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Interest rate modeling with applications to counterparty risk

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Problem Description

The main objective of this study is to implement and analyze a Monte Carlo simulation model for counterparty exposure arising from interest rate transactions. The underlying interest rate model is a LIBOR market model, and this will be used for both interest rate evolution and pricing of derivatives. Exposure simulations are time consuming and we will use variance reduction techniques to speed up convergence. Furthermore, we want to study how to best structure simulations with concern to time discretization. We will also compare our implemented simulation model to the current exposure method provided by the Bank for International Settlements.

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Preface

This thesis was carried out at the Department of Mathematical Sciences at the Norwegian University of Science and Technology (NTNU), Trondheim, during the period February 2006 to June 2006, leading to the degree of Master of Science.

I would like to thank my supervisor Jacob Laading for providing useful information and giving constructive feedback.

Thanks to all students and colleagues at the Department of Mathematical Sciences for being helpful and for creating a positive and inspiring atmosphere.

Håvard Hegre

Abstract

This thesis studies the estimation of credit exposure arising from a portfolio of interest rate derivatives. The estimation is performed using a Monte Carlo simulation. The results are compared to the exposure obtained under the current exposure method provided by the Bank for International Settlements (BIS). We show that the simulation method provides a much richer set of information for credit risk managers. Also, depending on the current exposure and the nature of the transactions, the BIS method can fail to account for potential exposure. All test portfolios benefit significantly from a netting agreement, but the BIS approach tends to overestimate the risk reduction due to netting.

In addition we examine the impact of antithetic variates and different time-discretizations. We find that a discretization based on derivatives' start and maturity dates may reduce simulation time significantly without loosing generality in exposure profiles. Antithetic variates have a small effect.

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Chapter 1

Introduction

A *derivative* is a financial instrument whose value depends on the values of other, more basic underlying variables. Financial derivatives can be grouped into three general headings: Options, Forwards and Futures and Swaps. Bingham & Kiesel (2004) give a more precise academic definition:

Definition 1.1. *A derivative security, or contingent claim, is a financial contract whose value at expiration date T is determined exactly by the price of the underlying financial assets at time T.*

Very often the variables underlying derivatives are the prices of traded assets. A stock option, for example, is a derivative whose value is dependent on the price of a stock. However, derivatives can be dependent on almost any variable, from currencies to the amount of snow falling at a certain ski resort. Derivatives are themselves assets - they are traded, have value etc. - and so can be used as underlying for new contingent claims: options on futures, options on baskets of options etc. These developments give rise to so-called *exotic options*, demanding a sophisticated mathematical machinery to handle them.

Interest rate derivatives, which will be the main focus of this work, are instruments whose payoffs are dependent in some way on the level of interest rates. Interest rates themselves are notional assets, which cannot be delivered. Hedging exposure to interest rates is more complicated than hedging exposure to the price movements of a certain stock. Hull (2000) gives a number of reasons for this:

1. The behavior of an individual interest rate is more complicated than that of a stock price or an exchange rate. For instance, interest rates cannot grow unbounded. For large interest rates the rates will tend to decrease toward the mean, and vice versa.

2. For the valuation of many products, it is necessary to develop a model describing the behavior of the entire yield curve.

3. The volatilities of different points on the yield curve are different. Short-term rates have greater volatility than long-term rates.

4. In addition to defining the payoff from the derivative, interest rates are also used for discounting.

Financial derivatives are basically traded in two ways: on organized exchanges and over-the-counter (OTC). Organized exchanges are subject to regulatory rules, require a certain degree of standardization of the traded instruments and have a physical location at which trade takes place. The over-the-counter market is an important alternative to exchanges and, measured in terms of total volume of trading, has become much larger[1] than the exchange-traded market. It is a telephone- and computer-linked network of dealers, who do not physically meet. Trades are usually between two financial institutions or between a financial institution and one of its corporate clients. An advantage of the OTC market is that the terms of a contract do not have to be those specified by an exchange. Market participants are free to negotiate any mutually attractive deal. However, there is usually some credit risk in an OTC trade, i.e. there is a risk that the contract will not be honored (Hull 2000).

The outstanding volume of OTC derivatives has grown exponentially over the past 18 years. Market surveys[2] conducted by the International Swaps and Derivatives Association (ISDA) show outstanding notional amounts of interest rate and currency swaps reaching US$866 billion in 1987 and US$213.2 trillion in 2005. Derivatives have expanded the opportunities to transfer risk. Counterparty risk is the risk that a party to an OTC derivatives contract may fail to perform on its contractual obligations, causing losses to the other party. Counterparty exposure is defined as the larger of zero and the market value of the portfolio of derivative positions with a counterparty that would be lost if the counterparty were to default and there were zero recovery. The value of an OTC derivatives portfolio, which depends on market variables such as interest rates or exchange rates, will change when those variables change. As a result, counterparty credit exposure will change in the future even if no new positions are added to the portfolio.

The exposure from derivative transactions is very different and more complicated as compared to the exposure from the loan business. If a counterparty defaults only minutes after the confirmation of an OTC derivative transaction, losses are minimal because the transaction can be replaced with another transaction at approximately the same market rates. However, if default occurs at a later point of time and the market variables have changed, the replacement costs could be significant.

With increased competition and tighter spreads, banks must accurately quantify the credit risk they are facing so that they can appropriately price their products, set the proper level of capital reserves and manage their credit line efficiently. Currently, different banks employ different credit risk measurement techniques, ranging from applying a fixed percentage to the notionals of the transactions, to creating distributions of future credit exposure and credit loss.

The Basel Committee on Banking Supervision provides a forum for regular cooperation on banking supervisory matters. Over the recent years, it has developed into a standard-setting body on all aspects of banking supervision. The Committee's Secretariat is provided by the Bank for International Settlements (BIS) in Basel. The best known publication is the Basel II framework, also known as the Revised Framework, in which the Committee presents three different approaches for measuring exposure at default: an internal model method (IMM), the standardized method (SM) and the current exposure method (CEM). These three methods

[1]OTC derivatives statistics can be found at www.bis.org/statistics/derstats.html
[2]See published surveys on ISDA's website, www.isda.org.

represent a continuum of sophistication in risk management practices, where CEM is the simplest approach (Bas 2005*a*).

Although no particular form of model is required under the IMM, we will restrict ourselves to a simulation model. Monte Carlo (MC) simulation is a powerful risk analysis tool because it alone can accurately and clearly adjust risk estimates for optionality and convexity. Banks can employ Monte Carlo techniques to understand and evaluate current market pricing as well as their economic value at risk. This technique provides banks with a valuable tool for measuring and managing interest rate risk.

1.1 Thesis Outline

The thesis is organized as follows. In the next chapter we describe the basic ideas of Monte Carlo simulations and variance reduction techniques. In Chapter 3 we present derivative pricing theory. This includes a closer look at arbitrage and risk-neutral pricing, term structure of interest rates, interest rate derivatives and finally a detailed description of the LIBOR market model. We investigate the LIBOR model dynamics under the spot- and the forward measure. Chapter 4 is the heart of the thesis. It describes how to measure and mark counterparty credit exposure. We present both a simple supervisory method and an implemented Monte Carlo simulation engine. Chapter 5 describes data used for calibration and testing. Results are presented and discussed in Chapter 6. Chapter 7 summarizes the thesis and gives some directions for further research. A probability background and various mathematical tools are given in Appendix A.

Chapter 2

Preliminaries

2.1 Monte Carlo Simulation

A very commonly used tool for pricing and risk management of financial derivatives is the Monte Carlo method. The main advantage of the method is that it is easy to understand and implement but still very powerful and has a broad spectra of applications. The convergence order is $\mathcal{O}(n^{1/2})$, where n is the number of replications. Since the convergence order does not depend on the dimensionality of the problem the Monte Carlo method is popular in a wide range of high-dimensional problems, from atom physics to finance. However, it can be very slow since an additional factor 4 increase in the number of realizations only provides an additional improvement in accuracy by a factor 2.

Monte Carlo Integration

Valuing a derivative security by Monte Carlo typically involves simulating paths of stochastic processes used to describe the evolution of underlying asset prices, interest rates, model parameters, and other factors relevant to the security in question.

In general Monte Carlo is used for simulation and optimization. However, in the context of financial derivatives pricing with the LIBOR market model the focus will be on integration problems. Consider a square-integrable function $f \in L^2(0,1)$ and a uniformly distributed random variable $x \in [0,1]$. The integral of f over $[0,1]$ can be expressed as an expectation of the function value

$$C := \mathbb{E}[f(x)] = \int_0^1 f(x)dx,$$

which yields an unbiased estimator of the integral. Consider a sequence x_i sampled from $U[0,1]$. An empirical approximation of the expectation is then

$$\hat{C}_n := \mathbb{E}[f(x)] \approx \frac{1}{n}\sum_{i=1}^{n} f(x_i),$$

where we suppose that $\mathrm{Var}[C_i] = \sigma_C^2 < \infty$. The Strong Law of Large Numbers implies that this approximation is convergent with probability one, i.e.

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(x_i) = \int_0^1 f(x)dx,$$

and the Monte Carlo integration error is defined as

$$\epsilon_n := \hat{C}_n - C = \frac{1}{n}\sum_{i=1}^{n} f(x_i) - \int_0^1 f(x)dx.$$

The central limit theorem asserts that as the number of replications n increases, the standardized estimator $(\hat{C}_n - C)/(\sigma_C/\sqrt{n})$ converges in distribution to the standard normal,

$$\frac{\hat{C}_n - C}{\sigma_n/\sqrt{n}} \Rightarrow N(0,1) \qquad \text{or} \qquad \sqrt{n}[\hat{C}_n - C] \Rightarrow N(0,\sigma_C^2).$$

Here, $\Rightarrow$ denotes convergence in distribution and $N(a,b^2)$ denotes the normal distribution with mean a and variance b^2. The central limit theorem tells us something about the distribution of the error in our simulation estimate,

$$\hat{C}_n - C \sim N(0,\sigma_C^2/n).$$

The same limit holds if σ_C is replaced with the sample standard deviation s_C, which is important since σ_C is rarely known in practice. The fact that we can replace σ_C with s_C without changing the limit in the distribution follows from the fact that $s_C/\sigma_C \to 1$ as $n \to \infty$ and general results on convergence in distribution (Casella & Berger 2002).

Random Number Generation

At the core of nearly all Monte Carlo simulations is a sequence of random numbers used to drive the simulations. This driving engine will supply variates which in the limit of infinitely many draws satisfy a given joint multivariate distribution density function. Typically, the density function is obtained by transformation of draws from the uniform distribution function on the interval $[0,1]$.

In our setting it will be necessarily to generate samples from univariate normal distributions. Box & Muller (1958) and Marsaglia & Bray (1964) describe two methods. Since these algorithms are quite straightforward they will not be dealt with here. Note that the number generator is an important link in the chain that comprises a Monte Carlo method and that the reliability of it is crucial. One should always have more than one number generator available. Instead of rerunning a calculation with a new seed one could make the computation using a different number generator.

2.1.1 Variance Reduction Techniques

To overcome the rather slow convergence of Monte Carlo simulations we can use several methods to reduce the variance of simulation estimates. These methods draw on two broad strategies: taking advance of tractable features of a model to adjust or to correct simulations output, and reducing the variability in simulation inputs. Control variates, antithetic variates, stratified sampling and importance sampling are some examples of these methods. However, one disadvantage with these variance reduction techniques is that they have to be specially designed to each problem and for many problems it might be hard, or impossible, to find nicely working techniques.

The greatest gains in efficiency from variance reduction techniques result from exploiting specific features of a problem, rather than from generic applications of generic methods. In order to supplement a reduced-variance estimator with a valid confidence interval, we sometimes need to sacrifice some of the potential variance reduction.

Antithetic Variates

The method of antithetic variates is based on the simple observation that if Z has a standard normal distribution so does $-Z$. The idea is that random inputs obtained from the collection of antithetic pairs $\{(Z_i, -Z_i)\}_{i=1}^{I}$ are more regular distributed than a collection of $2I$ independent samples.

If the Z_i are used to simulate the increments of a Brownian path, then the $-Z_i$ simulate the increments of the reflection of the path about the origin. Suppose our objective is to calculate $E[Y]$ and that some implementation of antithetic variates produces a sequence of i.i.d. pairs of observations $(Y_1, \tilde{Y}_1), (Y_2, \tilde{Y}_2), \dots, (Y_n, \tilde{Y}_n)$. For each i, Y_i and $\tilde{Y}_i$ have the same distribution, though ordinarily they are not independent. The antithetic variates estimator is the average of all $2n$ observations,

$$\hat{Y}_{AV} = \frac{1}{2n}\left(\sum_{i=1}^{n} Y_i + \sum_{i=1}^{n} \tilde{Y}_i\right) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{Y_i + \tilde{Y}_i}{2}\right). \tag{2.1}$$

Thus, Y_{AV} is the sample mean of the n independent observations $(Y_1 + \tilde{Y}_1/2)$, $(Y_2 + \tilde{Y}_2/2)$, …, $(Y_n + \tilde{Y}_n/2)$. The central limit theorem therefore applies and gives

$$\frac{\hat{Y}_{AV} - E[Y]}{\sigma_{AV}/\sqrt{n}} \Rightarrow N(0,1) \tag{2.2}$$

with $\sigma_{AV}^2 = \mathrm{Var}[(Y_i + \tilde{Y}_i)/2]$.

The question is under what conditions is an antithetic variates estimator to be preferred to an ordinary Monte Carlo estimator based on independent replications? To make this comparison we start with the assumption that the computational effort required to generate a pair $(Y_i + \tilde{Y}_i)$ is approximately twice the effort required to generate Y_i. Thus, the effort required to compute Y_{AV} is approximately that required to compute the sample mean of $2n$ independent replications. Using antithetics reduces variance if

$$\text{Var}[Y_{AV}] < \text{Var}[\frac{1}{2n}\sum_{i=1}^{2n} Y_i],$$

i.e., if $\text{Var}[Y_i + \tilde{Y}_i] < 2\text{Var}[Y_i]$. Using the fact that Y_i and $\tilde{Y}_i$ have the same variance if they have the same distribution, we can write the variance on the left as

$$\begin{aligned} \text{Var}[Y_i + \tilde{Y}_i] &= \text{Var}[Y_i] + \text{Var}[\tilde{Y}_i] + 2\text{Cov}[Y_i, \tilde{Y}_i] \\ &= 2\text{Var}[Y_i] + 2\text{Cov}[Y_i, \tilde{Y}_i]. \end{aligned}$$

Therefore, the condition for antithetic sampling to reduce variance becomes

$$\text{Cov}[Y_i, \tilde{Y}_i] < 0.$$

This condition requires that negative dependence in the inputs (Z and $-Z$) produce negative correlation between the outputs of paired replications. A sufficient condition that ensure this is monotonicity of the mapping from inputs to outputs defined by a simulation algorithm. However, this requirement is rarely satisfied exactly (Glasserman 2004).

2.2 Order Statistics

The outcome of a Monte Carlo simulation is typically a random sample X_i, $i = 1, \ldots, n$. Sample values such as the smallest, largest, or middle observation from a random sample can provide useful summary information. These are all examples of order statistics. Casella & Berger (2002) give a precise definition:

Definition 2.1. *The order statistics of a random sample $X_1, \ldots, X_n$ are the sample values placed in ascending order. They are denoted by $X_{(1)}, \ldots, X_{(n)}$.*

The order statistics are random variables that satisfy $X_{(1)} \leq \cdots \leq X_{(n)}$. In particular,

$$\begin{aligned} X_{(1)} &= \min_{1 \leq i \leq n} X_i, \\ X_{(2)} &= \text{second smallest } X_i, \\ &\vdots \\ X_{(n)} &= \max_{1 \leq i \leq n} X_i. \end{aligned}$$

Chapter 3

Derivative Pricing

In order to calculate exposure at future dates, all trades with a counterparty must be priced at each date. That task consumes large computational resources. Imagine a swap dealer's book with 50000 positions. With average maturity of seven years and 5000 market scenarios generated every three month, one would need to perform 7.0 billion pricings. Unnecessary pricing complexity should be avoided: one should avoid refining within the margin of error. This is particularly important for long-dated trades because the volatilities, correlations and probabilistic assumptions used in constructing long-dated scenarios are themselves surrounded by considerable uncertainty.

This chapter deals with the pricing of interest rate derivatives. A probability background is given in Appendix A. For proofs and a more detailed description see Glasserman (2004) and Bingham & Kiesel (2004).

3.1 Basic Principles

The mathematical theory of derivatives pricing is both elegant and remarkably practical. A proper development of the full theory would require a book-length treatment. We will therefore highlight only some principles of the theory, especially those that bear on the application of Monte Carlo to the calculation of prices. To apply Monte Carlo simulation we must find a more convenient representation of derivative prices. In particular, we would like to represent derivative prices as expectations of random objects that we can simulate. Three principles are particularly important:

1. If a derivative security can be hedged through trading in other assets, the the price of the derivative security is the cost of the replacing trading strategy.

2. Discounted asset prices are martingales under a probability measure associated with the choice of discount factor. Prices are the expectations of discounted payoffs under such a martingale measure.

3. In a complete market, any payoff can be synthesized through a trading strategy, and the martingale measure associated with the numeraire is unique.

The first of the principles gives us a way of thinking what the price of a derivative security ought to be, but says little of how it might be evaluated. The second principle is the main link between pricing and Monte Carlo because it tells us how to represent prices as expectations. Expectations lend themselves to evaluation through Monte Carlo and other numerical methods. A complete market is one in which all risks can be perfectly hedged. The third principle may be viewed as describing conditions under which the price of a derivative security is determined by the prices of other assets so that the first and second principles apply. A more detailed description is given below.

3.2 Arbitrage and Risk-Neutral Pricing

One of the fundamental concepts underlying the theory of financial derivative pricing and hedging is that of *arbitrage*. That is, there are never[1] any opportunities to make an instantaneous risk-free profit. Bingham & Kiesel (2004) define the concept of arbitrage in mathematical terms:

Definition 3.1. *Let $\tilde{\Phi} \subset \Phi$ be a set of self-financing strategies. A strategy $\phi \in \tilde{\Phi}$ is called an arbitrage opportunity or arbitrage strategy with respect to $\tilde{\Phi}$ if $\mathbb{P}\{V_\phi(0) = 0\} = 1$, and the terminal wealth of ϕ satisfies*

$$\mathbb{P}\{V_\phi(T) \geq 0\} = 1 \text{ and } \mathbb{P}\{V_\phi(T) > 0\} > 0.$$

We say that a market $\mathcal{M}$ is arbitrage-free if there are no arbitrage opportunities in the class Φ of trading strategies. Before arriving at the first central theorem in this section we will need the definition of equivalent martingale measures.

Definition 3.2. *A probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F}_T)$ equivalent to $\mathbb{P}$ is called a martingale measure for $\tilde{S}$ if the process $\tilde{S}$ follows a $\mathbb{P}^*$-martingale with respect to the filtration $\mathbb{F}$. We denote by $\mathcal{P}(\tilde{S})$ the class of equivalent martingale measures.*

Here, Ω represents the sample space and the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t=0}^T$ represents the information, or knowledge, available to us at time t.

Theorem 3.1. ***(No-arbitrage Theorem)*** *The market $\mathcal{M}$ is arbitrage-free if and only if there exists a probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ under which the discounted d-dimensional asset price process $\tilde{S}$ is a $\mathbb{P}^*$-martingale.*

We say that a contingent claim is *attainable* if there exists a replicating strategy $\phi \in \Phi$ such that

$$V_\phi(T) = X.$$

Working with discounted values we have

[1] More correctly, such opportunities cannot exist for a significant length of time before prices move to eliminate them.

$$\beta(T)X = \tilde{V}_\phi(T) = V(0) + \tilde{G}_\phi(T),$$

where $\tilde{G}_\phi$ is the *gain process* of a trading strategy ϕ. In other words, the discounted value of a contingent claim is given by the initial cost of setting up a replication strategy and the gains from trading. Bingham & Kiesel (2004) show that, in an arbitrage-free market $\mathcal{M}$, any attainable contingent claim X is uniquely replicated in $\mathcal{M}$. Uniqueness allows us to define the important concept of an arbitrage price process.

Definition 3.3. *Suppose the market is arbitrage-free. Let X be any attainable contingent claim with time T maturity. Then the arbitrage price process $\pi_X(t), 0 \leq t \leq T$ is given by the value process of any replicating strategy ϕ for X.*

Analysing the arbitrage-pricing approach we observe that any attainable contingent claim must be independent of all preferences that do not admit arbitrage. Any set of risk preferences can, therefore, be used when evaluating V. In particular, the very simple assumption that all investors are risk-neutral can be made.

Proposition 3.1. *The arbitrage price process of any attainable contingent claim X is given by the risk-neutral valuation formula*

$$\pi_X(t) = \beta(t)^{-1}\mathbb{E}^*(X\beta(T) \mid \mathcal{F}_t) \qquad \forall t = 0, 1, \ldots, T,$$

where $\mathbb{E}^*$ is the expectation operator with respect to an equivalent martingale measure $\mathbb{P}^*$.

All attainable contingent claims can be priced using an equivalent martingale measure. A desired property of the market $\mathcal{M}$ would be if all contingent claims are attainable. We continue with the definition of completeness.

Definition 3.4. *A market $\mathcal{M}$ is complete if every contingent claim is attainable, i.e for every $\mathcal{F}_T$-measurable variable X there exists a replicating self-financing strategy $\phi \in \Phi$ such that $V_\phi(T) = X$.*

Theorem 3.2. ***(Completeness Theorem)*** *An arbitrage-free market $\mathcal{M}$ is complete if and only if there exists a unique probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ under which discounted asset prices are martingales.*

By combining the No-arbitrage Theorem and the Completeness Theorem we state the fundamental theorem of asset pricing.

Theorem 3.3. ***(Fundamental theorem of asset pricing)*** *In an arbitrage-free complete market $\mathcal{M}$, there exists a unique equivalent martingale measure $\mathbb{P}^*$.*

Pricing contingent claims is our central task, and for pricing purposes $\mathbb{P}^*$ is vital and $\mathbb{P}$ itself irrelevant. We therefore focus attention on $\mathbb{P}^*$ which is called the *risk-neutral* probability measure. Throughout we will assume the existence of *risk-free* investments that give a guaranteed return with no chance of default. Such an investment could be a government bond or a bank deposit. In a world where investors are risk-neutral, the expected return on all securities is the risk-free rate of interest, r. The reason for this is that risk-neutral investors do not

1. replace drift μ_i with the risk-free interest rate and simulate paths
2. calculate payoff of derivative security on each path
3. discount payoff at the risk-free rate
4. calculate average over paths

Figure 3.1: A summary of risk-neutral pricing by Monte Carlo.

require a premium to induce them to take risks. Also, the present value of any cash flow in a risk-neutral world can be obtained by discounting its expected value at the risk-free rate. The assumption that the world is risk neutral does, therefore, considerably simplifies the analysis of derivatives. The solutions that are obtained are valid in all worlds, not just those where investors are risk-neutral (Hull 2000). To summarize, we have:

Theorem 3.4. ***(Risk-neutral Pricing Formula)*** *In an arbitrage-free complete market $\mathcal{M}$, arbitrage prices of contingent claims are their discounted expected values under the risk-neutral (equivalent martingale) measure $\mathbb{P}^*$.*

Consider a derivative security with a payoff at time T specified through a function f of the prices of the underlying asset. To price the derivative we model the dynamics of the underlying asset under the risk-neutral measure, ensuring that discounted asset prices are martingales. The price is then given by $\mathrm{E}_\beta[\mathrm{e}^{rTf(S(T))}]$. To evaluate this expression, we simulate paths of the underlying asset over the time interval $[0, T]$. Next we calculate the discounted payoff $\mathrm{e}^{rT}f(S(T))$ and the average across paths is our estimate of the derivative's price. An overly simplified summary of risk-neutral pricing by Monte Carlo is given in Figure 3.1.

3.3 Term Structure of Interest Rates

The term structure of interest rates refers to the dependence of interest rates on maturity. There are several equivalent ways of recording this relationship - through the prices of or yields of zero-coupon bonds, through forward rates, and through swap rates, to name just a few examples.

Let $B(t, T)$ denote the price at time t of a security making a single payment of 1 at time $T, T \geq t$. This is a *zero-coupon bond* with maturity T. With constant continuously compounded interest rate an investor could replicate a zero-coupon bond with maturity T by investing e^{-RT} in a bearing account at time 0 and letting it grow to a value of 1 at time T. Thus, $B(0, T) = e^{-RT}$. More generally, if the continuously compounded rate at time t is given by a stochastic process $r(t)$, an investment of 1 at time 0 grows to a value of

$$\beta(t) = \exp\left(\int_0^t r(u)du\right)$$

at time t. It follows that the price of a bond is given by

$$B(0,T) = \mathrm{E}\left[\exp\left(\int_0^T r(t)dt\right)\right],$$

where the expectation is taken under the risk-neutral measure.

The continuously compounded *yield* $Y(t,T)$ of a zero-coupon bond maturing at T is defined as

$$B(t,T) = e^{Y(t,T)(T-t)} \qquad \text{or} \qquad Y(t,T) = -\frac{1}{T-t}\log B(t,T). \tag{3.1}$$

The yield of a coupon-bearing bond is similarly defined by discounting the coupons as well as the principal payment.

A *forward rate* is an interest rate set today for borrowing or lending at some date in the future. Consider the case of simple interest and let $F(t,T_1,T_2)$ denote the forward rate set at time t for the interval $[T_1,T_2]$. An investor entering into an agreement at time t to borrow 1 at time T_1 and repay at time T_2 pays interest at rate $F(t,T_1,T_2)$. Glasserman (2004) shows that forward rates are determined by bond prices through the relation

$$F(t,T_1,T_2) = \frac{1}{T_2-T_1}\left(\frac{B(t,T_1)-B(t,T_2)}{B(t,T_2)}\right). \tag{3.2}$$

3.4 Interest Rate Derivatives

An *interest rate cap* places a ceiling on a floating rate of interest on a specific notional principal amount for a specific term. The buyer of a cap uses the cap contract to limit his maximum interest rate. The cap premium charged by the seller depends upon the market's assessment of the probability that rates will move through the cap strike over the time horizon of the deal. The cap premium takes the form of an up front charge that is usually expressed in basic points as percentage of the notional principal amount. Any period that the cap is in the money, the cap buyer's effective rate is equal to the cap strike plus the amortized cap premium in basic points. Otherwise, the effective rate equals the floating rate plus the amortized cap premium. A cap could be divided into *caplets*, which can be interpreted as a call option on a simple forward rate. An option on a cap is called a caption.

An *interest rate floor* places a minimal value on a floating rate of interest on a specific notional principal amount for a specific term. The buyer of the floor uses the floor contract to limit his minimum interest rate. The seller of the floor accepts a minimum on the interest rate it will pay in return for the floor premium. Often the floor premium is used to offset the cost of a purchased interest rate cap. If the floating rate drops below the floor strike, the floor provides for payments from the seller to the buyer for the difference between the floor strike and the floating rate. A floor could be divided into *floorlets* and an option on a floor is called a floortion.

An *interest rate swap* is an agreement with a counterparty to exchange interest payments on some notional amount at fixed periods of time. In a *payer* swap on our tenor structure, the holder pays a fixed rate of interest K over each period $\delta_n = T_{n+1} - T_n$. The holder then receives a floating rate of interest fixed at the beginning of each period. Payments are exchanged at the end of each time period. A swaption is an option on a swap.

A *forward rate agreement* (FRA) is an agreement that a certain interest rate will apply to a certain principal during a specified future period of time. The FRA does not involve any transfer of principal. It is settled at maturity in cash, representing the profit or loss resulting from the difference in the agreed rate and the settlement rate at maturity.

3.5 The LIBOR Market Model

The LIBOR market model describes the arbitrage-free dynamics of the term structure of interest rates through the evolution of forward rates. We will work in terms of *simple* forward rates, i.e. rates that really are quoted in the market. For much of the financial industry, the most important benchmark interest rates are the London Inter-Bank Offered Rates or LIBOR. LIBOR is calculated daily through an average of rates offered by selected banks in London. The forward LIBOR rate $L(0,T)$ is the rate set at time 0 for the interval $[T, T+\delta]$. If we enter a contract at time 0 to borrow 1 at time T and repay it with interest at time $T+\delta$, the interest due will be $\delta L(0,T)$. The term "market model" is often used to describe an approach to interest rate modeling based on observable market rates.

3.5.1 LIBOR Market Model Dynamics

We start this section by looking at the relation between forward LIBOR rates and bond prices. From equation (3.2) we have

$$L(0,T) = \frac{B(0,T) - B(0,T+\delta)}{\delta B(0,T+\delta)}, \tag{3.3}$$

with a fixed length δ.

I should be noted that we treat the forward LIBOR rates as though they were risk-free rates. LIBOR rates are based on quotes by banks which could default and the risk is presumably reflected in the rates. Note also that the argument leading to (3.3) may not hold exactly if the bonds on one side and the forward rate on the other reflect different levels of credit-worthiness.

Although (3.3) apply in principle to a continuum of maturities, we will consider models in which a finite set of maturities or *tenor dates*

$$0 = T_0 < T_1 < \cdots < T_M < T_{M+1}$$

are fixed in advance. Let

$$\delta_i = T_{i+1} - T_i, \qquad i = 0, \ldots, M$$

denote the lengths of the intervals between tenor dates. For each date T_n we let $B_n(t)$ denote the time-t price of a bond maturing at $T_n, 0 \leq t \leq T_n$. Similarly, we write $L_n(t)$ for the forward rate as of time t for the accrual period $[T_n, T_{n+1}]$. Thus, we get

$$L_n(t) = \frac{B_n(t) - B_{n+1}(t)}{\delta B_{n+1}(t)}, \qquad 0 \leq t \leq T_n, \; n = 0, 1, \ldots, M. \tag{3.4}$$

An illustration showing the relation between bond prices, forward rates and the interest rate tenor structure is given in Figure 3.2.

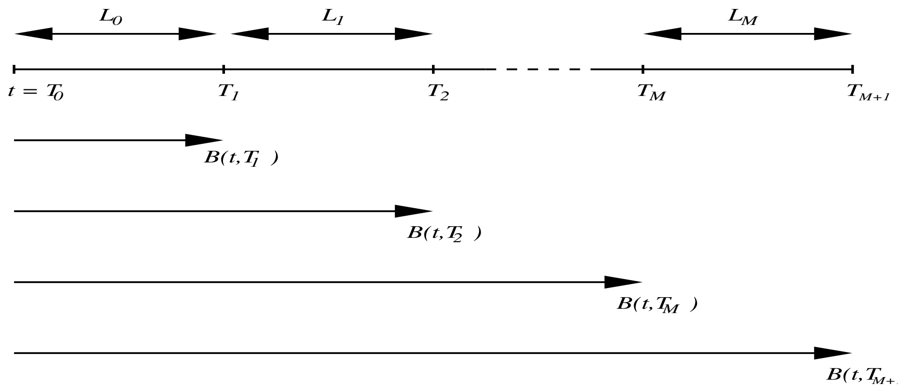


Figure 3.2: An $M + 1$ period interest rate tenor structure.

We see that bond prices determine the forward rates. At a tenor date, T_i, the relation can be inverted to produce

$$B_n(T_i) = \prod_{j=1}^{n-1} \frac{1}{1 + \delta_j L_j(T_i)}, \qquad n = i + 1, \ldots, M + 1. \tag{3.5}$$

However, one problem arises. At an arbitrary date t, the forward rates do not determine the bond prices because they do not determine the discount factor for interval shorter than the accrual periods. Suppose that $T_i < t < T_{i+1}$ and we want to find the price $B_n(t)$ for some $n > i + 1$. The factor

$$\prod_{j=i+1}^{n-1} \frac{1}{1 + \delta_j L_j(t)}$$

discounts the bond's payment at $\delta_j L_j(t, T_n)$ back to time T_{i+1}, and not the discount factor from T_{i+1} to t. To handle this problem we define a function $\eta : [0, T_{M+1}) \rightarrow \{1, \ldots, M + 1\}]$ by taking $\eta(t)$ to be the unique integer satisfying

$$T_{\eta(t)-1} \leq t \leq T_{\eta(t)}.$$

Thus, $\eta(t)$ gives the index of the next tenor date at time t. With this notation, we have

$$B_n(T_i) = B_{\eta(t)}(t) \prod_{j=\eta(t)}^{n-1} \frac{1}{1+\delta_j L_j(T_j)}, \qquad 0 \le t \le T_n. \tag{3.6}$$

Spot Measure

We seek a model were the forward LIBOR rates are described by a system of SDEs of the form

$$\frac{dL_n(t)}{L_n(t)} = \mu_n(t)dt + \sigma_n(t)^\top dW(t), \qquad 0 \le t \le T_n, \qquad n = i+1, \ldots, M+1, \tag{3.7}$$

where W is a d-dimensional standard Brownian motion. The coefficients μ_n and σ_n may depend on the current forward rates as well as the current time. The σ_n in (3.7) is the proportional volatility because we have divided by L_n on the left.

The numeraire associated with the risk-neutral measure is $\beta(t) = \exp(\int_o^t r(u)du)$. But in the LIBOR setting we are developing a model based on the simple rates $L_n(t)$, and are therefore not interested in the spot-rate process $r(t)$. Thus, we avoid the usual risk-neutral measure and instead focus on a numeraire asset better suited to the tenor dates T_i. A counterpart of $\beta(t)$ is derived by starting with 1 unit of account and reinvest in bonds with different maturities. The initial investment at time 0 grows to a value of

$$B^*(t) = B_{\eta(t)}(t) \prod_{j=0}^{\eta(t)-1} [1 + \delta_j L_j(T_j)]$$

at time t. We take this as the numeraire asset and call the associated measure the *spot measure*.

The absence of arbitrage means that bond prices must be martingales when *deflated*[2] by the numeraire asset. From (3.6) we get that the deflated bond price $D_n(t) = B_n(t)/B^*(t)$ is given by

$$D_n(t) = \left(\prod_{j=0}^{\eta(t)-1} \frac{1}{1+\delta_j L_j(T_j)} \right) \prod_{j=\eta(t)}^{n-1} \frac{1}{1+\delta_j L_j(t)}, \qquad 0 \le t \le T_n. \tag{3.8}$$

The spot measure numeraire B^* cancels the factor $B_{\eta(t)}(t)$, and we are left with an expression defined purely in terms of the LIBOR rates. Notice that the first factor in (3.8) is constant between tenor dates T_i. Starting with the requirement that deflated bond prices be positive martingales Glasserman (2004) arrives at

[2]We use the term 'deflated' rather than 'discounted' because we are dividing by the numeraire asset and not discounting at a continuously compounded rate.

$$\mu_n(t) = \sum_{j=\eta(t)}^{n} \frac{\delta_j L_j(t)\sigma_n(t)^\top \sigma_j(t)}{1+\delta_j L_l(t)}, \tag{3.9}$$

as the drift parameter in (3.7). Thus,

$$\frac{dL_n(t)}{L_n(t)} = \sum_{j=\eta(t)}^{n} \frac{\delta_j L_j(t)\sigma_n(t)^\top \sigma_j(t)}{1+\delta_j L_l(t)} dt + \sigma_n(t)^\top dW(t), \qquad 0 \leq t \leq T_n, \tag{3.10}$$

$n = 1, \ldots, M$, describes the arbitrage-free dynamics of forward LIBOR rates under the spot measure.

Forward Measure

We may formulate a LIBOR market model under the forward measure P_{M+1} for maturity T_{M+1}. The numeraire asset in this case is the bond B_{M+1}. The deflated bond $D_n(t) = B_n(t)/B_{M+1}(t)$ simplifies to

$$D_n(t) = \prod_{j=n+1}^{M} (1 + \delta_j L_j(t)). \tag{3.11}$$

Once again do the numeraire asset cancel the factor $B_{\eta(t)}$, leaving an expression that depends purely on the forward LIBOR rates.

The dynamics of the forward LIBOR rates under the forward measure could be derived using the Girsanov Theorem, Appendix A.3. Alternatively, we could start from the requirement that the D_n in (3.11) be martingales and proceed by induction to derive restrictions on the evolution of the L_n. Either way, the arbitrage-free dynamics of the L_n, $n = 1, \ldots, M$, under the forward measure P_{M+1} are given by

$$\frac{dL_n(t)}{L_n(t)} = -\sum_{j=n+1}^{M} \frac{\delta_j L_j(t)\sigma_n(t)^\top \sigma_j(t)}{1+\delta_j L_j(t)} dt + \sigma_n(t)^\top dW^{M+1}(t), \qquad 0 \leq t \leq T_n, \tag{3.12}$$

with W^{M+1} a standard d-dimensional Brownian motion under P_{M+1}. If we take $n = M$, we have

$$\frac{dL_M(t)}{L_M(t)} = \sigma_n(t)^\top dW^{M+1}(t),$$

so that L_M is martingale under the forward measure for maturity T_{M+1}. Moreover, if σ_M is deterministic it turns out that $L_M(t)$ has a log-normal distribution $LN(-\hat{\sigma}_M^2(t)/2, \hat{\sigma}_M^2(t))$ with

$$\hat{\sigma}_M(t) = \sqrt{\frac{1}{t}\int_0^t ||\sigma_M(u)||^2 du}.$$

In fact, each L_n is martingale under the forward measure P_{M+1}.

3.5.2 Simulation

We will use simulation to price interest rate derivatives in the LIBOR market model. Exact simulation is generally infeasible and some discretization error is inevitable. Because we have a finite set of maturities we only need to discretize the time argument.

Discretization and Numerical Schemes

We fix a time grid $0 = t_0 < t_1 < \cdots < t_m < t_{m+1}$ over which to simulate. It is sensible to include the tenor dates $T_1, \ldots, T_{M+1}$ among these. A natural choice of time discretization would be to simulate the exposure according to the tenor structure, i.e. $t_i = T_i$. However, this choice of time discretization will produce only a few points on the exposure profile. Although interpolation could be used to make a full exposure profile we might not capture shifts over small periods of time. Thus, we must require a more general partition of the time axis. Note that there are two time axes; one representing the exposure horizon and one representing the forward rate structure. We will use the notation t_i for the former and T_i for the latter.

Simulation of forward LIBOR rates is a special case of the general problem of simulating a system of SDEs. There are many choices of variables to discretize and many choices of probability measures under which to simulate. We will restrict ourselves to an Euler scheme to $\log L$ under the spot and the forward measure. This because an Euler scheme to L could potentially produce negative rates. Discretizing the SDE (3.10), gives

$$\begin{aligned}&\hat{L}_n(t_{i+1}) = \hat{L}_n(t_i)\times \\ &\exp\left(\left[\mu_n(\hat{L}(t_i), t_i) - \frac{1}{2}||\sigma_n(t_i)||^2\right][t_{i+1} - t_i] + \sqrt{t_{i+1} - t_i}\,\sigma_n(t_i)^\top Z_{i+1}\right),\end{aligned} \tag{3.13}$$

where

$$\mu_n(\hat{L}(t_i), t_i) = \sum_{j=\eta(t)}^{n} \frac{\delta_j \hat{L}_j(t_i)\sigma_n(t_i)^\top \sigma_j(t_i)}{1 + \delta_j \hat{L}_j(t_i)}, \tag{3.14}$$

and $Z_1, Z_2, \ldots$ independent $N(0, 1)$ random vectors in $\Re^d$. We use hats to identify discretized variables. We are given an initial set of bond prices $B_1(0), \ldots, B_{M+1}(0)$ and initialize simulation by setting

$$\hat{L}_n(0) = \frac{B_n(0) - B_{n+1}(0)}{\delta B_{n+1}(0)}, \qquad n = 1, \ldots, M,$$

in accordance with (3.4). Under the forward measure the drift is given by

$$\mu_n(\hat{L}(t_i), t_i) = -\sum_{j=n+1}^{M} \frac{\delta_j \hat{L}_j(t_i)\sigma_n(t_i)^\top \sigma_j(t_i)}{1+\delta_j \hat{L}_j(t_i)}. \tag{3.15}$$

Monte Carlo Pricing

In order to price a derivative security, we simulate, say N, paths of the discretized variables $\hat{L}_1, \ldots, \hat{L}_M$. Suppose we want to price a derivative with a payoff of $f(L(T_n))$ at time T_n. Under the spot measure, we simulate to time T_n and then calculate the deflated payoff

$$\hat{C} = f(\hat{L}(T_n)) \times \prod_{j=0}^{n-1} \frac{1}{1+\delta_j \hat{L}_j(T_j)}. \tag{3.16}$$

The Law of Large Numbers guarantees that averaging over independent replications will give the derivative's price at time 0. That is,

$$C = \mathbb{E}[\hat{C}] \approx \frac{1}{N}\sum_{k=1}^{N} \hat{C}_k.$$

3.5.3 Volatility Structure

So far we have taken the volatility factors as inputs without indicating how they might be specified. These coefficients are in practice chosen to calibrate the model to market prices of actively traded derivatives. The LIBOR market model considered in this section is automatically calibrated to the market through the relations (3.4) and (3.5).

The LIBOR market models do not specify interest rates over accrual periods shorter than the intervals $[T_i, T_{i+1}]$. It is therefore natural to choose the functions $\sigma_n(t)$ to be constant between tenor dates. To match previous notation, we take the σ_n's to be right-continuous and thus denote by $\sigma_n(T_i)$ its value over the interval $[T_i, T_{i+1})$.

Under the forward measure with volatilities, σ_n, deterministic, $L_n(T_n)$ has a log-normal distribution. In this case the caplet price is given by the *Black formula* (after Black (1976)),

$$C_n(0) = BC(L_n(0), \overline{\sigma}_n(T_n), T_n, K, \delta_n B_{n+1}(0)), \tag{3.17}$$

with

$$\begin{aligned} &BC(F,\sigma,T,K,b) = \\ &b\left(F\Phi\left(\frac{log(F/K)+\sigma^2 T/2}{\sigma\sqrt{T}}\right) - K\Phi\left(\frac{log(F/K)+\sigma^2 T/2}{\sigma\sqrt{T}}\right)\right), \end{aligned} \tag{3.18}$$

where Φ is the cumulative normal distribution. This formula could also be used in reverse direction. Given the market price of a caplet, one can solve for the 'implied volatility' that makes the formula match the market. Once it has been calibrated it can be used to price less liquid instruments.

However, we will use historical data to compute the volatility structure of the forward rates. If we have forward rate time series data going back a few years we can calculate the covariances between the *relative changes* in the forward rates at different maturities. We may have, for example, the rates for 3, 6, 9 month, 1, 2, 3, 5 and 10 year maturities. The covariance matrix will then be a 8×8 symmetric matrix.

Principal component analysis (PCA) is a technique for finding common movement in a set of variables, essentially for finding the eigenvalues and eigenvectors of the covariance/correlation matrix. Although p components are required to reproduce the total system variability, often much of this variability can be accounted for a small number d of the principal components. If so, there is almost as much information in the d components as there is in the original p variables. We expect to find that a large part of the movement of the forward rate curve is common between rates. That is, the first eigenvector is expected to be a parallel shift in the rates. The next most important movement would be a twisting of the curve, followed by a bending. The eigenvalues, λ_i, and eigenvectors, $\mathbf{v}_i$, corresponding to the covariance matrix, $\mathbf{M}$, satisfy

$$\mathbf{M}\mathbf{v}_i = \lambda_i \mathbf{v}_i.$$

The eigenvector associated with the largest eigenvalue is the first principal component. It gives the dominant part of the movement of the forward rate curve. Its first entry represents the movement of the 3-month rate. Its eigenvalue is the variance of these movements. Note that eigenvectors are orthogonal, there is no correlation between principal components (Johnson & Wichern 2002).

The result of this analysis, is that the volatility factors are given by

$$\sigma_n(t_i) = \sqrt{\lambda_n}(\mathbf{v}_n)_i.$$

To get the volatility of other maturities we will need some interpolation. Note that we scale time according to years. To get the proper scaling we must multiply the covariance matrix by the number of trading days a year (Wilmott 2000).

3.5.4 Algorithms

Algorithms for simulating forward rates and payoff are given in Algorithm 1, 2, 3 and 4. These four algorithms could obviously be combined, but keeping them separate could help clarify the various steps.

Algorithm 1 simulates vectors of forward rates at time t_{i+1} based on the rates from time t_i. The function $\eta(t)$ is the one described in Section 3.5.1 and gives the index of the next tenor date at time t. Line 3 and 4 make use of the antithetic variates technique described in Section

2.1.1. Initial data are the last observation of forward rates available, i.e. the LIBOR market model is initially calibrated to market prices of actively traded bonds.

Algorithm 2 and 3 calculate the drift coefficients under the risk-neutral measure and the forward measure respectively. The drift coefficients are functions of simulated forward rates, tenor- and volatility structure.

Algorithm 4 calculates the value of a portfolio based on a market scenario. The portfolio is specified as a matrix where each row represents a derivative. Derivatives are further specified by start date, maturity date, frequency of payments, strike, notional amount and type of derivative. For instance, a 5 year 100 million swap-contract starting in 2 years, with quarterly payments and strike 3.5% will be represented by the following matrix:

$$\text{Portfolio} \;<< 2 << 7 << 0.25 << 0.035 << 100 << \text{swap} \;.$$

Note that time arguments are relative to "today". Consider a cap spanning over a period $[T_i, T_{i+n}]$. This cap consists of $n-1$ caplets, each with a payoff $C_n = \delta_n(L_n - K_n)^+$. The cap price is the sum over all caplets

$$\text{Cap}(t) = \sum_{i=1}^{n-1} C_i(t).$$

Caplet payoffs are made at different times, and must therefore be discounted at different rates. We start with the payoff from the last caplet, C_{i+n-1} and discount back to time T_{i+n-1} using the simulated forward rate $\hat{L}_{i+n-1}$. Next we add the payoff made from the caplet C_{i+n-2} and discount the sum back to time T_{i+n-2} using the simulated forward rate $\hat{L}_{i+n-2}$. We continue doing this until all payoffs made are discounted back to time T_i. The cap's price at time t_i is then calculated using the general expression above, equation (3.16).

Since the LIBOR market model is based on simple forward rates, at any point T_i^* between the tenor dates T_i there is no corresponding forward rate. This is a drawback with the LIBOR market model compared to the HJM model (Heath, Jarrow & Morton 1990) which works in terms of instantaneous continuously compounded forward rates. Thus, between tenor dates, interpolation will be needed using the LIBOR model. We restrict ourselves to a linear interpolation technique.

If the cap matures at a date later than the last tenor date, T_M, the rates are, for $t \geq T_M$, set equal to the last simulated forward rate $\hat{L}_M$. Pricing a floor is trivial once a cap is priced. The only modification needed is to replace $\max(L_n(T_n) - K_n, 0)$ by $\max(K_n - L_n(T_n), 0)$ in the payoff function. In the swap case the difference between the simulated forward rates $L_n(T_n)$ and some fixed interest rate K_n is modeled. Removing the max-argument we are left with the swap payoff function of the form $(L_n(T_n) - K_n)$.

Algorithm 1 LIBOR - Calculation of forward rates as of time $i+1$ based on the forward rates set at time i.

Inputs: time axis $timeaxis = (t_1, \dots, t_N)$, volatility factors $\hat{\sigma} = (\hat{\sigma}_1, \dots, \hat{\sigma}_d)_n$, eta $\eta(t)$ and forward rates $L = (L_1, \dots, L_M)$

generate $Z1 \leftarrow (Z_1, \dots, Z_d) \sim N(0,1)$

$Z2 \leftarrow (-Z_1, \dots, -Z_d)$

for $n = \eta, \dots, M$ **do**

  $L1(n) \leftarrow L(n) + \hat{\mu}_n L(n)[t_{i+1} - t_i] + L(n)\sqrt{t_{i+1} - t_i}\hat{\sigma}_n^\top Z1_{i+1}$

  $L2(n) \leftarrow L(n) + \hat{\mu}_n L(n)[t_{i+1} - t_i] + L(n)\sqrt{t_{i+1} - t_i}\hat{\sigma}_n^\top Z2_{i+1}$

end for

return $L1, L2$

Algorithm 2 Drift - Risk-neutral measure.

Inputs: simulated forward rates $L = (L_{t_i}, \dots, L_M)$, volatility factors $\hat{\sigma} = (\hat{\sigma}_1, \dots, \hat{\sigma}_d)_n$, time between tenor dates $\delta = (\delta_1, \dots, \delta_M)$, eta η and n

for $j = \eta, \dots, n$ **do**

  $\hat{\mu}_n = \hat{\mu}_n + (\delta_j L_j \hat{\sigma}_n^\top \hat{\sigma}_j)/(1 + \delta_j L_j)$

end for

return $\hat{\mu}_n$

Algorithm 3 Drift - Forward measure.

Inputs: simulated forward rates $L = (L_{t_i}, \dots, L_M)$, volatility factors $\hat{\sigma} = (\hat{\sigma}_1, \dots, \hat{\sigma}_d)_n$, time between tenor dates $\delta = (\delta_1, \dots, \delta_M)$, eta η and n

for $j = n+1, \dots, M$ **do**

  $\hat{\mu}_n = \hat{\mu}_n + (\delta_j L_j \hat{\sigma}_n^\top \hat{\sigma}_j)/(1 + \delta_j L_j)$

end for

return $-\hat{\mu}_n$

Algorithm 4 Payoff - Calculation of payoff based on a market scenario.

```
inputs: vector of simulated forward rates L̂ = (L̂_1, ..., L̂_M), volatility-factors σ̂ = (σ̂_1, ..., σ̂_d)_n and portfolio Portfolio

P ← 0 {vector of payoffs}
D ← 1 {discounting factor}
index ← 0 {keeps track of what interval to interpolate}
fRate ← 0 {forward rates calculated by linear interpolation}

for j = 1, ..., Portfolio.Nrows do

    Portfolio.Column(1) ← max(Portfolio.Column(1), timeaxis(i))

    if timeaxis(i)! = Portfolio(j, 1) then
        D ← discounting factor until start of derivative
    end if

    k = Portfolio(j, 2) − Portfolio(j, 3)
    for k, ..., Portfolio(j, 1) do

        if K <= T_M then
            fRate = L_M
        else
            index = returnIndex(Tenor, k)
            fRate = linInterpolation(index)
        end if

        P(j) ← P(j)+ max(fRate − Portfolio(j, 4), 0) · Portfolio(j, 3) {cap}
        P(j) ← P(j)+ max(Portfolio(j, 4) − fRate, 0) · Portfolio(j, 3) {floor}
        P(j) ← P(j)+ (fRate − Portfolio(j, 4), 0) · Portfolio(j, 3) {swap}

        P(j) ← P(j)/(1 + fRate ∗ Portfolio(j, 3)) {discount payoff}
    end for
    P(j) ← D · P(j) · Portfolio(j, 5) {discount payoff back to time t_i}
end for

return P
```

Chapter 4

Counterparty Risk

We define credit risk as the risk of loss that will be incurred in the event of default by a counterparty. Default occurs if the counterparty fails to honor its contractual payments. Calculating potential credit losses requires the modeling of three processes that define: (i) the counterparty exposure at time of default; (ii) the probability that default will occur; and (iii) the amount that will be recovered after default. Effective credit risk management has gained an increased focus in the recent years, largely due to the fact that inadequate risk policies are still the main source of serious problems within the banking industry. Managing credit risk thus remains an essential and challenging corporate function. The chief goal of an effective credit risk management policy must be to maximize a bank's risk-adjusted rate of return by maintaining credit exposure within acceptable limits. Moreover, one need to manage credit risk in the entire portfolio as well as the risk in individual transactions.

The Enron bankruptcy in 2001 highlights the destructive potential of credit events. At the time of Enron's bankruptcy filing, the aggregate exposure to Enron of all its counterparties was estimated at $6.3 billion (Rich & Tange 2003).

4.1 Credit Risk Management

Effective credit risk management is a critical component of a bank's overall risk management strategy and is essential to the long term success of any banking organization. Key components of effective credit risk management include:

Robust technology and business processes - Robust technology is a critical component of effective credit risk management. It is though to help banks identify, measure, mange and validate counterparty risk, although it is of little value without effective credit risk policies and business processes in place.

Policies - Having a comprehensive and strategic vision for credit policy is vital as it sets guidelines for businesses. These guidelines include a set of general principles that apply to all credit risk situations, as well as specific situations applicable to some types of counterparties and/or transactions.

Exposures - The ability to measure, monitor and forecast potential credit risk exposure across the entire firm on both counterparty level and portfolio level is vital.

Robust analytics - Efficient and accurate credit analytics enable risk managers to make better and more informed decisions. Better information combined with timeliness in its delivery leads to more effective balancing of risk and reward and the possibility of higher long-term profitability.

Other - The other ingredients of effective credit risk management include factors such as stress testing and efficient credit risk reporting just to name a few.

Implementing a scalable and consistent enterprise risk management framework is a challenging task for many banks. To measure price and manage credit risk effectively a variety of disparate systems have to be integrated. These systems should be able to collect substantial quantities of data on credit ratings, credit transactions, loss experiences, rating and default histories in addition to a variety of other relevant credit information.

Calculating counterparty credit risk enhances credit risk management capabilities but poses a variety of challenges. Current practice in credit risk management consists of expected and unexpected loss measures, and portfolio management measures, which require current and potential exposure calculations. The purpose of exposure calculations is to support the assessment of portfolio and firm compliance with policies and guidelines, and to assess credit concentrations (Lepus 2004).

A risk managing program must offer a mechanism for limiting the size of a firm's exposure to its counterparties. The limits tend to be wider for short terms and tighter for long terms. Traditionally, such credit limits are tied to the credit rating of the counterparty, and are often identified in terms of the amount of exposure. In the process of permissioning new trades, the exposure profile to a counterparty is re-computed including the new trades. The exposure profile is then compared with the limit schedule.

Another application of exposure models is the calculation of economic capital to support the risk of a portfolio of counterparties. That is, the amount of capital that a firm should hold to protect itself from insolvency to a given degree of confidence over a specific time interval. What matters most when calculating economic capital is the expected level of losses and the volatility of those losses. Ideally, a full simulation model should be used to compute the economic capital of the portfolio of counterparties, i.e. a model in which market and credit risk variables are simulated simultaneously. An alternative is to use expected exposure profiles, sometimes grossed up by a multiplicative factor to proxy for the increased risk of variable exposure. The gross-up factor depends on the characteristics of the portfolio of counterparties, with typical values in the range 1 to 1.4.

The presence of credit risk in a transaction generates a cost. A model similar to the one that calculates economic capital can be employed to calculate the cost of credit risk over the entire life of a transaction. When exposures are uncorrelated with the credit quality of the counterparty, the unconditional expected exposure profile is used for valuation (Canabarro & Duffie 2003).

4.2 Counterparty Exposure - Definitions

Counterparty exposure is the cost of replacing or hedging the contract at the time of default assuming no recovery. Since default is an uncertain event that can occur at any time during the life of the contract, we consider potential changes in the exposure during the contract's life. This is particularly important for derivative contracts whose values can change substantially over time according to the state of the market (Aziz & Charupat 1998).

Expected exposure (EE) is the average exposure on a future date. The curve EE(t) provides the expected exposure profile. Effective expected exposure at a specific date is the maximum exposure that occurs at that date or any prior date. The average of EE(t) for t in a certain interval, is referred to as expected positive exposure (EPE).

Potential future exposure (PFE) is the maximum exposure estimated to occur on a future date at a high level of statistical confidence. For instance, the 95% PFE is the level of potential exposure that is exceeded with only 5% probability. The curve PFE(t), as t varies over future dates, is the potential exposure profile. A simulation model is usually used for the computation of PFE(t): for each future date the value of the portfolio of trades with a counterparty is simulated. The peak of PFE(t) is referred to as maximum potential future exposure (MPFE).

The Basel Committee defines rollover risk as the amount by which expected positive exposure is understated when future transactions with a counterparty are expected to be conducted on an ongoing basis, but the additional exposure from those future transactions is not included in calculation of expected positive exposure.

A company is said to be exposed to wrong-way risk if future exposure to a counterparty is expected to be high when the counterparty's probability of default is also high. For instance, a company writing put options on its own stock creates wrong-way exposure for the buyer. The Basel Committee distinguishes between general- and specific wrong-way risk. The former arises when probability of default of counterparties is positively correlated with general market risk factors, while the latter arises when the exposure to a particular counterparty is positively correlated with the probability of default of the counterparty due to the nature of the transactions with the counterparty.

4.3 The Basel Committee on Banking Supervision

The Basel Committee provides a forum for regular cooperation on banking supervisory matters. It has, over recent years, developed into a standard-setting body on all aspects of banking supervision. The Basel Committee produces publications relating subjects as capital adequacy, credit risk and securitisation among others. The Committee's Secretariat is provided by the Bank for International Settlements in Basel. BIS is an international organization which fosters international monetary and financial cooperation and serves as a bank for central banks.

Internal Model Method

A bank, meaning the individual legal entity or a group, that wishes to adopt an internal modeling method to measure exposure at default (EAD) for regulatory capital purposes must seek approval from its supervisor. This approval requires that institutions meet certain model validation and operational requirements. When using an internal model, the exposure amount is calculated as the product of alpha times Effective EPE,

$$\text{EAD} = \alpha \times \text{Effective EPE}. \tag{4.1}$$

The motivation for this calculation is the concern that EE and EPE may not capture rollover risk or may underestimate the exposures of OTC derivatives with short maturities. In order to calculate Effective EPE we need Effective EE which is computed recursively as

$$\text{Effective EE}_{t_i} = \max(\text{Effective EE}_{t_{i-1}}, \text{EE}_{t_i}), \tag{4.2}$$

where exposure is measured at future dates $t_1, t_2, t_3, \dots$ and Effective EE_{t_0} equals current exposure. If all contracts in the netting set mature before one year, EPE is the average of expected exposure until all contracts in the netting set mature,

$$\text{Effective EPE} = \sum_{i=1}^{\min(\text{1year, maturity})} \text{Effective EE}_{t_i} \times \Delta t_i. \tag{4.3}$$

The weights $\Delta t_i = t_i - t_{i-1}$ allow for the case when future exposure is calculated at dates that are not equally spaced over time. Effective EPE will always lie somewhere between EPE and peak EE. In general, the earlier that EE peaks, the closer Effective EPE will be to peak EE; the later EE peaks, the closer effective EPE will be to EPE. With prior approval of the supervisor, a measure that is more conservative than Effective EPE for every counterparty, i.e. a measure based on peak exposure, can be used in place of Effective EPE in equation (4.1). An illustration of different exposure measures is shown in Figure 4.1.

The alpha multiplier provides a means of conditioning internal estimates of EPE on a “bad state” of the economy. In addition, it acts to adjust internal EPE estimates for both correlations of exposures across counterparties and the potential lack of granularity across a firm’s counterparty exposures. In addition, the alpha multiplier provides as a method to offset model error or estimation error. Analysis from the industry and supervisors suggest that alpha may range from approximately 1.1 for large global portfolios to more than 2.5 for new users of derivatives with little or no current exposure. The Basel Committee requires institutions to use a specified alpha factor of 1.4, with the ability to seek approval from their supervisors to compute internal estimates of alpha subject to a floor of 1.2.

Current Exposure Method

Banks that do not have an approval for using the internal model method may use the current exposure method. This method is to be applied to OTC derivatives only. The existing CEM

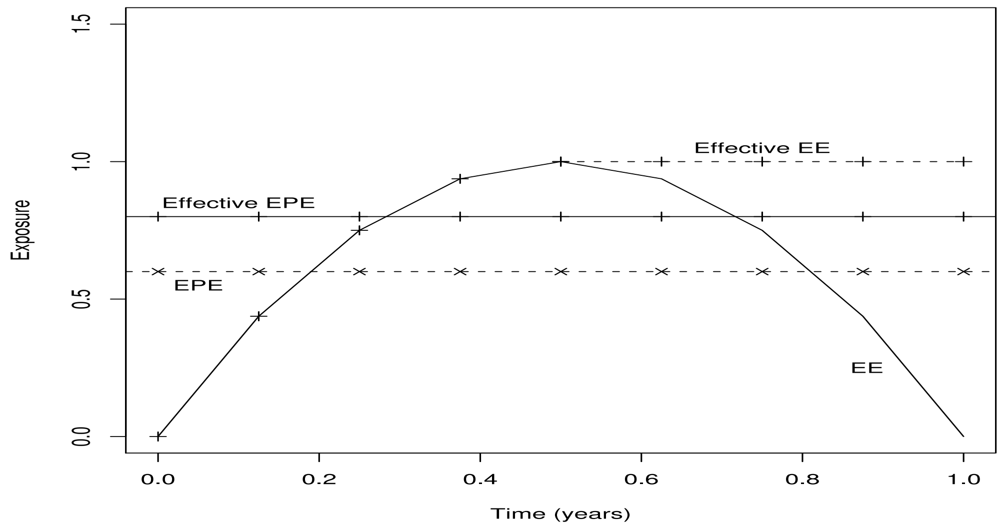


Figure 4.1: Illustration of different exposure measures under the internal method model.

takes the form of the following equation:

$$\begin{aligned}\text{Counterparty Capital Charge } &= [(\text{RC} + \text{add-on}) \text{ - volatility adjusted collateral}] \\ &\quad \times \text{Risk Weight} \times 8\%,\end{aligned} \tag{4.4}$$

where

RC = current replacement cost,

Add-on = the estimated amount of potential future exposure,

Volatility adjusted collateral = the value of collateral,

Risk weight = the risk weight of the counterparty.

Under the CEM, exposure amount or EAD is equal to [(RC + add-on) - volatility adjusted collateral]. The add-on factors are given in Table 4.1, and are intended to account for the possibility that future exposures exceed current exposures. Thus, the EAD is defined only at the current time. When effective bilateral netting contracts are in place, RC will be the net replacement cost and the add-on will be A_{net},

$$\text{A}_{\text{net}} = 0.4 * \text{A}_{\text{gross}} + 0.6 * \text{NGR} * \text{A}_{\text{gross}}, \tag{4.5}$$

where A_{gross} is the gross add-on for the netting set. NGR equals (net current replacement cost)/(gross replacement cost), and is calculated as

	Interest Rates	FX and Gold	Equities	Precious Metals Except Gold	Other Commodities
One year or less	0.0%	1.0%	6.0%	7.0%	10.0%
One year to five years	0.5%	5.0%	8.0%	7.0%	12.0%
Over five years	1.5%	7.5%	10.0%	8.0%	15.0%

Table 4.1: Credit conversion factors under the current exposure method.

$$\text{NGR} = \frac{\max\{0, \sum_i V_i(0)\}}{\sum_i \max\{0, V_i(0)\}},$$

where $V_i(0)$ is the current mark-to-market value of transaction number i.

4.4 Monte Carlo Exposure Calculation

A Monte Carlo model for counterparty exposure involves simulations of future market scenarios. The interest rate market is the dominant contributor to long-term exposure due to the possible long-term maturities of interest rate transactions. Therefore, we will model Norwegian forward rates using the LIBOR market model described in Section 3.5. Based on a large number of scenarios the portfolio's exposure distribution is calculated at each future date,

$$f(\text{Exposure}_t \mid f(F_t)). \tag{4.6}$$

The expectation of these distributions creates the expected exposure profile,

$$\text{EE}(t) = \mathbb{E}[f(\text{Exposure}_t)], \tag{4.7}$$

while order statistics give the desired statistical confidence on the potential future exposure profile,

$$\text{PFE}_t(\alpha) : Pr(\text{Exposure}_t < \text{PFE}_t(\alpha)) = \alpha. \tag{4.8}$$

Exposure distributions, expected exposure and potential future exposure profiles are illustrated in Figure 4.2. The exposure profiles identify the periods in which counterparty default would be most financial damaging (Gibson 2005).

At time 0, initial forward rates are given and current exposure deterministic. To model exposure at future dates $t_1, t_2, t_3, \ldots$ one must take into account that, as time passes by, interest rates change. Therefore, the distribution of forward rates at time $t = t^*$ depends on all possible paths from 0 to t^*,

$$f(F_{t=t^*} \mid \forall F_{t<t^*}). \tag{4.9}$$

The key point is that the drift term in the simulations must give an expectation that match future marked development, i.e. the forward rates. Under the risk-neutral measure the drift is biased because one has subtracted a term which Wilmott (2000) refers to as the "market price of risk". The market price of risk could be interpreted as the excess above the risk-free rate for accepting a certain level of risk, and must be included when modeling the real drift.

Under the forward measure P_{M+1} the final caplet is priced without discretization error by the Euler scheme for $\log L_n$. Switching measure just changes the relative likelihood of a particular path being chosen while the diffusion components remain unaffected. Thus, when modeling the evolution of forward rates one should work under the forward measure which has a drift term given by equation (3.15). The risk-neutral measure is used for pricing only (Glasserman & Zhao 2000).

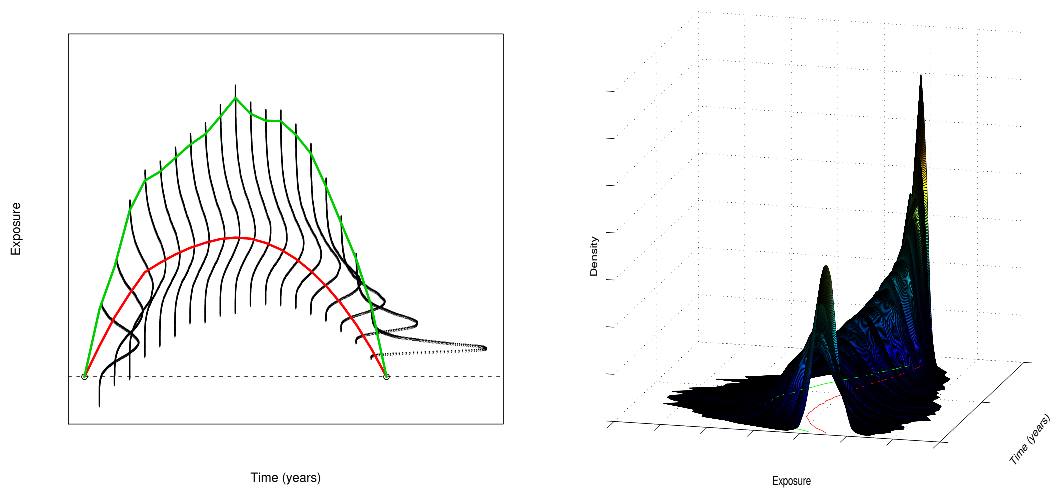


Figure 4.2: Distributions of exposure at each time-step provided by Monte Carlo simulations. The left panel shows a two-dimensional illustration whereas the left panel shows the same situation in three dimensions. Red curves represent expected exposure (EE) and green curves represent potential future exposure (PFE).

The main challenge is to structure simulations in such a way that it is manageable, both with concern to computer memory and time, without limiting the sample space too much. Figure 4.3 and Algorithm 5 show how simulations are structured. For each time-step t_i, $i = 1, \dots, T$, paths from time 0 to t_i are simulated. With use of antithetic variates, two paths are simulated simultaneously, exploring opposite areas of the sample space. Note again that the simulated paths are realizations of the real rates using the real drift parameter. Once a path is simulated, we switch measure and price the portfolio under the risk-neutral measure. Exposures are measured in money value according to time t_i. Complexity is $\mathcal{O}(m \cdot n)$, where m is the number of replications and n the number of pricing scenarios. Chosing large m, n and number of time-steps this algorithm can be very time-consuming. For financial institutions with many positions in their trading book, such a simulation will typically run overnight.

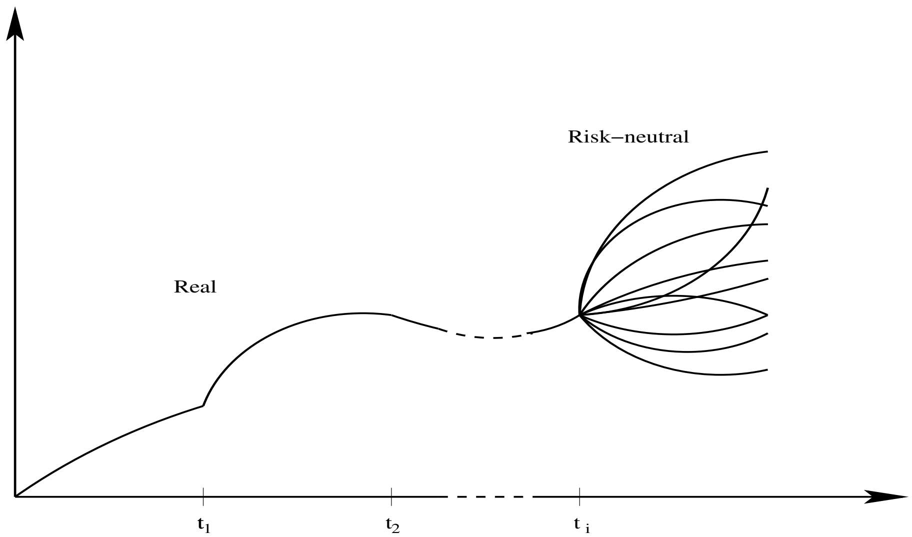


Figure 4.3: Paths showing the structure of Algorithm 5. Paths are simulated up to current time t_i using the real drift parameter. Exposure is calculated given the simulated paths under the risk-neutral measure. Complexity is $\mathcal{O}(m \cdot n)$, where m is the number of replications (real) and n the number of future market scenarios (risk-neutral).

4.5 Credit risk mitigants

Credit risk mitigants are designed to reduce credit exposures. They include netting rights, collateral agreements, and early settlement provisions. These agreements add another layer of complexity because future collateral amounts and margin calls must also be modeled.

Netting agreements allow trades to be offset when determining the net payable amount upon the default of the counterparty. Without netting, the positions of the non-defaulting party would be a loss of the full value of the out-of-the-money trades against a claim on the total value of the in-the-money trades. However, with netting, positives and negatives are added first to determine the net payment due. For example, if one has purchased from the counterparty a currency option with a market value of 100 and sold a forward contract with a current market value of -60, then the exposure is $100 - 60 = 40$. The extent to which netting is legally enforceable is an outstanding issue in some jurisdictions. Cross-product netting refers to the inclusion of transactions of different product categories within the same netting set. For rules and legal criteria see Bas (2005*b*).

Collateral agreements require counterparties to periodically, e.g., every week, mark to market their positions and provide collateral to each other as exposures exceed pre-established thresholds. Usually the the threshold is a function of the credit rating of the counterparties. Collateralization requires the two parties to agree on a validation model for the contract and to agree to a rate of interest paid on the collateral. Even on fully collateralized positions there are potential exposure associated with changes in the market value of the collateral relative to the market value of the positions.

Downgrade triggers are clauses in contracts with counterparties that state that if the credit rating of the counterparty falls below a certain level, then the contract is closed out using a

Algorithm 5 Exposure - Calculation of exposure over the time horizon $0 - T$.

Inputs: number of replications *replications*, number of iterations *niter* and statistical confidence α.

for $k = 1, \ldots, replications$ **do**
  for $i = 1, \ldots, T$ **do**
    $\text{path}_1(0,t_i)$ = LIBOR using real drift {Algorithm 1 and 3}
    $\text{path}_2(0,t_i)$ = LIBOR using real drift {antithetic pair}
    for $j = 1, \ldots, niter$ **do**
      $frate_1$ = future forward rates | $\text{path}_1(0,t_i)$ {Algorithm 1 and 2}
      $frate_2$ = future forward rates | $\text{path}_2(0,t_i)$ {Algorithm 1 and 2}
      $Exposure(i, j)$ =Payoff($frate_1$) {Algorithm 4}
      $Exposure(i, j + 1)$ =Payoff($frate_2$) {Algorithm 4}
    end for
  end for
end for
return $sortExposure.Row(\alpha * niter * replications)$ {order statistics}

pre-determined formula with one side paying a cash amount to the other side. These clauses lead to significant reduction in credit risk, but they do not completely eliminate all credit risk (Canabarro & Duffie 2003).

Chapter 5

Data Description

This chapter presents data used to calibrate and test the implemented Monte Carlo exposure model.

5.1 Historical Data

The data used in this paper are

1. Daily zero-coupon Norwegian yields from DnBNOR. They cover the period 1 October 1998 to 29 August 2005. The data consist of a total of 1758 trading days[1] and yields with maturity 3, 6, 9 months, 1, 2, 3, 5 and 10 years. The data are shown in Figure 5.1.

The yield curve is the plot of $Y(t,T)$ against time to maturity $T-t$, and it typically comes in one of three distinct shapes associated with different economic conditions:

- *increasing* - Usually, short-term bonds carry lower yields to reflect the fact that an investor's money is under less risk. The longer one tie up cash, the more one should be rewarded for the risk one is taking. A normal yield curve, therefore, slopes gently upward as maturities lengthen and yields rise.
- *decreasing* - This is typical of periods when the short rate is high but expected to fall. Long-term investors will settle for lower yields now if they think rates and the economy are going even lower in the future.
- *humped* - For the yield curve to be inverted it must pass through a period where long-term yields are the same as short-term yields. When that happens the shape will appear to be flat or, more commonly, a little raised in the middle.

Different yield curves are given in Figure 5.2.

[1]Meaning all days except weekends and holidays

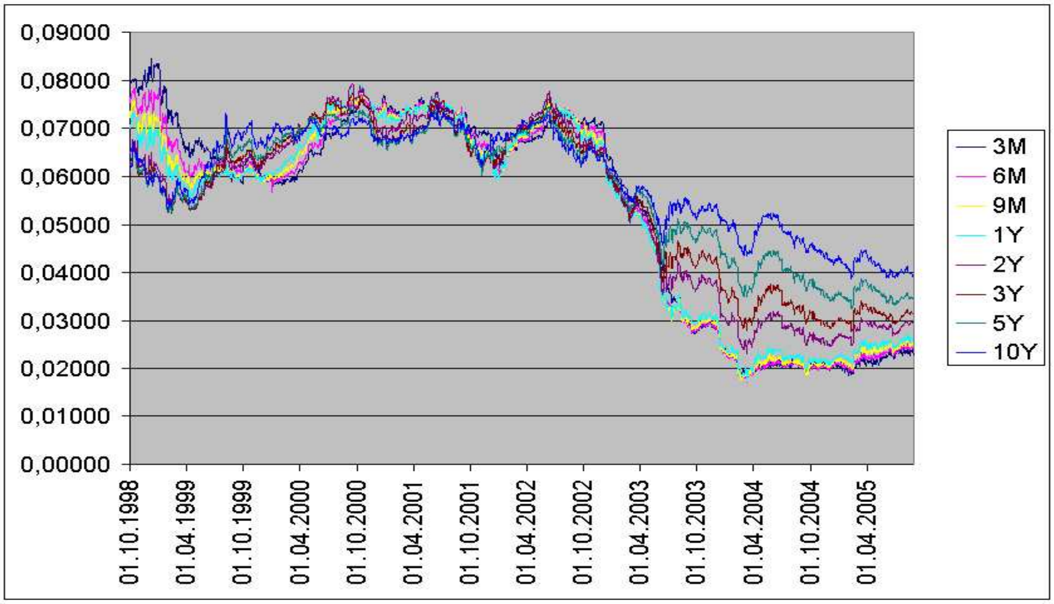


Figure 5.1: Time-series of zero-coupon yields covering the period from 1 October 1998 to 29 August 2005. The eight time-series are yields with 3, 6, 9 month, 1, 2, 3, 5 and 10 year maturities.

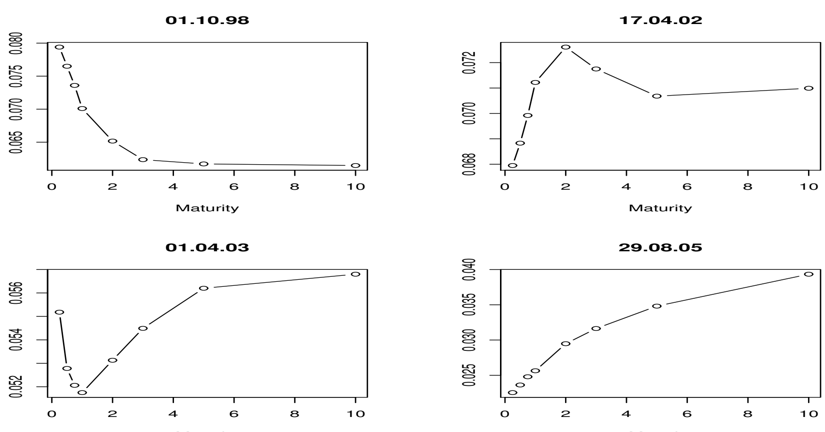


Figure 5.2: Typical yield curves. Data are taken from 1 October 1998, 17 April 2002, 1 April 2003 and 29 August 2005 respectively.

Zero-coupon yields are converted to zero-coupon bond prices through equation (3.1). Forward rates are determined by bond prices through equation (3.2). The data are used to compute the volatility structure of forward rates as discussed in Section 3.5.3. Simulations are initialized by the last observation of forward rates. These are taken from 29 August 2005 and given in Table 5.2.

Mean and standard deviations of historical forward rates are given in Table 5.1. Short-term rates have greater volatility than long-term rates. A surface-plot of the correlation matrix of historical forward rates is given in Figure 5.3. The forward rates are strongly positive correlated.

Period	Mean	St.dev
0-3 months	0.05369	0.02209
3-6 months	0.05247	0.02158
6-9 months	0.05260	0.02072
9 months - 1 year	0.05362	0.01974
1-2 years	0.05772	0.01684
2-3 years	0.05928	0.01253
3-5 years	0.06359	0.01045
5-10 years	0.07323	0.01092

Table 5.1: Mean and standard deviations of historical forward rates.

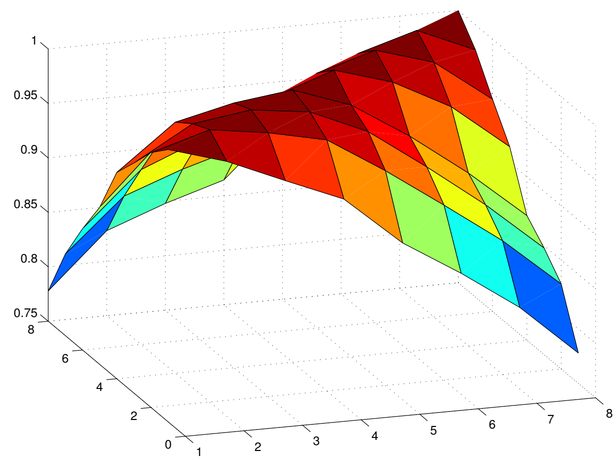


Figure 5.3: Correlation matrix of historical forward rates.

5.2 Portfolio

This section describes the construction of a test portfolio used throughout this thesis. The portfolio will be used to test and highlight some key results on exposure calculations.

We construct a sample portfolio which contains portfolios for four counterparties, denoted A, B, C and D, with four type of interest rate transactions - caps, floors, swaps and forward rate agreements. The portfolio is given in Figure 5.4. Note that start and maturity dates are

	Start	Maturity	Forward LIBOR rate (%)	ΔT
L_1	29.08.05	29.11.05	2.26	0.25
L_2	29.11.05	28.02.06	2.48	0.25
L_3	28.02.06	29.05.06	2.72	0.25
L_4	29.05.06	29.08.06	2.83	0.25
L_5	29.08.06	29.08.07	3.39	1.00
L_6	29.08.07	29.08.08	3.65	1.00
L_7	29.08.08	29.08.10	4.12	2.00
L_8	29.08.10	29.08.15	4.91	5.00

Table 5.2: Market data used in pricing. Data are taken from 29 August 2005.

given relative to 29 August 2005. IRR represents interest rate receiver swap while IRP means interest rate payer swap. FRB is a forward rate agreement where a forward rate is purchased. All transactions allow for netting agreements.

The day count defines the way in which interest accrues over time and is usually expressed as X/Y. When calculating the interest earned between two dates, X defines the way in which the number of days between the two dates is calculated, ant Y defines the way in which the total number of days in the reference period is measured. The interest earned between two dates is

$$\frac{\text{Number of days between dates}}{\text{Number of days in reference period}} \times \text{Interest earned in reference period}$$

The most commonly used day count conventions are:

1. Actual/actual (in period)
2. 30/360
3. Actual/360.

The use of 30/360 indicates that we assume 30 days per month and 360 days per year when carrying out calculations. Note that the interest earned in a whole year of 365 days is 365/360 times the quoted rate.

The trades to counterparty A include 17 swaps, both IRR and IRP, one floor and two caps with a total notional amount of NOK 7450 million. Most of the derivatives have quarterly payments but semiannual and annual do exist. The portfolio is a mixture of both long- and short-dated contracts. Counterparty B, five contracts, has total notional amount of NOK 1950 million. To be noted is the short-dated cap contract which alone contributes to nearly half the total notional. Counterparty C has a mixture of all interest rate derivatives considered, which all are relatively short-dated. Total notional amount is NOK 1920 million. Counterparty D has only a NOK 300 million swap contract in its trading book.

Counterparty A

IR Swaps

Product	CCY	Start	Mat	Notional	Rate	Pay freq	Basis
IRR	NOK	3.8	11.8	300 000 000	4,30 %	Q	30 360
IRP	NOK	4.0	9.85	500 000 000	2,70 %	Q	ACT360
IRP	NOK	2.1	4.7	500 000 000	2,50 %	S	30 360
IRR	NOK	6.6	11.5	150 000 000	3,10 %	1Y	ACT360
IRR	NOK	10.8	14.2	300 000 000	6,00 %	Q	30 360
IRR	NOK	0.0	5.0	500 000 000	3,75 %	Q	ACT360
IRR	NOK	3.0	4.7	500 000 000	3,95 %	1Y	30 360
IRP	NOK	0.0	2.7	350 000 000	2,40 %	S	ACT360
IRR	NOK	8.0	14.5	600 000 000	3,50 %	Q	30 360
IRR	NOK	6.6	14.5	100 000 000	4,50 %	Q	ACT360
IRP	NOK	0.8	3.7	400 000 000	3,70 %	Q	30 360
IRR	NOK	0.7	3.8	350 000 000	2,30 %	Q	ACT360
IRR	NOK	1.0	3.0	600 000 000	6,10 %	S	30 360
IRR	NOK	6.6	14.5	100 000 000	3,20 %	Q	ACT360
IRP	NOK	4.0	9.85	500 000 000	4,20 %	Q	30 360
IRR	NOK	1.0	3.0	550 000 000	6,10 %	Q	ACT360

IR options

Product	CCY	Start	Mat	Notional	Strike	Buy/sell	Pay freq	Basis
Floor	NOR	2.0	7.5	100 000 000	5,00 %	Sell	Quarterly	30 360
Cap	NOR	3.0	13.0	450 000 000	4,00 %	Buy	Annual	ACT360
Cap	NOR	0.0	3.7	400 000 000	3,00 %	Buy	Quarterly	30 360
Cap	NOR	0.0	14.5	200 000 000	10,00 %	Buy	Quarterly	ACT360

Counterparty B

IR Swaps

Product	CCY	Start	Mat	Notional	Rate	Pay freq	Basis
IRR	NOK	0.0	10.0	350 000 000	4.8%	Q	30 360
IRP	NOK	0.0	4.5	250 000 000	2.8%	Q	ACT360
IRR	NOK	0.0	2.5	350 000 000	2.4%	Q	30 360
IRR	NOK	7.0	10.0	200 000 000	5.0%	Q	ACT360

IR options

Product	CCY	Start	Mat	Notional	Strike	Buy/sell	Pay freq	Basis
Cap	NOR	4.5	5.5	800 000 000	3.5%	Buy	Quarterly	30 360

Counterparty C

IR Swaps

Product	CCY	Start	Mat	Notional	Rate	Pay freq	Basis
IRP	NOK	2.0	6.5	350 000 000	2.4%	Q	30 360
IRP	NOK	0.8	3.7	400 000 000	3.7%	Q	ACT360
IRR	NOK	0.7	7.0	350 000 000	2.3%	Q	30 360

FRA

Product	CCY	Start	Mat	Notional	Rate		Basis
FRB	NOK	2.0	5.5	220 000 000	3,50 %		ACT360
FRB	NOK	1.6	4.5	100 000 000	4,50 %		ACT360

IR options

Product	CCY	Start	Mat	Notional	Strike	Buy/sell	Pay freq	Basis
Floor	NOK	2.2	5.5	100 000 000	3,00 %	Sell	Quarterly	30 360
Cap	NOK	0.0	3.7	400 000 000	5,00 %	Buy	Quarterly	ACT360

Counterparty D

IR Swaps

Product	CCY	Start	Mat	Notional	Rate	Pay freq	Basis
IRR	NOK	0.0	10.0	300 000 000	4,20 %	Q	30 360

Figure 5.4: Portfolio used to test the implemented exposure model.

Chapter 6

Result and Discussion

We have implemented a LIBOR market model as presented in Section 3.5. The model has been calibrated using the data described in Section 5.1. All program code is written in the programming language C++. In order to handle matrices effectively we have taken use of the C++ package *Newmat* (Davies 2005).

This chapter starts with results and discussion from the calibration of and pricing by the LIBOR market model. Next we focus on exposure calculations. We present exposure profiles and make a comparison between the Monte Carlo method and the BIS approach. This includes a closer look at the benefit from netting agreements. Finally, we present the results obtained from simulation efficiency improvements.

6.1 Calibration and LIBOR Market Model

Models calibrated to historical data tend to project future values based on the statistical regularities observed in the past, while models calibrated to market prices tend to reflect forward-looking views. There are positive and negative aspects of each method. Historical calibration implies that the process generating future market behavior is the same that was observed in the past. The model may be slow to react to changes in market conditions and structure, even if a time-decay factor is used to over-weight more recent observations.

The first three eigenvectors of the covariance matrix of relative daily changes in forward rates are given in Figure 6.1. Recall from Section 3.5.3 that the first principal component is expected to be a parallel shift in the forward rates, the second a twisting of the curve and the third a bending. Ideally, for the first principal component to be a parallel shift all entries should have the same sign, representing a common movement in one direction.

Previous findings such as in Attaoui (2004) and Litterman & Scheinkman (1991) indicate that the first three factors explain close to 100% of the total variance. From Figure 6.2 it is clear that the first three principal components explain only 74.73% and to capture more than 90% of the variance, six principal components must be included. An explanation for the discrepancy from previous finding is that the Norwegian interest rate market is not that liquid as for instance the English, American or German markets, meaning that there are relatively few interest rate products traded. As a result, the whole covariance matrix is used, meaning a 8-factor LIBOR market model. The computational difference to that of a three factor model is about 10-14%.

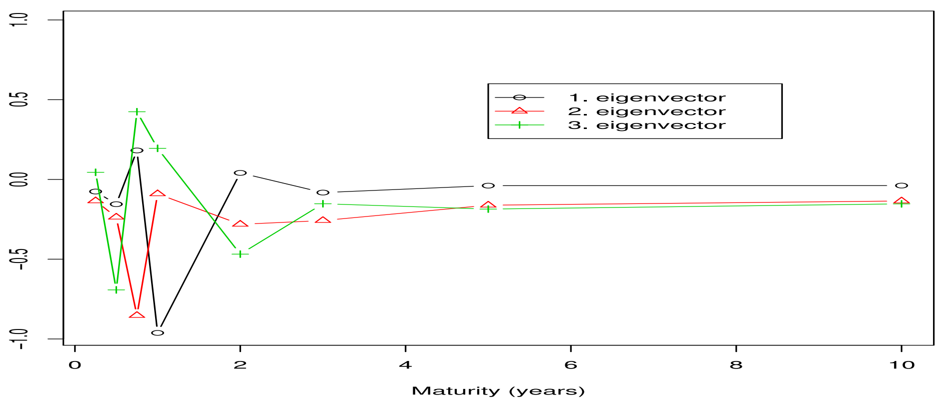
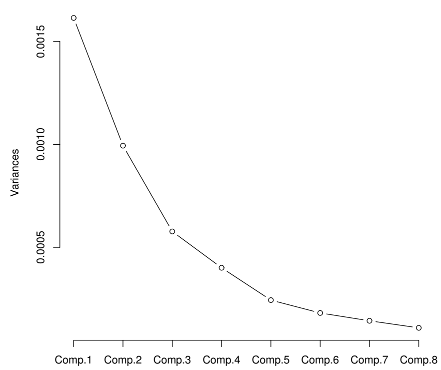


Figure 6.1: The three most significant eigenvectors.



Eigenvalue	Contribution (%)	Cumulative (%)
λ_1	37.87	37.87
λ_2	23.32	61.19
λ_3	13.54	74.73
λ_4	9.40	84.12
λ_5	5.71	89.84
λ_6	4.25	94.09
λ_7	3.36	97.45
λ_8	2.55	100.00

Figure 6.2: Scree plot (left) and contributions to the total variance (right) of principal components.

All derivatives in our test portfolio are priced using the eight factor LIBOR model. Price distributions at time 0 of the first swap, cap and floor for counterparty A are given in Figure 6.3. The cap and floor prices can not get negative, therefore, the distributions look much like log-normal distributions. The swap can have both negative and positive values, resulting in the greatest variance.

To check convergence we have shown cumulative mean in Figure 6.3. Convergence occurs after approximately 10000 iterations. However, a portfolio of derivatives will converge slower than a single derivative. Also, the quantiles in the exposure distributions require several more iterations than the expectation. This point out that we should generate considerably more market scenarios at each time-step in the exposure calculations. Antithetic variates have been used to speed up convergence. We will return to the effects of variance reduction techniques in Section 6.3.

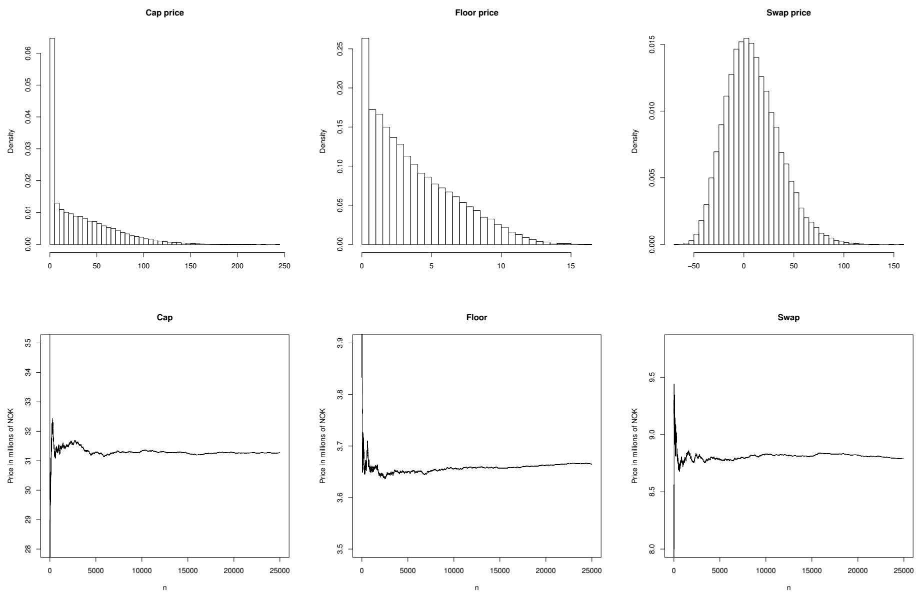


Figure 6.3: Price distributions of an interest rate cap, floor and swap, upper row. Patterns of cumulative mean are given in bottom row. Convergence occurs after approximately 10000 iterations.

6.2 Exposure Calculations

In this section we present the results from exposure calculations. First, we present the details for the test portfolio based on the Monte Carlo approach. We examine information and highlight several significant aspects of the counterparty exposure profiles. Finally, we contrast these results with those derived from the BIS approach. The mark-to-market values of the transactions are calculated using the data on interest rates as of 29 August 2005.

Monte Carlo based credit exposures

To generate distributions of credit exposure at future points of time, we create 5000 scenarios of the real interest rate path and 400 pricings scenarios for each path using the implemented 8-factor LIBOR market model.

Table 6.1 and Figure 6.4 summarize the results of an analysis based on the Monte Carlo approach. Total mark-to-market value is negative. Two counterparties - C and D - have positive current exposure. The total current exposure represents the portfolio's replacement cost should both counterparties default today. Counterparty D has the greatest relative exposure with a worst case scenario of nearly 11% of total notional amount. The other three counterparties hold opposite positions in their trading book, resulting in lower relative exposure.

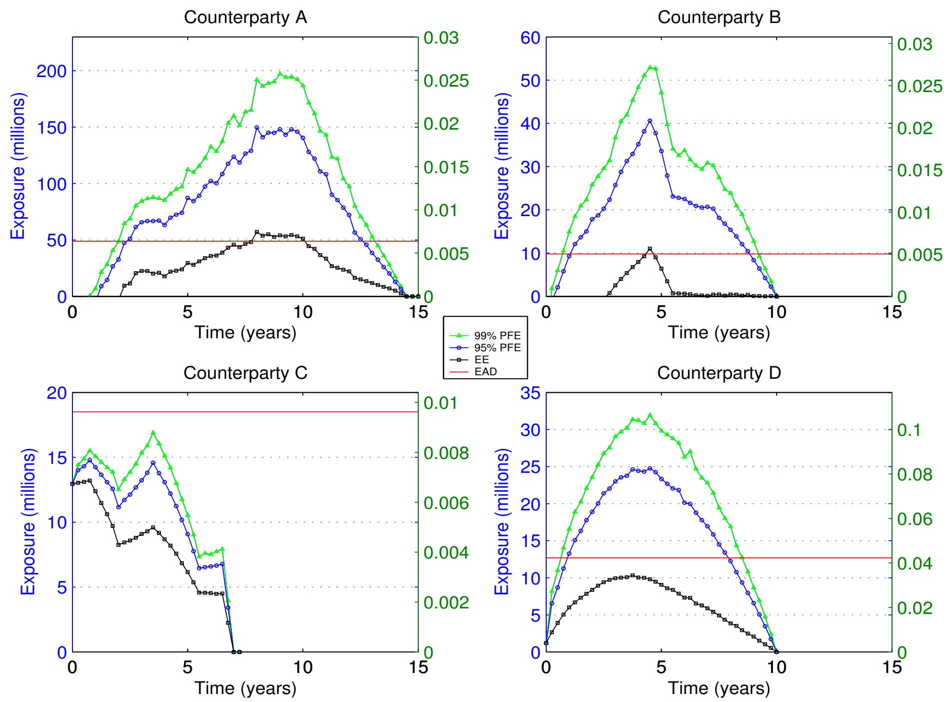


Figure 6.4: EAD, expected exposure, 95% and 99% PFE profiles for counterparty A, B, C and D. Right y-axes represent exposure relative to total notional amount.

The tabulated effective EPE and MPFEs suggest that future exposures are greater than current exposure in each case. This suggestion is confirmed by the counterparty exposure profiles. Each of the three exposure profiles for each counterparty increases from current value and remains positive for some period during the simulation. At the portfolio level, effective EPE suggests that future credit exposure is, on average, 453% higher than current exposure. The maximum scenario, MPFE(99%), is almost 21 times higher. As a result, the decision taken on the appropriate confidence level significantly affects the amount of reserves that should be set.

The exposure profiles of counterparty A peak late compared to the other counterparties. This is an effect of long-dated and late-starting derivatives. Assumptions for long-dated scenarios must be considered significantly uncertain. A sudden shift in market conditions will affect both the magnitude and the shape of exposure profiles. As a result, one should monitor exposure to counterparty A carefully in the future.

Consider the exposure profiles of counterparty B. The PFE(95%) is initially 0 but raises almost linearly and reaches its peak after approximately five years. Then it drops almost 50% during a year period. Taking a look at the portfolio we discover that the peak is a result of a short but profitable cap contract.

The difference between exposure profiles for counterparty C is small. This indicates that trades held against C offset each other. By taking opposite positions one can insure oneself against adverse movements of interest rates and reduce exposure to risk one already faces.

Counterparty	Mark-to-market Value, V	Current Exposure, CE	Effective EPE	Maximum Potential Future Exposure	
				MPFE(95%)	MPFE(99%)
A	-35.94	0	34.91	149.69	197.16
B	-9.58	0	7.00	40.59	52.92
C	12.96	12.96	13.20	14.78	16.90
D	1.20	1.20	9.10	24.77	31.92
Total	-31.36	14.16	64.21	229.83	298.9

Table 6.1: Monte Carlo credit exposures in millions of NOK.

This strategy is broadly known as hedging and its main focus is to reduce the sensitivity of a portfolio to the movement of the underlying. For hedging to be possible there must be traders that take opposite positions to hedgers. These traders are known as speculators.

Exposure profiles for the 10 year swap contract, counterparty D, are smooth and peak after approximately four years. The difference between EE, PFE(95%) and PFE(99%) profiles reflects the volatility of interest rates.

Note that Monte Carlo simulations, like all other risk measurements systems, are only as good as the data and assumptions underlying the analysis. Two critical assumptions in the Monte Carlo analysis are the process used to derive the interest rate paths and the cash flow relationship developed for each interest rate path. If these assumption are faulty, the result of the simulations will be suspect. The alpha multiplier in equation (4.1) provides as a method to offset these errors.

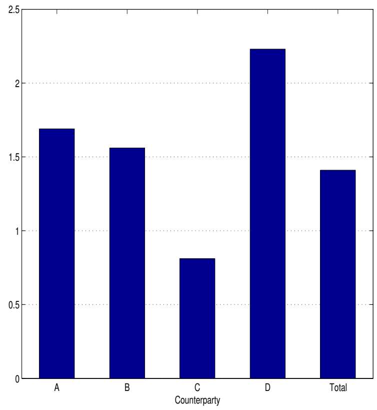
Comparison of methods

In this section, we compare measures of credit exposure under the Monte Carlo approach to the CEM under the BIS methodology. The BIS approach does not take into account the evolution of exposure through time. Therefore, its resulting EAD can be either too low or too high, depending of the nature of transactions.

The ratios of MC/CEM exposure are given in Figure 6.5. They range from 0.81 for counterparty C to 2.23 for counterparty D, confirming that CEM exposures can be higher or lower than the simulated values. In general, it appears that the CEM exposures are lower when the counterparty's current exposure is zero, i.e. A, B and D, but not C. This is to be expected since the BIS approach applies the same add-on regardless of the moneyness of the current position.

Next, we consider the capability of the methods to account for the impact of netting. We compare the ratios of the Monte Carlo approach with and without netting to the ratios of the CEM with and without netting.

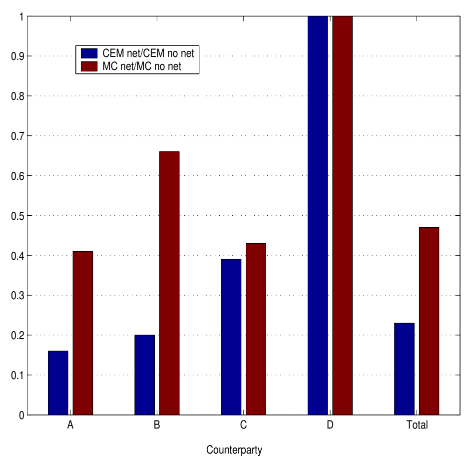
Figure 6.6 shows that all counterparties except D can benefit significantly from netting agreements. The ratios of counterparty D are 1 because counterparty D holds only a single swap contract. At portfolio level, the total effect of netting is close to 50%. The CEM ratios are for all counterparties lower than the Monte Carlo ratios, implying that the BIS approach over-



	A	B	C	D	Total
CEM	28.9	6.3	22.7	5.7	63.6
MC	34.9	7.0	13.2	9.1	64.2
α·MC	48.9	9.8	18.5	12.7	89.9
Ratio	1.69	1.56	0.81	2.23	1.41

Figure 6.5: MC/CEM exposure ratios.

estimates the benefit of netting. The overestimation is particularly visible for counterparty A and B whose current exposures are zero. While a Monte Carlo simulation over time captures the changing characteristics of the portfolio and the reduced netting benefits, the static BIS approach does not recognize any changes in the exposure profiles over time and overestimates the risk reduction due to netting.



	Netting/no netting ratios				
	A	B	C	D	Total
CEM	0.16	0.20	0.39	1.00	0.23
MC	0.41	0.66	0.43	1.00	0.47

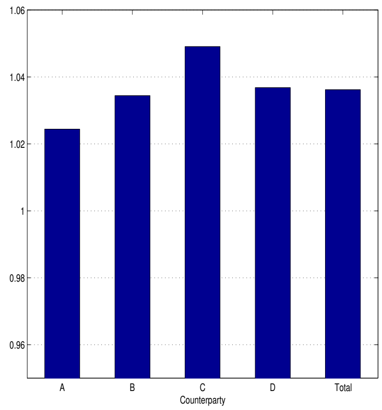
Figure 6.6: Netting/no netting exposure ratios.

6.3 Simulation Efficiency Improvements

In this section we discuss two different methods to reduce simulation time. First we present the effects of variance reduction techniques. Next, we discuss different time discretizations.

Variance Reduction

Reducing variance of simulation estimates will reduce number of iterations needed to obtain accurate results, i.e. faster convergence. With use of antithetic variates we explore opposite areas of the sample space simultaneously. We have used the method both with concern to the evaluation of the real interest rate paths and the risk-neutral pricing. The effects are given in Figure 6.7. At portfolio level, the average effect is 3.62%. Antithetic variates do not increase computer time. Therefore, even though the effects are small, we recommend the use of this method.



Counterparty	Effect (%)
A	2.44
B	3.44
C	4.91
D	3.68
Total	3.62

Figure 6.7: Effects of antithetic variates for each counterparty. The bar plot shows the ratios without/with antithetic variates.

Time Discretization

Limiting the number of evaluation points on the exposure time-horizon is a valuable tool for decreasing simulation time. A bisection of time steps will reduce simulation time approximately by a factor 1/2. This section compares equally spaced time grids to that of a carefully chosen time discretization. Carefully chosen means in this setting to choose a time discretization such that number of points are minimized but still captures the overall structure of exposure profiles.

Quarterly discretization, $h = 0.25$, is a natural choice since most of the derivatives considered have quarterly payments. Over a 15 year time horizon this will cause $15/0.25 = 60$ evaluation points. An annual approach, $h = 1.0$, reduces number of time steps by a factor 1/4. However, this discretization may not capture shifts or peaks over small periods of time, leading to potential large errors. The magnitude of errors depends on the portfolios' exposure profiles.

Alternatively, one can locate dates of interest in advance and fix a time grid according to these. Typical points will be derivatives' start, maturity and payoff dates. It may also be useful to include nearby points, $t_i \pm \Delta t$, for some fixed Δt. Possible time discretizations take the form shown in Figure 6.8.

We assume that using quarterly time-steps provides the most correct picture, and that the the

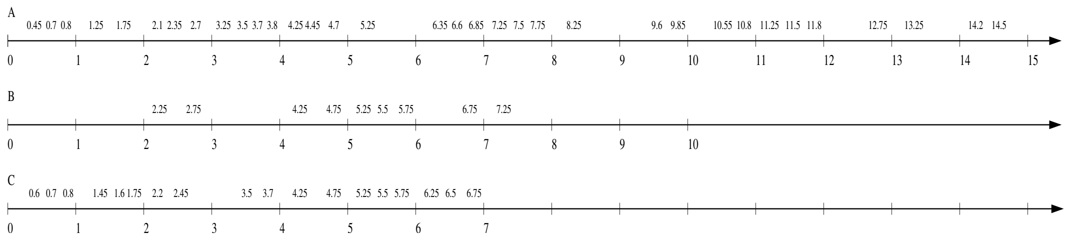


Figure 6.8: Different time discretizations based on derivatives' start and maturity dates. Some surrounding points are also included.

Counterparty	Method	Red. of sim. time (%)	Maximum Error (%)	Fraction underestimation (%)
A	$h = 1.00$	77	10.5	68
	smart	17	9.5	39
B	$h = 1.00$	79	17.1	87
	smart	55	5.9	36
C	$h = 1.00$	80	45.6	81
	smart	21	1.5	59
D	$h = 1.00$	78	4.2	95
	$h = 0.50$	50	2.4	76

Table 6.2: Different discretizations of PFE(95%) profiles for counterparty A, B, C and D. All results are measured relatively to the quarterly approach, $h = 0.25$.

errors compared to the true profiles are small. Therefore, the errors in the "smart" and annual discretization are measured relatively to the quarterly approach. Since expected exposure profiles are zero for some periods of time, counterparty A and B, we use the PFE(95%) profiles to make the comparison. Figure 6.9 and Table 6.2 summarize the results of this analysis.

Simulation time is reduced by 77-80% by the annual discretization compared to the quarterly approach. This is to be expected, i.e. not 75%, since the effort of simulating the real interest rate paths increases with increasing time step. Reduction of simulation time by the "smart" discretization ranges from 55% to 17% for counterparty B and A respectively.

Since counterparty D holds only one single quarterly paying swap contract, the smart discretization equals the quarterly discretization. The so-called smart discretization is in this case replaced with a semiannual approach. However, the three methods do not differ much due to the smooth nature of the swap contract.

Annual discretization provides relatively large errors. In particular, the peaks of counterparty B and C are greatly underestimated. However, the largest error occurs after approximately 7 years for portfolio C with nearly 50% underestimation. This points out the danger of taking too large time-steps.

The "smart" discretization captures the overall trend considerably better than the annual approach. For counterparty B, C and D there are small or no errors. However, the peaks of counterparty A are not entirely captured. This suggests that we should have included some more points, especially in the interval 5-10 years.

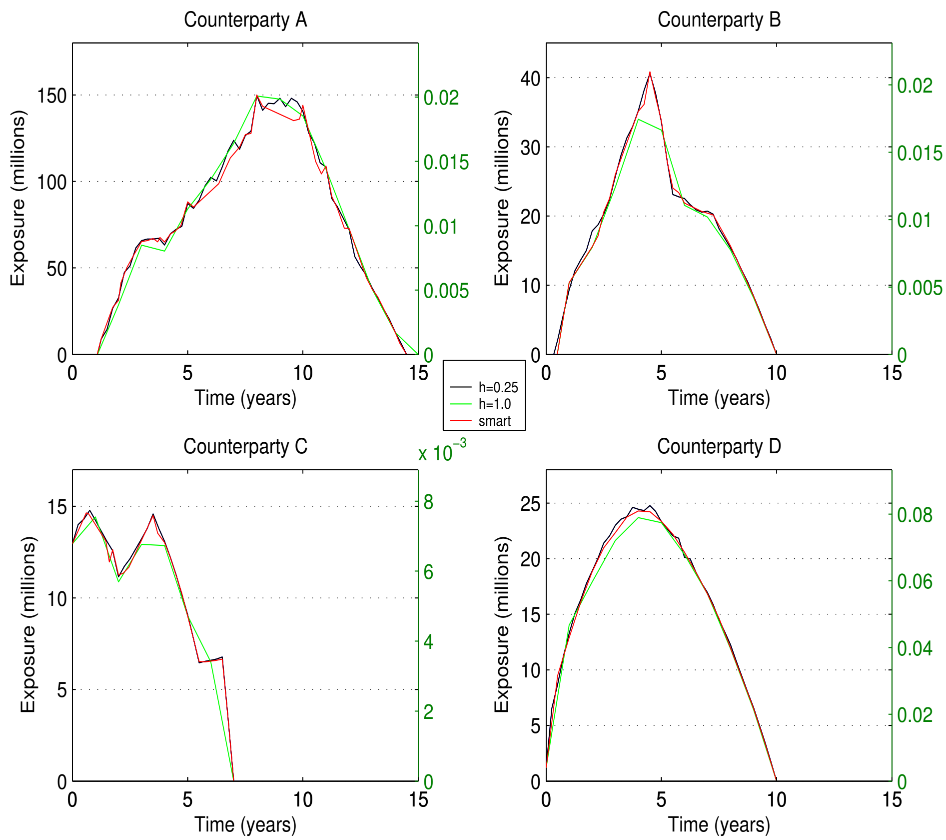


Figure 6.9: 95% PFE profiles obtained from quarterly, annual and smart time discretizations. Right y-axes represent exposure relative to total notional amount.

Note that for large portfolios it makes little sense to discretize according to start, maturity and payoff dates. However, we may locate substantial peaks and shifts and discretize such that these are for sure captured. Generally, one would have a finer partition of the time axis for the first couple of years than for the long time horizon. This is due to the fact that the assumptions in constructing long-dated scenarios are surrounded by considerable uncertainty.

In conclusion, it would have been optimal to use strategy "smart" to portfolio types B, C and D, whereas A is a case of doubt.

Chapter 7

Concluding Remarks

This chapter summarizes the thesis. In addition, some directions for further work are pointed out.

7.1 Conclusion

The LIBOR market model is a sophisticated underlying interest rate model which gives a great flexibility with regard to exotic instruments and different interest rates. The implemented Monte Carlo method estimates future exposure based on a set of market scenarios over an appropriate simulation horizon. This leads to a precise evaluation of the impact of netting and moneyness of the position. Moreover, the Monte Carlo method provides a rich set of information which credit managers can use to explain the causes of the exposure.

Under the Monte Carlo method, the capital reserves are set to cover a maximum loss calculated at some level of confidence. The level chosen will significantly affect the amount of reserve that should be set. The BIS approach does not take into account the evolution of the exposure through time. Therefore, depending on the nature of the transactions and the portfolio composition, its resulting reserve can be either too low or too high.

A simulation model for counterparty exposure is time-consuming and there is a fine line between accuracy and effort. Taking too large time-steps when evaluating exposure profiles may produce large errors. An underestimation of exposures can be disastrous in the case of default by the counterparty. A carefully chosen time discretization can reduce simulation time considerably without loosing the overall structure of exposures. However, the number of transactions to a counterparty will affect the possibility to reduce time-steps. With use of antithetic variates simulation time can be reduced by a few additional percent.

7.2 Further Work

Due to the rich and sophisticated nature of the LIBOR market model we can add layers of complexity without changing the foundation. To include more exotic instruments or other interest rates we only need to specify payoff functions. Also, simulation efficiency improvements and different calibrations can be modeled independently of the LIBOR market model. This gives us an enormous freedom and numerous possibilities for further developments.

We have studied the calculation of counterparty exposure. Calculating potential credit losses requires, in addition, the modeling of the probability of default and the amount that can be recovered after default occurs. Aziz & Charupat (1998) suggest that the default probabilities could be calculated based on average one-year transition matrices for banks and for corporations. The cumulative probability that a company will default by the end of n years can be obtained by multiplying the appropriate transition matrix by itself n times.

Furthermore, simulation time could be additionally reduced by using other variance reduction techniques such as control variates, stratified sampling and importance sampling. Glasserman, Heidelberger & Shahabuddin (1999) discuss these methods in the HJM framework, which in some extent can be generalized to the LIBOR market model. Also, Kajsajuntti (2004) and Glasserman (2004) investigate the use of quasi-Monte Carlo and low-discrepancy sequences to find that faster convergence holds in high dimensions.

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Appendix A

Probability, Martingales and Various Mathematical Tools

The aim of this appendix is to provide mathematical tools that will be useful in order to derive and understand the LIBOR market model and its applications. The theory is based on Glasserman (2004) and Bingham & Kiesel (2004).

A.1 Probability and Stochastic Processes

Measure

The language of modeling financial markets involves that of probability, which in turn involves that of measure theory. Let Ω be a set.

Definition A.1. *A collection $\mathcal{F}_0$ of subsets of Ω is called an algebra on Ω if:*

i $\Omega \in \mathcal{F}_0$,

ii $F \in \mathcal{F}_0 \Rightarrow F^c = \Omega \backslash F \in \mathcal{F}_0$,

iii $F_1, F_2 \in \mathcal{F}_0 \Rightarrow F_1 \cup F_2 \in \mathcal{F}_0$.

Definition A.2. *An algebra $\mathcal{F}$ of subsets of Ω is called a σ-algebra on Ω if for any sequence $F_n \in \mathcal{F}$, $(n \in \mathbb{N})$, we have*

$$\bigcup_{n=1}^{\infty} F_n \in \mathcal{F}.$$

Such a pair $(\Omega, \mathcal{F})$ is called a measurable space.

Definition A.3. *A measure $\mathbb{P}$ on a measurable space $(\Omega, \mathcal{F})$ is called a probability measure if*

$$\mathbb{P}(\Omega) = 1.$$

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space.

Probability

To describe a random experiment we begin with the sample space Ω, the set of all possible outcomes. Each point ω of Ω represents a possible outcome of performing the random experiment. For a set $A \subseteq \Omega$ of points ω we want to know the probability $\mathbb{P}(A)$. We want

1 $\mathbb{P}(\emptyset) = 0, \mathbb{P}(\Omega) = 1$,

2 $\mathbb{P}(A) \geq 0$ for all A,

3 If $A_1, A_2, \ldots$ are disjoint, $\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$

4 If $B \subseteq A$ and $\mathbb{P}(A) = 0$, then $\mathbb{P}(B) = 0$ (completeness).

Definition A.4. *A probability space, or Kolmogorov triple, is a triple* $(\Omega, \mathcal{F}, \mathbb{P})$ *satisfying Kolmogorov axioms* $(1), (2), (3), (4)$ *above.*

Information and Filtration

Information is the most important determinant of success in financial life. As time passes, new information becomes available, and we need a framework that handles dynamic situations. In particular we need to be able to speak in terms of "the information available at time n". In addition we need to be able to increase n, i.e. talk about the information flow over time. The mathematical language to model this information flow is provided by the idea of filtration.

A *stochastic process* $X = \{X_n : n \in I\}$ is a family of random variables, defined on some common probability space. The process $X = (X_n)_{n=0}^{\infty}$ is said to be *adapted* to the filtration $\mathbb{F} = (\mathcal{F}_n)_{n=0}^{\infty}$ if

$$X_n \text{ is } \mathcal{F}_n \text{ - measurable for all } n.$$

Thus, if X is adapted we will know the value of X_n at time n. If

$$\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$$

we call $(\mathcal{F}_n)$ the natural filtration of X.

Martingale

Definition A.5. *A process* $X = (X_n)$ *is called martingale relative to* $(\{\mathcal{F}_n\}, \mathbb{P})$ *if*

i X *is adapted (to* $\{\mathcal{F}_n\}$*),*

ii $E[| X_n |] < \infty$ *for all* n*,*

iii $E[X_n \mid \mathcal{F}_{n-1}] = X_{n-1}$.

A.2 Mathematical Finance in Discrete Time

We specify a time horizon T, which is the terminal date of all economic activities considered. The filtration $\mathbb{F} = \mathcal{F}$ consists of σ-algebras $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \mathcal{F}_T$. The financial market contains $d+1$ financial assets in a market with prices $S_0(t), \ldots, S_d(t)$. It will be essential to assume that the price process of at least one asset follows a strictly positive process.

Definition A.6. *A numeraire is a price process $(X(t))_{t=0}^T$ which is strictly positive for all $t \in \{0, 1, \ldots, T\}$.*

Numeraires can be used to express all prices in an economy. Suppose the asset price process $S_1(t)$ is chosen as numeraire. The prices of other assets expressed in $S_1(t)$ are called relative prices and are denoted by $S_i^{'}(t) = S_i(t)/S_1(t)$.

A trading strategy ϕ is a R^{d+1} vector stochastic process $\phi = (\phi(t))_{t=1}^T = ((\phi_0(t, \omega),\ \phi_1(t, \omega),\ \ldots,\ \phi_d(t, \omega))')_{t=1}^T$ which is predictable. That is, each $\phi_i(t)$ is $\mathcal{F}_{t-1}$ measurable for $t \geq 1$.

Definition A.7. *The value of the portfolio at time t is the scalar product*

$$V_\phi(t) = \phi(t) \cdot S(t) := \sum_{i=0}^{d} \phi_i(t) S_i(t), \qquad (t = 1, 2, \ldots, T) \text{ and } V_\phi(0) = \phi(1) \cdot S(0).$$

The process $V_\phi(t, \omega)$ is called the wealth or value process of the trading strategy ϕ.

The initial wealth $V_\phi(0)$ is called the *initial investment*. Special classes of trading strategies are the self-financing.

Definition A.8. *The strategy ϕ is self-financing, $\phi \in \Phi$, if*

$$\phi(t) \cdot S(t) = \phi(t+1) \cdot S(t), \qquad (t = 1, 2, \ldots, T-1).$$

Definition A.9. *The gains process G_ϕ of a trading strategy ϕ is given by*

$$G_\phi := \sum_{\tau=1}^{t} \phi(t) \dot{(} S(\tau) - S(\tau - 1)), \qquad (t = 1, 2, \ldots, T).$$

A.3 Some Useful Stochastic Calculus

With use of the martingale pricing equation one can calculate the value of a derivative security. The price must be independent of the choice of numeraire but the pricing procedure might be more convenient in some measures than others. Moreover, the derivatives' price process might be stated in one measure but one would like to price it using another measure. It is therefore useful to provide theory on how to connect different measures and define ways to express expectations under one measure in terms of another.

Consider two different numeraires $N(t)$ and $M(t)$ connected with the equivalent martingale measures $\mathbb{Q}^N$ and $\mathbb{Q}^M$. Since the prices must be independent of the choice of numeraire one can state

$$N(t)\mathbb{E}^N\left[\frac{X(T)}{N(T)} \mid \mathcal{F}_t\right] = M(t)\mathbb{E}^M\left[\frac{X(T)}{M(T)} \mid \mathcal{F}_t\right].$$

We would like to derive an expression for the random variables in the left hand side expectation in terms of the right hand side expectation. Let $G(T) = X(T)/N(T)$ and state

$$\mathbb{E}^N[G(T) \mid \mathcal{F}_t] = \mathbb{E}^M\left[\frac{N(T)/N(t)}{M(T)/M(t)} \mid \mathcal{F}_t\right].$$

Thus, the expectation of the martingale G under $\mathbb{Q}^N$ is equal to the expectation of G times the random variable $\frac{N(T)/N(t)}{M(T)/M(t)}$ under the measure $\mathbb{Q}^M$. This random variable is known as the *Radon-Nikodym derivative* and is denoted by $d\mathbb{Q}^N/d\mathbb{Q}^M$. This is summarized below.

Theorem A.1. *Let $\mathbb{Q}^N$ and $\mathbb{Q}^M$ be equivalent measures with respect to the numeraires N(t) and M(t). The Radon-Nikodym derivative that changes the equivalent measure $\mathbb{Q}^N$ into $\mathbb{Q}^M$ is given by*

$$\frac{d\mathbb{Q}^N}{d\mathbb{Q}^M} = \frac{N(T)/N(t)}{M(T)/M(t)}.$$

The most well known result from stochastic calculus is the Girsanov's Theorem. Girsanov's Theorem provides a tool to determine the effect of a change of measure on a stochastic process.

Theorem A.2. ***(Girsanov)*** *For any stochastic process $k(t)$ such that*

$$\mathbb{P}\left(\int_0^t k^2(s)ds < \infty\right) = 1$$

consider the Radon-Nikodym derivative $\frac{dQ^*}{dQ} = \rho(t)$ given by,

$$\rho(t) = exp\left\{\int_0^t k(s)dW(s) - \frac{1}{2}\int_0^t k(s)ds\right\},$$

where W is a Brownian motion under the measure Q. Under the measure Q^* the process

$$W^*(t) = W(t) - \int_0^t k(s)ds$$

is a Brownian motion.

The main consequence of the Girsanov theorem is that when one changes measures the drift component is affected but the diffusion component remains unaffected. One can say that switching from one measure to another just changes the relative likelihood of a particular path being chosen.

Itô's Lemma

Itô's Lemma is to functions of random variables what Taylor's theorem is to functions of deterministic variables, in that it relates the small changes in a function of random variables to the small changes in the random variables itself. Suppose that a function f depends on the n variables $x_1, x_2, \ldots, x_n$ and time t. Suppose further that x_i follows an Itô process with instantaneous drift a_i and instantaneous variance b_i^2 $(1 \leq i \leq n)$, that is,

$$dx_i = a_i dt + b_i dz_i \tag{A.1}$$

where dz_i is a Wiener process $(1 \leq i \leq n)$. Each function a_i and b_i may be any function of all the x_i's and t. A Taylor series expansion of f gives

$$\Delta f = \sum_i \frac{\partial f}{\partial x_i}\Delta x_i + \frac{\partial f}{\partial t}\Delta t + \frac{1}{2}\sum_i\sum_j \frac{\partial^2 f}{\partial x_i \partial x_j}\Delta x_i \Delta x_j + \frac{1}{2}\sum_j \frac{\partial^2 f}{\partial x_i \partial t}\Delta x_i \Delta t + \cdots \tag{A.2}$$

Equation (A.1) can be discretized as

$$\Delta x_i = a_i \Delta t + b_i \epsilon_i \sqrt{\Delta t}$$

where ϵ_i is a random sample from a standardized normal variable. The correlation ρ_{ij} between dz_i and dz_j is defined as the correlation between ϵ_i and ϵ_j. We can show that

$$\lim_{\Delta t \to 0} \Delta x_i \Delta x_j = b_i b_j \rho_{ij} dt.$$

As $\Delta t \to 0$, the first three terms in the expansion of Δf in equation (A.2) are of order Δt. All other terms are of higher order. Hence,

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i + \frac{\partial f}{\partial t} dt + \frac{1}{2}\sum_i\sum_j \frac{\partial^2 f}{\partial x_i \partial x_j} b_i b_j \rho_{ij} dt.$$

This is the generalized version of Ito's lemma. Substituting for dx_i from equation (A.1) gives

$$df = \left(\sum_i \frac{\partial f}{\partial x_i} a_i + \frac{\partial f}{\partial t} + \frac{1}{2}\sum_i\sum_j \frac{\partial^2 f}{\partial x_i \partial x_j} b_i b_j \rho_{ij}\right) dt + \sum_i \frac{\partial f}{\partial x_i} b_i dz_i. \tag{A.3}$$

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