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STABILITY STRUCTURES FOR ABELIAN AND TRIANGULATED CATEGORIES

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Problem Description

There are several ways to define algebraic stability structures. In any case semistable subcategories and a kind of Harder-Narisimhan filtration should be defined. The goal of the paper was to compare existing approaches to stability for abelian and triangulated categories. Then to study the quivers where distinguished slope or stability data can be defined.

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Abstract. This thesis is intended to present some developments in the theory of algebraic stability. The main topics are stability for triangulated categories and the distinguished slopes of Hille and de la Peña for quiver representations.

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Contents

Introduction

The notion of stability from algebraic geometry gave rise to several similar tools that can be used to study abelian or triangulated categories. In [Rud97] a definition of stability was made for arbitrary abelian categories, the articles [Bri02] and [GKR04] give definitions of stability for triangulated categories.

The aim of this thesis is to give an understandable presentation of the stability notions of [Bri02] and [GKR04] for triangulated categories. We also consider the distinguished slopes of [HdlP01] for categories of representations of wild quivers with no oriented cycles.

In chapter 1 we review the main results of stability for abelian categories. Since the preproject [Ste06] was closely connected to this topic the proofs are omitted, but they can be found in [Rud97]. Chapter 2 is a short introduction to triangulated categories with emphasis on t-structures. Hopefully this gives the reader the background he needs to appreciate the stability notions of [Bri02] and [GKR04] which are presented in chapter 3. Chapter 4 is not so tightly related to triangulated categories and can be read independently of Chapters 2 and 3. There we present the distinguished stability conditions for quiver representations of [HdlP01]. In the spirit of Bridgeland's central charge we order the representations of a quiver Q by the angles of their images through a special group homomorphism from the Grothendieck group into the complex numbers. The homomorphism is obtained via the eigenvectors of the coxeter matrix of Q. In this chapter we assume that the reader is familiar with some of the theory of quiver representations, an introduction can be found in [ARS97]. We also use the reflection functors of [BGP73] for quiver representations to see that even when we don't have distinguished slopes, we sometimes still have some stability data.

Notation and Basic Definitions. Let us make some remarks about the notation that is used in this thesis. For an abelian category $\mathcal{A}$ we denote the Grothendieck group of $\mathcal{A}$ by $K_0(\mathcal{A})$. With a slight abuse of notation we denote the grothendieck group of the representations of a quiver Q by $K_0(Q)$. To symbolise intervals of real numbers we use the brackets (,) for open boundaries and [,] are used for closed boundaries. So $(x, y] = \{t \in \mathbb{R} | x < t \leq y\}$ for x and y in $\mathbb{R}$. It should be mentioned that the letters $\mathcal{T}$ and τ are different and will have different meanings in this text. The letter $\mathcal{T}$ will be used for the shift functor of a triangulated category, while τ will be used for the Auslander-Reiten translate DTr of quiver representations. We use $\subset$ for strict inclusions and $\subseteq$ for non-strict inclusions, examples are ended by —— on the right hand side of the page.

Thanks to. There are many people I would like to thank for their help and support during my studies at NTNU. First of all let me thank my friends and family, especially my brother Vegard Bertelsen who got me interested in mathematics. Secondly let me thank my fellow students for their good company and helpful remarks. I would also like to thank my supervisor Alexei Rudakov who has influenced the way I think about mathematics alot during the last couple of years, and whose feedback has been essential for concluding this paper.

1. Stability conditions on abelian categories

In this chapter we review some of the main results concerning stability of abelian categories. We shall introduce several similar but different definitions that may motivate the two different definitions for stability of triangulated categories later. The main reference is [Rud97].

1.1. **Slope stability.**

Definition 1.1.1. Let $\mathcal{A}$ be an abelian category, $K_0(\mathcal{A})$ its Grothendieck group and $\theta, \kappa \colon K_0(\mathcal{A}) \to \mathbb{R}$ group homomorphisms such that for each object $A \in \mathcal{A}$ we have $\kappa([A]) > 0$. Then the map

$$\begin{aligned} \mu \colon K_0(\mathcal{A}) &\to \mathbb{R} \\ [X] &\mapsto \frac{\theta([X])}{\kappa([X])} \end{aligned}$$

is called a *slope* on $K_0(\mathcal{A})$.

Each slope μ on $K_0(\mathcal{A})$ gives us some notion of order on the non-zero objects in $\mathcal{A}$ by $A \leq_\mu B \iff \mu([A]) \leq \mu([B])$. This ordering satisfies the seesaw property of [Rud97] that for any short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of non-zero objects in $\mathcal{A}$ the only possible orderings of A, B and C are $(A <_\mu B <_\mu C)$, $(A >_\mu B >_\mu C)$ or $(A =_\mu B =_\mu C)$.

Definition 1.1.2 (Slope stability)**.** For a slope μ on $K_0(\mathcal{A})$, we say that an object X of $\mathcal{A}$ is *μ-semistable* if for each subobject $Y \subset X$ we have $Y \leq_\mu X$. If the strict inequality $Y <_\mu X$ holds for each $Y \subset X$ we say that X is *μ-stable.*

We get a set of full abelian subcategories of $\mathcal{A}$ by considering the μ-semistable objects of a given value in μ

$$ob(S_\mu(r)) = \{X \in \mathcal{A} | X \ \mu\text{-semistable and } \mu([X]) = r\} \cup \{0\}.$$

Some of these categories are just zero categories and may be disregarded. The morphisms between these categories behave very well, for $r_1 < r_2 \in \mathbb{R}$ we have $\mathrm{Hom}(S_\mu(r_2), S_\mu(r_1)) = 0$. If the category $\mathcal{A}$ is both noetherian and artinian we get for each non-zero object $A \in \mathcal{A}$ a filtration

$$\begin{array}{ccccccccccccc} A = F_0 & \hookleftarrow & F_1 & \hookleftarrow & F_2 & \hookleftarrow & \cdots & \hookleftarrow & F_n & \hookleftarrow & 0 \\ & \searrow & & \searrow & & \searrow & & & & \searrow & \\ & & F_0/F_1 & & F_1/F_2 & & F_2/F_3 & & \cdots & & F_n \end{array}$$

where the factors F_i/F_{i+1} are μ-semistable and ordered

$$F_0/F_1 <_\mu F_1/F_2 <_\mu \cdots <_\mu F_{n-1}/F_n <_\mu F_n.$$

This filtration will be called the Harder-Narisimhan filtration (or HN-filtration for short).

1.2. **Order stability.** As we consider the properties of slope stability we may notice that to get the categories $\{S_\mu(r)\}$ and existence of HN-filtrations we do not really need the slope. What is crucial is the ordering of the non-zero objects of $\mathcal{A}$ satisfying the seesaw property. We make definitions very similar to definition 1.1.2.

Definition 1.2.1. An ordering $\leq_\mu$ of the non-zero objects of $\mathcal{A}$ is called a *stability ordering* on $\mathcal{A}$ if the following conditions are satisfied:

(i) $A =_\mu B$ whenever $A \simeq B$,

(ii) the seesaw property holds.

Definition 1.2.2 (Order stability)**.** Let $\leq_\mu$ be a stability ordering on $\mathcal{A}$. We say that an object X is $\leq_\mu$-semistable if for any subobject $Y \subset X$ we have that $Y \leq_\mu X$. If $Y <_\mu X$ for any $Y \subset X$ then we say that X is $\leq_\mu$-stable.

We can define the full semistable categories $S_{\leq_\mu}(A)$ by their objects

$$ob(S_{\leq_\mu}(A)) = \{X | X \leq_\mu\text{-semistable and } X =_\mu A\} \cup \{0\}.$$

The good behaviour of the morphisms and the existence of HN-filtration for each non-zero object follows by the same arguments as for slope stability (see [Rud97]).

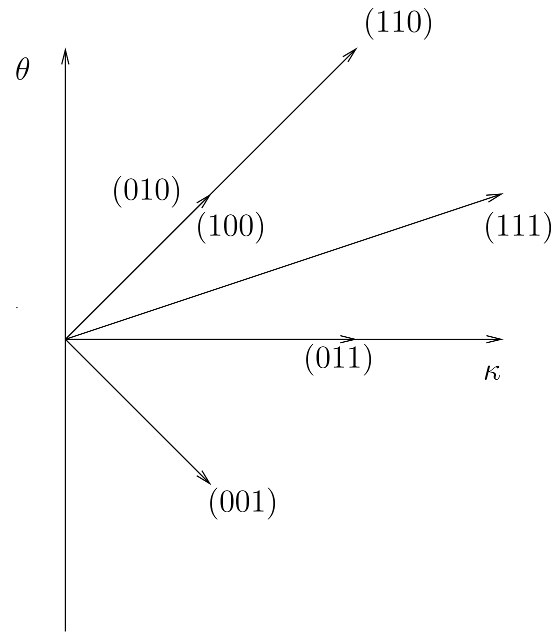
The ordering induced by $\leq_\mu$ on the semistable categories is a total ordering. It is natural to say that two stability orderings are *equivalent* if they induce the same semistable categories and the total order on these categories are the same. In [Ste06] we saw that for the category of representations of the quiver

$$Q: \ 1 \longrightarrow 2 \longrightarrow 3$$

every order stability is equivalent to a slope stability. In general however it is not clear whether or not any such result can be true.

Note that whenever we have a slope μ given by the group homomorphisms θ and κ as above, we can get equivalent stability conditions by considering the group homomorphism $\mu' \colon K_0(\mathcal{A}) \to \mathbb{C}$ defined by $\mu'(X) = \kappa(X) + i\theta(X)$ and ordering the objects by their angle from the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ in $\mathbb{C}$.

Example. Let us consider the quiver $1 \to 2 \to 3$ and the slope given by ratio $\mu = \theta/\kappa$ of the additive functions $\theta(d_1, d_2, d_3) = d_1 + d_2 - d_3$ and $\kappa(d_1, d_2, d_3) = d_1 + d_2 + d_3$. We get an ordering by the angles as illustrated in the picture.



$$(001) <_\mu (011) <_\mu (111) <_\mu (100) =_\mu (010) =_\mu (110).$$

1.3. **Stability data.** We can of course also define stability abstractly by just assuming our desired results to be satisfied, this we will call stability data.

Definition 1.3.1 (Stability data)**.** For an abelian category $\mathcal{A}$ we say that a set $\{\Pi_\phi\}_{\phi\in\Phi}$ of full abelian subcategories indexed by a totally ordered set Φ gives *stability data* for $\mathcal{A}$ if

(i) $\phi > \psi \in \Phi$ implies $\mathrm{Hom}(\Pi_\phi, \Pi_\psi) = 0$,
(ii) each non-zero object A in $\mathcal{A}$ have a HN-filtration

$$\begin{array}{ccccccccccc} A = F_0 & \hookleftarrow & F_1 & \hookleftarrow & F_2 & \hookleftarrow & \cdots & \hookleftarrow & F_n & \hookleftarrow & 0 \\ & \searrow & & \searrow & & \searrow & & & & \searrow & \\ & & \pi_{\phi_0} & & \pi_{\phi_1} & & \pi_{\phi_2} & & \cdots & & \pi_{\phi_n} \end{array}$$

with $\pi_{\phi_i} \in \Pi_{\phi_i}$ and $\phi_0 < \phi_1 < \cdots < \phi_n$.

Clearly we can define an equivalence relation on stability data the same way as we did for stability orderings. We shall see in chapter 4 that there are examples of stability data that are not equivalent to any slope stability.

2. Triangulated categories

In this chapter we will go through the essential properties of triangulated categories, it seems appropriate to give a short introduction to these categories before we investigate their stabilty in chapter 3. Our main focus will be on t-structures which are tightly connected to our stability conditions. We leave out much of the technical details about the derived categories. For a more precise and detailed introduction to triangulated categories I strongly recommend [KS90, Chapter I].

2.1. **Axioms.**

Definition 2.1.1. A *triangulated category* is an additive category $\mathcal{C}$ together with an automorphism $\mathcal{T}:\mathcal{C} \to \mathcal{C}$ called the *shift functor*, and a class of sequences $\{A \to B \to C \to \mathcal{T}A | A, B, C \in \mathcal{C}\}$ that we shall call triangles. The triangles should satisfy the following axioms:

$Tr1$:

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & \mathcal{T}A \\ \downarrow{\scriptstyle \phi_A} & & \downarrow{\scriptstyle \phi_B} & & \downarrow{\scriptstyle \phi_C} & & \downarrow{\scriptstyle \mathcal{T}\phi_A} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & \mathcal{T}A' \end{array}$$

If $A \to B \to C \to \mathcal{T}A$ is a triangle and the maps ϕ_A, ϕ_B and ϕ_C are isomorphisms making the diagram commutative, then $A' \to B' \to C' \to \mathcal{T}A'$ is also a triangle.

$Tr2$: The sequence $A = A \longrightarrow 0 \longrightarrow \mathcal{T}A$ is a triangle for any $A \in \mathcal{C}$.

$Tr3$: For any morphism $f : A \to B$ there is a triangle

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow \mathcal{T}A$$

for some $C \in \mathcal{C}$.

$Tr4$: The sequence $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \mathcal{T}A$ is a triangle if and only if $B \xrightarrow{g} C \xrightarrow{h} \mathcal{T}A \xrightarrow{-\mathcal{T}f} \mathcal{T}B$ is a triangle.

$Tr5$: If $A \xrightarrow{f} B \longrightarrow C \longrightarrow \mathcal{T}A$ and $A' \xrightarrow{f'} B' \longrightarrow C' \longrightarrow \mathcal{T}A'$ are two triangles and

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\ A' & \xrightarrow{f'} & B' \end{array}$$

commutes, then there is a commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \longrightarrow & C & \longrightarrow & \mathcal{T}A \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \mathcal{T}u} \\ A' & \xrightarrow{f'} & B' & \longrightarrow & C' & \longrightarrow & \mathcal{T}A'. \end{array}$$

We will call such a diagram a *morphism* of triangles, if u, v and w are all isomorphisms we say that the diagram is an *isomorphism* of triangles.

$Tr6:$ (Octahedral axiom) If

$$A \xrightarrow{f} B \xrightarrow{p} C' \longrightarrow \mathcal{T}A,$$

$$B \xrightarrow{g} C \xrightarrow{q} A' \longrightarrow \mathcal{T}B,$$

$$A \xrightarrow{gf} C \xrightarrow{r} B' \longrightarrow \mathcal{T}A$$

are three triangles, then there is a triangle

$$C' \longrightarrow B' \longrightarrow A' \longrightarrow \mathcal{T}C'$$

making the following diagram commutative:

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{p} & C' & \longrightarrow & \mathcal{T}A \\ \| & & \downarrow{\scriptstyle g} & & \downarrow & & \| \\ A & \xrightarrow{gf} & C & \xrightarrow{r} & B' & \longrightarrow & \mathcal{T}A \\ \downarrow{\scriptstyle f} & & \| & & \downarrow & & \downarrow{\scriptstyle \mathcal{T}f} \\ B & \xrightarrow{g} & C & \xrightarrow{q} & A' & \longrightarrow & \mathcal{T}B \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle r} & & \| & & \downarrow{\scriptstyle \mathcal{T}p} \\ C' & \longrightarrow & B' & \longrightarrow & A' & \longrightarrow & \mathcal{T}C' \end{array}$$

Notice that being an automorphism of categories is stricter than just being an equivalence. For an automorphism $\mathcal{T}$ we require the existence of an additive functor $\mathcal{T}^{-1}$ such that both compositions $\mathcal{T}\mathcal{T}^{-1}$ and $\mathcal{T}^{-1}\mathcal{T}$ equals the identity functor, while for an equivalence we are satisfied with the compositions being naturally isomorphic to the identity functor.

2.2. **Basic properties.**

Definition 2.2.1. A *cohomological functor* from a triangulated category $\mathcal{C}$ is an additive functor $F : \mathcal{C} \to \mathcal{A}$ to an abelian category $\mathcal{A}$ such that for any triangle $A \to B \to C \to \mathcal{T}A$ the image $F(A) \to F(B) \to F(C)$ (or $F(C) \to F(B) \to F(A)$ in the contravariant case) is an exact sequence.

By $Tr4$ any covariant cohomological functor F will give us long exact sequences

$$\cdots \to F(\mathcal{T}^{-1}C) \to F(A) \to F(B) \to F(C) \to F(\mathcal{T}A) \to \cdots,$$

and similarly for contravariant cohomological functors.

Lemma 2.2.2. *Let*

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \mathcal{T}A$$

be a triangle, then $gf = 0$

Proof. We have the commutative diagram:

$$\begin{array}{ccc} A & = & A \\ \| & & \downarrow f \\ A & \xrightarrow{f} & B \end{array}$$

By $Tr5$ it can be extended to the morphism of triangles

$$\begin{array}{ccccccc} A & = & A & \longrightarrow & 0 & \longrightarrow & \mathcal{T}A \\ \| & & \downarrow f & & \downarrow & & \downarrow \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & \mathcal{T}A \end{array}$$

so gf factors through 0.

□

Proposition 2.2.3. *For any object X in a triangulated category $\mathcal{C}$ the functors* $\operatorname{Hom}(X,-)$ *and* $\operatorname{Hom}(-,X)$ *are cohomological functors.*

Proof. We only do the contravariant case, the covariant case is done by similar arguments. Let

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \mathcal{T}A$$

be a triangle. Since $\operatorname{Hom}(-,X)\colon \mathcal{C} \to \mathrm{Ab}$ is a functor we get from Lemma 2.2.2 that $\operatorname{Im}(\operatorname{Hom}(g,X)) \subseteq \operatorname{Ker}(\operatorname{Hom}(f,X))$. We only have to show that $\operatorname{Ker}(\operatorname{Hom}(f,X)) \subseteq \operatorname{Im}(\operatorname{Hom}(g,X))$. Let $\phi \in \operatorname{Ker}(\operatorname{Hom}(f,X))$, in other words ϕ is a map from B to X such that $\phi f = 0$. We get the following morphism of triangles:

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \mathcal{T}A \\ \downarrow & & \downarrow \phi & & \downarrow \psi & & \downarrow \\ 0 & \longrightarrow & X & = & X & \longrightarrow & 0 \end{array}$$

Note that $0 \longrightarrow X \xrightarrow{Id} X \longrightarrow 0$ is a triangle by $Tr2$, $Tr4$ and $Tr1$, and that there is a map ψ by $Tr5$. It follows that $\operatorname{Hom}(g,X)(\psi) = \phi$ so $\operatorname{Ker}(\operatorname{Hom}(f,X)) \subseteq \operatorname{Im}(\operatorname{Hom}(g,X))$ and we get the exact sequence

$$\operatorname{Hom}(C,X) \to \operatorname{Hom}(B,X) \to \operatorname{Hom}(A,X).$$

$\square$

The following important result can also be found in [GM96]. It helps us to study the morphisms between triangles.

Lemma 2.2.4. *Let*

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \mathcal{T}A, \\ & & \downarrow & & & & \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & \mathcal{T}A' \end{array}$$

be two triangles in a triangulated category, and $\phi\colon B \to B'$ *a morphism.*

(i) *If* $g'\phi f = 0$ *then* ϕ *extends to a morphism of triangles,*
(ii) *if in addition* $\mathrm{Hom}(A, \mathcal{T}^{-1}C') = 0$ *then this extension is unique.*

Proof. We are using that $\mathrm{Hom}(A, -)$ is a cohomological functor (Proposition 2.2.3) by applying it to the sequence $A' \to B' \to C' \to \mathcal{T}A'$. This yields the exact sequence

$$\mathrm{Hom}(A, A') \to \mathrm{Hom}(A, B') \to \mathrm{Hom}(A, C').$$

Since ϕf is in the kernel of $\mathrm{Hom}(A, g')$ it must also be in the image of $\mathrm{Hom}(A, f')$. In other words there is a morphism $\psi\colon A \to A'$ such that $f'\psi = \phi f$. By axiom $Tr5$ of definition 2.1.1 we can get from this a morphism of triangles

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \mathcal{T}A \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \mathcal{T}\phi} \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & \mathcal{T}A'. \end{array}$$

To verify uniqueness in the case when $\mathrm{Hom}(A, \mathcal{T}^{-1}C') = 0$ notice that ψ is unique modulo elements from $\mathrm{Ker}(\mathrm{Hom}(A, f'))$ which is the same as $\mathrm{Im}(\mathrm{Hom}(A, \mathcal{T}^{-1}h'))$. Clearly $\mathrm{Hom}(A, \mathcal{T}^{-1}h')$ is the zero morphism, so ψ is uniquely determined. To see that also γ is unique we apply the functor $\mathrm{Hom}(-, C')$ to the triangle $A \to B \to C \to \mathcal{T}A$ and obtain the exact sequence

$$\mathrm{Hom}(\mathcal{T}A, C') \to \mathrm{Hom}(C, C') \to \mathrm{Hom}(B, C').$$

As before γ is unique modulo elements from $\mathrm{Ker}(\mathrm{Hom}(g, C'))$ which is the same as $\mathrm{Im}(\mathrm{Hom}(h, C'))$, and since $\mathrm{Hom}(\mathcal{T}A, C') = 0$ we have that γ is unique. $\square$

Corollary 2.2.5. *Consider the triangle* $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \mathcal{T}A$ *, if* $\mathrm{Hom}(A, \mathcal{T}^{-1}C) = 0$ *then the only morphism from this triangle to* *itself that extends the identity morphism on B is*

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \mathcal{T}A \\ \| & & \| & & \| & & \| \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & \mathcal{T}A. \end{array}$$

2.3. **t-Structures.** A t-structure on a triangulated category $\mathcal{C}$ can be considered as a technical tool used to find abelian subcategories of $\mathcal{C}$, but as we shall see later the bounded t-structures can also be viewed as the crudest kind of stability on $\mathcal{C}$.

Definition 2.3.1. A *t-structure* on a triangulated category $\mathcal{C}$ is a pair of full subcategories $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ satisfying the list of properties below. The categories $\mathcal{C}^{\leq l}$ and $\mathcal{C}^{\geq l}$ are defined by $\mathcal{T}^{-l}(\mathcal{C}^{\leq 0})$ and $\mathcal{T}^{-l}(\mathcal{C}^{\geq 0})$ respectively.

- (i) $\mathcal{C}^{\leq 0}$ and $\mathcal{C}^{\geq 0}$ are closed under isomorphisms.
- (ii) $\mathcal{C}^{\leq 0} \subseteq \mathcal{C}^{\leq 1}$ and $\mathcal{C}^{\geq 1} \subseteq \mathcal{C}^{\geq 0}$.
- (iii) $\mathrm{Hom}(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 1}) = 0$.
- (iv) Any object $X \in \mathcal{C}$ has a triangle

$$X^{\leq 0} \longrightarrow X \longrightarrow X^{\geq 1} \longrightarrow \mathcal{T}X^{\leq 0}$$

with $X^{\leq 0} \in \mathcal{C}^{\leq 0}$ and $X^{\geq 1} \in \mathcal{C}^{\geq 1}$.

If in addition for any object $X \in \mathcal{C}$ there are two integers m and n such that $X \in \mathcal{C}^{\leq m} \cap \mathcal{C}^{\geq n}$ we say that the t-structure is *bounded*.

Proposition 2.3.2. *Let $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ be a t-structure, then* $\mathcal{C}^{\leq 0} = \mathrm{Ker}(\mathrm{Hom}(-, \mathcal{C}^{\geq 1}))$ *and* $\mathcal{C}^{\geq 1} = \mathrm{Ker}(\mathrm{Hom}(\mathcal{C}^{\leq 0}, -))$.

Proof. We already know by the definition of t-structure that $\mathcal{C}^{\leq 0} \subseteq \mathrm{Ker}(\mathrm{Hom}(-, \mathcal{C}^{\geq 1}))$ and $\mathcal{C}^{\geq 1} \subseteq \mathrm{Ker}(\mathrm{Hom}(\mathcal{C}^{\leq 0}, -))$, so it is enough to show the opposite inclusions. For an object X in $\mathrm{Ker}(\mathrm{Hom}(-, \mathcal{C}^{\geq 1}))$ we have the triangle

$$X^{\leq 0} \longrightarrow X \longrightarrow X^{\geq 1} \longrightarrow \mathcal{T}X^{\leq 0}$$

from the defininition. Since $\mathrm{Hom}(X, X^{\geq 1}) = 0$ we have a morphism of triangles

$$\begin{array}{ccccccc} X & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow & \mathcal{T}X \\ \downarrow & & \| & & \downarrow & & \downarrow \\ X^{\leq 0} & \longrightarrow & X & \longrightarrow & X^{\geq 1} & \longrightarrow & \mathcal{T}X^{\leq 0} \end{array}$$

by Lemma 2.2.4. Similarly we have a morphism

$$\begin{array}{ccccccc} X^{\leq 0} & \longrightarrow & X & \longrightarrow & X^{\geq 1} & \longrightarrow & \mathcal{T}X^{\leq 0} \\ \downarrow & & \| & & \downarrow & & \downarrow \\ X & \longrightarrow & X & \longrightarrow & 0 & \longrightarrow & \mathcal{T}X \end{array}$$

and by Corollary 2.2.5 the compositions must be the identity morphisms, hence $X \simeq X^{\leq 0}$. The other inclusion is done by similar arguments. ☐

Corollary 2.3.3. *Let* $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ *be a t-structure, and*

$$A \longrightarrow B \longrightarrow C \longrightarrow \mathcal{T}A$$

a triangle.

(*i*) *If* A *and* C *are in* $\mathcal{C}^{\leq i}$, *then also* B *is in* $\mathcal{C}^{\leq i}$.
(*ii*) *If* A *and* C *are in* $\mathcal{C}^{\geq i}$, *then also* B *is in* $\mathcal{C}^{\geq i}$.
(*iii*) *If* A *and* B *are in* $\mathcal{C}^{\leq i}$, *then also* C *is in* $\mathcal{C}^{\leq i}$.
(*iv*) *If* B *and* C *are in* $\mathcal{C}^{\geq i}$, *then also* A *is in* $\mathcal{C}^{\geq i}$.

Proof. It follows immediately from Proposition 2.3.2 and the long exact sequences obtained by Proposition 2.2.3. ☐

Definition 2.3.4. The *heart* of a t-structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ on a triangulated category $\mathcal{C}$ is the full subcategory of $\mathcal{C}$ with objects in $\mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}$.

Proposition 2.3.5. *The heart* $\mathcal{A}$ *of a* t*-structure* $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ *is an abelian category and the short exact sequences of this category are given by*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

Where

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \mathcal{T}A$$

is a triangle in $\mathcal{C}$ *with* A, B *and* C *in* $\mathcal{A}$.

Proof. The existence of a zero object and direct sums can be verified by the additivity of the triangulated category, Corollary 2.3.3 and the existence of a triangle

$$A \longrightarrow A \oplus C \longrightarrow C \longrightarrow \mathcal{T}A$$

for each pair (A, C) of objects in $\mathcal{A}$. We need to find the kernels and cokernels to complete the proof.

Let $Y \xrightarrow{g} Z$ be a morphism of $\mathcal{A}$, we can complete it to a triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow \mathcal{T}X$$

in $\mathcal{C}$. The morphism $f \colon X \to Y$ is a good candidate for being the kernel of g. Let $t \colon T \to Y$ be a morphism in $\mathcal{A}$ such that $gt = 0$, then by Lemma 2.2.4 we have a morphism of triangles

$$\begin{CD} T @= T @>>> 0 @>>> \mathcal{T}T \\ @VVV @VV{t}V @VVV @VVV \\ X @>{f}>> Y @>{g}>> Z @>>> \mathcal{T}X. \end{CD}$$

So t factors through f, and since $T \in \mathcal{C}^{\leq 0}$ and $\mathcal{T}^{-1}Z \in \mathcal{C}^{\geq 1}$ we have $\operatorname{Hom}(T, \mathcal{T}^{-1}Z) = 0$ and hence the factorisation is unique. The only problem is that X is not necessarily in $\mathcal{A}$. We want a morphism ϕ with source in $\mathcal{A}$ and target X, such that any morphism $S \to X$ with S in $\mathcal{A}$ factors uniquely through ϕ. By the definition of a t-structure we have a special triangle for X.

$$X^{\leq 0} \longrightarrow X \longrightarrow X^{\geq 1} \longrightarrow \mathcal{T}X^{\leq 0}$$

By using Corollary 2.3.3 we have $X \in \mathcal{C}^{\geq 0}$, and since $X^{\geq 1} \in \mathcal{C}^{\geq 1} \subseteq \mathcal{C}^{\geq 0}$ it follows that $X^{\leq 0}$ is in $\mathcal{A}$. We get the following diagram

$$\begin{array}{ccccccc} S & = & S & \longrightarrow & 0 & \longrightarrow & \mathcal{T}T \\ & & \downarrow & & & & \\ X^{\leq 0} & \longrightarrow & X & \longrightarrow & X^{\geq 1} & \longrightarrow & \mathcal{T}X^{\leq 0} \end{array}$$

Since $\mathrm{Hom}(S, X^{\geq 1}) = 0$ we have unique factorisation through $X^{\leq 0} \to X$. Hence the composition $X^{\leq 0} \to X \to Y$ must be a kernel of g in $\mathcal{A}$.

To see why every monomorphism has to be a kernel let $Y \overset{g}{\hookrightarrow} Z$ be a monomorphism in $\mathcal{A}$. We extend this morphism to a triangle

$$X \xrightarrow{f} Y \overset{g}{\hookrightarrow} Z \xrightarrow{h} \mathcal{T}X$$

and consider the composition $X^{\leq 0} \longrightarrow X \xrightarrow{f} Y$ which must be zero since $gf = 0$ and g is mono (in $\mathcal{A}$). It follows from the diagram of triangles below and from our results Lemma 2.2.4 and Corollary 2.2.5 that $X^{\leq 0} = 0$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & Y & = & Y & \longrightarrow & 0 \\ \downarrow & & \| & & \downarrow & & \downarrow \\ X^{\leq 0} & \xrightarrow{0} & Y & \longrightarrow & U & \longrightarrow & \mathcal{T}X \\ \downarrow & & \| & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Y & = & Y & \longrightarrow & 0 \end{array}$$

Here the middle triangle is the triangle extending the zero map from $X^{\leq 0}$ to Y. Since $X^{\leq 0} = 0$ we have that X is in $\mathcal{C}^{\geq 1}$ and hence $\mathcal{T}X \in \mathcal{A}$ and g is the kernel of $Z \to \mathcal{T}X$. Similar arguments can be made for the cokernels and epimorphisms. $\square$

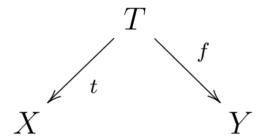
2.4. **The bounded derived category.** So far we have seen some of the nice but a bit abstract theory of triangulated categories. For someone new to these concepts it might seem like the motivation for studying these categories is absent. The reason why these caregories have been so extensively studied is the construction of derived categories that Alexander Grothendieck made in an atempt to create a more "natural setting" for homological algebra. The bounded derived category is a specific subcategory of the derived category. Sadly the theory of derived categories is quite technical so we must leave much of the details to the curiousity of the reader and references like [GM96] and [KS90].

Theorem 2.4.1. *For any abelian category $\mathcal{A}$ there exist a triangulated category $\mathcal{D}^b(\mathcal{A})$ such that $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}^b(\mathcal{A})$.*

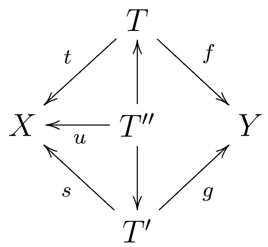
The category $\mathcal{D}^b(\mathcal{A})$ is called the bounded derived category of $\mathcal{A}$, its objects are bounded complexes

$$\cdots 0 \longrightarrow X^m \xrightarrow{d_X^m} X^{m+1} \xrightarrow{d_X^{m+1}} \cdots \xrightarrow{d_X^{n-2}} X^{n-1} \xrightarrow{d_X^{n-1}} X^n \longrightarrow 0 \cdots$$

over $\mathcal{A}$. A morphism from a complex X to a complex Y is an equivalence class of pairs of chain maps



where t is a quasi isomorphism (i.e. $H^i(t)$ is an isomorphism for each $i \in \mathbb{Z}$). The equivalence relation is given by $(t, f) \sim (s, g)$ if and only if there is an object T'' and a commutative diagram of chain maps



and u is a quasi isomorphism.

This construction makes a category "similar" to the category $\mathbf{C}^b(\mathcal{A})$ of bounded complexes and chain maps just with the quasi isomorphisms made invertible. This kind of formally inverting morphisms of a category is called localisation of the category, and it is closely related to localisations of rings. We also have the universal factorisation property.

Theorem 2.4.2. *There is a functor $Q\colon \mathbf{C}^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{A})$ with the property that any functor from $\mathbf{C}^b(\mathcal{A})$ that maps the quasi isomorphisms to isomorphisms can be factored uniquely through Q.*

The functor Q is the natural choice for a functor $\mathbf{C}^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{A})$, it acts on objects as the identity functor and on morphisms by $Q(f) = (\mathrm{id}, f)$ for a morphism f in $\mathbf{C}^b(\mathcal{A})$.

In order to define the triangulated structure on this category we must define a shift functor $\mathcal{T}$ and specify our triangles. The shift functor $\mathcal{T}$ can be defined on complexes by $(\mathcal{T}X)^i = X^{i+1}$ and $d^i_{\mathcal{T}X} = -d^{i+1}_X$. Suppose a morphism in $\mathcal{D}^b(\mathcal{A})$ is represented by a pair (t, f), the image of this morphism through the shift functor is represented by (t', f') where $t'^i = t^{i+1}$ and $f'^i = f^{i+1}$. To define the triangles of $\mathcal{D}^b(\mathcal{A})$ let us consider what is called the cone of a map.

Definition 2.4.3. Let $f\colon X \to Y$ be a morphism of the category $\mathbf{C}(\mathcal{A})$. The *mapping cone* of f is a complex denoted $M(f)$ and defined by $M(f)^i = X^{i+1} \oplus Y^i$, and $d^i_{M(f)} = \begin{pmatrix} -d^{i+1}_X & 0 \\ f & d^i_Y \end{pmatrix}$.

The cone $M(f)$ comes together with two maps $\alpha(f)\colon Y \to M(f)$ and $\beta(f)\colon M(f) \to \mathcal{T}X$.

$$\alpha(f)^i = \begin{pmatrix} \mathrm{id}_{Y^i} \\ 0 \end{pmatrix}, \quad \beta(f)^i = \begin{pmatrix} 0 & \mathrm{id}_{X^{i+1}} \end{pmatrix}.$$

The triangles of $\mathcal{D}^b(\mathcal{A})$ are the sequences

$$A \longrightarrow B \longrightarrow C \longrightarrow \mathcal{T}A$$

in $\mathcal{D}^b(\mathcal{A})$ that are isomorphic in the sense of $Tr1$ (definition 2.1.1) to the image through Q of a sequence

$$A \xrightarrow{f} B \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} \mathcal{T}A$$

for some f in $\mathbf{C}^b(\mathcal{A})$.

We have seen in this chapter that for any abelian category $\mathcal{A}$ we can construct a triangulated category $\mathcal{C} = \mathcal{D}^b(\mathcal{A})$ such that $\mathcal{A}$ sits inside $\mathcal{C}$ in a very special way. It is interesting to see if we can extend our stability conditions on $\mathcal{A}$ to $\mathcal{C}$.

3. Stability conditions on triangulated categories

The notion of stability on a triangulated category was first introduced by Tom Bridgeland in his article [Bri02]. Inspired by this Rudakov, Gorodentscev and Kuleshov made a similar but more general definition in [GKR04]. In many cases, however, our stability conditions have more structure than what is required by the latter notion of stability. For example if our stability is induced by a slope on the heart of a bounded t-structure we already have a "central charge" so in this case it is convenient to use Bridgeland's stability notion.

3.1. Stability data.

For a triangulated category $\mathcal{C}$ we will call the general stability conditions of [GKR04] stability data. Let us define it and verify some of the results that follow from these conditions.

Definition 3.1.1. Let $\mathcal{C}$ be a triangulated category, we shall say that a subcategory $\mathcal{S}$ is *extension closed* if for any triangle $A \to B \to C \to \tau A$ with A and C in $\mathcal{S}$ also B is in $\mathcal{S}$. The category $\mathcal{S}$ will be called a *strict* subcategory if it is closed under isomorphisms.

Definition 3.1.2. *Stability data* on a triangulated category $\mathcal{C}$ is a set of strict, full and extension closed subcategories $\{\Pi_\phi\}_{\phi\in\Phi}$ indexed by a totally ordered set Φ and satisfying the follwing properties:

(i) There exist an automorphism of totally ordered sets $t\colon \Phi \to \Phi$ such that $\mathcal{T}\Pi_\phi = \Pi_{t(\phi)}$ and $t(\phi) \geq \phi$,
(ii) $\phi > \psi \in \Phi \Rightarrow \mathrm{Hom}(\Pi_\phi, \Pi_\psi) = 0$,
(iii) for any object $X \in \mathcal{C}$ there is a "system of triangles" called the HN-filtration of X

$$
\begin{array}{ccccccccccc}
X = F_0 & \longleftarrow & F_1 & \longleftarrow & F_2 & \longleftarrow & \cdots & \longleftarrow & F_n & \longleftarrow & 0 \\
 & \searrow & \uparrow & \searrow & \uparrow & & & \searrow & \uparrow & \searrow & \uparrow \\
 & & X_{\phi_0} & & X_{\phi_1} & & \cdots & & X_{\phi_{n-1}} & & X_{\phi_n}
\end{array}
$$

with $X_{\phi_k} \in \Pi_{\phi_k}$ and $\phi_0 < \phi_1 < \cdots < \phi_n$.

Notice that if all the objects of the HN-filtration above are contained in the heart $\mathcal{A}$ of a t-structure, each triangle will be a short exact sequence in $\mathcal{A}$ by Proposition 2.3.5. So if additionally there exist some stability data on $\mathcal{A}$ such that the factors of the HN-filtration are semistable and ordered $X_{\phi_0} < X_{\phi_1} < \cdots < X_{\phi_n}$ with respect to this stability data, then it will also be the HN-filtration of X in $\mathcal{A}$.

Our next aim is to show that the HN-filtration of an object X in a triangulated category $\mathcal{C}$ with a fixed stability data is unique up to some set of unique isomorphisms. But to follow the proof of [GKR04] we will first introduce the following Proposition.

Proposition 3.1.3. *Let*

$$\begin{array}{ccccccccccc} X = F_0 & \longleftarrow & F_1 & \longleftarrow & F_2 & \longleftarrow & \cdots & \longleftarrow & F_n & \longleftarrow & 0 \\ & \searrow & \uparrow & \searrow & \uparrow & & & \searrow & \uparrow & \searrow & \uparrow \\ & & X_{\phi_0} & & X_{\phi_1} & & \cdots & & X_{\phi_{n-1}} & & X_{\phi_n} \end{array}$$

be a HN-filtration of X. Then

(*i*) $\mathrm{Hom}(X, \Pi_\phi) = 0$ *for all* $\phi < \phi_0$,
(*ii*) $\mathrm{Hom}(F_i, \Pi_\phi) = 0$ *for all* $\phi \leq \phi_i$,
(*iii*) $\mathrm{Hom}(\Pi_\psi, X) = 0$ *for all* $\psi > \phi_n$;
(*iv*) *if*

$$\begin{array}{ccccccccccc} Y = G_0 & \longleftarrow & G_1 & \longleftarrow & G_2 & \longleftarrow & \cdots & \longleftarrow & G_n & \longleftarrow & 0 \\ & \searrow & \uparrow & \searrow & \uparrow & & & \searrow & \uparrow & \searrow & \uparrow \\ & & Y_{\psi_0} & & Y_{\psi_1} & & \cdots & & Y_{\psi_{n-1}} & & Y_{\psi_n} \end{array}$$

is a HN-filtration for Y such that $\phi_n < \psi_0$, then $\mathrm{Hom}(Y, X) = 0$.

Proof. All the statements can be done by application of the cohomological Hom functors. We only do statement (*i*). For any object $\pi_\phi \in \Pi_\phi$ we can apply the functor $\mathrm{Hom}(-, \pi_\phi)$ to the triangles of the HN-filtration above, thereby obtaining long exact sequences.

$$\cdots \longrightarrow \mathrm{Hom}(X_{\phi_i}, \pi_\phi) \longrightarrow \mathrm{Hom}(F_i, \pi_\phi) \longrightarrow \mathrm{Hom}(F_{i+1}, \pi_\phi) \longrightarrow \cdots$$

We have $\phi < \phi_0 \leq \phi_i$ so the left term is always zero. In the case when $(i+1) = n$ the right hand side is also zero since $F_n \in \Pi_{\phi_n}$. It follows that $\mathrm{Hom}(F_{n-1}, \pi_\phi) = 0$, and now we can just continue down to $\mathrm{Hom}(F_0, \pi_\phi) = 0$ by induction. □

Theorem 3.1.4 (Uniqueness of HN-filtration)**.** *Let $\mathcal{C}$ be a triangulated category with stability data $\{\Pi_\phi\}_{\phi \in \Phi}$ and X a non-zero object in $\mathcal{C}$. If*

$$\begin{array}{ccccccccccc} X = F^0 & \longleftarrow & F^1 & \longleftarrow & F^2 & \longleftarrow & \cdots & \longleftarrow & F^n & \longleftarrow & 0 \\ & \searrow & \uparrow & \searrow & \uparrow & & & \searrow & \uparrow & \searrow & \uparrow \\ & & X_{\phi_0} & & X_{\phi_1} & & \cdots & & X_{\phi_{n-1}} & & X_{\phi_n} \end{array}$$

and

$$\begin{array}{ccccccccccc} X = G^0 & \longleftarrow & G^1 & \longleftarrow & G^2 & \longleftarrow & \cdots & \longleftarrow & G^m & \longleftarrow & 0 \\ & \searrow & \uparrow & \searrow & \uparrow & & & \searrow & \uparrow & \searrow & \uparrow \\ & & X_{\psi_0} & & X_{\psi_1} & & \cdots & & X_{\psi_{n-1}} & & X_{\psi_n} \end{array}$$

are both HN-filtrations of X, then $n = m$, $\phi_k = \psi_k$ and there is a unique set of triangle isomorphisms

$$\begin{array}{ccccccc} F^{k+1} & \longrightarrow & F^k & \longrightarrow & X_{\phi_k} & \longrightarrow & \mathcal{T}F^{k+1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ G^{k+1} & \longrightarrow & G^k & \longrightarrow & X_{\psi_k} & \longrightarrow & \mathcal{T}G^{k+1}. \end{array}$$

Proof. First of all we can apply the functor $\operatorname{Hom}(-, X_{\phi_0})$ to the triangle

$$F_1 \longrightarrow F_0 \longrightarrow X_{\phi_0} \longrightarrow \mathcal{T}F_1$$

to obtain an exact sequence

$$\operatorname{Hom}(\mathcal{T}F_1, X_{\phi_0}) \to \operatorname{Hom}(X_{\phi_0}, X_{\phi_0}) \to \operatorname{Hom}(F_0, X_{\phi_0}) \to \operatorname{Hom}(F_1, X_{\phi_0}) .$$

Note that by Proposition 3.1.3 $\operatorname{Hom}(\mathcal{T}F_1, X_{\phi_0}) = 0$ and $\operatorname{Hom}(F_1, X_{\phi_0}) = 0$ since $\phi_0 < \phi_1 \leq t\phi_1$. In other words we have a group isomorphism $\operatorname{Hom}(X_{\phi_0}, X_{\phi_0}) \to \operatorname{Hom}(F_0, X_{\phi_0})$. Since the identity of X_{ϕ_0} maps to a non-zero morphism $X \to X_{\phi_0}$ we get by Proposition 3.1.3 (i) that $\phi_0 \geq \psi_0$, but by the same arguments on the other HN-filtration we get $\psi_0 \geq \phi_0$. Now that we have established that $\phi_0 = \psi_0$ we can verify that there is a unique extension of the diagram

$$\begin{array}{ccccccc} F_1 & \longrightarrow & F_0 & \longrightarrow & X_{\phi_0} & \longrightarrow & \mathcal{T}F_1 \\ & & \Vert & & & & \\ G_1 & \longrightarrow & G_0 & \longrightarrow & X_{\psi_0} & \longrightarrow & \mathcal{T}G_1 \end{array}$$

to a morphism of triangles in each direction (Lemma 2.2.4). Also since $\operatorname{Hom}(F_1, X_{\phi_0}) = 0$ and $\operatorname{Hom}(G_1, X_{\psi_0}) = 0$ their compositions must be the identity morphisms by Corollary 2.2.5. We do the same procedure for the next triangle and we can proceed by induction. Notice that since the triangles are isomorphic in each step we must also have that $m = n$. □

We mentioned in chapter 2 that a bounded t-structure on a triangulated category $\mathcal{C}$ can be viewed as a coarse stability condition. Let us verify that it indeed defines stability data on $\mathcal{C}$.

Lemma 3.1.5. *Let $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ be a bounded t-structure on a triangulated category $\mathcal{C}$ and $\mathcal{A}$ its heart, then the categories $\Pi_i = \mathcal{T}^i\mathcal{A}$ (ordered naturally by $\mathbb{Z}$) defines stability data on $\mathcal{C}$.*

Proof. We verify the conditions for stability data as they were introduced in definition 3.1.2.

(i) The shift functor $\mathcal{T}$ behaves well with respect to the automorphism $t\colon \mathbb{Z} \to \mathbb{Z}$ defined by $n \mapsto n + 1$.

(ii) Let $i > j \in \mathbb{Z}$, then $\operatorname{Hom}(\Pi_i, \Pi_j) \simeq \operatorname{Hom}(\mathcal{A}, \mathcal{T}^{i-j}\mathcal{A})$ since $\mathcal{T}$ is an automorphism, and since $\mathcal{A} \subseteq \mathcal{C}^{\leq 0}$ and $\mathcal{T}^{i-j}\mathcal{A} \subseteq \mathcal{C}^{\geq 1}$ we must have that $\operatorname{Hom}(\Pi_i, \Pi_j) = 0$.

(*iii*) To get the existence of HN-filtration for each non-zero object X recall that since our t-structure is bounded we have $X \in \mathcal{C}^{\leq n} \cap \mathcal{C}^{\geq m}$ for some $n, m \in \mathbb{Z}$. Notice that $n \geq m$ or we would have $\mathrm{Hom}(X, X) = 0$. It is clear from the definition of t-structure that we can get a triangle

$$X^{\leq n-1} \longrightarrow X \longrightarrow X^{\geq n} \longrightarrow \mathcal{T} X^{\leq n-1}$$

with $X^{\leq n-1} \in \mathcal{C}^{\leq n-1}$ and $X^{\geq n} \in \mathcal{C}^{\geq n}$. By Corollary 2.3.3 we have that $X^{\geq n} \in \Pi_{(-n)}$ since $X^{\leq n-1} \in \mathcal{C}^{\leq n}$ and $X \in \mathcal{C}^{\leq n}$. Now we do the same procedure for $X^{\leq n-1}$. To verify that this process has to stop it is enough to observe that $X^{\leq n-1} \in \mathcal{C}^{\leq n-1} \cap \mathcal{C}^{\geq m}$ so that after $(n - m)$ iterations of this procedure we will have a HN-filtration.

□

The preceding Lemma can provide us with a way to induce "more refined" stability data on a triangulated category $\mathcal{C}$ from stability data on the heart of a bounded t-structure on $\mathcal{C}$. Especially stability data on an abelian category $\mathcal{A}$ induces stability data on its bounded derived category $\mathcal{D}^b(\mathcal{A})$.

Proposition 3.1.6. *Let $\mathcal{A}$ be the heart of a bounded t-structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ of a triangulated category $\mathcal{C}$ and let $\{\Pi_\phi\}_{\phi \in \Phi}$ define stability data on $\mathcal{A}$. Then the categories $\{P_{(i,\phi)}\}_{(i,\phi) \in \mathbb{Z} \times \Phi}$ defined by $P_{(i,\phi)} = \mathcal{T}^i \Pi_\phi$ and ordered lexicographically give stability data on $\mathcal{C}$.*

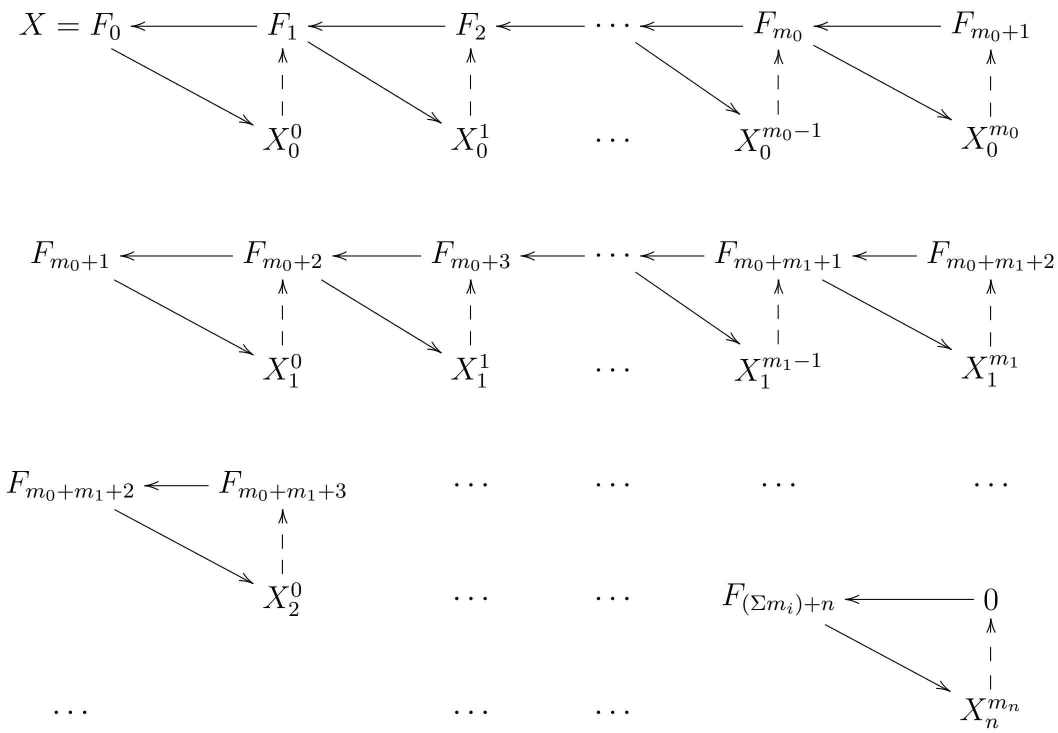
Sketch of the proof. The only non-trivial part is to show existence of HN-filtration for each non-zero object. By Lemma 3.1.5 each non-zero object X has a filtration

$$\begin{array}{ccccccccccc} X = F_0X & \longleftarrow & F_1X & \longleftarrow & F_2X & \longleftarrow & \cdots & \longleftarrow & F_nX & \longleftarrow & 0 \\ & \searrow & \uparrow & \searrow & \uparrow & & & \searrow & \uparrow & \searrow & \uparrow \\ & & X_0 & & X_1 & & \cdots & & X_{n-1} & & X_n \end{array}$$

with X_k in $\mathcal{T}^{i_k}\mathcal{A}$ for some $i_0 < i_1 < \cdots < i_n \in \mathbb{Z}$. Furthermore the HN-filtration of $\mathcal{T}^{-i_k} X_k$ in $\mathcal{A}$ induces a filtration

$$\begin{array}{ccccccccccc} X_k = F_0X_k & \longleftarrow & F_1X_k & \longleftarrow & F_2X_k & \longleftarrow & \cdots & \longleftarrow & F_{m_k}X_k & \longleftarrow & 0 \\ & \searrow & \uparrow & \searrow & \uparrow & & & \searrow & \uparrow & \searrow & \uparrow \\ & & X_k^0 & & X_k^1 & & \cdots & & X_k^{m_k-1} & & X_k^{m_k} \end{array}$$

of X_k with factors in $X_k^i \in P_{(i_k,\phi_i)}$ for some $\phi_0 < \phi_1 < \cdots < \phi_{m_k} \in \Phi$. We want to make the filtration



through multiple uses of the octahedral axiom (definition 2.1.1). Denote the object X by F_0.

Step 1: Consider the following two triangles:

$$F_1X_0 \longrightarrow F_0X_0 \longrightarrow X_0^0 \longrightarrow \mathcal{T}F_1X_0$$

$$F_0X_0 \longrightarrow \mathcal{T}F_1X \longrightarrow \mathcal{T}F_0X \longrightarrow \mathcal{T}X_0.$$

We can extend the composition of the two leftmost maps $F_1X_0 \to F_0X_0 \to \mathcal{T}F_1X$ to a triangle. Let us define the object F_1 to be such that this triangle can be written as

$$F_1X_0 \longrightarrow \mathcal{T}F_1X \longrightarrow \mathcal{T}F_1 \longrightarrow \mathcal{T}F_1X_0.$$

Then by the octahedral axiom we have a triangle

$$F_1 \longrightarrow F_0 \longrightarrow X_0^0 \longrightarrow \mathcal{T}F_1.$$

Step 2: To see how the procedure goes we do the next step. Consider the next triangle in the HN-filtration of X_0

$$F_2X_0 \longrightarrow F_1X_0 \longrightarrow X_0^1 \longrightarrow \mathcal{T}F_2X_0,$$

and the triangle obtained by composition in the previous step:

$$F_1X_0 \longrightarrow \mathcal{T}F_1X \longrightarrow \mathcal{T}F_1 \longrightarrow \mathcal{T}F_1X_0$$

We make composition, apply octahedral axiom and define F_2 suitably to obtain our step 2 triangle

$$F_2 \longrightarrow F_1 \longrightarrow X_0^1 \longrightarrow \mathcal{T}F_2.$$

We can continue like this until step (m_0+1). To see how we go from the last part of the HN-filtration of X_0 to the first part of the HN-filtration of X_1 we should mention that $F_{m_0+1} \simeq F_1X$. So that on step (m_0+2) we need to define F_{m_0+2} and make a triangle

$$F_{m_0+2} \longrightarrow F_1X \longrightarrow X_1^0 \longrightarrow \mathcal{T}F_{m_0+2}.$$

Notice how similar this is to what we had in step 1, the process repeats itself. □

On the other hand stability data gives rise to a collection of t-structures.

Proposition 3.1.7. *Let $(\Phi, \{\Pi_\phi\}_{\phi\in\Phi})$ be stability data on a triangulated category $\mathcal{C}$ and t an automorphism on Φ such that $\Pi_{t(\phi)} = \mathcal{T}\Pi_\phi$. Then each element $\phi \in \Phi$ defines a t-structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0})$ by*

- $\mathcal{C}_\phi^{\leq 0}$**:** *the smallest full extension closed subcategory of $\mathcal{C}$ that contains Π_ψ for every $\psi > \phi$,*
- $\mathcal{C}_\phi^{\geq 0}$**:** *the smallest full extension closed subcategory of $\mathcal{C}$ that contains Π_ψ for every $\psi \leq t(\phi)$.*

Proof. We verify the axioms for t-structures as they were defined in definition 2.3.1. Remember that the automorphism t has to satisfy $t(\phi) \geq \phi$ (definition 3.1.2). For simplicity we omit the subscript ϕ from $\mathcal{C}_\phi^{\leq i}$ and $\mathcal{C}_\phi^{\geq i}$ throughout the proof.

(i) Clearly $\mathcal{C}^{\leq 0}$ and $\mathcal{C}^{\geq 0}$ are strict subcategories of $\mathcal{C}$ since each Π_ψ is strict.

(ii) We have $\mathcal{C}^{\leq 0} \subseteq \mathcal{C}^{\leq 1}$ since $\psi > \phi$ implies $\psi > t^{-1}(\phi)$ and $\mathcal{C}^{\geq 1} \subseteq \mathcal{C}^{\geq 0}$ since $\psi \leq \phi$ implies $\psi \leq t(\phi)$.

(iii) We need to verify that $\mathrm{Hom}(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 1}) = 0$. To see this we start by the observation that if X' is semistable in $\mathcal{C}^{\leq 0}$ and Y' is semistable in $\mathcal{C}^{\geq 0}$, then $\mathrm{Hom}(X', Y') = 0$. Next assume that we have the triangle

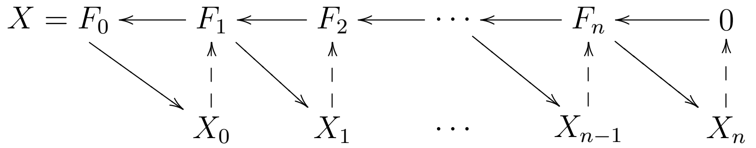
$$X' \longrightarrow X \longrightarrow X'' \longrightarrow \mathcal{T}X',$$

where $\mathrm{Hom}(X', Y') = 0$ and $\mathrm{Hom}(X'', Y') = 0$ for every semistable Y' in $\mathcal{C}^{\geq 1}$. We apply the cohomological hom functors

$$\mathrm{Hom}(X'', Y') \longrightarrow \mathrm{Hom}(X, Y') \longrightarrow \mathrm{Hom}(X', Y')$$

and see that also $\mathrm{Hom}(X, Y') = 0$ for every semistable Y' in $\mathcal{C}^{\geq 1}$. Similar arguments allow us to step from semistable objects of $\mathcal{C}^{\geq 1}$ to arbitrary objects of $\mathcal{C}^{\geq 1}$ and thus $\mathrm{Hom}(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 1}) = 0$.

(iv) To get the special triangles $X^{\leq 0} \to X \to X^{\geq 1} \to \mathcal{T}X^{\leq 0}$ of a t-structure we can use the HN-filtration. Let the object X of $\mathcal{C}$ have the HN-filtration below.

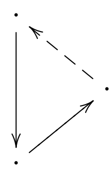


If all the HN-factors are in $\mathcal{C}^{\leq 0}$ it follows that X is in $\mathcal{C}^{\leq 0}$, if the HN-factors are in $\mathcal{C}^{\geq 1}$ then also X is in $\mathcal{C}^{\geq 1}$. In either case we can make a trivial triangle. The only case left to prove is if there is some $0 \leq k < n$ such that $X_i \in \mathcal{C}^{\geq 1}$ for $i \leq k$ and $X_i \in \mathcal{C}^{\leq 0}$ for $i > k$. In this case define $X^{\leq 0} = F_{k+1}$, it is in $\mathcal{C}^{\leq 0}$ since all of its HN-factors $X_{k+1}, \cdots, X_n$ are. We can complete the map $X^{\leq 0} \to X$ to a triangle

$$X^{\leq 0} \longrightarrow X \longrightarrow X^{\geq 1}.$$

What is left is to show that $X^{\geq 1}$ is really in $\mathcal{C}^{\geq 1}$. This can be done by similar arguments of those in Proposition 3.1.6, for $i = 0, \cdots, k+1$ we have objects G_i and triangles

$$F_{k+1} \longrightarrow F_i \longrightarrow G_i \longrightarrow \mathcal{T}F_{k+1}$$
$$F_i \longrightarrow F_{i-1} \longrightarrow X_{i-1} \longrightarrow \mathcal{T}F_i$$
$$F_{k+1} \longrightarrow F_{i-1} \longrightarrow G_{i-1} \longrightarrow \mathcal{T}F_{k+1}$$



where the third triangle is the composition $F_{k+1} \to F_i \to F_{i-1}$ extended to a triangle. We can use the octahedral axiom and get a HN-filtration of $X^{\geq 1}$ with $X_0, \cdots, X_k$ as factors, hence $X^{\geq 1}$ is in $\mathcal{C}^{\geq 1}$.

□

3.2. **Central charge stability.** The initial stability notion that Bridgeland introduced will be called central charge stability in this thesis.

Definition 3.2.1. Let $\mathcal{C}$ be a triangulated category. The Grothendieck group $K_0(\mathcal{C})$ of $\mathcal{C}$ is the free group generated by the isomorphism classes of objects of $\mathcal{C}$ modulo the relations $[B] = [A] + [C]$ for every triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow \mathcal{T}A.$$

Notice that the relations of this definition imply $[\mathcal{T}A] = -[A]$ since triangles can be rotated ($Tr4$ of definition 2.1.1). So for a heart $\mathcal{A}$ of a bounded t-structure on $\mathcal{C}$ we can identify $K_0(\mathcal{A})$ with $K_0(\mathcal{C})$.

Definition 3.2.2 (Central charge stability)**.** Let $\mathcal{C}$ be a triangulated category. *Central charge stability* on $\mathcal{C}$ is defined by a group homomorphism $Z\colon K_0(\mathcal{C}) \to \mathbb{C}$ called the Central charge and a set of full additive subcategories $\{\mathcal{P}(\phi) | \phi \in \mathbb{R}\}$ of $\mathcal{C}$ satisfying the following axioms.

- (i) $E \in \mathcal{P}(\phi)$ implies $Z(E) = m(E)e^{i\pi\phi}$ for some positive $m(E) \in \mathbb{R}$,
- (ii) for all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{T}\mathcal{P}(\phi)$,
- (iii) if $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j)$ then $\mathrm{Hom}_{\mathcal{C}}(A_1, A_2) = 0$,
- (iv) for each non-zero object $E \in \mathcal{C}$ there is system of triangles

$$\begin{array}{ccccccccccccc} E = E_0 & \longleftarrow & E_1 & \longleftarrow & E_2 & \longleftarrow & \cdots & \longleftarrow & E_{n-1} & \longleftarrow & E_n & \longleftarrow & 0 \\ & \searrow & \uparrow & \searrow & \uparrow & & & \searrow & \uparrow & \searrow & \uparrow & \searrow & \uparrow \\ & & A_0 & & A_1 & & \cdots & & A_{n-2} & & A_{n-1} & & A_n \end{array}$$

for some $n \in \mathbb{N}$, with $A_i \in \mathcal{P}(\phi_i)$ and $\phi_0 < \phi_1 < \cdots < \phi_{n-1} < \phi_n$.

Clearly by enlarging the categories $\mathcal{P}(\phi)$ to strict extension closed categories we get stability data. It follows that the uniqueness of HN-filtration for stability data applies also for central charge stability.

Remember from the example in chapter 1 that a slope $\mu = \theta/\kappa$ gives us a map $(\kappa + i\theta)\colon K_0(\mathcal{A}) \to \mathbb{C}$ and that ordering by angle in $\mathbb{C}$ is equivalent to the slope ordering. We would like to see that a slope on the heart $\mathcal{A}$ of a bounded t-structure on $\mathcal{C}$ induces central charge stability on $\mathcal{C}$.

Proposition 3.2.3. *Let $\mathcal{A}$ be the heart of a bounded t-structure $(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 1})$ on the triangulated category $\mathcal{C}$ and let $\mu = \theta/\kappa$ be a slope on $\mathcal{A}$. The stability data induced on $\mathcal{C}$ by μ is equivalent to a central charge stability with $Z = (\kappa + i\theta)\colon K_0(\mathcal{C}) \to \mathbb{C}$.*

Proof. From Proposition 3.1.6 the semistable categories of our induced stability data are shifts $\{\mathcal{T}^n\Pi_r\}_{(n,r)\in\mathbb{Z}\times\mathbb{R}}$ of the μ-semistable categories $\{\Pi_r\}_{r\in\mathbb{R}}$ in $\mathcal{A}$, ordered lexicographically. We define the categories $\{\mathcal{P}(\phi)\}_{\phi\in\mathbb{R}}$ as follows: For any $\phi \in \mathbb{R}$ we can in a unique way write ϕ as $n + \phi'$ where ϕ' is on the interval $[-\frac{1}{2}, \frac{1}{2})$ and n is an integer. We define $\mathcal{P}(\phi)$ to be the category $\mathcal{T}^n\Pi_{\tan(\pi\phi')}$ if $\tan(\pi\phi')$ is defined, and the zero category if $\tan(\pi\phi')$ is not defined. Notice that when $\mathcal{P}(\phi) = \mathcal{T}^n\Pi_x$ and $\mathcal{P}(\psi) = \mathcal{T}^m\Pi_y$ are non-zero categories we have $\phi < \psi$ in $\mathbb{R} \iff (n, x) <_{\text{lex}} (m, y)$ in $\mathbb{Z} \times \mathbb{R}$. Also $\mathcal{P}(\phi)$ was made so that $Z(\mathcal{P}(\phi))$ has angle $(-1)^n\pi\phi'$ in $\mathbb{C}$ and $\mathcal{P}(\phi + 1) = \mathcal{T}\mathcal{P}(\phi)$. The rest of the properties follow from the fact that $\{\mathcal{T}^n\Pi_r\}$ makes stability data. $\square$

4. Distinguished stability

In this chapter we are going to consider some "distinguished" orderings that can be considered in the case when our abelian category $\mathcal{A}$ is the category $Rep_{\mathbb{K}}(Q)$ of finite dimensional representations of some finite wild quiver Q with no oriented cycles over some fixed field $\mathbb{K}$. It is known that in some cases the distinguished orderings provide us with stability conditions. Since Q has no oriented cycles we can order the vertices Q_0 of Q from 1 to n in such a way that there are no arrows $i \to j$ for $i > j$. Let us first introduce some useful results about dimension vectors that we can use to define our stability conditions. The distinguished stability conditions can be found in the article [HdlP01] that was based on the results of [dlPT90].

4.1. **Linear algebra on dimension vectors.** It is well known that the Grothendieck group $K_0(Q)$ of a quiver Q with no oriented cycles is isomorphic to the free group with the simple representations of Q as a basis. Therefore we can identify it with $\mathbb{Z}^n$ where n is the number of vertices in Q. The class $[M]$ in $K_0(Q)$ of a representation M can be identified with the dimension vector of M, which is the vector $(d_1, d_2, \cdots, d_n)$ in $\mathbb{Z}^n$ where d_i is the dimension of the vectorspace of M corresponding to vertex i of Q. We will always view vectors as column vectors. It is convenient to include these vectors into $\mathbb{R}^n$ so we can apply our techniques from linear algebra.

Denote by (SI) the matrix with the entry $(SI)_{ij}$ equal to the number of different paths from i to j in Q (trivial paths are also counted). By the ordering we made to the vertices of Q this matrix is upper triangular, and has ones on the diagonal. Clearly (SI) is of full rank.

Example.

$$Q' : \qquad 1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; 2 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; 3$$

$$(SI) = \begin{pmatrix} 1 & 3 & 9 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

———

Notice that left multiplication by the matrix (SI) corresponds to the linear transformation mapping simple representations to their injective envelopes with respect to the simple basis on $K_0(Q)$. Likevise we can see that the transpose matrix $(SI)^{tr} = (SP)$ corresponds to the transformation mapping simple representations to their projective covers. We will denote $(SI)^{-1}$ by (IS) since it corresponds to the transformation mapping indecomposable injectives to simples, and $(SP)^{-1}$ by (PS) which sends indecomposable projectives to simples. The matrix (IS) is often called the Cartan matrix of Q.

There is a remarkable, yet relatively simple and well known result that for any pair of representations M, N of Q the number

$$\dim_{\mathbb{K}} \operatorname{Hom}(M,N) - \dim_{\mathbb{K}} \operatorname{Ext}^1(M,N)$$

only depends on the dimension vectors $[M]$ and $[N]$ in $K_0(Q)$.

Lemma 4.1.1. *The number* $\dim_{\mathbb{K}} \operatorname{Hom}(M,N) - \dim_{\mathbb{K}} \operatorname{Ext}^1(M,N)$ *is given by the formula*

$$[M]^{tr}(IS)[N].$$

Proof. Since the category $\mathcal{A}$ of representations of Q is hereditary we have a "short" projective resolution of M

$$0 \longrightarrow P' \longrightarrow P \longrightarrow M \longrightarrow 0.$$

It follows that $[M] = [P] - [P']$ in $K_0(Q)$. Recall that any direct summand of a projective is again projective so P and P' can both be written as direct sums of the n indecomposable projectives, say $P = \oplus P_i^{a_i}$ and $P' = \oplus P_i^{b_i}$. So $[M] = \Sigma(a_i - b_i)[P_i]$, it follows that $(PS)[M] = \Sigma(a_i - b_i)[S_i]$. Applying the contravariant functor $\operatorname{Hom}(-, N)$ to the sequence above gives an exact sequence

$$0 \to \operatorname{Hom}(M,N) \to \operatorname{Hom}(P,N) \to \operatorname{Hom}(P',N) \to \operatorname{Ext}^1(M,N) \to 0.$$

The righthand zero can be justified by noting that P is projective. This sequence gives us what we need.

$$\begin{aligned}
\dim_{\mathbb{K}} \operatorname{Hom}(M,N) - \dim_{\mathbb{K}} \operatorname{Ext}^1(M,N) &= \\
\dim_{\mathbb{K}} \operatorname{Hom}(P,N) - \dim_{\mathbb{K}} \operatorname{Hom}(P',N) &= \\
&= \Sigma(a_i - b_i) \dim_{\mathbb{K}} \operatorname{Hom}(P_i, N) \\
&= \Sigma(a_i - b_i)[S_i]^{tr}[N] \\
&= [M]^{tr}(PS)^{tr}[N] \\
&= [M]^{tr}(IS)[N].
\end{aligned}$$

□

Definition 4.1.2. The map $([M],[N]) \mapsto [M]^{tr}(IS)[N]$ is called the *Euler form*.

Another matrix that is important to us is the Coxeter matrix $C = -(SI)(PS)$. The Coxeter matrix is tightly connected to the Auslander Reiten translate $\tau = \mathrm{DTr}$ for representations. Let us say that $[M]$ is *positive* if $[S_i]^{tr}[M] \geq 0$ for all i and not all of them are zero. Likewise $[M]$ is negative if $[S_i]^{tr}[M] \leq 0$ for all i and they are not all zero. The next Lemma can be found in [ARS97, Proposition 2.2, page 270].

Lemma 4.1.3. *Let M be an indecomposable representation of Q, then we have the following:*

(*i*) *$C[M]$ is either positive or negative.*
(*ii*) *If M is non-projective then $C[M] = [\tau M]$.*

(*iii*) M *is projective if and only if* $C[M]$ *is negative.*
(*iv*) $C^{-1}[M]$ *is either positive or negative.*
(*v*) *If* M *is non-injective then* $C^{-1}[M] = [\tau^{-1}M]$.
(*vi*) M *is injective if and only if* $C^{-1}[M]$ *is negative.*

We also need the following result from [dlPT90] before we can define the distinguished orderings.

Proposition 4.1.4. *For a finite connected wild quiver* Q *with no oriented cycles, assume that the associated Coxeter matrix* $C = -(SI)(PS)$ *has spectral radius* ρ. *Then* ρ *and* $\frac{1}{\rho}$ *are eigenvalues of* C, *moreover eigenvectors* v^+ *and* v^- *corresponding to* ρ *and* $\frac{1}{\rho}$ *can be chosen such that* v^+ *and* v^- *are positive.*

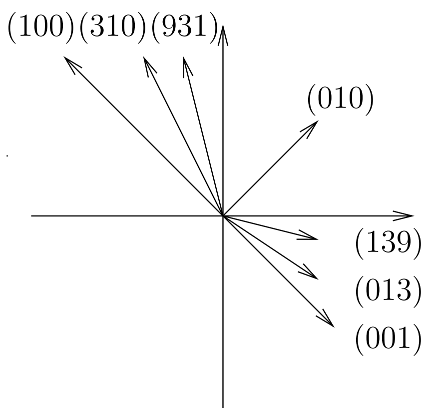
Definition 4.1.5. Let Q be a finite wild connected quiver with no oriented cycles. Define
(*i*) $X : K_0(Q) \to \mathbb{R}$ by $X([M]) = [M]^{tr}(IS)v^+$,
(*ii*) $Y : K_0(Q) \to \mathbb{R}$ by $Y([M]) = (v^-)^{tr}(IS)[M]$,
(*iii*) $X + iY : K_0(Q) \to \mathbb{C}$ by $(X + iY)([M]) = X([M]) + iY([M])$.
The *distinguished ordering* on the category of finite dimensional representations of Q is given by ordering the representations that are not in the kernel of $X + iY$ by the angle on the interval $(-\pi, \pi]$ of their image in $\mathbb{C} \setminus \{0\}$.

The maps of the previous definiton obviously depend on our scaling of the vectors v^- and v^+, but the distinguished ordering is independent of how we choose v^- and v^+ as long as they are positive eigenvectors as in Proposition 4.1.4.

Example. Let Q' be the quiver

$$1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; 2 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; 3$$

as above. The angles of the indecomposable simple, injective and projective dimension vectors for some choice of v^+ and v^- are sketched below.



Notice that all the indecomposable projective representations are mapped to the quadrant with angles $(-\frac{\pi}{2}, 0)$, and the indecomposable injective are mapped to the quadrant $(\frac{\pi}{2}, \pi)$. This holds true in general, in fact we have a result from [dlPT90] about the maps X and Y. The follwing results are independent of the scaling of v^+ and v^-.

Theorem 4.1.6. *Let M be an indecomposable Q representation*

(i) *M is preprojective if and only if $Y([M]) < 0$,*
(ii) *If M is not preprojective then $Y([M]) > 0$,*
(iii) *M is preinjective if and only if $X([M]) < 0$,*
(iv) *If M is not preinjective then $X([M]) > 0$.*

The next Lemma describes how the Auslander Reiten-translate τ behaves together with the map $X + iY$.

Lemma 4.1.7. *Let M be an indecomposable representation of Q.*

(i) *If M is non-projective, then*

$$X([\tau M]) + i(Y([\tau M])) = \frac{1}{\rho}X([M]) + i(\rho Y([M])).$$

(ii) *If M is non-injective, then*

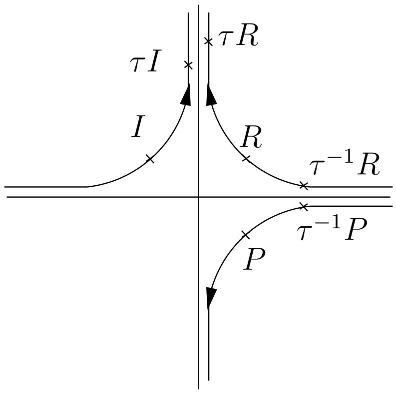
$$X([\tau^{-1}M]) + i(Y([\tau^{-1}M])) = \rho X([M]) + i(\frac{1}{\rho}Y([M])).$$

Proof. By Lemma 4.1.3 it is enough to show that $X(C[M]) = \frac{1}{\rho}X([M])$, and $Y(C[M]) = \rho Y([M])$. We only do the first equality, the second is done by similar computations.

$$\begin{aligned} X(C[M]) &= -((SI)(PS)[M])^{tr}(IS)v^+ \\ &= -[M]^{tr}((SI)(PS))^{tr}(IS)v^+ \\ &= -[M]^{tr}(IS)C^{-1}v^+ \\ &= \frac{1}{\rho}[M]^{tr}(IS)v^+ \\ &= \frac{1}{\rho}X([M]). \end{aligned}$$

□

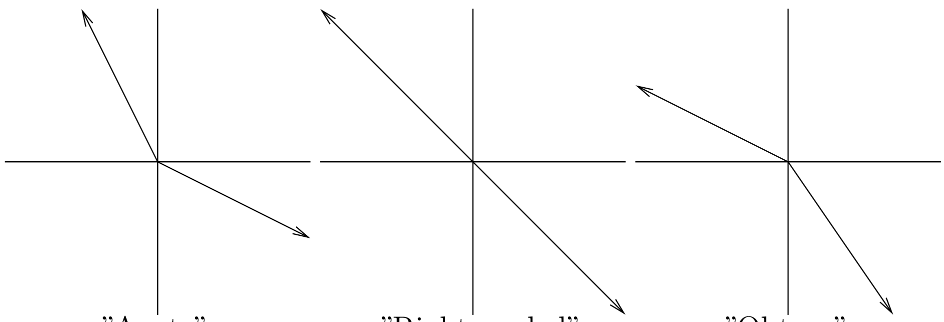
So τ sends indecomposable representations along trajectories as illustrated in the picture below.



The representations P, R and I are respectively preprojective, regular and preinjective. Observe that this implies that for any indecomposable representation M there is a projective representation that is smaller that M, and an injective representation that is greater than M with respect to the distinguished ordering. As a consequence there always exist a projective representation P and an injective representation I such that for any indecomposable representation M we have $P \leq M \leq I$.

4.2. **Distinguished stability ordering.**

Definition 4.2.1. Let Q be a finite wild quiver with no oriented cycles, and let P and I be indecomposable representations such that the distinguished ordering is $P \leq M \leq I$ for any indecomposable representation M. Admitting a slight abuse of the standard terminology we will say that Q is *acute* if the angle of I minus the angle of P is smaller than π, if it is equal to π we say that Q is *right angled*, and if it is greater than π we will say that Q is *obtuse*.



Recall that we consider the angle of a representation as a number on the interval $(-\pi, \pi]$. The diffence of angles in the definition above is a number on the inteval $(0, \frac{3\pi}{2})$ since P has angle on $(-\frac{\pi}{2}, 0)$ and I has angle on $(\frac{\pi}{2}, \pi)$. It is not difficult to se that the definition above is independent of scaling of the vectors v^+ and v^-.

Lemma 4.2.2. *If Q is an acute quiver, then the distinguished ordering can be extended to a stability ordering.*

Proof. We have to extend the distinguished ordering from indecomposable to arbitrary representations. Since Q is acute we have that every representation is mapped to the same halfplane by $(X+iY)\colon K_0(Q) \to \mathbb{C}$. Thus we can order all the non-zero representations by the angle of their images, and since $(X+iY)$ is a homomorphism of abelian groups the seesaw property is reduced to the fact that the angle of the sum of two vectors in our half-plane is between the angles of its sumands in $\mathbb{C}$. □

In fact one can make a slope with equivalent ordering by considering some linear combinations $\theta = (a_X X - a_Y Y)$ and $\kappa = (b_X X + b_Y Y)$ such that $\kappa([M]) > 0$ for any non-zero representation M. This was done in [HdlP01] and the following set of the properties was emphasized.

Theorem 4.2.3. *Let Q be an acute quiver and let P, R and I be respectively preprojective, regular and preinjective representations. The distinguished stability ordering $\leq_\mu$ satisfies the following properties:*

(*i*) $P <_\mu R <_\mu I$.

(*ii*) *If R is indecomposable, then R is $\leq_\mu$-stable if and only if τR is $\leq_\mu$-stable.*

(*iii*) *If M is an indecomposable representation, then $\tau M >_\mu M$ if and only if M is regular.*

(*iv*) *If M and N are representations with no projective summands, then $N <_\mu M$ if and only if $\tau N <_\mu \tau M$.*

Proof. These properties are immediate consequences of Theorem 4.1.6 and Lemma 4.1.7. Note that when M is a representation with no projective summands we can get the identity $(X([\tau M]), Y([\tau M])) = (\frac{1}{\rho}X([M]), \rho Y([M]))$ from Lemma 4.1.7. □

We have seen that for acute quivers (definition 4.2.1) the distinguished ordering gives us order stability conditions. In the case when the quivers are right angled or obtuse the situation is more complicated. In the examples above we have considered the quiver

$$Q' : \qquad 1 \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} 2 \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} 3.$$

With some simple calculations one can show that the quiver Q' is a right angled quiver. The representation P_3 and the representation I_1 are mapped to the same line in $\mathbb{C}$ by the map $X+iY$. In fact, for this quiver we get that the representation $(P_3)\oplus(I_1)$ maps to zero in $\mathbb{C}$. This is troublesome. Where should this representation be in an extention of the distinguished ordering? The following proposition shows that in general we cannot hope to get slope stability for right angled or obtuse quivers.

Proposition 4.2.4. *The quivers*

$$Q' : \qquad 1 \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} 2 \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} 3$$

and

$$Q'' : \qquad 1 \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; 2 \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; 3 \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; 4$$

are examples of respectively right angled and obtuse quivers, on these two quivers the distinguished ordering can not be extended to a slope.

Proof. For the quiver Q' we have that I_1 is the largest indecomposable, and P_3 is the smallest indecomposable with respect to the distinguished ordering. Since Q' is right angled we have that the angle of $[P_3]$ is equal to the angle of $-[I_1]$, it follows that P_1 is equal $\tau^{-1}P_3$ and that τI_1 is equal to I_3 in distinguished ordering. Suppose that we have a slope μ given by $\mu(d_1, d_2, d_3) = \frac{\theta_1 d_1 + \theta_2 d_2 + \theta_3 d_3}{\kappa_1 d_1 + \kappa_2 d_2 + \kappa_3 d_3}$ that agrees with the distinguished ordering on the indecomposable representations. The two identities above give the following equations.

$$3 \begin{vmatrix} \theta_1 & \kappa_1 \\ \theta_2 & \kappa_2 \end{vmatrix} + 8 \begin{vmatrix} \theta_1 & \kappa_1 \\ \theta_3 & \kappa_3 \end{vmatrix} - 3 \begin{vmatrix} \theta_2 & \kappa_2 \\ \theta_3 & \kappa_3 \end{vmatrix} = 0$$

$$3 \begin{vmatrix} \theta_1 & \kappa_1 \\ \theta_2 & \kappa_2 \end{vmatrix} - 8 \begin{vmatrix} \theta_1 & \kappa_1 \\ \theta_3 & \kappa_3 \end{vmatrix} - 3 \begin{vmatrix} \theta_2 & \kappa_2 \\ \theta_3 & \kappa_3 \end{vmatrix} = 0$$

Subtracting one equation from the other gives

$$\begin{vmatrix} \theta_1 & \kappa_1 \\ \theta_3 & \kappa_3 \end{vmatrix} = 0.$$

But this is precisely the condition that $P_3 =_\mu I_1$, which contradicts the assumption since I_1 is greater than P_3 in the distinguished ordering.

To verify the statement for the quiver Q'' we can do similar calculations with gaussian elimination on the equations coming from the following observations:

$$\begin{aligned} P_3 &=_d \tau^{-1} P_1 \\ P_4 &=_d \tau^{-1} P_2 \\ I_1 &=_d \tau I_3 \\ I_2 &=_d \tau I_4 \end{aligned}$$

Here $=_d$ means that they are equal in distinguished ordering. □

4.3. **Reflection functors and stability data.** The reflection functors of [BGP73] could help us to study the distinguished orderings of right angled and obtuse quivers. Let Q be a quiver and the vertex $x \in Q_0$ be a sink or a source of Q, then we can make a *reflected quiver* $Q^{(x)}$ of Q by inverting the direction of every arrow that begins or ends in x. Some important properties of reflection functors are listed in the following theorem.

Theorem 4.3.1. *Let Q be a quiver with no oriented cycles and x a sink or source of Q. Then there is a functor*

$$F_x : Rep_{\mathbb{K}}(Q) \longrightarrow Rep_{\mathbb{K}}(Q^{(x)})$$

satisfying the following list of properties:

(i) *Denote by* S_x *the simple representation of* Q *and* $S_x^{(x)}$ *the simple representation of* $Q^{(x)}$ *corresponding to the vertex* x. F_x *can be restricted to an equivalence of categories*

$$Rep_{\mathbb{K}}(Q) \setminus S_x \longrightarrow Rep_{\mathbb{K}}(Q^{(x)}) \setminus S_x^{(x)} .$$

Here $Rep_{\mathbb{K}}(Q) \setminus S_x$ *is the full subcategory of* $Rep_{\mathbb{K}}(Q)$ *having as objects the representations of* Q *with no summands isomorphic to* S_x *and* $Rep_{\mathbb{K}}(Q^{(x)}) \setminus S_x^{(x)}$ *the category with no summands isomorphic to* $S_x^{(x)}$. *The inverse equivalence is the restriction of the functor*

$$(F_x)_x \colon Rep_{\mathbb{K}}(Q^{(x)}) \longrightarrow Rep_{\mathbb{K}}(Q).$$

(ii) *Suppose* $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ *is a short exact sequence in* $Rep_{\mathbb{K}}(Q) \setminus S_x$, *then*

$$0 \longrightarrow F_x(A) \longrightarrow F_x(B) \longrightarrow F_x(C) \longrightarrow 0$$

is a short exact sequence in $Rep_{\mathbb{K}}(Q^{(x)}) \setminus S_x^{(x)}$.

(iii) *Let* V *be a representation in* $Rep_{\mathbb{K}}(Q) \setminus S_x$ *with dimension vector* $(v_1, v_2, \cdots, v_n)^{tr}$, *then* $F_x(V)$ *has dimension vector* $(w_1, w_2, \cdots, w_n)^{tr}$ *where* $w_i = v_i$ *for* $i \neq x$ *and*

$$w_x = \begin{cases} (\sum\limits_{\alpha \colon i \to x} v_i) - v_x & \text{if } x \text{ is a sink,} \\ (\sum\limits_{\alpha \colon x \to i} v_i) - v_x & \text{if } x \text{ is a source.} \end{cases}$$

Much of the details concerning these functors are written out in [Tep06].

Example. We consider the quiver

$$Q' : \qquad 1 \Rrightarrow 2 \Rrightarrow 3$$

and make reflection in the sink 3 to obtain a new quiver.

$$Q'^{(3)} : \qquad 1 \Rrightarrow 2 \Lleftarrow 3$$

We have functors $F_3 \colon Rep_{\mathbb{K}}(Q') \to Rep_{\mathbb{K}}(Q'^{(3)})$ and $(F_3)_3 \colon Rep_{\mathbb{K}}(Q'^{(3)}) \to Rep_{\mathbb{K}}(Q')$. These functors can be restricted to inverse equivalences $Rep_{\mathbb{K}}(Q') \setminus S_3 \simeq Rep_{\mathbb{K}}(Q'^{(3)}) \setminus S_3^{(3)}$ and the dimension vectors of representations from these subcategories are transformed by the linear map $K_0(Rep_{\mathbb{K}}(Q')) \leftrightarrow K_0(Rep_{\mathbb{K}}(Q'^{(3)}))$ given by the matrix

$$R^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & -1 \end{pmatrix}.$$

Observe that the quiver $Q'^{(3)}$ is acute whereas Q' is right angled. Also the matrix R_3 is its own inverse. ——

Lemma 4.3.2. *The restriction of the reflection functor F_3 to an equivalence $Rep_{\mathbb{K}}(Q') \setminus S_3 \to Rep_{\mathbb{K}}(Q'^{(3)}) \setminus S_3^{(3)}$ respects the distinguished ordering.*

Proof. For simplicity let us fix the eigenvectors v^+ and v^- for Q' as in Proposition 4.1.4. Notice that the Cartan matrix $(IS)^{(3)}$ of the quiver $Q'^{(3)}$ can be obtained by the transformation

$$(IS)^{(3)} = R_3^{tr}(IS)R_3$$

of the Cartan matrix (IS) of Q'. It follows that the Coxeter matrix of $Q'^{(3)}$ can be written as $C^{(3)} = R_3^{-1}CR_3 = R_3CR_3$, where C is the Coxeter matrix of Q'. We get corresponding positive eigenvectors of $C^{(3)}$ by defining

$$v^+_{(3)} = R_3v^+ \quad , \quad v^-_{(3)} = R_3v^-.$$

With these choices of eigenvectors it is easy to see that the equivalences $Rep_{\mathbb{K}}(Q') \setminus S_3 \leftrightarrow Rep_{\mathbb{K}}(Q'^{(3)} \setminus S_3^{(3)}$ preserve distinguished ordering. In fact, for any $M \in Rep_{\mathbb{K}}(Q') \setminus S_3$ we have

$$\begin{aligned} X([M]) &= [M]^{tr}(IS)v^+ \\ &= [M]^{tr}(R_3^{tr})^2(IS)(R_3)^2v^+ \\ &= [F_3(M)]^{tr}R_3^{tr}(IS)R_3(R_3v^+) \\ &= [F_3(M)]^{tr}(IS)^{(3)}v^+_{(3)} \\ &= X_{(3)}([F_3(M)]) \end{aligned}$$

and likewise $Y([M]) = Y_{(3)}([F_3(M)])$. Here X, Y, $X_{(3)}$ and $Y_{(3)}$ are the homomorphisms from definition 4.1.5. □

Definition 4.3.3. Let Q be a quiver with no oriented cycles and $\leq_d$ its distinguished ordering. For an indecomposable representation M of Q We define the subcategory $\Pi(M) \subset Rep_{\mathbb{K}}(Q)$ by its objects

$$\{R \in Rep_{\mathbb{K}}(Q) | R =_d M \text{ and } [S \subset R \Rightarrow S \leq_d R]\} \cup \{0\}.$$

If the categories $\{\Pi(M)\}$ ordered by $\leq_d$ give stability data on $Rep_{\mathbb{K}}(Q)$ we will call this the *distinguished stability data* on $Rep_{\mathbb{K}}(Q)$.

In particular, if $R \in \Pi(M)$ the distinguished ordering should be defined for every non-zero subrepresentation $S \subseteq R$ so $(X+iY)(S) \neq 0$ (see definition 4.1.5). Let us keep the notation of the example above for the following Lemma.

Lemma 4.3.4. *Let M be an indecomposable representation of Q' and let $M' \in \Pi(M)$ be a representation with no summands isomorphic to S_3. Then $F_3(M')$ is semistable in the distinguished stability ordering of $Q'^{(3)}$.*

Proof. We know that $F_3(M')$ is in $Rep_{\mathbb{K}}(Q'^{(3)}) \setminus S_3^{(3)}$. Since $S_3^{(3)}$ is injective it follows that $S_3^{(3)}$ is not a subrepresentation of $F_3(M')$, and no subrepresentation of $F_3(M')$ can have $S_3^{(3)}$ as a summand. The Lemma follows from the definition of $\Pi(M)$ and Lemma 4.3.2. $\square$

Proposition 4.3.5. *The quiver*

$$Q' : \qquad 1 \rightrightarrows 2 \rightrightarrows 3$$

has distinguished stability data.

Proof. (*i*) Suppose M and N are indecomposable representations of Q' and $M >_d N$, we have to show that $\mathrm{Hom}(\Pi(M), \Pi(N)) = 0$. Let us assume $M' \in \Pi(M)$ and $N' \in \Pi(N)$. Let us first consider the case when M' and N' are also in $Rep_{\mathbb{K}}(Q') \setminus S_3$. In this case we have by the previous Lemmas that $F_3(M') >_{d_{(3)}} F_3(N')$ are semistable representations, so $\mathrm{Hom}(F_3(M'), F_3(N')) = 0$. It follows that in this case $\mathrm{Hom}(M', N') = 0$.

Next we want to show that if $M' \in \Pi(M)$ is not in $Rep_{\mathbb{K}}(Q') \setminus S_3$, then $M = S_3$ and M' is a direct sum of copies of S_3. To see this we suppose that M' can be written as $M'' \oplus S_3$. M'' can not have S_1 as a summand or M' would have $S_1 \oplus S_3 = I_1 \oplus P_3$ as a summand which contradicts $M' \in \Pi(M)$. Now it should be clear that if $M' >_d S_3$, then $M'' >_d M'$ which also contradicts $M' \in \Pi(M)$. Hence $M' =_d M =_d S_3$, it follows that $M = S_3$ and that M' is a direct sum of S_3. Since S_3 is simple projective we have that $\mathrm{Hom}(\Pi(M), \Pi(S_3)) = 0$ for any indecomposable $M \not\simeq S_3$.

(*ii*) We need to show the existence of HN-filtrations. Let R be a representation of Q', we can write R as $R = R' \oplus (S_3)^n$, with R' in $Rep_{\mathbb{K}}(Q') \setminus S_3$ for some $n \in \mathbb{N}$. The idea is to get a HN-filtration

$$\begin{array}{ccccccccccc} F_3(R') & \hookleftarrow & G_1^{(3)} & \hookleftarrow & G_2^{(3)} & \hookleftarrow & \cdots & \hookleftarrow & G_n^{(3)} & \hookleftarrow & 0 \\ & \searrow & & \searrow & & \searrow & & & & \searrow & \\ & & R_0'^{(3)} & & R_1'^{(3)} & & R_2'^{(3)} & & \cdots & & R_n'^{(3)} \end{array}$$

of $F_3(R')$ and then apply the inverse equivalence to this filtration to obtain a filtration

$$\begin{array}{ccccccccccccc} R & \hookleftarrow & R' & \hookleftarrow & G_1 & \hookleftarrow & G_2 & \hookleftarrow & \cdots & \hookleftarrow & G_n & \hookleftarrow & 0 \\ & \searrow & & \searrow & & \searrow & & \searrow & & & & \searrow & \\ & & (S_3)^n & & R_0' & & R_1' & & R_2' & & \cdots & & R_n' \end{array}$$

of R. Since $S_3^{(3)}$ is injective and maximal in $\leq_{d_{(3)}}$ we have that the whole HN-filtration of $F_3(R')$ is in $Rep_{\mathbb{K}}(Q'^{(3)}) \setminus S_3^{(3)}$. We must check that each R'_i is in some category $\Pi(M)$, and that the factors are ordered $(S_3)^n <_d R'_0 <_d R'_1 <_d \cdots <_d R'_n$. By Lemma 4.3.2 the ordering of the factors is clear. Let us work with a factor R'_i and $R'^{(3)}_i = F_3(R'_i)$. Since $R'^{(3)}_i$ is in $Rep_{\mathbb{K}}(Q^{(3)}) \setminus S_3^{(3)}$ we know that $R'_i \in Rep_{\mathbb{K}}(Q) \setminus S_3$. Remark that for any indecomposable summand Z of $R'^{(3)}$ we have $Z =_{d_{(3)}} R'^{(3)}$ so there is an indecomposable representation M of Q' such that $M =_d R'$. Lastly we have that any subrepresentation of R' in $Rep_{\mathbb{K}}(Q) \setminus S_3$ are smaller or equal to R' in the distinguished ordering. Since S_3 is minimal in $\leq_d$ it follows that R' is in $\Pi(M)$. So the filtration of R is a HN-filtration.

Since $\Pi(M)$ is equivalent to $F_3(\Pi(M))$ for $M \not\simeq S_3$ it is an abelian category. It is not hard to see that also $\Pi(S_3)$ is abelian since its objects are just the direct sums of S_3, hence Q' has distinguished stability data.

□

Finally we have seen an example of stability data for an abelian category that is not equivalent to any slope stability. It seems that similar arguments can be made for quite a large family of right angled and obtuse quivers.

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