On the convexity of yield and potential surfaces in rotational hardening critical state models

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ABSTRACT: This article shows that a common formulation often used for an anisotropic modified cam clay yield and potential surface with a modified Lode angle dependency may become concave for high values of anisotropy. Concave surfaces are undesirable in plasticity theory and could lead to numerical problems. To remediate this problem the article suggests a formulation for the Lode angle dependency that will not suffer from concavity. The suggested formulation is discussed. The formulation does not introduce any additional parameters for Lode angle dependency than that used to describe Lode angle dependency of an isotropic yield surface. In this paper, a generalized continuous Mohr–Coulomb criterion is used that allows a π-plane cross-section to take the shape of several criteria including Mohr-Coulomb, Matsuoka-Nakai and Lade-Duncan.

1 INTRODUCTION

Many of the state-of-the-art soil models today, in particular for clays, are based on the critical state (CS) concept (Schofield and Wroth, 1968) with the Modified Cam-Clay (MCC) model (Roscoe and Burland, 1968) as formulation basis. New features have been added such as rotational hardening. A common formulation defines a yield and potential surface which when rotated changes its shape and become "sheared", as first proposed by Dafalias (Dafalias, 1986) (Figure 1). This model is often referred to as the Anisotropic Modified Cam Clay Model (AMCCM). The formulation was originally derived considering plastic dissipation in triaxial coordinates. In general, stress space the yield surface may be given by:

$$p_{eq} = p' \cdot \left(1 + \frac{3}{2} \cdot \left| \frac{\boldsymbol{\sigma}_d}{p'} - \boldsymbol{\alpha}_d \right|^2 \cdot g^2\left(\theta^\alpha, \alpha\right) \right) \tag{1}$$

$$F = p_{eq} - p_{mi} = 0$$

Where $\boldsymbol{\sigma}_d$ and $\boldsymbol{\alpha}_d$ are the deviatoric stress and rotational vector respectively (see appendix), p' is the mean effective stress and p_{mi} is the size of the surface and represents the pre-consolidation pressure. The function $g(\theta^\alpha,\alpha)$ is the subject of this paper. The modified Lode angle, θ^α, is defined as:

$$\sin 3\theta^\alpha = -\frac{3\sqrt{3} \cdot J_3^\alpha}{2 \cdot \left(J_2^\alpha\right)^{3/2}} \tag{2}$$

where J_2^α and J_3^α is the second and third invariant of the modified deviatoric stress vector, $\boldsymbol{\sigma}_d - p' \cdot \boldsymbol{\alpha}_d$. The function, $g(\theta^\alpha, \alpha)$, represents the shape of the surface in a plane normal to the α-line (Figure 2, eq. (3)) and includes a dependency of a critical state ratio, M. If M is a constant then the shape will be circular, but most often it varies in accordance with a failure criteria such as Mohr-Coulomb, Matsuoka-Nakai (Matsuoka and Nakai, 1974), Lade-Duncan (Lade and Duncan, 1975), or a generalized form (Grimstad et al., 2018). In general M must be a function of the modified Lode angle, $M = M(\theta^\alpha)$.

$$\alpha = \sqrt{\frac{3}{2} \cdot \boldsymbol{\alpha}_d^{\,T} \cdot \boldsymbol{\alpha}_d} \tag{3}$$

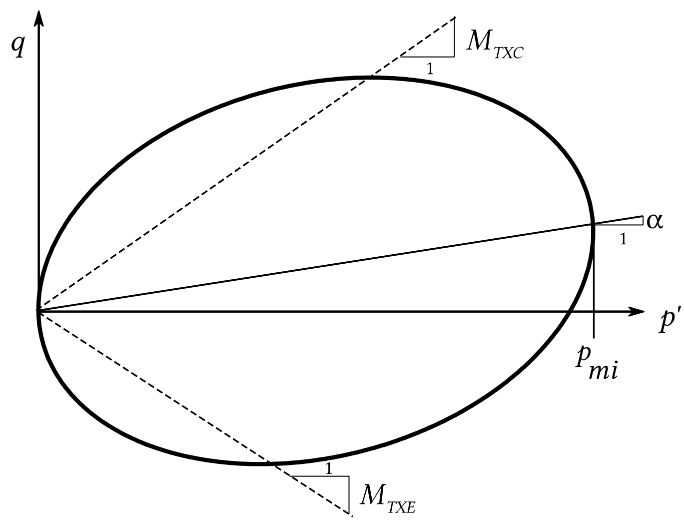

Figure 1 Yield surface for AMCCM in p'-q space

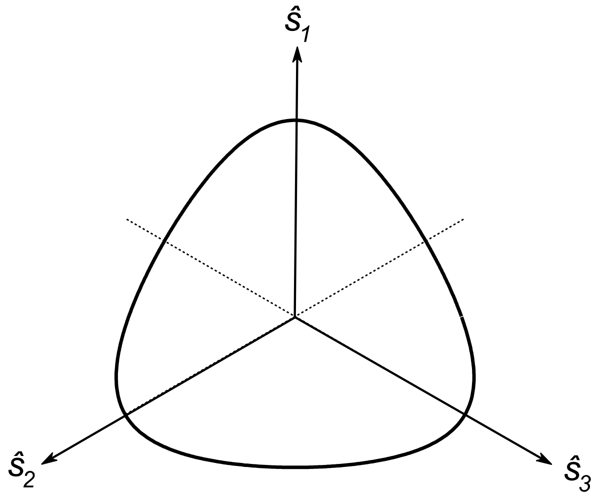

Figure 2 Typical projected cross section of the yield surface in the translated π-plane. Here $\hat{s}_i = \sigma_i - \alpha_i \cdot p$.

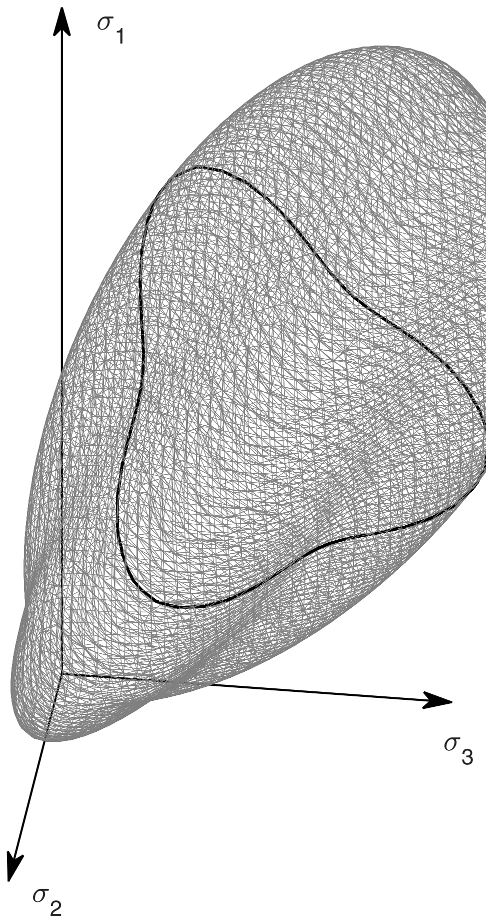

Figure 3 Yield surface for AMCCM in principal stress space for high values of α

2 PROBLEM DESCRIPTION

In many formulations found in the literature the function $g^2(\theta^\alpha, \alpha)$, from eq. (1), is defined as:

$$g^2\left(\theta^\alpha, \alpha\right) = \frac{1}{M^2\left(\theta^\alpha\right) - \alpha^2} \tag{4}$$

In this expression, M is directly a function of the modified Lode angle, while α might change due to kinematic hardening. It can be seen in the equation above that whenever α approaches M (i.e. due to kinematic hardening) the expression $g(\theta^\alpha, \alpha)$ becomes very large and approaches infinity as a limit. As stated in (Crouch and Wolf, 1995) the introduction of a dependency of the anisotropic bounding surface on the Lode angle is not straightforward. In the work on this subject it is discovered that due to the specific form of eq. (4) and (1), the shape of the surface could actually become concave as it is "sheared" (Figure 3 and Figure 4). This may happen even if the shape is convex for α equal to zero. In general, concave surfaces are undesirable in plasticity theory and could lead to numerical problems.

It can be shown that convexity is ensured if eq. (5) is satisfied.

$$g''\left(\theta^\alpha\right) + g\left(\theta^\alpha\right) \geq 0 \tag{5}$$

Solving this equation at the limit, for any θ^α, gives the maximum rotation, i.e. the limit for α, for which the surface remains convex. For the Lade-Duncan criterion the solution will follow the line LD in Figure 5. For other criteria such as generalized Mohr-Coulomb, the curve may be lower or higher up, this is dependent on the shape parameters used. Typically, for sharper corners and straighter segments the curve is lower than the one for LD.

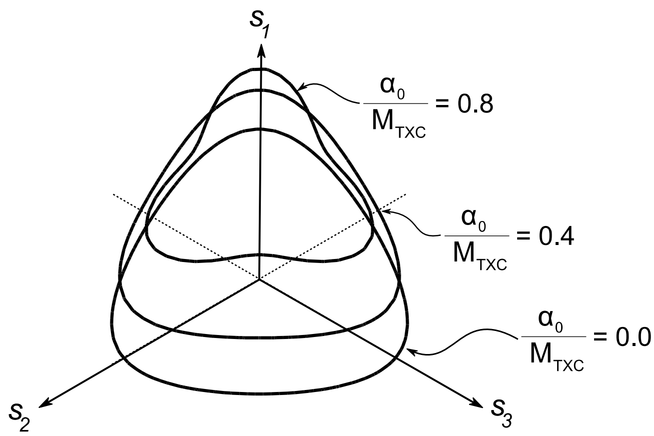

Figure 4 Cross section of the Modified Lode angle dependent AMCCM yield surface in the π-plane

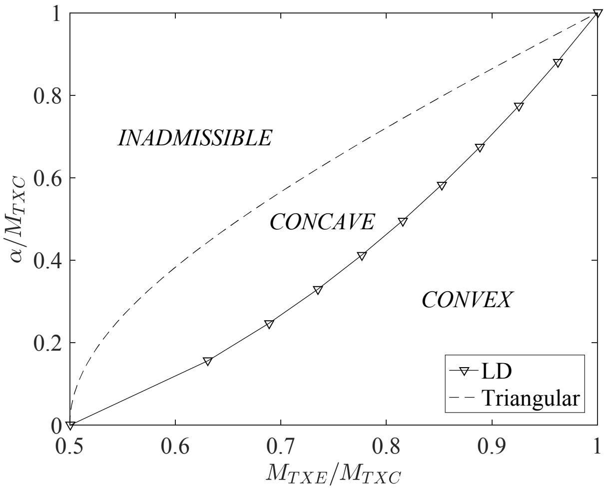

Figure 5 Limit value for α using the LD criterion for critical state description, the line for "Triangular" to refer to eq. (10), restricting the required shape to not be "beyond" triangular.

In order to investigate if the concavity is a problem in practice one need to see if typical material parameters for a typical clay material could lead to concavity.

It is common to use the earth pressure coefficient at rest under virgin loading, K_0^{NC}, to define the initial value of α. Dafalias provided an exact expression for the initial value α_0 corresponding to virgin compression, which, when disregarding elastic strain contributions, simplifies to:

$$\alpha_0 \approx \frac{\eta_0^{\ 2} + 3\cdot\eta_0 - M_{TXC}^{\ \ 2}}{3} \qquad \eta_0 = \frac{3\cdot\left(1-K_0^{NC}\right)}{1+2\cdot K_0^{NC}} \tag{6}$$

For soft Scandinavian clays typical values for K_0^{NC} is in the range 0.55 to 0.65, which for an effective friction angle of 25° to 32° might, in a worst case combination, give a value of $\alpha_0 \sim 0.465 \cdot M_{TXC}$, where M_{TXC} is the value of M for $\theta^\alpha = -\pi/6$ (i.e. for triaxial compression). For such an extreme case, concavity of the surfaces may develop initially. However, a low effective friction angle will normally results in higher value for K_0^{NC}, and a value of $\alpha_0 \sim 0.2$ will be a typical case. Then concavity does not have to be a problem. If this is the case or not will depend on the type of critical state (failure) criterion used. For boundary value problems where soil is to failure in shear, concavity will normally increase as loading is applied since α increases due to kinematic hardening. Concavity may be totally avoided, or eventually become negligible for volumetrically dominated deformation, i.e. a state resulting in $\alpha \rightarrow 0$. One should notice that concavity would often not be experienced for K_0 loading since the rotational hardening is modest in this case, $\alpha \approx \alpha_0$ (if this is the case or not is again dependent on the failure criterion deployed).

3 PROPOSED SOLUTION

Some simple adjustments are suggested to avoid concavity. Eq. (7) is proposed as an alternative to eq. (4) . In Eq. (7) the triaxial compression value, M_{TXC}, is used instead of the function $M(\theta^\alpha)$ (in eq. (4)). The function $h(\theta^\alpha,\alpha)$ (eq. (8)) can ensure that the ratio between the critical state line for compression and extension states remains constant when the function $M(\theta^\alpha,\alpha)$ is carefully selected.

$$g^2\left(\theta^\alpha,\alpha\right) = \frac{h^2\left(\theta^\alpha,\alpha\right)}{M_{TXC}^{\ \ 2} - \alpha^2} \tag{7}$$

Where:

$$h\left(\theta^\alpha,\alpha\right) = \frac{M_{TXC}}{M\left(\theta^\alpha,\alpha\right)} \tag{8}$$

In order to find a proper function $h(\theta^\alpha,\alpha)$ (i.e. $M(\theta^\alpha,\alpha)$), that is consistent with critical state in triaxial extension, one need to establish a relation between the input value of M_{TXE} and the value for M at $\theta^\alpha = \pi/6$ (mod. Lode angle equivalent to triaxial extension). Eq. (9) gives the required value of $M(\theta^\alpha = \pi/6, \alpha)$ such that the critical state line in triaxial extension is equal to the desired value M_{TXE}. If limiting oneself to a shape of the failure criterion reproducible in six sectors (i.e. defined in the sector $-\pi/6 < \theta^\alpha < \pi/6$), the ratio between triaxial compression (TXC) and triaxial extension (TXE) strengths are limited by a triangular shape in order not to become concave. This means that it will be impossible to maintain a convex surface if the ratio between $M(\theta^\alpha = \pi/6)$ and $M(\theta^\alpha = -\pi/6)$ is less than ½. Eq. (10) gives the limit for M_{TXE} in this case. Figure 5 gives a graphical representation of eq. (10), separating admissible from inadmissible (dotted line).

$$M\left(\theta^\alpha = \frac{\pi}{6},\alpha\right) = M_{TXC}\cdot\sqrt{\frac{M_{TXE}^{\ \ 2}-\alpha^2}{M_{TXC}^{\ \ 2}-\alpha^2}} \tag{9}$$

$$M_{TXE} > \sqrt{\frac{M_{TXC}^{\ \ 2} + 3\cdot\alpha^2}{4}} \tag{10}$$

For moderate values of α there is moderate difference between the necessary $M(\theta^\alpha = \pi/6)$ and the value for M_{TXE}. Hence, it might be practical to leave the formulation without this correction, and instead directly apply the Lode angle dependency to $M(\theta^\alpha, [\alpha = 0])$, eq. (11). The great benefit in doing this is that the yield surface will never become concave as long as it is not concave for the isotropic MCCM ($\alpha = 0$).

$$h\left(\theta^\alpha\right) = \frac{M_{TXC}}{M\left(\theta^\alpha\right)} \tag{11}$$

By using this function the maximum ratio between the input values M in extension and compression,

M_{TXE}/M_{TXC}, is as "normal" the factor of ½. The downside of eq. (11) is an over-prediction of the experienced critical state line in extension (when compared to the input value of M_{TXE}). Eq. (12) gives the ratio between the experienced triaxial extension critical state, M_{TXE}^{exp}, compared to the input value, M_{TXE}^{inp}, when eq. (11) is used for the function h.

$$\frac{M_{TXE}^{\exp}}{M_{TXE}^{inp}} = \sqrt{1 + \left(\frac{1}{\left(\frac{M_{TXE}^{inp}}{M_{TXC}} \right)^2} - 1 \right) \cdot \left(\frac{\alpha}{M_{TXC}} \right)^2} \tag{12}$$

The equation for $M(\theta^\alpha)$ can be selected to provide surfaces like Mohr-Coulomb (MC), Lade-Duncan (LD) or Matsouka-Nakai (MN), or other surfaces.

Graphical illustrations of eq. (7) with use of eq. (11) are given in Figure 6 and Figure 7. Here parameters in the generalized criterion (Grimstad et al., 2018) equivalent to the Lade-Duncan criterion are used.

Figure 8 shows a graphical representation of eq. (12). As seen in the figure, for "normal" input, the difference is typically less than 10% between M_{TXE}^{exp} and M_{TXE}^{inp}.

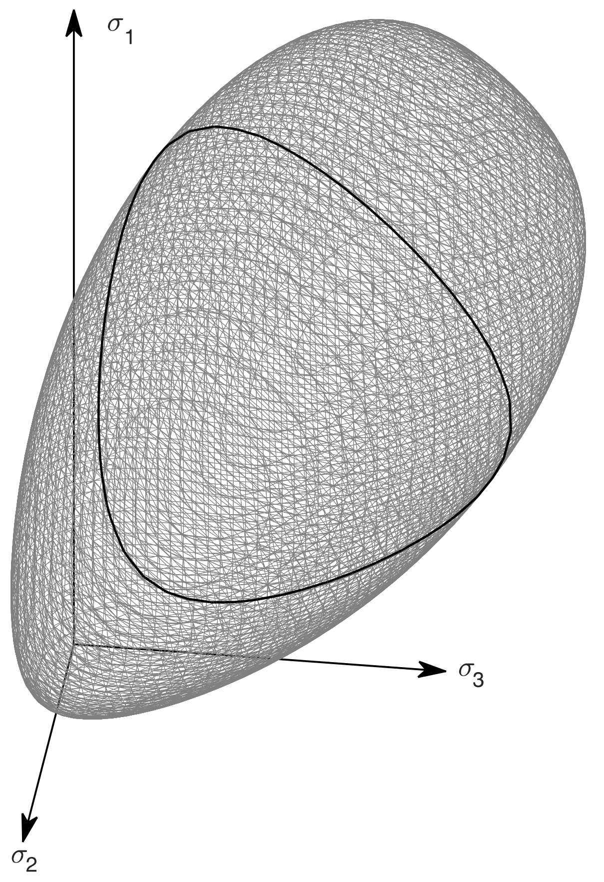

Figure 6 Modified yield surface for AMCCM in principal stress space

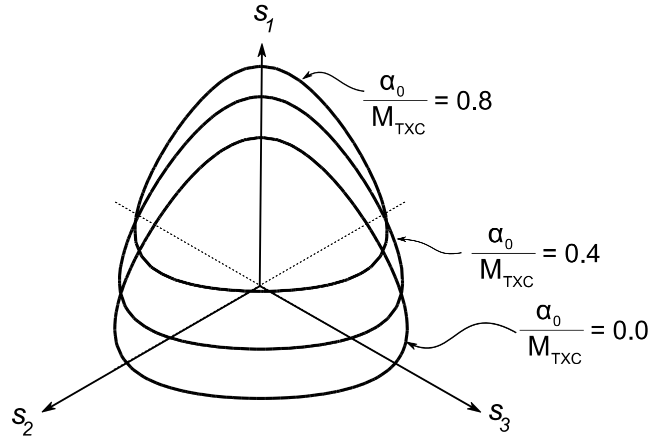

Figure 7 Cross section of the Modified Lode angle dependent AMCCM yield surface in the π-plane with proposed modification

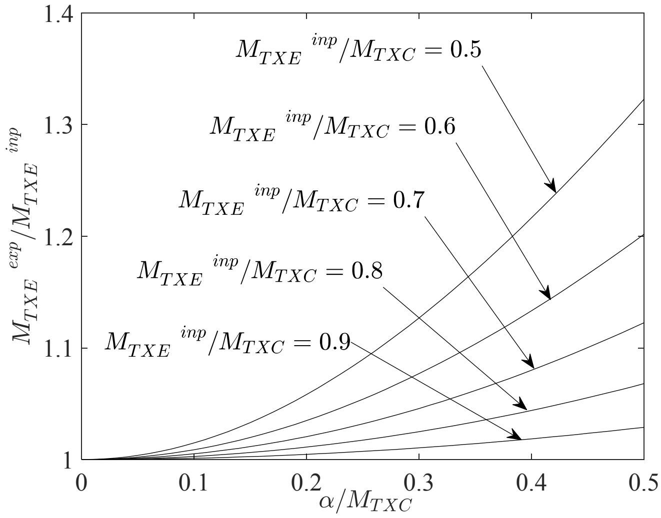

Figure 8 Graphical representation of eq. (12), showing the experienced value of critical state in extension compared to the input value as a function of α

4 FINAL FORMULATION

As mentioned above the equation for $M(\theta^\alpha)$ can be selected to provide surfaces like Mohr-Coulomb (MC), Lade-Duncan (LD) or Matsouka-Nakai (MN), or other surfaces. In the following the GMC criterion from (Grimstad et al., 2018) is used to provide a description of the $h(\theta^\alpha)$. Eq. (13) to (15) is taken from Grimstad et al. where the terms c_θ ans s_θ is modified to be a function of the Modified Lode angle, θ^α.

$$M\left(\theta^\alpha\right) = \frac{3 \cdot \sin\varphi_0}{\sqrt{3} \cdot \mathrm{c}_{\theta^\alpha} + \mathrm{s}_{\theta^\alpha} \cdot \frac{\sin\varphi_0}{a_2}} \tag{13}$$

$$\mathrm{c}_{\theta^\alpha} = \cos\left(\frac{1}{3} \cdot \arcsin\left(a_1 \cdot \sin 3\theta^\alpha \right) \right) \tag{14}$$

$$\mathrm{s}_{\theta^\alpha} = \sin\left(\frac{1}{3} \cdot \arcsin\left(a_1 \cdot \sin 3\theta^\alpha \right) \right) \tag{15}$$

Where $\sin\varphi_0$ is the friction angle for modified Lode angle of 0° and the two new parameters a_1 and a_2 are used to control the shape of the cross section in the π-plane (the Lode angle dependency).

In order to use the criterion the parameter $\sin\varphi_0$ must be linked to the value of M_{TXC} used in eq. (11). By inserting the lode angle for triaxial compression into eq. (13) the following relationship is obtained:

$$\sin\varphi_0 = \frac{\sqrt{3}\cdot\cos\beta\cdot M_{TXC}}{3+\dfrac{\sin\beta\cdot M_{TXC}}{a_2}} \tag{16}$$

Where: $\beta = \dfrac{1}{3}\cdot\arcsin\left(a_1\right)$

Then inserting eq. (16) into eq. (13) gives:

$$M\left(\theta^{\alpha}\right) = \frac{M_{TXC}}{\dfrac{c_{\theta^{\alpha}}}{\cos\beta}+\left(c_{\theta^{\alpha}}\cdot\tan\beta+s_{\theta^{\alpha}}\right)\cdot\dfrac{M_{TXC}}{3\cdot a_2}} \tag{17}$$

Finally inserting eq. (17) into eq. (11) gives:

$$h\left(\theta^{\alpha}\right) = \frac{c_{\theta^{\alpha}}}{\cos\beta}+\left(c_{\theta^{\alpha}}\cdot\tan\beta+s_{\theta^{\alpha}}\right)\cdot\frac{M_{TXC}}{3\cdot a_2} \tag{18}$$

The parameters a_1 and a_2 can be selected based on knowledge of the actual Lode angle dependency. However, in most cases such information is not available for other states than triaxial compression and triaxial extension. In cases where only information on M_{TXC} is available e.g. the Matsouka-Nakai criterion could be employed. Then eq. (19) and (20) gives the parameters.

$$a_1 = \frac{27}{2\cdot\sqrt{3}}\cdot\frac{M_{TXC}\cdot\left(3+M_{TXC}\right)}{\sqrt{\left(9+3\cdot M_{TXC}+M_{TXC}^{2}\right)^{3}}} \tag{19}$$

$$a_2 = \frac{3\cdot M_{TXC}}{\sqrt{3\cdot\left(9+3\cdot M_{TXC}+M_{TXC}^{2}\right)}} \tag{20}$$

$$\sin\varphi_0 = a_2 \tag{21}$$

One may also express the relation between the parameters as:

$$a_1 \in \left\langle 0,1\right\rangle \tag{22}$$

$$a_2 = \frac{2}{3}\cdot\left(\frac{1}{M_{TXE}}-\frac{1}{M_{TXC}}\right)^{-1}\cdot\sin\beta \tag{23}$$

If one decides to use the GMC criterion with parameters equal to a rounded Mohr-Coulomb (instead of the MN formulation), then:

$$a_1 \rightarrow 1 \tag{24}$$

$$a_2 = 2\cdot\sin\beta \tag{25}$$

Giving:

$$h\left(\theta^{\alpha}\right) = \frac{c_{\theta^{\alpha}}}{\cos\beta}+\left(\frac{c_{\theta^{\alpha}}}{\cos\beta}+\frac{s_{\theta^{\alpha}}}{\sin\beta}\right)\cdot\frac{M_{TXC}}{6} \tag{26}$$

5 CONCLUSION

This paper shows that when considering the anisotropic Modified Cam Clay (AMCC) type of yield surface and introducing a modified Lode angle dependency for critical state, the yield/potential surface might become concave for high values of anisotropy. Therefore, this paper proposes a function, $g(\theta^{\alpha}, \alpha)$, that ensures convexity, for the Lode angle dependent AMCCM, for any $\alpha < M$, as long as the surface itself is convex for $\alpha = 0$. Avoiding concave yield/potential surfaces is important and the proposed remedy offers a simple solution to solve this problem. The simplified modification proposed will match the input value for the critical state line in triaxial compression. However, it will result in higher experienced values for critical state in extension that the value given for $\alpha = 0$. Typically, the difference is in the order of up to 10%.

Finally a generalized form of the function $h(\theta^{\alpha})$ is proposed that allow a variety of shapes of the surfaces. The formulation is shown to depend on the ratio between M_{TXC} and M_{TXE}. If the GMC criterion is used to give a rounded Mohr-Coulomb, then one additional parameter (that is just below 1.0) is needed.

REFERENCES

CROUCH, R. S. & WOLF, J. P. 1995. On a three-dimensional anisotropic plasticity model for soil. *Géotechnique,* 45, 301-305.

DAFALIAS, Y. F. 1986. An anisotropic critical state soil plasticity model. *Mechanics Research Communications,* 13**,** 341-347.

GRIMSTAD, G., RØNNINGEN, J. A. & NORDAL, S. Application of a generalized continuous Mohr–Coulomb criterion 9th Conference on Numerical Methods in Geotechnical Engineering (NUMGE), 25-27 June 2018 2018 Porto, Portugal.

LADE, P. V. & DUNCAN, J. M. 1975. Elastoplastic stress-strain theory for cohesionless soil. *ASCE Journal of the Geotechnical Engineering Division,* 101**,** 1037-1053.

MATSUOKA, H. & NAKAI, T. Stress-deformation and strength characteristics of soil under three different principal stresses. Proceedings of JSCE, 1974. 59–70.

ROSCOE, K. H. & BURLAND, J. 1968. On the generalized stress-strain behaviour of wet clay.

SCHOFIELD, A. & WROTH, P. 1968. Critical state soil mechanics.

APPENDIX

Mean effective stress:

$$p' = \frac{1}{3}\left(\sigma'_{11} + \sigma'_{22} + \sigma'_{33}\right) \tag{27}$$

Deviatoric stress vector:

$$\boldsymbol{\sigma}_d = \begin{Bmatrix} \sigma'_{11} - p' \\ \sigma'_{22} - p' \\ \sigma'_{33} - p' \\ \sqrt{2}\sigma'_{12} \\ \sqrt{2}\sigma'_{23} \\ \sqrt{2}\sigma'_{13} \end{Bmatrix} \tag{28}$$

Rotational vector:

$$\boldsymbol{\alpha}_d = \begin{Bmatrix} \alpha_{11} - 1 \\ \alpha_{22} - 1 \\ \alpha_{33} - 1 \\ \sqrt{2}\alpha_{12} \\ \sqrt{2}\alpha_{23} \\ \sqrt{2}\alpha_{13} \end{Bmatrix} \tag{29}$$

For cross ansiotropic condition (major direction as z-direction):

$$\boldsymbol{\sigma}_d = \begin{Bmatrix} \sigma'_{xx} - p' \\ \sigma'_{yy} - p' \\ \sigma'_{zz} - p' \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \sigma'_{2} - p' \\ \sigma'_{3} - p' \\ \sigma'_{1} - p' \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{30}$$

$$\boldsymbol{\alpha}_d = \begin{Bmatrix} -\frac{1}{3}\cdot\alpha \\ -\frac{1}{3}\cdot\alpha \\ \frac{2}{3}\cdot\alpha \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{31}$$