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Abstract

Ordinary Dirichlet series, of which the Riemann zeta function is the most important, play a prominent role in classical analysis and number theory, and in modern mathematics. It is well-known that the Riemann zeta function has a single pole at the point $s = 1$. The present thesis investigates both the behaviour of various zeta functions near this point and the function spaces of ordinary Dirichlet series they can be said to generate.

Chapter 1 gives a comprehensive overview of the thesis and offers brief surveys of related results.

Chapter 2 introduces a new scale of function spaces of Dirichlet series and explains the local behaviour of the reproducing kernels and establishes local embeddings into classical function spaces. Other such spaces are also considered, of which the Dirichlet-Hardy spaces are the most important.

Chapter 3 determines the spaces spanned by the real parts of the boundary functions and distributions in the different settings.

Chapter 4 characterises the local interpolating sequences for the Hilbert spaces under consideration. In the non-Hilbert spaces only partial results are obtained.

Chapter 5 deals with a family of zeta functions corresponding to subsets of the integers. A complete characterisation of their behaviour close to the point $s = 1$ is given in terms of lower norm bounds of integral operators with the zeta functions as kernels.

Chapter 6 considers the results of the previous chapter under the additional hypothesis of arithmetic structure. The characterisations become simpler and more can be said.

Preface

This thesis is submitted in partial fulfilment of the requirements for the degree of philosophiae doctor (PhD) at the Norwegian University of Science and Technology (NTNU) in Trondheim. The research was supported by the Research Council of Norway grant 160192/V30.

The thesis is based on the three papers:

[63] Jan-Fredrik Olsen. Modified zeta functions as kernels of integral operators. Preprint, 2009.

[64] Jan-Fredrik Olsen and Eero Saksman. Some local properties of functions in Hilbert spaces of Dirichlet series. Preprint, 2009.

[65] Jan-Fredrik Olsen and Kristian Seip. Local interpolation in Hilbert spaces of Dirichlet series. *Proc. Amer. Math. Soc.*, 136:203–212, 2008.

The work on the thesis began September 2005 and was concluded February 2009. The research was carried out at the Department of Mathematical Sciences, NTNU, with the exception of a nine month stay during fall 2007 and spring 2008 at the Washington University in St. Louis, Missouri, USA, and two stays of about two weeks each at the University of Helsinki, Finland during fall 2006 and 2008.

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Elin and me to their home. I also extend my gratitude to Professor Eero Saksman for inviting me twice to Finland to work on what has become a joint-paper, and letting me give a talk at the seminar there.

Next, I would like to thank all the graduate students, post. docs. and others I have crossed paths with during these years, both at NTNU, Wash. U., and all the conferences I have attended. I mention especially Geir Arne for his help mathematically and otherwise, both in Trondheim and in St. Louis, Jordi for having to suffer my periods of insanity and his thorough reading of my third paper, Shahaf for showing me the wonders of (QF)-systems and taking the time to read large parts of this thesis thoroughly, Josh and Mike for attending my seminars and helping me build muscles that vanished all to soon, Scott and Megan for teaching me the finer points of American culture, Nacho and Dani for making conferences a joy, Yurii and Polina for inviting us to their home in St. Petersburg, Inge and Marit for showing Elin and me that snow in the Rockies is not best enjoyed while stuck in traffic, Marcus and Lotta for the good times driving through the USA, Alejandro and Manuel for being wonderfully insane spanish mathematicians, and finally Connie, Yutaka and Lars for hanging out while in Helsinki.

I am grateful to the organiser of the Dirichlet Series, Problems and Analysis seminars at Washington University in St. Louis, Professor McCarthy, for letting me give several seminars in all three. Also, I would like to thank the participants as they all made interesting comments and suggestions. In addition to some of those mentioned above, these include the Professors Stefan Richter, Richard Rochberg, Guido Weiss and Albert Baernstein, II.

Finally, I would like to thank my mother, my father and my brother for giving me the confidence to work hard and continuously for my goals and providing me with a safe haven up north at all times. Last but not least, I would like to thank my wonderful Elin for her continuing support and love, and especially for suffering through the final stages of the work on this thesis.

Trondheim, February 2009
Jan-Fredrik Olsen

Contents

1 Introduction

The theory of ordinary Dirichlet series, i.e. functions of the type

$$F(s) = \sum_{n\in\mathbb{N}} a_n n^{-s}, \quad s = \sigma + \mathrm{i}t, \tag{1.1}$$

is a classical field of study that dates back, in one form or the other, for more than 200 years. The primary motivation for the study of these functions has been their significance in analytic number theory. The connection is perhaps seen in its purest form through the Euler product formula [26]

$$\sum_{n\in\mathbb{N}} n^{-s} = \prod_{p\in\mathbb{P}} \left(\frac{1}{1-p^{-s}}\right), \quad \text{for } \sigma > 1.$$

This formula gives a deep connection between the multiplicative properties of the prime numbers $\mathbb{P}$ and the additive properties of the positive integers $\mathbb{N}$. Either side of this formula defines the Riemann zeta function which we denote by $\zeta(s)$.

Our ambition is to continue work begun by H. Helson in the sixties [37, 38]. What Helson did was to apply the tools of functional analysis to study the ordinary Dirichlet series[1]. We pursue two directions.

Function spaces of Dirichlet series

Our main focus is on a scale of Banach spaces analogue to the classical Hardy spaces H^p for $p \in [1, \infty)$. These spaces are called the Dirichlet-Hardy spaces. The definition for the important special case $p = 2$ first appeared in a paper by H. Hedenmalm, P. Lindqvist and K. Seip [35]. It is the Hilbert space that consists of the Dirichlet series with square summable coefficients.

[1] In what follows we refer to ordinary Dirichlet series as Dirichlet series.

In addition, we introduce a new scale of Hilbert spaces of Dirichlet series. We call these spaces the Dirichlet-Bergman spaces. Finally, we consider a scale of Hilbert spaces introduced by J. E. McCarthy [58]. Both McCarthy's spaces and the Dirichlet-Bergman spaces offer analogies to the classical scale of Hilbert function spaces on the half-plane which contain the Bergman-type spaces, the Hardy space H^2 and the Dirichlet-type spaces.

We study the boundary functions, local embedding properties, interpolating sequences and Carleson measures of both the Dirichlet-Hardy, Dirichlet-Bergman and McCarthy's spaces. Parts of this research has appeared in joint work with E. Saksman [64] and K. Seip [65].

Modified zeta functions

The second part of this thesis deals with a class of functions which we call the K-zeta functions. These are given by

$$\zeta_K(s) = \sum_{n \in K} n^{-s}, \quad K \subset \mathbb{N}, s = \sigma + \mathrm{i}t. \tag{1.2}$$

Note that for $K = \mathbb{N}$ the formula (1.2) yields the Riemann zeta function $\zeta(s)$. The first statement in B. Riemann's famous paper[2] [71, p. 145] is that even though the Riemann zeta function only converges absolutely for $\sigma > 1$, it has a meromorphic extension to the entire complex plane with a single pole of residue one at $s = 1$. Hence, there exists an entire function ψ such that

$$\zeta(s) = \frac{1}{s-1} + \psi(s). \tag{1.3}$$

By considering the K-zeta functions as kernels of certain integral operators on L^2-spaces over bounded intervals, we are able to characterise their behaviour near the point $s = 1$ in terms of densities of the subsets $K \subset \mathbb{N}$. These operators appear naturally in the study of the Dirichlet-Hardy spaces.

Part of this research has appeared in [63].

[2] See [25] for an English translation of the original paper.

1.1 Structure of the thesis

This work is roughly divided into two parts. In chapters 2, 3 and 4 we give a detailed account of the work relating to the function spaces of Dirichlet series, while chapters 5 and 6 deal with the modified zeta functions.

In the current chapter we proceed by giving a comprehensive survey of the work done in this thesis and indicate related results. We stress that we do not give an overview of all the research related to function spaces of Dirichlet series. The overall structure of the survey reflects that of the rest of the thesis in that every chapter is discussed in separate sections. Inside of each section we discuss the results of the relevant chapter and point out connections to related subjects.

The enumeration of our results, as they appear in the introduction, is identical to the enumeration in the various chapters. This may lead to jumps since not every lemma and corollary is stated in the introduction. In the case of results that have appeared in joint-works, they are accompanied with the appropriate names and year. Results due to other mathematicians are enumerated as they appear in the chapters, or without enumeration if they only appear in the introduction.

1.2 Function spaces of Dirichlet series and local embeddings

Chapter 2 establishes the fundamental connections between each of the three scales of spaces of Dirichlet series that we discuss and their classical counter-parts. These connections are the local embeddings and the local equivalence of the point evaluation functionals. We mention that the local embedding in the case[3] $p = 2$ for the Dirichlet-Hardy space was already known. In fact, it served as the starting point for this thesis. We begin with a rather detailed explanation of this case since it sheds light on the main ideas of this thesis.

[3]This was also known to have a trivial extension to the cases $p = 2\nu$ with $\nu \in \mathbb{N}$.

The space $\mathscr{H}^2$ and connections to $H^2(\mathbb{C}_{1/2})$

The Dirichlet-Hardy space $\mathscr{H}^2$ introduced by Hedenmalm, Lindqvist and Seip in [35] is the closure of the Dirichlet polynomials $\mathscr{P}$, i.e. the functions given by finite series of the type (1.1), in the norm

$$\|F\|_{\mathscr{H}^2} = \left(\sum_{n \in \mathbb{N}} |a_n|^2 \right)^{1/2}.$$

By the Cauchy-Schwarz inequality, this is seen to be a Hilbert space of Dirichlet series analytic on the half-plane[4] $\mathbb{C}_{1/2} = \{\sigma > 1/2\}$. Moreover, by inspection, one finds that the translate $\zeta(s + \bar{w})$ of the Riemann zeta function is the reproducing kernel at the point $w \in \mathbb{C}_{1/2}$, i.e. it is the unique function $k_w^{\mathscr{H}^2} \in \mathscr{H}^2$ such that $\langle F | k_w^{\mathscr{H}^2} \rangle = F(w)$ for all $F \in \mathscr{H}^2$.

A typical use of the formula (1.3) is that it implies that for $s + \bar{w}$ in a bounded subset of $\mathbb{C}$ we have[5]

$$k_w^{\mathscr{H}^2}(s) = \frac{1}{s + \bar{w} - 1} + \mathcal{O}(1).$$

It is well-known that the function $k_w^{H^2}(s) = (s + \bar{w} - 1)^{-1}$ is the reproducing kernel for the space $H^2(\mathbb{C}_{1/2})$. This space is contained in the scale of classical Hardy spaces of functions analytic on the half-plane $\mathbb{C}_{1/2}$ and finite in the norm

$$\|f\|_{H^p(\mathbb{C}_{1/2})}^p = \lim_{\sigma \to 1/2^+} \frac{1}{2\pi} \int_{\mathbb{R}} |f(\sigma + \mathrm{i}t)|^p \mathrm{d}t, \quad p \in [1, \infty).$$

The function theory of these spaces is very rich and they are considered to be well understood. See for instance [22, 31, 51, 57]. This explains our first observation, namely that for $s + \bar{w}$ in a bounded set

$$k_w^{\mathscr{H}^2}(s) = k_w^{H^2}(s) + \mathcal{O}(1). \tag{1.4}$$

[4] We give a brief discussion of the general convergence properties of Dirichlet series in the preliminaries part of chapter 2.

[5] The relation $g(x) = \mathcal{O}(f(x))$ is short for the statement that there exists some constant $C > 0$ such that $|g(x)| \leq C f(x)$ for x in some specified range.

The second fundamental connection between the spaces $H^2(\mathbb{C}_{1/2})$ and $\mathscr{H}^2$ is the following embedding that was found in the context of analytic number theory by H. L. Montgomery in [59, p. 140] and in the context of the space $\mathscr{H}^2$ in [35].

Lemma 2.2 (Montgomery 1994, Hedenmalm, Lindqvist and Seip 1997)**.** *For $F \in \mathscr{H}^2$ and every bounded interval I there exists a constant $C > 0$, depending only on the length of I, such that*

$$\lim_{\sigma \to \frac{1}{2}^+} \int_I |F(\sigma + \mathrm{i}t)|^2 \, \mathrm{d}t \leq C \|F\|^2_{\mathscr{H}^2}. \tag{1.5}$$

We sketch a short proof of Lemma 2.2 since the idea will be of importance throughout this work[6]. This proof is different from the ones given in [59] and [35]. Let χ_I denote the indicator function of the interval I of the real line $\mathbb{R}$, and consider the embedding operator defined on the Dirichlet polynomials by

$$E_I : \sum_{n \in \mathbb{N}} a_n n^{-s} \in \mathscr{P} \longmapsto \chi_I \sum_{n \in \mathbb{N}} a_n n^{-1/2 - \mathrm{i}t}. \tag{1.6}$$

The operator E_I is densely defined from $\mathscr{H}^2$ to $L^2(I)$. We use the convention

$$\|g\|^2_{L^2(I)} = \int_I |g(t)|^2 \mathrm{d}t.$$

We stress that throughout this thesis we view $L^2(I)$ as the subspace of $L^2(\mathbb{R})$ consisting of functions with support in I. With this, we find the adjoint operator E_I^* and show that[7]

$$E_I E_I^* g = \lim_{\sigma \to 1^+} \chi_I (g * \zeta_{1+\sigma}).$$

By using the fact (1.3) that the Riemann zeta function is meromorphic with a pole at $s = 1$ this implies that

$$E_I E_I^* g = 2\pi \chi_I P_+ g + \chi_I (g * \psi_1),$$

[6] We give this argument in full in chapter 2.

[7] For functions of a complex variable $s = \sigma + \mathrm{i}t$ we will frequently use the notation $\zeta(\sigma + \mathrm{i}t) = \zeta_\sigma(t)$ in order to determine which variable is used in the convolution on $\mathbb{R}$ which we denote by $*$.

where P_+ denotes the Riesz projection which is a bounded operator from $L^2(\mathbb{R})$ to $H^2(\mathbb{C}_{1/2})$ and $\psi_1(t) = \psi(1+it)$ is the restriction of an entire function. Hence E_I extends to a bounded operator from $\mathscr{H}^2$ to $L^2(I)$ and Lemma 2.2 follows.

An immediate consequence of Lemma 2.2 is that if $F \in \mathscr{H}^2$ then $F(s)/s \in H^2(\mathbb{C}_{1/2})$. The trick is to apply the lemma to the inequality[8]

$$\int_{\mathbb{R}} \left| \frac{F(\sigma + \mathrm{i}t)}{\sigma + \mathrm{i}t} \right|^2 \mathrm{d}t \lesssim \sum_{m \in \mathbb{Z}} \frac{1}{m^2+1} \int_0^1 |F(\sigma + \mathrm{i}t - \mathrm{i}m)|^2 \mathrm{d}t. \tag{1.7}$$

Note that the norm of $\mathscr{H}^2$ is invariant under vertical translations. In particular, this implies that F has non-tangential limits almost everywhere on the abscissa $\sigma = 1/2$, giving sense to the notation $F(1/2 + \mathrm{i}t)$.

We remark that Hedenmalm and Saksman [36] discovered an analogue of L. Carleson's celebrated convergence theorem for Fourier series with square summable coefficients [15].

Theorem (Hedenmalm and Saksman 2003)**.** *Let* $\sum_{n \in \mathbb{N}} |a_n|^2 < \infty$. *Then the series* $\sum_{n=1}^{\infty} a_n n^{-1/2-\mathrm{i}t}$ *converges for almost every* $t \in \mathbb{R}$.

A shorter argument that uses Carleson's result directly was found by S. V. Konyagin and H. Queffélec [50].

We mention that there is an active research effort on composition operators in the setting of the space $\mathscr{H}^2$. We refer the reader to the recent survey of Queffélec [68] and the references in it.

Analytic number theory and the Montgomery-Bourgain conjectures

The study of local embeddings of function spaces of Dirichlet series has mainly been motivated by the role Dirichlet series play in analytic number theory. As was briefly mentioned above, Lemma 2.2 was discovered by the analytic number theorist H. L. Montgomery who gave an argument based

[8]The relation $f(x) \lesssim g(x)$ is taken to mean that there exists some constant $C > 0$ such that $f(x) \leq Cg(x)$ for x in some specified range. In the corresponding way we define $f(x) \gtrsim g(x)$. If both hold, we say $f(x) \simeq g(x)$.

on the following inequality due to Montgomery and R. C. Vaughan[9] [59, p. 140].

Theorem(Montgomery and Vaughan 1994)**.** *Let $\lambda_1, \ldots, \lambda_n$ be distinct real numbers and set $\delta_n = \min_{m\neq n} |\lambda_n - \lambda_m|$. Then*

$$\sum_{\substack{n,m=1\\ n\neq m}}^{N} \frac{x_n y_m}{\lambda_n - \lambda_m} \leq \gamma_0 \left(\sum_{n=1}^{N} \frac{|x_n|^2}{\delta_n}\right)^{1/2} \left(\sum_{n=1}^{N} \frac{|y_n|^2}{\delta_n}\right)^{1/2}, \tag{1.8}$$

where[10] $\gamma_0 \leq 3.2$.

The advantage of Montgomery's argument is that he obtains explicit estimates of the constant C, not only in terms of the length of I, but also the degree of the Dirichlet polynomial. Montgomery's motivation was a series of conjectures that he gave on the growth of the quantities

$$\int_0^T \left|\sum_{n=1}^{N} a_n n^{-\mathrm{i}t}\right|^p \mathrm{d}t, \quad p \geq 1. \tag{1.9}$$

These conjectures were later reformulated by J. Bourgain [12] who conjectured that for $\epsilon > 0$, $\nu \in (1,2)$ and $T > N$ it holds that

$$\int_0^T \left|\sum_{n=N}^{2N} a_n n^{-\mathrm{i}t}\right|^{2\nu} \mathrm{d}t \lesssim N^\nu T^\epsilon (T + N^\nu) \max_{N\leq n\leq 2N} |a_n|^{2\nu}. \tag{1.10}$$

It is clear that Lemma 2.2 implies the estimate (1.10) for $\nu = 1$ since the constant of the lemma only depends on the length of the bounded interval I. We remark that the inequality (1.10) also holds for $p = 2\nu$ with $\nu \in \mathbb{N}$. This trivial extension may be seen either from a direct application of the inequality (1.10) in the case $\nu = 1$, or from Lemma 2.2, which has an analogue extension which we mention in the following subsection.

[9] We reproduce what is essentially this argument in chapter 3 where we state this inequality as lemma 3.13.

[10] Montgomery reports in his book [59, p. 145] that this estimate was given by A. Selberg.

If true in general, Montgomery showed that this conjecture would imply the density hypothesis on the zeroes of the Riemann zeta function. Also, Bourgain showed that a closely related conjecture implies the Kakeya conjecture.

The Dirichlet-Hardy spaces $\mathscr{H}^p$ and the embedding problem

For general $p \in [1, \infty)$ the Dirichlet-Hardy space $\mathscr{H}^p$ were defined by F. Bayart [3] to be the closure of the Dirichlet polynomials F in the norm

$$\|F\|_{\mathscr{H}^p} = \lim_{T\to\infty} \left(\frac{1}{2T} \int_{-T}^{T} \left| \sum_{n\in\mathbb{N}} a_n n^{-\mathrm{i}t} \right|^p \mathrm{d}t \right)^{1/p}. \tag{1.11}$$

Following [35, 3] we prefer not to define the spaces $\mathscr{H}^p$ directly by this norm, instead choosing a somewhat indirect approach that we indicate below.

The similarity between (1.9) and (1.11) is striking, however an exact connection has yet to be established. E.g. it is an open question if for Dirichlet polynomials and general $p \in [1, \infty)$ there exist constants $C_p > 0$ such that

$$\int_0^T \left| \sum_{n\in\mathbb{N}} a_n n^{-1/2-\mathrm{i}t} \right|^p \mathrm{d}t \le C_p \left\| \sum_{n\in\mathbb{N}} a_n n^{-s} \right\|_{\mathscr{H}^p}^p. \tag{1.12}$$

We remark that this inequality holds for $p = 2k$ with $k \in \mathbb{N}$. This is an immediate consequence of Lemma 2.2. Indeed, one simply uses the fact that $F \in \mathscr{H}^{2k}$ implies $F^k \in \mathscr{H}^2$ on both sides of the inequality.

The problem of establishing the inequalities (1.12) has become known as the $\mathscr{H}^p$ embedding problem[11]. The difficulty involved becomes apparent through the use of an idea due to H. Bohr. He observed [9] that it is possible to consider Dirichlet series as power series in infinite variables. Specifically, the idea is to identify the Dirichlet monomial p_n^{-s}, where p_n denotes the n'th prime number, with the n'th coordinate of the infinite dimensional torus

$$\mathbb{T}^\infty = \{(z_1, \dots) : z_j \in \mathbb{T}\}.$$

[11] See [74] for a discussion of the $\mathscr{H}^p$ embedding problem.

This results in the one to one Bohr correspondence

$$\mathcal{B} : \sum_{n\in\mathbb{N}} a_n n^{-s} \longmapsto \sum_{n\in\mathbb{N}} a_n z_1^{\nu_1} \cdots z_k^{\nu_k},$$

where $n = p_1^{\nu_1} \cdots p_k^{\nu_k}$ is the unique prime number factorisation of the integer n. Since $\mathbb{T}^\infty$ is a compact abelian group, it has a unique normalised Haar measure ρ. This allows one to define the spaces $L^p(\mathbb{T}^\infty)$ and their subspaces $H^p(\mathbb{T}^\infty)$. In particular, in [35, 3], it is established, using ergodic theory that for Dirichlet polynomials F it holds that[12]

$$\int_{\mathbb{T}^\infty} |\mathcal{B}F(\chi)|^p \mathrm{d}\rho(\chi) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |F(\mathrm{i}t)|^p \mathrm{d}t.$$

In this way $\mathscr{H}^p$ is realised as the inverse image of $H^p(\mathbb{T}^\infty)$ under the linear isometry $\mathcal{B}$. The point we want to make here is that one now sees that the norm (1.11) expresses an integral over the infinite dimensional torus, while (1.9) deals with an integral over a 1-dimensional complex curve on this huge domain.

Saksman and Seip [74] found an equivalent condition for the $\mathscr{H}^p$ embedding to hold for $p \geq 2$ using a a Fatou type of theorem for the spaces $H^p(\mathbb{T}^\infty)$. In order to state the condition we introduce some notation. For $\chi \in \mathbb{T}^\infty$ we let $\chi(p_n)$, where p_n is the n'th prime number, denote the n'th coordinate. Let $T_t(\chi) = (2^{-\mathrm{i}t}\chi(2), 3^{-\mathrm{i}t}\chi(3), 5^{-\mathrm{i}t}\chi(5), \ldots)$ denote the Kroenecker flow. The infinite dimensional polydisk is given by

$$\mathbb{D}^\infty = \{(z_1, z_2, \ldots) : z_n \in \mathbb{D}\}.$$

B. J. Cole and T. W. Gamelin [19] established that an element $f \in H^p(\mathbb{T}^\infty)$ has bounded point evaluations for[13] $z \in \mathbb{D}^\infty \cap \ell^2$, thereby giving sense to the space $H^p(\mathbb{D}^\infty)$ in analogy to the one dimensional case. What Saksman and Seip showed is that there exists boundary functions on the natural boundary of $\mathbb{D}^\infty \cap \ell^2$. To find the correct approach region set $b_\theta(\chi) = (2^{-\theta}\chi(2), 3^{-\theta}\chi(3), 5^{-\theta}\chi(5), \ldots)$. Now their Fatou type result may

[12] We give a more elementary proof due to Saksman and Seip in chapter 2.

[13] See Lemma 2.3 which we state below.

be stated as follows. Given $f \in H^p(\mathbb{D}^\infty)$ there exists $\widetilde{f} \in L^p(\mathbb{T}^\infty)$ such that

$$\widetilde{f}(\chi) = \lim_{\theta \to 1/2} f(b_\theta(\chi)), \quad \text{for almost every } \chi \in \mathbb{T}^\infty. \tag{1.13}$$

We may now formulate the following.

Theorem(Saksman and Seip 2008)**.** *Let $p \geq 2$. Then the inequality* (1.12) *holds if and only if for $f \in H^p(\mathbb{T}^\infty)$ and every $\chi \in \mathbb{T}^\infty$ it holds that*

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\widetilde{f}((T_t(\chi))|^p \mathrm{d}t = \|\widetilde{f}\|^p_{L^p(\mathbb{T}^\infty)}, \tag{1.14}$$

where $\widetilde{f}$ satisfies (1.13).

We remark that by the Birkhoff-Khinchin ergodic theorem [20, p. 11-12] the identity (1.14) always holds for almost every $\chi \in \mathbb{T}^\infty$. Moreover, for $p = 2$, Lemma 1.2 implies that (1.14) holds for all $\chi \in \mathbb{T}^\infty$. For $\chi = (1, 1, \ldots)$ the identity reduces to

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |F(1/2 + it)|^2 \mathrm{d}t = \sum_{n \in \mathbb{N}} |a_n|^2 n^{-1}.$$

This is a special case of a theorem of F. Carlson [16]. We give another equivalent condition for the $\mathscr{H}^p$ embedding problem in Theorem 4.17.

Some consequences of the identification of $\mathscr{H}^p$ with $H^p(\mathbb{T}^\infty)$

In operator theoretic terms the inequalities (1.12) say that the operator E_I defined on the Dirichlet polynomials by (1.6) extends to a bounded operator from $\mathscr{H}^p$ to $L^p(I)$. As we remarked, this is known to hold true only for $p = 2k$ where $k \in \mathbb{N}$. The following result by S. Ebenstein[14] [24] implies that there are no bounded projections from $L^p(\mathbb{T}^\infty)$ to $H^p(\mathbb{T}^\infty)$.

[14] We thank J. Marzo for pointing out this reference. Also, the proof of Ebenstein relies crucially on the fact that for $p \neq 2$ the norm of the Riesz projection on $L^p(\mathbb{T})$ is strictly greater than one. We thank H. Queffélec for pointing out that the reference Ebenstein gives does not seem to be accurate, and that the exact norms were computed in [41].

This explains why it is problematic to use interpolation theory[15] to deduce that E_I extends to a bounded operator from $\mathscr{H}^p$ to $L^p(I)$ for general $p \geq 2$ and thereby solving the $\mathscr{H}^p$ embedding problem.

Theorem (Ebenstein 1972). *Let $p \in [1, \infty)$. If $p \neq 2$ then $H^p(\mathbb{T}^\infty)$ is uncomplemented as a subspace of $L^p(\mathbb{T}^\infty)$.*

It is remarked in [74] that by similar arguments as may be used to prove Ebenstein's result, one may establish that in the natural duality the inclusion $\mathscr{H}^q \subset (\mathscr{H}^p)'$ is always strict when $p \neq 2$.

A positive consequence of the identification of $\mathscr{H}^p$ with $H^p(\mathbb{T}^\infty)$ is the following. It was shown by Cole and Gamelin [19] in the context of the spaces $H^p(\mathbb{T}^\infty)$, and applied to the spaces $\mathscr{H}^p$ by Bayart [3].

Lemma 2.3 (Cole and Gamelin 1985, Bayart 2002). *Let $p \in [1, \infty)$. Then the norm of the point evaluation in $\mathscr{H}^p$ at the point $s = \sigma + \mathrm{it}$ in $\mathbb{C}_{1/2}$ equals $\zeta(2\sigma)^{1/p}$.*

By the formula (1.3) we get a relation analogue to (1.4). Indeed, let $\omega_{\mathscr{H}^p}(s)$ and $\omega_{H^p}(s)$ denote the norms of the point evaluation functionals of $\mathscr{H}^p$ and $H^p(\mathbb{C}_{1/2})$, respectively, then

$$\omega_{\mathscr{H}^p}(s)^p = C\omega_{H^p}(s)^p + \mathcal{O}(1), \quad \sigma \to 1/2^+, \tag{1.15}$$

with the constant $C > 0$ depending on p. It is a curious fact that even though the norm (1.11) is evaluated by integral means over the imaginary axis, the elements in the closure of the $\mathscr{H}^p$ norm are in general only defined on the half-plane $\mathbb{C}_{1/2}$. We remark that this jump of $1/2$ appears several places in the theory of Dirichlet series and served as the theme of a paper by Konyagin and Queffélec [50].

[15] By this we mean the theory which includes the theorems of Marcinkiewicz and Riesz-Thorin.

McCarthy's spaces and the Dirichlet-Bergman spaces

On the unit disc, there is scale of spaces $D_\alpha(\mathbb{D})$ defined as the closure of the complex polynomials in the norm

$$\|\sum a_n z^n\|_{D_\alpha(\mathbb{D})} = \left(\sum_{n\in\mathbb{N}} |a_n|^2 (n+1)^\alpha\right)^{1/2}.$$

This scale includes the classical Bergman ($\alpha = -1$), Hardy ($\alpha = 0$) and Dirichlet ($\alpha = 1$) spaces.

In [58] McCarthy studied several Hilbert spaces of Dirichlet series. In particular, he defined a family of spaces that resemble the scale $D_\alpha(\mathbb{D})$. These are the spaces

$$\mathscr{H}^2_\alpha = \left\{\sum_{n\in\mathbb{N}} a_n n^{-s} : \sum_{n\in\mathbb{N}} |a_n|^2 \log^\alpha(n+1) < \infty\right\}, \quad \alpha \in \mathbb{R}.$$

By the Cauchy-Schwarz inequality the elements of these spaces are analytic on $\mathbb{C}_{1/2}$.

We establish local embeddings and local equivalences of reproducing kernels connecting the spaces $\mathscr{H}^2_\alpha$ to the classical counter-parts of the spaces $D_\alpha(\mathbb{D})$ on the half-plane $\mathbb{C}_{1/2}$. Let $\mathrm{d}m$ denote Lebesgue area measure. We denote these spaces by $D_\alpha(\mathbb{C}_{1/2})$. For $\alpha < 0$ these spaces are defined to be the functions analytic on $\mathbb{C}_{1/2}$ and finite in the norm

$$\|f\|^2_{D_\alpha} = \frac{1}{\pi}\int_{\mathbb{C}_{1/2}} |f(s)|^2 \left(\sigma - \frac{1}{2}\right)^{-\alpha-1} \mathrm{d}m(s).$$

For $0 < \alpha < 2$ we let $D_\alpha(\mathbb{C}_{1/2})$ be the Dirichlet-type of space of functions analytic in $\mathbb{C}_{1/2}$ such that[16] $f(\sigma) \to 0$ when $\sigma \to \infty$ and finite in the norm

$$\|f\|^2_{D_\alpha} = \frac{1}{\pi}\int_{\mathbb{C}_{1/2}} |f'(s)|^2 \left(\sigma - \frac{1}{2}\right)^{-\alpha+1} \mathrm{d}m(s).$$

[16] We make a technical adjustment to the norm in the case $\alpha = 1$ in chapter 2. We also note that it is possible to define these spaces for $\alpha \geq 2$ using higher order derivatives.

As another suggestion of a natural analogue to the spaces $D_\alpha(\mathbb{D})$, we introduce the family of Hilbert function spaces $\mathscr{D}_\alpha$. For $\alpha \in \mathbb{R}$ we set

$$\mathscr{D}_\alpha = \left\{ \sum_{n\in\mathbb{N}} a_n n^{-s} : \sum_{n\in\mathbb{N}} |a_n|^2 d(n)^\alpha < \infty \right\}.$$

The function $d(n)$ is the divisor function defined by[17] $d(n) = \sum_{k|n} 1$. Since $d(n) = \mathcal{O}(n^\epsilon)$ for all $\epsilon > 0$ it follows by the Cauchy-Schwarz inequality that these Dirichlet series converge absolutely on the half-plane $\mathbb{C}_{1/2}$. In the limiting case $\alpha \to \infty$, it is natural to define the space

$$\mathscr{D}_\infty = \left\{ \sum_{p\in\mathbb{P}} a_p p^{-s} : \sum_{p\in\mathbb{P}} |a_p|^2 < \infty \right\}.$$

We observe that the Bohr correspondence extends to an isometric isomorphism from these spaces to the spaces $D_\alpha(\mathbb{D}^\infty)$. In terms of reproducing kernels and local embeddings, we show that while the natural counter-part of the space $\mathscr{H}^2_\alpha$ on $\mathbb{C}_{1/2}$ is $D_\alpha(\mathbb{C}_{1/2})$, the natural counter-part of $\mathscr{D}_\alpha$ is the space $D_{1-2^{-\alpha}}(\mathbb{C}_{1/2})$. In the same sense $\mathscr{D}_\infty$ corresponds to[18] $D_1(\mathbb{C}_{1/2})$. We remark that by the irregularity of the function $d(n)$ it follows that if $\alpha \neq 0$ then neither $\mathscr{D}_\alpha \subset \mathscr{H}^2_\alpha$ nor $\mathscr{H}^2_\alpha \subset \mathscr{D}_\alpha$ holds.

Helson's conjecture

Let $\mathcal{K}$ denote the projective tensor product space $\mathscr{H}^2 \otimes \mathscr{H}^2$. The elementary tensors are the products fg where $f, g \in \mathscr{H}^2$. The tensor space is then the closure of finite sums of these elementary tensors in the norm

$$\|F\|_{\mathcal{K}} = \inf \left\{ \sum_{\text{finite}} \|f_i\|_{\mathscr{H}^2} \|g_i\|_{\mathscr{H}^2} : F = \sum_{\text{finite}} f_i g_i,\ f_i, g_i \in \mathscr{H}^2 \right\}.$$

[17] The expression $k|n$ should be read as k divides n. We treat the divisor function in more detail on page 51.

[18] From this point on by the space $D_{1-2^{-\alpha}}(\mathbb{C}_{1/2})$ for $\alpha = \infty$ we mean the space $D_1(\mathbb{C}_{1/2})$.

In other words, the infimum is taken over all representations of F as a finite sum of elementary tensors. It follows from the definition that $\mathcal{K} \subset \mathscr{H}^1$. We remark that the space $\mathcal{K}$ appears naturally in chapter 3.

Helson conjectured in [38] that $\mathcal{K} = \mathscr{H}^1$. An equivalent statement of this conjecture is that the Nehari problem for $H^2(\mathbb{T}^\infty)$ has a positive solution. I.e. given a bounded form[19]

$$\Big((a_n),(b_n)\Big) \in \ell^2 \times \ell^2 \longmapsto \sum_{j,k\in\mathbb{N}} a_j b_k \varrho_{jk},$$

then the function

$$\sum_{n\in\mathbb{N}} \varrho_n z^{\nu_1} \cdots z^{\nu_r}$$

would be bounded, i.e. belong to[20] $H^\infty(\mathbb{T}^\infty)$. The converse always holds. Helson solved the Nehari problem in the special case of Hilbert-Schmidt operators, i.e. under the additional hypothesis that $\sum |\varrho_n|^2 < \infty$, in [40] (see also [39]). Helson obtained this result by establishing the following remarkable property of the space $\mathscr{D}_{-1}$, improving upon earlier results by Bayart [3].

Theorem (Helson 2005)**.** *If $F \in \mathscr{H}^1$ then $\|F\|_{\mathscr{D}_{-1}} \leq \|F\|_{\mathscr{H}^1}$.*

In other words, the space $\mathscr{H}^1$ embeds contractively in the space $\mathscr{D}_{-1}$. The crucial observation is that this inequality holds in one dimension. This is due to D. Vukotic [81].

Theorem (Vukotic 2003)**.** *Suppose $f \in H^1(\mathbb{D})$. Then*

$$\left(\int_{\mathbb{D}} |f(z)|^2 \frac{\mathrm{d}x\mathrm{d}y}{\pi}\right)^{1/2} \leq \int_{\mathbb{T}} |f(\mathrm{e}^{\mathrm{i}\theta})| \frac{\mathrm{d}\theta}{2\pi}.$$

[19] These are the Hankel forms on $H^2(\mathbb{T}^\infty)$. On $H^2(\mathbb{T})$ they are given by $\sum_{j,k\in\mathbb{N}} a_j b_k \varrho_{j+k}$. In one dimension it is a well-known theorem by Z. Nehari [62] and in two dimensions the problem was solved by S. Ferguson and M. Lacey [27].

[20] By the Bohr correspondence the space $H^\infty(\mathbb{T}^\infty)$ is identified with the space $\mathscr{H}^\infty$ of all Dirichlet series uniformly bounded on the half-plane $\sigma > 0$. It was shown in [35] that this space is the multiplier algebra of $\mathscr{H}^2$. In [3] this was generalised to all $p \geq 1$. We point out that that this is another manifestation of the jump of 1/2 mentioned on page 11. Also, in [77] the local interpolating sequences were determined. Since we present no new results concerning this space we do not discuss it further.

We remark that Seip [78] recently observed the following local theorem.

Theorem (Seip 2009)**.** *Suppose* $\alpha > 0$*. For* $F \in \mathscr{D}_\alpha$ *and every bounded interval* I *there exists a constant* $C > 0$ *depending only on the length of* I*, such that*

$$\int_I |F(\sigma + \mathrm{i}t)|^{2^{\alpha+1}} \mathrm{d}t \leq C\|F\|_{\mathscr{D}_\alpha}^2.$$

Moreover, this embedding is sharp.

This follows from a duality argument and a classical inequality due to Hardy and Littlewood which says that for $g \in L^p(\mathbb{T})$ and $p \in (1, 2)$ then

$$\sum_{k \in \mathbb{Z}} |\hat{g}(k)|^2 k^{-(p/2-1)} \leq \|g\|_p^2.$$

1.3 Boundary functions

Suppose $p \in [1, \infty)$ and $v \in L^p(\mathbb{R})$ is a real valued function. It is well-known that there exists $f \in H^p(\mathbb{C}_{1/2})$ such that the real parts of the non-tangential limits of f on the abscissa $\sigma = 1/2$ coincide with v almost everywhere. One such function is given by

$$f(s) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{v(t)}{s - \mathrm{i}t - \frac{1}{2}} \mathrm{d}t.$$

In chapter 3 we investigate the corresponding problem for the spaces $\mathscr{H}^p$, $\mathscr{H}_\alpha^2$ and $\mathscr{D}_\alpha$.

The main result deals with the space $\mathscr{H}^2$ and is given as Theorem 3.5. In this section we present it as theorems 3.5a and 3.5b. We give explicit quantitative estimates in theorems 3.10a and 3.10b. We compare this to a result due to W. H. J. Fuchs. In addition, we obtain similar results for the spaces $\mathscr{H}_\alpha^2$ and $\mathscr{D}_\alpha$.

Boundary functions for $\mathscr{H}^2$

The question of finding $F \in \mathscr{H}^2$ that matches given real L^2 boundary data over bounded intervals is in broad strokes answered in the following theorem.

Theorem 3.5a (Olsen and Saksman 2009)**.** *Given a bounded interval I and a real valued function $v \in L^2(I)$ there exists $F \in \mathscr{H}^2$ such that the real part of the non-tangential values of F on the abscissa $\sigma = 1/2$ coincide with v for almost every $t \in I$.*

Let $\mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$. The theorem may be considered as a dual statement of Lemma 2.2. Indeed, we prove it by establishing a lower norm bound of the adjoint of what could be called the real embedding operator, namely

$$R_I : (a_n)_{n\in\mathbb{Z}^*} \in \ell^2 \longmapsto \chi_I \sum_{n\in\mathbb{N}} \frac{a_n n^{-\mathrm{i}t} + a_{-n} n^{\mathrm{i}t}}{\sqrt{n}} \in L^2(I). \tag{1.16}$$

It is not hard to see that such a lower bound is reasonable. Indeed, for $g \in L^2(I)$ one has

$$\|R_I^* g\|^2_{\mathscr{H}^2} = 2\pi \sum_{n\in\mathbb{N}} \frac{|\hat{g}(\log n)|^2 + |\hat{g}(-\log n)|^2}{n}, \tag{1.17}$$

where

$$\mathcal{F} : g \longmapsto \hat{g}(\xi) = \frac{1}{\sqrt{2\pi}} \int_I g(\tau)\mathrm{e}^{-\mathrm{i}\tau\xi}\mathrm{d}\tau$$

is the Fourier transform of g. The expression (1.17) is a Riemann sum for the $L^2(\mathbb{R})$ integral of $\hat{g}(\xi)$, hence it approximates the quantity $\|\hat{g}\|^2_{L^2(\mathbb{R})}$, which by Plancherel's formula equals $\|g\|^2_{L^2(I)}$. It is worth noting that while the embedding of Lemma 2.2 extends to the spaces $\mathscr{H}^{2k}$ for $k \in \mathbb{N}$, Theorem 3.5a may be extended to the spaces[21] $\mathscr{H}^{2/k}$.

We make the following definition. Given a real valued function $v \in L^2(I)$ and a bounded interval $I \subset \mathbb{R}$ we set

$$\mathscr{V}_{v,I} = \left\{ F \in \mathscr{H}^2 : \lim_{\sigma\to 1/2^+} \operatorname{Re} F(\sigma + \mathrm{i}t) = v(t) \text{ for almost every } t \in I \right\}.$$

Here $\operatorname{Re} F$ denotes the real part of F. By the theory of convex sets, it is clear that there exists a unique smallest element in $\mathscr{V}_{v,I}$. By the Schwarz

[21] The extension to the space $\mathscr{H}^1$ is given in Corollary 3.8. As for the case $p < 1$ we refer to the remark at the beginning of section 4.6.

reflection principle we deduce the following consequence of the previous theorem. For a bounded interval I we let $\mathbb{C}_I = \{s \in \mathbb{C} : \mathrm{i}(1/2-s) \notin \mathbb{R}\backslash I\}$, i.e. the complex plane with two rays on the abscissa $\sigma = 1/2$ removed. We denote by $\mathrm{Hol}(\mathbb{C}_I)$ the set of functions holomorphic in $\mathbb{C}_I$.

Theorem 3.5b (Olsen and Saksman 2009)**.** *Let $I \subset \mathbb{R}$ be a bounded interval and $f \in H^2(\mathbb{C}_{1/2})$. Set $v = \chi_I \operatorname{Re} f(1/2+\mathrm{i}t)$ for $t \in I$. For any element $F \in \mathscr{V}_{v,I}$ there exists $\phi \in \mathrm{Hol}(\mathbb{C}_I)$ such that $f = F + \phi$. Moreover, there is a constant $C_I > 0$ that only depends on the length of I such that the minimal element $F \in \mathscr{V}_{v,I}$ satisfies*

$$\|F\|^2_{\mathscr{H}^2} \leq C_I \|f\|^2_{H^2(\mathbb{C}_{1/2})}.$$

In addition, for every set $\Omega \subset \mathbb{C}_I$ at a positive distance from $\mathbb{C}\backslash\mathbb{C}_I$, the minimal element satisfies

$$\|\phi\|^2_{L^\infty(\Omega)} \leq D_{\Omega,I}\left(1 + \frac{C_I}{|I|}\right)\|f\|^2_{H^2},$$

where

$$D_{\Omega,I} \leq \sup_{s\in\Omega}\left|\frac{1+2s}{2\pi}\right|^2 \int_{\mathbb{R}\backslash I} \frac{\mathrm{d}t}{|s-\frac{1}{2}-\mathrm{i}t|^2}.$$

A dual statement in terms of frames and explicit estimates

We consider the sequence of vectors

$$\mathscr{G} = \left(\ldots, \frac{(-n)^{\mathrm{i}t}}{\sqrt{(-n)}}\ldots, 1, \ldots, \frac{n^{-\mathrm{i}t}}{\sqrt{n}}\ldots\right),$$

where n is understood to run through $\mathbb{Z}^*$, to be elements of the space $L^2(I)$ by multiplying each with the characteristic function χ_I. We recall that a sequence (f_n) in a Hilbert space H is called a frame [22] for H if for every $f \in H$ there exists constants $A, B > 0$ such that

$$A\|f\|^2_H \leq \sum |\langle f|f_n\rangle|^2 \leq B\|f\|^2_H.$$

[22] We give more information on frames in chapter 3.

Hence, to say that (1.17) is bounded above and below in norm is to say that $\mathscr{G}$ is a frame for $L^2(I)$. Using the general theory of frames, we are able to express the unique smallest element of $\mathscr{V}_{I,v}$ explicitly in terms of the operator $R_I R_I^*$, where R_I is defined as in (1.16). This yields the following.

Theorem 3.10a (Olsen and Saksman 2009)**.** *Let $I \subset \mathbb{R}$ be a bounded interval. Then there exist constants $c_I, C_I > 0$ such that for real valued $v \in L^2(I)$ the set $\mathscr{V}_{v,I}$ admits a unique smallest element $F \in \mathscr{H}^2$ satisfying*

$$c_I \|v\|^2_{L^2(I)} \leq \|F\|^2_{\mathscr{H}^2} \leq C_I \|v\|^2_{L^2(I)}.$$

By the theory of entire functions we are able to estimate the upper and lower frame bounds of the sequence $\mathscr{G}$. This yields the following asymptotic estimate in terms of $\mathscr{V}_{v,I}$.

Theorem 3.10b (Olsen and Saksman 2009)**.** *The constants c_I, C_I of Theorem 3.10a satisfy the following. For every $\epsilon > 0$ there exist constants such that for $|I| \geq 1$ we have*

$$|I|^{(1-\epsilon)\frac{|I|}{\pi}\log \pi} \lesssim C_I \lesssim |I|^{(1+\epsilon)\frac{6|I|}{\pi}\log 2}.$$

For all I it holds that

$$\frac{1}{|I|+d} \leq c_I \leq \frac{1}{|I|}.$$

Also,

$$\lim_{|I|\to 0} C_I = \frac{2}{\pi},$$

and

$$\frac{1}{\pi} \leq \liminf_{|I|\to 0} c_I, \quad \limsup_{|I|\to\infty} c_I \leq \frac{2}{\pi}.$$

The computation concerning C_I may be compared to a quantitative version of the uncertainty principle[23] due to Fuchs [30].

[23] See also [80] for a survey on this and related topics.

Theorem(Fuchs 1964). *Given $\epsilon > 0$ there exists a constant only depending on ϵ such that for $T > 0$ and $f \in L^2(-T,T)$ then*

$$\int_{(-K,K)^C} |\hat{f}(\xi)|^2 \mathrm{d}\xi \gtrsim \mathrm{e}^{-(1+\epsilon)TK} \|f\|^2_{L^2(-T,T)},$$

and this is best possible.

The comparison becomes clearer when we state explicitly that as an intermediate step in proving the asymptotic bounds for C_I, we show that given $\epsilon > 0$ and $T > 1$ there exists a constant only depending on ϵ such that for $f \in L^2(-T,T)$ we get

$$\sum_{\log n \geq \log T} \frac{|\hat{f}(\log n)|^2 + |\hat{f}(-\log n)|^2}{n} \gtrsim T^{-c(1+\epsilon)T} \|f\|^2_{L^2(I)},$$

with $c \in (\frac{2\log\pi}{\pi}, \frac{12\log 2}{\pi}) = (0.729\ldots, 2.290\ldots)$. The left-hand side of this inequality is a Riemann sum for the integral

$$\int_{(-\log T, \log T)^C} |\hat{f}(\xi)|^2 \mathrm{d}\xi.$$

So by Fuchs' result, a natural conjecture would be that $C_I \simeq |I|^{|I|}$ as $|I| \to \infty$.

Boundary functions for the spaces $\mathscr{H}^2_\alpha$ and $\mathscr{D}_\alpha$

Let the spaces $D_\alpha(\mathbb{C}_{1/2})$ be as in section 1.2. For the spaces $\mathscr{H}^2_\alpha$ we prove the following.

Theorem 3.18 (Saksman and Olsen 2009). *Let $I \subset \mathbb{R}$ be a bounded and open interval and $\alpha < 2$. Then for every $f \in D_\alpha(\mathbb{C}_{1/2})$ there exists an $F \in \mathscr{H}^2_\alpha$ such that $f - F$ continues analytically to all of $\mathbb{C}_I$ with $\mathrm{Re}(f - F)(1/2 + \mathrm{i}t) = 0$ on I. There exists a unique $F \in \mathscr{H}^2_\alpha$ of minimal norm satisfying this. Moreover, there exists a constant $C > 0$ depending only on α and the length of I such that the minimal element satisfies*

$$\|F\|^2_{\mathscr{H}^2_\alpha} \leq C \|f\|^2_{D_\alpha(\mathbb{C}_{1/2})}.$$

The analogue theorem for the spaces $\mathscr{D}_\alpha$ is as follows.

Theorem 3.22. *Let $I \subset \mathbb{R}$ be an open and bounded interval and $\alpha \in \mathbb{R} \cup \{+\infty\}$. Then for every $f \in D_{1-2^{-\alpha}}(\mathbb{C}_{1/2})$ there exists an $F \in \mathscr{D}_\alpha$ such that $f - F$ continues analytically to all of $\mathbb{C}_I$ with $\mathrm{Re}(f-F)(1/2+\mathrm{i}t) = 0$ on I. There exists a unique $F \in \mathscr{D}_\alpha$ of minimal norm satisfying this. Moreover, there exists a constant C depending only on α and the length of I such that he minimal element satisfies*

$$\|F\|^2_{\mathscr{D}_\alpha} \leq C\|f\|^2_{D_{1-2^{-\alpha}}(\mathbb{C}_{1/2})}.$$

1.4 Interpolating sequences and Carleson measures

Chapter 4 deals with interpolating sequences and Carleson measures for the function spaces of Dirichlet series with emphasis on the spaces $\mathscr{H}^p$. In this section we recall the relevant definitions and some results in the classical setting before moving on to describe the results we obtain.

Definitions and some classical results

Let H be a Hilbert space of functions on $\mathbb{C}_{1/2}$. A positive measure μ on $\mathbb{C}_{1/2}$ is called a Carleson measure for H if there is a constant $C > 0$ such that

$$\int |f(s)|^2 \mathrm{d}\mu(s) \leq C\|f\|^2_H, \quad \text{for all } f \in H.$$

The smallest such number $C > 0$ is called the norm of the Carleson measure and is denoted by $\|\mu\|_{\mathrm{CM}(H)}$. Assume, in addition, that H admits a reproducing kernel k_w for every $w \in \mathbb{C}_{1/2}$. Then a sequence $S = (s_n)$ of points in $\mathbb{C}_{1/2}$ is called a (universal) interpolating sequence if the following operator is (bounded and) onto ℓ^2,

$$f \in H \longmapsto \left(\frac{f(s_n)}{\|k_{s_n}\|_H} \right).$$

Note that if the mapping is bounded, then by the open mapping theorem there exists a constant $C > 0$ such that for all sequences $(c_n/\|k_{s_n}\|_H) \in \ell^2$

there is a function $f \in H$ that interpolates $f(s_n) = c_n$ with $\|f\|_H \leq C\|(c_n/\|k_{s_n}\|_H)\|_H$. The smallest such constant is called the constant of interpolation for the sequence S. Note that the of the norms of Carleson measures and constants of interpolating sequences remain true in the various Banach space settings we introduce below.

For the Banach spaces in the scale $H^p(\mathbb{C}_{1/2})$ these definitions are extended as follows. Recall that $\omega_{H^p}(s)$ denotes the norm of the bounded point evaluation at the point $s \in \mathbb{C}_{1/2}$. We say that a sequence of distinct points $S = (s_n)$ in $\mathbb{C}_{1/2}$ is a (universal) interpolating sequence for the space $H^p(\mathbb{C}_{1/2})$ if the operator defined by $f \in H^p(\mathbb{C}_{1/2}) \mapsto (f(s_n)/\omega_{H^p}(s_n))$ is (bounded and) onto the space ℓ^p. Moreover, we say that a positive measure $\mathrm{d}\mu$ on $\mathbb{C}_{1/2}$ is a Carleson measure for the space $H^p(\mathbb{C}_{1/2})$ if there exists a constant $C > 0$ such that for all $f \in H^p(\mathbb{C}_{1/2})$ it holds that $\int_{\mathbb{C}_{1/2}} |f(s)|^p \mathrm{d}\mu(s) \leq C\|F\|_{H^p}^p$.

Finally, for the spaces $\mathscr{H}^p$ these the definitions are extended in the same way as for the spaces $H^p(\mathbb{C}_{1/2})$. We say that $S = (s_n)$ is a (universal) interpolating sequence for $\mathscr{H}^p$ if the operator sending $F \in \mathscr{H}^p \mapsto (F(s_n)/\omega_{\mathscr{H}^p}(s))$ is (bounded and) onto ℓ^p. We say that a positive measure μ on $\mathbb{C}_{1/2}$ is a Carleson measure for $\mathscr{H}^p$ if there exists some constant $C > 0$ such that $\int |F(s)|^p \mathrm{d}\mu(s) \leq C\|F\|_{\mathscr{H}^p}^p$ for all $F \in \mathscr{H}^p$.

Note that we call a Carleson measure local if it has bounded support, and we call a sequence S local if it is bounded. The Carleson measures for $H^p(\mathbb{C}_{1/2})$ were characterised by Carleson [14].

Lemma 4.1 (Carleson 1962). *Let $p \in [1, \infty)$. A positive measure μ is a Carleson measure for the space $H^p(\mathbb{C}_{1/2})$ if there exists $C > 0$ such that for every square $Q \subset \mathbb{C}_{1/2}$ it holds that*

$$\mu(Q) \leq C|Q|,$$

where $|Q|$ denotes the length of a side of the square Q.

The interpolating sequences for the spaces $H^p(\mathbb{C}_{1/2})$ were characterised by H. S. Shapiro and A. L. Shields [79] in their generalisation of Carleson's classical interpolation theorem [13].

Lemma 4.2 (Carleson 1958, Shapiro and Shields 1961)**.** *Let $p \in [1, \infty)$. A sequence $S = (s_n)$, where $s_n = \sigma_n + \mathrm{i}t_n$, is a interpolating sequence for $H^p(\mathbb{C}_{1/2})$ if and only the measure $\mu = \sum \delta_{s_n}(2\sigma_n - 1)$ is a Carleson measure for $H^p(\mathbb{C}_{1/2})$ and if there is a number $\eta > 0$ such that*

$$\inf_{n \neq m} \left| \frac{s_n - s_m}{s_n + \bar{s}_m - 1} \right| \geq \eta.$$

Note that an interpolating sequence $S = (s_n)$ for $H^p(\mathbb{C}_{1/2})$ is universal if and only if the measure

$$\mu_S = \sum \delta_{s_n}(2\sigma_n - 1) \tag{1.18}$$

is a Carleson measure for $H^p(\mathbb{C}_{1/2})$. As a consequence, the interpolating and universal interpolating sequences for the spaces $H^p(\mathbb{C}_{1/2})$ coincide. Moreover, it is clear that both the Carleson measures and interpolating sequences are the same for all the spaces $H^p(\mathbb{C}_{1/2})$.

Local interpolation in the spaces $\mathscr{H}^2$, $\mathscr{H}^2_\alpha$ and $\mathscr{D}_\alpha$

The following theorem says that Lemma 4.2 completely characterises the local interpolating sequences of the space $\mathscr{H}^2$.

Theorem 4.3 (Olsen and Seip 2008)**.** *Suppose S is a bounded sequence of distinct points from $\mathbb{C}_{1/2}$. Then S is an interpolating sequence for $\mathscr{H}^2$ if and only if it is an interpolating sequence for $H^2(\mathbb{C}_{1/2})$.*

The essential ingredient of the proof is the following lemma proved by R. P. Boas [6]. By this lemma the theorem on local interpolation follows readily once the local embedding of $\mathscr{H}^2$ into $H^2(\mathbb{C}_{1/2})$ and the local equivalence of the reproducing kernels of the spaces $H^2(\mathbb{C}_{1/2})$ and $\mathscr{H}^2$ have been established. These were given as Lemma 2.2 and (1.4), respectively.

Lemma 4.4 (Boas 1941)**.** *Suppose (f_n) is a sequence of unit vectors in a Hilbert space H. Then the moment problem $\langle f | f_n \rangle_H = a_n$ has a solution f in H for every sequence (a_n) in ℓ^2 if and only if there is a positive constant $C > 0$ such that*

$$\left\| \sum c_j f_j \right\| \geq C \|(c_j)\|_{\ell^2}$$

for every finite sequence of scalars (c_j).

The advantage of this approach is that the proof is readily generalised to other Hilbert function spaces. A problem, however, is that by arguing by duality in this way, it is difficult to extend the result to the spaces $\mathscr{H}^p$. We discuss an alternative proof in the next subsection.

We are able to establish the corresponding local equivalences between the spaces $\mathscr{H}^2_\alpha$ for $\alpha \leq 1$ and the classical scale of spaces $D_\alpha(\mathbb{C}_{1/2})$. This yields the following more general version of the previous theorem.

Theorem 4.5 (Olsen and Seip 2008)**.** *Suppose* S *is a bounded sequence of distinct points from* $\mathbb{C}_{1/2}$ *and assume* $\alpha \leq 1$. *Then* S *is a (universal) interpolating sequence for* $\mathscr{H}^2_\alpha$ *if and only if it is a (universal) interpolating sequence for* $D_\alpha(\mathbb{C}_{1/2})$.

Finally, since we are able to make the same connections between $\mathscr{D}_\alpha$ and $D_{1-2^{-\alpha}}(\mathbb{C}_{1/2})$ we get the following.

Theorem 4.6. *Suppose* S *is a bounded sequence of distinct points from* $\mathbb{C}_{1/2}$ *and* $\alpha \in \mathbb{R} \cup \{+\infty\}$. *Then* S *is a (universal) interpolating sequence for* $\mathscr{D}_\alpha$ *if and only if it is an interpolating sequence for* $D_{1-2^{-\alpha}}(\mathbb{C}_{1/2})$.

We remark that in the case of bounded interpolating sequences for $H^2(\mathbb{C}_{1/2})$, there is no reason to make a distinction between interpolating sequences and universal interpolating sequences. The same holds true for $D_\alpha(\mathbb{C}_{1/2})$ when $\alpha < 0$. However, for $D_\alpha(\mathbb{C}_{1/2})$ with $0 < \alpha \leq 1$ this is no longer the case [5], [56].

Also, we remark that there exist geometric descriptions of the (universal) interpolating sequences for all $\alpha \leq 1$. For $\alpha < 0$, Beurling-type density theorems were proved by Seip [75]. Descriptions in terms of Carleson measures were found by W. Cohn in the case $0 < \alpha < 1$ [18] and independently by C. Bishop and by D. Marshall and C. Sundberg in the case $\alpha = 1$ [5], [56]. For further information, we refer to the monograph by Seip [76].

Constructive proof for $\mathscr{H}^2$

In addition to the proof of Theorem 4.3 indicated in the previous section, we give a proof in the case $\mathscr{H}^2$ which is more constructive and which does not use duality in the construction itself. For $f \in H^2(\mathbb{C}_{1/2})$ it is known by the Paley-Wiener theorem that there exists $g \in L^2(0,\infty)$ such that

$$f(s) = \int_0^\infty g(\xi) \mathrm{e}^{-(s-1/2)\xi} \mathrm{d}\xi. \tag{1.19}$$

This representation may be related to the Dirichlet series

$$F(s) = \sum_{n\in\mathbb{N}} \int_0^\infty g(\xi) n^{-(s-1/2)} \mathrm{d}\xi.$$

Using this idea we are able to construct a sequence of Dirichlet series converging in $\mathscr{H}^2$ to a solution of any local interpolation problem solvable in $H^2(\mathbb{C}_{1/2})$.

Interpolating sequences for the spaces $\mathscr{H}^p$

As we have indicated, for general $p \in [1,\infty)$ it seems difficult to give the same local characterisation as in the case $p = 2$. (Another indication is given in Theorem 4.17 below.) However, we are able to get the following general necessary condition.

Theorem 4.8 (Olsen and Saksman 2009)**.** *Let $p \in [1,\infty)$ and assume that S is a sequence of distinct points in $\mathbb{C}_{1/2}$. If S is a universal interpolating sequence for $\mathscr{H}^p$ then it is an interpolating sequence for $H^p(\mathbb{C}_{1/2})$.*

The idea of the proof is straight-forward. It is simply to show that both conditions of Lemma 4.2 hold for such a sequence S.

It is clear, however, that the condition of Theorem 4.8 is by no means sufficient. To illustrate this point, we give an example by constructing the following sequence. For each positive integer j pick points equi-distributed on the line segment $\sigma = 1/2 + 2^{-j}$, $0 \le t \le 1$, i.e., choose

$$s_{j,l} = \frac{1}{2} + 2^{-j} + \mathrm{i}\frac{l}{j}, \quad l = 1, 2, ..., j.$$

Then Carleson's theorem along with our Theorem 4.3 shows that $(s_{j,l})$ is an interpolating sequence for $\mathscr{H}^2$. In particular, the measure given by (1.18) is a Carleson measure for both $\mathscr{H}^2$ and $H^2(\mathbb{C}_{1/2})$. Now if we move the points vertically and far apart, this may fail in the space $\mathscr{H}^2$. This is a consequence of the almost periodicity of $t \mapsto \zeta(\sigma + \mathrm{i}t)$. If we measure the distance between two points in terms of the angle between the corresponding reproducing kernels, this almost periodicity implies that points that are far apart in the hyperbolic sense of the half-plane may be arbitrarily close in the geometry induced by $\mathscr{H}^2$.

The question still remains if the necessary condition of Theorem 4.8 is also sufficient in the case of local interpolating sequences. As we saw this was the case for $\mathscr{H}^2$. In fact, we may deduce the following from the theorem for $\mathscr{H}^2$ using the fact that $F \in \mathscr{H}^2$ implies $F^2 \in \mathscr{H}^1$.

Theorem 4.12 (Olsen and Saksman 2009)**.** *Let S be a bounded sequence of distinct points in $\mathbb{C}_{1/2}$. If S is interpolating for H^1 then it is also interpolating for $\mathscr{H}^1$.*

However, this theorem does not give a complete characterisation for the space $\mathscr{H}^1$, as we do not know if the sequence will be a universal interpolating sequence for $\mathscr{H}^1$, and therefore does not imply a solution of the $\mathscr{H}^1$ embedding problem.

By what amounts to a third proof of the local interpolation theorem of the space $\mathscr{H}^2$ we establish the following theorem.

Theorem 4.13 (Olsen and Saksman 2009)**.** *Let S be a bounded sequence of distinct points in $\mathbb{C}_{1/2}$. Suppose that for all $k \in \mathbb{N}$ and open intervals $I \subset \mathbb{R}$ there exists $C_k > 0$, depending on $k \in \mathbb{N}$, such that the following holds: Given $f \in H^{2k}(\mathbb{C}_{1/2})$ there exists $F \in \mathscr{H}^{2k}$ such that $F - f \in \mathrm{Hol}(\mathbb{C}_I)$ and $\|F\|_{\mathscr{H}^{2k}} \leq C_k \|f\|_{H^{2k}}$. Under these assumptions it holds that if S is an interpolating sequence for the spaces $H^p(\mathbb{C}_{1/2})$ then S is an interpolating sequence for the spaces $\mathscr{H}^q$ for all $q \in \mathbb{Q}$.*

The proof of this theorem has one novel feature compared to the previous proofs of the local interpolation type theorems. This is that the duality arguments are contained in the function theoretic statement on matching boundary values.

The space $\mathcal{K}$

Let $\mathcal{K}$ denote the projective the tensor product space $\mathscr{H}^2 \otimes \mathscr{H}^2$ defined on page 13. In the proof of Theorem 4.12 we note that we actually solve the boundary value problem using functions from the space $\mathcal{K}$.

Theorem 4.16 (Olsen and Saksman 2009)**.** *Let S be a bounded sequence. Then S is an interpolating sequence for $\mathcal{K}$ if and only it is interpolating for H^1. Moreover, the local interpolating and universal interpolating sequences of $\mathcal{K}$ coincide.*

We point out that the definition of interpolating sequences for $\mathcal{K}$ is defined in the same way as for the spaces $\mathscr{H}^1$ and $H^1(\mathbb{C}_{1/2})$.

Carleson measures and the embedding problem for $\mathscr{H}^p$

For a Dirichlet polynomial F write

$$\lim_{\sigma \to 1/2^+} \int_I |F(\sigma + \mathrm{i}t)|^p \mathrm{d}t = \lim_{\sigma \to 1/2^+} \int_{\mathbb{C}_{1/2}} |F(s)|^p \mathrm{d}\mu_\sigma(s),$$

where μ_σ denotes the singular measure on $\mathbb{C}_{1/2}$ with support on the segment $\sigma + \mathrm{i}I$ of the complex plane. It is clear that the embedding holds for the space $\mathscr{H}^p$ if and only if the family of measures $\{\mu_\sigma\}$ is uniformly bounded as Carleson measures on $\mathscr{H}^p$ for $\sigma \in [1/2, 1)$. By definition, this family of measures has this property as Carleson measures for the space $H^p(\mathbb{C}_{1/2})$. We obtain the following result.

Theorem 4.17 (Olsen and Saksman 2009)**.** *Let $p \in [1, \infty)$. Then the following statements are equivalent.*

(a) For every bounded interval $I \subset \mathbb{R}$ there exists a constant $C > 0$ such that for all finite sequences (a_n) of complex numbers it holds that

$$\int_I \left| \sum a_n n^{-\frac{1}{2} - \mathrm{i}t} \right|^p \mathrm{d}t \le C \left\| \sum a_n n^{-s} \right\|_{\mathscr{H}^p}^p .$$

(b) Every local Carleson measure for $H^p(\mathbb{C}_{1/2})$ is also a Carleson measure for $\mathscr{H}^p$.

(c) *There exists a constant $D > 0$ such that every local Carleson measure for $H^p(\mathbb{C}_{1/2})$ of the form*

$$\mu_S = \sum \delta_{s_n}(2\sigma_n - 1), \quad s_n = \sigma_n + \mathrm{i}t_n,$$

is also a Carleson measure for $\mathscr{H}^p$ with

$$\int |F(s)|^p \mathrm{d}\mu_S(s) \leq D\|\mu\|_{\mathrm{CM}^p(H^p)}\|F\|^p_{\mathscr{H}^p} \quad \forall F \in \mathscr{H}^p.$$

1.5 A class of modified zeta functions

In chapters 5 and 6 we look at the behaviour of the K-zeta functions near the point $s = 1$. Recall that these are the modified zeta functions given by

$$\zeta_K(s) = \sum_{n \in K} \frac{1}{n^s}, \quad K \subset \mathbb{N}.$$

The infinite series defining these functions converge absolutely on the half-plane $\sigma > 1$. We study them by considering the operator[24]

$$\mathscr{Z}_{K,I} : g \in L^2(I) \longmapsto \lim_{\delta \to 0} \frac{\chi_I}{\pi}(g * \operatorname{Re}\zeta_{K,1+\delta}) \in L^2(I),$$

where $I \subset \mathbb{R}$ is an interval of the form $I = (-T, T)$.

In this section we describe the results obtained in chapter 5. However, we begin by giving two brief surveys of related work. The first discusses convergence properties on the abscissa $\sigma = 1$ obtained by J.-P. Kahane and Queffélec. The second discusses work by Kahane which may be said to be related to the operator $\mathscr{Z}_{K,I}$.

Convergence properties on $\sigma = 1$

A natural question when dealing with the behaviour of the K-zeta functions on the abscissa $\sigma = 1$ is whether it is reasonable, in general, to expect

[24] Recall that we use the notation $\zeta_K(\sigma + \mathrm{i}t) = \zeta_{K,\sigma}(t)$ to clarify the use of the convolution which we denote by $*$.

analytic continuations past this abscissa. This question was answered in the negative by Kahane [46] and Queffélec [67]. They considered Dirichlet series of the type

$$\sum_{n\in\mathbb{N}} \varepsilon_n a_n n^{-s}, \tag{1.20}$$

where $(\varepsilon_n)_{n\in\mathbb{N}}$ is a sequence with $\varepsilon_n \in \{-1, 1\}$ for each $n \in \mathbb{N}$. In other words the sequence belongs to the infinite product $\mathcal{W} = \{-1, 1\}^{\infty}$.

The set $\mathcal{W}$ may be equipped with two reasonable notions of probability. In the product topology one may use the probability of coin tosses, with -1 representing heads and 1 representing tails. In this way $\mathcal{W}$ is a probability space. We say that any property valid for (1.20) for $(\varepsilon_n)_{n\in\mathbb{N}}$ outside of a subset of measure zero holds almost surely. The second way is to use Baire categories. Since $\mathcal{W}$ is the infinite product of compact sets the Baire category theorem says that any countable union of nowhere dense sets is nowhere dense. Such a union is said to be of the first category. We say that any property valid for (1.20) on the complement of a set of the first category holds quasi surely.

The following result is proved in greater generality in both of the aforementioned papers.

Theorem (Kahane 1973)**.** *Let a_n be a sequence of positive real numbers such that $\sigma = 1$ is the smallest number for which $\sum_{n\in\mathbb{N}} a_n n^{-\sigma} = \infty$. Then the Dirichlet series $\sum_{n\in\mathbb{N}} \varepsilon_n a_n n^{-s}$ has the abscissa $\sigma = 1$ as a natural boundary quasi surely.*

It now follows immediately that if $K \subset \mathbb{N}$ is such that $\sum_{n\in K} n^{-1} = \infty$ then quasi surely the function

$$\sum_{n\in K} \varepsilon_n n^{-s} \tag{1.21}$$

has $\sigma = 1$ as a natural boundary. If we split $K = K_1 \cup K_2$ according to the sign of (ε_n) we may write the function (1.21) as a difference of two K-zeta functions, i.e. $\zeta_{K_1}(s) - \zeta_{K_2}(s)$. Hence, for each point on the abscissa $\sigma = 1$ either the function ζ_{K_1} or ζ_{K_2} does not have an analytic continuation.

This reasoning also holds true when the zeta function is given by the Euler product

$$\prod_{p\in\mathbb{P}} \frac{1}{1-\varepsilon_p p^{-s}}$$

It was noticed by Queffélec that by expressing this function as $\phi(s)\mathrm{e}^{\sum \varepsilon_n p^{-s}}$, where $\phi(s)$ is analytic on the half-plane defined by $\sigma > 1/2$, and applying Kahane's theorem to the function $\sum_{p\in\mathbb{P}} \varepsilon_p p^{-s}$, it follows that even if ζ_K admits an Euler product it may have the abscissa $\sigma = 1$ as a natural boundary. It is interesting to note that Kahane also proved that the function (1.21) almost surely admits an analytic continuation up to the abscissa $\sigma = 1/2$.

We remark that there has also been done work in finding conditions for analytic continuation to hold, see e.g. T. Kurokawa [53].

A Fourier formula for the prime numbers

The operator $\mathcal{Z}_{K,I}$ indirectly appears in a series of articles by Kahane in the form of the functional

$$\phi \in \mathcal{C}_0^\infty(I) \longmapsto \int_{\mathbb{R}} \phi(\tau)\zeta_{\mathbb{P}}(1+\mathrm{i}\tau)\mathrm{d}\tau. \tag{1.22}$$

In [47] he uses this functional to give a proof of the Prime number theorem

$$\pi_{\mathbb{P}}(x) \sim \frac{x}{\log x},$$

where $\pi_{\mathbb{P}}$ is the counting function of the prime numbers and $f(x) \sim g(x)$ means that $f(x)/g(x) \to 1$ as $x \to +\infty$. The key observation is that the right-hand side of (1.22) is equal to

$$\sum_{p\in\mathbb{P}} \frac{\hat{\phi}(\log p)}{p}. \tag{1.23}$$

Kahane calls the resulting identity the Fourier formula for prime numbers. He carefully selects a function $\phi = \phi_x$ that depends on a parameter $x > 0$

to make the expression (1.23) asymptotically comparable to $\mathrm{e}^{-x}\pi(\mathrm{e}^x)$. By using information about the function $\zeta_{\mathbb{P}}(1+\mathrm{i}t)$ he essentially estimates (1.22) to be asymptotically comparable to $x^{-1}(1+o(1))$, thereby proving the Prime number theorem.

In [48] the same approach is used in the more general context of A. Beurling's theory on generalised prime numbers. This field of research was initiated by Beurling in [4]. The setup is to consider any increasing sequence $R=(r_i)_{i\in\mathbb{N}}$ of real numbers as being a substitute for the primes and then formally define a zeta function by

$$\zeta_R(s)=\prod_{r\in R}\frac{1}{1-r^{-s}}.$$

Let N be the multiplicative semi-group that R generates, i.e. the collection of finite products of elements of R. The set N is called the Beurling integers. The point is to establish theorems connecting the asymptotic behaviour of the counting function of the Beurling primes, $\pi_R(x)$, with the one of the Beurling integers, $\pi_N(x)$. In the paper in question, Kahane essentially makes the same choice for the functions ϕ_x as in [47], making the analogue of (1.23) asymptotically comparable to $\pi_R(x)$. The strength of the paper lies in that he discusses a wide class of conditions that one may impose on the function $\zeta_{\mathbb{R}}(1+\mathrm{i}t)$ to extract enough information to prove asymptotic estimates for $\pi_R(x)$. In particular, he settles a long standing conjecture made by P. T. Bateman and H. G. Diamond [2] in proving that the condition

$$\int_1^\infty\left|\frac{\pi_N(x)-cx}{x}\right|^2\log^2 x\,\frac{dx}{x}<+\infty$$

implies the Prime number theorem for π_R, i.e. that

$$\pi_R(x)\sim\frac{x}{\log x}.$$

Two questions about the operator $\mathcal{Z}_{K,I}$

We recall some known results on the K-zeta functions. For $K = \mathbb{N}$ we get the classical Riemann zeta function $\zeta_{\mathbb{N}} = \zeta$ which satisfies he formula

$$\zeta(s) = \frac{1}{s-1} + \psi(s), \tag{1.24}$$

for some entire function ψ. This formula essentially says that the local behaviour at $s = 1$ is an analytic, and therefore small, perturbation of a pole with residue one. In general, the K-zeta functions may have the abscissa $\sigma = 1$ as a natural boundary, even when they admit an Euler product representation, as was shown in [46, 67]. Therefore it is not reasonable to expect the type of nice behaviour displayed by (1.24) for general $K \subset \mathbb{N}$. However, consider the following example. For $K = \mathbb{N}$ the formula (1.24) implies that

$$\operatorname{Re}\zeta_{\mathbb{N}}(1+\delta+\mathrm{i}t) = \frac{\delta}{\delta^2+t^2} + \operatorname{Re}\psi(1+\delta+\mathrm{i}t),$$

whence

$$\mathcal{Z}_{\mathbb{N},I} = \mathrm{Id} + \Psi_{\mathbb{N},I}, \tag{1.25}$$

where $\Psi_{\mathbb{N},I}$ is a compact operator on $L^2(I)$ for all intervals of the form $I = (-T, T)$. We point out that we consider (5.6) to be a generalisation of the formula (5.1). This leads us to pose the following question.

(1) For which $K \subset \mathbb{N}$ does the formula $\mathcal{Z}_{K,I} = A\mathrm{Id} + \Psi_{K,I}$, with $\Psi_{K,I}$ a compact operator, hold for some $A \geq 0$?

We can pose this question in a more general manner. In chapter 3 what is essentially the formula (5.6) is used to prove that the operator $\mathcal{Z}_{\mathbb{N},I}$ is bounded below in norm on $L^2(I)$. This relies on the following is a result from Semi-Fredholm theory. It is a classical result shown in e.g. [49, p. 238, thm. 5.26].

Lemma 5.2 (Second stability theorem of Semi-Fredholm theory)**.** *Let X, Y be Banach spaces and $Z : X \to Y$ a continuous linear operator that is bounded below. If $\Phi : X \to Y$ is a compact operator and $Z + \Phi$ is injective, then it follows that $Z + \Phi$ is bounded below.*

If we assume that there are no problems in deciding when $\mathcal{Z}_{K,I}$ is injective or not, this result implies that whenever (1) holds, $\mathcal{Z}_{K,I}$ is bounded below as an operator on $L^2(I)$. Letting this be an indicator of when $\zeta_K(s)$ has a pole-like behaviour at $s = 1$ we are led to pose the following general question:

(2) For which $K \subset \mathbb{N}$ is $\mathcal{Z}_{K,I}$ bounded below?

We now outline the solutions to these two questions which are given in full detail in chapter 5.

A general formula

Given any $K \subset \mathbb{N}$ we observe that

$$\mathcal{Z}_{K,I}g(t) = \frac{\chi_I}{\sqrt{2\pi}} \sum_{n \in K} \frac{\hat{g}(\log n)n^{\mathrm{i}t} + \hat{g}(-\log n)n^{-\mathrm{i}t}}{\sqrt{n}}.$$

We consider the right-hand side to be a Riemann sum. To express the integral it approximates we set

$$L = \bigcup_{k \in K} \Big(\big(-\log(n+1), -\log n \big] \cup \big[\log n, \log(n+1) \big) \Big). \tag{1.26}$$

With this notation we see that the operator $\mathcal{Z}_{K,I}$ gives an approximation in the sense of Riemann sums of the bounded operator

$$g \in L^2(I) \longmapsto \chi_I \mathcal{F}^{-1} \chi_L \mathcal{F} g \in L^2(I).$$

In this way, we obtain the following theorem which is of importance in answering both of the aforementioned questions. This generalises the formula (1.25) for any $K \subset \mathbb{N}$.

Theorem 5.4. *Let $K \subset \mathbb{N}$ be arbitrary, $I \subset \mathbb{R}$ be a bounded symmetric interval and let L be defined by* (1.26). *Then there exists a compact operator $\Phi_{K,I}$ such that*

$$\mathcal{Z}_{K,I} = \chi_I \mathcal{F}^{-1} \chi_L \mathcal{F} + \Phi_{K,I}.$$

The Wiener-Ikehara-Korevaar theorem and the first question

The answer to question (1) is indicated by a tauberian theorem due to J. Korevaar [52] that generalises a classical result of N. Wiener and S. Ikehara [43]. We state a special case.

Lemma 5.6 (Korevaar 2005). *Let $K \subset \mathbb{N}$ and $A \geq 0$. Then the function defined by*

$$\psi_K(s) = \frac{1}{s}\zeta_K(s) - \frac{A}{s-1} \tag{1.27}$$

extends to a pseudo-function on $\sigma = 1$ if and only if

$$\lim_{x\to\infty} \frac{\pi_K(x)}{x} = A.$$

Here $\pi_K(x)$ denotes the counting function of the integers K. We call the distributional Fourier transform of L^∞ functions with decay at infinity pseudo-functions. Any function ϕ that is analytic on $\sigma > 1$ and for which $\phi_\sigma(t) = \phi(\sigma + \mathrm{i}t)$ converges in terms of distributions to a pseudo-function as $\sigma \to 1^+$ is said to extend to a pseudo-function on $\sigma = 1$. Hence, the behaviour of these K-zeta functions are captured by the pole with residue A up to a perturbation by a pseudo-function. The significance of the pseudo-functions in our situation is explained by the following observation. If ϕ is analytic on $\sigma > 1$ and extends to a pseudo-function on $\sigma = 1$. Then the operator defined by

$$g \in L^2(I) \longmapsto \lim_{\sigma\to 1^+} \chi_I(g * \phi_\sigma)$$

is a compact operator. Based on this and the Wiener-Ikehara-Korevaar Lemma the sufficiency part of the next result follows in a straight-forward manner. The necessity follows by an application of Theorem 5.4.

Theorem 5.7. *Let $K \subset \mathbb{N}$ be arbitrary and $A \geq 0$. Then the operator defined by*

$$\Psi_{K,I} = \mathcal{Z}_{K,I} - A\mathrm{Id}$$

is compact for all intervals of the form $I = (-T, T)$ if and only if

$$\lim_{x\to\infty} \frac{\pi_K(x)}{x} = A.$$

Panejah's theorem and the second question

We now turn to the second question. A crucial result in the study of this operator is due to B. Panejah [66].

Lemma 5.8 (Panejah 1966)**.** *Let $L \subset \mathbb{R}$. Then the operator $\chi_L \mathcal{F}$ is bounded below from $L^2(I)$ to $L^2(\mathbb{R})$ if and only if there exists $\delta > 0$ such that*

$$\inf_{\xi \in \mathbb{R}} |L \cap (\xi - \delta, \delta)| > 0.$$

By combining this with Lemma 5.2 and Theorem 5.4 we get the following theorem[25] which answers question (2).

Theorem 5.10*. *Let $K \subset \mathbb{N}$ be arbitrary, $I \subset \mathbb{R}$ be a bounded symmetric interval and $L \subset \mathbb{R}$ be defined by 1.26. Then the following conditions are equivalent.*

$$\mathcal{Z}_{K,I} \text{ is bounded below on } L^2(I).$$

$$\chi_I \mathcal{F}^{-1} \chi_L \mathcal{F} \text{ is bounded below on } L^2(I).$$

$$\text{There exists } \delta \in (0,1) \text{ such that } \liminf_{x \to \infty} \frac{\pi_K(x) - \pi_K(\delta x)}{x} > 0.$$

1.6 Modified zeta functions and prime numbers

In chapter 6 we continue the study of the K-zeta functions under the additional assumption of arithmetic structure on the subsets $K \subset \mathbb{N}$. In effect, K consists of the integers whose prime number decomposition contain only elements of some specified subset of the prime numbers.

We describe the results obtained in the chapter.

Arithmetic structure implies asymptotic density

A fundamental fact is that $K \subset \mathbb{N}$ with arithmetic structure in the sense described above always admit an asymptotic density[26].

[25] Theorem 5.10* is a simplified statement of Theorem 5.10.

[26] In chapter 6 we provide a proof of Lemma 6.1 using the Wiener-Ikehara-Korevaar tauberian theorem.

Lemma 6.1. *Let $Q \subset \mathbb{P}$ generate $K \subset \mathbb{N}$, and let J denote the integers generated by the prime numbers not in Q. Then*

$$\lim_{x\to\infty} \frac{\pi_K(x)}{x} = \lim_{\sigma\to 1^+} \frac{1}{\zeta_J(\sigma)}.$$

Hence, if J denotes the integers generated by the primes not in Q, then the condition of Theorem 5.7 always holds with $A = \lim_{\sigma\to 1^+} \zeta_J^{-1}(\sigma)$. By the Euler product representation

$$\zeta_J(s) = \prod_{p\in\mathbb{P}\setminus Q} \left(\frac{1}{1-p^{-s}}\right)$$

it is seen that $\zeta_J(1) < \infty$ if and only if

$$\sum_{p\in\mathbb{P}\setminus Q} \frac{1}{p} < \infty. \tag{1.28}$$

This means that we get the following simpler form of the theorem.

Theorem 6.2. *Let $Q \subset \mathbb{P}$ generate the integers K, and J be the integers generated by the primes not in Q. Then*

$$\mathcal{Z}_{K,I} = \zeta_J^{-1}(1)\mathrm{Id} + \Psi_{K,I},$$

for a compact operator $\Psi_{K,I}$. Moreover, the operator $\mathcal{Z}_{K,I}$ is bounded below on $L^2(I)$ if and only if

$$\sum_{p\in\mathbb{P}\setminus Q} \frac{1}{p} < \infty.$$

This can be compared with the following theorem of F. Moricz which is stated in more generality in [61].

Theorem (Moricz 1999)**.** *Let $K \subset \mathbb{N}$ be arbitrary and set $\phi(x) = |x|^p$. Then*

$$\int_1^2 \phi(\zeta_K(\sigma))\mathrm{d}\sigma \simeq \sum_{n=1}^{\infty} \frac{1}{n^2}\phi\left(\sum_{m=1}^{n} \frac{\pi_K(2^m)}{2^m}\right).$$

In particular, for $\phi(x) = |x|$, it follows that

$$\int_1^2 |\zeta_K(\sigma)| \mathrm{d}\sigma \simeq \sum_{n=1}^{\infty} \frac{1}{n} \frac{\pi_K(2^n)}{2^n}.$$

Also, when we combine Theorem 6.2 with Theorem 5.8, we obtain that if $Q \subset \mathbb{P}$ generates $K \subset \mathbb{N}$, then

$$\chi_I \mathcal{F}^{-1} \chi_L \mathcal{F} = \frac{1}{\zeta_J(1)} \mathrm{Id} + \Upsilon_{K,I},$$

where $\Upsilon_{K,I}$ denotes a compact operator for all intervals $I = (-T, T)$. However, more can be said about the formula (1.27). We state a special case of Theorem 6.3.

Theorem 6.3*. *Let $Q \subset \mathbb{P}$ generate $K \subset \mathbb{N}$. Then the function ψ_K of formula* (1.27) *extends to a L^1_{loc} function on $\sigma = 1$ if and only if*

$$\sum_{p \in \mathbb{P} \setminus Q} \frac{\log \log p}{p} < \infty.$$

Prime number theorems for Beurling primes

Let $R = (r_i)$ be an increasing sequence of real numbers greater than one and let N be the multiplicative semi-group it generates. Recall that R may be said to be the Beurling prime numbers for the Beurling integers N. In our case Q corresponds to the Beurling primes and K to the Beurling integers. We say that the prime number theorem holds for Q if

$$\pi_Q(x) \sim \frac{x}{\log x}.$$

We show the following lemma.

Lemma 6.5. *Let $Q \subset \mathbb{P}$ generate $K \subset \mathbb{N}$ and let J denote the integers generated by the primes not in Q. Then the prime number theorem holds for the set K if and only if*

$$\sum_{p \in \mathbb{P} \setminus Q \cap (\delta x, x)} \frac{\log p}{p} = o\left(1\right), \quad \textit{for all } \delta \in (0,1). \tag{1.29}$$

We check if the asymptotic information about Q given by the condition (1.28) is related to a prime number theorem being true for Q. By comparing it to the condition (1.29) we prove that neither condition implies the other.

Theorem 6.6. *Let $Q \subset \mathbb{P}$ generate $K \subset \mathbb{N}$. Then the prime number theorem for Q neither implies nor is implied by the lower boundedness of the operator $\mathcal{Z}_{K,I}$.*

This completes our survey.

2 Function spaces of Dirichlet series and local embeddings

In this chapter we explain the basic analogies between the spaces $\mathscr{H}^p$, $\mathscr{H}^2_\alpha$ and $\mathscr{D}_\alpha$ and their classical counterparts, the spaces H^p and D_α. These are given in terms of the Bohr correspondence, point evaluations and local embedding properties. The reader should consult the previous chapter for more on the background and motivation for the study of the spaces $\mathscr{H}^p$

First we give some basic information on the convergence properties of Dirichlet series in general, and the Riemann zeta function in particular.

2.1 Preliminaries

By Dirichlet series we refer to functions of the type

$$F(s) = \sum_{n\in\mathbb{N}} a_n n^{-s}, \quad s = \sigma + \mathrm{i}t \tag{2.1}$$

which are assumed to converge for some $s \in \mathbb{C}$. Finite sums of this type are called Dirichlet polynomials. We denote the set of Dirichlet polynomials by $\mathscr{P}$.

Classical theory of convergence

By a direct application of summation by parts it follows that if a Dirichlet series converges at a point $s_0 = \sigma_0 + \mathrm{i}t_0$ then it also converges for all points on the half-plane $\operatorname{Re} s > \sigma_0$. For a Dirichlet series of the type (2.1) it therefore makes sense to define the abscissa of convergence,

$$\sigma_c(F) = \inf\left\{\sigma : \sum_{n\in\mathbb{N}} a_n n^{-\sigma}. \text{ converges.}\right\}$$

By applying this to the Dirichlet series with coefficients $|a_n|$ one establishes in the same way the existence of an abscissa of absolute convergence, denoted by $\sigma_a(F)$.

These abscissae are in a sense similar to the radii of convergence of complex polynomials. Unlike complex polynomials, however, the abscissae of convergence and absolute convergence do not coincide in general. An example of this is given by the alternating zeta function

$$\sum_{n\in\mathbb{N}}(-1)^n n^{-s}.$$

This Dirichlet series converges for $\sigma > 0$, but only converges absolutely for $\sigma > 1$. This indicates that the convergence properties of Dirichlet series are more delicate than those of Taylor series.

We will not need these concepts in our investigations but for completeness we briefly mention that in the same way, it is possible to define the corresponding abscissae of uniform convergence, σ_u, and convergence to a bounded function, σ_b. It is easily seen to hold true that

$$\sigma_c \leq \sigma_a \leq \sigma_c + 1, \quad \text{and} \quad \sigma_b \leq \sigma_u \leq \sigma_a.$$

It was shown by H. Bohr [9, 10] that $\sigma_u = \sigma_b$ and that $\sigma_a \leq 1/2 + \sigma_u$. H. F. Bohnenblust and E. Hille [8] proved that the second estimate is sharp.

For more on the classical theory of Dirichlet series one may consult H. Bohr's thesis [11] or the classical account of G. H. Hardy and M. Riesz [33].

Meromorphic continuation of the Riemann zeta function

The Riemann zeta function is given by

$$\zeta(s) = \sum_{n\in\mathbb{N}} n^{-s}, \quad s = \sigma + \mathrm{i}t.$$

It is clear that the series on the right-hand side converges absolutely on the half-plane $\sigma > 1$ and diverges at the point $\sigma = 1$. For us, the essential property of the Riemann zeta function is contained in the following classical result [71, p. 145].

Lemma 2.1 (Riemann 1859)**.** *There exists an entire function ψ such that*

$$\zeta(s) = \frac{1}{s-1} + \psi(s). \tag{2.2}$$

We give two short arguments that show how this extension is realised on the half-plane $\sigma > 0$. The first relies on a trick. Since $2^{-s}\zeta(s) = \sum_{n\in\mathbb{N}}(2n)^{-s}$ it follows that

$$(2^{1-s} - 1)\zeta(s) = \sum_{n\in\mathbb{N}} (-1)^n n^{-s}.$$

It is clear that the alternating sum on the right-hand side converges for $s = \sigma > 0$. Hence, the right-hand side converges for all complex s with real part $\sigma > 0$.

The second approach is related to the techniques we use in chapters 5 and 6. We let $\pi_{\mathbb{N}}(x)$ denote the counting function of the integers, i.e.

$$\pi_{\mathbb{N}}(x) = \sum_{\substack{n\in\mathbb{N}\\ n\le x}} 1.$$

We may now write

$$\begin{aligned}\zeta(s) &= \int_1^\infty x^{-s}\mathrm{d}\pi_{\mathbb{N}}(x) = s\int_1^\infty x^{-s-1}\pi_{\mathbb{N}}(x)\mathrm{d}x\\ &= \frac{1}{s-1} + 1 + \int_1^\infty x^{-s-1}(\pi_{\mathbb{N}}(x) - x)\mathrm{d}x.\end{aligned}$$

Since $|x - \pi_{\mathbb{N}}(x)| \le 1$ for $x \le 1$ it follows that the last term on the right hand side defines an analytic function for s with $\sigma > 0$. Note that by the second equality $\zeta(s)/s$ is what is called the Mellin transform of $\pi_{\mathbb{N}}$.

2.2 The Dirichlet-Hardy spaces $\mathscr{H}^p$

The space $\mathscr{H}^2$ was introduced in [35] and is defined to be the closure of the Dirichlet polynomials in the norm

$$\left\|\sum_{n\in\mathbb{N}} a_n n^{-s}\right\|_{\mathscr{H}^2} = \left(\sum_{n\in\mathbb{N}} |a_n|^2\right)^{1/2}.$$

By the Cauchy-Schwarz inequality it is seen that $\mathscr{H}^2$ defines a Hilbert space of functions analytic on the half-plane $\mathbb{C}_{1/2}$. For $p \in [1, \infty)$ the spaces $\mathscr{H}^p$ may be defined [3] to be the closure of the Dirichlet polynomials in the norm

$$\left\| \sum_{n\in\mathbb{N}} a_n n^{-s} \right\|_{\mathscr{H}^p} = \lim_{T\to\infty} \left(\frac{1}{2T} \int_{-T}^{T} \left| \sum_{n\in\mathbb{N}} a_n n^{-\mathrm{i}t} \right|^p \mathrm{d}t \right)^{1/p}. \tag{2.3}$$

Following [35, 3] we proceed to explain how to define these spaces using the Bohr correspondence.

The spaces $L^p(\mathbb{T}^\infty)$ and $H^p(\mathbb{T}^\infty)$

The infinite dimensional torus may be defined as the set sequences

$$\mathbb{T}^\infty = \{(z_1, z_2, \dots) : z_n \in \mathbb{T}\}.$$

With the group action of termwise multiplication and the product topology, $\mathbb{T}^\infty$ is a compact abelian group. This means that it has a unique normalised Haar measure ρ. In fact, it may be seen that this measure is nothing but the infinite product of the normalised Lebesgue measure on each copy of $\mathbb{T}$. We define the spaces $L^p(\mathbb{T}^\infty)$ to be the closure of the trigonometric polynomials on $\mathbb{T}^\infty$ in the norm

$$\|g\|_{L^p(\mathbb{T}^\infty)} = \left(\int_{\mathbb{T}^\infty} |g(\chi)|^p \mathrm{d}\rho(\chi) \right)^{1/p}.$$

To do Fourier analysis on these spaces we need to use the dual group of $\mathbb{T}^\infty$. This group is the positive rational numbers $\mathbb{Q}_+$ under multiplication, equipped with the discrete topology. For $\chi \in \mathbb{T}^\infty$ we let the n'th coordinate be given by $\chi(p_n)$. With this notation, for $r \in \mathbb{Q}_+$ with the prime number decomposition $r = p_1^{\gamma_1} \cdots p_n^{\gamma_n}$ for $\gamma_1, \dots, \gamma_n \in \mathbb{Z}$ and $\chi \in \mathbb{T}^\infty$, the duality is realised by

$$r : \chi \longmapsto \chi(r) = z_1^{\gamma_1} \cdots z_n^{\gamma_n}.$$

This means that for $g \in L^p(\mathbb{T}^\infty)$ we have the standard identification

$$g(\chi) \sim \sum_{r\in\mathbb{Q}_+} c_r(g)\chi(r),$$

where

$$c_r(g) = \int_{\mathbb{T}^\infty} g(\chi)\overline{\chi(r)}\mathrm{d}\rho(\chi).$$

See W. Rudin's book [72] for more on Fourier analysis on groups. We say that a function $g \in L^p(\mathbb{T}^\infty)$ belongs to the subspace $H^p(\mathbb{T}^\infty)$ if and only if $c_r(g) = 0$ for $r \in \mathbb{Q}_+ \backslash \mathbb{N}$,

An equivalent definition of the space $\mathscr{H}^p$

We now define $\mathscr{H}^p$ to be the Banach space of formal Dirichlet series obtained as the inverse image of $H^p(\mathbb{T}^\infty)$ under the Bohr correspondence

$$\mathcal{B} : \sum_{n\in\mathbb{N}} a_n n^{-s} \longmapsto \sum_{n\in\mathbb{N}} a_n \chi(n).$$

The topology is given by the norm

$$\|F\|^p_{\mathscr{H}^p} = \int_{\mathbb{T}^\infty} |\mathcal{B}F(\chi)|^p \mathrm{d}\rho(\chi).$$

To see that this definition coincides with the one given by (2.3) one may use ergodic theory as we mentioned in the introduction. However, an argument using only K. Weierstrass' polynomial approximation theorem was given by Saksman and Seip in [74]. The idea is first to establish that for Dirichlet polynomials F it holds that

$$\int_{\mathbb{T}^\infty} \mathcal{B}F(\chi)^n \overline{\mathcal{B}F(\chi)}^m \mathrm{d}\rho(\chi) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} F(\mathrm{i}t)^n \overline{F(\mathrm{i}t)}^m \mathrm{d}t.$$

This follows by multiplying out both sides and integrating term by term. By the triangle inequality both functions $\mathcal{B}F$ and F are uniformly bounded by some constant $C > 0$ on the respective domains of integration. So by the Weierstrass approximation theorem we may approximate the function $p(z) = |z|^p$ uniformly for $|z| \leq 2C$ by a sequence of polynomials

$$p_N(z) = \sum_{|n|,|m|\leq N} c_{n,m} z^n \overline{z}^m.$$

It now follows that as $N \to \infty$ we get

$$\begin{array}{ccc} \int_{\mathbb{T}^\infty} p_N(\mathcal{B}F(\chi))\mathrm{d}\rho(\chi) & = & \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} p_N(F(\mathrm{i}t))\mathrm{d}t \\ \downarrow & & \downarrow \\ \int_{\mathbb{T}^\infty} |\mathcal{B}F(\chi)|^p\mathrm{d}\rho(\chi) & & \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |F(\mathrm{i}t)|^p\mathrm{d}t \end{array}$$

Hence, the Bohr correspondence $\mathcal{B}$ extends to an isometry between the spaces $\mathscr{H}^p$ and $H^p(\mathbb{T}^\infty)$.

Connections to to the classical range of Hardy spaces $H^p(\mathbb{C}_{1/2})$

We first consider the case $p = 2$. We have already remarked that the functions in $\mathscr{H}^2$ converge absolutely on the half-plane $\mathbb{C}_{1/2}$. By inspection, the point evaluation functional at $w \in \mathbb{C}_{1/2}$, which is given by the reproducing kernel $k_w^{\mathscr{H}^2}(s)$, is seen to be a translate of the Riemann zeta function, namely

$$k_w^{\mathscr{H}^2}(s) = \sum_{n\in\mathbb{N}} n^{-s-\bar{w}} = \zeta(s+\bar{w}). \tag{2.4}$$

It is well-known that the reproducing kernel for[1] $H^2(\mathbb{C}_{1/2})$ is given by

$$k_w^{H^2}(s) = \frac{1}{s+\bar{w}-1}.$$

By (2.4) and the formula for the Riemann zeta function (2.2), it follows that for bounded $s+\bar{w}$ it holds that

$$k_w^{\mathscr{H}^2}(s) = k_w^{H^2}(s) + \mathcal{O}(1). \tag{2.5}$$

A property that will be of crucial importance for us is the local embedding property of the space $\mathscr{H}^2$. We give a proof of this embedding using an argument that will appear as a central idea of several results in this work. This argument is different from the ones presented in [59] and [35].

[1] See page 4 for the definition of the space $H^p(\mathbb{C}_{1/2})$.

Lemma 2.2 (Montgomery 1994, Hedenmalm, Lindqvist and Seip 1997). *For $F \in \mathscr{H}^2$ and every bounded interval $I \subset \mathbb{R}$ there exists a constant $C > 0$, only depending on the length of I, such that*

$$\lim_{\sigma \to 1/2^+} \int_I |F(\sigma + \mathrm{i}t)|^2 \mathrm{d}t \leq C \|F\|^2_{\mathscr{H}^2}. \tag{2.6}$$

Proof. Recall that $\mathscr{P}$ denotes the set of Dirichlet polynomials and that we use the convention $\|f\|^2_{L^2(I)} = \int_I |g(t)|^2 \mathrm{d}t$. By the invariance under vertical translations of the norm $\mathscr{H}^2$ it suffices to consider the case $I = (-T, T)$ for some $T > 0$. We begin by defining the operator

$$E_I : \sum_{n \in \mathbb{N}} a_n n^{-s} \in \mathscr{P} \longmapsto \chi_I(t) \sum_{n \in \mathbb{N}} a_n n^{-1/2 - \mathrm{i}t}, \tag{2.7}$$

where χ_I denotes the characteristic function of the interval I. We wish to show that E_I extends to a bounded operator from $\mathscr{H}^2$ to $L^2(I)$. To do this, let $\epsilon_k \to 0$ be some sequence of positive real numbers and define the sequence of operators

$$E_k : \sum_{n \in \mathbb{N}} a_n n^{-s} \in \mathscr{P} \longmapsto \chi_I(t) \sum_{n \in \mathbb{N}} a_n n^{-1/2 - \epsilon_k - \mathrm{i}t}.$$

It is clear that $E_k F \to E_I F$ in norm for $F \in \mathscr{P}$. Hence, it suffices to prove that $E_k E_k^*$ converges in the strong operator topology to a bounded operator from $\mathscr{H}^2$ to $L^2(I)$. By a straight-forward computation

$$E_k^* g = \sqrt{2\pi} \sum_{n \in \mathbb{N}} \frac{\hat{g}(-\log n)}{n^{\frac{1}{2} + \epsilon - k}} n^{-s}.$$

By applying the operator E_k to this expression, and then pulling the integral sign in the expression of the Fourier transform $\hat{g}$ outside of the sum sign, we get

$$E_k E_k^* g = \chi_I (g * \zeta_{1+\epsilon_k}), \tag{2.8}$$

where $*$ denotes convolution on $\mathbb{R}$, the function $\zeta_\sigma(t) = \zeta(\sigma + \mathrm{i}t)$ is the Riemann-zeta function and $g \in L^2(I)$ is extended to all of $\mathbb{R}$ by setting it equal to zero outside of I. Inserting the formula (2.2) for the Riemann-zeta function in the expression for $E_k E_k^*$ and taking the limit as $k \to \infty$, we establish that

$$E_I E_I^* g = 2\pi \chi_I P_+ g + \chi_I (g * \psi_1). \tag{2.9}$$

Here $\psi_1(t) = \psi(1 + \mathrm{i}t)$ and P_+ denotes the Riesz projection $L^2(\mathbb{R}) \to H^2(\mathbb{C}_1)$ given by

$$P_+ g(t) = \lim_{\sigma \to 1} \frac{1}{2\pi} \int_{\mathbb{R}} \frac{g(\tau)}{(\sigma - 1) + \mathrm{i}(t - \tau)} \mathrm{d}\tau. \tag{2.10}$$

Since the Riesz projection is bounded[2] on $L^2(\mathbb{R})$, the lemma now follows. □

We note that it now follows that if $F \in \mathscr{H}^2$ then[3] $F(s)/s \in H^2(\mathbb{C}_{1/2})$. In particular this implies that F has non-tangential boundary limits on the abscissa $\sigma = 1/2$ for almost every $t \in \mathbb{R}$.

We turn to general $p \in [1, \infty)$. Bayart [3] used the sharp results on point evaluations in the space $H^p(\mathbb{T}^\infty)$ obtained by Cole and Gamelin [19] to show that the formal Dirichlet series in $\mathscr{H}^p$ converge on the half-plane $\mathbb{C}_{1/2}$.

Lemma 2.3 (Cole and Gamelin 1985, Bayart 2002)**.** *Let $p \in [1, \infty)$. Then the norm of the point evaluation in $\mathscr{H}^p$ at the point $s = \sigma + \mathrm{i}t$ in $\mathbb{C}_{1/2}$ equals $\zeta(2\sigma)^{1/p}$.*

For $s \in \mathbb{C}_{1/2}$ let the norm of the point evaluation functional $F \in \mathscr{H}^p \longmapsto F(s)$ be denoted by $\omega_{\mathscr{H}^p}$. We now use the fact that the Riemann zeta function is meromorphic with a pole at $s = 1$ to get

$$\omega_{\mathscr{H}^p}(s)^p = \frac{1}{\sigma - 1/2} + \mathcal{O}(1), \quad \sigma \to 1/2^+. \tag{2.11}$$

For $f \in H^p(\mathbb{C}_{1/2})$ it is well known that a reproducing formula holds,

$$f(s) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{f\left(\frac{1}{2} + \mathrm{i}t\right)}{s - \frac{1}{2} - \mathrm{i}t} \mathrm{d}t.$$

[2] See for instance [51, p. 128] for more on the Riesz projection.
[3] We gave a demonstration of this fact on page 6.

From this it follows that the norm $\omega_{H^p}(s)$ of the point evaluation at the point $s \in \mathbb{C}_{1/2}$ satisfies $\omega_{H^p}(s) = C_p(2\sigma - 1)^{-1/p}$, where the constant C_p only depends on p. By the relation (2.11) this implies

$$\omega_{\mathscr{H}^p}(s)^p = C_p \omega_{H^p}(s)^p + \mathcal{O}(1), \qquad \sigma \to 1/2^+. \tag{2.12}$$

As we remarked in the introduction, the inequality of Lemma 2.2 is only known to hold for $p = 2k$ where $k \in \mathbb{N}$. Indeed, it follows immediately from the fact that $F \in \mathscr{H}^{2k} \Rightarrow F^k \in \mathscr{H}^2$ that for bounded intervals I and the same constant C_I as in the lemma,

$$\lim_{\sigma \to 1/2^+} \int_I |F(\sigma + \mathrm{i}t)|^{2k} \mathrm{d}t \le C_I \|F\|^{2k}_{\mathscr{H}^{2k}}.$$

2.3 McCarthy's spaces $\mathscr{H}^2_\alpha$

The spaces $\mathscr{H}^2_\alpha$ were introduced in [58]. They consist of the Dirichlet series finite in the norm

$$\left\| \sum_{n \in \mathbb{N}} a_n n^{-s} \right\|_{\mathscr{H}^2_\alpha} = \left(\sum_{n \in \mathbb{N}} |a_n|^2 \log^\alpha(n+1) \right)^{1/2}.$$

By the Cauchy-Schwarz inequality, we observe that every space $\mathscr{H}^2_\alpha$ consists of functions analytic in the half-plane $\sigma > 1/2$.

A lemma on weighted zeta functions

We observe that the reproducing kernel at $w \in \mathbb{C}_{1/2}$ may be expressed in terms of a weighted zeta-function,

$$k_w^{\mathscr{H}_\alpha}(s) = \sum_{n \in \mathbb{N}} \frac{n^{-s-\bar{w}}}{\log^\alpha(n+1)}.$$

The following lemma describes the local behaviour of these weighted zeta functions. Note that the Gamma function is given by

$$\Gamma(x) = \int_0^\infty \mathrm{e}^u u^{x-1} \mathrm{d}u.$$

Lemma 2.4 (Olsen and Seip 2008)**.** *For $\alpha < 1$, we have*

$$\sum_{n\in\mathbb{N}} \frac{n^{-s}}{\log^\alpha(n+1)} = \Gamma(1-\alpha)(s-1)^{\alpha-1} + \mathcal{O}(1)$$

for $s \in \mathbb{C}_{1/2}$ as $s \to 1$. In the limiting case $\alpha = 1$ we have

$$\sum_{n\in\mathbb{N}} \frac{n^{-s}}{\log(n+1)} = \log\frac{1}{s-1} + \mathcal{O}(1)$$

for $s \in \mathbb{C}_{1/2}$ as $s \to 1$.

Proof. The proof is a calculation analogous to the one for the Riemann zeta-function found for instance in A. Ivic's book [44]. To begin with,

$$\begin{aligned}\sum_{n\in\mathbb{N}} \frac{n^{-s}}{\log^\alpha(n+1)} &= \int_1^\infty \frac{x^{-s}}{\log^\alpha(x+1)}\mathrm{d}[x] \\ &= \int_1^\infty \frac{x^{-s-1}[x]}{\log^\alpha(x+1)}\left(s + \frac{\alpha}{\log(x+1)}\frac{x}{x+1}\right)\mathrm{d}x.\end{aligned}$$

The integral

$$\int_1^\infty \frac{x^{-s-1}[x]}{\log^\alpha(x+1)}\left(s + \frac{\alpha}{\log(x+1)}\frac{x}{x+1}\right) - \frac{x^{-s}}{\log^\alpha(x+1)}\left(s + \frac{\alpha}{\log(x+1)}\right)\mathrm{d}x$$

converges absolutely and defines an analytic function in the right half-plane. We may therefore pass from $[x]$ to x in our integral, and ignore the factor $x/(x+1)$. For a similar reason, we may replace $\log(x+1)$ by $\log x$ and if necessary change the lower limit of integration.

When $\alpha < 1$, we make the following computation:

$$\int_1^\infty \frac{x^{-s}}{\log^\alpha x}\mathrm{d}x = \Gamma(1-\alpha)(s-1)^{\alpha-1}.$$

This gives the desired result for $0 < \alpha < 1$. When $\alpha < 0$, we find, using the functional equation $\Gamma(s+1) = s\Gamma(s)$ for the gamma-function, that

$$\int_1^\infty \frac{x^{-s}}{\log^\alpha x}\left(s + \frac{\alpha}{\log x}\right)\mathrm{d}x = \Gamma(1-\alpha)(s-1)^{\alpha-1}$$

as well. In the limiting case $\alpha = 1$, we find that

$$\int_2^\infty \frac{x^{-s}}{\log x}\mathrm{d}x = \log\frac{1}{s-1} + \mathcal{O}(1)$$

as $s \to 1$. $\square$

Connections to the classical range of Hilbert spaces $D_\alpha(\mathbb{C}_{1/2})$

The natural counterparts to the spaces $\mathscr{H}^2_\alpha$ in the half-plane $\sigma > 1/2$ are the spaces $D_\alpha(\mathbb{C}_{1/2})$. For $\alpha < 2$ these were defined on page 12. In order to give an exact representation of the reproducing kernels, we redefine the norm for $\alpha = 1$ (the space remains the same) to be given by

$$\|f\|^2_{D_1} = \frac{1}{2\pi}\int_{-\infty}^\infty \left|f\left(\frac{1}{2}+\mathrm{i}t\right)\right|^2 \frac{\mathrm{d}t}{1+t^2} + \frac{1}{\pi}\int_{\mathbb{C}_{1/2}} |f'(s)|^2 \mathrm{d}m(s).$$

Recall that $\mathrm{d}m$ denotes Lebesgue area measure. The reproducing kernels for $D_\alpha(\mathbb{C}_{1/2})$ at $w \in \mathbb{C}_{1/2}$ is

$$k_w^{D_\alpha}(s) = c_\alpha(\bar{w} + s - 1)^{\alpha-1} \tag{2.13}$$

when $\alpha < 1$, with $c_\alpha = (-\alpha)2^{-\alpha-1}$ for $\alpha < 0$ and $c_\alpha = 2^{\alpha-1}(1-\alpha)^{-1}$ for $0 < \alpha < 1$. In the limiting case $\alpha = 1$, we now have

$$k_w^{D_1}(s) = \frac{3+2\bar{w}}{1+2\bar{w}}\frac{3+2s}{1+2s}\left(\log\frac{(1+2\bar{w})(1+2s)}{2^3} + \log\frac{1}{\bar{w}+s-1}\right).$$

What is essential in this case is that for $s + \bar{w}$ in a bounded set we have

$$k_w^{D_1}(s) = \log\frac{1}{s+\bar{w}-1} + \mathcal{O}(1).$$

For $\alpha > 1$ the reproducing kernels are uniformly bounded and are therefore not of interest to us.

By combining these considerations with Lemma 2.4 we get for all $\alpha \leq 1$ the relations

$$k_w^{\mathscr{H}_\alpha^2}(s) = c_\alpha^{-1} k_w^{D_\alpha}(s) + \mathcal{O}(1). \tag{2.14}$$

Moreover, for each $\alpha < 2$ the inequality (2.6) can now be transformed into a version that reveals the local boundary behavior of functions in $\mathscr{H}_\alpha$. To this end, for a bounded interval I let Q_I denote the half-strip $\sigma > 1/2$, $t \in I$.

Lemma 2.5 (Olsen and Seip 2008)**.** *Suppose $F \in \mathscr{H}_\alpha^2$ and let I be a bounded interval. There exist constants C only depending on the length of I and α such that for $\alpha < 0$ we have*

$$\int_{Q_I} |F(s)|^2 \left(\sigma - \frac{1}{2}\right)^{-\alpha-1} \mathrm{d}m(s) \leq C\|F\|^2_{\mathscr{H}_\alpha}. \tag{2.15}$$

Similarly, for $0 < \alpha < 2$ we have

$$\int_{Q_I} |F'(s)|^2 \left(\sigma - \frac{1}{2}\right)^{-\alpha+1} \mathrm{d}m(s) \leq C\|F\|^2_{\mathscr{H}_\alpha}. \tag{2.16}$$

Proof. Using Lemma 2.2 to write (2.6), for $\sigma > 1/2$, as

$$\int_I |F(\sigma + \mathrm{i}t)|^2 \, \mathrm{d}t \leq C \sum_{n=1}^{\infty} |a_n|^2 n^{2\sigma-1},$$

multiplying both sides by $(\sigma - \frac{1}{2})^{-\alpha-1}$, and integrating from $\frac{1}{2}$ to $+\infty$, we get the desired inequality for $\alpha < 0$. By a similar computation the inequality for $0 < \alpha < 2$ is obtained. □

2.4 The Dirichlet-Bergman spaces $\mathscr{D}_\alpha$

In this section we introduce the spaces $\mathscr{D}_\alpha$. For $\alpha \in \mathbb{R}$ the space $\mathscr{D}_\alpha$ consists of the Dirichlet series finite in the norm

$$\left\| \sum_{n\in\mathbb{N}} a_n n^{-s} \right\|_{\mathscr{D}_\alpha} = \left(\sum_{n\in\mathbb{N}} |a_n|^2 d(n)^\alpha \right)^{1/2}, \quad \alpha \in \mathbb{R}. \tag{2.17}$$

In the limiting case $\alpha = +\infty$, the space consists of the Dirichlet series $\sum_{p\in\mathbb{P}} a_p p^{-s}$ that are finite in the norm

$$\left\| \sum_{p\in\mathbb{P}} a_p p^{-s} \right\|_{\mathscr{D}_\infty} = \left(\sum_{p\in\mathbb{P}} |a_p|^2 \right)^{1/2}.$$

Here $d(n)$ denotes the number of divisors of the integer n and p_n the n'th prime number. It follows by the properties of this divisor function which we list below that the spaces $\mathscr{D}_\alpha$ for $\alpha \in \mathbb{R} \cup \{+\infty\}$ consist of functions analytic on $\mathbb{C}_{1/2}$.

The divisor function

It is convenient to express the divisor function $d(n)$ as

$$d(n) = \sum_{k|n} 1.$$

Some basic results on this function is given in G. H. Hardy and E. M. Wright's book [34]. The divisor function is multiplicative in the sense that if m, n are relatively prime integers, then $d(nm) = d(n)d(m)$. It is easily computed that $d(p^n) = n + 1$ for prime numbers p. A basic estimate is that for all $\epsilon > 0$ it holds that

$$d(n) = \mathcal{O}\left(n^\epsilon\right). \tag{2.18}$$

In particular, this implies that the series (2.17) converge absolutely for s with $\sigma > 1/2$. The inequality implied by (2.18) is best possible in the sense that one may show that the divisor function grows faster than any power of the logarithm. It follows that the divisor function is very irregular since prime numbers and highly composite numbers may occur side by side among the natural numbers. In particular this implies that neither $\mathscr{D}_\alpha \subset \mathscr{H}^2_\alpha$ nor $\mathscr{H}^2_\alpha \subset \mathscr{D}_\alpha$. However, the function $d(n)$ does have some good properties expressed both in terms of averages and zeta functions. For instance, it is immediately verified that

$$\zeta(s)^2 = \sum_{n\in\mathbb{N}} d(n) n^{-s}, \qquad \text{and} \qquad \frac{\zeta(s)^4}{\zeta(2s)} = \sum_{n\in\mathbb{N}} d(n)^2 n^{-s},$$

while the following identity dates back to P. G. L. Dirichlet[4],

$$D(n) := \sum_{k=1}^{n} d(k) = n \log n + (2\gamma - 1)n + \mathcal{O}\left(\sqrt{n}\right).$$

Here γ is Euler's constant. More generally, the following identities were announced by S. Ramanujan in [70] and later proved by B. M. Wilson in [82].

$$D_\beta(n) = \sum_{k=1}^{n} d(n)^\beta = Cn(\log n)^{2^\beta - 1} + \mathcal{O}\left(n(\log n)^{2^\beta - 2}\right), \quad \beta \in \mathbb{R}, \tag{2.19}$$

for some constant C depending on α, and

$$\zeta(s)^{2^\beta} \phi_\beta(s) = \sum_{n \in \mathbb{N}} d(n)^\beta n^{-s}, \quad \beta \in \mathbb{R}, \tag{2.20}$$

where ϕ is some function converging absolutely on the half-plane defined by $\sigma > 1/2$. It can be shown that $\phi_\beta(s)$ is non-zero in a neighbourhood of $s = 1$.

Let $\pi_\mathbb{P}$ denote the prime counting function. By the celebrated prime number theorem due to C. J. de la Vallée Poussin [21] and J. Hadamard [32] it is known that

$$\pi_\mathbb{P}(x) = \frac{x}{\log x} + \mathcal{O}\left(\frac{x}{(\log x)^2}\right).$$

Moreover, it is known that close to $s = 1$ it holds that

$$\zeta_\mathbb{P}(s) = \sum_{p \in \mathbb{P}} p^{-s} = \log\left(\frac{1}{s-1}\right) + \mathcal{O}(1). \tag{2.21}$$

[4]The determination of the optimal Big-oh function is called the Dirichlet divisor problem. The function $\sqrt{n}$ that appears here is due to Dirichlet himself. Currently M. N. Huxley [42] has the best estimate which is $n^{131/416}(\log n)^{26947/8320}$.

Relation to classical spaces on polydisks

Recall that the classical space $D_\alpha(\mathbb{D})$ is be the closure of the complex polynomials of one variable in the norm

$$\left\| \sum_{\nu \in \mathbb{N}} a_\nu z^\nu \right\|_{D_\alpha(\mathbb{D})} = \left(\sum_{\nu \in \mathbb{N}} |a_\nu|^2 (\nu+1)^\alpha \right)^{1/2}.$$

We define these spaces on the finite polydisks $\mathbb{D}^k$ to be the closure of complex polynomials of k variables in the norm

$$\left\| \sum_{\nu_1,\ldots,\nu_k \in \mathbb{N}} a_{\nu_1,\ldots,\nu_k} z_1^{\nu_1} \cdots z_k^{\nu_1} \right\|_{D_\alpha(\mathbb{D}^k)} = \left(\sum_{\nu_1,\ldots,\nu_k \in \mathbb{N}} |a_{\nu_1,\ldots,\nu_k}|^2 (\nu_1+1)^\alpha \cdots (\nu_k+1)^\alpha \right)^{1/2}. \quad (2.22)$$

Finally, we define the space[5] $D_\alpha(\mathbb{D}^\infty)$ by declaring that it is the closure of the polynomials on

$$\mathbb{D}^\infty = \{(z_1, z_2, \ldots) : z_n \in \mathbb{D}\},$$

in the norm $\|\cdot\|_{D_\alpha(\mathbb{D}^\infty)}$. This norm is defined on each subset of polynomials only depending on the k first variables by $\|\cdot\|_{D_\alpha(\mathbb{D}^k)}$. We identify the k-tuple $(\nu_1, \ldots, \nu_k)$ with the natural number $n = p_1^{\nu_1} \cdots p_k^{\nu_k}$, and according to this we set $a_n = a_{\nu_1,\ldots,\nu_k}$. Since $d(p^m) = m+1$ for any prime number p, it follows by comparing (2.17) and (2.22) that for Dirichlet polynomials

$$\|\mathcal{B}F\|_{D_\alpha(\mathbb{D}^\infty)} = \|F\|_{\mathscr{D}_\alpha}. \quad (2.23)$$

Hence, the Bohr identification extends to an isometric isomorphism between $D_\alpha(\mathbb{D}^\infty)$ and $\mathscr{D}_\alpha$. Let

$$D_\infty(\mathbb{D}^\infty) = \left\{ \sum_{n \in \mathbb{N}} a_n z_n : \sum_{n \in \mathbb{N}} |a_n|^2 < \infty \right\}.$$

[5] For an in depth explanation of $\mathbb{D}^\infty$ in terms of analyticity and function spaces consult the paper [19]. In our case it suffices to consider the spaces $D_\alpha(\mathbb{D}^\infty)$ as sequence spaces.

It is clear by comparing the norms formally that (2.23) also holds for $\alpha = +\infty$.

Relation to classical spaces on half-planes

Recall that in the case of McCarthy's spaces, the space $\mathscr{H}^2_\alpha$ was analogue to $D_\alpha(\mathbb{C}_{1/2})$. In this case the situation is slightly different. It is clear by the formula for the norm that for $\alpha \in \mathbb{R}$ the reproducing kernel for the space $\mathscr{D}_\alpha$ is given by

$$k_w^{\mathscr{D}_\alpha}(s) = \sum_{n\in\mathbb{N}} \frac{n^{-s-\bar{w}}}{d(n)^\alpha}.$$

By the formula (2.20), it follows that for bounded $s + \bar{w}$ it holds that

$$k_w^{\mathscr{D}_\alpha}(s) = C_\alpha k_w^{D_{1-2^{-\alpha}}}(s) + \mathcal{O}(1), \tag{2.24}$$

for constants $C_\alpha > 0$ that only depend on α. For the space $\mathscr{D}_\infty$ the reproducing kernel is given by

$$k_w^{\mathscr{D}_\infty}(s) = \sum_{p\in\mathbb{P}} p^{-s-\bar{w}}.$$

By (2.21) it satisfies for bounded $s + \bar{w}$ that

$$k_w^{\mathscr{D}_\infty}(s) = C k_w^{D_1}(s) + \mathcal{O}(1), \tag{2.25}$$

where $C > 0$ is an absolute constant. The next Lemma establishes corresponding embeddings[6]. Let Q_I be the half-strip $\sigma > 1/2$ and $t \in I$, for some bounded interval I.

Lemma 2.6. *Suppose $F \in \mathscr{D}_\alpha$ and let I be a bounded interval. Then there exists constants C only depending on the length of I and α such that for $\alpha < 0$ we have*

$$\int_{Q_I} |F(s)|^2 \left(\sigma - \frac{1}{2}\right)^{2^{-\alpha}-2} \mathrm{d}m(s) \leq C\|F\|^2_{\mathscr{D}_\alpha}. \tag{2.26}$$

[6] In the case $\alpha < 0$, the following is reproduced with permission from an unpublished note by K. Seip [78].

For $\alpha > 0$ we have

$$\int_{Q_I} |F'(s)|^2 \left(\sigma - \frac{1}{2}\right)^{2-\alpha} \mathrm{d}m(s) \leq C\|F\|^2_{\mathscr{D}_\alpha}. \tag{2.27}$$

Similarly, for $\alpha = \infty$ and $F(s) = \sum_{p\in\mathbb{P}} a_p p^{-s}$ we have

$$\int_{Q_I} |F'(s)|^2 \mathrm{d}m(s) \leq C\|F\|^2_{\mathscr{D}_\infty}. \tag{2.28}$$

Proof. Let F be a Dirichlet polynomial. By duality,

$$\begin{aligned}\left(\int_I |F(\sigma + \mathrm{i}t)|^2 \,\mathrm{d}t\right)^{1/2} &= \sup_{\substack{g\in L^2\\ \|g\|=1}} \int_I F(\sigma+\mathrm{i}t)\, g(\mathrm{i}t)\mathrm{d}t \\ &= \sup_{\substack{g\in L^2\\ \|g\|=1}} \sum_{n=1}^N a_n n^{-\sigma} \int_I g(\mathrm{i}t) n^{-\mathrm{i}t}\mathrm{d}t \\ &= \sqrt{2\pi} \sup_{\substack{g\in L^2\\ \|g\|=1}} \sum_{n=1}^N a_n \frac{\hat{g}(\log n)}{n^\sigma}.\end{aligned} \tag{2.29}$$

Suppose $\alpha < 0$. Set $\beta = -\alpha$. We multiply and divide each term by $\sqrt{(\log n)^{2\beta-1}/d(n)^\beta}$, and apply the Cauchy-Schwarz inequality to see that this is less than or equal to

$$\left(\sum_{n=1}^N \frac{|a_n|^2}{d(n)^\beta} \frac{(\log n)^{2\beta-1}}{n^{2\sigma-1}}\right)^{1/2} \underbrace{\sup_{\substack{g\in L^2\\ \|g\|=1}} \left(\sum_{n=1}^N |\hat{g}(\log n)|^2 \frac{d(n)^\beta}{n(\log n)^{2\beta-1}}\right)^{1/2}}_{(*)}.$$

Since g has compact support there exists $C > 0$ such that

$$\sup_{\xi\in(k,k+1)} |\hat{g}(\xi)| \leq \|\hat{g}'\|_{L^\infty(\mathbb{R})} |\hat{g}(k)| \leq C\|g\|_{L^2(\mathbb{R})} |\hat{g}(k)|. \tag{2.30}$$

In this way,

$$(*) \leq \sup_{\substack{g\in L^2 \\ \|g\|=1}} \sum_{k\in\mathbb{N}} \sum_{n\in(e^k,\mathrm{e}^{k+1})} |\hat{g}(\log n)|^2 \frac{d(n)^\beta}{n(\log n)^{2^\beta-1}}$$

$$\lesssim \sup_{\substack{g\in L^2 \\ \|g\|=1}} \sum_{k\in\mathbb{N}} |\hat{g}(k)|^2 \sum_{n\in(e^k,\mathrm{e}^{k+1})} \frac{d(n)^\beta}{n(\log n)^{2^\beta-1}} \lesssim 1.$$

In the last inequality we used the fact that $\sum_{n=1}^k d(n)^\beta = \mathcal{O}\left(k(\log k)^{2^\beta-1}\right)$. Hence,

$$\int_{1/2}^\infty \left(\int_I |F(\sigma+\mathrm{i}t)|^2 \mathrm{d}t\right)\left(\sigma-\frac{1}{2}\right)^{2^\beta-2} \mathrm{d}\sigma$$
$$\lesssim \sum \frac{|a_n|^2}{d(n)^\beta}(\log n)^{2^\beta-1}\int_{1/2}^\infty n^{-(2\sigma-1)}\left(\sigma-\frac{1}{2}\right)^{2^\beta-2}\mathrm{d}\sigma \simeq \|F\|^2_{\mathscr{D}_{-\beta}}.$$

The proof for $\alpha > 0$ and $\alpha = \infty$ follow in a similar way. We explain briefly how the latter case is obtained. Let F be a Dirichlet polynomial of the form $F(s) = \sum_{p\in\mathbb{P}} a_p p^{-s}$. We apply (2.29) to the derivative $F'(s)$, and much as before we multiply and divide by $\sqrt{\log p_n}$ and then use the Cauchy-Schwarz inequality to get

$$\int_I |F'(\sigma+\mathrm{i}t)|^2\mathrm{d}t$$
$$\leq \left(\sum_{n\in\mathbb{N}} \frac{|a_n|^2}{p^{2\sigma-1}}\log p_n\right) \underbrace{\sup_{\substack{g\in L^2(I) \\ \|g\|=1}} \left(\sum_{n\in\mathbb{N}} \frac{|\hat{g}(\log p_n)|^2}{p_n}\log p_n\right)}_{(**)}. \quad (2.31)$$

Let $\pi_{\mathbb{P}}(x)$ denote the counting function of the prime numbers. By the prime number theorem and the inequality (2.30), it now follows that

$$(**) \lesssim \sum_{k\in\mathbb{N}} |\hat{g}(k)|^2 \sum_{p_n\in(e^k,\mathrm{e}^{k+1})} \frac{\log p_n}{p_n} \lesssim 1.$$

So integrating (2.31) from 1/2 to ∞ with respect to σ yields

$$\int_{1/2}^{\infty}\left(\int_I |F'(\sigma+\mathrm{i}t)|^2 \mathrm{d}t\right)\mathrm{d}\sigma \lesssim \|F\|^2_{\mathscr{D}_\infty}.$$

□

We remark that this implies that for $\alpha \in \mathbb{R}$ the spaces $\mathscr{D}_\alpha$ embed locally into the spaces $D_{1-2^{-\alpha}}(\mathbb{C}_{1/2})$. It follows by a theorem in section 3.8 that this embedding cannot be improved.

3 Boundary functions

In this chapter the main result is that given any function $f \in H^2(\mathbb{C}_{1/2})$ and bounded interval on the abscissa $\sigma = 1/2$ there exists a function $F \in \mathscr{H}^2$ such that $F - f$ extends analytically over the abscissa on this interval. We observe that the dual formulation of this result is that the sequence

$$\left(\ldots, \frac{(-n)^{\mathrm{i}t}}{\sqrt{-n}}, \ldots, \frac{2^{\mathrm{i}t}}{\sqrt{2}}, 1, 1, \frac{2^{-\mathrm{i}t}}{\sqrt{2}}, \ldots, \frac{n^{-\mathrm{i}t}}{\sqrt{n}}, \ldots\right),$$

where n is understood to run through $\mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$, is a frame in $L^2(I)$ for any bounded interval $I \subset \mathbb{R}$. Taking advantage of this point of view we are able to find explicit asymptotic estimates on upper and lower bounds for the norm of the function F in terms of the function f.

We also give similar results for the spaces $\mathscr{H}^2_\alpha$ and $\mathscr{D}_\alpha$. In addition we note that the result holds for the space $\mathscr{H}^1$ and the projective tensor space $\mathcal{K} = \mathscr{H}^2 \otimes \mathscr{H}^2$.

3.1 Preliminaries

We recall some results and definitions about frames and a result from Semi-Fredholm theory.

Frame

The following information about frames may be found in any general textbook on the subject. In particular, the three following lemmas are standard. See for instance [17, ch. 5].

Let H be a Hilbert space. A sequence of vectors (f_n) in H is called a frame for H with lower frame bound A and upper frame bound B if

$$A\|g\|^2 \leq \sum |\langle g|f_n\rangle|^2 \leq B\|g\|^2 \quad \forall g \in H. \tag{3.1}$$

The constants A, B in the inequalities are called the lower and upper bounds for the frame in question. A frame may be seen as a sort of over-complete basis. In fact, the following holds true.

Lemma 3.1. *A sequence (f_n) is a frame for a Hilbert space H if and only if for any element $f \in H$ there exists a sequence $a_n \in \ell^2$ such that $f = \sum a_n f_n$ and $\|f\|_H^2 \simeq \sum |a_n|^2$.*

It should be noted that the choice of the coefficients is not in general unique. If it is, we call $\{f_n\}$ a Riesz basis for H. Define the operator

$$\mathcal{S} : (a_n) \in \ell^2 \longmapsto \sum a_n f_n \in H. \tag{3.2}$$

By the defining inequalities (3.1) it is not hard to see that the adjoint operator $\mathcal{S}^*$ is bounded and bounded below if and only if (f_n) is a frame for H. The previous lemma then basically states the fact that the operator $\mathcal{S}$ is surjective in this case[1]. In the literature the operator $\mathcal{S}\mathcal{S}^*$ is often called the frame operator. It is practical to use since it is readily verified that for every $f \in H$ it holds that

$$A\|f\| \leq \|\mathcal{S}\mathcal{S}^* f\| \leq B\|f\|, \tag{3.3}$$

with the same constants as in (3.1). We state this as a lemma for easy reference.

Lemma 3.2. *A sequence (f_n) is a frame for a Hilbert space H if and only if the frame operator $\mathcal{S}\mathcal{S}^*$ is bounded and bounded below. Moreover, the upper and lower bounds coincide with the upper and lower frame bounds of (f_n), respectively.*

For a frame (f_n) the sequence $((\mathcal{S}\mathcal{S}^*)^{-1} f_n)$ is also a frame, and it is called the canonical dual frame.

Lemma 3.3. *Let (f_n) be a frame in a Hilbert space H that has frame bounds $A, B > 0$ as in (3.1) and let (g_n) denote its canonical dual frame. Then for all $f \in H$ we have*

$$\frac{1}{B}\|f\|^2 \leq \sum | \langle f|g_n \rangle |^2 \leq \frac{1}{A}\|f\|^2,$$

[1] The reader may wish to consult [73, p. 97] for this standard result.

and the representation

$$f = \sum \langle f|g_n\rangle f_n.$$

Moreover, if (a_n) *is a sequence such that* $f = \sum a_n f_n$ *then*

$$\sum |\langle f|g_n\rangle|^2 < \sum |a_n|^2.$$

We note that if (f_n) is a Riesz basis, then the canonical dual frame is unique and also a Riesz basis.

Semi-Fredholm theory

We also need the following lemma on Banach spaces which is a special case of [49, p. 238, thm. 5.26].

Lemma 3.4 (Second stability theorem of Semi-Fredholm theory)**.** *Let* X, Y *be Banach spaces and* $Z : X \to Y$ *a continuous linear operator that is bounded below. If* $\Phi : X \to Y$ *is a compact operator and* $Z + \Phi$ *is injective, then it follows that* $Z + \Phi$ *is bounded below.*

3.2 Boundary functions for $\mathscr{H}^2$ and $\mathscr{H}^1$

We noted on page 46 that every function in $\mathscr{H}^2$ has non-tangential limits almost everywhere on the abscissa[2] $\sigma = 1/2$. This gives sense to the notation $F(1/2+\mathrm{i}t)$ in the following theorem. We let $\mathbb{C}_I = \{s \in \mathbb{C} : \mathrm{i}(1/2-s) \notin \mathbb{R}\backslash I\}$,i.e. the complex plane with two rays on the abscissa $\sigma = 1/2$ removed, and by $\mathrm{Hol}(\mathbb{C}_I)$ we denote the set of functions holomorphic in $\mathbb{C}_I$.

Theorem 3.5 (Olsen and Saksman 2009)**.** *Let* $I \subset \mathbb{R}$ *be a bounded interval. Then for every* $f \in H^2(\mathbb{C}_{1/2})$ *there exists an* $F \in \mathscr{H}^2$ *such that* $f - F$ *continues analytically to all of* $\mathbb{C}_I$ *with* $\mathrm{Re}(f - F)(1/2 + \mathrm{i}t) = 0$ *on* I*. There exists a unique* $F \in \mathscr{H}^2$ *of minimal norm satisfying this. For this* F *the following holds:*

[2] In fact, on page 6 of the introduction we stated a theorem by Hedenmalm and Saksman which says that the Dirichlet series of functions in $\mathscr{H}^2$ converge almost everywhere on $\sigma = 1/2$.

1. *There exists constants* $C_I > 0$ *such that*

$$\|F\|^2_{\mathscr{H}^2} \leq C_I \|f\|^2_{H^2(\mathbb{C}_{1/2})}. \tag{3.4}$$

2. *Given a bounded subset* $\Omega \subset \mathbb{C}_I$ *at a positive distance from* $\mathbb{C}\backslash\mathbb{C}_I$ *then*

$$\|f - F\|^2_{L^\infty(\Omega)} \leq D_{\Omega,I} \left(1 + \frac{C_I \pi}{|I|}\right) \|f\|^2_{H^2(\mathbb{C}_{1/2})}, \tag{3.5}$$

where

$$D_{\Omega,I} \leq \sup_{s\in\Omega} \left|\frac{1+2s}{2\pi}\right|^2 \int_{\mathbb{R}\backslash I} \frac{\mathrm{d}t}{|s - \frac{1}{2} - \mathrm{i}t|^2}.$$

To prove this theorem we define an operator analogue to the operator E_I which we defined by (2.7) in chapter 2. As before we may assume that $I = (-T, T)$ for some $T > 0$ since the norm of $\mathscr{H}^2$ is invariant under vertical translations. Let $\mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$, and set

$$R_I : (a_n)_{n\in\mathbb{Z}^*} \in \ell^2(\mathbb{Z}^*) \longmapsto \chi_I(t) \sum_{n\in\mathbb{N}} \frac{a_n n^{-\mathrm{i}t} + a_{-n} n^{\mathrm{i}t}}{\sqrt{n}} \in L^2(I).$$

The following lemma gives an analogue of formula (2.8). It is the key to proving Theorem 3.5.

Lemma 3.6 (Olsen and Saksman 2009)**.** *Let* $R_I : \ell^2(\mathbb{Z}^* \to L^2(I)$ *be the operator defined above. Then*

$$R_I R_I^* g = 2\pi g + 2\chi_I (g * \operatorname{Re}\psi_1), \tag{3.6}$$

where $\psi_1(t) = \operatorname{Re}\psi(1+\mathrm{i}t)$ *and* ψ *is the entire function of the formula* (2.2) *for the Riemann zeta function. In particular, the operator* R_I *is bounded.*

Proof of Lemma 3.6. We define, for a sequence $\epsilon_k \to 0$ of positive real numbers, the sequence of operators

$$R_{I,k} : (a_n)_{n\in\mathbb{Z}^*} \in \ell^2(\mathbb{Z}^*) \longmapsto \chi_I(t) \sum_{n\in\mathbb{Z}^*} \frac{a_n n^{-\mathrm{i}t} + a_{-n} n^{\mathrm{i}t}}{n^{\frac{1}{2}+\epsilon_k}} \in L^2(I).$$

It is readily checked that $R_{I,k}(a_n) \to R_I(a_n)$ in norm for (a_n) where only finitely many coefficients are non-zero, and that

$$R_{I,k}R_{I,k}^* g(t) = \chi_I(t) \int_I g(\tau) 2 \operatorname{Re} \zeta(1 + \epsilon_k + \mathrm{i}(t - \tau)) \mathrm{d}\tau.$$

By the formula (2.2) for the Riemann zeta function,

$$\operatorname{Re} \zeta(1 + \epsilon_k + \mathrm{i}t) = \frac{\epsilon_k}{\epsilon_k^2 + t^2} + \operatorname{Re} \psi(1 + \epsilon_k + \mathrm{i}t).$$

Hence, taking limits, we get

$$R_I R_I^* g(t) = 2\pi g(t) + \chi_I(t) \int_I g(\tau) 2 \operatorname{Re} \psi(1 + \mathrm{i}(t - \tau)) \mathrm{d}\tau.$$

□

Combining this lemma with Lemma 3.4 we are able to turn the key, so to speak, and we obtain the following.

Lemma 3.7 (Olsen and Saksman 2009)**.** *The operator R_I is onto $L^2(I)$.*

Note that the operator R_I is exactly of the form (3.2). This means that the operator $R_I R_I^*$ is the frame operator for the sequence

$$\mathscr{G}_I^* = \left(\dots, \frac{(-n)^{\mathrm{i}t}}{\sqrt{-n}}, \dots, \frac{2^{\mathrm{i}t}}{\sqrt{2}}, 1, 1, \frac{2^{-\mathrm{i}t}}{\sqrt{2}}, \dots, \frac{n^{-\mathrm{i}t}}{\sqrt{n}}, \dots \right), \tag{3.7}$$

and the lemma may be interpreted as saying that this sequence is a frame for $L^2(I)$. We will come back to this in section 3.3.

Proof of Lemma 3.7. In light of the lemmas 3.1 and 3.2 the operator R_I is onto $L^2(I)$ if and only if $R_I R_I^*$ is bounded below in norm. We consider the formula (3.6). The first term on the right hand side is a constant multiple of the identity operator which is bounded below. In order to use Lemma 3.4 to show that this implies that $R_I R_I^*$ is bounded below, it suffices to show that the operator $R_I R_I^*$ is injective, and that the operator $\Psi : g \longmapsto \chi_I(g * \operatorname{Re} \psi_1)$ is compact. The last assertion follows essentially from the Riemann-Lebesgue lemma, but we give an explicit argument by approximating it with finite rank operators in the strong norm topology. We make a small adjustment to the function ψ. The operator Ψ does not change if we replace $\operatorname{Re}\psi$ by $\phi(t) = \chi_{2I}(t)\operatorname{Re}\psi(t)$, where $2I$ is the interval symmetric around the origin with twice the Lebesgue measure as I. We consider the Fourier expansion of the function $\phi \in L^2(2I)$, say $\phi(t) = \sum a_n e_n(t)$, where the $e_n(t)$ are the Fourier characters of $2I$. By the Riemann-Lebesgue lemma $a_n \to 0$ as $n \to \infty$. Choose $N \in \mathbb{N}$ so large that $|a_n| \le \epsilon$ for all $n \ge N$. Set ϕ_N equal to the N'th symmetric partial Fourier sum. For $g \in L^2(I)$ we have

$$\begin{aligned}\|\chi_I(g * \phi_{2I}) - \chi_I(g * \phi_N)\|_{L^2(I)} &= \|\sum_{n=N}^{\infty} a_n(g, \chi_I e_n)\chi_I e_n\|_{L^2(I)} \\ &\le \|\sum_{n=N}^{\infty} a_n(g, \chi_I e_n) e_n\|_{L^2(2I)} \\ &\le \epsilon\|g\|_{L^2(2I)} = \epsilon\|g\|_{L^2(I)}.\end{aligned}$$

Since convolution with ϕ_N yields a finite dimensional operator on $L^2(2I)$ we deduce the desired compactness.

We show that $R_I R_I^*$ is injective. Since R_I is always injective on the image of R_I^* it suffices to check that R_I^* is injective. To show this, we need to check that for $g \in L^2(I)$ the condition $\hat{g}(\pm\log n) = 0$ for all $n \in \mathbb{N}$ implies $g = 0$. To get a contradiction, assume that the function g is non-zero. The function $\hat{g}$ is entire and of exponential type $|I|/2$. In particular it is bounded on $\mathbb{R}$ and is therefore of the Cartwright class. A basic property of functions in this class (see [54, lesson 17]) is that the number of zeroes with modulus less than $r > 0$, which we denote by $\lambda(r)$, has to satisfy

$$\lim_{r\to\infty} \frac{\lambda(r)}{r} = \frac{|I|}{\pi}.$$

In our case $\lambda(r) \simeq \mathrm{e}^r$ and it is clear that we get a contradiction. □

With these preparations we are ready to give the proof of our main theorem.

Proof of Theorem 3.5. Let $f \in H^2(\mathbb{C}_{1/2})$ and $I = (-T, T)$. Set $v = \chi_I \operatorname{Re} f(1/2 + \mathrm{i}t)$. Since R_I is surjective and $v \in L^2(I)$ there exists a sequence $(\gamma_n)_{n\in\mathbb{Z}^*}$ such that

$$v = \sum_{n\in\mathbb{N}} (\gamma_n n^{-1/2-\mathrm{i}t} + \gamma_{-n} n^{-1/2+\mathrm{i}t}),$$

with the convergence of the sum being in the sense of $L^2(I)$. It now follows that the function

$$F(s) = \sum_{n\in\mathbb{N}} (\gamma_n + \overline{\gamma_{-n}}) n^{-s}$$

is in $\mathscr{H}^2$ and satisfies $\operatorname{Re} E_I F = v$, where $E_I : \mathscr{H}^2 \to L^2(I)$ is the embedding operator we defined in the introduction. It follows that the function $f - F$ has vanishing real parts on I. It is well known that under these conditions this function has a Schwarz reflection that extends it to all of $\mathbb{C}_I$. Since we need an explicit expression for this extension to prove statement (2), we proceed to explain how one is obtained. For $g \in H^2(\mathbb{C}_{1/2})$ it is well-known that a representation formula based on the real boundary values holds,

$$g(s) = \frac{1}{2\pi\mathrm{i}} \int_{\sigma=1/2} \frac{\operatorname{Re} g(w)}{s - w} \mathrm{d}w. \tag{3.8}$$

If $f - F$ were an element of $H^2(\mathbb{C}_{1/2})$, then $f - F$ could have been extended analytically to all of $\mathbb{C}_I$ using this representation. However, we only know that $(f - F)/s \in H^2(\mathbb{C}_{1/2})$. Since the function $1/s$ is not real valued on the abscissa $\sigma = 1/2$ this does not immediately remedy the situation, although it does offer a way around the problem. Indeed, consider the conformal mapping $\vartheta : \mathbb{D} \to \mathbb{C}_{1/2}$ given by

$$\vartheta(z) = \frac{3 - z}{2 + 2z}. \tag{3.9}$$

It now follows that $(f - F) \circ \vartheta \in H^2(\mathbb{D})$ since by a change of variables we get

$$\int_{\mathbb{T}} \left| (f - F) \circ \vartheta(\mathrm{e}^{\mathrm{i}\theta}) \right|^2 \mathrm{d}\theta = \int_{\mathbb{R}} \left| (f - F) \left(\frac{1}{2} + \mathrm{i}t \right) \right|^2 \frac{2\mathrm{d}t}{1 + t^2} < \infty.$$

For functions in $H^1(\mathbb{D})$, and therefore a fortiori for functions in $H^2(\mathbb{D})$, a representation similar to (3.8) holds, which in our case takes the form

$$(f-F)\circ\vartheta(z) = \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{T}} \frac{\operatorname{Re}(f-F)\circ\vartheta(\xi)}{\xi - z}\mathrm{d}\xi. \tag{3.10}$$

Letting $s = \vartheta(z)$ and making a change variables now yields the representation

$$(f-F)(s) = \frac{1+2s}{2\pi\mathrm{i}}\int_{\sigma=\frac{1}{2}, t\notin I} \frac{\operatorname{Re}(f-F)(w)}{w-s}\frac{\mathrm{d}w}{1+2w}. \tag{3.11}$$

Since $\operatorname{Re}(f-F)(1/2+\mathrm{i}t)$ vanishes almost everywhere for $t \in I$ it follows that the integral defines an analytic function for all $s \in \mathbb{C}_I$. We denote this extension by $f - F$, and we note that it satisfies

$$(f-F)(s) = -\overline{(f-F)(1-\overline{s})} \quad s \in \mathbb{C}.$$

We turn to statement (1). Since R_I is onto and bounded it follows by the open mapping theorem that there exists a constant $C > 0$ such that the sequence (γ_n) above may be chosen to satisfy $\|(\gamma_n)\|_{\ell^2}^2 \leq C\|v\|_{L^2(I)}^2$. Define F in the same way as above. Since $\|v\|_{L^2(I)} \leq \|f\|_{H^2(\mathbb{C}_{1/2})}$ it follows that

$$\|F\|_{\mathscr{H}^2}^2 \leq 2\|\gamma_n\|_{\ell^2}^2 \leq 2C\|f\|_{L^2(I)}^2 \leq 2C\|f\|_{H^2(\mathbb{C}_{1/2})}^2.$$

The existence and uniqueness of the element of minimal norm follows from the general theory of convex sets: a closed convex set of a Hilbert space always has a unique element of minimal norm (see, for instance, section 2.2 of [23]). In particular, the set

$$\left\{F \in \mathscr{H}^2 : \operatorname{Re} F(1/2+\mathrm{i}t) = \operatorname{Re} f(1/2+\mathrm{i}t) \text{ as functions in } L^2(I)\right\}$$

is clearly closed and convex in $\mathscr{H}^2$.

Statement (2) is a consequence of being able to choose the element F prescribed by statement (1) and the representation (3.11). So assume that F satisfies $\|F\|_{\mathscr{H}^2}^2 \leq C_I\|f\|_{H^2(\mathbb{C}_{1/2})}^2$. Applying the Cauchy-Schwarz inequality to (3.11) now gives

$$|(f-F)(s)|^2 \leq \left|\frac{1+2s}{2\pi}\right|^2 \left\|\frac{\operatorname{Re}(f-F)(w)}{1+2w}\right\|_{L^2\left(\sigma=\frac{1}{2}\right)}^2 \int_{\sigma=\frac{1}{2}, t\notin I}\left|\frac{\mathrm{d}w}{(s-w)^2}\right|.$$

Since

$$\left\| \frac{\operatorname{Re}(f-F)(w)}{1+2w} \right\|^2_{L^2\left(\sigma=\frac{1}{2}\right)} \leq \left(1+\frac{\pi C_I}{|I|}\right) \|f\|^2_{H^2(\mathbb{C}_{1/2})},$$

and clearly

$$\sup_{s\in\Omega} \int_{\sigma=\frac{1}{2}, t\notin I} \left| \frac{\mathrm{d}w}{(s-w)^2} \right| < \infty,$$

we get the desired estimate. □

A similar result holds for $\mathscr{H}^1$. For convenience we state it for $I = (-T, T)$.

Corollary 3.8 (Olsen and Saksman 2009)**.** *Let $I = (-T, T)$. Then for every $f \in H^1(\mathbb{C}_{1/2})$ there exists $F \in \mathscr{H}^1$ such that $f - F$ continues analytically to all of $\mathbb{C}_I$ with* $\operatorname{Re}(f-F)(1/2+\mathrm{i}t) = 0$ *on I. Moreover, the function F may be chosen in such a way that the following holds.*

1. *For $\epsilon \in (0,1)$ let $T \geq 1$ and $I_\epsilon = (-(1+\epsilon)T, (1+\epsilon)T)$ Then*

$$\|F\|_{\mathscr{H}^1} \lesssim \frac{\sqrt{|I|}(|I| + C_{I_\epsilon})}{\epsilon} \|f\|_{H^1(\mathbb{C}_{1/2})}.$$

2. *Given a bounded subset $\Omega \subset \mathbb{C}_I$ at a positive distance from $\mathbb{C}\backslash\mathbb{C}_I$ then*

$$\|f-F\|_{L^\infty(\Omega)} \lesssim D_{\Omega,I} \frac{\sqrt{|I|}(|I| + C_{I_\epsilon})}{\epsilon} \|f\|_{H^1(\mathbb{C}_{1/2})},$$

where

$$D_{\Omega,I} \leq \sup_{s\in\Omega} \left| \frac{1+2s}{\pi} \right| \left\| \frac{1+\mathrm{i}\tau}{\frac{1}{2}+\mathrm{i}\tau - s} \right\|_{L^\infty(\mathbb{R}\backslash I)}.$$

We note that the implicit constants are absolute.

Proof. Fix the interval $I \subset \mathbb{R}$. For $f \in H^1(\mathbb{C}_{1/2})$ let $f = JO$ be its unique factorisation into an inner function J and an outer function O. Set $g = JO^{1/2}$ and $h = O^{1/2}$. We have both $g, h \in H^2$, so by Theorem

3.5 there exists functions $G, H \in \mathscr{H}^2$ and $\phi_g, \phi_h \in \mathrm{Hol}(\mathbb{C}_{I_\epsilon})$ such that $g = G + \phi_g$ and $h = H + \phi_h$. Using this,

$$f - GH = g\phi_h + h\phi_g - \phi_g\phi_h.$$

In particular $v = \chi_I \operatorname{Re}(f - GH) \in L^2(I)$. Let

$$\widetilde{v}(s) = \frac{1}{2\pi \mathrm{i}} \int_I v(\tau) \frac{1}{s - \mathrm{i}\tau - \frac{1}{2}} \mathrm{d}\tau.$$

Then $\widetilde{v} \in H^2(\mathbb{C}_{1/2})$ and so we may find $V \in \mathscr{H}^2$ and $\phi_v \in \mathrm{Hol}(\mathbb{C}_I)$ such that $\widetilde{v} = V + \phi_v$. Let $F = GH + V$. It now follows that $F \in \mathscr{H}^1$ and moreover that $\operatorname{Re}(f - F)(1/2 + \mathrm{i}t) = 0$ on I.

We turn to the norm bound of statement (1). First, note that $\widetilde{v}(1/2 + \mathrm{i}t) = P_+ v$ almost everywhere on for $t \in \mathbb{R}$, where P_+ denotes the Riesz projection which we defined in (2.10). Since the Riesz projection is a contraction, we get

$$\|\widetilde{v}\|_{H^2} = \|P_+ v\|_{L^2(\mathbb{R})} \le \|v\|_{L^2(I)}.$$

In addition, we may choose the functions G, H in such a way that they each satisfy an inequality of the type (3.4) and (3.5). Due to the special choice of the functions g, h in terms of the inner-outer factorisation of the function $f \in H^1$ we get

$$\|GH\|_{\mathscr{H}^1} \le \|G\|_{\mathscr{H}^2} \|H\|_{\mathscr{H}^2} \le C_{I_\epsilon} \|g\|_{H^2} \|h\|_{H^2} = C_{I_\epsilon} \|f\|_{H^1}.$$

Combining this with the inequality $\|V\|_{\mathscr{H}^1} \le \|V\|_{\mathscr{H}^2} \le \sqrt{C_I} \|\widetilde{v}\|_{H^2}$, we get

$$\|F\|_{\mathscr{H}^1} = \|V + GH\|_{\mathscr{H}^1} \le \sqrt{C_I} \|v\|_{L^2(I)} + C_{I_\epsilon} \|f\|_{H^1}.$$

What remains is to estimate

$$\|v\|_{L^2(I)} \le \|\phi_g\|_{L^\infty(I)} \|h\|_{L^2(I)} + \|\phi_h\|_{L^\infty(I)} \|g\|_{L^2(I)} + \sqrt{I} \|\phi_g \phi_h\|_{L^\infty(I)}.$$

We use the inequality (3.5) with $\Omega = (\mathrm{i}I + 1/2)$ to find bounds for ϕ_g and ϕ_h. Since $\epsilon \in (0, 1)$ and $T \ge 1$ we get

$$D_{(\mathrm{i}I + \frac{1}{2}), I_\epsilon} \le \frac{1 + T^2}{\pi^2} \frac{4(1 + \epsilon)}{\epsilon(2\epsilon + 1)T} \lesssim \frac{T}{\epsilon}.$$

Hence,

$$\|\phi_g\|^2_{L^\infty(I)} \lesssim \frac{1}{\epsilon}\left(|I| + C_{I_\epsilon}\right)\|g\|^2_{H^2}.$$

A corresponding upper bound is valid for $\|\phi_h\|_{L^\infty(I)}$ and so

$$\|v\|_{L^2(I)} \lesssim \sqrt{|I|}\frac{|I| + C_{I_\epsilon}}{\epsilon}\|f\|_{H^1},$$

where the implicit constant is absolute.

Statement (2) follows much as before and we only give a sketch of how to proceed. Let ϑ be the conformal map defined by (3.9). By the formula $F = V + GH$ one may show that $(f - F) \circ \vartheta$ belongs to $H^1(\mathbb{D})$. This implies that the representations (3.10) and (3.11) holds for $(f - F) \circ \vartheta$. Hence, we get the Schwarz reflection of $(f - F)$ and the inequality

$$|(f - F)(s)| \leq \int_{\sigma=\frac{1}{2}, t\notin I}\left|\frac{\mathrm{Re}(f - F)(w)}{(1 + 2w)^2}\right||\mathrm{d}w| \\ \times \sup_{s\in\Omega}\left|\frac{1 + 2s}{2\pi}\right|\left\|\frac{1 + 2w}{w - s}\right\|_{L^\infty\left(\sigma=\frac{1}{2}, t\notin I\right)}.$$

Finally, we note that

$$\int_{\sigma=\frac{1}{2}, t\notin I}\left|\frac{\mathrm{Re}(f - F)(w)}{(1 + 2w)^2}\right||\mathrm{d}w| \lesssim (1 + C_{I_\epsilon})\|f\|_{H^1} + \sqrt{C_{I_\epsilon}}\|v\|_{L^2(I)}.$$

Using the previous estimates, this ends the proof. □

3.3 A dual formulation and asymptotic calculations

In this section we consider a dual formulation of Theorem 3.5. In particular this will help us determine the asymptotic behaviour of the constant C_I of Theorem 3.5. Recall that the operator R_I was defined by

$$R_I : (a_n)_{n\in\mathbb{Z}^*} \in \ell^2 \longmapsto \chi_I(t)\left(\sum_{n\in\mathbb{N}} a_n\frac{n^{-\mathrm{i}t}}{\sqrt{n}} + a_{-n}\frac{n^{-\mathrm{i}t}}{\sqrt{n}}\right) \in L^2(I).$$

By comparing this to the operator defined by (3.2) we see that R_I is essentially the frame operator in the space $L^2(I)$ for the set of vectors

$$\mathscr{G}_I = \left(\ldots, \frac{(-n)^{\mathrm{i}t}}{\sqrt{(-n)}}, \ldots, \frac{2^{\mathrm{i}t}}{\sqrt{2}}, 1, \frac{2^{-\mathrm{i}t}}{\sqrt{2}} \ldots, \frac{n^{-\mathrm{i}t}}{\sqrt{n}}, \ldots \right),$$

where n runs through $\mathbb{Z}^*$. Lemma 3.7 says that R_I is onto, which by Lemma 3.2 establishes that the sequence $\mathscr{G}_I^*$ defined by (3.7) for which it is a frame operator really is a frame. Note, however, that the operator R_I counts the constant function of $L^2(I)$ twice. Since we are interested in finding explicit bounds for the upper and lower frame constants we find it convenient to only count this function once. We let $\mathcal{S}_I$ denote the frame operator for $\mathscr{G}_I$.

Theorem 3.9 (Olsen and Saksman 2009)**.** *Let $I \subset \mathbb{R}$ be an interval. Then $\mathscr{G}_I$ forms a frame for $L^2(I)$. I.e. there exists optimal frame bounds $A_I, B_I > 0$, depending only on the length of I, such that for all $f \in L^2(I)$ we have*

$$A_I \|f\|^2 \leq 2\pi \left(|\hat{f}(0)|^2 + \sum_{n \in \mathbb{N} \setminus \{1\}} \frac{|\hat{f}(\log n)|^2 + |\hat{f}(-\log n)|^2}{n} \right) \leq B_I \|f\|^2.$$

Moreover,

$$|I| \leq B_I \leq |I| + d, \tag{3.12}$$

where $d \leq 13.3138\ldots$. Let $\epsilon > 0$ be given. Then for $|I| \geq 1$ there exists constant only depending on ϵ such that

$$|I|^{-(1+\epsilon)\frac{6|I|}{\pi}\log 2} \lesssim A_I \lesssim |I|^{-(1-\epsilon)\frac{I}{\pi}\log \pi}, \tag{3.13}$$

Also,

$$\lim_{|I| \to 0} A_I = \lim_{|I| \to 0} B_I = 2\pi. \tag{3.14}$$

We immediately state and prove the following theorem.

Theorem 3.10 (Olsen and Saksman 2009)**.** *Let $I \subset \mathbb{R}$ be a bounded interval. Then there exists constants c_I, C_I only depending on the length of I such that given $f \in H^2(\mathbb{C}_{1/2})$ the following holds. If $F \in \mathscr{H}^2$ is the unique minimal element such that $\operatorname{Re} E_I F = \operatorname{Re} f(1/2 + \mathrm{i}t)$ holds, then*

$$c_T \|\chi_I \operatorname{Re} f(1/2 + \mathrm{i}t)\|^2_{L^2(I)} \leq \|F\|^2_{\mathscr{H}^2} \leq C_T \|\chi_I \operatorname{Re} f(1/2 + \mathrm{i}t)\|^2_{L^2(I)}.$$

For $|I| > 0$ it holds that

$$\frac{1}{|I| + d} \leq c_I \leq \frac{2}{|I|}.$$

and given $\epsilon > 0$ and $|I| > 1$ there are constants only depending on ϵ such that

$$|I|^{(1-\epsilon)\frac{|I|}{\pi}\log \pi} \lesssim C_I \lesssim |I|^{(1+\epsilon)\frac{6|I|}{\pi}\log 2}.$$

Also,

$$\lim_{|I| \to 0} C_I = \frac{2}{\pi},$$

and

$$\frac{1}{\pi} \leq \liminf_{|I| \to 0} c_I, \quad \limsup_{|I| \to \infty} c_I \leq \frac{2}{\pi}.$$

Proof. Let $f \in H^2(\mathbb{C}_{1/2})$ and $I = (-T, T)$. Set $v = \chi_I \operatorname{Re} f(1/2 + \mathrm{i}t)$. Since $R_I R_I^*$ is the frame operator for the sequence $\mathscr{G}_I^*$ it follows by Lemma 3.3 that for the choice $\gamma_n = \left\langle v | (R_I R_I^*)^{-1} n^{-1/2-\mathrm{i}t} \right\rangle$ for $n \geq 1$ and correspondingly for $n \leq -1$ we have

$$v = \sum_{n \in \mathbb{N}} (\gamma_n n^{-1/2-\mathrm{i}t} + \gamma_{-n} n^{-1/2-\mathrm{i}t}).$$

Following the proof of Theorem 3.5 we find that the function

$$F(s) = \sum_{n \in \mathbb{N}} (\gamma_n + \overline{\gamma}_{-n}) n^{-s}$$

satisfies $\operatorname{Re} E_I F = v$. It is readily checked that $\gamma_n = \overline{\gamma_{-n}}$, and so

$$\|F\|^2_{\mathscr{H}^2} = 2\|\gamma_n\|^2_{\ell^2(\mathbb{Z}^*)}. \tag{3.15}$$

We check that this choice of coefficients is optimal. Let $G(s) = \sum z_n n^{-s}$ be such that $\operatorname{Re} E_I G = v$. Write $z_n = x_n + \mathrm{i}y_n$. Then

$$v = \sum_{n\in\mathbb{N}} \left(\frac{x_n + \mathrm{i}y_n}{2} n^{-1/2-\mathrm{i}t} + \frac{x_n - \mathrm{i}y_n}{2} n^{-1/2+\mathrm{i}t} \right)$$

gives an expansion for v in the frame $\mathscr{G}^*$. Since the (γ_n) is the coefficients of v relative to the frame $\mathscr{G}_I^*$ given by the canonical dual frame, Lemma 3.3 implies that

$$\sum_{n\in\mathbb{Z}^*} |\gamma_n|^2 < \sum_{n\in\mathbb{N}} \frac{|x_n + \mathrm{i}y_n|^2}{4} + \frac{|x_n - \mathrm{i}y_n|^2}{4}.$$

It now follows that $\|F\|_{\mathscr{H}^2} < \|G\|_{\mathscr{H}^2}$. This establishes the optimality of the choice of coefficients.

We turn to the explicit bounds of the corollary. Using the definition of the sequence $(\gamma_n)_{n\in\mathbb{Z}^*}$ and Lemma 3.3, we get

$$\frac{1}{\widetilde{B}_I} \|v\|^2_{L^2(I)} \le \sum_{n\in\mathbb{Z}^*} |\gamma_n|^2 \le \frac{1}{\widetilde{A}_I} \|v\|^2_{L^2(I)} \tag{3.16}$$

where $\widetilde{A}_I$ and $\widetilde{B}_I$ are the lower and upper frame bounds for $\mathscr{G}_I^*$ when restricted to the real vector space $L^2_{\mathbb{R}}(I)$, respectively. It is easily checked that $\mathscr{G}_I^*$ has lower frame bound A_I and upper frame bound is less than $2B_I$. But since the frame operator $R_I R_I^*$ is self-adjoint and preserves real functions, it follows that it has the same upper and lower norm bounds on $L^2(I)$ as when restricted to $L^2_{\mathbb{R}}(I)$. Hence $A_I = \widetilde{A}_I$ and $B_I \le \widetilde{B}_I \le 2B_I$ holds. By (3.15) this implies that the following inequalities hold

$$c_I \|v\|^2_{L^2(I)} \le \|F\|^2_{\mathscr{H}^2} \le C_I \|v\|^2_{L^2(I)},$$

with $B_I^{-1} \le c_I \le 2B_I^{-1}$ and $C_I = 2A_I^{-1}$. By the previous theorem, the result follows. □

We turn to the proof of Theorem 3.9. What remains is to show the quantitative statements on the frame bounds. We denote the operator $\mathcal{S}_I$ by $\mathcal{S}_{2T}$. The inequalities (3.13) are an immediate consequence of the two following lemmas.

Lemma 3.11 (Olsen and Saksman 2009). *Let $\epsilon \in (0,1)$. For $T > 2\pi$ we define the parameter $\mu > 1$ through $T = (1+\epsilon)\pi\mu^{-1}\mathrm{e}^{\mu}$ and let*

$$G(x) = \sin \pi x \prod_{k=2}^{[\mathrm{e}^{\mu}]} \frac{\sin \frac{\pi}{\log k} x}{\frac{\pi}{\log k} x}.$$

For T large enough, we have $g = \mathcal{F}^{-1}G \in L^2(-T,T)$ and

$$\frac{\|\mathcal{S}_{2T}^* g\|_{\ell^2}^2}{\|g\|_2^2} \leq T^{-(1-\epsilon)\frac{2T}{\pi}\log \pi}.$$

Proof. This is proved in section 3.4 □

Lemma 3.12 (Olsen and Saksman 2009). *Let $\epsilon > 0$ be given. For T large enough and $f \in L^2(-T,T)$ it is possible to choose a set of frequencies Λ in such a way that both inequalities*

$$\sum_{n \in \mathbb{N}} \frac{|\hat{f}(\log n)|^2}{n} \geq \frac{1}{2T} \sum_{\lambda \in \Lambda} |\hat{f}(\lambda)|^2, \tag{3.17}$$

and

$$\|f\|_2^2 \leq T^{(1+\epsilon)\frac{12T}{\pi}\log 2} \sum_{\lambda \in \Lambda} |\hat{f}(\lambda)|^2 \tag{3.18}$$

are satisfied.

Proof. This is proved in section 3.5. □

To get the limits (3.14) we consider the operator $\mathcal{S}_I$. Recall that the upper and lower norm bounds for the operator $\mathcal{S}_I\mathcal{S}_I^*$ give exactly the upper and lower frame bounds for $\mathscr{G}_I$. Next, it is not hard to see that for $g \in L^2(I)$ we get

$$\mathcal{S}\mathcal{S}_I^* g = R_I R_I^* g - \chi_I \int_I g(\tau)\mathrm{d}\tau.$$

Since the norm of the operator defined by the right-most term goes to zero as $|I|$ goes to zero, it follows by the formula (3.6) that $\|\mathcal{S}_I\mathcal{S}_I^* - 2\pi\mathrm{Id}\|$, where Id denotes the identity operator on $L^2(I)$, tends to zero as $|I|$ tends to zero. The desired limits then follow.

We turn to the inequalities (3.12). Let $(a_n)_{n\in\mathbb{Z}^*\setminus\{-1\}}$ be a sequence with only finitely many non-zero coefficients. For some $N \in \mathbb{N}$ we have

$$\|\mathcal{S}_I(a_n)\|^2 = \int_I \left| \sum_{n=1}^{N} a_n n^{-1/2-\mathrm{i}t} + \sum_{n=2}^{N} a_{-n} n^{-1/2+\mathrm{i}t} \right|^2 \mathrm{d}t. \tag{3.19}$$

Finding the upper bound of this amounts to finding the upper frame bound. This was essentially calculated by Montgomery using an inequality due to Montgomery and Vaughan [59, eq. (27) p. 140]. We state this inequality without proof and show how the bound follows from it.

Lemma 3.13 (Montgomery and Vaughan 1994)**.** *For $N \in \mathbb{N}$ let $\lambda_1, \ldots, \lambda_N$ be distinct real numbers and $\delta_n = \min_{m\le N, m\ne n} |\lambda_m - \lambda_n|$. Then*

$$\sum_{n,m=1}^{N} \frac{x_n y_m}{\lambda_n - \lambda_m} \le \gamma_0 \left(\sum_{n=1}^{N} \frac{|x_n|^2}{\delta_n} \right)^{1/2} \left(\sum_{n=1}^{N} \frac{|y_n|^2}{\delta_n} \right)^{1/2}, \tag{3.20}$$

where $\gamma_0 \le 3.2$.

We follow Montgomery's argument. By applying the triangle inequality to (3.19), it suffices to consider $G(s) = \sum_{n=1}^{N} a_n n^{-s}$ and $I = (-T, T)$. Multiplying out $(\sum a_n n^{-s})(\overline{\sum a_n n^{-s}})$ and integrating, we get

$$\int_{-T}^{T} \left| G\left(\frac{1}{2} + \mathrm{i}t\right) \right|^2 \mathrm{d}t = 2T \sum_{n=1}^{N} \frac{|a_n|^2}{n} + 2 \sum_{n\ne m} \frac{a_n \overline{a}_m}{\sqrt{nm}} \frac{\sin(T \log \frac{n}{m})}{\log \frac{n}{m}}.$$

We apply inequality (3.20) to the second term on the right hand side to find that this is less than or equal to

$$\left(2T + \frac{2\gamma_0}{\log 2}\right) \sum_{n=1}^{N} |a_n|^2.$$

Taking into account $|I| = 2T$, we get the upper bound of (3.12). The lower bound follows by applying this to the constant function.

3.4 Proof of Lemma 3.11

The point of the proof is to construct a function $g \in L^2(-T,T)$ for which the mass $\int_{-T}^{T} |\hat{g}(\xi)|^2 \mathrm{d}\xi$ is the biggest possible under the restriction $\hat{g}(\pm\log n) = 0$ for $\pm\log n \in (-T,T)$.

Let $\epsilon > 0$ be fixed. It is readily checked that

$$\operatorname{supp} \mathcal{F}^{-1} \prod_{k=2}^{\mathrm{e}^\mu} \frac{\sin \frac{\pi}{\log k} x}{\frac{\pi}{\log k} x} \subset \left(-\pi \sum_{k=2}^{\mathrm{e}^\mu} \frac{1}{\log k}, \pi \sum_{k=2}^{\mathrm{e}^\mu} \frac{1}{\log k} \right).$$

We note that $\sum_{k=2}^{[\mathrm{e}^\mu]} \log^{-1} k / (\mu^{-1}\mathrm{e}^\mu) \to 1$ as $\mu \to \infty$. Hence, there is a number $\mu_0 > 0$ such that for $\mu > \mu_0$ we have

$$\operatorname{supp} \mathcal{F}^{-1} G \subset \Big(-(1+\epsilon)\pi\mu^{-1}\mathrm{e}^\mu, (1+\epsilon)\pi\mu^{-1}\mathrm{e}^\mu \Big).$$

Since the parameter μ is chosen by requiring that $T = (1+\epsilon)\pi\mu^{-1}\mathrm{e}^\mu$, the function $g = \mathcal{F}^{-1}G$ then satisfies $\operatorname{supp} g = (-T,T)$. The rest of the proof is split into two parts. First we find a lower bound for $\|g\|_2$, then we compute an upper bound for $\|\mathcal{S}_{2T}^* g\|_{\ell^2}$.

1) Our lower bound for $\|g\|_2$ is crude and based on Bernstein's inequality. This inequality says that for functions in the Bernstein space, i.e. functions $F \in L^\infty(\mathbb{R})$ which satisfy $\mathcal{F}F \in L^\infty(-T,T)$, it holds that $\|F'\|_\infty \le T\|F\|_\infty$. We let $F(x) = G(x)/\sin \pi x$. Clearly F is in the Bernstein space and satisfies $F(0) = 1$ and $\|F\|_\infty \le 1$. By the Bernstein inequality $\|F'\|_\infty \le T$, and so

$$|F(x)| \ge 1 - Tx$$

for $x \in (-T^{-1}, T^{-1})$. Since $\sin \pi x \ge 2x$ for $x \in (0, 1/2)$, we get for $T > 2$ the estimate

$$\|G\|_{L^2(\mathbb{R})}^2 \ge 8 \int_0^{T^{-1}} x^2 (1 - Tx)^2 \mathrm{d}x = \frac{4}{15T^3}. \tag{3.21}$$

2) We check the upper bound for $\|\mathcal{S}_{2T}^* g\|_{\ell^2}$. For $n \in \mathbb{N}$ such that $\log n > \mu$, we simply use $|\sin x| \le 1$ and the inequality $\sum_{k=2}^{[\mathrm{e}^\mu]} \log\log k \le \mathrm{e}^\mu \log \mu$

to get

$$|G(\log n)| \le \prod_{k=2}^{[\mathrm{e}^\mu]} \frac{\log k}{\pi \log n} \le \frac{\mu^{[\mathrm{e}^\mu]}}{(\pi \log n)^{[\mathrm{e}^\mu]-1}}.$$

Summing over $\log n \ge \mu$, we use the formula $\int_{\mathrm{e}^\mu}^\infty x^{-1} \log^{-\alpha} x \mathrm{d}x = (\alpha - 1)^{-1}\mu^{1-\alpha}$ for $\alpha > 1$ to find that

$$\sum_{\log n \ge \mu} \frac{|G(\log n)|^2}{n} \le \frac{\mu^{2[\mathrm{e}^\mu]}}{\pi^{2[\mathrm{e}^\mu]-2}} \sum_{\log n \ge \mu} \frac{1}{n \log n^{2[\mathrm{e}^\mu]-2}}$$
$$\le \exp\left\{-2\log\pi\left(1-\frac{\epsilon}{2}\right)\mathrm{e}^\mu\right\}.$$

Since G is an odd function satisfying $G(\log n) = 0$ for n such that $\log n \le \mu$, we may use the relation $T = (1+\epsilon)\pi\mu^{-1}\mathrm{e}^\mu$ and, again, play a game of epsilons to conclude that

$$\|\mathcal{S}_{2T}^* g\|_{\ell^2} \le T^{-(1-\epsilon)\frac{\log \pi}{\pi} T}.$$

The lemma now follows by combining this with the estimate (3.21).

3.5 Proof of Lemma 3.12

We give a short outline of how we proceed. Given $f \in L^2(-T, T)$ we construct the set Λ from the harmonic frequencies of the space $L^2(-W, W)$ where $W = (1+\eta)T$ with $\eta \in (0, 1)$. We perturb these frequencies so that they coincide with members of the set $\pm \log \mathbb{N} = \big\{ \log n \in \mathbb{N} : n \in \mathbb{N}\big\} \cup \big\{ -\log n : n \in \mathbb{N}\big\}$ while minimising the size of $\hat{f}$ on certain intervals. In this way we ensure that (3.17) holds.

The inequality (3.18) is harder to establish. We combine an approach found in [28] with a well-known stability theorem due to M. I. Kadec [45], and some growth estimates due to S. A. Avdonin [1]. The point is to ensure that the perturbation process used to construct Λ is done in such a way we get a representation of the type

$$\hat{f}(x) = \sum_{\lambda \in \Lambda} \hat{f}(\lambda)\Psi_\lambda(x) \tag{3.22}$$

making it possible to consider estimates of the type

$$\|f\|^2_{L^2(-T,T)} \le \left(\sum_{\lambda\in\Lambda} |\hat{f}(\lambda)|^2\right)\left(\sum_{\lambda\in\Lambda} \|\Psi_\lambda(x)\|^2_{L^2(\mathbb{R})}\right).$$

It is in making these functions Ψ_λ decay fast enough that we need the over-sampling that we achieve by considering $L^2(-T,T)$ as a subspace of $L^2(-W,W)$.

Constructing the set Λ

Fix $f \in L^2(-T,T)$. We remark that although the set Λ depends on f, the estimates below will be uniform over such sets. Our starting point is the set of harmonic frequencies $\{\pi k/W\}_{k\in\mathbb{Z}}$ of the space $L^2(-W,W)$. The first fundamental fact is Kadec's 1/4-Theorem which we state in the following lemma. (See e.g. [83, p. 36, thm. 14] for a proof.)

Lemma 3.14 (Kadec's 1/4-Theorem, 1964)**.** *Let $\{\mu_n\}_{n\in\mathbb{Z}}$ be a sequence of real numbers such that $|\mu_n - n| \le \delta < 1/4$, then $\{\mathrm{e}^{\mathrm{i}\mu_n t}\}_{n\in\mathbb{Z}}$ forms a Riesz base for $L^2(-\pi,\pi)$ with bounds only depending on $\delta > 0$.*

In particular, by scaling, we find that if $\{\mu_n\}_{n\in\mathbb{Z}}$ is such that $|\mu_n - \pi n/W| \le \rho < \pi/4W$ then $\{\mathrm{e}^{\mathrm{i}\mu_n t}\}_{n\in\mathbb{Z}}$ forms a Riesz base for $L^2(-W,W)$ satisfying

$$W\|g\|^2_{L^2(-W,W)} \simeq \sum_{n\in\mathbb{Z}} |\left\langle g|\mathrm{e}^{\mathrm{i}\mu_n t}\right\rangle|^2, \tag{3.23}$$

with the implicit constants only depending on $W\rho > 0$.

We split the construction of Λ into two steps:

1) For each harmonic frequency $\pi k/W$ let I_k denote the open interval of radius $\rho = 1/2W < \pi/4W$ centered on this frequency, i.e.

$$I_k = \left(\frac{\pi k - 1/2}{W}, \frac{\pi k + 1/2}{W}\right).$$

For $n \in \mathbb{N}$ the distance between $\log n$ and $\log(n+1)$ is less than n^{-1}. This means that for $k \in \mathbb{Z}$ such that $|\pi k/W| \ge \log W$, the neighbourhoods I_k

will always contain a member of $\pm\log\mathbb{N}$. We define a threshold

$$k_0 = \left[\frac{W\log W}{\pi}\right].$$

Here the brackets denote the integer function. Now it follows that for each $k > k_0$ we may choose $n_k \in \mathbb{N}$ such that $\log n_k \in I_k$ and for which

$$|\hat{f}(\log n_k)| = \min_{n\in\mathbb{N}:\log n\in I_k} |\hat{f}(\log n)|.$$

We do the corresponding selection for the negative frequencies. I.e. for $k < 0$ we choose $n_k \in \mathbb{N}$ such that $\log n_k \in I_{-k}$ and for which

$$|\hat{f}(-\log n_k)| = \min_{n\in\mathbb{N}:\log n\in I_k} |\hat{f}(-\log n)|.$$

Define

$$\lambda_k = \begin{cases} \log n_k & \text{if } k > k_0 \\ -\log n_k & \text{if } k < -k_0 \end{cases},$$

and let $\Lambda_w = \left\{\lambda_k\right\}_{|k|>k_0}$. In particular

$$U = \{\pi k/W\}_{|k|\le k_0} \cup \Lambda_w \tag{3.24}$$

satisfies the Kadec theorem with $\rho W = 1/2$. Hence, $E_U = \left\{\mathrm{e}^{\mathrm{i}\mu t}\right\}_{\mu\in U}$ forms a Riesz base for $L^2(-W, W)$ with the same bounds as in (3.23).

2) Let L_k denote the open interval of radius $1/2W$ centered on the point $\pi(k + k_0 + 1/2)/W$ for $k \ge 0$ and at $\pi(k - k_0 - 1/2)/W$ for $k < 0$. The intersection of the sets L_k with $\pm\log\mathbb{N}$ is non-empty. For $|k| \le k_0$ choose the number n_k minimising the value of $|\hat{f}(\mathrm{sgn}(n)\log|n|)|$ on $L_k \cap \pm\log\mathbb{N}$. Let

$$\lambda_k = \begin{cases} \log n_k & \text{if } \quad 0 \le k \le k_0 \\ -\log n_k & \text{if } \ -k_0 \le k < 0 \end{cases}.$$

We denote $\Lambda_0 = \left\{\lambda_k\right\}_{|k|\le k_0}$ and set

$$\Lambda = \Lambda_w \cup \Lambda_0. \tag{3.25}$$

We show in the next subsection that the property of being a Riesz basis is stable under the arbitrary perturbation of a finite number of points as long as the new set of points is separated. This is a special case by a theorem of Avdonin in [1]. Note that the intervals I_k and L_k are disjoint, and moreover that we have the separation

$$\min_{\substack{n\in\mathbb{Z}\\ n\neq m}} |\lambda_m - \lambda_n| \geq \frac{1}{2W}. \tag{3.26}$$

A sampling formula

For $f \in L^2(-T,T)$ a well-known sampling formula is the Whittaker-Kotel'nikov-Shannon formula

$$\hat{f}(x) = \sum_{k\in\mathbb{N}} \hat{f}\left(\frac{\pi k}{T}\right) \frac{\sin Tx}{(-1)^k T\left(x - \frac{\pi k}{T}\right)}.$$

The formula follows by taking the $L^2(\mathbb{R})$ Fourier transform of both sides of the Fourier series

$$f = (\sqrt{2\pi}/2T)\chi_{(-T,T)} \sum_{k\in\mathbb{N}} \hat{f}(\pi k/T)\mathrm{e}^{\mathrm{i}\pi kt/T}.$$

There are two problems with this formula; we want to represent the function f in terms of the frequencies Λ, and it does not converge fast enough to separate the sampled coefficients from the rest of the terms in the way indicated above. Faced with a similar problem, it was realised by K. M. Flornes, Y. Lyubarskii and Seip in [28] that the correct replacement for this formula is the Boas-Bernstein formula (see e.g. [7, p. 193]). The formula says that if $\{\lambda_k\}_{k\in\mathbb{Z}}$ is a sequence of real numbers such that $\sup_{k\in\mathbb{Z}} |\lambda_k - k| < \infty$, then for some $l \in \mathbb{N}$ large enough, it holds that

$$\hat{f}(x) = \sum_{k\in\mathbb{Z}} \hat{f}(\lambda) h_l(x - \lambda_k) \frac{G(x)}{G'(\lambda_k)(x - \lambda_k)}, \tag{3.27}$$

where

$$G(x) = \prod_{\lambda\in\mathbb{Z}}{}' \left(1 - \frac{x}{\lambda_k}\right)$$

should be thought of as a sine type function while the factor

$$h_l(x) = \left(\frac{\sin \eta x/l}{\eta x/l}\right)^l$$

helps the sum converge. The symbol $\prod'$ means that whenever $\lambda_k = 0$ the corresponding factor is taken to be z. Clearly the hypothesis is satisfied for the sequence Λ. Our function G is slightly more complicated than what was studied in [28].

We now explain for the readers convenience why this formula holds in our case, and at the same time we collect some explicit estimates that we need. Recall that by Kadec's 1/4-Theorem the set of frequencies U defined by (3.24) gives a Riesz base for $L^2(-W, W)$. Let

$$S(z) = \prod_{\lambda \in U}{}' \left(1 - \frac{z}{\lambda}\right). \tag{3.28}$$

It is not hard to see that $S(z)$ is a function of exponential type. For instance, denote $|z| = r$. Then for any $\epsilon > 0$ there exists constants $R > 0$, $K > 0$ and a polynomial $P(r)$ such that for $r > R$ we have

$$\begin{aligned} |S(z)| \leq P(r) \prod_{k \geq K} \left(1 + \frac{(1+\epsilon)^2 W^2 r^2}{\pi^2 k^2}\right) \\ \leq P(r) \sin\left(\mathrm{i}(1+\epsilon)Wr\right) \lesssim \mathrm{e}^{(1+\epsilon)Wr}. \end{aligned} \tag{3.29}$$

We note that in this computation, the implicit constant depends polynomially on r. This implies that $S(z)$ is at most of exponential type W. By comparing it to the sine function with approximately the same zeroes, one is able to estimate the growth along lines parallel to the real axis. This is the content of the following technical lemma by Avdonin [1, Lemma 4].

Lemma 3.15 (Avdonin 1979). *Let $\Phi(z)$ be a function of the same type as (3.28) with zeroes $m + \delta_m$ satisfying $\sup_{n \in \mathbb{N}} |\delta_m| < \infty$. Then given $h > 0$ there exists absolute constants such that*

$$\left|\frac{\sin \pi(x + \mathrm{i}h)}{\Phi(x + \mathrm{i}h)}\right| \simeq \exp\left\{\sum_{|m| \leq 2|x|} \frac{\delta_m}{m} + \frac{\delta_m}{x + \mathrm{i}h - m}\right\}, \quad x \in \mathbb{R}.$$

In our case we apply this lemma to $\Phi(x) = S(\pi x/W)$ with $|\delta_k| \leq 1/2\pi$. A simple computation now shows that

$$\frac{1}{(1+|x|)^{1/\pi}} \lesssim |\Phi(x+\mathrm{i})| \lesssim (1+|x|)^{1/\pi}. \tag{3.30}$$

In particular, this implies that

$$|\Phi(x)| \lesssim (1+|x|)^{1/\pi}. \tag{3.31}$$

Indeed,

$$\left|\frac{\Phi(x)}{\Phi(x+\mathrm{i})}\right|^2 = \prod_{m\in\mathbb{N}}{}' \left(1 - \frac{1}{1+(m+\delta_m - x)^2}\right).$$

Splitting this product into two parts depending on whether $x \leq m$ or $x < m$, it is clear that it is bounded by some constant independent of x. We may now infer from the definition of $\Phi(x)$ and (3.31) that

$$|S(x)| \lesssim (1+|Wx|)^{1/\pi}. \tag{3.32}$$

Since $\pi^{-1} < 1/2$ it follows that $S(x)/(S'(\lambda)(x-\lambda))$ is in $L^2(\mathbb{R})$. Hence, by the Paley-Wiener theorem, this is the Fourier transform of a function $s_\lambda \in L^2(-W,W)$ that satisfies

$$\langle s_\lambda | \mathrm{e}^{\mathrm{i}\mu t}\rangle = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & if\ \lambda \neq \mu \end{cases}.$$

Using this bi-orthogonality and the fact that e_λ gives Riesz basis we immediately get the estimate

$$\int_{\mathbb{R}} \left|\frac{S(x)}{S'(\lambda)(x-\lambda)}\right|^2 \mathrm{d}x \lesssim W. \tag{3.33}$$

Moreover, we get the representation for $u \in L^2(-W,W)$,

$$\hat{u}(x) = \sum_{\lambda\in\Lambda} \hat{u}(\lambda)\frac{S(x)}{S'(\lambda)(x-\lambda)}. \tag{3.34}$$

If we set

$$G(z) = S(z) \prod_{|k|<k_0}{}' \left(\frac{1 - \frac{z}{\lambda_k}}{1 - \frac{z}{\pi k/W}} \right), \tag{3.35}$$

it follows for the same reasons that $G(z)/(z-\lambda)$ is the Fourier transform of a function $g_\lambda \in L^2(-W, W)$ and that $\{g_\lambda\}$ is bi-orthogonal to $\{e^{i\lambda t}\}_{\lambda\in\Lambda}$. By (3.35) and the representation (3.34) we also get the formula

$$\hat{u}(x) = \sum_{\lambda\in\Lambda} \hat{u}(\lambda) \frac{G(x)}{G'(\lambda)(x-\lambda)}. \tag{3.36}$$

We now use the oversampling. Since $f \in L^2(-T, T)$ it follows that the function $\hat{f}(x)h(y-x)$, where

$$h_l(x) = \left(\frac{\sin \eta x/l}{\eta x/l} \right)^l,$$

is the Fourier transform of a function in $L^2(-W, W)$. We may apply formula (3.36) on the function $\hat{f}(x)h(y-x)$, since by the relation between T, W and η it is the Fourier transform of an element of $L^2(-W, W)$. As we substitute $y = x$ the Boas-Bernstein formula (3.27) follows.

The inequality (3.17)

Let $f \in L^2(-T, T)$ and $W = (1+\eta)T$. Let Λ be the set of frequencies constructed in the previous section. It is clear that

$$\sum_{n\in\mathbb{N}} \frac{|\hat{f}(\log n)|^2}{n} \geq \sum_{\log n \in \cup L_k} \frac{|\hat{f}(\log n)|^2}{n} + \sum_{\log n \in \cup I_k} \frac{|\hat{f}(\log n)|^2}{n}.$$

By the choice of the frequencies Λ_w, followed by elementary estimates, this is seen to be bigger than

$$\left(\sum_{|k|\leq k_0} |\hat{f}(\lambda_k)|^2 \sum_{\log n\in L_k} \frac{1}{n} \right) + \left(\sum_{|k|>k_0} |\hat{f}(\lambda_k)|^2 \sum_{\log n\in I_k} \frac{1}{n} \right) \geq \frac{1}{4T} \sum_{\lambda\in\Lambda} |\hat{f}(\lambda)|^2,$$

as was to be shown.

The inequality (3.18)

Recall that we made the choice $W = (1+\eta)T$ for $\eta \in (0,1)$. Now we specify that we want η to be such that $1+\eta = (1+\epsilon/2)/(1+\epsilon/3)$. With this choice let

$$F_0 = \sum_{\lambda\in\Lambda_0} \hat{f}(\lambda) h_2(x-\lambda)\frac{G(x)}{G'(\lambda)(x-\lambda)} \quad \text{and}$$
$$F_w = \sum_{\lambda\in\Lambda_w} \hat{f}(\lambda) h_2(x-\lambda)\frac{G(x)}{G'(\lambda)(x-\lambda)}.$$

From this point on we write $h_2 = h$. By the Boas-Bernstein formula (3.27) we have $f = F_0 + F_w$, from which it follows that $\|f\|_2 \le \|F_0\|_2 + \|F_w\|_2$. The inequality (3.18) will follow from the estimates

$$\|F_0\|^2 \lesssim T^{(1+\epsilon)\frac{12T}{\pi}\log 2} \sum_{\lambda\in\Lambda_0} |\hat{f}(\lambda)|^2,$$

and

$$\|F_w\|^2 \lesssim T^{(1+\epsilon)\frac{12T}{\pi}\log 2} \sum_{\lambda\in\Lambda_w} |\hat{f}(\lambda)|^2. \tag{3.37}$$

These estimates are valid for $T > 1$ and the implicit constants depend on ϵ. The proofs are essentially the same, so we only explain how to get (3.37). We collect some technical estimates in the following two lemmas.

Lemma 3.16. *For $\lambda \in \Lambda_w \cup \left\{\frac{\pi k}{W}\right\}_{|k_0|\le k}$ we have*

$$\frac{1}{(W\lambda)^{1/\pi}} \lesssim \left|S'(\lambda)\right| \lesssim (W\lambda)^{1/\pi}. \tag{3.38}$$

Proof. The estimate follows from Lemma 3.15 in a similar way as the inequality (3.32). As before, we set $\Phi(x) = S(\pi x/W)$. We write the zeroes of Φ in the form $\mu_m = m + \delta_m$. Given $\lambda \in \Lambda_w \cup \left\{\pi k/W\right\}_{|k|\le k_0}$ it follows that $\mu_m = (W/\pi)\lambda_m$ and for $m \in \mathbb{Z}$, and $|\delta_m| \leq 1/2\pi$. In addition set $\Phi_m(x) = \Phi(x)/(x - \mu_m)$. We get

$$S'(\lambda) = \lim_{x\to\lambda} \frac{S(x)}{x-\lambda} = \Phi_m(\mu_m).$$

Moreover, we observe that

$$\Phi_m(\mu_m + \mathrm{i}y) = \frac{\Phi(\mu_m + \mathrm{i}y)}{\mu_m + \mathrm{i}y - \mu_m} = \frac{1}{\mathrm{i}y}\Phi(\mu_m + \mathrm{i}y).$$

Combining these two formulas we find

$$S'(\lambda) = \frac{1}{\mathrm{i}y}\frac{\Phi_m(\mu_m)}{\Phi_m(\mu_m + \mathrm{i}y)}\Phi(\mu_m + \mathrm{i}y).$$

We fix $y = 1$. By the inequalitites (3.30), the formula (3.38) follows as soon as we know that

$$\left|\frac{\Phi_m(\mu_m)}{\Phi_m(\mu_m + \mathrm{i})}\right| \simeq 1,$$

with the implicit constants independent of $m \in \mathbb{N}$. But this follows by simply expanding the left hand side into an infinite product:

$$\prod_{k\neq m}{}' \left|\frac{1 - \frac{\mu_m}{\mu_k}}{1 - \frac{\mu_m + \mathrm{i}}{\mu_k}}\right| = \left(\prod_{k\neq m} 1 + \frac{1}{(\mu_k - \mu_m)^2}\right)^{-1/2}$$
$$\simeq \left(\prod_{k\neq m} 1 + \frac{1}{(k-m)^2}\right)^{-1/2} = \left(\prod_{k=1}^{\infty} 1 + \frac{1}{k^2}\right)^{-1}.$$

□

We establish some notation. Let $J_m = \left[(m - \frac{1}{2})\frac{\pi}{W}, (m + \frac{1}{2})\frac{\pi}{W}\right)$. This means that

$$J = \bigcup_{|m|\leq k_0} J_m = \left[\lambda_{-k_0} - \frac{\pi}{2W}, \lambda_{k_0} + \frac{\pi}{2W}\right).$$

Since $J_m \cap J_n = \emptyset$ this is a partition of the interval J.

Lemma 3.17. *Let $\epsilon_1 > 0$. For $\lambda_k \in \Lambda_w$ and $W > 1$ we have*

$$\left| \prod_{\lambda_n \in \Lambda_0} \frac{\lambda_k - \frac{\pi n}{W}}{\lambda_k - \lambda_n} \right| \lesssim W^{(1+\epsilon_1)\frac{2W}{\pi}\log 2}. \tag{3.39}$$

$$\left\| \prod_{\lambda_n \in \Lambda_0} \frac{x - \lambda_n}{x - \frac{\pi n}{W}} \right\|_{L^\infty(\mathbb{R}\setminus J)} \lesssim W^{(1+\epsilon_1)\frac{W}{\pi}\log \frac{3^3}{2^4}} \tag{3.40}$$

For $m \in \{-k_0, \dots, k_0\}$ we have

$$\left\| \prod_{\substack{\lambda_n \in \Lambda_0 \\ n \neq m}} \frac{x - \lambda_n}{x - \frac{n\pi}{W}} \right\|_{L^\infty(J_m)} \lesssim W^{(1+\epsilon_1)\frac{4W}{\pi}\log 2}. \tag{3.41}$$

We note that here the constants depend on ϵ_1.

Proof. The arguments for all the inequalities are basically the same, so we only give the one for (3.41). We first consider the case $m = 0$.

$$\begin{aligned} \left\| \prod_{\substack{\lambda_n \in \Lambda_0 \\ n \neq m}} \frac{x - \lambda_n}{x - \frac{n\pi}{W}} \right\|_{L^\infty(J_m)} &\leq \prod_{\substack{\lambda_n \in \Lambda_0 \\ n > 0}} \frac{\frac{\pi}{2W} - \lambda_n}{\frac{\pi}{2W} - \frac{\pi}{W}n} \prod_{\substack{\lambda_n \in \Lambda_0 \\ n < 0}} \frac{-\frac{\pi}{2W} - \lambda_n}{-\frac{\pi}{2W} - \frac{\pi}{W}n} \\ &\leq \prod_{n=1}^{k_0} \frac{(n + k_0 + \delta_n)(n + k_0 - \delta_{-n})}{(n - \frac{1}{2})^2} \leq 4\left\{\frac{(2k_0+1)!}{(k_0+1)!(k_0-1)!}\right\}^2 \end{aligned}$$

By Stirling's formula $n! \simeq (n/\mathrm{e})^n (2\pi n)^{1/2}$, for any $\epsilon_1 > 0$ this is smaller than some constant times $\mathrm{e}^{(1+\epsilon_1)4k_0 \log 2}$. Using $k_0 = [W \log W/\pi]$ gives us the desired inequality. Elementary considerations imply that the largest value is attained by $m = 0$. Hence, the inequality follows for $m \in \{-k_0, \dots, k_0\}$. □

We now resume the proof of the inequality (3.18). The first thing to notice is that the factor h allows us to use the Cauchy-Schwarz inequality.

This is essentially why we constructed the set Λ to be over-sampling for $L^2(-T,T)$. We get

$$
\begin{aligned}
\|F_w\|^2_{L^2(\mathbb{R})} &= \int_{\mathbb{R}} \left| \sum_{\lambda\in\Lambda_w} \hat{f}(\lambda)h(x-\lambda)\frac{G(x)}{G'(\lambda)(x-\lambda)} \right|^2 \mathrm{d}x \\
&\leq \int_{\mathbb{R}} \sum_{\lambda\in\Lambda_w} |\hat{f}(\lambda)|^2 |h(x-\lambda)| \left| \frac{G(x)}{G'(\lambda)(x-\lambda)} \right|^2 \times \sum_{\mu\in\Lambda_w} |h(x-\mu)| \mathrm{d}x \\
&\leq \left\| \sum_{\mu\in\Lambda_w} h(x-\mu) \right\|_{L^\infty(\mathbb{R})} \sum_{\lambda\in\Lambda_w} |\hat{f}(\lambda)|^2 \underbrace{\int_{\mathbb{R}} |h(x-\lambda)| \left| \frac{G(x)}{G'(\lambda)(x-\lambda)} \right|^2 \mathrm{d}x}_{(I)} .
\end{aligned}
\tag{3.42}
$$

It is readily checked that the factor outside of the sum is less than some absolute constant, so it only remains to deal with (I). For $\lambda_k \in \Lambda_w$ we get by expanding $G(x)$ in terms of $S(x)$ and using the inequality (3.39) that

$$
\begin{aligned}
(I) &= \int_{\mathbb{R}} \left| \frac{S(x)}{S'(\lambda_k)(x-\lambda_k)} \right|^2 \\
&\qquad \times |h(x-\lambda_k)| \left| \prod_{\lambda_n\in\Lambda_0} \frac{x-\lambda_n}{x-\frac{\pi n}{W}} \prod_{\lambda_n\in\Lambda_0} \frac{\lambda_k - \frac{\pi n}{W}}{\lambda_k-\lambda_n} \right|^2 \mathrm{d}x \\
&\lesssim W^{(1+\frac{\epsilon}{3})\frac{4W}{\pi}\log 2} \underbrace{\int_{\mathbb{R}} \left| \frac{S(x)}{S'(\lambda_k)(x-\lambda_k)} \right|^2 |h(x-\lambda_k)| \prod_{\lambda_n\in\Lambda_0} \left| \frac{x-\lambda_n}{x-\frac{\pi n}{W}} \right|^2 \mathrm{d}x}_{(II)} .
\end{aligned}
\tag{3.43}
$$

Here and below, the inequalities are valid for $W > 1$ and the implicit constants depend on ϵ. Recall that $J_m = \left((m-1/2)\frac{\pi}{W}, (m+1/2)\frac{\pi}{W}\right)$ and

$J = \cup_{m=-k_0}^{k_0} J_m$. Then

$$(II) = \underbrace{\int_{\mathbb{R}\setminus\cup J_m} \left|\frac{S(x)}{S'(\lambda_k)(x-\lambda_k)}\right|^2 |h(x-\lambda_k)| \prod_{\lambda_n\in\Lambda_0} \left|\frac{x-\lambda_n}{x-\frac{\pi n}{W}}\right|^2 \mathrm{d}x}_{(III)}$$
$$+ \underbrace{\sum_{|m|\le k_0} \left|\frac{S'(\frac{\pi m}{W})}{S'(\lambda_k)}\right|^2 \int_{J_m} \left|\frac{S(x)}{S'(\frac{\pi m}{W})(x-\frac{\pi m}{W})}\right|^2 \times |h(x-\lambda_k)| \left|\frac{x-\lambda_m}{x-\lambda_k} \prod_{\substack{\lambda_n\in\Lambda_0\\ n\neq m}} \frac{x-\lambda_n}{x-\frac{\pi n}{W}}\right|^2 \mathrm{d}x}_{(IV)}.$$

By the bound $|h(x)| \le 1$ and the inequalities (3.33) and (3.40) this implies that $(III) \lesssim W^{(1+\frac{\epsilon}{3})\frac{2W}{\pi}\log\frac{3^3}{2^4}}$. It is readily checked that for $|k| > k_0$ then $\|h(x-\lambda_k)\|_{L^\infty(J)} \lesssim (|k|-k_0)^{-2}$ for some absolute constant. Using this, in addition to the estimates (3.33), (3.38) and (3.41) we get

$$(IV) \le \frac{\|h(x-\lambda_k)\|_{L^\infty(\cup J_m)}}{|S'(\lambda_k)|^2} \sum_{|m|\le k_0} |S'(\frac{\pi m}{W})|^2 \left\|\frac{x-\lambda_n}{x-\lambda_k}\right\|_{L^\infty(J_m)}$$
$$\times \left\|\prod_{\substack{n\neq m\\ \lambda_n\in\Lambda_0}} \frac{x-\lambda_n}{x-\frac{\pi n}{W}}\right\|_{L^\infty(J_m)}^2 \int_{\mathbb{R}} \left|\frac{S(x)}{S'(\frac{\pi m}{W})(x-\frac{\pi m}{W})}\right|^2 \mathrm{d}x$$
$$\lesssim W^{(1+\frac{\epsilon}{3})\frac{8W}{\pi}\log 2}.$$

By combining the inequalities for (I)-(IV) with (3.42) we get for $W > 1$ the estimate

$$\|F_w\|^2 \lesssim W^{(1+\frac{\epsilon}{3})\frac{12W}{\pi}\log 2} \sum_{\lambda_k\in\Lambda} |\hat{f}(\lambda_k)|^2.$$

The implicit constant depends on ϵ. By using $W = (1+\frac{\epsilon}{2})T/(1+\frac{\epsilon}{3})$ we establish

$$\begin{aligned}\|F_w\|^2 &\lesssim (1+\eta)^{(1+\frac{\epsilon}{2})\frac{12T}{\pi}\log 2} T^{(1+\frac{\epsilon}{2})\frac{12T}{\pi}\log 2} \sum_{\lambda_k\in\Lambda} |\hat{f}(\lambda_k)|^2 \\ &= \frac{(1+\eta)^{(1+\frac{\epsilon}{2})\frac{12T}{\pi}\log 2}}{T^{\frac{\epsilon}{2}\frac{12T}{\pi}\log 2}} T^{(1+\epsilon)\frac{12T}{\pi}\log 2} \sum_{\lambda_k\in\Lambda} |\hat{f}(\lambda_k)|^2.\end{aligned}$$

Since the first factor of the last expression is bounded for $T > 1$ by some constant depending on ϵ, the inequality (3.37) follows.

3.6 Boundary functions for $\mathscr{H}^2_\alpha$

In this section we prove an analogue of Theorem 3.5 for McCarthy's spaces $\mathscr{H}^2_\alpha$. Recall that we established in section 2.3 that for $\alpha < 2$ the space $\mathscr{H}^2_\alpha$ is embedded locally in $D_\alpha(\mathbb{C}_{1/2})$. We stress that by the notation $f(1/2+\mathrm{i}t)$ we mean the boundary distribution, and not point-wise values.

Theorem 3.18 (Saksman and Olsen 2009)**.** *Let $I \subset \mathbb{R}$ be a bounded and open interval and $\alpha < 2$. Then for every $f \in D_\alpha(\mathbb{C}_{1/2})$ there exists an $F \in \mathscr{H}^2_\alpha$ such that $f - F$ continues analytically to all of $\mathbb{C}_I$ with* $\mathrm{Re}(f - F)(1/2 + \mathrm{i}t) = 0$ *on I. There exists a unique $F \in \mathscr{H}^2_\alpha$ of minimal norm satisfying this. Moreover, there exists a constant $C > 0$ depending only on $\alpha \in \mathbb{R}$ and the length of I such that the minimal element satisfies*

$$\|F\|^2_{\mathscr{H}^2_\alpha} \le C\|f\|^2_{D_\alpha}.$$

Much like in section 3.2 we establish this result by considering the operator defined on finite sequences by

$$R_I : (a_n)_{n\in\mathbb{Z}^*} \longmapsto \left(\sum_{n\in\mathbb{N}} \frac{a_n n^{-\mathrm{i}t} + a_{-n} n^{\mathrm{i}t}}{\sqrt{n}} \right)\Bigg|_I .$$

In order to determine the proper domain and target spaces for this operator we need to introduce Sobolev spaces that in general consist of distributions.

Note that since multiplying distributions with the indicator function is in general problematic, we consider restrictions instead.

It is well-known that the functions in the spaces $D_\alpha(\mathbb{C}_{1/2})$ have distributional boundary values that belong to the Sobolev spaces $W^{\alpha/2}(I)$ on bounded and open intervals $I \subset \mathbb{R}$. By the local embeddings the same holds true for the spaces $\mathscr{H}^2_\alpha$. In order to define $W^\beta(I)$ for $\beta \in \mathbb{R}$ we introduce the weights $w_\beta(\xi) = (1+|\xi|^2)^{\beta/2}$ and denote the space of tempered distributions by $\mathcal{S}'(\mathbb{R})$. Now we define the unrestricted Sobolev space

$$W^\beta(\mathbb{R}) = \left\{ u \in \mathcal{S}'(\mathbb{R}) : \int_{\mathbb{R}} |\hat{u}(\xi)|^2 w_{2\beta}(\xi) \mathrm{d}\xi < \infty \right\}.$$

For a bounded and open interval $I \subset \mathbb{R}$, we let $W^\beta_0(I)$ be the subspace of $W^\beta(\mathbb{R})$ that consists of distributions having support in I. We may now define the restricted Sobolev space to be the quotient space

$$W^\beta(I) = W^\beta(\mathbb{R})/W^\beta_0(\mathbb{R}\backslash\bar{I}^C).$$

This space may be said to contain the restrictions of distributions in $W^\beta(\mathbb{R})$ to the interval I with the norm

$$\|u\|^2_{W^\beta(I)} = \inf_{\substack{v \in W^\beta(\mathbb{R}) \\ v|_I = u}} \|v\|^2_{W^\beta(\mathbb{R})}.$$

Under the natural pairing $(u,v) = \int_{\mathbb{R}} \hat{u}(\xi)\hat{v}(\xi)\mathrm{d}\xi$, the dual space of $W^\beta(I)$ is isometric to $W^{-\beta}_0(I)$ as is readily verified. Note that the closure of $\mathcal{C}^\infty_0(I)$ in the norm of $W^\beta(\mathbb{R})$ coincides with the space $W^\beta_0(I)$. This may be seen by a scaling and mollifying argument.

Note that we may express the McCarthy spaces as,

$$\mathscr{H}^2_\alpha = \left\{ \sum_{n\in\mathbb{N}} a_n n^{-s} : \sum_{n\in\mathbb{N}} |a_n|^2 w_\alpha(\log n) < \infty \right\}.$$

Closely related to this space is the sequence space $\ell^2_\alpha(\mathbb{Z}^*)$. We define it to be the sequences of complex numbers (a_n) finite in the norm

$$\|(a_n)\|^2_{\ell^2_\alpha} = \sum_{n\in\mathbb{N}} |a_n|^2 w_\alpha(\log n) + |a_{-n}|^2 w_\alpha(-\log n).$$

The following lemma is analogue to Lemma 3.7.

Lemma 3.19. *The operator R_I extends to a bounded and onto operator from ℓ^2_α to $W^{\alpha/2}(I)$.*

Let R_I^* denote the adjoint operator of R_I with respect to the natural pairing of the Sobolev spaces and of ℓ^2_α. We remark that Lemma 3.19 says that $R_I : \ell^2_\alpha \longrightarrow W^{\alpha/2}(I)$ is both bounded and surjective. This is equivalent to saying that the operator $R_I^* : W_0^{-\alpha/2}(I) \to \ell^2_{-\alpha}$ is bounded and bounded below in norm. Since

$$\|R_I^* g\|^2_{\ell^2_{-\alpha}} = \sum_{n\in\mathbb{N}} \frac{|\hat{g}(\log n)|^2 w_{-\alpha}(\log n) + |\hat{g}(-\log n)|^2 w_{-\alpha}(\log n)}{n},$$

we note that Lemma 3.19 may be formulated as follows.

Lemma 3.20 (Saksman and Olsen 2009)**.** *Let $I \subset \mathbb{R}$ be a bounded interval. Then there exist constants depending on the length of I such that*

$$\sum \frac{|\hat{f}(\log n)|^2 w_\beta(\log n) + |\hat{f}(-\log n)|^2 w_\beta(\log n)}{n} \simeq \|f\|^2_{W_0^{\beta/2}(I)}. \quad (3.44)$$

We return to the proof of this in section 3.7. We remark that for For $\beta < 1/2$ this means that the sequence

$$\mathscr{G}_\beta = \left(\dots, (-n)^{-\mathrm{i}t}\frac{w_{\beta/2}(\log(-n))}{\sqrt{(-n)}}, \dots, 1, \dots, n^{-\mathrm{i}t}\frac{w_{\beta/2}(\log n)}{\sqrt{n}}, \dots\right),$$

where n is understood to run through $\mathbb{Z}^*$, is a frame for $W_0^{\beta/2}(I)$ when restricted to the interval I.

Proof of Theorem 3.18. Let $f(1/2 + \mathrm{i}t)$ denote the boundary distribution of $f \in D_\alpha(\mathbb{C}_{1/2})$. Moreover, let v be the real part of $f(1/2+\mathrm{i}t)$ considered as an element of $W^{\alpha/2}(2I)$. We now argue essentially in the same way as in the proof of Theorem 3.5. Since $R_{2I} : \ell^2_\alpha \to W^{\alpha/2}(2I)$ is surjective, there exists a sequence $(\gamma_n)_{n\in\mathbb{Z}^*}$ such that

$$v = \sum_{n\in\mathbb{N}} (\gamma_n n^{-1/2-\mathrm{i}t} + \gamma_{-n} n^{-1/2+\mathrm{i}t})$$

with the convergence being in the sense of $W^{\alpha/2}(2I)$. It now follows that the function

$$F(s) = \sum_{n\in\mathbb{N}} (\gamma_n + \overline{\gamma_{-n}}) n^{-s}$$

is in $\mathscr{H}^2_\alpha$ and satisfies

$$\lim_{\sigma\to 1/2^+} \operatorname{Re} F(\sigma + \mathrm{i}t) = v$$

as distributions on $2I$. Hence the function $F - f$ is analytic on $\mathbb{C}_{1/2}$ and has vanishing real parts on $2I$ in the sense of distributions.

The analytic continuation of the function $F - f$ to all of $\mathbb{C}_I$ is obtained in much the same way as in Theorem 3.5. Indeed, we may use same conformal mapping γ which sends the unit disc $\mathbb{D}$ to the half-plane $\mathbb{C}_{1/2}$. By the local embedding of $\mathscr{H}^2_\alpha$ into $D_\alpha(\mathbb{C}_{1/2})$ it follows that the function $(f - F)\circ\gamma$ is contained in the spaces $D_\alpha(\mathbb{D})$. The Schwarz reflection of this function may now be obtained directly by considering limits of $H^2(\mathbb{D})$ type representations of the function $(f - F)\circ\gamma$ using the functions $\operatorname{Re}(f - F)\circ\gamma(rz)$ as $r \to 1^-$.

The assertion about the smallest element and the existence of a norm constant follow exactly as in the proof of Theorem 3.5. □

3.7 Proof of Lemma 3.19

In place of Lemma 3.4 we use the following lemma. As we are not able to find a reference, we provide a proof.

Lemma 3.21. *Let X, Y be Hilbert spaces and $Z : X \to Y$ be a bounded and injective operator. If there exist a subspace $M \subset X$ of finite co-dimension such that Z is bounded below as an operator on M and a constant $C > 0$, such that $\|Zf\| \geq C\|f\|$ for all $f \in M$, then Z is bounded below on all of X.*

Proof of lemma 3.21. Assume that there exists a sequence of vectors (f_n) such that $\|f_n\| \equiv 1$ and $\|Zf_n\| \leq n^{-1}$. We seek a contradiction by showing that the sequence (Zf_n) converges to some non-zero $h \in X$.

Let P_M and $P_{M^\perp}$ be the orthogonal projections onto M and $M^\perp$, respectively. Since $M^\perp$ is of finite dimension, we assume that $P_{M^\perp} f_n$ converges to some vector $g \in H$. Moreover, by the triangle inequality,

$$\|ZP_M f_n + Zg\| \leq \frac{1}{n} + \|Z(P_{M^\perp} f_n - g)\|.$$

So, since Z is bounded it follows that $ZP_M f_n$ converges to $-Zg$, and in particular $(ZP_M f_n)$ has to be a Cauchy sequence. The final step is to observe that the lower boundedness of Z on M implies

$$\|ZP_M(f_n - f_m)\| \geq C\|P_M f_n - P_M f_m\|$$

for some $C > 0$. I.e. $(P_M f_n)$ is a Cauchy sequence. And so, since $f_n = P_M f_n + P_{M^\perp} f_n$, we get that (f_n) is the sum of two Cauchy sequences, and is therefore a Cauchy sequence. Observe that $\lim_{n\to\infty} \|f_n\| = 1$ whence $\lim_{n\to\infty} Zf_n \neq 0$. This yields a contradiction. □

We are now ready to prove the lemmas of the previous section.

Proof of Lemma 3.19. Since Lemma 3.19 and Lemma 3.20 are equivalent, it follows that it is enough to prove the relation (3.44). For $f \in C_0^\infty(I)$ it is clear that this expression converges. Since $n^{-1} \log 2 \leq \log(1 + n^{-1})$ and $w_\beta(\log n) \leq w_\beta(\xi)$ for $\xi \in (\log n, \log(n+1))$, it follows that the left hand side of (3.44) is less than the constant $1/\log 2$ times

$$\sum_{n\in\mathbb{N}} \left(\int_{\log n}^{\log(n+1)} |\hat{f}(\log n)|^2 w_\beta(\xi) \mathrm{d}\xi + \int_{-\log(n+1)}^{-\log n} |\hat{f}(-\log n)|^2 w_\beta(\xi) \mathrm{d}\xi \right).$$

Adding and subtracting by $\hat{f}(\xi) w_{\beta/2}(\mathrm{e}^\xi)$ within the absolute value signs, and using the inequality $|x+y|^2 \leq 2(|x|^2 + |y|^2)$ shows that this again is smaller than the constant 2 times

$$\int_{\mathbb{R}} |\hat{f}(\xi)|^2 w_\beta(\xi) \mathrm{d}\xi + \sum_{n\in\mathbb{N}} \left(\int_{\log n}^{\log(n+1)} |\hat{f}(\xi) - \hat{f}(\log n)|^2 w_\beta(\xi) \mathrm{d}\xi \right.$$
$$\left. + \int_{-\log(n+1)}^{-\log n} |\hat{f}(\xi) - \hat{f}(-\log n)|^2 w_\beta(\xi) \mathrm{d}\xi \right). \quad (3.45)$$

The first term is simply $\|f\|^2_{W_0^{\beta/2}}$. We need to show that the second term is also controlled by this norm. By expanding the Fourier transform, we find that

$$
\begin{aligned}
|\hat{f}(\log n) - \hat{f}(\xi)| &= \left| \int_{\log n}^{\xi} \hat{f}'(\tau)\mathrm{d}\tau \right| \\
&\leq \left(\frac{C_\beta}{n\, w_\beta(\log n)} \int_{\log n}^{\log(n+1)} |\hat{f}'(\tau)|^2 w_\beta(\tau)\mathrm{d}\tau \right)^{1/2},
\end{aligned}
\tag{3.46}
$$

Here C_β is a constant depending on β and which may be adjusted at the appropriate steps below. Inserting this into the last term of (3.45) yields the upper bound

$$
\begin{aligned}
C_\beta \sum_{n\in\mathbb{N}} \frac{1}{n^2} &\left(\int_{\log n}^{\log(n+1)} |\hat{f}'(\tau)|^2 w_\beta(\tau)\mathrm{d}\tau + \int_{-\log n}^{-\log(n+1)} |\hat{f}'(\tau)|^2 w_\beta(\tau)\mathrm{d}\tau \right) \\
&\leq C_\beta \int_{\mathbb{R}} |\hat{f}'(\tau)|^2 w_\beta(\tau)\mathrm{d}\tau = C_\beta \|tf\|^2_{W_0^{\beta/2}(I)}.
\end{aligned}
$$

The last equality follows by the rule for differentiating a Fourier transform and using the definitions of the norms. Since multiplication by t is continuous on $W_0^{\beta/2}(I)$ it follows that $\|tf\|_{W_0^{\beta/2}(I)} \lesssim \|f\|_{W_0^{\beta/2}(I)}$. This proves the upper inequality.

We turn to the lower inequality. We need to be slightly more careful. By the same argument as above, we find that the left hand side of (3.44) is greater than

$$
\|f\|^2_{W_0^{\beta/2}(I)} - C_\beta \|\hat{f}'\|^2_{W_0^{\beta/2}(I)}.
$$

However, for general β and I this leaves us with something negative. The solution is for some sufficiently large $N \in \mathbb{N}$ to leave the terms with $|n| < N$ out of the sum on the left hand side of (3.44). This only makes the sum smaller, and running through the same argument as before, (3.44) is seen to be greater than

$$\|f\|^2_{W_0^{\beta/2}(I)} - \int_{-\log N}^{\log N} |\hat{f}(\xi)|^2 w_\beta(\xi) \mathrm{d}\xi - C_\beta \sum_{|n|\geq N} \frac{1}{n^2} \left(\int_{\log n}^{\log(n+1)} |\hat{f}'(\tau)|^2 w_\beta(\tau)^2 \mathrm{d}\tau + \int_{-\log(n+1)}^{-\log n} |\hat{f}'(\tau)|^2 w_\beta(\tau)^2 \mathrm{d}\tau \right).$$

By the continuity of multiplication by the independent variable, given any $\epsilon > 0$, we may choose N large enough so that this is greater than

$$(1-\epsilon)\|f\|^2_{W_0^{\beta/2}(I)} - \int_{-\log N}^{\log N} |\hat{f}(\xi)|^2 w_\beta(\xi) \mathrm{d}\xi,$$

Next, we explain how to use Lemma 3.21 to conclude. The lemma says that it is sufficient to find a subspace $\mathcal{M} \subset W_0^{\beta/2}(I)$ with finite co-dimension such that for all $f \in \mathcal{M}$ we have

$$\int_{-\log N}^{\log N} |\hat{f}(\xi)|^2 w_\beta(\xi) \mathrm{d}\xi \leq \frac{1}{2}\|f\|^2_{W_0^{\beta/2}(I)}. \tag{3.47}$$

For $\eta > 0$ and $K > \frac{1}{2\eta}$ it is possible to choose a finite sequence of strictly increasing real numbers $(\xi_n)_{n=1}^{K+1}$ such that $\xi_1 = -\log N$, $\xi_{K+1} = \log N$ and $\inf_{n\in J} |\xi - \xi_n| < \eta$. We set

$$\mathcal{M} = \left\{ f \in W_0^{\beta/2}(I) : \hat{f}(\xi_n) = 0, \quad \text{for } 1 \leq n \leq K+1 \right\}.$$

Clearly this is a subspace of finite co-dimension in $W_0^{\beta/2}(I)$. Moreover, by choosing η small enough an estimate of the type (3.46) now implies (3.47). Indeed, for $f \in \mathcal{M}$, the left-hand side is equal to

$$\sum_{n=1}^{K} \int_{\xi_n}^{\xi_{n+1}} |\hat{f}(\xi) - \hat{f}(\xi_n)|^2 \mathrm{d}\xi \leq C_\beta \eta^2 \sum_{n=1}^{K} \frac{1}{w_\beta(\xi_n)} \int_{\xi_n}^{\xi_{n+1}} |\hat{f}'(\tau)|^2 w_\beta(\tau) \mathrm{d}\tau \leq C_\beta \eta^2 \|tf\|^2_{W_0^{\beta/2}(I)}.$$

By the continuity of multiplication by the independent variable, the assertion now follows.

□

3.8 Boundary functions for $\mathscr{D}_\alpha$

In this section we prove an analogue of Theorem 3.5 for the Dirichlet-Bergman spaces $\mathscr{D}_\alpha$. Recall from section 2.4 that the spaces $\mathscr{D}_\alpha$ are embedded locally in the spaces $D_{1-2^{-\alpha}}(\mathbb{C}_{1/2})$. Recall that we use the convention that for $\alpha = +\infty$ then $1 - 2^{-\alpha} = 1$.

Theorem 3.22. *Let $I \subset \mathbb{R}$ be an open and bounded interval and $\alpha \in \mathbb{R} \cup \{+\infty\}$. Then for every $f \in D_{1-2^{-\alpha}}(\mathbb{C}_{1/2})$ there exists $F \in \mathscr{D}_\alpha$ such that $f - F$ continues analytically to all of $\mathbb{C}_I$ with $\mathrm{Re}(f - F)(1/2 + \mathrm{i}t) = 0$ on I. There exists a unique $F \in \mathscr{D}_\alpha$ satisfying this. Moreover, there exists a constant C depending only on the length I and α such that the minimal element satisfies*

$$\|F\|^2_{\mathscr{D}_\alpha} \leq C\|f\|^2_{D_{1-2^{-\alpha}}(\mathbb{C}_{1/2})}$$

To establish a lemma analogue to Lemma 3.20 we recall that by $d(n)$ we denote the divisor function for which the following formula holds,

$$D_\alpha(n) = \sum_{k=1}^{n} d(k)^\alpha = A_\alpha n \log^{2^\alpha - 1} n + \mathcal{O}\left(n(\log n)^{2^s - 2}\right), \qquad (3.48)$$

where A_α are constants depending on α. Also, the Prime number theorem says that

$$\pi_{\mathbb{P}}(x) = \frac{x}{\log x} + \mathcal{O}\left(\frac{x}{(\log x)^2}\right). \qquad (3.49)$$

Lemma 3.23. *Let $I \subset \mathbb{R}$ be a bounded and open interval and $\beta \in \mathbb{R}$. Then there exist constants depending only on the length of I and β such that for $f \in W_0^{\frac{2^\beta - 1}{2}}(I)$ we have*

$$\sum_{n \in \mathbb{N}} \frac{|\hat{f}(\log n)|^2 d(n)^\beta + |\hat{f}(-\log n)|^2 d(n)^\beta}{n} \simeq \|f\|^2_{W_0^{\frac{2^\beta - 1}{2}}(I)}. \qquad (3.50)$$

Also, there exists a constant only depending on the length of I such that for $f \in W_0^{-1}(I)$ we have

$$\sum_{n\in\mathbb{N}} \frac{|\hat{f}(\log p_n)|^2 + |\hat{f}(-\log p_n)|^2}{p_n} \simeq \|f\|^2_{W_0^{-1/2}(I)}. \tag{3.51}$$

We remark that this means that for $\beta \in \mathbb{R}$ the sequence

$$\left(\dots, (-n)^{-\mathrm{it}}\sqrt{\frac{d(-n)^\beta}{(-n)}}, \dots, 1, \dots, n^{-\mathrm{it}}\sqrt{\frac{d(n)^\beta}{n}}, \dots\right),$$

where n is understood to run through $\mathbb{Z}$, is a frame for $W_0^{\frac{2^\beta-1}{2}}(I)$ when restricted to the interval I.

Proof. The upper and lower bounds can be proved in similar ways. Therefore, we only give the demonstration for the lower bound, since it is the more difficult one.

Let $(a_n), (b_n)$ be two sequences of complex numbers. Denote their n'th partial sums by the symbols A_n and B_n, respectively. Then summation by parts says that

$$\sum_{n=N}^{M} a_n b_n = a_M B_M - a_N B_{N-1} + \sum_{n=N}^{M-1} (a_n - a_{n+1}) B_n. \tag{3.52}$$

Suppose $\beta \in \mathbb{R}$. Let $f \in C_0^\infty(I)$. Using summation by parts and the decay of $\hat{f}$ we find that for $N \in \mathbb{N}$ the left-hand side of (3.50) is greater than

$$\sum_{n\geq N} \frac{|\hat{f}(\log n)|^2 d(n)^\beta + |\hat{f}(-\log n)|^2 d(n)^\beta}{n}$$
$$= \underbrace{\sum_{n\geq N} \left(\frac{|\hat{f}(\log n)|^2}{n} - \frac{|\hat{f}(\log(n+1))|^2}{n+1} \right) D_\beta(n)}_{(I)}$$
$$+ \underbrace{\sum_{n\geq N} \left(\frac{|\hat{f}(-\log n)|^2}{n} - \frac{|\hat{f}(-\log(n+1))|^2}{n+1} \right) D_\beta(n)}_{(II)}$$
$$- \underbrace{\frac{|\hat{f}(-\log N)|^2}{N} D_\beta(N-1) - \frac{|\hat{f}(\log N)|^2}{N} D_\beta(N-1)}_{(III)}.$$

We are going to handle the expressions (I) and (II) in similar ways, while we carry the expression (III) along until the end of the proof. By elementary estimates and Stirling's formula,

$$n(\log n)^{2^\beta - 1} - \sum_{k=1}^{n} (\log k)^{2^\beta - 1} = \mathcal{O}\left(n(\log n)^{2^\beta - 2} \right).$$

Combined with the formula (3.48) this yields

$$(I) = \underbrace{\sum_{n\geq N} \left(\frac{|\hat{f}(\log n)|^2}{n} - \frac{|f(\log(n+1))|^2}{n+1} \right) \left\{ \sum_{k=1}^{n} (\log k)^{2^\beta - 1} \right\}}_{(I_a)}$$
$$+ \underbrace{\sum_{n\geq N} \left(\frac{|\hat{f}(\log n)|^2}{n} - \frac{|\hat{f}(\log(n+1))|^2}{n+1} \right) h(n)}_{(I_b)},$$

with $h(n) = \mathcal{O}\left(n(\log n)^{2^\beta - 2}\right)$. Here (I_a) may be thought of as the main term, while (I_b) is the error term. In a similar way (II) may be split up into (II_a) and (II_b). By partial summation

$$(I_a) + (II_a) = \sum_{n \geq N} \frac{|\hat{f}(\log n)|^2 (\log n)^{2^\beta - 1} - |\hat{f}(-\log n)|^2 (\log n)^{2^\beta - 1}}{n} + \left(\frac{|\hat{f}(\log N)|^2}{N} + \frac{|\hat{f}(-\log N)|^2}{N}\right)(\log N - 1)^{2^\beta - 1}.$$

Since $\log^{2^\beta - 1}$ is comparable to $w_{\frac{2^\beta - 1}{2}}(\log)$ for $\xi \geq 1$, Lemma 3.20 implies that

$$(I_a) + (II_a) \gtrsim \|f\|^2_{W_0^{\frac{2^\beta - 1}{2}}(I)} - C_N \sum_{1 \leq n \leq N} \left(|\hat{f}(\log n)|^2 + |\hat{f}(-\log n)|^2\right), \tag{3.53}$$

where C_N depends on N while the implicit constant is independent of N.

We turn to (I_b). By using the identity $a^2 - b^2 = (a+b)(a-b)$ and expanding the Fourier transform, we get

$$\left|\frac{|\hat{f}(\log n)|^2}{n} - \frac{|\hat{f}(\log(n+1))|^2}{n+1}\right| = \left(\frac{|\hat{f}(\log n)|}{\sqrt{n}} + \frac{|\hat{f}(\log(n+1))|}{\sqrt{n+1}}\right) \times \left|\int_I f(\tau)(n^{-\mathrm{i}\tau - 1/2} - (n+1)^{-\mathrm{i}\tau - 1/2})\mathrm{d}\tau\right|$$

By Fubini's theorem, the integral expression is bounded by some absolute constant independent of n multiplied by

$$\frac{1}{n}\left(\int_{\log n}^{\log(n+1)} |\hat{f}'(x) + \hat{f}(x)|^2\right)^{1/2}.$$

Since $h(n) = \mathcal{O}\left(n(\log n)^{2^\beta - 2}\right)$ it now follows that

$$
\begin{aligned}
(I_b) &\lesssim \sum_{n \geq N} \frac{|\hat{f}(\log n)|}{\sqrt{n}} \frac{(\log n)^{2^\beta - 1}}{\log n} \left(\int_{\log n}^{\log(n+1)} |\hat{f}'(x) + \hat{f}(x)|^2 \right)^{1/2} \\
&\lesssim \frac{1}{\log N} \left(\sum_{n \geq N} \frac{|\hat{f}(\log n)|^2}{n} (\log n)^{2^\beta - 1} \right)^{1/2} \\
&\qquad \times \left(\int_{\log N}^{\infty} |\hat{f}'(x) + \hat{f}(x)|^2 (1 + |x|^2)^{2^\beta - 1} \right)^{1/2} .
\end{aligned}
$$

Multiplication by the independent variable is continuous on $W^\beta(I)$ for all β. So, by Lemma 3.20,

$$
(I_b) + (II_b) \lesssim \frac{1}{\log N} \|f\|^2_{W_0^{\frac{2^\beta - 1}{2}}(I)} . \tag{3.54}
$$

Given $\epsilon > 0$ we may choose N sufficiently large, not depending on f, so that we may combine this with (3.53) to get

$$
\begin{aligned}
&\sum_{n \geq N} \frac{|\hat{f}(\log n)|^2 d(n)^\beta + |\hat{f}(-\log n)|^2 d(n)^\beta}{n} \\
&\qquad \gtrsim (1 - \epsilon) \|f\|^2_{W_0^{\frac{2^\beta - 1}{2}}(I)} - C \sum_{1 \leq n \leq N} \left(|\hat{f}(\log n)|^2 + |\hat{f}(-\log n)|^2 \right) .
\end{aligned}
$$

We are now in a position to apply Lemma 3.21. We choose $\mathcal{M}$ to be the subspace of $W_0^{\frac{2^\beta - 1}{2}}(I)$ whose Fourier transforms vanish on the points $\pm \log n$ for $|n| \leq N$. We conclude that the inequality (3.50) holds for all $f \in W_0^{\frac{2^\beta - 1}{2}}(I)$.

We explain how to prove the theorem in the case $\beta = -\infty$. We define the weight

$$
d(n)^{-\infty} = \begin{cases} 1 & n \text{ is a prime number} \\ 0 & \text{otherwise} \end{cases} .
$$

We may now use the same notation as before since we may express (3.51) as

$$\sum_{n\in\mathbb{N}} \frac{|\hat{f}(\log n)|^2 d(n)^{-\infty} + |\hat{f}(-\log n)|^2 d(n)^{\infty}}{n} \simeq \|f\|^2_{W_0^{-1/2}(I)}.$$

In this case, we have $D_{-\infty}(n) = \pi_{\mathbb{P}}(n)$. As before,

$$\frac{n}{\log n} - \sum_{k=1}^{n} \frac{1}{\log k} = \mathcal{O}\left(\frac{n}{(\log n)^2}\right).$$

By the Prime number theorem this implies that

$$D_{-\infty}(n) = \sum_{k=1}^{n} \frac{1}{\log k} + \mathcal{O}\left(\frac{n}{(\log n)^2}\right).$$

The proof now follows exactly as before. □

4 Interpolating sequences and Carleson measures

In this chapter we give some results on interpolating sequences for the spaces $\mathscr{H}^p$, $\mathscr{H}^2_\alpha$ and $\mathscr{D}$. For bounded interpolating sequences we are able to give a complete characterisation for the space $\mathscr{H}^2$. This characterisation extends easily to the spaces $\mathscr{H}^2_\alpha$ for $\alpha \leq 1$ and the spaces $\mathscr{D}_\alpha$ for $\alpha \in \mathbb{R} \cup \{+\alpha\}$. For general $\mathscr{H}^p$ we are only able to give a necessary condition and a partial sufficient condition. We also discuss interpolation in the projective tensor space $\mathcal{K} = \mathscr{H}^2 \otimes \mathscr{H}^2$ introduced by Helson.

First, we recall some notions which will be of importance throughout this chapter.

4.1 Preliminaries

We restate the following definitions from section 1.4.

Let H be a Hilbert space of functions on $\mathbb{C}_{1/2}$. A positive measure μ on $\mathbb{C}_{1/2}$ is called a Carleson measure for H if there is a constant $C > 0$ such that

$$\int |f(s)|^2 \mathrm{d}\mu(s) \leq C\|f\|_H^2, \quad \text{for all } f \in H.$$

The smallest such number $C > 0$ is called the norm of the Carleson measure and is denoted by $\|\mu\|_{\mathrm{CM}(H)}$. Assume, in addition, that H admits a reproducing kernel k_w for every $w \in \mathbb{C}_{1/2}$. Then a sequence $S = (s_n)$ of points in $\mathbb{C}_{1/2}$ is called a (universal) interpolating sequence if the following operator is (bounded and) onto ℓ^2,

$$f \in H \longmapsto \left(\frac{f(s_n)}{\|k_{s_n}\|_H} \right).$$

Note that if the mapping is bounded, then by the open mapping theorem there exists a constant $C > 0$ such that for all sequences $(c_n/\|k_{s_n}\|_H) \in \ell^2$ there is a function $f \in H$ that interpolates $f(s_n) = c_n$ with $\|f\|_H \leq C\|(c_n/\|k_{s_n}\|_H)\|_H$. The smallest such constant is called the constant of interpolation for the sequence S. Note that the notions of norms of Carleson measures and constants of interpolating sequences are the same in the various Banach space settings we introduce below.

Note that we call a Carleson measure local if it has bounded support, and we call a sequence S local if it is bounded.

For the Banach spaces in the scales $H^p(\mathbb{C}_{1/2})$ and $\mathscr{H}^p$ these definitions are extended by using the appropriate bounded point evaluations in place of the reproducing kernels, and the sequence spaces ℓ^p in place of the space ℓ^2. We refer the reader to page 20 for the exact definitions. The Carleson measures for $H^p(\mathbb{C}_{1/2})$ were characterised in [14].

Lemma 4.1 (Carleson 1962)**.** *Let $p \in [1, \infty)$. A positive measure μ is a Carleson measure for the space $H^p(\mathbb{C}_{1/2})$ if there exists $C > 0$ such that for every square $Q \subset \mathbb{C}_{1/2}$ it holds that*

$$\mu(Q) \leq C|Q|,$$

where $|Q|$ denotes the length of a side of the square Q.

The interpolating sequences for the spaces $H^p(\mathbb{C}_{1/2})$ were characterised in [79] in a generalisation of a previous result [13].

Lemma 4.2 (Carleson 1958, Shapiro and Shields 1961)**.** *Let $p \in [1, \infty)$. A sequence $S = (s_n)$, where $s_n = \sigma_n + \mathrm{i}t_n$, is an interpolating sequence for $H^p(\mathbb{C}_{1/2})$ if and only the measure $\mu = \sum \delta_{s_n}(2\sigma_n - 1)$ is a Carleson measure for $H^p(\mathbb{C}_{1/2})$ and if there is a number $\eta > 0$ such that*

$$\inf_{n \neq m} \left| \frac{s_n - s_m}{s_n + \bar{s}_m - 1} \right| \geq \eta.$$

Note that an interpolating sequence $S = (s_n)$ for $H^p(\mathbb{C}_{1/2})$ is universal if and only if the measure

$$\mu_S = \sum \delta_{s_n}(2\sigma_n - 1) \tag{4.1}$$

is a Carleson measure for $H^p(\mathbb{C}_{1/2})$. As a consequence, the interpolating and universal interpolating sequences for the spaces $H^p(\mathbb{C}_{1/2})$ coincide. Moreover, it is clear that both the Carleson measures and interpolating sequences are the same for all the spaces $H^p(\mathbb{C}_{1/2})$.

4.2 Local interpolation in $\mathscr{H}^2$

In section 2.2 we established that for the spaces $H^2(\mathbb{C}_{1/2})$ and $\mathscr{H}^2$ bounded point evaluations are given by reproducing kernels $k_w^{\mathscr{H}^2}$ and $k_w^{H^2}$, respectively, that satisfy the relation

$$k_w^{\mathscr{H}^2}(s) = k_w^{H^2}(s) + \psi(s + \bar{w}). \tag{4.2}$$

Here ψ denotes the entire function of the formula (1.3). Also, Lemma 2.2 states that for $F \in \mathscr{H}^2$ the following inequality holds for some constant $C > 0$, only depending on the length of a bounded interval $I \subset \mathbb{R}$,

$$\lim_{\sigma \to 1/2^+} \int_I |F(\sigma + \mathrm{i}t)|^2 \mathrm{d}t \leq C\|F\|^2_{\mathscr{H}^2}. \tag{4.3}$$

We prove the following result.

Theorem 4.3 (Olsen and Seip 2008)**.** *Suppose S is a bounded sequence of distinct points from $\mathbb{C}_{1/2}$. Then S is an interpolating sequence for $\mathscr{H}^2$ if and only if it is an interpolating sequence for $H^2(\mathbb{C}_{1/2})$.*

Needless to say, now Lemma 4.2 gives a geometric description of the bounded interpolating sequences for $\mathscr{H}^2$.

One implication is immediate from (4.2) and the fact that $F(s)/s$ is in $H^2(\mathbb{C}_{1/2})$ whenever F is in $\mathscr{H}^2$. Namely, when we solve the problem $F(s_j) = a_j$ with F in $\mathscr{H}^2$, we simultaneously solve the problem $f(s) = a_j/s_j$ with f in $H^2(\mathbb{C}_{1/2})$. Also, since S is bounded, $(a_j/\|k_{s_j}^{H^2}\|_{H^2})_{j=1}^\infty$ is in ℓ^2 if and only if $(s_j a_j/\|k_{s_j}^{\mathscr{H}}\|_{\mathscr{H}^2})_{j=1}^\infty$ is in ℓ^2.

Let us now assume that the bounded sequence S is an interpolating sequence for $H^2(\mathbb{C}_{1/2})$. We wish to prove that then S is also an interpolating sequence for $\mathscr{H}^2$. To begin with, we observe that it suffices to show that the subsequence

$$S_\epsilon = \left\{ s_j = \sigma_j + \mathrm{i}t_j \in S : \ \frac{1}{2} < \sigma_j \leq \frac{1}{2} + \epsilon \right\}$$

is an interpolating sequence for $\mathscr{H}^2$ for some small ϵ. Indeed, it is clear that $S \setminus S_\epsilon$ is a finite sequence, which we may write as

$$S \setminus S_\epsilon = (s_j)_{j=1}^N.$$

The finite interpolation problem $F_0(s_j) = a_j$, $j = 1, ..., N$ can be solved explicitly as follows. Choose primes $p_1, ..., p_N$ (not necessarily distinct) such that the product

$$B(s) = \prod_{j=1}^N \left(1 - p_j^{s_j - s}\right)$$

has simple zeros at the points $s_1, ..., s_N$. If we set $B_j(s) = B(s)/(1-p_j^{s_j-s})$ then the finite interpolation problem has solution

$$F_0(s) = \sum_{j=1}^N a_j \frac{B_j(s)}{B_j(s_j)}.$$

To solve the full interpolation problem $F(s_j) = a_j$, we can now solve

$$F_\epsilon(s_j) = \frac{a_j - F_0(s_j)}{B(s_j)}, \quad s_j \in S_\epsilon,$$

so that we obtain the final solution $F = BF_\epsilon + F_0$. Clearly,

$$\left(\frac{F_\epsilon(s_j)}{\|k_{s_j}^{\mathscr{H}^2}\|_{\mathscr{H}^2}} \right)_{s_j \in S_\epsilon} \in \ell^2 \iff \left(\frac{a_j}{\|k_{s_j}^{\mathscr{H}^2}\|_{\mathscr{H}^2}} \right)_{s_j \in S_\epsilon} \in \ell^2,$$

so that we have reduced the problem to showing that S_ϵ is an interpolating sequence for $\mathscr{H}^2$.

Our reason for making the transition from S to S_ϵ is that it will allow us to make use of the fact that

$$\lim_{\epsilon \to 0} \sum_{s_j \in S_\epsilon} \left(\sigma_j - \frac{1}{2} \right) = 0. \tag{4.4}$$

We note that (4.4) is just a consequence of the trivial fact that an interpolating sequence for $H^2(\mathbb{C}_{1/2})$ is a Blaschke sequence in $\mathbb{C}_{1/2}$. Since S is a bounded sequence, this means that

$$\sum_{s_j \in S} \left(\sigma_j - \frac{1}{2} \right) < +\infty.$$

We argue by duality, using the following lemma of R. P. Boas [83], [6].

Lemma 4.4 (Boas 1941)**.** *Suppose $(f_j)_{j=1}^\infty$ is a sequence of unit vectors in a Hilbert space H. Then the moment problem $\langle f, f_j \rangle_H = a_j$ has a solution f in H for every sequence $(a_j)_{j=1}^\infty$ in ℓ^2 if and only if there is a positive constant m such that*

$$\left\| \sum_j c_j f_j \right\|_H \geq m \|(c_j)\|_{\ell^2} \tag{4.5}$$

for every finite sequence of scalars (c_j).

Thus we need to prove (4.5) with $H = \mathscr{H}^2$ and $f_j = k_{s_j}^{\mathscr{H}^2} / \|k_{s_j}^{\mathscr{H}^2}\|_{\mathscr{H}^2}$ for s_j in S_ϵ.

To simplify the writing, we set $k_s = k_s^{\mathscr{H}^2}$ and suppress the index in the norm setting $\|F\| = \|F\|_{\mathscr{H}^2}$. Let T be a positive number such that $|t_j| \leq T - 1$ for every $s_j = \sigma_j + \mathrm{i}t_j$ in S. We start by using the embedding 4.3:

$$\left\| \sum_{s_j \in S_\epsilon} c_j \frac{k_{s_j}}{\|k_{s_j}\|} \right\|^2 \geq m_T \int_{-T}^{T} \left| \sum_{s_j \in S_\epsilon} c_j \frac{k_{s_j}(\mathrm{i}t + \frac{1}{2})}{\|k_{s_j}\|} \right|^2 \mathrm{d}t.$$

The trick is now to replace the kernels of $\mathscr{H}^2$ by the kernels of $H^2(\mathbb{C}_{1/2})$. We use (2.5) and the triangle inequality:

$$\left(\int_{-T}^{T}\left|\sum_{s_j\in S_\epsilon} c_j \frac{k_{s_j}(\mathrm{i}t+\frac{1}{2})}{\|k_{s_j}\|}\right|^2 \mathrm{d}t\right)^{1/2} \geq \left(\int_{-T}^{T}\left|\sum_{s_j\in S_\epsilon} c_j \frac{\|k_{s_j}\|^{-1}}{\mathrm{i}t+\bar{s}_j-\frac{1}{2}}\right|^2 \mathrm{d}t\right)^{1/2} - \left(\int_{-T}^{T}\left|\sum_{s_j\in S_\epsilon} c_j\psi\left(\mathrm{i}t+\bar{s}_j+\frac{1}{2}\right)\|k_{s_j}\|^{-1}\right|^2 \mathrm{d}t\right)^{1/2}. \tag{4.6}$$

We split the first term on the right into two pieces:

$$\int_{-T}^{T}\left|\sum_{s_j\in S_\epsilon} c_j \frac{\|k_{s_j}\|^{-1}}{\mathrm{i}t+\bar{s}_j-\frac{1}{2}}\right|^2 \mathrm{d}t = \left(\int_{-\infty}^{\infty} - \int_{|t|>T}\right)\left|\sum_{s_j\in S_\epsilon} c_j \frac{\|k_{s_j}\|^{-1}}{\mathrm{i}t+\bar{s}_j-\frac{1}{2}}\right|^2 \mathrm{d}t. \tag{4.7}$$

The point is now that the first term on the right in (4.7) is just

$$2\pi\left\|\sum_{s_j\in S_\epsilon} c_j \frac{k_{s_j}}{\|k_{s_j}\|}\right\|_{H^2}^2,$$

so that by using the hypothesis on S and Lemma 4.4, we arrive at the inequality

$$\frac{1}{m_T}\left\|\sum_{s_j\in S_\epsilon} c_j \frac{k_{s_j}}{\|k_{s_j}\|}\right\| \geq m'\|(c_j)\|_{\ell^2} - \left(\int_{-T}^{T}\left|\sum_{s_j\in S_\epsilon} c_j\psi\Big(\mathrm{i}t+\bar{s}_j+\frac{1}{2}\Big)\|k_{s_j}\|^{-1}\right|^2 \mathrm{d}t\right) - \left(\int_{|t|>T}\left|\sum_{s_j\in S_\epsilon} c_j \frac{\|k_{s_j}\|^{-1}}{\mathrm{i}t+\bar{s}_j-\frac{1}{2}}\right|^2 \mathrm{d}t\right)^{1/2}.$$

The two terms that are subtracted on the right are easily estimated. Indeed, by applying the triangle inequality to get the sums outside of the integrals and then the Cauchy–Schwarz inequality, we see that the first term is bounded by

$$\sum_{s_j \in S_\epsilon} |c_j| \|k_{s_j}\|^{-1} \left(\int_{-T}^{T} \left| \psi \left(\mathrm{i}t + \bar{s}_j - \frac{1}{2} \right) \right|^2 \mathrm{d}t \right)^{1/2} \\ \leq B \left(\sum_{s_j \in S_\epsilon} \|k_{s_j}\|^{-2} \right)^{1/2} \|(c_j)\|_{\ell^2}$$

with B depending only on h, S, and T. The second term is treated in a similar way, and we find that it is bounded by

$$\sum_{s_j \in S_\epsilon} |c_j| \|k_{s_j}\|^{-1} \left(\int_{|t|>T} \left| \frac{1}{\bar{s}_j - \frac{1}{2} + \mathrm{i}t} \right|^2 \mathrm{d}t \right)^{1/2} \\ \leq 2 \left(\sum_{s_j \in S_\epsilon} \|k_{s_j}\|^{-2} \right)^{1/2} \|(c_j)\|_{\ell^2}.$$

By (4.2), $\|k_{s_j}\|^{-2} \leq C(\sigma_j - 1/2)$ with C depending only on S. In view of (4.4), we obtain (4.5) by choosing ϵ sufficiently small. This completes the proof of the theorem.

4.3 Local interpolation in the spaces $\mathscr{H}^2_\alpha$ and $\mathscr{D}_\alpha$

In this section we explain how the previous theorem is extended to the spaces $\mathscr{H}^2_\alpha$ for $\alpha \leq 1$ and $\mathscr{D}_\alpha$ for $\alpha \in \mathbb{R} \cup \{+\infty\}$. The key observation is that along with Lemma 4.4 one only needed the relation (4.2) and the estimate (4.3). Note that for $\alpha > 1$ the reproducing kernels for $\mathscr{H}^2_\alpha$ are uniformly bounded and there are no non-trivial interpolating sequences.

The appropriate analogues for the spaces $\mathscr{H}^2_\alpha$ were found in section 2.3 and are given by the relation (2.14) and the estimates (2.15) and (2.16).

The proof of the following extension of Theorem 4.3 is now essentially a plain rewriting of the proof in the previous section. We trust that the interested reader may check the details.

Theorem 4.5 (Olsen and Seip 2008)**.** *Suppose S is a bounded sequence of distinct points from $\mathbb{C}_{1/2}$ and assume $\alpha \leq 1$. Then S is a (universal) interpolating sequence for $\mathscr{H}^2_\alpha$ if and only if it is a (universal) interpolating sequence for $D_\alpha(\mathbb{C}_{1/2})$.*

In section 2.5 we found the relations (2.24) and (2.25), and the estimates (2.26), (2.27) and (2.28) as appropriate analogues to (4.2) and (4.3) for the spaces $\mathscr{D}_\alpha$ for $\alpha \in \mathbb{R} \cup \{+\infty\}$. This yields the following theorem.

Theorem 4.6. *Suppose S is a bounded sequence of distinct points from $\mathbb{C}_{1/2}$ and $\alpha \in \mathbb{R} \cup \{+\infty\}$. Then S is a (universal) interpolating sequence for $\mathscr{D}_\alpha$ if and only if it is a (universal) interpolating sequence for $D_{1-2^{-\alpha}}(\mathbb{C}_{1/2})$.*

We refer the reader to page 23 for references to the characterisations of interpolating sequences for the spaces $D_\alpha(\mathbb{C}_{1/2})$.

4.4 A constructive approach to local interpolation

In this section we give a second proof of the non-trivial implication of Theorem 4.3, namely that if a sequence $S = (s_n)$ is interpolating for $H^2(\mathbb{C}_{1/2})$ then it is also interpolating for $\mathscr{H}^2$. The idea is simple and uses the Paley-Wiener theorem as follows. Write $s_n = \sigma_n + \mathrm{i}t_n$ and suppose that (c_n) is a sequence such that $(c_n/\|k^{H^2}_{s_n}\|) \in \ell^2$. Then there exists $f \in H^2(\mathbb{C}_{1/2})$ that satisfies $f(s_n) = c_n$. The Paley-Wiener theorem[1] now says that there exists $g \in L^2(0,\infty)$ such that

$$f(s) = \frac{1}{\sqrt{2\pi}} \int_0^\infty g(\xi) \mathrm{e}^{-(s-1/2)\xi} \mathrm{d}\xi. \tag{4.8}$$

In fact, $\mathcal{F}g$ is equal almost everywhere to the non-tangential limits of f on $\sigma = 1/2$. In particular this implies that $\|g\|_{L^2(0,\infty)} = \|f\|_{H^2}$. The point is

[1]See Rudin's book [73][p. 180].

that by considering Riemann sums of this formula, one is led to consider the Dirichlet series

$$F(s) = \sum_{n\in\mathbb{N}} \frac{1}{\sqrt{2\pi}} \int_{\log n}^{\log(n+1)} g(\xi) n^{-(s-1/2)} \mathrm{d}\xi. \tag{4.9}$$

By a simple application of the Cauchy-Schwarz inequality we obtain that $\|F\|_{\mathscr{H}^2} \leq \|f\|_{H^2(\mathbb{C}_{1/2})}$. We use this idea to construct a sequence of Dirichlet series in $\mathscr{H}^2$ that converges to a solution of the interpolating problem $s_n \mapsto c_n$.

Approximating by Dirichlet series

In order to establish approximating properties of Dirichlet series of the type (4.9), we note the following trick. Given $f \in H^2(\mathbb{C}_{1/2})$ we multiply both sides of (4.8) by $\mathrm{e}^{-(s-1/2)A}$. By making a change of variables in the integral, this yields

$$\mathrm{e}^{-(s-1/2)A} f(s) = \frac{1}{\sqrt{2\pi}} \int_A^\infty g(\xi - A) \mathrm{e}^{-(s-1/2)\xi} \mathrm{d}\xi.$$

Let F_A denote the approximation of the type (4.9) of the function defined by this expression. The following lemma describes in quantitative terms the quality of this approximation.

Lemma 4.7. *Let $A > 0$ and $f \in H^2(\mathbb{C}_{1/2})$. If F_A is the approximation of $\mathrm{e}^{-(s-1/2)A} f(s)$ given by* (4.9) *then $\|F_A\|_{\mathscr{H}^2} \leq \|f\|_{H^2}$ and*

$$|\mathrm{e}^{-(s-1/2)A} f(s) - F_A(s)| \leq |s - 1/2| \mathrm{e}^{-A} \|f\|_{H^2(\mathbb{C}_{1/2})}. \tag{4.10}$$

Proof. Let $A > 0$, f and F_A be as in the hypothesis. Then

$$\begin{aligned} &|\mathrm{e}^{-(s-\frac{1}{2})A} f(s) - F_A(s)| \\ &\quad = \frac{1}{\sqrt{2\pi}} \left| \sum_{n\geq \mathrm{e}^A} \int_{\log n}^{\log(n+1)} g(\xi - A)(\mathrm{e}^{-(s-1/2)\xi} - n^{-(s-1/2)}) \mathrm{d}\xi \right| \end{aligned}$$

By expressing the difference $\mathrm{e}^{-(s-1/2)\xi} - n^{-(s-1/2)}$ as an integral, and applying the Cauchy-Schwarz inequality repeatedly, we find that this is less than $(2\pi)^{-1/2}$ multiplied by

$$\begin{aligned}
&|s-1/2| \sum_{n \geq \mathrm{e}^A} \frac{1}{n^{\sigma+1/2}} \int_{\log n}^{\log(n+1)} |g(\xi - A)| \mathrm{d}\xi \\
&\qquad \leq |s-1/2| \sum_{n \geq \mathrm{e}^A} \frac{1}{n^{\sigma+1}} \left(\int_{\log n}^{\log(n+1)} |g(\xi - A)|^2 \mathrm{d}\xi \right)^{1/2} \\
&\qquad \leq |s-1/2| \left(\sum_{n \geq \mathrm{e}^A} \frac{1}{n^3} \right)^{1/2} \left(\sum_{n \geq \mathrm{e}^A} \int_{\log n}^{\log(n+1)} |g(\xi - A)|^2 \mathrm{d}\xi \right)^{1/2} \\
&\qquad \leq |s-1/2| \mathrm{e}^{-A} \|f\|_{H^2}.
\end{aligned}$$

The lemma now follows. □

The construction[2]

To simplify notation we write $k_{s_n}^{H^2} = k_n$ and drop subscripts on the norms. Suppose that $S = (s_j)$ is a bounded sequence of points in $\mathbb{C}_{1/2}$ which is interpolating for $H^2(\mathbb{C}_{1/2})$. We now describe an iterative process which produces a sequence of Dirichlet series which converges in $\mathscr{H}^2$ to a solution for any given data $(c_j / \|k_{s_j}^{H^2}\|_{H^2}) \in \ell^2$.

First, we note that by the proof of Theorem 4.3 we only need to consider sequences $S = (s_j)$ for which $\sigma_j < 1$ for all $n \in \mathbb{N}$. We assume that the s_j are enumerated in such a way that $\sigma_j \geq \sigma_{j+1}$. Next, recall that since the sequence $S = (s_j)$ is interpolating for $H^2(\mathbb{C}_{1/2})$ it has a constant of interpolation $C > 0$. Based on this constant we fix some number $A > 0$ (to be determined later). Let f_1 be the solution of minimal norm of the interpolation problem

$$f_1 : s_j \longmapsto \mathrm{e}^{(s_j - 1/2)A} c_j.$$

[2] We thank H. Queffélec for pointing out that the iterative part of this construction is contained in Lemma 4.14.

By the definition of the constant of interpolation it follows that

$$\|f_1\|_{H^2} \leq C\mathrm{e}^{(\sigma_1-1/2)A} \left\| \left(\frac{c_j}{\|k_j\|} \right) \right\|_{\ell^2}.$$

We let F_1 be the Dirichlet series of the type (4.9) that approximates the function $\mathrm{e}^{-(s-1/2)A} f_1$. It follows by Lemma 4.7 that $\|F_1\| \leq \|f_1\|$. We set $\Delta c_j = \mathrm{e}^{-(s_j-1/2)} f_1(s_j) - F_1(s_j)$. By the same lemma, we get the bound

$$|\Delta c_j| \leq C|s_j - 1/2|\mathrm{e}^{(\sigma_1-3/2)A} \left\| \left(\frac{c_j}{\|k_j\|} \right) \right\|_{\ell^2}.$$

We now iterate this procedure, trying to capture as much of the error Δc_j as possible. At the n'th iteration we find the function f_n of minimal norm solving

$$f_n : s_j \longmapsto \mathrm{e}^{(s_j-1/2)A} \Delta^{(n-1)} c_j.$$

We let F_n be the Dirichlet series of the type (4.9) that approximates the function $\mathrm{e}^{-(s-1/2)A} f_n$. We set $\Delta^{(n)} c_j = \mathrm{e}^{-(s_j-1/2)A} f_n(s_j) - F_n(s_j)$. Lemma 4.7 now give the bounds

$$\|F_n\| \leq C^n \sup_{j\in\mathbb{N}} \left| s_j - \frac{1}{2} \right|^{n-1} \mathrm{e}^{n(\sigma_1-3/2)A+A} \left\| \left(\frac{1}{\|k_j\|} \right) \right\|_{\ell^2}^{n-1} \left\| \left(\frac{c_j}{\|k_j\|} \right) \right\|_{\ell^2},$$

and

$$|\Delta^{(n)} c_j| \leq C^n \left| s_j - \frac{1}{2} \right|^{n} \mathrm{e}^{n(\sigma_1-3/2)A} \left\| \left(\frac{1}{\|k_j\|} \right) \right\|_{\ell^2}^{n-1} \left\| \left(\frac{c_j}{\|k_j\|} \right) \right\|_{\ell^2}.$$

We sum up the functions F_n at the point s_j. This gives a telescoping sum.

$$\begin{aligned} &F_1(s_j) + F_2(s_j) + \cdots + F_n(s_j) \\ &\qquad = (c_j - \Delta c_j) + (\Delta c_j - \Delta^{(2)} c_j) \cdots + (\Delta^{(n-1)} c_j - \Delta^{(n)} c_j) \\ &\qquad\qquad = c_j - \Delta^{(n)} c_j. \end{aligned}$$

We set $G_n = \sum_{k=1}^{n} F_k$. The norm of this function has the bound

$$\|G_n\| \le C\mathrm{e}^{(\sigma_1 - 1/2)A} \left\| \left(\frac{c_j}{\|k_j\|} \right) \right\|_{\ell^2} \times \sum_{k=1}^{n} \left(C \sup_{j\in\mathbb{N}} |s_j - 1/2| \mathrm{e}^{(\sigma_0 - 3/2)A} \left\| \left(\frac{1}{\|k_j\|} \right) \right\|_{\ell^2} \right)^{k-1}.$$

The sequence G_n converges and $\Delta^{(n)} c_j \to 0$ for all $j \in \mathbb{N}$ if and only if

$$A > \frac{1}{\frac{3}{2} - \sigma_1} \log \left(C \sup_{j\in\mathbb{N}} |s_j - 1/2| \left\| \left(\frac{1}{\|k_j\|} \right) \right\|_{\ell^2} \right).$$

Recall that we reduced to the case $\sigma_1 < 1$. Hence, we conclude that the function $G = \lim_{n\to\infty} G_n$ satisfies $G(s_j) = c_j$ and $G \in \mathscr{H}^2$.

4.5 A necessary condition for interpolation in $\mathscr{H}^p$

Suppose S is a local interpolating sequence for $\mathscr{H}^2$. As we saw in the proof of Theorem 4.3, it follows essentially immediately from the embedding (4.3) and the relation (4.2) that S is also interpolating for $H^2(\mathbb{C}_{1/2})$.

In this section we show that for universal interpolating sequences an embedding is not needed, neither is the restriction to bounded sequences. We recall from section 2.2 that by Lemma 2.3 the norm of the point evaluation of the space $\mathscr{H}^p$ at the point $s \in \mathbb{C}$, which we denote by $\omega_{\mathscr{H}}(s)$, satisfies

$$\omega_{\mathscr{H}^p}(s)^p = \zeta(2\sigma). \tag{4.11}$$

By using the formula (2.2) for the Riemann zeta function, this implies

$$\omega_{\mathscr{H}^p}(s)^p = C_p \omega_{H^p}(s)^p + \mathcal{O}(1), \quad \sigma \to 1/2^+, \tag{4.12}$$

where $C_p > 0$ are constants only depending on p.

Theorem 4.8 (Olsen and Saksman 2009)**.** *Let $p \in [1, \infty)$ and assume that S is a sequence of points in $\mathbb{C}_{1/2}$. If S is a universal interpolating sequence for $\mathscr{H}^p$ then it is an interpolating sequence for $H^p(\mathbb{C}_{1/2})$.*

The theorem follows by Carleson's geometric description of the interpolating sequences for the spaces $H^p(\mathbb{C}_{1/2})$ given in Lemma 4.1 in combination with the two following lemmas.

Lemma 4.9 (Olsen and Saksman 2009)**.** *Let $p \in [1,\infty)$ and assume that μ is a positive measure on $\mathbb{C}_{1/2}$. If μ is a Carleson measure for $\mathscr{H}^p$ then μ is a Carleson measure for $H^p(\mathbb{C}_{1/2})$.*

Lemma 4.10 (Olsen and Saksman 2009)**.** *Let $p \in [1,\infty)$ and assume that $S = (s_n)$ is a sequence of points in $\mathbb{C}_{1/2}$. If S is a universal interpolating sequence for $\mathscr{H}^p$ then S is separated in the pseudo-hyperbolic metric, i.e.*

$$\inf_{n \neq m} \left| \frac{s_n - s_m}{s_n + \bar{s}_m - 1} \right| > 0.$$

Proof of Lemma 4.9. Assume that μ is a Carleson measure for $\mathscr{H}^p$ with constant $C > 0$. Let Q be a small Carleson box in $\mathbb{C}_{1/2}$. Let s_0 be the mid-point of the right edge of the box. Next we compute the norm of $\zeta_{s_0}^{2/p}(s) = \zeta^{2/p}(s + \overline{s_0})$ in $\mathscr{H}^p$. Consider the function

$$F(z_1, \cdots) = \prod \left(\frac{1}{1 - p_n^{-\overline{s_0}} z_n} \right)^{2/p}$$

on the infinite dimensional torus. We compute,

$$\|F\|_{L^p(\mathbb{T}^\infty)}^p = \prod_{n \in \mathbb{N}} \left\| \left(\frac{1}{1 - p_n^{-\overline{s_0}} z_n} \right)^{2/p} \right\|_{L^p(\mathbb{T})}^p = \prod_{n \in \mathbb{N}} \left(\frac{1}{1 - p_n^{-2\sigma_0}} \right) = \zeta(2\sigma_0).$$

It follows by the Bohr correspondence that $\mathcal{B}^{-1} F = \zeta_{s_0}^{2/p} \in \mathscr{H}^p$ with $\|\zeta_{s_0}^{2/p}\|_{\mathscr{H}^p}^p = \|\zeta_{s_0}\|_{\mathscr{H}^2}^2 = \zeta(2\sigma_0)$.

Next, we combine this with the fact that μ is a Carleson measure for $\mathscr{H}^p$ to get

$$\int_Q \frac{|\zeta_{s_0}^{2/p}(s)|^p}{\|\zeta_{s_0}\|_{\mathscr{H}^2}^2} \mathrm{d}\mu \leq C \frac{1}{\|\zeta_{s_0}\|_{\mathscr{H}^2}^2} \|\zeta_{s_0}^{2/p}\|_{\mathscr{H}^p}^p = C. \tag{4.13}$$

On the other hand, by the formula (2.2) for the Riemann zeta function it follows that $\zeta(2\sigma_0)^{-1} = \|\zeta_{s_0}\|_2^{-2} = (2\sigma_0 - 1)(1 + o(1))$ as $\sigma_0 \to 1/2$. So, for s_0 close to the abscissa $\sigma = 1/2$, the left hand side of (4.13) is greater than some constant times

$$\begin{aligned}(2\sigma_0 - 1) &\int_Q |\zeta(s+\overline{s_0})|^2 \,\mathrm{d}\mu \\ &= (2\sigma_0 - 1)\int_Q \left|\frac{1}{s+\overline{s_0}-1} + \psi(s+\sigma_0)\right|^2 \mathrm{d}\mu \\ &\geq \frac{2\sigma_0 - 1}{2}\int_Q \frac{1}{|s+\overline{s_0}-1|^2}\mathrm{d}\mu + \mathcal{O}\left(2\sigma_0 - 1\right). \quad (4.14)\end{aligned}$$

By geometric considerations, for $s \in Q$ it follows that $|s + \overline{s_0} - 1|^2 \leq (5/4)(1 - 2\sigma_0)^2$. Hence, the expressions in (4.14) are greater than

$$\frac{2}{5}\frac{1}{2\sigma_0 - 1}\mu(Q) + \mathcal{O}\left(2\sigma_0 - 1\right).$$

It follows that there is some constant $D > 0$ such that for any Carleson box with $\sigma_0 < 1$ we have

$$\mu(Q) \leq D(2\sigma_0 - 1).$$

We verify that this implies that μ is a Carleson measure for $\mathscr{H}^p$. Since $1 \in \mathscr{H}^p$ it follows that $\mu(\mathbb{C}_{1/2}) \leq C$. So for any Carleson box Q_2 with sides $\sigma_0 \geq 1$, it follows that

$$\mu(Q_2) \leq C \leq C(2\sigma_0 - 1).$$

Hence, by Carleson's characterisation of Carleson measures (Lemma 4.1) μ is a Carleson measure for the spaces $H^p(\mathbb{C}_{1/2})$ with constant smaller than $\max\{2D, 2C\}$ □

Proof of Lemma 4.10. Assume that $S = (s_j)$ is a universal interpolating sequence for $\mathscr{H}^p$ with constant of interpolation $C > 0$ and which is not separated in the pseudo-hyperbolic metric. Without loss of generality we assume that the s_j are enumerated with decreasing real parts.

Let $s, w \in \mathbb{C}_{1/2}$ satisfy $\operatorname{Re} s := \sigma_s \leq \operatorname{Re} w$. Then

$$|F(s) - F(w)| = |\int_s^w F'(z)\mathrm{d}z| \leq |s - w| \sup_{z\in(s,w)} |F'(z)|.$$

By (s,w) we mean the straight line with endpoints in s and w. Let D denote the cigar-shaped contour (two semi-circles with centres in s and w connected by lines fg to (s,w)) that holds a constant distance $(2\sigma_s-1)/4$ from the line (s,w). By Cauchy's formula, we get the estimate

$$\begin{aligned}|F'(z)| &\leq \frac{1}{2\pi}\int_D \frac{|F(\xi)|}{|\xi-z|^2}|\mathrm{d}\xi| \leq \frac{8}{\pi}\frac{|D|}{(2\sigma_s-1)^2}\sup_{\xi\in D}|F(\xi)| \\ &\leq \frac{8}{\pi}\frac{|D|}{(2\sigma_s-1)^2}\zeta\left(\sigma_s+\frac{1}{2}\right)^{1/p}\|F\|_{\mathscr{H}^p}.\end{aligned}$$

Here we used (4.11) which says that the norm of the point evaluation at $s\in\mathbb{C}_{1/2}$ in $\mathscr{H}^p$ is $\omega_{s,p}=\zeta(2\sigma)^{1/p}$. Note that the symbol $|D|$ denotes the arc-length of D. If both s and w are in a pseudo-hyperbolic ball of radius ϵ, an elementary argument shows that the diameter of this ball is less than $\frac{2\epsilon}{(1-\epsilon)^2}(2\sigma_s-1)$. The combined length of the semi-circular parts is $\frac{\pi}{2}(2\sigma_s-1)$, and so

$$|D| \leq \left(\frac{\pi}{2}+\frac{2\epsilon}{(1-\epsilon)^2}\right)(2\sigma_s-1)$$

Hence, for $\epsilon>0$ small enough

$$|F(s)-F(w)| \lesssim \frac{|s-w|}{2\sigma_s-1}\zeta\left(\sigma_s+\frac{1}{2}\right)^{1/p}\|F\|_{\mathscr{H}^p}.$$

The implicit constant depends only on ϵ. Recall that $C>0$ denotes the constant of interpolation for S. Hence, for the data (δ_{mn}) there exists $F_m\in\mathscr{H}^p$ that solves the problem $F_m(s_n)=\delta_{mn}$ with norm $\|F_m\|_{\mathscr{H}^p}\leq C\zeta(2\sigma_m)^{-1/p}$. This implies

$$\begin{aligned}1=|F_m(s_m)-F_m(s_{m-1})| &\lesssim \frac{|s_m-s_{m-1}|}{2\sigma_m-1}\left(\frac{\zeta(\sigma_m+\frac{1}{2})}{\zeta(2\sigma_m)}\right)^{1/p} \\ &\lesssim \frac{|s_m-s_{m-1}|}{2\sigma_m-1}.\end{aligned}$$

The last inequality follows again by considering the formula $\zeta(s)=(s-1)^{-1}+\psi(s)$, where ψ is some entire function. By elementary considerations, it now follows that if $\rho(s_m, s_{m-1}) \to 0$ in the pseudo-hyperbolic metric then the right hand side gets arbitrarily small. This leads to a contradiction. $\square$

4.6 A sufficient condition for interpolation in $\mathscr{H}^p$

We turn to the converse problem. If a sequence S is interpolating for $H^p(\mathbb{C}_{1/2})$, under what circumstances may we claim that it is also interpolating for some of the spaces in the range $\mathscr{H}^p$? We remark that by the theory in [19] it is possible to define the quasi Banach spaces $\mathscr{H}^p$ for $0 < p < 1$ along the same lines as for $p \geq 1$, extending the identity $\omega_{\mathscr{H}^p}(s)^p = \zeta(2\sigma)$ to $0 < p < 1$. We therefore tacitly draw conclusions about interpolation in these spaces in the following lemma and in Theorem 4.13.

Lemma 4.11 (Olsen and Saksman 2009)**.** *Let $p \in [1, \infty)$ and assume that S is a bounded sequence of points in $\mathbb{C}_{1/2}$. If S is interpolating for $\mathscr{H}^p$ then it is interpolating for $\mathscr{H}^{p/k}$ for all $k \in \mathbb{N}$.*

Proof. Let $S = (s_n)$ and recall that the bounded point evaluation in $\mathscr{H}^{p/k}$ at s_n satisfies $\omega_{\mathscr{H}^{p/k}}(s_n)^{p/k} = \zeta(2\sigma_n)$. Let (c_n) be a sequence of complex numbers such that

$$\sum \frac{|c_n|^{p/k}}{\omega_{\mathscr{H}^{p/k}}(s_n)^{p/k}} = \sum \frac{|c_n^{1/k}|^p}{\omega_{\mathscr{H}^p}(s_n)^p} < \infty. \tag{4.15}$$

By the hypothesis there exists $G \in \mathscr{H}^p$ such that $G(s_n) = c_n^{1/k}$. This means that $F = G^k$ satisfies $F(s_n) = c_n$. It is clear that $F \in \mathscr{H}^{p/k}$. $\square$

As an immediate consequence we get the following result.

Theorem 4.12 (Olsen and Saksman 2009)**.** *Let S be a bounded sequence of points in $\mathbb{C}_{1/2}$. If S is interpolating for H^1 then it is also interpolating for $\mathscr{H}^1$.*

From the embedding 4.3 it follows trivially that for $k \in \mathbb{N}$ and $F \in \mathscr{H}^{2k}$ we have

$$\int_I \left| F\left(\frac{1}{2} + \mathrm{i}t\right) \right|^{2k} \mathrm{d}t \le C_I \|F\|_{\mathscr{H}^{2k}}^{2k}. \tag{4.16}$$

In particular this implies that if μ is a local Carleson measure for $H^{2k}(\mathbb{C}_{1/2})$ it is also one for $\mathscr{H}^{2k}$. In addition, it implies that the bounded interpolating sequences in the weak sense for $\mathscr{H}^{2k}$ are interpolating for H^{2k}. The next result says that if we could show an analogue of Theorem 3.5 for exactly these spaces, then we can in fact find the local interpolating sequences for the spaces $\mathscr{H}^p$ for all $p \in \mathbb{Q}$. Recall that $\mathbb{C}_I = \{s \in \mathbb{C} : \mathrm{i}(s - 1/2) \notin \mathbb{R}\backslash I\}$.

Theorem 4.13 (Olsen and Saksman 2009)**.** *Let S be a bounded sequence of points in $\mathbb{C}_{1/2}$. Suppose that for all $k \in \mathbb{N}$ and open intervals $I \subset \mathbb{R}$ there exists $C_k > 0$, depending on $k \in \mathbb{N}$, such that the following holds: Given $f \in H^{2k}(\mathbb{C}_{1/2})$ there exists $F \in \mathscr{H}^{2k}$ such that $F - f \in \mathrm{Hol}(\mathbb{C}_I)$ and $\|F\|_{\mathscr{H}^{2k}} \le C_k \|f\|_{H^{2k}}$. Under these assumptions it holds that if S is an interpolating sequence for the spaces $H^p(\mathbb{C}_{1/2})$ then S is an interpolating sequence for the spaces $\mathscr{H}^q$ for all $q \in \mathbb{Q}$.*

We need the following lemma. Since we are unable to find a reference in the literature[3], we give a short proof.

Lemma 4.14. *Suppose that X, Y are Banach spaces and that $Z : X \to Y$ is bounded and linear. Let B_X and B_Y denote the unit balls of X and Y, respectively. If $M \subset Y$ is such that $\sup_{y \in M} \|y\|_Y \le 1/2$ and $ZB_X + M \supset B_Y$, then the operator Z is surjective.*

Proof of Lemma 4.14. The hypothesis implies that $M \subset \frac{1}{2} B_Y$, whence

$$ZB_X + \frac{1}{2} B_Y \supset B_Y.$$

The idea of the proof is to use the fact that by iterating this relation we get

$$\frac{1}{2^n} ZB_X + \frac{1}{2^{n+1}} B_Y \supset \frac{1}{2^n} B_Y.$$

[3] We thank H. Queffélec for bringing to our attention that this Lemma may be found in [69, chaptre 6, p. 202].

To be more precise, take $y \in B_Y$. Then there exists $x_0 \in B_X$ such that $y_1 = Zx_0 + y \in 2^{-1}B_Y$. Reiterating, we find $x_1 \in 2^{-1}B_X$ such that $y_2 = Zx_1 + y_1 \in 2^{-2}B_Y$. After n steps we find $x_n \in 2^{-n}B_X$ such that $y_{n+1} = Zx_n + y_n \in 2^{-n-1}B_Y$. Summing up, this implies that

$$Z(x_0 + x_1 + \cdots + x_n) = -y + y_{n+1}.$$

Since everything converges as $n \to \infty$, this means that for $y \in B_Y$ we have found $x \in X$ such that $Zx = y$, hence Z is surjective. □

Proof of Theorem 4.13. We begin by noting that by the assumption of the theorem, it follows that there exists a constant $D > 0$ such that given $f \in H^{2k}$ and a bounded subset Γ that is a positive distance from $\mathbb{C}\backslash\mathbb{C}_I$ then the inequality $\|F\| \leq C\|\phi\|$ implies that

$$\sup_{s\in\Gamma} |\phi(s)| \leq D\|f\|_{H^{2k}}.$$

Indeed, the same argument as was used in the proof of Theorem 3.5 may be used word by word.

We turn to the proof proper. Assume that $S = (s_n)_{n\in\mathbb{N}}$ is a bounded interpolating sequence for $H^{2k}(\mathbb{C}_{1/2})$. By Lemma 4.2 the interpolating and universal interpolating sequences for the spaces $H^{2k}(\mathbb{C}_{1/2})$ coincide. This means that the interpolation operator

$$\mathcal{T} : f \in H^{2k}(\mathbb{C}_{1/2}) \longmapsto \left(\frac{f(s_n)}{\omega_{H^{2k}}(s_n)}\right)_{n\in\mathbb{N}} \in \ell^{2k}$$

is bounded and onto ℓ^{2k}. By the inequality (4.16) and the equivalence of the point evaluations, it follows that $\mathcal{T}$ is also bounded as an operator from $\mathscr{H}^{2k}$ into ℓ^{2k}. Without loss of generalisation, we may assume that the sequence $(s_n)_{n\in\mathbb{N}}$ satisfies $\sigma_{n+1} \geq \sigma_n$. With this in mind we let $S_N = (s_n)_{n\geq N}$ and define the family of operators

$$\mathcal{T}_N : f \in \mathscr{H}^{2k}(\mathbb{C}_{1/2}) \longmapsto \left(\frac{f(s_n)}{\omega_{\mathscr{H}^{2k}}(s_n)}\right)_{n\geq N} \in \ell^{2k}.$$

By same procedure as in the proof of Theorem 4.3 it follows that if $\mathcal{T}_N$ is surjective, then the operator $\mathcal{T}$ is also surjective.

We now show that Lemma 4.14 implies that for N large enough the operator $\mathcal{T}_N$ is onto ℓ^{2k}. Note that the constants of interpolation of the sequences S_N is at most that of S, say $c_0 > 0$. Let $B_{H^{2k}}$ denote the unit ball of H^{2k}. Then it is clear that $c_0 \mathcal{T} B_{H^{2k}} \supset B_{\ell^{2k}}$. Next, let Γ be an open ball in $\mathbb{C}_{1/2}$ that contains S, and let $I \subset \mathbb{R}$ be some interval, symmetric with respect to the origin, such that $\sup\big\{|\Im s| : s \in I\big\} \geq 2\sup\big\{|\Im s| : s \in \Gamma\big\}$. By the hypothesis and the remark at the start of the proof, there exist constants $C, D > 0$ such that for every $f \in H^{2k}$ there is an $F \in \mathscr{H}^{2k}$ and $\phi \in \mathrm{Hol}(\mathbb{C}_I)$ such that $f = F + \phi$, with $\|F\|_{\mathscr{H}^{2k}} \leq C\|f\|_{H^{2k}}$ and $\|\phi\|_{L^\infty(\Gamma)} \leq D\|f\|_{H^{2k}}$. In other words, $B_{H^{2k}} \subset CB_{\mathscr{H}^{2k}} + DB_{\mathcal{C}(\Gamma)}$. Applying $\mathcal{T}_N$, this implies

$$B_{\ell^{2k}} \subset c_0 C\mathcal{T}_N B_{\mathscr{H}^{2k}} + c_0 D\mathcal{T}_N B_{\mathcal{C}(\Gamma)}. \tag{4.17}$$

Set $A_N = c_0 C\mathcal{T}_N$ and $K_N = c_0 D\mathcal{T}_N B_{\mathcal{C}(\Gamma)}$. Then we may express (4.17) as

$$B_{\ell^{2k}} \subset A_N B_{\mathscr{H}^{2k}} + K_N.$$

We need to show that the set K_N is compact for all $N \in \mathbb{N}$ and that for N large enough, we have

$$\sup_{(a_n)\in K_N} \|(a_n)\|_{\ell^{2k}} \leq 1/2. \tag{4.18}$$

The set $\mathcal{T}_N B_{\mathcal{C}(\Gamma)}$ is contained in the set $\big\{(a_n)_{n\geq N} \in \ell^{2k} : |a_n| \leq 1/w_{2k,n}\big\}$. It is readily checked that this set is both closed and totally bounded in ℓ^{2k} and therefore compact. (See, e.g., appendix A.4 in [73].) Since $K_N = D\mathcal{T}_N B_{\mathcal{C}(\Gamma)}$ (changing the constant if necessary), for $(a_n) \in K_N$ there exists $\phi \in B_{\mathcal{C}}(\Gamma)$ such that $(a_n) = D\mathcal{T}_N\phi$. From this it follows that

$$\|(a_n)\|_{\ell^{2k}}^{2k} = D^{2k} \sum_{n\geq N} \frac{|\phi(s_n)|^{2k}}{\omega_{\mathscr{H}^{2k}}(s_n)^{2k}} \lesssim \sum_{n\geq N} \frac{1}{\omega_{\mathscr{H}^{2k}}(s_n)^{2k}}.$$

However, since this sum is finite, it follows that we can make it arbitrarily small by increasing N. In particular, we can choose N large enough for (4.18) to hold, and so by Lemma 4.14 the operator $\mathcal{T}_N$ is onto ℓ^{2k}.

The result now follows from Lemma 4.11.

□

4.7 A distinguished subspace of $\mathscr{H}^1$

Let $\mathcal{K}$ denote the projective tensor space $\mathscr{H}^2 \otimes \mathscr{H}^2$. Recall that this space is the closure of the Dirichlet polynomials in the norm

$$\|F\|_{\mathcal{K}} = \inf \left\{ \sum_{\text{finite}} \|f_i\|_{\mathscr{H}^2} \|g_i\|_{\mathscr{H}^2} : F = \sum_{\text{finite}} f_i g_i,\ f_i, g_i \in \mathscr{H}^2 \right\}.$$

In other words, the infimum is taken over all representations of F as a finite sum of elementary tensors. It follows from the definition that $\mathcal{K} \subset \mathscr{H}^1$. We mentioned in the introduction that Helson conjectured that $\mathcal{K} = \mathscr{H}^1$. The truth of this conjecture would be convenient as the space $\mathcal{K}$ is easier to work with than $\mathscr{H}^1$. As a case in point, the embedding

$$\int_I \left| F\left(\frac{1}{2} + \mathrm{i}t\right) \right| \mathrm{d}t \le C_{|I|} \|F\|_{\mathcal{K}}$$

follows immediately by the corresponding embedding for $\mathscr{H}^2$. We note that the norm of the point evaluation in $\mathcal{K}$, which we denote by $\omega_{\mathcal{K}}$, is the same as the one in $\mathscr{H}^1$.

Lemma 4.15. *The function F in Corollary 3.8 may be chosen from the space $\mathcal{K}$ in such a way that (1) and (2) of the corollary hold with $\|F\|_{\mathcal{K}}$ replacing $\|F\|_{\mathscr{H}^1}$ where appropriate.*

Proof. The function F chosen in the proof of the theorem is clearly from $\mathcal{K}$. Since $\mathcal{K}$ is a Banach space and $FG \in \mathcal{K}$ for $F, G \in \mathscr{H}^2$ the statements (1) and (2) follow as before. □

Theorem 4.16 (Olsen and Saksman 2009)**.** *Let S be a bounded sequence. Then S is an interpolating sequence for $\mathcal{K}$ if and only it is interpolating for H^1. Moreover, the local interpolating and universal interpolating sequences of $\mathcal{K}$ coincide.*

Proof. One implication follows immediately by the embedding for $\mathcal{K}$. Indeed, assume that $S = (s_n)$ is bounded and interpolating for $\mathcal{K}$ and that S is indexed by decreasing real parts. We denote the norm of the point evaluation in $\mathcal{K}$ and H^1 at the point s_n by $\omega_{\mathcal{K}}(s_n)$ and $\omega_{H^1}(s_n)$, respectively.

For a sequence $(c_n) \subset \mathbb{C}$ such that $(c_n\omega_{H^1}(s_n)^{-1}) \in \ell^1$ we need to find $f \in H^1(\mathbb{C}_{1/2})$ such that $f(s_n) = c_n$. Recall that by (4.12) it holds that $\omega_{H^1}(s) \simeq \omega_{\mathscr{H}^1}(s)$ as $\sigma \to 1/2^+$, so by the remark immediately preceding Lemma 4.15 we have $\omega_{H^1}(s) \simeq \omega_{\mathcal{K}}(s)$ under the same limit. This means that $(s_n^2 c_n \omega_{\mathcal{K}}(s_n)^{-1}) \in \ell^1$. By hypothesis there exists $F \in \mathcal{K}$ such that $F(s_n) = s_n^2 c_n$. Hence, $f(s) = F(s)/s^2 \in H^1(\mathbb{C}_{1/2})$ solves the problem.

Next we note that by the interpolation theorem in [65] it follows that if a bounded sequence S is interpolating for H^1 then it is also interpolating for $\mathscr{H}^2$. By Lemma 4.11 this implies that S is also interpolating for $\mathscr{H}^1$. However, by examining the proof, it is seen that given a sequence (c_n) such that $(c_n/\omega_{H^1}(s_n)) \in \ell^1$ then it is solved by $F = G^2$, where $G \in \mathscr{H}^2$. In particular, this implies $F \in \mathcal{K}$. □

4.8 Carleson measures and the $\mathscr{H}^p$ embedding problem

In this section we prove the following theorem.

Theorem 4.17 (Olsen and Saksman 2009). *Let $p \in [1, \infty)$. Then the following statements are equivalent.*

(a) For every bounded interval $I \subset \mathbb{R}$ there exists a constant $C > 0$ such that for all finite sequences (a_n) of complex numbers it holds that

$$\int_I \left|\sum a_n n^{-\frac{1}{2}-\mathrm{i}t}\right|^p \mathrm{d}t \leq C \left\|\sum a_n n^{-s}\right\|_{\mathscr{H}^p}^p .$$

(b) Every local Carleson measure for $H^p(\mathbb{C}_{1/2})$ is also a Carleson measure for $\mathscr{H}^p$.

(c) There exists a constant $D > 0$ such that every local Carleson measure for $H^p(\mathbb{C}_{1/2})$ of the form

$$\mu_S = \sum \delta_{s_n}(2\sigma_n - 1), \quad s_n = \sigma_n + \mathrm{i}t_n,$$

is also a Carleson measure for $\mathscr{H}^p$ with

$$\int |F(s)|^p \mathrm{d}\mu_S(s) \leq D\|\mu\|_{\mathrm{CM}^p(H^p)}\|F\|_{\mathscr{H}^p}^p \quad \forall F \in \mathscr{H}^p.$$

Before we prove this theorem, we give a lemma on which the implication $(b) \Rightarrow (c)$ hinges. To do this we need to make some definitions. Let Γ be some bounded subset of $\mathbb{C}_{1/2}$ and let $M(\Gamma)$ denote the complex measures with support in the closure of Γ. This forms a Banach space under the norm $\|\mu\| = \int_\Gamma \mathrm{d}|\mu|$. For fixed $p \in [1,\infty)$ let X denote either of the spaces $H^p(\mathbb{C}_{1/2})$ or $\mathscr{H}^p$. By $\mathrm{CM}_\Gamma^p(X)$ we denote the space of all signed measures supported on a bounded subset $\Gamma \subset \mathbb{C}_{1/2}$ equipped with the norm

$$\|\mu\|_{\mathrm{CM}_\Gamma^p(X)} = \sup_{\|f\|_X=1} \int_\Gamma |f(s)|^p \mathrm{d}|\mu(s)|.$$

Here $\nu = |\mu|$ denotes the total variation measure of μ (see [29, p. 93]).

Lemma 4.18. *For fixed $p \in [1,\infty)$ let X denote either of the spaces $H^p(\mathbb{C}_{1/2})$ or $\mathscr{H}^p$. Then the space $\mathrm{CM}_\Gamma^p(X)$ is a Banach space.*

Proof. Since $1 \in \mathscr{H}^p$ and $s^{-2} \in H^p$ it follows that

$$\|\mu_n\|_{M(\Gamma)} = \int_\Gamma \mathrm{d}|\mu_n| \lesssim \|\mu_n\|_{CM_\Gamma^p(X)}. \tag{4.19}$$

Assume that $(\mu_n) \subset \mathrm{CM}_\Gamma^p(X)$ is such that $\sum \|\mu_n\|_{CM_\Gamma^p(X)} < \infty$. It suffices to show that $\sum \mu_n$ is convergent in $CM_\Gamma^p(X)$. By the inequality (4.19), we have

$$\sum \|\mu_n\|_{M(\Gamma)} \leq \sum \|\mu_n\|_{CM_\Gamma^p(X)} < +\infty,$$

and so $\sum \mu_n$ converges some element $\mu \in M(\Gamma)$. Moreover $\mu \in CM_\Gamma^p(X)$. Indeed, for any polynomial D,

$$\int |D(s)|^p \mathrm{d}|\mu| \leq \|D\|_X^p \sum \|\mu_n\|_{CM_\Gamma^p(X)}.$$

Finally, we confirm that $\mu_n \to \mu$ in the sense of $CM_\Gamma^p(X)$. But this follows immediately, since $\|\mu - \sum_{n=1}^N \mu_n\| \leq \sum_{n>N} \|\mu_n\|_{CM_\Gamma^p(X)}$. In conclusion, $CM_\Gamma^p(X)$ is a Banach space. □

Proof of Theorem 4.17. It is clear that $(a) \Rightarrow (b)$. We proceed to show $(b) \Rightarrow (c)$ and $(c) \Rightarrow (a)$.

$(b) \Rightarrow (c)$: Let Γ be some bounded subset of $\mathbb{C}_{1/2}$. Consider the operator

$$\mathcal{I} : \mu \in CM_\Gamma^p(H^p) \longmapsto \mu \in CM_\Gamma^p(\mathscr{H}^p).$$

By the hypothesis and Lemma 4.18 the operator $\mathcal{I}$ is well-defined. It suffices to show that it is continuous. By the closed graph theorem this follows if it has a closed graph. Assume that $\mu_n \to \mu$ in $CM_\Gamma^p(H^p)$ and that $\mu_n \to \nu$ in $CM_\Gamma^p(\mathscr{H}^p)$. By (4.19) , this implies that both $\mu_n \to \mu$ and $\mu_n \to \nu$ in the topology of $M(\Gamma)$, and so $\mu = \nu$ as measures. Hence $\mathcal{I}$ has a closed graph. Finally, (c) is just a special case of boundedness of $\mathcal{I}$ applied to sums of the point masses δ_{s_n}.

$(c) \Rightarrow (a)$: Let $F \in \mathscr{H}^p$ be a Dirichlet polynomial and consider

$$\int_0^T \left| F\left(\frac{1}{2} + \epsilon + \mathrm{i}t\right)\right|^p \mathrm{d}t.$$

For $\epsilon > 0$ small enough, the above is less than

$$\begin{aligned}\sum_{n=0}^{T[\epsilon^{-1}]} \int_{\epsilon n}^{\epsilon(n+1)} \left| F\left(\frac{1}{2} + \epsilon + \mathrm{i}t\right)\right|^p \mathrm{d}t &= \sum \int_0^\epsilon \left| F\left(\frac{1}{2} + \epsilon + \mathrm{i}t + \mathrm{i}n\epsilon\right)\right|^p \mathrm{d}t \\ &= \frac{1}{\epsilon}\int_0^\epsilon \int_{\mathbb{C}} |F(s)|^p \,\mathrm{d}\mu_{\epsilon,t}(s)\mathrm{d}t,\end{aligned}$$

where $\mu_{\epsilon,t} = \epsilon \sum_{n=0}^{T[\epsilon^{-1}]} \delta_{\frac{1}{2}+\epsilon+\mathrm{i}t+\mathrm{i}n\epsilon}$. By Carleson's geometric characterisation of Carleson measures (Lemma 4.1), the quantities $\|\mu_{\epsilon,t}\|_{\mathrm{CM}^p(H^p)}$ are uniformly bounded for $\epsilon \in (0,1)$. Let $\Gamma \subset \mathbb{C}_{1/2}$ be a bounded subset of $\mathbb{C}_{1/2}$ such that the supports of the measures $\mu_{\epsilon,t}$ for $\epsilon \in (0,1)$ are contained in Γ. Then the uniform boundedness also holds in the norm of $\mathrm{CM}_\Gamma^p(H^p)$. By (c), this implies that for $\epsilon \in (0,1)$ we have

$$\int_{\mathbb{C}} |F(s)|^p \,\mathrm{d}\mu_{\epsilon,t}(s) \lesssim \|F\|_{\mathscr{H}^p}^p.$$

Hence,

$$\int_0^T \left| F\left(\frac{1}{2} + \epsilon + \mathrm{i}t\right)\right|^p \mathrm{d}t \lesssim \|F\|_{\mathscr{H}^p}^p$$

as $\epsilon \to 0$, and the embedding theorem holds for $\mathscr{H}^p$. □

5 A class of modified zeta functions

In this chapter we study a class of modified zeta functions that we call the K-zeta functions. Recall that the Riemann zeta function has a meromorphic extension to the entire complex plane with a single pole at $s = 1$. Inspired by this, we completely characterise the behaviour of the K-zeta functions close to the point $s = 1$.

We begin by recalling some important notions and stating some preliminary results.

5.1 Preliminaries

For arbitrary $K \subset \mathbb{N}$ set

$$\zeta_K(s) = \sum_{n \in K} \frac{1}{n^s}, \quad s = \sigma + \mathrm{i}t.$$

We call the Dirichlet series obtained in this manner the K-zeta functions. By the triangle inequality it is clear that they are functions analytic on the half-plane $\sigma > 1$. The choice $K = \mathbb{N}$ yields the Riemann zeta function which satisfies the formula

$$\zeta(s) = \frac{1}{s-1} + \psi(s), \tag{5.1}$$

for some entire function ψ. It follows from [46, 67] that for general $K \subset \mathbb{N}$ these functions may have the abscissa $\sigma = 1$ as their natural boundary.

Counting functions and Mellin transforms

For a subset $K \subset \mathbb{N}$ we define the counting function

$$\pi_K(x) = \sum_{\substack{n \in K \\ n \leq x}} 1.$$

It enables us to write the K-zeta functions in the form

$$\frac{1}{s}\zeta_K(s) = \int_1^\infty x^{-s-1}\pi_K(x)\mathrm{d}x. \tag{5.2}$$

We call the right-hand side the Mellin transform of the counting function π_K. The formula follows by integrating the right hand side by parts and identifying the resulting expression with the sum defining the K-zeta function.

Pseudo functions and pseudo measures

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz class equipped with the usual topology. We call the dual elements $T \in \mathcal{S}'(\mathbb{R})$ distributions and denote their distributional Fourier transform by $\widehat{T}$. Following [52], we say that T is a pseudo-measure if $\widehat{T} \in L^\infty(\mathbb{R})$. If, in addition, $\widehat{T}$ decays as $|x| \to \infty$ we say that T is a pseudo-function.

By doing a change of variables, we observe that (5.2) may be expressed as

$$\frac{1}{s}\zeta_K(s) = \sqrt{2\pi}\mathcal{F}\left\{\mathrm{e}^{-(\sigma-1)u}\frac{\pi_K(\mathrm{e}^u)}{\mathrm{e}^u}\right\}(t).$$

It is readily checked that this implies that $\zeta_K(s)/s$ converges as $\sigma \to 1$, in the sense of distributions, to a pseudo-measure. We say that $\zeta_K(s)/s$ extends to a pseudo-measure on $\sigma = 1$.

Compact operators

We deal with operators of the type

$$g \in L^2(I) \longmapsto \chi_I \int_I g(\tau)\phi(t-\tau)\mathrm{d}\tau \in L^2(I), \tag{5.3}$$

where $I = (-T, T)$ for some $T > 0$ and $\phi \in L^1(2I)$. These operators are convolution operators followed by the projection onto $L^2(I)$. The following lemma is standard.

Lemma 5.1. *Let $\phi \in L^1(2I)$. Then (5.3) defines a compact operator on $L^2(I)$. More generally, if $(\phi_\delta)_{\delta\in(0,1)}$ is a net of functions in $L^1_{\mathrm{loc}}(\mathbb{R})$ converging in the sense of distributions to a pseudo-function ϕ, then the operator*

$$\Phi g(t) = \lim_{\delta\to 0}\int_I g(\tau)\phi_\delta(t-\tau)\mathrm{d}\tau$$

is bounded and compact on $L^2(I)$.

Proof. Let $e_n(t)$ denote the Fourier characters of $L^2(2I)$, and let the Fourier expansion of ϕ on $L^2(2I)$ be

$$\phi(t) = \sum_{n\in\mathbb{Z}} c_n e_n(t).$$

Hence, for $g \in L^2(I)$,

$$\Phi g(t) = |2I|^{1/2}\sum_{n\in\mathbb{Z}} c_n (g, e_n)_{L^2(I)} e_n(t). \tag{5.4}$$

By the Riemann-Lebesgue lemma it follows that $|c_n| \to 0$ as $|n| \to \infty$ and the operator Φ is seen to be compact.

We turn to the second part of the statement. Let $g \in \mathcal{C}_0^\infty(I)$. Then

$$\begin{aligned}\lim_{\delta\to 0}\int_{\mathbb{R}} g(t-\tau)\phi_\delta(\tau)\mathrm{d}\tau &= (g(t-\cdot),\phi)\\ &= \int_{\mathbb{R}} \hat{g}(\xi)\hat{\phi}(\xi)\mathrm{e}^{\mathrm{i}t\xi}\mathrm{d}\xi.\end{aligned}$$

By the dual expression of the $L^2(I)$ norm this is seen to be bounded by some constant times the L^2 norm of g. To see that it is compact, define an operator on $\mathcal{C}_0^\infty(I)$ by

$$\Phi_N g(t) = \int_{\mathbb{R}} \hat{g}(\xi)\hat{\phi}_N(\xi)\mathrm{e}^{\mathrm{i}t\xi}\mathrm{d}\xi,$$

with $\hat{\phi}_N = \chi_N\hat{\phi}$. Since $\mathcal{F}\hat{\phi}_N \in L^1(2I)$ this is a compact operator by the first part of the lemma. Moreover,

$$\|\Phi g - \Phi_N g\|_{L^2} \le \|g\|_{L^2}\|\hat{\phi}\|_{L^\infty(|\xi|>N)}.$$

Hence the sequence of compact operators Φ_N approximates Φ in the uniform operator topology as $N \to \infty$. $\square$

We also restate the following lemma from chapter 3.

Lemma 5.2 (Second stability theorem of Semi-Fredholm theory)**.** *Let X, Y be Banach spaces and $Z : X \to Y$ a continuous linear operator that is bounded below. If $\Phi : X \to Y$ is a compact operator and $Z + \Phi$ is injective, then it follows that $Z + \Phi$ is bounded below.*

5.2 The operator $\mathcal{Z}_{K,I}$ and two questions

Let $K \subset \mathbb{N}$ and $I \subset \mathbb{R}$ be a bounded interval symmetric about the origin. Define the operator

$$\mathcal{Z}_{K,I} : g \in L^2(I) \longmapsto \lim_{\delta \to 0} \frac{\chi_I}{\pi} \int_I g(\tau) \operatorname{Re} \zeta_K(1 + \delta + \mathrm{i}(t - \tau)) \mathrm{d}\tau$$

As in the previous chapters, we use the notation

$$\mathcal{Z}_{K,I} g = \lim_{\delta \to 0} \frac{\chi_I}{\pi} (g * \operatorname{Re} \zeta_{K,1+\delta}).$$

For $K = \mathbb{N}$ this is a constant multiple of the operator $R_I R_I^*$ of chapter 3. We define the more general operator on finite sequences

$$R_{I,K} : (a_n)_{n \in K \cup (-K)} \longmapsto \chi_I \sum_{n \in K} \frac{a_n n^{-\mathrm{i}t} + a_{-n} n^{\mathrm{i}t}}{\sqrt{n}}.$$

Since

$$\|R_{I,K}^* g\|_{\ell^2}^2 = 2\pi \sum_{n \in K} \frac{|\hat{g}(\log n)|^2 + |\hat{g}(-\log n)|^2}{n}, \tag{5.5}$$

the boundedness of R_I implies the boundedness of $R_{I,K}$ for all $K \subset \mathbb{N}$. By exactly the same argument used to establish Lemma 3.6 we get the following result, which in particular implies that $\mathcal{Z}_{K,I}$ is a bounded operator on $L^2(I)$ for all $K \subset \mathbb{N}$ and all intervals of the type $I = (-T, T)$ for $T < \infty$.

Lemma 5.3. *Let $K \subset \mathbb{N}$ and $I = (-T, T)$. Then for $g \in L^2(I)$ we have*

$$R_{I,K} R_{I,K}^* g = 2\pi \mathcal{Z}_{K,I} g.$$

In the case $K = \mathbb{N}$, this can by pushed a bit further since by the formula (5.1) it holds for $g \in L^2(I)$ that

$$\mathcal{Z}_{\mathbb{N},I} g = g + \frac{\chi_I}{\pi}(g * \operatorname{Re}\psi_1),$$

where $\psi_1(t) = \psi(1 + \mathrm{i}t)$. Let Id denote the identity operator on $L^2(I)$. It follows from Lemma 5.1 that we may rewrite this as

$$\mathcal{Z}_{\mathbb{N},I} = \mathrm{Id} + \Psi_{\mathbb{N},I}, \tag{5.6}$$

for a compact operator $\Psi_{\mathbb{N},I}$. Recall that in the introduction we used this to motivate the following questions.

(1) For which $K \subset \mathbb{N}$ does the formula $\mathcal{Z}_{K,I} = A\mathrm{Id} + \Psi_{K,I}$, with $\Psi_{K,I}$ a compact operator, hold for some $A \geq 0$?

(2) For which $K \subset \mathbb{N}$ is $\mathcal{Z}_{K,I}$ bounded below?

The following Theorem will be instrumental in answering both questions. It provides a formula which may be thought of as a generalisation of (5.6) to arbitrary $K \subset \mathbb{N}$.

Theorem 5.4. *Let $K \subset \mathbb{N}$ be arbitrary, $I \subset \mathbb{R}$ be a bounded and symmetric interval, and set*

$$L = \bigcup_{k \in K} \Big(\big(-\log(n+1), -\log n \big] \cup \big[\log n, \log(n+1) \big) \Big). \tag{5.7}$$

Then there exists a compact operator $\Phi_{K,I}$ such that

$$\mathcal{Z}_{K,I} = \chi_I \mathcal{F}^{-1} \chi_L \mathcal{F} + \Phi_{K,I}. \tag{5.8}$$

Proof. To make the notation easier we fix $K \subset \mathbb{N}$ and $I \subset \mathbb{R}$, and drop subscripts indicating dependence on these sets. The argument centres around the operator

$$R : (a_n)_{n \in K \cup (-K)} \longmapsto \chi_I \sum_{n \in K} \frac{a_n n^{-\mathrm{i}t} + a_{-n} n^{\mathrm{i}t}}{\sqrt{n}}.$$

It is straight-forward to compute that for $g \in L^2(I)$ we have

$$RR^* g(t) = \sqrt{2\pi} \sum_{n \in K} \left(\frac{\hat{g}(\log n)}{n} n^{\mathrm{i}t} + \frac{\hat{g}(-\log n)}{n} n^{-\mathrm{i}t} \right). \tag{5.9}$$

Note that for $g \in \mathcal{C}_0^\infty(I)$ this sum converges absolutely since $\hat{g}(\xi) = \mathcal{O}\left((1+\xi^2)^{-1}\right)$. Lemma 5.3 says exactly that $RR^* = 2\pi \mathcal{Z}$. Hence, the formula (5.8) in the statement of the theorem follows if we show that for $g \in \mathcal{C}_0^\infty(I)$ the difference

$$\sum_{n \in K} \left(\frac{\hat{g}(\log n)}{n} n^{\mathrm{i}t} + \frac{\hat{g}(-\log n)}{n} n^{-\mathrm{i}t} \right) - \int_L \hat{g}(\xi) \mathrm{e}^{\mathrm{i}t\xi} \mathrm{d}\xi$$

is given by a compact operator Φ. In order to simplify notation we set $L_+ = L \cap (0, \infty)$ and consider the difference of only the positive frequencies,

$$\sum_{n \in K} \frac{\hat{g}(\log n)}{n} n^{\mathrm{i}t} - \int_{L_+} \hat{g}(\xi) \mathrm{e}^{\mathrm{i}t\xi} \mathrm{d}\xi. \tag{5.10}$$

It suffices to show that this is given by a compact operator, say $2\pi\Phi_+$. The same argument then works on the negative frequencies by taking complex conjugates, giving us a compact operator $2\pi\Phi_-$. By choosing $\Phi = \Phi_+ + \Phi_-$ the proof is complete.

By adding and subtracting intermediate terms we see that the difference

(5.10) can be expressed as

$$\underbrace{\sum_{n\in K}\frac{1}{n\log(1+\frac{1}{n})}\int_{L_n}\Big(\hat{g}(\log n)n^{\mathrm{i}t}-\hat{g}(\xi)\mathrm{e}^{\mathrm{i}t\xi}\Big)\,\mathrm{d}\xi}_{(I)}$$
$$+\underbrace{\sum_{n\in K}\left(\frac{1}{n\log(1+\frac{1}{n})}-1\right)\int_{L_n}\hat{g}(\xi)\mathrm{e}^{\mathrm{i}t\xi}\mathrm{d}\xi}_{(II)}\,.$$

We want to interchange the integral and sum signs in these expressions. For (I), it suffices to show that

$$\sum_{n\in K}\int_{L_n}|\hat{g}(\log n)n^{\mathrm{i}t}-\hat{g}(\xi)\mathrm{e}^{\mathrm{i}\xi t}|\mathrm{d}\xi\le C\|g\|_{L^2(I)}. \tag{5.11}$$

for some constant $C>0$. Note that by expressing the difference inside the absolute value as a definite integral, we have

$$\int_{L_n}|n^{\mathrm{i}t}-\mathrm{e}^{\mathrm{i}t\xi}|\mathrm{d}\xi\le|t|\frac{1}{n^2}.$$

Pulling the absolute value sign inside of the expression for the Fourier transforms in combination with this, gives us the bound

$$\begin{aligned}\int_{L_n}|\hat{g}(\log n)n^{\mathrm{i}t}-\hat{g}(\xi)\mathrm{e}^{\mathrm{i}\xi t}|\mathrm{d}\xi&\le\int_I|g(\tau)|\int_{L_n}|n^{\mathrm{i}(t-\tau)}-\mathrm{e}^{\mathrm{i}(t-\tau)\xi}|\mathrm{d}\xi\mathrm{d}\tau\\&\le\frac{1}{n^2}\int_I|t-\tau||g(\tau)|\mathrm{d}\tau\le\frac{2|I|}{n^2}\left(\int_I|g(\tau)|^2\mathrm{d}\tau\right)^{1/2}.\end{aligned}$$

Taking the sum, and using the Cauchy-Schwarz inequality, we get (5.11) with constant $C=2|I|\zeta(4)^{1/2}$. Interchanging the integral and sum signs, we get

$$(I)=\int_I g(\tau)\alpha(t-\tau)\mathrm{d}\tau,$$

where

$$\alpha(\tau) = \frac{1}{\sqrt{2\pi}} \sum_K \frac{1}{n \log(1 + 1/n)} \int_{L_n} (n^{\mathrm{i}\tau} - \mathrm{e}^{\mathrm{i}\xi\tau}) \mathrm{d}\xi.$$

By the same bound we used above, this sum converges absolutely and therefore the function $\alpha(t)$ is continuous on I. Similar arguments show that

$$(II) = \int_I g(\tau)\beta(t - \tau)\mathrm{d}\tau,$$

where

$$\beta(\tau) = \sum_K \left(\frac{1}{n \log(1 + 1/n)} - 1 \right) \int_{L_n} \mathrm{e}^{\mathrm{i}t\xi} \mathrm{d}\xi$$

is a continuous function on I. Hence,

$$2\pi\Phi_+ g(t) = \frac{1}{\sqrt{2\pi}} \int_I g(\tau) \Big(\alpha(t - \tau) + \beta(t - \tau) \Big) \mathrm{d}\tau,$$

and so the compactness of Φ_+ follows from Lemma 5.1. By the comments of the first half of the proof this implies that Φ is also a compact operator. □

5.3 Korevaar's theorem and the first question

The following theorem of J. Korevaar [52, Theorem 1.1] generalises the Wiener-Ikehara tauberian theorem [43].

Lemma 5.5 (Korevaar 2005)**.** *Let $S(t)$ be a non-decreasing function with support in $(0, \infty)$, and such that the Laplace transform*

$$F(s) = \mathcal{L}S(s) = \int_0^\infty \frac{S(u)}{\mathrm{e}^u} \mathrm{e}^{-(s-1)u} \mathrm{d}u$$

exists for $\sigma > 1$. For some constant A, let

$$g(s) = F(s) - \frac{A}{s - 1}.$$

Then $g(s)$ extends to a pseudo-function on $\sigma = 1$ if and only if

$$\lim_{t \to \infty} \frac{S(u)}{\mathrm{e}^u} = A.$$

In the setting of the K-zeta functions the Ikehara-Wiener-Korevaar Theorem has the following consequence.

Lemma 5.6. *Let $K \subset \mathbb{N}$ and $A \geq 0$. Then the function defined by*

$$\psi_K(s) = \frac{1}{s}\zeta_K(s) - \frac{A}{s-1}$$

extends to a pseudo-function on $\sigma = 1$ if and only if

$$\lim_{x\to\infty} \frac{\pi_K(x)}{x} = A. \tag{5.12}$$

Proof. This follows by setting $S(u) = \pi_K(\mathrm{e}^u)$ in the Wiener-Ikehara-Korevaar Theorem. □

In terms of the operator $\mathcal{Z}_{K,I}$ this points the way to an answer to question (1). Indeed, when we combine Lemma 5.6 with Lemma 5.1 it follows immediately that if the condition (5.12) holds for some $A \geq 0$ then there exists a compact operator $\Psi_{K,I}$ such that

$$\mathcal{Z}_{K,I} = A\mathrm{Id} + \Psi_{K,I}.$$

We prove the converse statement as an application of Theorem 5.4

Theorem 5.7. *Let $K \subset \mathbb{N}$ be arbitrary and $A \geq 0$. Then the operator defined by*

$$\Psi_{K,I} = \mathcal{Z}_{K,I} - A\mathrm{Id}$$

is compact for all intervals of the form $I = (-T, T)$ if and only if

$$\lim_{x\to\infty} \frac{\pi_K(x)}{x} = A.$$

Proof. What remains is to prove the necessity of the density condition. By Theorem 5.4 we have the identity

$$\mathcal{Z}_{K,I} - A\mathrm{Id} = \chi_I \mathcal{F}^{-1} \chi_L \mathcal{F} - A\mathrm{Id} + \Phi_{K,I},$$

for some compact operator $\Phi_{K,I}$. Since the identity operator on $L^2(I)$ can be expressed as $\mathrm{Id} = \chi_I \mathcal{F}\mathcal{F}^{-1}$ it follows from the hypothesis that the operator

$$\widetilde{\Psi} : g \in L^2(I) \longmapsto \chi_I \int_{\mathbb{R}} (\chi_L - A)\hat{g}(\xi)\mathrm{e}^{\mathrm{i}\xi t}\mathrm{d}\xi$$

is compact on $L^2(I)$ for all bounded and symmetric $I \subset \mathbb{R}$. It is known that compact operators map sequences that converge weakly to zero to sequences that converge to zero in norm. We use this to show that for all $\delta > 0$ it holds that

$$\frac{|L \cap (\xi - \delta, \xi)|}{\delta} - A \to 0, \quad \text{as } \xi \to \infty. \tag{5.13}$$

We let $\epsilon > 0$, write $I = (-T, T)$, for some $T > 0$, and for $\xi \in \mathbb{R}$ define the $L^2(-T, T)$ functions

$$g_\xi(t) = \chi_{(-T,T)}\mathcal{F}^{-1}\{\chi_{(\xi-\delta,\xi)}\}(t) = \sqrt{\frac{2}{\pi}}\mathrm{e}^{\mathrm{i}t(\xi-\frac{\delta}{2})}\frac{\sin(\frac{\delta}{2}t)}{t}.$$

It is clear that by choosing $T > 0$ large the real valued functions $\hat{g}_\xi$ approximate the characteristic functions $\chi_{(\xi-\delta,\xi)}$ to an arbitrary degree of accuracy in $L^2(\mathbb{R})$ uniformly in ξ. In particular, we may choose $T > 0$ such that

$$\frac{1}{2}\delta \leq \|g_\xi\|_{L^2(I)} \leq 2\delta.$$

We fix some sequence $|\xi_n| \to \infty$. It follows readily that the functions g_{ξ_n} converge weakly to zero in $L^2(I)$. Hence, $\|\Psi g_{\xi_n}\| \to 0$ as $n \to \infty$. To obtain the connection to the set L, we use the dual expression for the norm of $\widetilde{\Psi} g_{\xi_n}$ to get

$$\begin{aligned}
\|\widetilde{\Psi} g_{\xi_n}\|_{L^2(I)} &\geq \frac{1}{\|g_n\|_{L^2(I)}} \left| \int_{\mathbb{R}} (\chi_L - A) \hat{g}_{\xi_n}(\xi)^2 \mathrm{d}\xi \right| \\
&\geq \underbrace{\frac{1}{2\delta} \left| \int_{\mathbb{R}} (\chi_L - A) \chi_{(\xi_n - \delta, \xi_n)}(\xi) \mathrm{d}\xi \right|}_{(*)} \\
&\quad - \underbrace{\frac{1}{2\delta} \left| \int_{\mathbb{R}} (\chi_L - A) \Big(\hat{g}_{\xi_n}(\xi)^2 - \chi_{(\xi_n - \delta, \xi_n)}(\xi) \Big) \mathrm{d}\xi \right|}_{(**)}.
\end{aligned}$$

It is clear that

$$(*) = \frac{1}{2} \left| \frac{|L \cap (\xi_n - \delta, \xi_n)|}{\delta} - A \right|.$$

Since $|\chi_L - A| \leq 1$ and $\chi_{(\xi_n - \delta, \xi_n)} = \chi^2_{(\xi_n - \delta, \xi_n)}$ we can use the formula $(a^2 - b^2) = (a+b)(a-b)$ and the Cauchy-Schwarz inequality to get

$$\begin{aligned}
(**) &\leq \frac{1}{2\delta} \|\hat{g}_{\xi_n} + \chi_{(\xi_n - \delta, \xi_n)}\|_{L^2(I)} \|\hat{g}_{\xi_n} - \chi_{(\xi_n - \delta, \xi_n)}\|_{L^2(I)} \\
&\leq \frac{3}{2} \|\hat{g}_{\xi_n} - \chi_{(\xi_n - \delta, \xi_n)}\|_{L^2(I)}.
\end{aligned}$$

By choosing $T > 0$ large enough, we can make $(**) \leq \epsilon/6$. Hence,

$$\left| \frac{|L \cap (\xi_n - \delta, \xi_n)|}{\delta} - A \right| \leq 2 \|\widetilde{\Psi} g_{\xi_n}\|_{L^2(I)} + \frac{\epsilon}{2}.$$

By choosing n sufficiently large for $\|\widetilde{\Psi} g_{\xi_n}\|_{L^2(I)} < \epsilon/4$, this establishes (5.13).

To get a contradiction, we assume that $\pi_K(x)/x$ does not tend to the limit A. Without loss of generality, we assume that there exists a number $\kappa > 0$ such that

$$\limsup_{x \to \infty} \frac{\pi_K(x)}{x} = A + \kappa.$$

This means that for any number $\eta \in (0,1)$ we may find a strictly increasing sequence of positive numbers ξ_n, with arbitrarily large separation, such that $\xi_n \to \infty$ as $n \to \infty$ and

$$\frac{\pi_K(\mathrm{e}^{\xi_n})}{\mathrm{e}^{\xi_n}} > A + \eta\kappa \quad \text{for } n \in \mathbb{N},$$

Moreover, since the counting function π_K changes slowly there exists a number $\delta_0 > 0$ such that for $n \in \mathbb{N}$ and $\xi \in (\xi_n - \delta_0, \xi_n)$ we have

$$\frac{\pi_K(\mathrm{e}^{\xi})}{\mathrm{e}^{\xi}} - A > \kappa/2.$$

Next, for $\xi_n > 2$, it holds that

$$\begin{aligned} |L \cap (\xi_n - \delta_0, \xi_n)| &\gtrsim \sum_{\substack{n \in (\mathrm{e}^{\xi_n - \delta_0}, \mathrm{e}^{\xi_n} - 1) \\ n \in K}} \log\left(1 + \frac{1}{n}\right) \\ &\gtrsim \sum_{\substack{n \in (\mathrm{e}^{\xi_n - \delta_0}, \mathrm{e}^{\xi_n} \\ n \in K}} \frac{1}{n} = \int_{\mathrm{e}^{\xi_n - \delta_0}}^{\mathrm{e}^{\xi_n}} \frac{1}{x} \mathrm{d}\pi_K(x) \\ &= \frac{\pi_K(\mathrm{e}^{\xi_n})}{\mathrm{e}^{\xi_n}} - \frac{\pi_K(\mathrm{e}^{\xi_n - \delta_0})}{\mathrm{e}^{\xi_n - \delta_0}} + \int_{\mathrm{e}^{\xi_n - \delta_0}}^{\mathrm{e}^{\xi_n}} \frac{1}{x} \frac{\pi_K(x)}{x} \mathrm{d}x. \end{aligned}$$

The last line follows from partial integration, and the implicit constants are absolute. By the choices of ξ_n, this implies that

$$|L \cap (\xi_n - \delta_0, \xi_n)| \gtrsim -(1-\eta)\kappa + \left(A + \frac{\kappa}{2}\right)\delta = A\delta + \left(\eta + \frac{\delta}{2} - 1\right)\kappa.$$

By choosing $\eta = (4-\delta)/4$, we find that for $\xi_n > 2$ we have

$$\frac{|L \cap (\xi_n - \delta_0, \xi_n)|}{\delta_0} - A > \frac{\kappa}{4}.$$

This contradicts (5.13). □

5.4 Panejah's theorem and the second question

The following result is a theorem due to B. Panejah . For a proof we refer the reader to [66]. Recall that $\mathcal{F}$ denotes the Fourier transform on $L^2(\mathbb{R})$ and that $L^2(I)$ is considered as a subspace of $L^2(\mathbb{R})$.

Lemma 5.8 (Panejah 1966)**.** *Let $L \subset \mathbb{R}$. Then the operator $\chi_L \mathcal{F}$ is bounded below from $L^2(I)$ to $L^2(\mathbb{R})$ if and only if there exists a $\delta > 0$ such that*

$$\inf_{\xi \in \mathbb{R}} |L \cap (\xi - \delta, \xi)| > 0.$$

This Lemma will be essential in answering the second question. We also include a technical lemma on the injectivity of the operator $\mathcal{Z}_{K,I}$. It generalises part of the argument of Lemma 3.7.

Lemma 5.9. *Let $K \subset \mathbb{N}$. A sufficient condition for $\mathcal{Z}_{K,I}$ to be injective is that*

$$\sum_{n \in K} \frac{1}{n} = +\infty.$$

Proof. Recall that $\mathcal{Z} = (2\pi)^{-1} R_{I,K} R_{I,K}^*$. Since an operator is always injective on the image of its adjoint it suffices to check that the hypothesis implies that $R_{I,K}^*$ is injective. In light of the expression (5.5) we need to check that for $g \in L^2(I)$ then $\hat{g}(\pm \log n) = 0$ for all $n \in K$ implies $g = 0$. To get a contradiction, assume that the function f is non-zero. The function $\hat{g}$ is entire and of exponential type $|I|/2$. In particular it is bounded on $\mathbb{R}$ and is therefore of the Cartwright class. A basic property of functions in this class (see [54, lesson 17]) is that the number of zeroes with modulus less than $r > 0$, which we denote by $\lambda(r)$, has to satisfy

$$\lim_{r \to \infty} \frac{\lambda(r)}{r} = \frac{|I|}{\pi}.$$

Let $\pi_K(x)$ be the counting function for K. Then $\lambda(r) \geq \pi_K(\mathrm{e}^r)$. The existence of the limit implies that $\pi_K(n) \leq C \log n$ for some $C > 0$. Summing by parts and using this estimate, we see that

$$\sum_{n \in K}^{N} \frac{1}{n} = \frac{\pi_K(N)}{N} + \sum_{n=1}^{N-1} \frac{\pi_K(n)}{n(n+1)} \leq 1 + C \sum_{n=1}^{N} \frac{\log n}{(n+1)^2}, \qquad (5.14)$$

which converges as $N \to +\infty$. Hence, we have a contradiction and so g has to equal zero, as was to be shown. $\square$

In order to state our theorem, we define the sequence

$$\mathscr{G}_K = \left(\dots, \frac{(-n)^{\mathrm{i}t}}{\sqrt{(-n)}}, \dots, \frac{n^{-\mathrm{i}t}}{\sqrt{n}}, \dots \right), \tag{5.15}$$

where n is understood to run through $K \cup (-K)$. In other words $\mathscr{G}_K$ is the two-sided sequence containing the elements $n^{\pm \mathrm{i}t}/\sqrt{n}$ for $n \in K$.

Theorem 5.10. *Let $K \subset \mathbb{N}$ be arbitrary, $I \subset \mathbb{R}$ be a bounded symmetric interval, $\mathscr{G}_K$ be given by (5.15) and $L \subset \mathbb{R}$ by the relation (5.7). Then the following conditions are equivalent.*

$$\mathcal{Z}_{K,I} \text{ is bounded below on } L^2(I) \tag{a}$$

$$\mathscr{G}_K \text{ is a frame for } L^2(I) \tag{b}$$

$$\chi_I \mathcal{F}^{-1} \chi_L \mathcal{F} \text{ is bounded below on } L^2(I) \tag{c}$$

$$\text{There exists } \delta \in (0,1) \text{ such that } \liminf_{x \to \infty} \frac{\pi_K(x) - \pi_K(\delta x)}{x} > 0. \tag{d}$$

Proof. To make the notation easier we fix $K \subset \mathbb{N}$ and $I \subset \mathbb{R}$, and drop subscripts indicating dependence on these sets.

$(a) \iff (b)$. By Lemma 5.3 it holds that $RR^* = 2\pi\mathcal{Z}$. This implies that the operator $\mathcal{Z}$ may be considered to be the frame operator for the sequence $\mathscr{G}$, defined by (5.15). Therefore, by Lemma 3.2, the boundedness and boundedness below of $\mathcal{Z}$ is equivalent to the sequence $\mathscr{G}$ being a frame.

$(c) \iff (d)$. Condition (c) says that $\chi_I \mathcal{F}^{-1} \chi_L \mathcal{F}$ is bounded below on $L^2(I)$. We begin by establishing that this is equivalent to $\chi_L \mathcal{F}$ being bounded below from $L^2(I)$ to $L^2(\mathbb{R})$. Indeed, one direction is clear since

$$\|\chi_L \mathcal{F} g\|_{L^2(\mathbb{R})} = \|\mathcal{F}^{-1} \chi_L \mathcal{F} g\|_{L^2(\mathbb{R})} \geq \|\chi_I \mathcal{F}^{-1} \chi_L \mathcal{F} g\|_{L^2(I)}.$$

To prove the converse, assume that there exists some $\delta > 0$ such that for $g \in L^2(I)$

$$\|\chi_L \mathcal{F} g\|_{L^2(\mathbb{R})} \geq \delta \|g\|_{L^2(I)}. \tag{5.16}$$

Moreover, assume that for all $\epsilon > 0$ there exists an $g_\epsilon \in L^2(I)$ such that

$$\|\chi_I \mathcal{F}^{-1} \chi_L \mathcal{F} g_\epsilon\|_{L^2(I)} \leq \epsilon^2 \|g_\epsilon\|_{L^2(I)}.$$

This implies that

$$\begin{aligned} \|\chi_I \mathcal{F}^{-1} \chi_{L^C} \mathcal{F} g_\epsilon\|_{L^2(I)} &\geq \|g_\epsilon\|_{L^2(I)} - \|\chi_I \mathcal{F}^{-1} \chi_L \mathcal{F} g_\epsilon\|_{L^2(I)} \\ &\geq (1-\epsilon^2)\|g_\epsilon\|_{L^2(I)}. \end{aligned}$$

On the other hand, the inequality (5.16) implies that

$$\begin{aligned} \|\chi_I \mathcal{F}^{-1} \chi_{L^C} \mathcal{F} g_\epsilon\|^2_{L^2(I)} &\leq \|\mathcal{F}^{-1} \chi_{L^C} \mathcal{F} g_\epsilon\|^2_{L^2(\mathbb{R})} \\ &= \|g_\epsilon\|^2_{L^2(I)} - \|\mathcal{F}^{-1} \chi_L \mathcal{F} g_\epsilon\|^2_{L^2(\mathbb{R})} \\ &\leq (1-\delta^2)\|g_\epsilon\|^2_{L^2(I)}. \end{aligned}$$

Combining these two inequalities, we find that $\epsilon \geq \delta$. This leads to a contradiction since we may choose $\epsilon^2 = \delta^2/2$.

We now invoke Panejah's theorem which says that the lower norm bound of $\chi_L \mathcal{F}$ on $L^2(\mathbb{R})$ is equivalent to the condition that there exists a $\delta > 0$ such that

$$\inf_{\xi \in \mathbb{R}} |L \cap (\xi - \delta, \xi)| > 0.$$

Finally, this is equivalent to

$$\liminf_{\xi \to \infty} \frac{\pi_K(\mathrm{e}^{\xi-\delta}, \mathrm{e}^{\xi})}{\mathrm{e}^{\xi}} > 0,$$

which is exactly condition (d). Indeed, this is just a matter of observing that

$$\frac{\pi_K(\mathrm{e}^{\xi-\delta}, \mathrm{e}^{\xi})}{\mathrm{e}^{\xi}} \leq \sum_{\log k \in (\xi-\delta,\xi)} \frac{1}{k} \leq \mathrm{e}^{\delta} \frac{\pi_K(\mathrm{e}^{\xi-\delta}, \mathrm{e}^{\xi})}{\mathrm{e}^{\xi}}.$$

$(a) \iff (c)$. This equivalence follows essentially from the result from semi-Fredholm theory given as Lemma 5.2 and the identity $\mathcal{Z} = \mathcal{F}^{-1}\chi_L \mathcal{F} + \Phi$, where Φ is a compact operator on $L^2(I)$ and L is given by (5.7). What needs to be checked is that the lower bound of $\mathcal{Z}$ implies the injectivity of $\chi_I \mathcal{F}^{-1} \chi_L \mathcal{F}$, and vice versa.

By the equivalence of (c) and (d), which we just established, we know that if the operator $\mathcal{F}^{-1}\chi_L\mathcal{F}$ is bounded below, then there exists $\delta \in (0,1)$ such that $\inf_{x\in\mathbb{R}}(\pi_K(x) - \pi_K(\delta x))/x > 0$. This is readily seen to imply that $\sum_{n\in K} n^{-1} = \infty$. By Lemma 5.9 we know that this is sufficient for the operator $\mathcal{Z}$ to be injective. We now apply Lemma 5.2 to conclude that $\mathcal{Z}$ is bounded below on all of $L^2(I)$.

The same argument holds if we reverse the roles of $\mathcal{Z}$ and $\mathcal{F}^{-1}\chi_L\mathcal{F}$ since the latter operator is injective whenever K is non-empty. Indeed, if K is non-empty then the operator $\chi_I\mathcal{F}^{-1}\chi_L\mathcal{F}$ is injective on $L^2(I)$. Assume that $K \neq \emptyset$ and let $g \in L^2(I)$ be such that $g \neq 0$. Neither $\chi_L\mathcal{F}g$ nor $\mathcal{F}^{-1}\chi_L\mathcal{F}g$ can be equal to zero almost everywhere as functions in $L^2(\mathbb{R})$. To conclude, we use the Plancherel-Parseval formula. For assume that $\chi_I\mathcal{F}^{-1}\chi_L\mathcal{F}g = 0$. Since $g = \chi_I\mathcal{F}^{-1}\chi_{L^C}\mathcal{F}g + \chi_I\mathcal{F}^{-1}\chi_L\mathcal{F}g$, this implies $\chi_I\mathcal{F}^{-1}\chi_{L^C}\mathcal{F}g = g$. And so

$$\begin{aligned}
\|g\|^2_{L^2(I)} &= \|\mathcal{F}^{-1}\chi_L\mathcal{F}g\|^2_{L^2(\mathbb{R})} + \|\mathcal{F}^{-1}\chi_{L^C}\mathcal{F}g\|^2_{L^2(\mathbb{R})} \\
&\geq \|\mathcal{F}^{-1}\chi_L\mathcal{F}g\|^2_{L^2(\mathbb{R})} + \|\chi_I\mathcal{F}^{-1}\chi_{L^C}\mathcal{F}g\|^2_{L^2(I)} \\
&= \|\mathcal{F}^{-1}\chi_L\mathcal{F}g\|^2_{L^2(\mathbb{R})} + \|g\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

But from what is already established $\|\mathcal{F}^{-1}\chi_L\mathcal{F}g\|_{L^2(\mathbb{R})} > 0$, which leads to a contradiction. This concludes the proof of the theorem. □

6 Modified zeta functions and prime numbers

In this chapter we study the consequences of the results obtained in the previous chapter under the additional hypothesis of arithmetic structure on the subset $K \subset \mathbb{N}$. It turns out that the question of the lower boundedness of the operator $\mathcal{Z}_{K,I}$ becomes easier to determine. However, in this case more can be said about the function ψ_K appearing in Korevaar's theorem. We also make a remark in the context of Beurling's generalised prime numbers.

6.1 Preliminaries

We say that K has arithmetic structure if for some subset Q of the prime numbers $\mathbb{P}$ we have

$$K = \{p_1^{\nu_1} \cdots p_n^{\nu_n} : p_1, \ldots, p_n \in Q\}.$$

In other words, K is the multiplicative semi-group generated by Q. We simply say that Q generates K.

Euler products

Suppose that $Q \subset \mathbb{P}$ generates K. This means that we may write

$$\zeta_K(s) = \prod_{q \in Q} \left(\frac{1}{1 - q^{-s}} \right),$$

i.e. ζ_K admits an Euler product. Let J be the integers generated by the prime numbers not in Q. Then it follows that

$$\zeta_K(s) = \frac{\zeta(s)}{\zeta_J(s)}.$$

Moreover, it follows by a simple computation that we have

$$\lim_{\sigma\to 1^+} \zeta_J(\sigma) < \infty \iff \sum_{p\in\mathbb{P}\setminus Q} \frac{1}{p} < \infty.$$

Asymptotic density

A fundamental fact is that every set K with arithmetic structure has an asymptotic density.

Lemma 6.1. *Let $Q \subset \mathbb{P}$ generate the integers $K \subset \mathbb{N}$, and J be the integers generated by the primes not in Q. Then*

$$\lim_{x\to\infty} \frac{\pi_K(x)}{x} = \lim_{\sigma\to 1^+} \frac{1}{\zeta_J(\sigma)}.$$

Proof. This lemma seems to be folklore, indeed for finite $\mathbb{P}\setminus Q$ it is readily known that it holds. See for instance [60, theorem 3.1]. An immediate consequence is that for infinite $\mathbb{P}\setminus Q$, then

$$\limsup_{x\to\infty} \frac{\pi_K(x)}{x} \leq \lim_{\sigma\to 1^+} \frac{1}{\zeta_J(\sigma)}.$$

In particular, if $\zeta_J(1)$ diverges, then $\pi_K(x)/x$ tends to zero. However, the remaining part of the lemma seems to be more difficult, and no analytic proof, or indication thereof, seems to be readily available in the literature. Therefore we show how one follows from the tauberian theorem of Korevaar given as Lemma 5.6.

Assume that $\zeta_J(1) < \infty$ and recall that $\zeta(s) = (s-1)^{-1} + \psi(s)$ for some entire function ψ. We need to calculate the distributional Fourier transform of

$$\begin{aligned} \frac{\zeta_K(s)}{s} - \frac{1}{\zeta_J(1)}\frac{1}{s-1} &= \frac{\zeta(s)}{s\zeta_J(s)} - \frac{1}{\zeta_J(1)}\frac{1}{s-1} \\ &= \frac{1}{s-1}\left(\frac{1}{s\zeta_J(s)} - \frac{1}{\zeta_J(1)}\right) + \frac{\psi(s)}{s\zeta_J(s)}, \end{aligned}$$

on $\sigma = 1$. Since $\zeta_J(1) < \infty$ it is not hard to use the Euler product formula to see that $\zeta_J(1+it)$ is bounded above and below in absolute value for all $t \in \mathbb{R}$. This means that the last term extends to a pseudo-function on $\sigma = 1$. Hence, the left hand side extends to a pseudo-function on $\sigma = 1$ if and only if the same holds true for the first term on the right hand side. It is readily seen that this function extends to a pseudo-function on $\sigma = 1$ if and only if the same is true for

$$\frac{1}{s-1}\left(\frac{\zeta_J(s)}{s} - \zeta_J(1)\right). \tag{6.1}$$

We calculate its distributional Fourier transform. Let ϕ be a test function. Since we may write

$$\frac{\zeta_J(s)}{s} = \frac{1}{s}\int_1^\infty x^{-s}\mathrm{d}\pi_J(x) = \int_0^\infty \frac{\pi_J(\mathrm{e}^u)}{\mathrm{e}^u}\mathrm{e}^{-(\sigma-1)-\mathrm{i}tu}\mathrm{d}u$$

it follows that

$$\lim_{\delta\to 0}\int_{\mathbb{R}} \hat{\phi}(t)\frac{1}{\delta+\mathrm{i}t}\left(\frac{\zeta_J(1+\delta+\mathrm{i}t)}{1+\delta+\mathrm{i}t} - \zeta_J(1)\right)\mathrm{d}t$$
$$= \lim_{\delta\to 0}\int_{\mathbb{R}} \frac{1}{\delta+\mathrm{i}t}\hat{\phi}(t)\int_0^\infty g(u)(\mathrm{e}^{-\delta u-\mathrm{i}ut}-1)\mathrm{d}u\mathrm{d}t,$$

where $g(u) = \pi_J(\mathrm{e}^u)e^{-u}$. Using the smoothness of ϕ, we change the order of integration,

$$\lim_{\delta\to 0}\int_0^\infty g(u)\int_{\mathbb{R}}\hat{\phi}(t)\frac{\mathrm{e}^{-\delta u-\mathrm{i}ut}-1}{\delta+\mathrm{i}t}\mathrm{d}t\mathrm{d}u$$
$$= \int_0^\infty g(u)\lim_{\delta\to 0}\int_{\mathbb{R}}\hat{\phi}(t)\frac{\mathrm{e}^{-\delta u-\mathrm{i}ut}-1}{\delta+\mathrm{i}t}\mathrm{d}t\mathrm{d}u$$
$$= \int_0^\infty g(u)F(u)\mathrm{d}u,$$

where $F'(u) = -\sqrt{2\pi}\phi(-u)$ and $F(0) = 0$. This means that

$$F(u) = -\sqrt{2\pi}\int_{-u}^0 \phi(x)\mathrm{d}x \quad \text{for } u \geq 0.$$

So,

$$\lim_{\delta\to 0}\int_{\mathbb{R}}\hat{\phi}(t)\frac{1}{\delta+\mathrm{i}t}\left(\frac{\zeta_J(1+\delta+\mathrm{i}t)}{1+\delta+\mathrm{i}t}-\zeta_J(1)\right)\mathrm{d}t$$
$$=-\sqrt{2\pi}\int_0^\infty g(u)\int_{-u}^0\phi(x)\mathrm{d}x\mathrm{d}u$$
$$=-\sqrt{2\pi}\int_{\mathbb{R}}\phi(x)\chi_{(-\infty,0)}(x)\int_{-x}^\infty g(u)\mathrm{d}u\mathrm{d}x.$$

Since $g(u)$ is integrable, this implies that

$$\chi_{(0,\infty)}(x)\int_x^\infty g(u)\mathrm{d}u$$

decays as $|x|\to\infty$ and so the Fourier transform of (6.1) on $\sigma=1$ is by definition a pseudo-function. □

6.2 Lower boundedness of $\mathcal{Z}_{K,I}$ and arithmetic structure

Let $K\subset\mathbb{N}$ have arithmetic structure. This means that Theorem 5.10 reduces to the following.

Theorem 6.2. *Let $Q\subset\mathbb{P}$ generate the integers K, and J be the integers generated by the primes not in Q. Then*

$$\mathcal{Z}_{K,I}=\zeta_J^{-1}(1)\mathrm{Id}+\Psi_{K,I},$$

for a compact operator $\Psi_{K,I}$. Moreover, the operator $\mathcal{Z}_{K,I}$ is bounded below on $L^2(I)$ if and only if

$$\sum_{p\in\mathbb{P}\setminus Q}\frac{1}{p}<\infty.$$

Proof. By Lemma 6.1 the limit

$$\lim_{x\to\infty}\frac{\pi_K(x)}{x}=A$$

always holds with $A = \zeta_J^{-1}(\sigma)$. With this, Theorem 5.7 implies the formula for $\mathcal{Z}_{K,I}$.

Finally, Theorem 5.10 says that $\mathcal{Z}_{K,I}$ is bounded below if and only if $A > 0$. By considering the Euler product of $\zeta_J(s)$ it follows that $\zeta_J(1) < +\infty$ is exactly the condition of the theorem. □

6.3 Boundary behaviour and arithmetic structure

In light of Lemma 6.1, we know that when the set K has arithmetic structure the limit $\lim \pi_K(x)/x$ always exists. This means that ψ_K always gives a pseudo-function. In fact, we are able to say more.

The proof of the following theorem is given in section 6.5.

Theorem 6.3. *Let $Q \subset \mathbb{P}$ generate the integers K, and J be the integers generated by the primes not in Q, and assume that*

$$\sum_{\mathbb{P}\setminus Q} \frac{1}{p} < \infty.$$

Then the pseudo-function defined by

$$\psi_K(s) = \frac{1}{s}\zeta_K(s) - \frac{\zeta_J^{-1}(1)}{s-1}$$

is locally in L^1 on the abscissa $\sigma = 1$ if and only if

$$\sum_{p\in\mathbb{P}\setminus Q} \frac{\log\log p}{p} < \infty.$$

For $q > 1$, it is locally in $L^q(I)$ on the abscissa $\sigma = 1$ if

$$\sum_{p\in\mathbb{P}\setminus Q} \frac{\log^{1/q'} p}{p} < \infty,$$

where $q' > 1$ is the real number satisfying $q^{-1} + q'^{-1} = 1$. Conversely, if

$$\sum_{p \in \mathbb{P} \setminus Q} \frac{\log^{1/q'} p}{p} = \infty,$$

then ψ is not locally in $L^r(I)$ for $r > q$.

6.4 Some remarks on the Prime number theorem for K

Let $Q \subset \mathbb{P}$ generate $K \subset \mathbb{N}$. In this section we consider such K in the more general context of Beurling prime numbers. As we mentioned in the introduction, the setup is to consider any increasing sequence $R = (r_i)_{i \in \mathbb{N}}$ of real numbers as being a substitute for the prime numbers. The multiplicative semi-group generated by these Beurling primes are called Beurling integers. In our case Q corresponds to the Beurling primes and K to the Beurling integers. We make a definition.

Definition 6.4. Let $Q \subset \mathbb{P}$ generate $K \subset \mathbb{N}$. We say that the prime number theorem for the set Q (or equivalently K) holds if

$$\pi_Q(x) \sim \frac{x}{\log x}.$$

We get the following characterisation of when such a prime number theorem holds.

Lemma 6.5. *Let $Q \subset \mathbb{P}$ generate $K \subset \mathbb{N}$ and let J denote the integers generated by the primes not in Q. Then the prime number theorem holds for the set K if and only if*

$$\sum_{p \in \mathbb{P} \setminus Q \cap (\delta x, x)} \frac{\log p}{p} = o\left(1\right), \quad \text{for all } \delta \in (0,1). \tag{6.2}$$

Proof. Let $P = \mathbb{P} \setminus Q$. It is clear that the prime number theorem holds for K if and only if $\pi_P(x) = o(x/\log x)$. Moreover, it is readily seen that

$$\frac{\log x}{x}(\pi_P(x) - \pi_P(\delta x)) \leq \sum_{p \in (\delta x, x)} \frac{\log p}{p} \leq \frac{\log \delta x}{\delta x}(\pi_P(x) - \pi_P(\delta x)).$$

So we have to show that $\pi_P(x) = o(x/\log x)$ is equivalent to the statement that for all $\delta > 0$ it holds that $\pi_P(x) - \pi_P(\delta x) = o(x/\log x)$. One direction is immediate. For the other, assume that $\pi_P(x) - \pi_P(\delta x) = o(x/\log x)$. Rewrite this assumption in the form

$$\pi_P(x)\frac{\log x}{x} = \delta\left(\pi_P(\delta x)\frac{\log \delta x}{\delta x}\right)\frac{\log x}{\log \delta x} + o(1).$$

Hence, for all $\delta > 0$, we have

$$\limsup_{x\to\infty} \pi_P(x)\frac{\log x}{x} < \delta.$$

□

Since the condition of this lemma resembles the condition of Theorem 6.2, it is natural to check if the validity of the prime number theorem for a set K with arithmetic structure is related to the lower bound of the operator $\mathcal{Z}_{K,I}$. We get the following theorem.

Theorem 6.6. *Let $Q \subset \mathbb{P}$ generate $K \subset \mathbb{N}$. Then the prime number theorem for Q neither implies nor is implied by the lower boundedness of the operator $\mathcal{Z}_{K,I}$.*

Proof. Recall that by Theorem 6.2, the lower boundedness of the operator $\mathcal{Z}_{K,I}$ is equivalent to the condition

$$\sum_{p\in\mathbb{P}\setminus Q} \frac{1}{p} < \infty, \tag{6.3}$$

and that by Lemma 6.1 this is equivalent to the condition

$$\liminf_{x\to\infty} \frac{\pi_K(x)}{x} > 0. \tag{6.4}$$

First we seek a set of primes Q for which Panejah's condition holds but the prime number theorem does not. This part of the theorem follows by comparing the condition (6.4) to the condition (6.2) of Lemma 6.5 in combination with a variant of Merten's formula (see e.g. [60][p. 50]):

$$\sum_{p\le x}\frac{\log p}{p} = \log x + \mathcal{O}(1)\,. \tag{6.5}$$

One the one hand, (6.5) implies that for $\delta > 0$ small enough, then

$$\liminf_{x\to\infty}\sum_{\delta x\le p\le x}\frac{\log p}{p} > 0.$$

We choose a sequence $(x_n)_{n\in\mathbb{N}}$ which realises this condition and for which the intervals $(\delta x_n, x_n)$ do not overlap. On the other hand, (6.5) implies that

$$\sum_{\delta x\le p\le x}\frac{1}{p} \lesssim \frac{1}{\log \delta x}.$$

Choose a sub-sequence of (x_{n_k}) for which $\sum_k (\log x_{n_k})^{-1} < \infty$. Let $P = \mathbb{P}\cap(\cup(\delta x_{n_k}, x_{n_k}))$, and set $Q = \mathbb{P}\backslash P$. This set does the job.

Next, we seek a set Q for which the prime number theorem holds, but Panejah's condition fails. Consider the consecutive intervals $I_k = (2^k, 2^{k+1})$. In each interval choose essentially the first $2^k/(k\log k)$ prime numbers. This is seen to be exactly possible for large k using the fact that the n'th prime $p_n \sim n\log n$. Denote the set of primes chosen in this way from the interval I_k by P_k. Set $P = \cup P_k$ and let $Q = \mathbb{P}\backslash P$. It now follows that the condition (6.3) does not hold, since

$$\sum_{p\in P}\frac{1}{p} \gtrsim \sum_{k\in\mathbb{N}}\frac{1}{k\log k} = \infty.$$

To see that the prime number theorem for Q holds, we let $\delta\in(0,1)$ and readily check that for $x > x_\delta$ we have

$$\sum_{p\in P\cap(\delta x, x)}\frac{\log p}{p} \lesssim \frac{\log x}{x}\frac{x}{\log x\log\log x} = \frac{1}{\log\log x}.$$

Hence the condition (6.2) of Lemma 6.5 holds.

□

6.5 Proof of Theorem 6.3

We give a series of lemmas before before turning to the proof of Theorem 6.3. The three first are standard exercises in Fourier analysis. Although we give an elementary proof, the fourth, Lemma 6.10, follows from a more general result by P. Malliavin [55]. Throughout this section we fix $T > 0$.

Lemma 6.7. *Let $\phi \in L^1(0,T)$. Then*

$$\int_0^u \hat{\phi}(t)\mathrm{d}t = \frac{1}{\sqrt{2\pi}} \int_0^T \phi(\xi)\frac{\mathrm{e}^{-\mathrm{i}u\xi} - 1}{-\mathrm{i}\xi}\mathrm{d}\xi.$$

Proof. This follows by applying Fubini's theorem to the left hand side. □

Lemma 6.8. *Let $\phi \in L^1(0,T)$ and $\zeta_J(1) < \infty$. Then*

$$\int_0^T \frac{\phi(t)}{t}\left(\zeta_J(1+\mathrm{i}t) - \zeta_J(1)\right)\mathrm{d}t = -\sqrt{2\pi}\sum_{n\in J}\frac{1}{n}\int_0^{\log n}\hat{\phi}(t)\mathrm{d}t.$$

Proof. We compute:

$$\int_0^T \frac{\phi(t)}{t}\left(\zeta_J(1+\mathrm{i}t) - \zeta_J(1)\right)\mathrm{d}t = \int_0^T \frac{\phi(t)}{t}\int_1^\infty \frac{x^{-\mathrm{i}t} - 1}{x}\mathrm{d}\pi_J(x)\mathrm{d}t$$

$$= \int_1^\infty \int_0^T \phi(t)\frac{x^{-\mathrm{i}t} - 1}{t}\mathrm{d}t\frac{\mathrm{d}\pi_J(x)}{x}$$

By Lemma 6.7 this is the same as

$$-\sqrt{2\pi}\int_1^\infty \int_0^{\log x}\hat{\phi}(t)\mathrm{d}t\frac{\mathrm{d}\pi_J(x)}{x} = -\sqrt{2\pi}\sum_{n\in J}\frac{1}{n}\int_0^{\log n}\hat{\phi}(t)\mathrm{d}t.$$

□

Lemma 6.9. *For $\phi \in L^\infty(0,T)$ we have*

$$\left|\int_0^x \hat{\phi}(t)\mathrm{d}t\right| \lesssim \|\phi\|_\infty \log x,$$

and for $1 < q < \infty$ *and* $\phi \in L^q(0,T)$ *we have*

$$\left|\int_0^x \hat{\phi}(t)\mathrm{d}t\right| \lesssim \|\phi\|_q x^{\frac{1}{q}}.$$

Moreover, this is best possible in the sense that there exist $\phi \in L^\infty(0,T)$ *for which*

$$\sqrt{2\pi}\int_0^x \hat{\phi}(t)\mathrm{d}t = \log x + \mathcal{O}\left(1\right),$$

and for all $r > q$ *there exists* $\phi \in L^q(0,T)$ *for which*

$$\sqrt{2\pi}\int_0^x \hat{\phi}(t)\mathrm{d}t = x^{\frac{1}{r}} + \mathcal{O}\left(1\right).$$

Proof. We only show the part of the statement dealing with $\phi \in L^\infty$. The rest is shown in a similar way. The upper-bound part runs as follows. By Lemma 6.7 and a change of variables, we get

$$\begin{aligned}\sqrt{2\pi}\left|\int_0^x \hat{\phi}(t)\mathrm{d}t\right| &= \left|\int_0^T \phi(u)\frac{\mathrm{e}^{-\mathrm{i}xu}-1}{u}\mathrm{d}u\right| \\ &= \left|\int_0^{Tx} \phi\left(\frac{-u}{x}\right)\frac{\mathrm{e}^{\mathrm{i}u}-1}{u}\mathrm{d}u\right| \le \|\phi\|_\infty \int_0^{Tx}\frac{|\mathrm{e}^{-\mathrm{i}u}-1|}{u}\mathrm{d}u.\end{aligned}$$

The integral in the last term is $\mathcal{O}\left(\log x\right)$. We claim that $\phi(x) = \chi_{(0,T)}(x)$ gives the last part of the statement. For with this choice

$$\sqrt{2\pi}\int_0^x \hat{\phi}(t)\mathrm{d}t = \int_0^T \frac{\mathrm{e}^{-\mathrm{i}tx}-1}{\mathrm{i}t}\mathrm{d}t = \int_0^{Tx}\frac{\mathrm{e}^{-\mathrm{i}t}-1}{\mathrm{i}t}\mathrm{d}t.$$

The result follows, since it is clear that

$$\int_0^{Tx}\frac{\mathrm{e}^{-\mathrm{i}t}-1}{t}\mathrm{d}t < \infty \quad \text{and} \quad \int_0^{Tx}\frac{1}{t}\mathrm{d}t = \log x + \log T.$$

□

Lemma 6.10. *Let $f : \mathbb{N} \to \mathbb{R}_+$ satisfy $f(nm) \leq f(n)+f(m)$ and $f(n) \geq 1$ for n big enough. If the primes P generate the integers J then*

$$\sum_{n\in J} \frac{f(n)}{n} < \infty \quad \Longleftrightarrow \quad \sum_{p\in P} \frac{f(p)}{p} < \infty.$$

Proof. One way to prove this is for $\sigma > 1$ to show the inequality

$$\sum_{n\in J} f(n)n^{-\sigma} \lesssim \left(\sum_{p\in P} f(p)p^{-\sigma} \right) \mathrm{e}^{\sum_p p^{-\sigma}},$$

and then conclude by the monotone convergence theorem. To prove this we study the linear map $D_f : \sum a_n n^{-\sigma} \to \sum a_n f(n) n^{-\sigma}$. It is not hard to show that the abscissa of absolute convergence is invariant under D_f. Moreover, for Dirichlet series F, G with positive coefficients it holds that $D_f(FG)(\sigma) \leq D_f(F)G(\sigma) + FD_f(G)(\sigma)$. We use this on the identity

$$\sum_{n\in J} n^{-\sigma} = \mathrm{e}^{\sum_{p\in P} p^{-\sigma}} \phi(\sigma),$$

where the function

$$\phi(\sigma) = \mathrm{e}^{-\sum_{p\in P} \log(1-p^{-\sigma}) - \sum_{p\in P} p^{-\sigma}}$$

is given by a Dirichlet series that converges absolutely for $\sigma > 1/2$. Here we used the Euler product formula for the function ζ_J. The inequality is now seen to hold since

$$D_f(\mathrm{e}^{\sum_{p\in P} p^{-\sigma}}) \leq \left(\sum_{p\in P} f(p)p^{-\sigma} \right) \mathrm{e}^{1+\sum_{p\in P} p^{-\sigma}}.$$

□

Proof of Theorem 6.3. The fact that we have the formula

$$\mathcal{Z}_{K,I} = \zeta_J^{-1}(1)\mathrm{Id} + \Psi_{K,I},$$

for a compact operator $\Psi_{K,I}$ and the statement that ψ_K extends to a pseudo-function on $\sigma = 1$ both follow immediately from Theorem 5.7.

Assume that $\zeta_J^{-1}(1) > 0$. By the factorisation $\zeta(s) = \zeta_K(s)\zeta_J(s)$ and the formula $\zeta(s) = (s-1)^{-1} + \psi(s)$ for the Riemann zeta function we have the identity

$$\begin{aligned}\psi_K(s) &= \frac{\zeta_K(s)}{s} - \frac{1}{\zeta_J(1)}\frac{1}{s-1} \\ &= \frac{1}{s-1}\left(\frac{1}{s\zeta_J(s)} - \frac{1}{\zeta_J(1)}\right) + \frac{\psi(s)}{s}.\end{aligned}$$

As mentioned in the proof of Lemma 6.1, it follows from the Euler product formula that $\zeta_J(1+\mathrm{i}t)$ is a continuous function bounded away from zero. Therefore

$$\frac{1}{t}\left(\frac{1}{\zeta_J(1+\mathrm{i}t)} - \frac{1}{\zeta_J(1)}\right) \in L^p_{\mathrm{loc}}(\mathbb{R}) \iff \frac{\zeta_J(1+\mathrm{i}t) - \zeta_J(1)}{t} \in L^p_{\mathrm{loc}}(\mathbb{R}).$$

We prove the L^1 condition. It follows from lemmas 6.8 and 6.9 that

$$\begin{aligned}\left\|\frac{\zeta_J(1+\mathrm{i}t) - \zeta_J(1)}{t}\right\|_{L^1(I)} &= \sqrt{2\pi}\sup_{\phi\in L^\infty(I)}\left|\sum_{n\in J}\frac{1}{n}\int_0^{\log n}\hat{\phi}(t)\mathrm{d}t\right| \\ &\le \sqrt{2\pi}\sum_{n\in J}\frac{1}{n}\log\log n < \infty.\end{aligned}$$

We apply the Lemma 6.10 to the function $f(n) = \log\log n$ to find that

$$\sum_{n\in J}\frac{\log\log n}{n} < \infty \iff \sum_{p\in\mathbb{P}\setminus Q}\frac{\log\log p}{p} < \infty.$$

This proves the sufficiency. As for the necessity, assume that we have $\sum_{p\in\mathbb{P}\setminus Q} p^{-1}\log\log p = \infty$. By the dual expression for the $L^1(I)$ norm and the optimality of Lemma 6.9, it is clear that

$$\frac{\zeta_J(1+\mathrm{i}t) - \zeta_J(1)}{t} \notin L^1(I).$$

Let $q > 1$ and $q' > 1$ be as given by the hypothesis. The necessity is exactly the same as when $p = 1$ and the sufficiency follows by a slight modification. ☐

Bibliography

[1] Sergei A. Avdonin. On Riesz bases of exponentials in L^2. *Vestnik Leningrad Univ. Math.*, 7:203–211, 1979.

[2] Paul T. Bateman and Harold G. Diamond. Asymptotic distribution of Beurling's generalized prime numbers. *Math. Assoc. Amer. (distributed by Prentice-Hall Englewoods Cliffs, N.J.)*, pages 152–210, 1969.

[3] Frédéric Bayart. Hardy spaces of Dirichlet series and their composition operators. *Monatsh. Math.*, 136(3):203–236, 2002.

[4] Arne Beurling. Analyse de la loi asymptotique de la distribution des nombres premiers généralisés I. *Acta Math.*, 68:255–291, 1937.

[5] Christopher J. Bishop. Interpolating sequences for the Dirichlet space and its multipliers. Preprint, 1994.

[6] Ralph P. Boas Jr. A general moment problem. *Amer. J. Math*, 63:361–370, 1941.

[7] Ralph P. Boas Jr. *Entire Functions.* Academic Press, 1954.

[8] H. F. Bohnenblust and Einer Hille. On the absolute convergence of Dirichlet series. *Ann. of Math.*, 32:600–622, 1931.

[9] Harald Bohr. Über die Bedeutung der Potenzreihen unendlich vieler Variabeln in der Theorie der Dirichletschen Reihen $\sum a_n/n^s$. *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl.*, pages 441–488, 1913.

[10] Harald Bohr. Über die gleichmässige Konvergenz Dirichletscher Reihen. *J. Reine Angew. Math.*, 143:203–211, 1913.

[11] Harald Bohr. *Collected Mathematical Works*, volume 1. Dansk Matematisk Forening, København, 1952.

[12] Jean Bourgain. *Harmonic Analysis and Combinatorics: How Much May They Contribute to Each Other?*, pages 13–32. Amer. Math. Soc., Providence, 2000.

[13] Lennart Carleson. An interpolation problem for bounded analytic functions. *Amer. J. Math*, 80:921–930, 1958.

[14] Lennart Carleson. Interpolations by bounded analytic functions and the Corona problem. *Ann. of Math.*, 76(3):547–559, 1962.

[15] Lennart Carleson. On convergence and growth of partial sums of Fourier series. *Acta Math.*, 116:135–157, 1966.

[16] Fritz Carlson. Contributions à la théorie des séries de Dirichlet, Note I. *Ark. Math.*, 16(18):1–19, 1922.

[17] Ole Christensen. *An Introduction to Frames and Riesz Bases*. Applied and Numerical Harmonic Analysis. Birkhäuser Boston, Inc., 2003.

[18] William S. Cohn. Interpolation and multipliers on Besov and Sobolev spaces. *Complex Variables Theory Appl.*, 22:35–45, 1993.

[19] Brian J. Cole and T. W. Gamelin. Representing measures and Hardy spaces for the infinite polydisk algebra. *Proc. London Math. Soc.*, 53(3):112–142, 1986.

[20] I. P. Cornfeld, S. V. Fomin, and Ya. G. Sinai. *Ergodic Theory*, volume 245 of *Grundlehren Math. Wiss.* Springer-Verlag, New York, 1982.

[21] Charles Jean de la Vallée Poussin. Reserches analytiques sur la théorie des nombres premiers. *Brux. S. sc.*, 21(B):183–256, 281–362, 363–297, 1896.

[22] Peter L. Duren. *Theory of H^p Spaces.* Academic press, 1970.

[23] Peter L. Duren and Alexander Schuster. *Bergman Spaces*. Number 100 in Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2004.

[24] Samuel E. Ebenstein. Some H^p spaces which are uncomplemented in L^p. *Pacific Journal of Mathematics*, 43(2):327–339, 1972.

[25] Harold M. Edwards. *Riemann's Zeta Function*, volume 58 of *Pure and Applied Mathematics*. Academic press, 1974.

[26] Leonard Euler. Variae observationes circa series infinitas. *Commentarii academiae scientiarum Petropolitanae*, 9:160–188, 1744. Catalogue number E072. English translation available at http://dewey.uab.es/lbibiloni/Euler/EulerObservat.pdf.

[27] Sarah H. Ferguson and Michael T. Lacey. A characterisation of product BMO by commutators. *Acta. Math.*, 189(2):143–160, 2002.

[28] Kristin M. Flornes, Yurii Lyubarskii, and Kristian Seip. A direct interpolation method for irregular sampling. *Appl. Comput. Harmon. Anal.*, 7(3):305–314, 1999.

[29] Gerald B. Folland. *Modern Techniques and Their Applications*. Pure and applied mathematics. John Wiley and Sons, second edition, 1999.

[30] Wolfgang H. J. Fuchs. On the eigenvalues of an integral equation arising in the theory of band-limited signals. *J. Math. Anal. Appl.*, 9:317–330, 1964.

[31] John B. Garnett. *Bounded Analytic Functions*. Academic Press, 1981.

[32] Jacques Hadamard. Sur la distribution des zéros de la fonction $\zeta(s)$ et ses conséquences arithmétiques. *S. M. F. Bull.*, 24:199–220, 1896.

[33] G. H. Hardy and M. Riesz. *The general theory of Dirichlet series*, volume 18 of *Cambridge Tracts in Mathematics and Mathematical Physics*. Cambridge University Press, 1915.

[34] G. H. Hardy and E. M. Wright. *An Introduction to the Theory of Numbers*. Oxford University Press, fourth edition, 1960.

[35] Håkan Hedenmalm, Peter Lindqvist, and Kristian Seip. A Hilbert space of Dirichlet series and systems of dilated functions in $L^2(0,1)$. *Duke Math. J.*, 86:1–37, 1997.

[36] Håkan Hedenmalm and Eero Saksman. Carleson's convergence theorem for Dirichlet series. *Pacific J. Math.*, 208(1):85–109, 2003.

[37] Henry Helson. Foundations of the theory of Dirichlet series. *Acta Math.*, 118:61–77, 1967.

[38] Henry Helson. *Dirichlet Series*. Regent press, 2005.

[39] Henry Helson. Lifting Hankel forms. preprint, 2005.

[40] Henry Helson. Hankel forms and sums of random variables. *Studia Math.*, 176(1):85–92, 2006.

[41] Brian Hollenbeck and Igor E. Verbitsky. Best constant for the Riesz projection. *J. Funct. Anal.*, 175(2):370–392, 2000.

[42] Martin N. Huxley. Exponential sums and lattice points III. *Proc. London Math. Soc.*, 87(3):591–609, 2003.

[43] Shikao Ikehara. An extension of Landau's theorem in the analytical theory of numbers. *J. of Math. Phys.*, 10:1–12, 1931.

[44] Aleksandar Ivic. *The Riemann Zeta-Function. Theory and Applications*. Dover Publications Inc., 2003.

[45] Mikhail I. Kadec. The exact value of the Paley-Wiener constant. *Sov. Math. Dokl.*, 5:559–561, 1964.

[46] Jean-Pierre Kahane. Sur les séries de Dirichlet $\sum_1^\infty \pm n^{-s}$. *C. R. Acad. Sc. Paris*, 276:739–742, 1973.

[47] Jean-Pierre Kahane. A Fourier formula for prime numbers. *Harmonic analysis and number theory. Papers in honour of Carl S. Herz. Proceedings of the conference, April 15–19, 1996, Montréal, Canada. Providence, RI: American Mathematical Society. CMS Conf. Proc*, 21:89–102, 1997.

[48] Jean-Pierre Kahane. Le rôle des algbèbres A de Wiener, A^∞ de Beurling et H^1 de Sobolev dans la théorie des nombres premiers généralisés de Beurling. *Ann. Inst. Fourier (Grenoble)*, 48(3):611–648, 1998.

[49] Tosio Kato. *Perturbation Theory for Linear Operators*, volume 132 of *Die Grundlehren der mathematischen Wissenschaften.* Springer-Verlag, New York, Inc., 1966.

[50] Sergei V. Konyagin and Hervé Queffélec. The translation $\frac{1}{2}$ in the theory of Dirichlet series. *Real Anal. Exchange*, 27(1):155–175, 2001/02.

[51] Paul Koosis. *Introduction to H_p Spaces.* Cambridge University Press, second edition edition, 1998.

[52] Jacob Korevaar. Distributional Wiener-Ikehara theorem and twin primes. *Indag. Math. (N.S.)*, 16(1):37–49, 2005.

[53] Nobushige Kurokawa. On certain Euler products. *Acta Arith.*, 48(1):49–52, 1987.

[54] Boris Yakovlevich Levin. *Lectures on Entire Functions*, volume 150 of *Translations of Mathematical Monographs.* Am. Math. Soc., Providence, Rhode Island, 1996.

[55] Paul Malliavin. Sur le reste de la loi asymptotique de répartition des nombres premiers généralisés de Beurling. *Acta Math.*, 106:281–298, 1961.

[56] Donald E. Marshall and Carl Sundberg. Interpolating sequences for the multipliers of the Dirichlet space. Preprint at http://www.math.washington.edu/˜marshall/preprints/preprints.html, 1993.

[57] Rubén P. Martínez-Avendaño and Peter Rosenthal. *An Introduction to Operators on the Hilbert-Hardy Space*, volume 237 of *Graduate Texts in Mathematics*. Springer-Verlag, 2007.

[58] John E. McCarthy. Hilbert spaces of Dirichlet series and their multipliers. *Trans. Amer. Math. Soc.*, 356(3):881–893, 2004.

[59] Hugh L. Montgomery. *Ten Lectures on the Interface Between Analytic Number Theory and Harmonic Analysis*, volume 84 of *CBMS Regional Conference Series in Mathematics*. AMS, 1994.

[60] Hugh L. Montgomery and R. C. Vaughan. *Multiplicative Number Theory I. Classical Theory*, volume 97 of *Cambridge studies in advanced mathematics*. Cambridge University Press, 2007.

[61] Ferenc Móricz. Exact estimates for integrals involving Dirichlet series with nonnegative coefficients. *Proceedings of the American Mathematical Society*, 127(8):2417–2422, 1999.

[62] Zeev Nehari. On bounded bilinear forms. *Ann. of Math.*, 65(2):153–162, 1957.

[63] Jan-Fredrik Olsen. Modified zeta functions as kernels of integral operators. Preprint, 2009.

[64] Jan-Fredrik Olsen and Eero Saksman. Some local properties of functions in Hilbert spaces of Dirichlet series. Preprint, 2009.

[65] Jan-Fredrik Olsen and Kristian Seip. Local interpolation in Hilbert spaces of Dirichlet series. *Proc. Amer. Math. Soc.*, 136:203–212, 2008.

[66] Boris P. Panejah. Certain inequalities for functions of exponential type and a priori estimates for general differential operators. (Russian). *Uspehi Mat. Nauk*, 21(3 (129)):75–114, 1966. English translation in Russian Math. Surveys 21 (1966), 75-114.

[67] Hervé Queffélec. Propriétés presque sûres et quasi-sûres des séries de Dirichlet et des produits d'Euler. *Can. J. Math.*, XXXII(3):531–558, 1980.

[68] Hervé Queffélec. Composition operators in the Dirichlet series setting. In *Perspectives in operator theory*, volume 75, pages 261–287. Banach Center Publ., 2007.

[69] Hervé Queffélec and Claude Zuily. *Analyse pour l'Agrégation*. Dunod, 3me edition, 2007.

[70] Srinivasan Ramanujan. Some formulæ in the analytic theory of numbers. *Messenger Math.*, 45:81–84, 1916. Available at http://www.imsc.res.in/˜rao/ramanujan/collectedindex.html.

[71] Bernhard Riemann. *Gesammelte Werke*. Teubner, Leipzig, 1892. Reprinted by Dover Books, New York, 1953.

[72] Walter Rudin. *Fourier Analysis on Groups*. Interscience Publishers (a division of John Wiley and Sons), New York-London, 1962.

[73] Walter Rudin. *Functional Analysis*. McGraw-Hill, Inc., 1973.

[74] Eero Saksman and Kristian Seip. Integral means and boundary limits of Dirichlet series. Preprint at arXiv.org, 2007.

[75] Kristian Seip. Beurling type density theorems in the unit disk. *Invent. Math.*, 113:21–39, 1993.

[76] Kristian Seip. *Interpolation and Sampling in Spaces of Analytic Functions*, volume 33 of *University Lecture Series*. American Mathematical Society, Providence, R. I., 2004.

[77] Kristian Seip. Interpolation by Dirichlet series in H^∞. Preprint at http://www.math.ntnu.no/˜seip/recentpapers.html, 2008.

[78] Kristian Seip. Embeddings associated with the divisor function. Unpublished note, 2009.

[79] Harold S. Shapiro and Allen L. Shields. On some interpolation problems for analytic functions. *Amer. J. Math*, 83:513–532, 1961.

[80] David Slepian. Some comments on Fourier analysis, uncertainty and modeling. *SIAM review*, 25(3):379–393, 1983.

[81] Dragan Vukotic. The isoperimetric inequality and a theorem of Hardy and Littlewood. *Amer. Math. Monthly*, 110:532–536, 2003.

[82] B. M. Wilson. Proofs of some formulae enunciated by Ramanujan. *London M. S. Proc.*, 21(2):235–255, 1922.

[83] Robert M. Young. *An Introduction to Nonharmonic Fourier Series.* Academic Press, New York, Revised First Edition edition, 2001.

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