Determination of constraint forces for an offshore crane on a moving base

Andrej Cibicik[1] and Olav Egeland[1]

***Abstract*— In this paper we propose an efficient method for calculating the forces of constraints in an open-chain multibody system, like a robot mounted on a vehicle with 6 degrees of freedom. The dynamical model is based on Kane's equation of motion, where screw theory is used to calculate the projection matrices from the link twists of the multibody system. This leads to a general modeling procedure relying on screw transformations that is presented in the paper. The procedure for determination of the constraint forces is given as an extension of the dynamical model and can be implemented after the equations of motion have been formulated and solved. We implement the described method for the specific case of a vessel with a heavy crane, and provide the simulation results. The method provides a basis for future work on the detailed modeling of friction in the joints of serial link mechanisms, and on the evaluation of potential fatigue consequences of different control solutions.**

I. Introduction

Cranes are often installed on different types of offshore vessels. When cranes were first installed on the facilities in the North Sea in the 1970's, they were based on the design of land cranes. This led to the number of structural accidents, as they were under-designed for offshore-related loads [1]. Now the vessel acceleration and other relative dynamic effects are considered during the design of cranes. However, there is an increasing demand for heavy offshore installations [2] and the requirements for the crane lifting capacity are rather high. The cranes become heavier and the total mass of the crane/payload assembly becomes significant relative to the vessel mass and dynamic coupling between both bodies becomes important. In this paper we derive a combined crane/vessel dynamical model and determine the constraint forces on the interface between the crane and the vessel. We model the restoring forces of the vessel by means of nonlinear springs. In this work we are looking at open-chain holonomic multibody systems with geometric constraints.

The importance of dynamic coupling between a crane and a vessel is discussed in [3], where the model of interconnected dynamics of a crane/vessel system based on Lagrange's method is derived. The authors highlight that relatively little research exist on the coupled behavior of crane/vessel systems. The research within the rigid multibody dynamics provides several methods for modeling such coupled behavior of the systems, as well as several methods for determination of constraint forces exist. Similarly, in [4] Lagrangian mechanics is used to derive the equation of motion and the controller for a 3-DOF link manipulator on a 6-DOF vessel. A comparative overview on nonlinear behavior of different floating cranes is given in [5]. An example of more involved analysis with coupled dynamics of a vessel, a crane and a flexible cable is presented in [6].

Kane's method for modeling multibody dynamics is based on the Newton-Euler formulation, where the principle of virtual work is used to eliminate the forces and moments of constraints by projecting the equations of motion to the directions defined by the generalized speeds [7]. This leads to a minimal set of ODEs, which are formulated in terms of the generalized speeds. The procedure for deriving these equations of motion in vector form can be formulated efficiently using projection matrices, as it was done in [8] for a spacecraft/manipulator system.

In [7] the authors describe the method of auxiliary generalized speeds for the determination of constraint forces (i.e. noncontributing forces) in holonomic systems. Auxiliary generalized speeds are the magnitudes of velocities which the body joint *would* have if, affected by the constraint forces, it lost the contact with the neighboring body. The advantage of this method is that the algebraic equations for constraint forces can be formulated and solved separately from the equations of motion, given as a minimal set of motion ODEs. The constraint forces are related to structural forces in the joints and can be used in the mechanism design process. A discussion on the application of the method of auxiliary generalized speeds for simple nonholonomic systems is presented in [9]. Another closely related method for determination of constraint forces is given in [10], where DOFs specifying prohibited motion in the joints are introduced. In [11] the author uses the concept of DOFs for prohibited motion and derives the closed-form expressions for prismatic and revolute joints. Alternatively, a classical method of undetermined Langrange multipliers can be also used for determination of constraint forces in holonomic systems [12], [13]. However, then the constraint forces are related to generalized coordinates and sometimes are also referred as generalized constraint forces [14]. Generalized constraint forces can be converted into actual constraint forces in the joints, however that might be impractical [15]. The method is very general and easy to implement computationally, however constraints are imposed on the acceleration (i.e. force) level, which might lead to the position drift during the simulation. A number of less conventional methods exist, the reader can refer to [9] for more details.

In [3] the authors propose that models with coupled

[1] The authors are with the Dept. of Mechanical and Industrial Engineering, Norwegian University of Science and Technology (NTNU), NO-7491 Trondheim, Norway. andrej.cibicik@ntnu.no

crane/vessel dynamics can be used for determination of safe operation limits under the given weather conditions and for the risk analysis of offshore hoisting operations. At this point it is important to add that structural integrity is also an important safety issue. In the case when dynamic coupling between a crane and a vessel is significant, the exact internal structural forces in a crane or a vessel can only be calculated from the model with coupled dynamics of both bodies. This is especially important when a force time history is of interest. In the field of structural mechanics, these forces are used for strength verification of the supporting structure or for evaluation of the deflection in mechanism joints. The structural strength of a mechanism can also be used as an optimization criteria for the control inputs. Alternatively, impact of different control solutions on the structural strength of a mechanism can be studied. The collected dynamic force data can also be used to get a better estimation of the fatigue lifetime of mechanisms. Fatigue is a form of structural failure when a crack develops in the steel parts under cyclic loading over time. It is a typical problem for mechanisms installed on moving bases, for example vessels. Constraint forces are also relevant if a detailed model of frictional behavior in the joints is of interest for the control purposes. Note that in the field of structural mechanics internal structural forces in the joints are the same as constraint forces in the field of multibody dynamics. In this work we will refer to both constraint forces and moments by just writing *constraint forces*.

The first contribution of this work is an extension of Kane's method formulation presented in [8] using Screw Theory, in particular twists and screw transformations. The second contribution is that we present a computational scheme for determination of constraint forces. The scheme is an extension of the dynamical model and is based on the method of auxiliary generalized speeds [7], [9]. In contrast to the original basis-independent formulation, we derive the equations using twists and screw transformations represented by column vectors and matrices. This leads to an elegant and simplified computational implementation. We first formulate and solve the minimal set of ODE for the dynamics in terms of the generalized speeds, and then determine the constraint forces related to the actual structural forces in the joints.

The paper is organized as follows. In Sect. II we introduce the twist notation and present derivation of the equation of motion using twists and screw transformations. Additionally, we present the method of auxiliary generalized speeds for determination of constraint forces. In Sect. III we apply the method for a crane on a moving base system and present the simulation results. The conclusions are given in Sect. IV.

II. Preliminaries

A. Introduction of a twist

Consider the homogeneous transformation matrix

$$\mathbf{T}^a_b = \begin{bmatrix} \mathbf{R}^a_b & \mathbf{p}^a_{ab} \\ \mathbf{0}^\mathrm{T} & 1 \end{bmatrix} \tag{1}$$

from Frame a to Frame b. The time derivative of the homogeneous transformation matrix is

$$\dot{\mathbf{T}}^a_b = \begin{bmatrix} \hat{\boldsymbol{\omega}}^a_{ab}\mathbf{R}^a_b & \mathbf{v}^a_{ab} \\ \mathbf{0}^\mathrm{T} & 0 \end{bmatrix}, \tag{2}$$

where $\hat{\boldsymbol{\omega}}$ is the skew-symmetric form of the vector $\boldsymbol{\omega}$. Here $\hat{\boldsymbol{\omega}}^a_{ab} = \dot{\mathbf{R}}^a_b(\mathbf{R}^a_b)^\mathrm{T}$ is the skew-symmetric form of the angular velocity of Frame b relative to Frame a in the coordinates of Frame a, and $\mathbf{v}^a_{ab} = \dot{\mathbf{p}}^a_{ab}$ is the velocity of the origin of Frame b relative to the origin of Frame a in the coordinates of Frame a. This can be written

$$\dot{\mathbf{T}}^a_b = \mathbf{T}^a_b \hat{\mathrm{t}}^b_{ab/b}, \tag{3}$$

where

$$\hat{\mathrm{t}}^b_{ab/b} = (\mathbf{T}^a_b)^{-1}\dot{\mathbf{T}}^a_b = \begin{bmatrix} \hat{\boldsymbol{\omega}}^b_{ab} & \mathbf{v}^b_{ab} \\ \mathbf{0}^\mathrm{T} & 0 \end{bmatrix} \tag{4}$$

is the matrix form of the twist

$$\mathrm{t}^b_{ab/b} = \begin{bmatrix} \boldsymbol{\omega}^b_{ab} \\ \mathbf{v}^b_{ab} \end{bmatrix} \tag{5}$$

of Frame b relative to Frame a referenced to b. A twist is a screw which satisfies the screw transformation [16]

$$\mathrm{t}^a_{ab/a} = \begin{bmatrix} \mathbf{R}^a_b & \mathbf{0} \\ \hat{\mathbf{p}}^a_{ab}\mathbf{R}^a_b & \mathbf{R}^a_b \end{bmatrix} \mathrm{t}^b_{ab/b}. \tag{6}$$

The resulting twist is

$$\mathrm{t}^a_{ab/a} = \begin{bmatrix} \boldsymbol{\omega}^a_{ab} \\ \hat{\mathbf{p}}^a_{ab}\boldsymbol{\omega}^a_{ab} + \mathbf{v}^a_{ab} \end{bmatrix}, \tag{7}$$

where the velocity term $\hat{\mathbf{p}}^a_{ab}\boldsymbol{\omega}^a_{ab} + \mathbf{v}^a_{ab}$ is the velocity of a point fixed in Frame b that passes through the origin of Frame a. The twist $\mathrm{t}^b_{ab/b}$ is called the body velocity of Frame b and $\mathrm{t}^a_{ab/a}$ is called the spatial velocity of Frame b in [17].

It is noted that the screw transformation can be performed in matrix form according to

$$\hat{\mathrm{t}}^a_{ab/a} = \mathbf{T}^a_b \hat{\mathrm{t}}^b_{ab/b} (\mathbf{T}^a_b)^{-1}. \tag{8}$$

Consider the composite displacement $\mathbf{T}^a_c = \mathbf{T}^a_b \mathbf{T}^b_c$. Then the twist of the composite displacement is given in matrix form by $\hat{\mathrm{t}}^c_{ac/c} = (\mathbf{T}^a_c)^{-1}\dot{\mathbf{T}}^a_c$, which gives

$$\hat{\mathrm{t}}^c_{ac/c} = (\mathbf{T}^b_c)^{-1}(\mathbf{T}^a_b)^{-1}\dot{\mathbf{T}}^a_b\mathbf{T}^b_c + (\mathbf{T}^b_c)^{-1}\dot{\mathbf{T}}^b_c. \tag{9}$$

It follows that

$$\hat{\mathrm{t}}^c_{ac/c} = \hat{\mathrm{t}}^c_{ab/c} + \hat{\mathrm{t}}^c_{bc/c}. \tag{10}$$

It is seen that the twist of a composite displacement is the sum of the twists of the individual displacements, where all twists are referenced to the origin of the same reference frame. Obviously, this also applies for the vector formulation, which is $\mathrm{t}^c_{ac/c} = \mathrm{t}^c_{ab/c} + \mathrm{t}^c_{bc/c}$.

A relevant example of a twist is shown in Fig. 1, where Frame a is fixed in Body A. The couple of vectors $\boldsymbol{\omega}^a_{0a}$ and $\mathbf{v}^a_{0a/m_a}$ constitute the twist t^a_{0a/m_a} of Frame a relative to Frame 0 and referenced to the point m_a, which is the COG (center of gravity) of Body A. Both vectors of the twist are given in the coordinates of Frame a. The twist is written

$$\mathrm{t}^a_{0a/m_a} = \begin{bmatrix} \boldsymbol{\omega}^a_{0a} \\ \mathbf{v}^a_{0a/m_a} \end{bmatrix}. \tag{11}$$

Body A is connected to Body B with a rotary joint of one degree of freedom, so that the joint axis is through the origin of Frame b, which is fixed in B. The point m_b is the COG of Body B. Let t^b_{ab/m_b} be the twist of of Frame b relative to Frame a with reference to point m_b. Then the twists t^a_{0a/m_a} and t^b_{ab/m_b} can be added if they are transformed so that they have the same reference point, and are given in the same coordinate frame. This can be done by transforming t^a_{0a/m_a} with the screw transformation

$$\mathrm{t}^b_{0a/m_b} = \begin{bmatrix} \mathbf{R}^b_a & \mathbf{0} \\ \hat{\mathbf{p}}^b_{m_b,m_a}\mathbf{R}^b_a & \mathbf{R}^b_a \end{bmatrix} \mathrm{t}^a_{0a/m_a} \tag{12}$$

where $\mathbf{p}_{m_b,m_a}$ is the position vector from m_b to m_a. Then the twists can be added as

$$\mathrm{t}^b_{0b/m_b} = \mathrm{t}^b_{0a/m_b} + \mathrm{t}^b_{ab/m_b}. \tag{13}$$

which gives

$$\mathrm{t}^b_{0b/m_b} = \begin{bmatrix} \boldsymbol{\omega}^b_{0b} \\ \mathbf{v}^b_{0b/m_b} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\omega}^b_{0a} + \boldsymbol{\omega}^b_{ab} \\ \mathbf{v}^b_{0a/m_b} + \mathbf{v}^b_{ab/m_b} \end{bmatrix}, \tag{14}$$

which is also shown graphically in Fig. 1. We will demonstrate later in this paper, that twists are useful and handy mathematical objects for defining the equation of motion of a multibody system, as they satisfy screw transformations. For more information about twists, the reader can refer to any book on Screw Theory, for example [18].

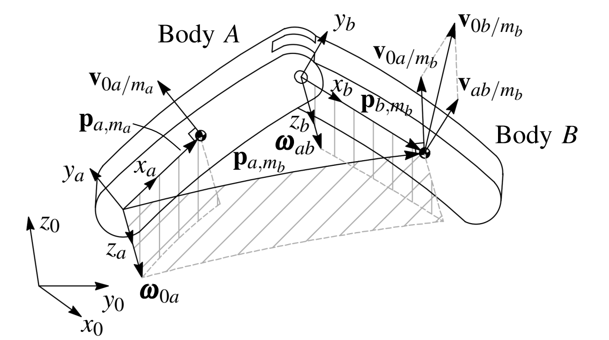

Fig. 1. An ordered couple of vectors of angular velocity and linear velocity constitute a twist. The hatched area spanned by the angular velocity and the distance vector is the magnitude of the linear velocity. Twists referenced to the same point and expressed in the coordinates of the same frame can be summed up.

B. System configuration and equation of motion

A crane is a system mechanically similar to a robotic manipulator arm. Therefore, in this section we present the preliminaries in general form, referring to manipulator links. In the following sections the theory will be applied for modeling of a crane on a moving base.

The configuration space of the system is defined by the set of degrees of freedom

$$\mathbf{q} = \begin{bmatrix} q_1 & .. & q_k & .. & q_n \end{bmatrix}^{\mathrm{T}}, \tag{15}$$

then the vector of generalized speeds [7] is given by

$$\mathbf{u} = \begin{bmatrix} u_1 & .. & u_k & .. & u_n \end{bmatrix}^{\mathrm{T}}, \tag{16}$$

where $u_k = \dot{q}_k$ are the speeds of the link joints and n is a number of links.

The equation of motion of each link is derived using the Newton-Euler approach. We describe dynamics about the center of gravity (COG) and formulate equations in a convenient matrix form as in [8]

$$\begin{bmatrix} \mathbf{M}^k_{m_k}\dot{\boldsymbol{\omega}}^k_{0k} + \hat{\boldsymbol{\omega}}^k_{0k}\mathbf{M}^k_{m_k}\boldsymbol{\omega}^k_{0k} - \mathbf{n}^k_{m_k} - \mathbf{n}^{k(c)}_k \\ m_k(\dot{\mathbf{v}}^k_{0k/m_k} + \hat{\boldsymbol{\omega}}^k_{0k}\mathbf{v}^k_{0k/m_k}) - \mathbf{f}^k_{m_k} - \mathbf{f}^{k(c)}_k \end{bmatrix} = \mathbf{0}, \tag{17}$$

where $\mathbf{v}^k_{0k/m_k}$ is the linear velocity of the COG of Body k relative to the inertial frame, $\boldsymbol{\omega}^k_{0k}$ is the angular velocity of Body k relative to the inertial frame, $\hat{\boldsymbol{\omega}}^k_{0k}$ denotes a skew-symmetric form of $\boldsymbol{\omega}^k_{0k}$, and $\mathbf{M}^k_{m_k}$ is the inertia matrix of Body k about its COG. The terms $\mathbf{f}^k_{m_k}$ and $\mathbf{n}^k_{m_k}$ are the equivalent force and moment with a line of action through the body COG, $\mathbf{f}^{k(c)}_k$ and $\mathbf{n}^{k(c)}_k$ are the forces and moments of constraints. All terms are given in the coordinates of Frame k which is fixed in Body k.

We define a link twist relative to the inertial frame (see Fig. 2), referenced to the origin of Frame k and expressed in the coordinates of Frame k as

$$\mathrm{t}^k_{0k/k} = \begin{bmatrix} \boldsymbol{\omega}^k_{0k} \\ \mathbf{v}^k_{0k/k} \end{bmatrix}. \tag{18}$$

The twist (18) can be transformed to be referenced to the COG point m_k by the screw transformation

$$\mathrm{t}^k_{0k/m_k} = \mathbf{U}^k_{k,m_k}\mathrm{t}^k_{0k/k}. \tag{19}$$

The term $\mathbf{U}^k_{k,m_k}$ is a screw transformation matrix which transforms the reference point of the twist. The matrix is defined by

$$\mathbf{U}^k_{k,m_k} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \hat{\mathbf{p}}^k_{m_k,k} & \mathbf{I} \end{bmatrix}, \tag{20}$$

where $\hat{\mathbf{p}}^k_{m_k,k}$ is a skew-symmetric form of the position vector $\mathbf{p}^k_{m_k,k}$ from the point m_k to the origin of Frame k. The twist in (18) can also be referenced to the origin of Frame $k+1$ by a screw transformation in the form of (19) using the $\mathbf{p}^k_{k+1,k}$ distance. It is initially expressed in the coordinates of Frame k and can be expressed in the coordinates of Frame $k+1$ by the coordinate transformation

$$\mathrm{t}^{k+1}_{0k/k+1} = \begin{bmatrix} \mathbf{R}^{k+1}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^{k+1}_k \end{bmatrix} \mathrm{t}^k_{0k/k+1}, \tag{21}$$

A simultaneous transformation of the reference and coordinates can be obtained by

$$\mathrm{t}^{k+1}_{0k/k+1} = \begin{bmatrix} \mathbf{R}^{k+1}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^{k+1}_k \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \hat{\mathbf{p}}^k_{k+1,k} & \mathbf{I} \end{bmatrix} \mathrm{t}^k_{0k/k}, \tag{22}$$

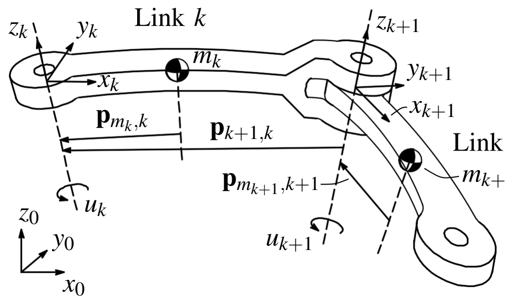

Fig. 2. A crane can be generalized as a robotic manipulator arm, where the links are connected by pivot joints with one actuated rotational degree of freedom in each joint.

which leads to the expression

$$\mathrm{t}^{k+1}_{0k/k+1} = \begin{bmatrix} \mathbf{R}^{k+1}_k & \mathbf{0} \\ \mathbf{R}^{k+1}_k \hat{\mathbf{p}}^k_{k+1,k} & \mathbf{R}^{k+1}_k \end{bmatrix} \mathrm{t}^k_{0k/k}. \tag{23}$$

The twist (23) can additionally be referenced to the COG of Link $k+1$ by the screw transformation

$$\mathrm{t}^{k+1}_{0k/m_{k+1}} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \hat{\mathbf{p}}^{k+1}_{m_{k+1},k+1} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{R}^{k+1}_k & \mathbf{0} \\ \mathbf{R}^{k+1}_k \hat{\mathbf{p}}^k_{k+1,k} & \mathbf{R}^{k+1}_k \end{bmatrix} \mathrm{t}^k_{0k/k}, \tag{24}$$

which gives

$$\mathrm{t}^{k+1}_{0k/m_{k+1}} = \begin{bmatrix} \mathbf{R}^{k+1}_k & \mathbf{0} \\ \hat{\mathbf{p}}^{k+1}_{m_{k+1},k+1} \mathbf{R}^{k+1}_k + \mathbf{R}^{k+1}_k \hat{\mathbf{p}}^k_{k+1,k} & \mathbf{R}^{k+1}_k \end{bmatrix} \mathrm{t}^k_{0k/k}. \tag{25}$$

By introducing the transformation matrix

$$\mathbf{V}^{k,k+1}_{k,m_{k+1}} = \begin{bmatrix} \mathbf{R}^{k+1}_k & \mathbf{0} \\ \hat{\mathbf{p}}^{k+1}_{m_{k+1},k+1} \mathbf{R}^{k+1}_k + \mathbf{R}^{k+1}_k \hat{\mathbf{p}}^k_{k+1,k} & \mathbf{R}^{k+1}_k \end{bmatrix}, \tag{26}$$

the transformation (25) can be written

$$\mathrm{t}^{k+1}_{0k/m_{k+1}} = \mathbf{V}^{k,k+1}_{k,m_{k+1}} \mathrm{t}^k_{0k/k}, \tag{27}$$

Twists expressed in the coordinates of the same frame and references to the same point can be summed up as in (13)

$$\mathrm{t}^{k+1}_{0,k+1/m_{k+1}} = \mathrm{t}^{k+1}_{0k/m_{k+1}} + \mathrm{t}^{k+1}_{k,k+1/m_{k+1}}. \tag{28}$$

A twist referenced to the COG of Link k can be expressed in term of generalized speeds as

$$\mathrm{t}^k_{0k/m_k} = \mathbf{P}_k \mathbf{u}, \tag{29}$$

where $\mathbf{P}_k$ is the projection matrix defined by

$$\mathbf{P}_k = \frac{\partial \mathrm{t}^k_{0k/m_k}}{\partial \mathbf{u}}. \tag{30}$$

In the point of contact of two rigid bodies, there is always the same set of constraint forces and moments acting on each body, but with the opposite sign. Therefore, the virtual work of the sum of constraint forces and moments is equal to zero. This is the principle of virtual work [8], which, in our case, can conveniently be formulated as

$$\sum_k \mathbf{P}^{\mathrm{T}}_k \begin{bmatrix} \mathbf{n}^{k(c)}_k \\ \mathbf{f}^{k(c)}_k \end{bmatrix} = 0. \tag{31}$$

The vector of generalized external forces (see [8]) is given by

$$\boldsymbol{\tau} = \sum_k \mathbf{P}^{\mathrm{T}}_k \begin{bmatrix} \mathbf{n}^k_{m_k} \\ \mathbf{f}^k_{m_k} \end{bmatrix}, \tag{32}$$

where the ordered couple of torque and force vectors is a wrench, which we flip for the convenience of the further development. It is notable that both wrenches and twists are screws and they satisfy the the screw transformation rules.

We formulate the equation of motion for the whole system by premultiplication of (17) with $\mathbf{P}^{\mathrm{T}}_k$, summing over all bodies k, and by utilizing the results (31), (32). This gives the result equivalent to the equation of motion given in [7]

$$\sum_k \mathbf{P}^{\mathrm{T}}_k \begin{bmatrix} \mathbf{M}^k_{m_k} \dot{\boldsymbol{\omega}}^k_{0k} + \hat{\boldsymbol{\omega}}^k_{0k} \mathbf{M}^k_{m_k} \boldsymbol{\omega}^k_{0k} \\ m_k (\dot{\mathbf{v}}^k_{0k/m_k} + \hat{\boldsymbol{\omega}}^k_{0k} \mathbf{v}^k_{0k/m_k}) \end{bmatrix} = \boldsymbol{\tau}. \tag{33}$$

Equation (33) can be written in the form

$$\sum_k \mathbf{P}^{\mathrm{T}}_k \left[\mathbf{D}_k \dot{\mathrm{t}}^k_{0k/m_k} + \mathbf{W}_k \mathbf{D}_k \mathrm{t}^k_{0k/m_k} \right] = \boldsymbol{\tau}, \tag{34}$$

where the matrices $\mathbf{D}_k$ and $\mathbf{W}_k$ are defined as

$$\mathbf{D}_k = \begin{bmatrix} \mathbf{M}^k_{m_k} & \mathbf{0} \\ \mathbf{0} & m_k \mathbf{I} \end{bmatrix}, \quad \mathbf{W}_k = \begin{bmatrix} \hat{\boldsymbol{\omega}}^k_{0k} & \mathbf{0} \\ \mathbf{0} & \hat{\boldsymbol{\omega}}^k_{0k} \end{bmatrix} \tag{35}$$

and $\boldsymbol{\omega}^k_{0k}$ is the first three rows of t^k_{0k/m_k}. By substitution of

$$\dot{\mathrm{t}}^k_{0k/m_k} = \mathbf{P}_k \dot{\mathbf{u}} + \dot{\mathbf{P}}_k \mathbf{u} \tag{36}$$

and (29) into (34) we get

$$\mathbf{M}\dot{\mathbf{u}} + \mathbf{C}\mathbf{u} = \boldsymbol{\tau}, \tag{37}$$

where

$$\begin{aligned} \mathbf{M} &= \sum_k \mathbf{P}^{\mathrm{T}}_k \mathbf{D}_k \mathbf{P}_k, \\ \mathbf{C} &= \sum_k \left[\mathbf{P}^{\mathrm{T}}_k \mathbf{D}_k \dot{\mathbf{P}}_k + \mathbf{P}^{\mathrm{T}}_k \mathbf{W}_k \mathbf{D}_k \mathbf{P}_k \right]. \end{aligned} \tag{38}$$

The matrices $\mathbf{M}$ and $\mathbf{C}$ have the property that $(\dot{\mathbf{M}} - 2\mathbf{C})$ is a skew-symmetric matrix [8].

C. Constraint forces

After the dynamics have been formulated, all constraint forces have been eliminated from the equation of motion, which is a consequence of the application of the principle of virtual work (31). However, in certain cases it can be necessary to bring those forces to evidence, see Sect. I. First, we decide on which constraint forces and moments (later referred just as *constraint forces*) we want to determine. Magnitudes of the constraint forces are collected in a vector as following

$$\boldsymbol{\rho}_{\mathbf{c}} = \begin{bmatrix} \rho_1 & \dots & \rho_j \end{bmatrix}^{\mathrm{T}}. \tag{39}$$

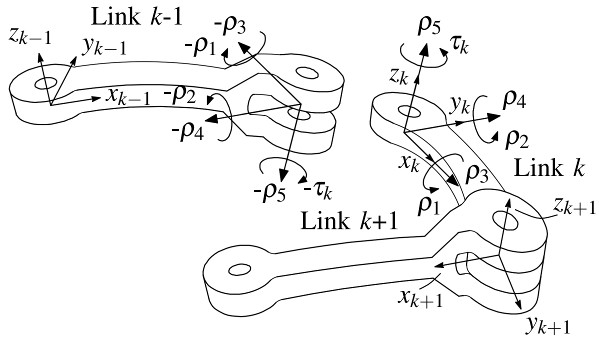

Fig. 3. For bringing noncontributing constraint forces to evidence, we imagine that Link k can lose the contact with Link $k-1$ and start moving with some additional velocities in the direction of the unknown constraint forces. Note that the gap between the links is only drawn for the demonstration purposes.

Then we define auxiliary generalized speeds associated with the unknown constraint forces

$$\mathbf{u_c} = \begin{bmatrix} u_{c1} & \dots & u_{cj} \end{bmatrix}^{\mathrm{T}}, \tag{40}$$

where each u_{ci} gives rise to an auxiliary velocity in the direction of each unknown constraint force ρ_i. Note that the relevant link, in fact, cannot posses those auxiliary velocities due to the constraints. Let us refer to the example given in Fig. 3. We *imagine* that Link k affected by the auxiliary velocities will lose the contact with Link $k-1$. In this way we can define the updated link velocities based on the contribution from the auxiliary velocities and the actual link velocities. The updated link twist based on the updated velocities is then defined by

$$\mathsf{t}^k_{\mathrm{c}0k/m_k} = \begin{bmatrix} \boldsymbol{\omega}^k_{\mathbf{c}0k} \\ \mathbf{v}^k_{\mathbf{c}0k/m_k} \end{bmatrix}. \tag{41}$$

It is worth noting that the updated link twist $\mathsf{t}_{\mathrm{c}0,k-1/m_{k-1}}$ of Link $k-1$ is not affected by the auxiliary velocities. The updated projection matrix is

$$\mathbf{P}_{\mathbf{c}k} = \frac{\partial \mathsf{t}^k_{\mathrm{c}0k/m_k}}{\partial \mathbf{u_c}}. \tag{42}$$

The updated vector of generalized external forces is given by

$$\boldsymbol{\tau}_{ft} = \boldsymbol{\tau}_f + \boldsymbol{\tau}_{\mathbf{c}}, \tag{43}$$

where $\boldsymbol{\tau}_f$ is the contribution from the actual external forces as in (32), and it is given by

$$\boldsymbol{\tau}_f = \sum_k \mathbf{P}^{\mathrm{T}}_{\mathbf{c}k} \begin{bmatrix} \mathbf{n}^k_{m_k} \\ \mathbf{f}^k_{m_k} \end{bmatrix}, \tag{44}$$

while $\boldsymbol{\tau}_{\mathbf{c}}$ is the contribution from the constraint forces, and it is given by

$$\boldsymbol{\tau}_{\mathbf{c}} = \big[\sum_k \mathbf{P}^{\mathrm{T}}_{\mathbf{c}k} \sum_p \bar{\mathbf{U}}^k_{p,m_k} \mathbf{N}_{pk}\big] \boldsymbol{\rho}_{\mathbf{c}}, \tag{45}$$

where p is a number of points on Body k where the constraint forces are applied. The matrix $\mathbf{N}_{pk}$ is a $6 \times n$ selection matrix. It is used to select the specific constraint forces acting at the point p from the vector of all unknown forces $\boldsymbol{\rho}_{\mathbf{c}}$ and form a six-dimensional vector of constraint forces and moments $[\mathbf{n}^{k(c)\mathrm{T}}_p \;\; \mathbf{f}^{k(c)\mathrm{T}}_p]^{\mathrm{T}}$ as in (44). Note that the sign and direction are important. In the special case shown in Fig. 3, $p=1$ for Links $k-1$, k and $p=0$ (i.e. $\mathbf{N}_{p,k+1} = \mathbf{0}$) for Link $k+1$.

The transformation matrix $\bar{\mathbf{U}}^k_{p,m_k}$ is introduced because the constraint forces will have a line of action through some point p, which, in general, will be different from the COG. It is convenient to substitute such a set of constraint forces by an equivalent set with a line of action through the COG, because the dynamics are formulated about it. The transformation matrix is given by

$$\bar{\mathbf{U}}^k_{p,m_k} = \begin{bmatrix} \mathbf{I} & \hat{\mathbf{p}}^k_{m_k,p} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \tag{46}$$

where $\mathbf{p}_{m_k,p}$ is the distance vector from the COG to the point p where the constraint forces are initially applied. The transformation matrix $\bar{\mathbf{U}}^k_{p,m_k}$ has a similar structure as (20), as we are using a flipped form of a wrench to define forces. It would be exactly the same if we have used a wrench in its correct form, where the force vector goes first and then followed by the torque vector.

To maintain the state of equilibrium, the external forces should always be equal to the inertial forces from the actual body dynamics. Premultiplication with $\mathbf{P}^{\mathrm{T}}_{\mathbf{c}k}$ can be interpreted as projection of the dynamics onto the direction of the unknown constraint forces. The final dynamic equilibrium can be written as

$$\sum_k \mathbf{P}^{\mathrm{T}}_{\mathbf{c}k} \big[\mathbf{D}_k \dot{\mathsf{t}}^k_{0k/m_k} + \mathbf{W}_k \mathbf{D}_k \mathsf{t}^k_{0k/m_k}\big] = \boldsymbol{\tau}_f + \boldsymbol{\tau}_{\mathbf{c}} \tag{47}$$

or, alternatively, as

$$\mathbf{M_f}\dot{\mathbf{u}} + \mathbf{C_f}\mathbf{u} = \boldsymbol{\tau}_f + \boldsymbol{\tau}_{\mathbf{c}}, \tag{48}$$

where $\mathbf{u}$ are taken from the actual system dynamics. The new matrix notations are defined by

$$\begin{aligned} \mathbf{M_f} &= \sum_k \mathbf{P}^{\mathrm{T}}_{\mathbf{c}k} \mathbf{D}_k \mathbf{P}_k, \\ \mathbf{C_f} &= \sum_k \big[\mathbf{P}^{\mathrm{T}}_{\mathbf{c}k} \mathbf{D}_k \dot{\mathbf{P}}_k + \mathbf{P}^{\mathrm{T}}_{\mathbf{c}k} \mathbf{W}_k \mathbf{D}_k \mathbf{P}_k\big]. \end{aligned} \tag{49}$$

Finally the vector of the unknown magnitudes of constraint forces can be found from

$$\boldsymbol{\rho}_{\mathbf{c}} = \big[\sum_k \mathbf{P}^{\mathrm{T}}_{\mathbf{c}k} \sum_p \bar{\mathbf{U}}^k_{p,m_k} \mathbf{N}_{pk}\big]^{-1} \big[\mathbf{M_f}\dot{\mathbf{u}} + \mathbf{C_f}\mathbf{u} - \boldsymbol{\tau}_f\big]. \tag{50}$$

It is seen that if the equations of motion have been developed in the form of (37), then the additional work to determine the certain forces of constraints as given by (50) is systematic and efficient, and where the steps of the procedure have a clear geometric interpretation.

III. A CRANE/VESSEL SYSTEM

A. Equation of motion

We consider a crane/vessel system, which consists of four bodies as shown in Fig. 4. Body 1 is a vessel, which is a moving base for the rest of the system. Body 2 is the

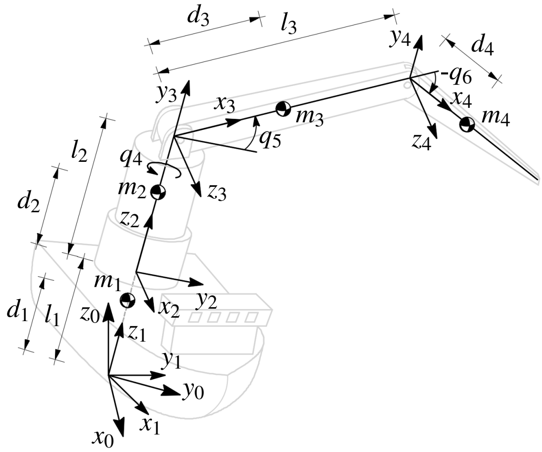

Fig. 4. A crane/vessel system consists of four bodies: a vessel, a crane king, an inner boom and an outer boom.

first link of the crane, which is often also referred as a king. Body 3 is an inner boom of the crane and Body 4 is an outer boom. In this work we do not aim to simulate the true motion of the vessel, but rather show the way of determination of the constraint forces between the crane and the vessel. Therefore, we model the vessel as a 3D body supported by three nonlinear springs in roll, pitch and yaw directions. This modeling technique provides simplified, yet realistic, model of restoring forces of the vessel.

We assume that the rotation center of the vessel is positioned in the origin of the inertial frame. Then the orientation of the vessel is described by three variables q_1, q_2, q_3: roll, pitch and yaw. The configuration of the crane is defined by the rotation angles q_4, q_5, q_6 in three active joints. The configuration of the whole system is defined by

$$\mathbf{q} = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 & q_5 & q_6 \end{bmatrix}^{\mathrm{T}}. \tag{51}$$

The vector of generalized speeds is given by

$$\mathbf{u} = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{bmatrix}^{\mathrm{T}}. \tag{52}$$

The inertial frame is given as Frame 0, the vessel body-fixed frame is Frame 1, the king body-fixed frame is Frame 2, the inner boom body-fixed frame is Frame 3 and the outer boom body-fixed frame is Frame 4.

The rotation matrix from Frame 0 to Frame 1 is given by

$$\mathbf{R}_1^0 = \mathbf{R}_z(q_3)\mathbf{R}_y(q_2)\mathbf{R}_x(q_1), \tag{53}$$

where the matrices $\mathbf{R}_x(\cdot)$, $\mathbf{R}_y(\cdot)$ and $\mathbf{R}_z(\cdot)$ are the rotation matrices about the x, y and z axes, respectively. The rotation matrix from Frame 1 to Frame 2 is given by

$$\mathbf{R}_2^1 = \mathbf{R}_z(q_4). \tag{54}$$

The rotation matrix from Frame 2 to Frame 3 is given by

$$\mathbf{R}_3^2 = \mathbf{R}_z(\pi/2)\mathbf{R}_x(\pi/2)\mathbf{R}_z(q_5). \tag{55}$$

The rotation matrix from Frame 3 to Frame 4 is given by

$$\mathbf{R}_4^3 = \mathbf{R}_z(q_6). \tag{56}$$

The relative body twists of the system bodies are

$$\begin{aligned} \mathrm{t}_{01/1}^1 &= \begin{bmatrix} \boldsymbol{\omega}_{01}^1 \\ \mathbf{0} \end{bmatrix}, \quad \mathrm{t}_{12/2}^2 = \begin{bmatrix} \boldsymbol{\omega}_{12}^2 \\ \mathbf{0} \end{bmatrix}, \\ \mathrm{t}_{23/3}^3 &= \begin{bmatrix} \boldsymbol{\omega}_{23}^3 \\ \mathbf{0} \end{bmatrix}, \quad \mathrm{t}_{34/4}^4 = \begin{bmatrix} \boldsymbol{\omega}_{34}^4 \\ \mathbf{0} \end{bmatrix}. \end{aligned} \tag{57}$$

The relative body twists are used to calculate the body twists relative to the inertial frame and referenced to the COG of each body using the screw transformation rules (19), (27) and (28).

The twist $\mathrm{t}_{01/1}^1$ is referenced to the COG of Body 1 and expressed in the coordinates of Frame 1 as in (19)

$$\mathrm{t}_{01/m_1}^1 = \mathbf{U}_{1,m_1}^1 \mathrm{t}_{01/1}^1. \tag{58}$$

The twist t_{02/m_2}^2 referenced to the COG of Body 2 and expressed in the coordinates of Frame 2 is calculated as

$$\begin{aligned} \mathrm{t}_{01/m_2}^2 &= \mathbf{V}_{1,m_2}^{12} \mathrm{t}_{01/1}^1, \\ \mathrm{t}_{12/m_2}^2 &= \mathbf{U}_{2,m_2}^2 \mathrm{t}_{12/2}^2, \\ \mathrm{t}_{02/m_2}^2 &= \mathrm{t}_{01/m_2}^2 + \mathrm{t}_{12/m_2}^2. \end{aligned} \tag{59}$$

An explicit definition of the transformation matrix $\mathbf{V}_{1,m_2}^{12}$ is

$$\mathbf{V}_{1,m_2}^{12} = \begin{bmatrix} \mathbf{R}_1^2 & \mathbf{0} \\ \hat{\mathbf{p}}_{m_2,2}^2 \mathbf{R}_1^2 + \mathbf{R}_1^2 \hat{\mathbf{p}}_{21}^1 & \mathbf{R}_1^2 \end{bmatrix}, \tag{60}$$

where $\mathbf{R}_1^2 = (\mathbf{R}_2^1)^{\mathrm{T}}$, $\mathbf{p}_{21}^1 = \begin{bmatrix} 0 & 0 & -l_1 \end{bmatrix}^{\mathrm{T}}$ and $\mathbf{p}_{m_2,2}^2 = \begin{bmatrix} 0 & 0 & -d_2 \end{bmatrix}^{\mathrm{T}}$. The twist t_{03/m_3}^3 referenced to the COG of Body 3 and expressed in the coordinates of Frame 3 is calculated as

$$\begin{aligned} \mathrm{t}_{02/2}^2 &= \mathbf{U}_{m_2,2}^2 \mathrm{t}_{02/m_2}^2, \\ \mathrm{t}_{02/m_3}^3 &= \mathbf{V}_{2,m_3}^{23} \mathrm{t}_{02/2}^2, \\ \mathrm{t}_{23/m_3}^3 &= \mathbf{U}_{3,m_3}^3 \mathrm{t}_{23/3}^3, \\ \mathrm{t}_{03/m_3}^3 &= \mathrm{t}_{02/m_3}^3 + \mathrm{t}_{23/m_3}^3. \end{aligned} \tag{61}$$

The twist t_{04/m_4}^4 referenced to the COG of Body 4 and expressed in the coordinates of Frame 4 is calculated as

$$\begin{aligned} \mathrm{t}_{03/3}^3 &= \mathbf{U}_{m_3,3}^3 \mathrm{t}_{03/m_3}^3, \\ \mathrm{t}_{03/m_4}^4 &= \mathbf{V}_{3,m_4}^{34} \mathrm{t}_{03/3}^3, \\ \mathrm{t}_{34/m_4}^4 &= \mathbf{U}_{4,m_4}^4 \mathrm{t}_{34/4}^4, \\ \mathrm{t}_{04/m_4}^4 &= \mathrm{t}_{03/m_4}^4 + \mathrm{t}_{34/m_4}^4. \end{aligned} \tag{62}$$

The equation of motion for the crane/vessel system can be formulated following the rest of the procedure presented in Sect. II-B.

B. Control

We propose a PD controller to asymptotically regulate the crane joint positions at the desired values. If $\boldsymbol{\tau}_a$ is a three-dimensional vector of the crane input torques, then the controller is given by

$$\boldsymbol{\tau}_a = -k_P \mathbf{q}_c - k_D \dot{\mathbf{q}}_c, \tag{63}$$

where $\mathbf{q}_c = \begin{bmatrix} q_4 & q_5 & q_6 \end{bmatrix}^{\mathrm{T}}$ and k_P, k_D are positive constants.

C. Constraint forces

To obtain the constraint forces and moments (or just *constraint forces*) in the joint between Body 1 and Body 2 (i.e. between the crane and the vessel), we follow the procedure presented in Sect. II-C. The vector of the unknown magnitudes of constraint forces is

$$\boldsymbol{\rho}_{\mathbf{c}} = \begin{bmatrix}\rho_1 & \rho_2 & \rho_3 & \rho_4 & \rho_5\end{bmatrix}^{\mathrm{T}}, \tag{64}$$

where ρ_1, ρ_2 are the constraint moments about x_2, y_2 axes and ρ_3, ρ_4, ρ_5 are the constraint forces in x_2, y_2, z_2 directions, see Fig. 4. We define auxiliary generalized speeds associated with the unknown constraint forces as

$$\mathbf{u_c} = \begin{bmatrix}u_{c1} & u_{c2} & u_{c3} & u_{c4} & u_{c5}\end{bmatrix}^{\mathrm{T}}. \tag{65}$$

The auxiliary generalized speeds (65) give rise to the auxiliary velocity of Body 2, which propagates to all successive bodies in the system, while it does not affect the velocity of Body 1, that is, $\mathsf{t}^1_{\mathrm{c}01/m_1} = \mathsf{t}^1_{01/m_1}$. For all other bodies in the system we define the updated velocities with contribution from the actual velocities and the auxiliary velocities (65). The updated twist referenced to the origin of Frame 2 and expressed in the coordinates of Frame 2 is

$$\mathsf{t}^2_{\mathrm{c}02/2} = \mathsf{t}^2_{02/2} + \tilde{\mathsf{t}}^2_{\mathrm{c}2}, \tag{66}$$

where

$$\tilde{\mathsf{t}}^2_{\mathrm{c}2} = \begin{bmatrix}u_{c1} & u_{c2} & 0 & u_{c3} & u_{c4} & u_{c5}\end{bmatrix}^{\mathrm{T}}. \tag{67}$$

The updated twist $\mathsf{t}^2_{\mathrm{c}02/m_2}$ referenced to the COG of Body 2 and expressed in the coordinates of Frame 2 is

$$\mathsf{t}^2_{\mathrm{c}02/m_2} = \mathbf{U}^2_{2,m_2}\mathsf{t}^2_{\mathrm{c}02/2}. \tag{68}$$

In this specific case we are only interested in the constraint forces in one joint and we have only introduced auxiliary velocities in one joint.

The updated twist $\mathsf{t}^3_{\mathrm{c}03/m_3}$ referenced to the COG of Body 3 and expressed in the coordinates of Frame 3 is

$$\begin{aligned}\mathsf{t}^3_{\mathrm{c}02/m_3} &= \mathbf{V}^{23}_{2,m_3}\mathsf{t}^2_{\mathrm{c}02/2},\\ \mathsf{t}^3_{\mathrm{c}03/m_3} &= \mathsf{t}^3_{\mathrm{c}02/m_3} + \mathsf{t}^3_{23/m_3}.\end{aligned} \tag{69}$$

The updated twist $\mathsf{t}^4_{\mathrm{c}04/m_4}$ referenced to the COG of Body 4 and expressed in the coordinates of Frame 4 is

$$\begin{aligned}\mathsf{t}^3_{\mathrm{c}03/3} &= \mathbf{U}^3_{m_3,3}\mathsf{t}^3_{\mathrm{c}03/m_3},\\ \mathsf{t}^4_{\mathrm{c}03/m_4} &= \mathbf{V}^{34}_{3,m_4}\mathsf{t}^3_{\mathrm{c}03/3},\\ \mathsf{t}^4_{\mathrm{c}04/m_4} &= \mathsf{t}^4_{\mathrm{c}03/m_4} + \mathsf{t}^4_{34/m_4}.\end{aligned} \tag{70}$$

The updated projection matrices $\mathbf{P}_{\mathbf{c}k}$ for $k = 1,2,3,4$ are calculated by (42), while the contribution from the external forces and moments $\boldsymbol{\tau}_f$ is obtained as in (44).

As the constraint forces $\boldsymbol{\tau}_{\mathbf{c}}$ (45) are only applied on Bodies 1 and 2 at one point $p = 1$, the selection matrices are

$$\mathbf{N}_{1,1} = \begin{bmatrix} -c_4 & s_4 & 0 & 0 & 0\\ -s_4 & -c_4 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -c_4 & s_4 & 0\\ 0 & 0 & -s_4 & -c_4 & 0\\ 0 & 0 & 0 & 0 & -1\end{bmatrix}, \tag{71}$$

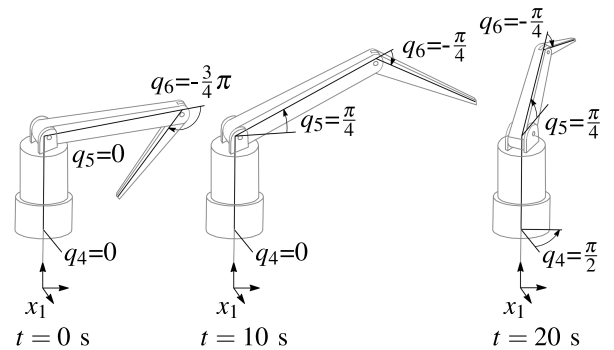

Fig. 5. Positions of the crane throughout the simulation represent the sequence of crane positions during a typical hoisting operation.

where $c_4 = \cos(q_4)$ and $s_4 = \sin(q_4)$, and

$$\mathbf{N}_{1,2} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}, \tag{72}$$

while $\mathbf{N}_{p,3} = \mathbf{N}_{p,4} = 0$. It is straightforward to calculate the rest of the terms in (50) and then determine the magnitudes of the constraint forces.

D. Simulation results

In this section we have considered a scaled crane/vessel system given in Fig. 4. The goal of the simulation was to determine the constraint forces in the interface between the crane and the vessel. The geometric constants and masses used in the model are given in Table I.

TABLE I
SYSTEM DIMENSIONS AND MASSES

Constant	l_1	l_2	l_3	d_1	d_2	d_3	d_4
Value	5.0m	2.0m	2.0m	2.5m	1.0m	1.0m	1.0m
Constant	m_1	m_2	m_3	m_4			
Value	1.2t	30kg	30kg	30kg			

The inertia matrices of the system bodies in the coordinates of the body-fixed frames are defined as

$$\begin{aligned}\mathbf{M}^1_{m_1} &= \mathrm{diag}(m_1 l_1^2/12,\ m_1 l_1^2/12,\ m_1 l_1^2/12),\\ \mathbf{M}^2_{m_2} &= \mathrm{diag}(m_2 l_2^2/12,\ m_2 l_2^2/12,\ 0.1 m_2 l_2^2/12),\\ \mathbf{M}^3_{m_3} &= \mathrm{diag}(0,\ m_3 l_3^2/12,\ m_3 l_3^2/12),\\ \mathbf{M}^4_{m_4} &= \mathrm{diag}(0,\ m_4 l_4^2/12,\ m_4 l_4^2/12).\end{aligned} \tag{73}$$

The nonlinear rotational spring elements were attached to the origin of Frame 1 to simulate the vessel motion. Torques in roll, pitch and yaw directions generated by these springs are formulated as $T_{si} = -k_i q_i^3$, where $i = 1,2,3$ and all $k_i = 8 \cdot 10^7$.

The vessel was initialized at its origin, that is $q_i = 0$ for $i = 1,2,3$ at $t = 0$. The crane variables were changed during the simulation according to Fig. 5 and 6. Such motion of the

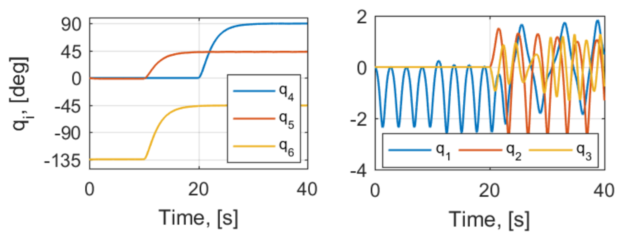

Fig. 6. The crane joint variables are regulated according to the motion sequence given in Fig. 5. The vessel is initialized at its origin. The motion of the vessel is excited by the crane motion.

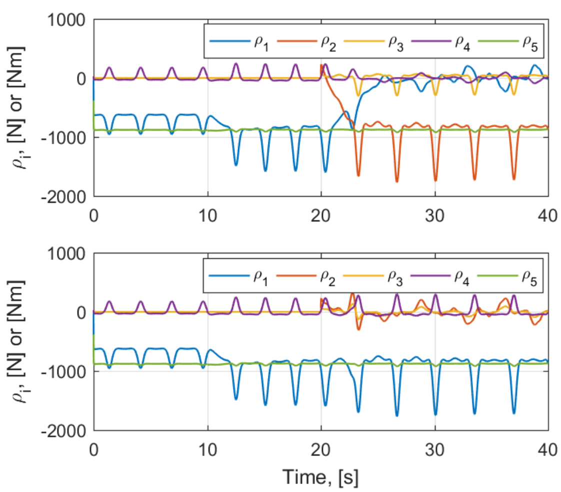

Fig. 7. The constraint forces and moments between the crane and the vessel are given in the coordinates of Frame 1 (on the top) and in the coordinates of Frame 2 (in the bottom).

crane can be considered as a typical sequence of the crane positions during a hoisting operation. The values ρ_1, ρ_2 are the constraint moments about x_2, y_2 axes and ρ_3, ρ_4, ρ_5 are the constraint forces in x_2, y_2, z_2 directions, see Fig. 4. They are given in the coordinates of Frame 1 and in the coordinates of Frame 2 in Fig. 7.

IV. Conclusions

We have presented the general procedure for modeling of the dynamics of multibody systems. The procedure is based on Kane's method and is relying on twists and screw transformations. Additionally, we have presented the procedure for determination of constraint forces, which is based on the method of generalized speeds and is given as an extension of the dynamical model.

The presented procedure was implemented for the dynamical model of a crane on a moving base system. We have demonstrated how 5 constraint forces in the interface between the crane and the vessel can be efficiently determined and the simulation results were provided. The constraint forces are related to the structural forces in the joints and can directly be used in the problems where modeling and control of friction is of interest. Constraint forces in this form can also be used in sensitivity studies on how different control algorithms affect the structural strength of mechanisms. This gives an opportunity for optimization of the control system by minimizing the impact on the structural strength and the fatigue lifetime, which is a relevant practical aspect in the production and operation of mechanisms.

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