & . & & & . & . \\
| & 0 & 0 & 0 & 1 & H^{A_4^{eq}} & - & - & - & & - & - & & - & - \\
| & 0 & 0 & 0 & 0 & | & 1 & . & . & & . & & & . & . \\
| & 0 & 0 & 0 & 0 & | & 0 & 1 & . & & . & & & . & . \\
| & 0 & 0 & 0 & 0 & | & 0 & 0 & 1 & & . & . & & . & . \\
| & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & H^{A_3^{eq}} & - & - & & - & - \\
| & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & | & 1 & . & & . & . \\
| & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & | & 0 & 1 & & . & . \\
| & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & | & 0 & 0 & H^{A_2^{eq}} & - & - \\
| & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & | & 0 & 0 & | & 1 & . \\
| & 0 & 0 & 0 & 0 & | & 0 & 0 & 0 & | & 0 & 0 & | & 0 & H^{A_1^{eq}}
\end{array}
\right),
$$

where

$$
H^{A_5^{eq}} =
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

$$
H^{A_4^{eq}} =
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

$$
H^{A_3^{eq}} =
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad
H^{A_2^{eq}} =
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix} \ and \ \ H^{A_2^{eq}} = (\ 1 \ ).
$$

So

$$
P^{A_5^{eq}} =
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1
\end{pmatrix},
$$

$$TrD(P^{A_4^{eq}}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix},$$

$$TrD^2(P^{A_3^{eq}}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix},$$

$$TrD^3(P^{A_2^{eq}}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \ and$$

$$TrD^4(P^{A_2^{eq}}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

$\triangle$

For the other Dynkin diagrams the number of times we must apply TrD is independent of the orientation on the underlying quiver:

$$H^{D_n} = \begin{pmatrix} P^{D_n} \\ TrD(P^{D_n}) \\ \cdot \\ \cdot \\ \cdot \\ (TrD)^{n-2}(P^{D_n}) \end{pmatrix},$$

$$H^{E_6} = \begin{pmatrix} P^{E_6} \\ TrD(P^{E_6}) \\ (TrD)^2(P^{E_6}) \\ (TrD)^3(P^{E_6}) \\ (TrD)^4(P^{E_6}) \\ (TrD)^5(P^{E_6}) \end{pmatrix},$$

$$H^{E_7} = \begin{pmatrix} P^{E_7} \\ TrD(P^{E_7}) \\ (TrD)^2(P^{E_7}) \\ (TrD)^3(P^{E_7}) \\ (TrD)^4(P^{E_7}) \\ (TrD)^5(P^{E_7}) \\ (TrD)^6(P^{E_7}) \\ (TrD)^7(P^{E_7}) \\ (TrD)^8(P^{E_7}) \end{pmatrix},$$

$$H^{E_8} = \begin{pmatrix} P^{E_8} \\ TrD(P^{E_8}) \\ (TrD)^2(P^{E_8}) \\ (TrD)^3(P^{E_8}) \\ (TrD)^4(P^{E_8}) \\ (TrD)^5(P^{E_8}) \\ (TrD)^6(P^{E_8}) \\ (TrD)^7(P^{E_8}) \\ (TrD)^8(P^{E_8}) \\ (TrD)^9(P^{E_8}) \\ (TrD)^{10}(P^{E_8}) \\ (TrD)^{11}(P^{E_8}) \\ (TrD)^{12}(P^{E_8}) \\ (TrD)^{13}(P^{E_8}) \\ (TrD)^{14}(P^{E_8}) \end{pmatrix},$$

where P^Q is a matrix which corresponds to the dimension of the Hom-spaces from the projectives of the AR-quiver of $Rep_K(Q)$ to all the others.

Proposition 2.4.3 *If Q is a quiver of finite representation type, then*

$$H^Q = (T^Q)^T.$$

Proof: Assume that $Rep_K(Q)$ contains t distinct isomorphism classes of indecomposable representations. After ordering the fixed representations corresponding to the isomorphism classes of indecomposable representations from 1 to t, we have to prove that entry (i,j) in H^Q, denoted $(i,j)_{H^Q}$, equals entry (j,i) in T^Q, denoted $(j,i)_{T^Q}$. By the definition of the matrices we know that

$$(i,j)_{H^Q} = dim_K \mathrm{Hom}_\Lambda(idb_i, idb_j), \ \forall \ 1 \le i, \ j \le t,$$

and

$$(j,i)_{T^Q} = dim_K(idb^j \otimes_\Lambda idb_i), \ \forall \ 1 \le i, \ j \le t.$$

From Section 1.6 we know that

$$dim_K D_K(idb_i \otimes_\Lambda idb^j) = dim_K \mathrm{Hom}_\Lambda(idb_i, D_K idb^j),$$

where $D_K = \mathrm{Hom}_K(-, K)$ and $idb^i := D_K idb_i$.

We see that

$$(j,i)_{T^Q} = dim_K(idb^j \otimes_\Lambda idb_i) = dim_K D_K(idb^j \otimes_\Lambda idb_i) = dim_K(\mathrm{Hom}_\Lambda(idb^j, D_K idb_i)) =$$

$$dim_K(\mathrm{Hom}_\Lambda(D_k idb_j, D_K idb_i)) = dim_K D_K(\mathrm{Hom}_\Lambda(D_K^2 idb_j, D_K^2 idb_i)) =$$

$$dim_K(\mathrm{Hom}_\Lambda(idb_i, idb_j)) = (i,j)_{H^Q},$$

which proves the assertion.

$$QED$$

Chapter 3

Orbit Closures Determined by Algebraic Equations

We start this chapter with some motivation in Section 3.1. In Section 3.2 we develop a general procedure to find the set $S(x)$ of algebraic equations, which determines the orbit closure of $X \in Rep_K(Q)$, where Q is Dynkin. In Section 3.3 we give some examples of such algebraic equations and in Section 3.4 we briefly relate degeneration to some algebraic geometry.

3.1 A Third Matrix R^Q

It is already known (see [14]) that to determine the orbit closure by algebraic equations is possible for all representations of Dynkin quivers. These equations are not unique. In this chapter we shall provide a different and perhaps more effective method for finding algebraic equations which determines the orbit closures. Before we do it stringently in Section 3.2, let us try a "naive" generalization of the more or less straightforward method used for A_n-quivers, and see what goes wrong. This could work as motivation for the more involved procedures developed in the next section.

In Section 1.3.2 we saw that $O(x)$ corresponds to the isomorphism class of X. So we want to classify the isomorphism class of X like we did in Chapter 2. We developed a machinery which solves this for all X in $Rep_K(Q)$, where Q is Dynkin. But this classification has a more natural variant for the A_n (with arbitrary orientation) quivers. This more straightforward procedure will be briefly introduced via an example. Let us first explain our notation. By $\begin{pmatrix} M_\alpha & M_\beta \end{pmatrix}$ we mean the map from $K^{s(\alpha)} \times K^{s(\beta)} \to K^{t(\alpha)}(= K^{t(\beta)}.)$ So it is the matrix consisting of the columns of M_α succeeded by the columns of M_β.

Example 3.1.1 *Let*

$$Q: \ 1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3 \xleftarrow{\gamma} 4 \ ,$$

and

$$X: \ K^{d_1} \xrightarrow{M_\alpha} K^{d_2} \xleftarrow{M_\beta} K^{d_3} \xleftarrow{M_\gamma} K^{d_4} \ .$$

The indecomposable representation in $Rep_K(Q)$ is:

$idb_1 = 0 \longrightarrow k \longleftarrow 0 \longleftarrow 0$, $idb_2 = k \xrightarrow{\ 1\ } k \longleftarrow 0 \longleftarrow 0$, $idb_3 = 0 \longrightarrow k \xleftarrow{1} k \longleftarrow 0$,

$idb_4 = 0 \longrightarrow k \xleftarrow{1} k \xleftarrow{1} k$, $idb_5 = k \xrightarrow{\ 1\ } k \xleftarrow{1} k \longleftarrow 0$, $idb_6 = 0 \longrightarrow 0 \longleftarrow k \longleftarrow 0$,

$idb_7 = k \xrightarrow{\ 1\ } k \xleftarrow{1} k \xleftarrow{1} k$, $idb_8 = k \longrightarrow 0 \longleftarrow 0 \longleftarrow 0$, $idb_9 = 0 \longrightarrow 0 \longleftarrow k \xleftarrow{1} k$

and $idb_{10} = 0 \longrightarrow 0 \longleftarrow 0 \longleftarrow k$.

One can see relatively easy that the isomorphism class of $X \in Rep_K(Q)$ with dimension vector $\mathbf{d} = (d_1, d_2, d_3, d_4)$ is given by the following computations:

$$\#idb_1 = d_2 - rank \begin{pmatrix} M_\alpha & M_\beta \end{pmatrix} ,$$

$$\#idb_2 = rank \begin{pmatrix} M_\alpha & M_\beta \end{pmatrix} - rank(M_\beta),$$

$$\#idb_3 = rank\begin{pmatrix} M_\alpha & M_\beta \end{pmatrix} - rank(M_\alpha),$$

$$\#idb_4 = rank\begin{pmatrix} M_\alpha & M_\gamma M_\beta \end{pmatrix} - rank(M_\alpha),$$

$$\#idb_5 = rank(M_\beta) + rank\begin{pmatrix} M_\alpha & M_\gamma M_\beta \end{pmatrix} - rank\begin{pmatrix} M_\alpha & M_\beta \end{pmatrix} - rank(M_\beta M_\gamma),$$

$$\#idb_6 = d_3 + rank(M_\beta M_\gamma) - rank(M_\beta) - rank(M_\gamma),$$

$$\#idb_7 = rank(M_\alpha) + rank(M_\beta M_\gamma) - rank\begin{pmatrix} M_\alpha & M_\gamma M_\beta \end{pmatrix},$$

$$\#idb_8 = d_1 - rank(M_\alpha),$$

$$\#idb_9 = rank(M_\gamma) - rank(M_\beta M_\gamma) \ and$$

$$\#idb_{10} = d_4 - rank(M_\gamma),$$

where $\#idb_i$ equals the multiplicity of the ith indecomposable summand in X.

Since the dimension vector $\mathbf{d}$ is fixed, we see that the isomorphism class of X only depends on the rank of the following set of matrices:

$$\{M_\alpha,\ M_\beta,\ M_\gamma,\ M_\beta M_\gamma,\ \begin{pmatrix} M_\alpha & M_\beta \end{pmatrix},\ \begin{pmatrix} M_\alpha & M_\gamma M_\beta \end{pmatrix}\}.$$

$$\triangle$$

We are looking for a similar set of matrices whereby we can determine the isomorphism class of X and therefore $O(x)$, when $X \in Rep_K(Q)$ and Q is Dynkin. As we shall see in the next example, the straightforward approach used in the previous example does not seem to work for a general Dynkin quiver.

Example 3.1.2 *Let*
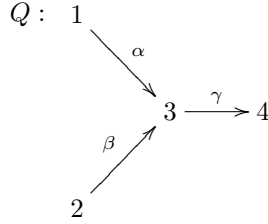
and
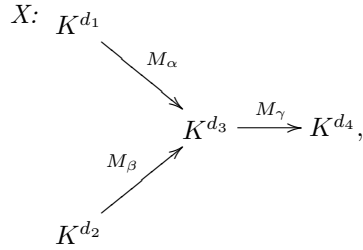
What we basically want to do, is to obtain a third matrix R^Q, which can determine the isomorphism class of a representation in $Rep_K(Q)$, where the conditions are as easy as possible. By easy, we here mean it in the sense that they are similar to the conditions in example 3.1.1, or the method one in general would use for a quiver of A_n-type. The following 11 independent conditions are thought of as nice and similar to the conditions for A_n.

$$a_1(X) = dimX(1) = d_1,\ a_2(X) = dimX(2) = d_2,\ a_3(X) = dimX(3) = d_3,\ a_4(X) = dimX(4) = d_4,$$

$$a_5(X) = rank(M_\alpha),\ a_6(X) = rank(M_\beta),\ a_7(X) = rank(M_\gamma),\ a_8(X) = rank(M_\gamma M_\alpha),\ a_9(X) = rank(M_\gamma M_\beta),$$

$$a_{10}(X) = rank\begin{pmatrix} M_\gamma M_\alpha & M_\gamma M_\beta \end{pmatrix},\ a_{11}(X) = rank\begin{pmatrix} M_\alpha & M_\beta \end{pmatrix},$$

where X varies over the indecomposables. These conditions are independent and will distinguish nearly all indecomposable summands in a representation in $Rep_K(Q)$. There are 12 isomorphism

classes of indecomposables in $Rep_K(Q)$, so it should be no surprise that one more condition is needed to obtain a matrix which will determine degeneration in $Rep_K(Q)$. It is the representations in the following almost split sequence which are not uniquely determined by the 11 conditions stated above:

$$0 \longrightarrow (1,1,2,1)_6 \longrightarrow (0,1,1,0)_7 \oplus (1,0,1,0)_8 \oplus (1,1,1,1)_9 \longrightarrow (1,1,1,0)_{10} \longrightarrow 0 \ .$$

The 11 conditions do not distinguish

$$(1,1,2,1)_6 \oplus (1,1,1,0)_{10} \ from \ (0,1,1,0)_7 \oplus (1,0,1,0)_8 \oplus (1,1,1,1)_9.$$

By reducing the matrix obtained by the set of 11 conditions, one can realize that the following tensor-condition

$$a_{12}(X) = dim_K(idb_6 \otimes_\Lambda X) = dim((e_3 + e_4)X) - rank \begin{pmatrix} M_\gamma M_\alpha & 0 \\ M_\alpha & M_\beta \end{pmatrix},$$

is not in $span\{a_1(X), a_2(X), a_3(X), a_4(X), a_5(X), a_6(X), a_7(X), a_8(X), a_9(X), a_{10}(X), a_{11}(X)\}$, and thus can be chosen to be the 12th condition. Since the dimension vector of X is fixed, the condition is equivalent to

$$a'_{12}(X) = rank \begin{pmatrix} M_\gamma M_\alpha & 0 \\ M_\alpha & M_\beta \end{pmatrix}.$$

These conditions are equivalent to both the Hom-*conditions and the tensor-conditions and are mentioned explicitly because of the "nice and intuitive" form of the 11 first conditions. We obtain the following matrix:*

$$R^Q = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 2 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$\triangle$

As just illustrated in the example above, a "naive" generalization of what is done for the representations with underlying quiver equal to A_n (with arbitrary orientation), is not a fruitful approach to classifying the isomorphism class of a representation when the underlying quiver is one of the other Dynkin quivers.

For H^Q the convention is that if $H^X \leq H^Y$, where X and Y are representations with equal dimension vector in $Rep_K(Q)$, then $X \preceq_{deg} Y$, this follows from Theorem 1.5.4. Due to the connections between T^Q and H^Q given in Section 1.6, we see that for T^Q the convention is that if $T^X \leq T^Y$, where X and Y are representations with equal dimension vector in $Rep_K(Q)$, then $X \preceq_\otimes Y \Leftrightarrow X \preceq_{hom} Y \Leftrightarrow X \preceq_{deg} Y$.

Note that the convention for the partial order in the Hasse-diagram with respect to degeneration must be the opposite for R^Q on $Rep_K(Q)$. That is if $R^X \leq R^Y$, where X and Y are representations with equal dimension vector in $Rep_K(Q)$, then $Y \preceq_{deg} X$. One way to explain this is to look at the tight connection between the R^Q-conditions and the T^Q-conditions. This will be done in the next section.

We end this section with a description of $O(x)$ by the R^Q- conditions, where the corresponding X is just as in example 3.1.2:

Example 3.1.3 *Let*

$$Q: \quad 1$$
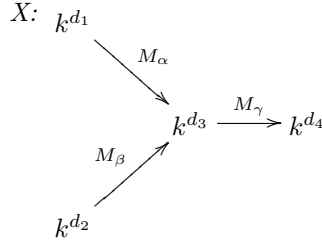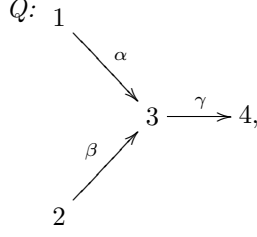
The orbit of the corresponding $x \in Rep_K(Q, \mathbf{d})$, where $\mathbf{d} = (d_1, d_2, d_3, d_4)$, is given by

$$O(x) = \{(N_\alpha, N_\beta, N_\gamma) \in Rep_K(Q, \mathbf{d}) \mid rank(N_\alpha) = rank(M_\alpha), \ rank(N_\beta) = rank(M_\beta), \ rank(N_\gamma) = rank(M_\gamma),$$

$$rank(N_\gamma N_\alpha) = rank(M_\gamma M_\alpha), \ rank(N_\gamma N_\beta) = rank(M_\gamma M_\beta), \ rank\begin{pmatrix} N_\gamma N_\alpha & N_\gamma N_\beta \end{pmatrix} = rank\begin{pmatrix} M_\gamma M_\alpha & M_\gamma M_\beta \end{pmatrix},$$

$$rank\begin{pmatrix} N_\alpha & N_\beta \end{pmatrix} = rank\begin{pmatrix} M_\alpha & M_\beta \end{pmatrix}, \ rank\begin{pmatrix} N_\gamma N_\alpha & 0 \\ N_\alpha & N_\beta \end{pmatrix} = rank\begin{pmatrix} M_\gamma M_\alpha & 0 \\ M_\alpha & M_\beta \end{pmatrix}\}.$$

$\triangle$

3.2 The General Procedure

Let Q be a Dynkin quiver. From a set-theoretic point of view one can say that the orbit closure $\overline{O(x)}$ of a point $x \in Rep_K(Q, \mathbf{d})$ corresponding to $X \in Rep_K(Q)$, can be determined by a finite set of algebraic equations (see Definition 1.1.1). In [14], there is a procedure for computing these algebraic equations explicitly. What we basically will do here is to show how such a set of algebraic equations can be computed explicitly in an alternative way which we believe is more efficient.

We start by defining what we mean by a universal representation.

Definition 3.2.1 *A universal representation Y with dimension vector $\mathbf{d}$ is defined to be $Y := \{Y_\alpha \mid \alpha \in Q_1\}$, where Y_α is a $d_{s(\alpha)} \times d_{t(\alpha)}$-matrix with algebraic variables as entries: $(Y_\alpha)_{(i,j)} = x^\alpha_{(i,j)}$.*

Assume that $Rep_K(Q)$ contains n distinct isomorphism classes of indecomposable representation. We take a projective resolution of idb^i, $1 \leq i \leq n$,

$$0 \longrightarrow Q_i = \oplus^t_{k=1} Q_{(i,k)} \xrightarrow{q_i} P_i = \oplus^s_{j=1} P_{(i,j)} \longrightarrow idb^i \longrightarrow 0,$$

where

$$q_i = \left(f_{(i,k,j)}\right), \text{ and } f_{(i,k,j)}: Q_{(i,k)} \longrightarrow P_{(i,j)}.$$

We then tensor the projective resolution with $(- \otimes_\Lambda X)$, where X has dimension vector $\mathbf{d}$. This gives

$$(Q_i \otimes_\Lambda X) \xrightarrow{(q_i \otimes_\Lambda id_X)} (P_i \otimes_\Lambda X) \longrightarrow (idb^i \otimes_\Lambda X) \longrightarrow 0.$$

We then see that

$$dim_K(idb^i \otimes_\Lambda X) = dim_K(P_i \otimes_\Lambda X) - rank(q_i \otimes_\Lambda id_X). \tag{3.1}$$

We define

$$a^x_i = -dim_K(idb^i \otimes_\Lambda X) + dim_K(P_i \otimes_\Lambda X) = rank(q_i \otimes_\Lambda id_X). \tag{3.2}$$

We know that $Q_{(i,j)} = \Lambda e_{(i,k)}$ and $P_{(i,j)} = \Lambda f_{(i,j)}$, where $e_{(i,k)}$ and $f_{(i,j)}$ are two idempotents in Λ. We see that $\text{Hom}_\Lambda(\Lambda e_{(i,k)}, \Lambda f_{(i,j)}) = e_{(i,k)} \Lambda f_{(i,j)} \cong K p_{(i,k,j)}$, if such a path $p = \alpha_1...\alpha_m$ exist, or 0 if there exist no such path p. Let $\alpha_i \in Q_1$, $1 \le i \le m$. A path p in X is then $M_p := M_{\alpha_1} M_{\alpha_2}...M_{\alpha_m}$. Since $f_{(i,k,j)}$ is given by multiplication $r_{(i,k,j)} p_{(i,k,j)}$, where $r_{(i,k,j)} \in K$, we have that

$$(q_i \otimes_\Lambda Id_X) = \left(r_{(i,k,j)} M_{p_{(i,k,j)}} \right). \tag{3.3}$$

Let Y be a universal representation with dimension vector $\mathbf{d}$. Let $S(x)$ be the union of the vanishing of the $(rank(q_i \otimes_\Lambda id_X) + 1)$-minors in the matrices $(q_i \otimes_\Lambda id_Y)$, which is a finite set of algebraic equations due to the fact that the matrices $(q_i \otimes_\Lambda id_Y)$ has algebraic expressions as entries (see equation 3.2), as i varies from 1 to n.

Theorem 3.2.2 *Let X be a representation in $Rep_K(Q)$ with dimension vector $\mathbf{d}$ and let Y be a universal representation in $Rep_K(Q)$ with dimension vector $\mathbf{d}$, where Q is a Dynkin quiver. Let x be the point corresponding to X in $Rep_K(Q, \mathbf{d})$. Then $\overline{O(x)}$ is determined by the vanishing of the $(rank(q_i \otimes_\Lambda id_X) + 1)$-minors of the matrices $(q_i \otimes_\Lambda id_Y)$. This vanishing constitute the finite set $S(x)$ of algebraic equations.*

Proof: From Corollary 1.6.2 we know that the $\preceq_\otimes$-order is equivalent with the $\preceq_{deg}$-order for algebras of finite representation type. Since $dim_K(P_i \otimes_\Lambda X)$ is independent of X with dimension vector $\mathbf{d}$, this can be regarded as just a constant term. Our definition of a_i^x (see Equation 3.2) involves multiplying Equation 3.1 by -1, so the degeneration order is reversed, that is

$$y \in \overline{O(x)} \Leftrightarrow a_i^y \le a_i^x, \ 1 \le i \le n.$$

We see that the the $\preceq_\otimes$-order is equivalent to the "a^x"-order.

It follows that $\overline{O(x)}$ is described by the vanishing of the $(rank(q_i \otimes_\Lambda id_X) + 1)$-minors in the matrices $(q_i \otimes_\Lambda id_Y)$, as i varies from 1 to n. This is the set $S(x)$, the assertion follows.

$$QED$$

For the quiver A_n^{eq} it is rather easy to determine the closure of the orbits by algebraic equations. Recall from Section 1.3.2 that $O(x)$ corresponds to the isomorphism class of X.

Example 3.2.3 *Let*

$$Q: \ 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-1}} n \ .$$

For a fixed dimension vector $\mathbf{d} = (d_1, ..., d_n)$, the orbits corresponds to the isomorphism class of X. So for the representation

$$X: \ K^{d_1} \xrightarrow{M_{\alpha_1}} K^{d_2} \xrightarrow{M_{\alpha_2}} \cdots \xrightarrow{M_{\alpha_{n-1}}} K^{d_n} \ ,$$

the orbit of the corresponding $x \in Rep_K(Q, \mathbf{d})$ can be characterized by (see 14)

$$O(x) = \{(N_{\alpha_1}, ..., N_{\alpha_{n-1}}) \in Rep_K(Q, \mathbf{d}) \mid rank(N_{\alpha_i}...N_{\alpha_j}) = rank(M_{\alpha_i}...M_{\alpha_j}) = c(i,j), \ \forall 1 \le i \le j \le n-1 \ \}.$$

By Theorem 3.2.2 the orbit closure is

$$\overline{O(x)} = \{(N_{\alpha_1}, ..., N_{\alpha_{n-1}}) \in Rep_K(Q, \mathbf{d}) \mid rank(N_{\alpha_i}...N_{\alpha_j}) \le c(i,j), \ \forall 1 \le i \le j \le n-1 \ \}.$$

$$\triangle$$

Let us revisit example 3.1.1.

Example 3.2.4 *Let*

$$Q: \ 1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3 \xleftarrow{\gamma} 4 \ ,$$

and

$$X: \ K^{d_1} \xrightarrow{M_\alpha} K^{d_2} \xleftarrow{M_\beta} K^{d_3} \xleftarrow{M_\gamma} K^{d_4} \ .$$

The orbit of the corresponding $x \in Rep_K(Q, \mathbf{d})$, where $\mathbf{d} = (d_1, d_2, d_3, d_4)$, was given by

$$O(x) = \{(N_\alpha, N_\beta, N_\gamma) \in Rep_K(Q, \mathbf{d}) \mid rank(N_\alpha) = rank(M_\alpha), \ rank(N_\beta) = rank(M_\beta), \ rank(N_\gamma) = rank(M_\gamma),$$

$$rank(N_\beta N_\gamma) = rank(M_\beta M_\gamma), \ rank\begin{pmatrix} N_\alpha & N_\beta \end{pmatrix} = rank\begin{pmatrix} M_\alpha & M_\beta \end{pmatrix},$$

$$rank\begin{pmatrix} N_\alpha & N_\gamma N_\beta \end{pmatrix} = rank\begin{pmatrix} M_\alpha & M_\gamma M_\beta \end{pmatrix}\}.$$

So the orbit closure is given by

$$\overline{O(x)} = \{(N_\alpha, N_\beta, N_\gamma) \in Rep_K(Q, \mathbf{d}) \mid rank(N_\alpha) \leq rank(M_\alpha), \ rank(N_\beta) \leq rank(M_\beta), \ rank(N_\gamma) \leq rank(M_\gamma),$$

$$rank(N_\beta N_\gamma) \leq rank(M_\beta M_\gamma) \ rank\begin{pmatrix} N_\alpha & N_\beta \end{pmatrix} \leq rank\begin{pmatrix} M_\alpha & M_\beta \end{pmatrix},$$

$$rank\begin{pmatrix} N_\alpha & N_\beta N_\gamma \end{pmatrix} \leq rank\begin{pmatrix} M_\alpha & M_\beta M_\gamma \end{pmatrix}\}.$$

$$\triangle$$

The following relatively extensive example will hopefully give a clear insight in how one in general can determine the orbit closures by algebraic equations for all representation finite algebras.

Example 3.2.5 *Let*

$$Q:$$

$$X:$$

Note that the injective indecomposables of the category $Rep_K(Q)$ corresponds to the dimension of the different vector spaces in the representation X. Since there are 36 isomorphism classes of indecomposable representations in the category $Rep_K(Q)$ and the dimension vector $\mathbf{d} = (d_1, d_2, d_3, d_4, d_5, d_6)$ is 6 dimensional and fixed, we need 36-6=30 conditions to determine the orbit of the corresponding $x \in Rep_K(Q, \mathbf{d})$. So to determine the orbit $O(x)$ we need 30 equations on the form

$$a_i^x = -dim_K(idb_i \otimes_\Lambda X) + dim_K(P \otimes_\Lambda X) = rank(f_{q^i} \otimes_\Lambda id).$$

These are:

$$a_1^x = -dim_K((0, 0, 0, 1, 0, 0) \otimes_\Lambda X) + dim_K(e_4 X) = rank\begin{pmatrix} M_\alpha & M_\gamma & M_\delta \end{pmatrix},$$

$$a_2^x = -dim_K((0, 0, 1, 1, 0, 0) \otimes_\Lambda X) + dim_K(e_4 X) = rank\begin{pmatrix} M_\alpha & M_\gamma M_\beta & M_\delta \end{pmatrix},$$

$$a_3^x = -dim_K((0, 1, 1, 1, 0, 0) \otimes_\Lambda X) + dim_K(e_4 X) - rank\begin{pmatrix} M_\alpha & M_\delta \end{pmatrix},$$

$$a_4^x = -dim_K((0, 0, 0, 1, 1, 0) \otimes_\Lambda X) + dim_K(e_4 X) = rank\begin{pmatrix} M_\alpha & M_\gamma & M_\delta M_\epsilon \end{pmatrix},$$

$$a_5^x = -dim_K((0, 0, 0, 1, 1, 1) \otimes_\Lambda X) + dim_K(e_4 X) = rank\begin{pmatrix} M_\alpha & M_\gamma \end{pmatrix},$$

$$a_6^x = -dim_K((1, 0, 0, 1, 0, 0) \otimes_\Lambda X) + dim_K(e_4 X) = rank\begin{pmatrix} M_\gamma & M_\delta \end{pmatrix},$$

$$a_7^x = -dim_K((1, 0, 1, 2, 1, 0) \otimes_\Lambda X) + 2dim_K(e_4 X) = rank\begin{pmatrix} M_\alpha & 0 & 0 & M_\delta & M_\delta M_\epsilon \\ -M_\alpha & M_\beta M_\gamma & M_\gamma & 0 & 0 \end{pmatrix},$$

$$a_8^x = -dim_K((1,1,1,2,1,0) \otimes_\Lambda X) + 2dim_K(e_4X) = rank \begin{pmatrix} M_\alpha & 0 & M_\delta & M_\delta M_\epsilon \\ -M_\alpha & M_\gamma & 0 & 0 \end{pmatrix},$$

$$a_9^x = -dim_K((1,0,0,1,1,0) \otimes_\Lambda X) + dim_K(e_4X) = rank \begin{pmatrix} M_\gamma & M_\delta M_\epsilon \end{pmatrix},$$

$$a_{10}^x = -dim_K((1,0,1,2,1,1) \otimes_\Lambda X) + 2dim_K(e_4X) = rank \begin{pmatrix} M_\alpha & 0 & 0 & M_\delta \\ -M_\alpha & M_\beta M_\gamma & M_\gamma & 0 \end{pmatrix},$$

$$a_{11}^x = -dim_K((1,0,1,1,0,0) \otimes_\Lambda X) + dim_K(e_4X)_K = rank \begin{pmatrix} M_\gamma M_\beta & M_\delta \end{pmatrix},$$

$$a_{12}^x = -dim_K((0,0,1,1,1,0) \otimes_\Lambda X) + dim_K(e_4X) = rank \begin{pmatrix} M_\alpha & M_\gamma M_\beta & M_\delta M_\epsilon \end{pmatrix},$$

$$a_{13}^x = -dim_K((1,1,2,3,2,1) \otimes_\Lambda X) + 3dim_K(e_4X) = rank \begin{pmatrix} M_\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & M_\alpha & 0 & M_\gamma & M_\delta & 0 \\ 0 & 0 & M_\gamma M_\beta & M_\gamma & M_\delta & M_\delta M_\epsilon \end{pmatrix},$$

$$a_{14}^x = -dim_K((1,0,1,2,2,1) \otimes_\Lambda X) + 2dim_K(e_4X) = rank \begin{pmatrix} M_\alpha & 0 & 0 & M_\delta M_\epsilon \\ -M_\alpha & M_\gamma M_\beta & M_\gamma & 0 \end{pmatrix},$$

$$a_{15}^x = -dim_K((0,0,1,1,1,1) \otimes_\Lambda X) + dim_K(e_4X) = rank \begin{pmatrix} M_\gamma M_\beta & M_\alpha \end{pmatrix},$$

$$a_{16}^x = -dim_K((1,1,2,2,1,0) \otimes_\Lambda X) + 2dim_K(e_4X) = rank \begin{pmatrix} M_\alpha & 0 & M_\delta & M_\delta M_\epsilon \\ -M_\alpha & M_\gamma M_\beta & 0 & 0 \end{pmatrix},$$

$$a_{17}^x = -dim_K((0,1,1,1,1,0) \otimes_\Lambda X) + dim_K(e_4X) = rank \begin{pmatrix} M_\alpha & M_\delta M_\epsilon \end{pmatrix},$$

$$a_{18}^x = -dim_K((1,1,1,2,1,1) \otimes_\Lambda X) + 2dim_K(e_4X) = rank \begin{pmatrix} M_\alpha & 0 & M_\delta \\ -M_\alpha & M_\gamma & 0 \end{pmatrix},$$

$$a_{19}^x = -dim_K((2,1,2,3,2,1) \otimes_\Lambda X) + 3dim_K(e_4X) = rank \begin{pmatrix} M_\alpha & M_\gamma M_\beta & 0 & 0 & M_\delta M_\epsilon \\ 0 & -M_\gamma M_\beta & 0 & M_\delta & 0 \\ 0 & 0 & M_\gamma & 0 & -M_\delta M_\epsilon \end{pmatrix},$$

$$a_{20}^x = -dim_K((1,1,2,2,1,1) \otimes_\Lambda X) + 2dim_K(e_4X) = rank \begin{pmatrix} M_\alpha & 0 & M_\delta \\ -M_\alpha & M_\gamma M_\beta & 0 \end{pmatrix},$$

$$a_{21}^x = -dim_K((1,1,1,1,0,0) \otimes_\Lambda X) + dim_K(e_4X) = rank \begin{pmatrix} M_\delta \end{pmatrix},$$

$$a_{22}^x = -dim_K((1,1,1,2,2,1) \otimes_\Lambda X) + 2dim_K(e_4X) = rank \begin{pmatrix} M_\alpha & 0 & M_\delta M_\epsilon \\ -M_\alpha & M_\gamma & 0 \end{pmatrix},$$

$$a_{23}^x = -dim_K((1,0,0,1,1,1) \otimes_\Lambda X) + dim_K(e_4X) = rank \begin{pmatrix} M_\gamma \end{pmatrix},$$

$$a_{24}^x = -dim_K((1,0,1,1,1,0) \otimes_\Lambda X) + dim_K(e_4X) = rank \begin{pmatrix} M_\gamma M_\beta & M_\delta M_\epsilon \end{pmatrix},$$

$$a_{25}^x = -dim_K((1,1,2,2,2,1) \otimes_\Lambda X) + 2dim_K(e_4X) = rank \begin{pmatrix} M_\alpha & 0 & M_\delta M_\epsilon \\ -M_\alpha & M_\gamma M_\beta & 0 \end{pmatrix},$$

$$a_{26}^x = -dim_K((1,1,1,1,1,0) \otimes_\Lambda X) + dim_K(e_4X) = rank \left(M_\delta M_\epsilon \right),$$

$$a_{27}^x = -dim_K((0,0,0,0,1,0) \otimes_\Lambda X) + dim_K(e_5X) = rank \left(M_\epsilon \right),$$

$$a_{28}^x = -dim_K((1,0,1,1,1,1) \otimes_\Lambda X) + dim_K(e_4X) = rank \left(M_\gamma M_\beta \right),$$

$$a_{29}^x = -dim_K((0,0,1,0,0,0) \otimes_\Lambda X) + dim_K(e_3X) = rank \left(M_\beta \right),$$

$$a_{30}^x = -dim_K((0,1,1,1,1,1) \otimes_\Lambda X) + dim_K(e_4X) = rank \left(M_\alpha \right).$$

The orbit is given by
$$O(x) = \{y \in Rep_K(Q, \mathbf{d}) | a_i^y = a_i^x, \ 1 \le i \le 30\}$$

and

$$\overline{O(x)} = \{y \in Rep_K(Q, \mathbf{d}) | a_i^y \le a_i^x, \ 1 \le i \le 30\}.$$

$\triangle$

3.3 Some Algebraic Equations

In this section we shall see some examples where the algebraic equations are given explicitly.

Example 3.3.1 *Let*
$$Q : 1 \xrightarrow{\alpha} 2,$$

and
$$X : K^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} K^2.$$

From Section 1.3.2 we have that the orbit of x consists of all points in $Rep_K(Q,(2,2))$ where the linear map M_α of the corresponding representation has rank two. So the orbit is determined by the following condition:
$$det(M_\alpha) = det \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = x_1 x_4 - x_2 x_3 \ne 0.$$

From Section 1.3.2 one also can also see that the orbit closure $\overline{O(x)} = Rep_K(Q,(2,2))$.

If
$$X : K^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} K^2.$$

Then
$$\overline{O(x)} = \{\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} | x_1 x_4 - x_2 x_3 = 0\}.$$

If
$$X : K^2 \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}} K^2.$$

Then
$$\overline{O(x)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

$\triangle$

For $X : K^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} K^2$, we had $\overline{O(x)} = \{ \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \mid x_1 x_4 - x_2 x_3 = 0 \}$.

Before we have another and more involved example; let us compare this algebraic equation by the set of algebraic equations obtained by using Bongartz method, given in the article by Weyman [14]. Before computing the set of algebraic equations, we first cite the relevant theorem without proof, and then briefly give a description of the method.

Theorem 3.3.2 *Let X and Y be two representations of dimension vector $\mathbf{d}$ of a Dynkin quiver Q. Then*
$$O(x) \subseteq \overline{O(y)} \Leftrightarrow dim_K(\mathrm{Hom}_\Lambda(idb_i, X)) \geq dim_K(\mathrm{Hom}_\Lambda(idb_i, Y)),$$
for all isomorphism classes of indecomposable representations in $Rep_K(Q)$, represented by all the idb_i's.

Let $r_i := dim_K(\mathrm{Hom}_\Lambda(idb_i, X))$. Then
$$\overline{O(x)} = \{ Y \mid dim_K(\mathrm{Hom}_\Lambda(idb_i, Y)) \geq r_i \}. \tag{3.4}$$

Let us define the map
$$d_Y^X : \oplus_{j \in Q_0} \mathrm{Hom}_K(X(j), Y(j)) \longrightarrow \oplus_{\alpha \in Q_1} \mathrm{Hom}_K(X(s(\alpha)), Y(e(\alpha)))$$

by the formula
$$d_Y^X(\phi(j))_{j \in Q_0} = (\phi(e(\alpha))X(\alpha) - Y(\alpha)\phi(s(\alpha))_{\alpha \in Q_1},$$

whose kernel is $\mathrm{Hom}_\Lambda(X, Y)$.

One can realize that set-theoretically equation 3.4 means that $\overline{O(x)}$ is given by the vanishing of $dim_K(\oplus_{j \in Q_0} \mathrm{Hom}_K(X(j), Y(j))) - r_i + 1$ size minors of the matrix $d_Y^{idb_i}$.

Now, let us use this method to compute a set of algebraic equations which determines $\overline{O(x)}$, for
$$X : K^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} K^2.$$

First note that there are three isomorphism classes of indecomposable representations, represented by
$$idb_1 : 0 \longrightarrow K, \ idb_2 : K \xrightarrow{1} K \text{ and } idb_3 : K \longrightarrow 0.$$

The matrices are
$$d_Y^{idb_1} = \begin{pmatrix} 0 & 0 \end{pmatrix},$$
$$d_Y^{idb_2} = \begin{pmatrix} x_1 & 1 & 0 & x_2 \\ x_3 & 0 & 1 & x_4 \end{pmatrix} \text{ and}$$
$$d_Y^{idb_3} = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}.$$

By the vanishing of the $dim_K(\oplus_{j \in Q_0} \mathrm{Hom}_K(X(j), Y(j))) - r_i + 1$ size minors of the matrix $d_Y^{idb_i}$, one can see that the following set of algebraic equations determines the orbit closure:
$$S_X = \{ x_1 = 0 \cup x_2 = 0 \cup x_3 = 0 \cup x_4 = 0 \cup 1 = 0 \cup x_1 x_4 - x_2 x_3 = 0 \}.$$

We already know that the elements in $\overline{O(x)}$ is determined by the single algebraic equation
$$x_1 x_4 - x_2 x_3 = 0.$$

So as we can see S_X contains many superfluous algebraic equations. The following is two interesting open questions:

Question 3.3.3 *Let Q be a Dynkin quiver. Let X be a representation in $Rep_K(Q)$ and x the corresponding point in $Rep_K(Q, \mathbf{d})$.*
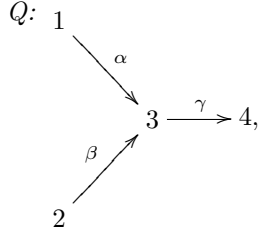$$\text{Is } S(x) \subseteq S_X \ ?$$

Question 3.3.4 *Let Q be a Dynkin quiver. Let X be a representation in $Rep_K(Q)$ and x the corresponding point in $Rep_K(Q, \mathbf{d})$. Can there exist a proper subset of $S(x)$ which also determines the orbit closure of x, or is $S(x)$ a minimal set in this sense?*
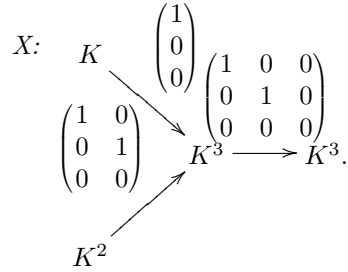
When determining $S(x)$ the projective resolution where chosen to be minimal. This might be related to these questions.

We continue with a more involved example.

Example 3.3.5 *Let*

$$Q: \quad \begin{array}{c} 1 \\ \end{array}$$

$$Q: \quad 1 \xrightarrow{\alpha} \quad 3 \xrightarrow{\gamma} 4,$$
$$\begin{array}{c} \beta \nearrow \\ 2 \end{array}$$

and

$$X: \quad K \xrightarrow{\begin{pmatrix}1\\0\\0\end{pmatrix}} \quad \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{pmatrix}$$
$$\begin{pmatrix}1 & 0\\0 & 1\\0 & 0\end{pmatrix} \searrow$$
$$K^3 \longrightarrow K^3.$$
$$K^2 \nearrow$$

From Section 3.1 we know by which rank-conditions we can determine the orbit closure of the corresponding $x \in Rep_K(Q, (1, 2, 3, 3))$.

$$\overline{O(x)} = \{ \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}, \begin{pmatrix}x_4 & x_5\\x_6 & x_7\\x_8 & x_9\end{pmatrix}, \begin{pmatrix}x_{10} & x_{11} & x_{12}\\x_{13} & x_{14} & x_{15}\\x_{16} & x_{17} & x_{18}\end{pmatrix} \in Rep_K(Q, (1, 2, 3, 3)) \mid 1: rank \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \le 1, \; 2: rank \begin{pmatrix}x_4 & x_5\\x_6 & x_7\\x_8 & x_9\end{pmatrix} \le 2,$$

$$3: rank \begin{pmatrix}x_{10} & x_{11} & x_{12}\\x_{13} & x_{14} & x_{15}\\x_{16} & x_{17} & x_{18}\end{pmatrix} \le 2, \; 4: rank \begin{pmatrix}x_{10}x_1 + x_{11}x_2 + x_{12}x_3\\x_{13}x_1 + x_{14}x_2 + x_{15}x_3\\x_{16}x_1 + x_{17}x_2 + x_{18}x_3\end{pmatrix} \le 1,$$

$$5: rank \begin{pmatrix}x_{10}x_4 + x_{11}x_6 + x_{12}x_8 & x_{10}x_5 + x_{11}x_7 + x_{12}x_9\\x_{13}x_4 + x_{14}x_6 + x_{15}x_8 & x_{13}x_5 + x_{14}x_7 + x_{15}x_9\\x_{16}x_4 + x_{17}x_6 + x_{18}x_8 & x_{16}x_5 + x_{17}x_7 + x_{18}x_9\end{pmatrix} \le 2,$$

$$6: rank \begin{pmatrix}x_{10}x_1 + x_{11}x_2 + x_{12}x_3 & x_{10}x_4 + x_{11}x_6 + x_{12}x_8 & x_{10}x_5 + x_{11}x_7 + x_{12}x_9\\x_{13}x_1 + x_{14}x_2 + x_{15}x_3 & x_{13}x_4 + x_{14}x_6 + x_{15}x_8 & x_{13}x_5 + x_{14}x_7 + x_{15}x_9\\x_{16}x_1 + x_{17}x_2 + x_{18}x_3 & x_{16}x_4 + x_{17}x_6 + x_{18}x_8 & x_{16}x_5 + x_{17}x_7 + x_{18}x_9\end{pmatrix} \le 2,$$

$$7: rank \begin{pmatrix}x_1 & x_4 & x_5\\x_2 & x_6 & x_7\\x_3 & x_8 & x_9\end{pmatrix} \le 2,$$

$$8: rank \begin{pmatrix}x_{10}x_1 + x_{11}x_2 + x_{12}x_3 & 0 & 0\\x_{13}x_1 + x_{14}x_2 + x_{15}x_3 & 0 & 0\\x_{16}x_1 + x_{17}x_2 + x_{18}x_3 & 0 & 0\\x_1 & x_4 & x_5\\x_2 & x_6 & x_7\\x_3 & x_8 & x_9\end{pmatrix} \le 3\}.$$

It follows from Theorem 3.2.2 that $\overline{O(x)}$ can be described by a set $S(x)$ of algebraic equations. If we look at the proof of Theorem 3.2.2 it is clear that

$S(x) = \{$ *3:* $x_{10}x_{14}x_{18} + x_{11}x_{15}x_{16} + x_{12}x_{13}x_{17} - x_{12}x_{14}x_{16} - x_{11}x_{13}x_{18} - x_{10}x_{15}x_{17} = 0 \cup$

6: $(x_{10}x_1 + x_{11}x_2 + x_{12}x_3)(x_{13}x_4 + x_{14}x_6 + x_{15}x_8)(x_{16}x_5 + x_{17}x_7 + x_{18}x_9) +$

$(x_{10}x_4 + x_{11}x_6 + x_{12}x_8)(x_{13}x_5 + x_{17}x_7 + x_{18}x_9)(x_{16}x_1 + x_{17}x_2 + x_{18}x_3) +$

$(x_{10}x_5 + x_{11}x_7 + x_{12}x_9)(x_{13}x_1 + x_{14}x_2 + x_{15}x_3)(x_{16}x_4 + x_{17}x_6 + x_{18}x_8) -$

$(x_{10}x_5 + x_{11}x_7 + x_{12}x_9)(x_{13}x_4 + x_{14}x_6 + x_{15}x_8)(x_{16}x_1 + x_{17}x_2 + x_{18}x_3) -$

$(x_{10}x_4 + x_{11}x_6 + x_{12}x_8)(x_{13}x_1 + x_{14}x_2 + x_{15}x_3)(x_{16}x_5 + x_{17}x_7 + x_{18}x_9) -$

$(x_{10}x_1 + x_{11}x_2 + x_{12}x_3)(x_{13}x_5 + x_{17}x_7 + x_{18}x_9)(x_{16}x_4 + x_{17}x_6 + x_{18}x_8) = 0 \cup$

7: $x_1x_6x_9 + x_4x_7x_3 + x_5x_2x_8 - x_5x_6x_3 - x_4x_2x_9 - x_1x_7x_8 = 0\}.$

$\triangle$

3.4 Algebraic Geometry

A point $v = (v_1, ..., v_n) \in \mathbb{A}^n$ corresponds to $M(v) = \{f \mid f(v_1, ..., v_n) = 0\}$ a maximal ideal in $K[\mathbb{A}^n] = K[x_1, ..., x_n]$. If $V \in \mathbb{A}^n$ is closed, V corresponds to an ideal $I(V) = \{f \mid f(v_1, ..., v_n) = 0, \ \forall \ v \in V\}$ in $K[\mathbb{A}^n]$.

Let I be an ideal in a commutative ring R. By the radical of I, $\sqrt{I}$, we mean

$$\sqrt{I} = \{y \in R \mid \exists n \text{ such that } y^n \in I\}.$$

Let $R = K[x_1, x_2, x_3, x_4]$ as in example [3.3.1], the $rank \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = 1$ case. We know that $\overline{O(x)}$ is determined by the algebraic equation $x_1x_4 - x_2x_3 = 0$. Let $V = \{(x_1, x_2, x_3, x_4) \mid x_1x_4 - x_2x_3 = 0\}$, it follows by the definition of the topology on $\mathbb{A}^n$ (see [1.1.1]), that V is closed. Let V correspond to the ideal $I(V)$. The equation $(x_1x_4 - x_2x_3)^2 = x_1^2x_4^2 - 2x_1x_2x_3x_4 + x_2^2x_3^2 = 0$ also determines $\overline{O(x)}$. Let J be the ideal generated by $(x_1x_4 - x_2x_3)^2$. The equation $x_1x_4 - x_2x_3 = 0$ is thought of as "nicer" because for $I(V)$, we have that $\sqrt{I(V)} = I(V)$, while $\sqrt{J} = I(V) \neq J$.

In general one is interested in finding algebraic equations determining $\overline{O(x)}$. If found, a natural question is whether the ideal I generated by the algebraic equations has the property that $\sqrt{I} = I$. If not, how can one find a set of algebraic equations determining $\overline{O(x)}$ which generates an ideal with this property, if possible?

We have the following question:

Question 3.4.1 *Let Q be a Dynkin quiver. Let X be a representation in $Rep_K(Q)$. Do the set $S(x)$, determining $\overline{O(x)}$, corresponds to an ideal $I(S(x))$, where $\sqrt{I(S(x))} = I(S(x))$?*

Using a different method, which possibly gives a different set of equations than $S(x)$, this question was settled for A^{eq} by V. Lakshmibai and P. Magyar (see [8]).

Chapter 4

Summary and Further Discussions

We have seen a general procedure for how one can obtain explicit algebraic equations describing the orbit closure of a point in the affine variety $Rep_K(Q, \mathbf{d})$, where Q is a quiver of finite representation type. Our procedure depends on the fact that $\preceq_{deg}$ and $\preceq_{hom}$ coincide for this class of algebras, and we have also seen that $\preceq_{hom}$ and $\preceq_{\otimes}$ coincide. For algebras of finite representation type we have also given a general procedure for how one can determine degeneration.

A natural question is how one can determine degeneration for other types of algebra. One of the natural candidates for such a further investigation would be the algebras of tame type.

Definition 4.0.2 *A quiver is tame if Q is of infinite representation type but its indecomposable representations occur in a given dimension either a finite number of times or in a finite number of one-parameter families.*

It is known that $\preceq_{deg}$ and $\preceq_{hom}$ (and therefore also $\preceq_{\otimes}$) coincide for all representations of tame quivers [4]. In general one can classify all indecomposable representations in $Rep_K(Q)$, when Q is tame, even though there is infinitely many of them. The problem with the method provided here when it comes to quivers of tame type, is that H^Q and T^Q will be infinitely large quadratic matrices. Since $\preceq_{deg}$ and $\preceq_{hom}$ (and therefore also $\preceq_{\otimes}$)coincide for all representations of tame quivers [4], these infinitely large quadratic matrices must be invertible and in principle determine degeneration for quivers of tame type, but they will not be upper triangular matrices. However, the preprojective and preinjective components of the AR-quiver will be directed (i. e. contains no loops) (see [1]). So if one is only considering these components and one choose a smart indexing of the fixed indecomposable representations (corresponding to isomorphism classes), the infinitely large quadratic matrices will be upper triangular with non-zero entries on the diagonal, and in principle determine degeneration for representations only containing indecomposable objects from these components of the AR-quiver.

For a quiver Q of tame type, the sets S_X and $S(x)$, determining the orbit closure of a representation X in $Rep_K(Q)$, will exist. But, $S(x)$ will be a infinite set of equations. We have assumed representations to be finite dimensional and the equations in $S(x)$ will be equations in a ring of polynomials over a field K. Thus the equations corresponds to ideals in a Noetherian ring. For all Noetherian rings we have that every non-empty set of ideals has a maximal element with respect to set inclusion. It follows that the orbit closure of a X can be described by a finite subset of $S(x)$. There might be a smart way to choose such a subset. Perhaps based on the isomorphism class of the representation one are considering? However, it is an open question how this could be done.

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