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Regularity results for the Dirac-Klein-Gordon equations

Thesis for the degree of Philosophiae Doctor

Trondheim, January 2009

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Faculty of Information Technology, Mathematics and Electrical Engineering
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To my mother,
Tigabe Mekonnen

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Achenef Tesfahun,
Trondheim, January 2009.

CONTENTS

This thesis consists of an introduction and the following three papers:

- **Paper** I: Low regularity local well-posedness for the 1+3 dimensional Dirac-Klein-Gordon system. A. Tesfahun. Electronic Journal of Differential Equations, Vol. 2007(2007), No. 162, pp. 126.

- **Paper** II: Low regularity well-posedness for the one-dimensional Dirac-Klein-Gordon system. S. Selberg and A. Tesfahun. Commun. Contemp. Math **10** (2008) No. 2, 181-194.

- **Paper** III: Global Well-posedness of the 1D Dirac-Klein-Gordon system in Sobolev spaces of negative index. A. Tesfahun. Submitted.

INTRODUCTION

The main goal of this thesis is to study low regularity solutions for nonlinear wave equations arising from relativistic quantum field theories. In particular, we study low regularity solutions for the the system of Dirac-Klein-Gordon equations in three space dimensions (3d) and one space dimension (1d).

The Dirac-Klein-Gordon system (DKG) arises in the so-called Yukawa interaction [3], and describes an interaction between a Dirac spinor ψ of mass $M \geq 0$ and a meson field ϕ of mass $m \geq 0$.

In n space dimensions, DKG reads

$$\begin{cases} (-i\gamma^\mu \partial_\mu + M)\,\psi = \phi\psi \\ \qquad (-\Box + m^2)\phi = \psi^\dagger \gamma^0 \psi, \end{cases} \tag{0.1}$$

where $\psi : \mathbb{R}^{1+n} \to \mathbb{C}^N$ is the Dirac spinor field regarded as a column vector in $\mathbb{C}^N$ and $\phi : \mathbb{R}^{1+n} \to \mathbb{R}$ represents a meson field. Points in Minkowski space-time $\mathbb{R}^{1+n}$ are denoted by (t, x), where $x = (x^1, \cdots, x^n)$; we also denote $t = x^0$ when convenient. For partial derivatives we write $\partial_\mu = \frac{\partial}{\partial x^\mu}$. Roman indices $j, k, \cdots$ range over $1, \cdots, n$, while Greek indices $\mu, \nu, \cdots$ over $0, 1, \cdots, n$, and repeated upper and lower indices are implicitly summed over these ranges. The wave operator $\Box = -\partial_t^2 + \Delta$, where Δ is the Laplace operator. The γ^μ's are $N \times N$ matrices which should satisfy (see [31])

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\mathrm{g}^{\mu\nu} I, \qquad (\gamma^0)^\dagger = \gamma^0, \quad (\gamma^j)^\dagger = -\gamma^j, \tag{0.2}$$

where $\mathrm{g}^{\mu\nu} = \mathrm{diag}(1, -1, \cdots, -1)$, and the superscript $\dagger$ denotes a conjugate transpose.

We can rewrite (0.1) in a slightly different form, multiplying the Dirac equation on the left by $\beta := \gamma^0$ (note that $\beta^2 = I$), and setting $\alpha^j := \gamma^0\gamma^j$ to get

$$\begin{cases} -i\,(\partial_t + \alpha \cdot \nabla)\,\psi = -M\beta\psi + \phi\beta\psi, \\ \qquad -\Box\phi = -m^2\phi + \psi^\dagger \beta \psi, \end{cases} \tag{0.3}$$

where $\alpha = (\alpha^1, \cdots, \alpha^n)$.

For the DKG system there are many conserved quantities which are not positive definite, such as the energy (see [17]). However, there is a known positive conserved quantity, namely the charge,

$$\|\psi(t,.)\|_{L^2} = \text{const.}$$

0.1. **Dirac-Klein-Gordon system in 3d.** In 3d the smallest possible dimension of the spin space, i.e., the smallest N for which the relations in (0.2) can take place, is $N = 4$, (see [31]). In other words, in 3d the matrices $\{\gamma^\mu\}_{\mu=0}^3$ should be 4×4 matrices, and hence the Dirac operator $(-i\gamma^\mu\partial_\mu + M)$ has to act on a 4-component column matrix $\psi \in \mathbb{C}^4$. The usual representation of $\{\gamma^\mu\}_{\mu=0}^3$ in 2×2 blocks is given by

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \qquad (j = 1, 2, 3), \tag{0.4}$$

where σ^j's are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

0.2. **Dirac-Klein-Gordon system 1d.** In 1d the smallest possible dimension of the spin space, i.e., the smallest N for which the relations in (0.2) can take place, is $N = 2$, (see [31]). In other words, in 1d $\{\gamma^\mu\}_{\mu=0}^1$ should be 2×2 matrices, and hence the Dirac operator $(-i\gamma^\mu \partial_\mu + M)$ has to act on a 2-component column matrix $\psi \in \mathbb{C}^2$. A representation for $\{\gamma^\mu\}_{\mu=0}^1$ can be either

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{0.5}$$

or

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{0.6}$$

both of which satisfy (0.2).

1. Historical background on Klein-Gordon and Dirac equations

Klein-Gordon and Dirac equations are two of the most important field equations arising in relativistic quantum mechanics. We briefly review the historical background of these equations in 3d (for details see, for example, [3]). In what follows the speed of light c and Planck's constant $\hbar$ are scaled to unity.

In quantum mechanics a particle is represented by a wave function $\phi = \phi(t, x)$ taking value in $\mathbb{C}$. A quantum mechanical description of a free particle results from applying the correspondence principle, which allows one to replace classical observables by quantum mechanical operators acting on the wave function ϕ.

Let m be the mass and $\mathbf{p} = \mathbf{p}(t, x)$ be the momentum of the particle ($\mathbf{p}$ is a vector field). In non-relativistic mechanics, the energy for a free particle

$$E = \frac{\mathbf{p}^2}{2m}$$

is quantized by the correspondence principle

$$E \to i\partial_t, \quad \mathbf{p} \to \frac{\nabla}{i} \tag{1.1}$$

to give the Schrödinger equation

$$i\partial_t \phi = -\frac{1}{2m}\Delta\phi.$$

The relativistic energy-momentum equation

$$E^2 = \mathbf{p}^2 + m^2, \tag{1.2}$$

is quantized by the substitution (1.1) to give the free Klein-Gordon equation

$$(-\Box + m^2)\phi = 0,$$

which is the relativistic analogue of the Schrödinger equation.

The four-current density $\{j_\mu\}_{\mu=0}^3$ associated with the Klein-Gordon equation for a particle is given by

$$j_\mu = \frac{i}{2m}(\overline{\phi}\partial_\mu \phi - \phi \partial_\mu \overline{\phi}),$$

and it is conserved,

$$\partial^\mu j_\mu = 0.$$

However, the density $\rho = -j_0$ is not positive definite, and hence cannot describe a probability density for a single particle. For this reason the Klein-Gordon equation was discarded initially as a single-particle equation until it was resurrected in quantum field theory, where it describes spin-0 particles.

To solve the negative probability density problem of the Klein-Gordon equation, Dirac tried to look for a relativistic equation which only contain first-order derivatives in both space and time. Dirac derived his operator in 1928 starting from the usual classical expression of the energy of a free relativistic particle (1.2). The operator should feature then a differential operator of the type

$$\mathcal{D} = i\gamma^\mu \partial_\mu, \tag{1.3}$$

where γ^μ's are matrices to be determined.

A starting point for a wave equation with only a first order time derivative is to write $E = \pm\sqrt{\mathbf{p}^2 + m^2}$. Application of the correspondence principle (1.1) leads to the wave equation

$$i\partial_t \psi = \pm\sqrt{-\Delta + m^2}\psi. \tag{1.4}$$

These two equations can be combined to give the Klein-Gordon equation:

$$\left(-\Box + m^2\right)\psi = \left(i\partial_t + \sqrt{-\Delta + m^2}\right)\left(i\partial_t - \sqrt{-\Delta + m^2}\right)\psi = 0. \tag{1.5}$$

Dirac achieved the linearization of the "square root operator" in the right hand side of (1.4) by factorizing, according to (1.5), the Klein-Gordon operator $(-\Box + m^2) = (\partial_\mu \partial^\mu + m^2)$ into

$$\partial_\mu \partial^\mu + m^2 = -\left(\mathcal{D} + m\right)\left(\mathcal{D} - m\right). \tag{1.6}$$

Inserting (1.3) into (1.6), and then comparing the left and right hand sides of (1.6) he concluded that the matrices $\{\gamma^\mu\}_{\mu=0}^3$ have to satisfy (0.2). Consequently, the Klein-Gordon equation can be factorized formally

$$\left(i\gamma^\mu \partial_\mu + m\right)\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0,$$

where $\{\gamma^\mu\}_{\mu=0}^3$ satisfy (0.2). From this equation one can conclude the famous free Dirac equation

$$\left(-i\gamma^\mu \partial_\mu + m\right)\psi = 0.$$

The Dirac equation provides a description of elementary spin-half particles, such as electrons. Note that any solution to the free Dirac equation is automatically a solution to the free Klein-Gordon equation but the converse is not true.

Now, the four-current density $\{j^\mu\}_{\mu=0}^3$ associated with the Dirac equation for a particle is given by

$$j^\mu = \psi^\dagger \gamma^0 \gamma^\mu \psi,$$

and it is conserved

$$\partial_\mu j^\mu = 0.$$

In particular, the probability density

$$\rho = j^0 = \psi^\dagger \psi = |\psi|^2$$

has the desired property of being positive definite, eliminating one of the problems of the Klein-Gordon equation.

2. Low regularity well-posedness of the DKG system

We complement the DKG system (0.1) with initial data

$$\psi(0,x)=\psi_0(x), \qquad \phi(0,x)=\phi_0(x), \quad \partial_t\phi(0,x)=\phi_1(x), \tag{2.1}$$

which have the regularity

$$(\psi_0,\phi_0,\phi_1)\in H^s\times H^r\times H^{r-1}$$

for $s,r\in\mathbb{R}$. Here $H^s=H^s(\mathbb{R}^n)$ is the standard Sobolev space with norm

$$\|f\|_{H^s}=\left\|\langle\xi\rangle^s\hat f(\xi)\right\|_{L^2_\xi},$$

where $\hat f(\xi)$ denotes the Fourier transform of $f(x)$ and $\langle\cdot\rangle=\sqrt{1+|\cdot|^2}$. We denote by $\dot H^s$ the corresponding homogeneous Sobolev space with norm

$$\|f\|_{\dot H^s}=\left\||\xi|^s\hat f(\xi)\right\|_{L^2_\xi}.$$

Our main interest concerning well-posedness of the Cauchy problem (0.1), (2.1) is to minimize the regularity assumption on the data (i.e., minimizing s and r) necessary to ensure well-posedness. In particular, we will be concerned with Local well-posedness of DKG in 3d, and both local and global well-posedness of DKG in 1d, given low regular initial data.

An important concept for the local existence problem is the critical exponent, which gives an idea about the minimal regularity of data required to ensure well-posedness. This is the unique exponent such that the homogeneous data space

$$\dot H^s\times\dot H^r\times\dot H^{r-1}$$

is invariant under the natural scaling of equation (0.1). For the non-massive case, $M=m=0$, the DKG system (0.1) in n space dimensions is invariant under the scaling

$$\psi_\lambda(t,x):=\lambda^{3/2}\psi\left(\lambda t,\lambda x\right), \quad \phi_\lambda(t,x):=\lambda\phi\left(\lambda t,\lambda x\right).$$

The scale invariant data space is therefore

$$(\psi_0,\phi_0,\phi_1)\in\dot H^{(n-3)/2}\times\dot H^{(n-2)/2}\times\dot H^{(n-4)/2},$$

and hence the critical Sobolev exponents are $(s_c,r_c)=((n-3)/2,(n-2)/2)$. In particular, the critical Sobolev exponents for DKG in 3d and 1d are $(s_c,r_c)=(0,1/2)$ and $(s_c,r_c)=(-1,-1/2)$, respectively. Heuristically, one cannot expect well-posedness below this regularity.

3. Local well-posedness of the DKG system in 3d

In 3d, one can prove using energy estimates and Sobolev embeddings that DKG is locally well-posed for data

$$(\psi_0,\phi_0,\phi_1)\in H^{1+\varepsilon}\times H^{3/2+\varepsilon}\times H^{1/2+\varepsilon}$$

for any $\varepsilon>0$. Bachelot [1] proved that the ε can be removed. By using Strichartz type estimates for the homogeneous wave equations one can prove local well-posedness for data

$$(\psi_0,\phi_0,\phi_1)\in H^{1/2+\varepsilon}\times H^{1+\varepsilon}\times H^{\varepsilon}$$

(see [25], [7]).

Bournaveas [7] proved local well-posedness in the energy class, i.e., for data

$$(\psi_0, \phi_0, \phi_1) \in H^{1/2} \times H^1 \times L^2.$$

His proof relies on a null structure he discovered in the Dirac part of the system; the quadratic nonlinearity in the Klein-Gordon part of the system, i.e., $\psi^\dagger \gamma^0 \psi$, was already known to be a null form (see [19], [2]). In order to take best advantage of the null structure in the system one should work in spaces of Bourgain-Klainerman-Machedon type, while this was not the case in [7]. Later, using these type of spaces, Fang and Grillakis [16] proved local well-posedness for data in

$$(\psi_0, \phi_0, \phi_1) \in H^s \times H^1 \times L^2.$$

for all $1/4 < s \leq 1/2$, improving the result in [7].

The Dirac-part null structure found by Bournaveas has the drawback that it involves squaring the Dirac equation, which seems to creat difficulties at very low regularity. Later, P. d'Ancona, D. Foschi and S. Selberg [13] proved, using a duality argument, that the null form $\psi^\dagger \gamma^0 \psi$ occurs not only in the KleinGordon part, but in fact also in the Dirac part of the system. Consequently, they proved [13] local well-posedness for data

$$(\psi_0, \phi_0, \phi_1) \in H^\varepsilon \times H^{1/2} \times H^{-1/2},$$

which is arbitrarily close to the minimal regularity predicted by the scaling ($\varepsilon = 0$).

Recently, the author proved, as part of his present thesis, local well-posedness for data

$$(\psi_0, \phi_0, \phi_1) \in H^s \times H^r \times H^{r-1}$$

with (s, r) in the region

$$s > 0, \quad \max\left(\frac{1}{2} + \frac{s}{3}, \frac{1}{3} + \frac{2s}{3}, s\right) < r < \min\left(\frac{1}{2} + 2s, 1 + s\right),$$

and moreover, (s, r) such that $r = 1 + s$ if $s > 1/2$ and $r = s$ if $s > 1$ are allowed. This result contains and extends the earlier known results for the same problem. The proof relies on the complete null structure in the system and interpolation of bilinear estimates of the wave equation in Bourgain-Klainerman-Machedon type spaces.

4. Local well-posedness of the DKG system in 1d

The one dimensional DKG is first studied by Chadam and Glassey in [8] and [9] where they proved global well-posedness for data

$$(\psi_0, \phi_0, \phi_1) \in H^1 \times H^1 \times L^2.$$

This result was improved by Bournaveas [5] (see also Fang [14]) who proved global well-posedness for data

$$(\psi_0, \phi_0, \phi_1) \in L^2 \times H^1 \times L^2.$$

Local well-posedness was shown by Fang [15] for data

$$(\psi_0, \phi_0, \phi_1) \in H^s \times H^r \times H^{r-1}$$

with (s, r) in the region

$$-\frac{1}{4} < s \leq \frac{1}{2}, \quad \frac{1}{2} < r \leq 1 + 2s.$$

Bournaveas and Gibbeson [6] proved global well-posedness for data

$$(\psi_0, \phi_0, \phi_1) \in L^2 \times H^r \times H^{r-1}$$

with $1/4 \leq r < 1/2$. The proof of local existence in [5], [14], [15] and [6] relies on a null form estimate of Klainerman and Machedon type for solutions of the wave equation which is adapted to the setting of the Dirac equation.

Machihara [22] and Pecher [23], who worked independently of each other, proved local well posedness for data

$$(\psi_0, \phi_0, \phi_1) \in H^s \times H^r \times H^{r-1}$$

with (s, r) in the region

$$-1/4 < s \leq 0, \quad 2|s| \leq r \leq 1 + 2s$$

in Machihara's case, whereas

$$s > -\frac{1}{4}, \quad r > 0, \quad |s| \leq r \leq 1 + s, \quad r < 1 + 2s.$$

in Pecher's result. Recently, S. Selberg and the present author [28] proved local well posedness for data

$$(\psi_0, \phi_0, \phi_1) \in H^s \times H^r \times H^{r-1}$$

with (s, r) in the region

$$s > -\frac{1}{4}, \quad r > 0, \quad |s| \leq r \leq 1 + s,$$

relaxing the condition $r < 1 + 2s$ imposed in Pecher's result. The result was also shown to be optimal if one works within the frame work of Bourgain-Klainerman-Machedon Fourier restriction norm method. This result [28] is included in the present thesis.

The results in [23] and [28] relies on the null structure that occurs in the quadratic nonlinearity in the Dirac part of the system, obtained by d'Ancona, Foschi and Selberg in [13]. More recently, Pecher [24] proved local well posedness for data

$$(\psi_0, \phi_0, \phi_1) \in \widehat{H^{s,p}} \times \widehat{H^{r,p}} \times \widehat{H^{r-1,p}}$$

with (s, r, p) in the region

$$s > -\frac{1}{2} + \frac{1}{2p}, \quad r > \frac{2}{p} - 1, \quad 1 < p \leq 2, \quad |s| \leq r \leq 1 + s,$$

generalizing the results for $p = 2$ by Selberg and Tesfahun. Here $\widehat{H^{s,p}}$ is a Sobolev space with norm

$$\|f\|_{\widehat{H^{s,p}}} = \left\| \langle \xi \rangle^s \hat{f}(\xi) \right\|_{L_\xi^{p'}},$$

where $1/p + 1/p' = 1$.

5. Global well-posedness of the DKG system in 1d

As indicated in the preceding section global well-posedness of DKG was first studied by Chadam [8] and Glassey [9]. Using the conservation of charge Bournaveas [5], Fang [14], and Bournaveas and Gibbeson [6] (in increasing order of improvements) able to lower the regularity requirements on the initial data which ensure global-in-time solutions (see the preceding section for the results). Later, Machihara [22] and Pecher [23] improved the earlier results by proving global well-posedness for data

$$(\psi_0, \phi_0, \phi_1) \in L^2 \times H^r \times H^{r-1}$$

with $0 < r < 1$; again, the conservation of charge is used in their proof.

Recently, using the method of Bourgain (see [4]), Selberg [27] proved global well-posedness for data

$$(\psi_0, \phi_0, \phi_1) \in H^s \times H^r \times H^{r-1}$$

with (s, r) in the region

$$-\frac{1}{8} < s < 0, \quad -s + \sqrt{s^2 - s} < r \leq 1 + s.$$

In this result the regularity of data needed for the spinor is below the charge norm, i.e., below L^2 norm, which greatly improved the earlier known results. More recently, using the theory of "almost conservation law" and "I-method" introduced by Colliander, Keel, Staffilani, Takaoka and Tao (see [10, 11, 12]), the present author proved global well-posedness for data

$$(\psi_0, \phi_0, \phi_1) \in H^s \times H^r \times H^{r-1}$$

with (s, r) in the region

$$-\frac{1}{8} < s < 0, \quad s + \sqrt{s^2 - s} < r \leq 1 + s,$$

improving Selberg's result. This result is part of the present thesis.

In what follows we briefly explain, using the cubic wave equation in 3d as an example, the ideas behind Bourgain's method and the theory of "almost conservation law" and "I-method" for proving global well-posedness of nonlinear wave equation for rough initial data (i.e., data with regularity below the conserved norm of the equation). In general, these methods apply to dispersive and wave equations, and the main steps in the application of these methods are the same.

Example 1. Consider the cubic wave equation in 3d

$$\Box u = u^3, \tag{5.1}$$

with data

$$u(0) = u_0, \qquad \partial_t u(0) = u_1. \tag{5.2}$$

It is known [20] that (5.1) is locally well-posed for all data $(u_0, u_1) \in \dot{H}^s \times \dot{H}^{s-1}$ with $s > 1/2$.

The cubic wave equation (5.1) enjoys the conservation of energy

$$E(u(t)) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + |\partial_t u|^2 + \frac{1}{2} u^4 \, dx = E(u_0).$$

This implies

$$\|u[t]\|_{\dot{H}^1} \leq C E(u_0)^{1/2}, \tag{5.3}$$

where we used the notation

$$\|u[t]\|_{H^s} = \|u(t)\|_{H^s} + \|\partial_t u(t)\|_{H^{s-1}}\,.$$

Combining (5.3) with the local well-posedness theory we immediately have global well-posedness of (5.1) for all data $(u_0, u_1) \in \dot{H}^1 \cap L^4 \times L^2$. We are interested on the question of global well-posedness of (5.1) for data whose norm is below the energy norm, i.e., $s < 1$. This is a difficult question since conservation of energy is unavailable for $s < 1$. To resolve such problems one can use the method of Bourgain or the theory of "almost conservation law" and "I-method".

5.1. **The method of Bourgain.** We briefly follow the proof by Kenig, Ponce and Vega [18] who showed that (5.1) is global well-posedness for all data $(u_0, u_1) \in \dot{H}^s \cap L^4 \times \dot{H}^{s-1}$ with $3/4 < s < 1$, which of course is below the energy class.

The basic idea here is to split the initial data into two parts corresponding to low and high frequencies, and then treat their evolution separately at each time step of iteration of the local result. For the Cauchy problem (5.1), (5.2) we assume that $u_0 \in \dot{H}^s \cap L^4$ where $s < 1$ (for simplicity we set $u_1 = 0$). We split

$$u_0 = f + g,$$

where

$$\widehat{f}(\xi) = \chi_{|\xi| \le N} \widehat{u_0}(\xi), \quad \widehat{g}(\xi) = \left(1 - \chi_{|\xi| \le N}\right) \widehat{u_0}(\xi),$$

for a characteristic function χ, and a large parameter $N \gg 1$ to be chosen later. One can immediately observe that the low frequency part, f, is smoother, but has a large norm:

$$\|f\|_{\dot{H}^1} \lesssim N^{1-s} \tag{5.4}$$

(note also that $\|f\|_{L^4} \lesssim \|u_0\|_{L^4} \sim 1 \lesssim N^{1/2(1-s)}$) while the high frequency part, g, clearly is no more regular than u_0, but its lower order norms are small:

$$\|g\|_{\dot{H}^a} \lesssim N^{a-s} \quad \text{for } a \le s.$$

Let u_l be the evolution of the low frequency part f under equation (5.1), and u_h be the evolution of the high frequency part g under the difference equation

$$\Box u_h = 3u_h u_l^2 + 3u_h^2 u_l + u_h^3. \tag{5.5}$$

Then we can write

$$u_h(t) = u_h^0(t) + U_h(t),$$

where u_h^0 is the homogenous part of u_h and $U_h = \Box^{-1}(3u_h u_l^2 + 3u_h^2 u_l + u_h^3)$ is the inhomogeneous part of u_h. Clearly, $u = u_l + u_h$. Since $f \in \dot{H}^1 \cap L^4$ we conclude by the conservation of energy that u_l exists globally in time. On the other hand, by local existence theory there exists $\Delta T = \Delta T(\|u_0\|_{\dot{H}^s \cap L^4}) > 0$ and a solution u_h to (5.5) for all $0 \le t \le \Delta T$.

Moreover, due to a nonlinear smoothing effect one has

$$U_h(t) \in \dot{H}^1 \cap L^4 \quad \text{for } 0 \le t \le \Delta T.$$

In particular, for $0 \le t \le \Delta T$,

$$\|U_h[t]\|_{\dot{H}^1} \lesssim N^{-\sigma} \quad \text{for some } \; \sigma = \sigma(s) > 0. \tag{5.6}$$

A key observation is that since the inhomogeneous part, $U_h \in \dot{H}^1 \cap L^4$, at the end of the time interval of existence (i.e., ΔT), U_h can be added to the evolution of the low-frequency data, and the whole process can be iterated. Thus, we start with new Cauchy problems

$$\begin{cases} \Box\tilde{u}_l = \tilde{u}_l^3 \\ \tilde{u}_l(\Delta T) = u_l(\Delta T) + U_h(\Delta T) \in \dot{H}^1 \cap L^4, \\ \partial_t\tilde{u}_l(\Delta T) = \partial_t u_l(\Delta T) + \partial_t U_h(\Delta T) \in L^2, \end{cases} \tag{5.7}$$

and

$$\begin{cases} \Box\tilde{u}_h = 3\tilde{u}_h\tilde{u}_l^2 + 3\tilde{u}_h^2\tilde{u}_l + \tilde{u}_h^3 \\ \tilde{u}_h(\Delta T) = u_h(\Delta T) \in \dot{H}^s, \\ \partial_t\tilde{u}_h(\Delta T) = \partial_t u_h(\Delta T) \in \dot{H}^{s-1}. \end{cases} \tag{5.8}$$

Fix arbitrary time $0 < T < \infty$. We shall divide this interval into subintervals of length ΔT, and show well-posedness on each subinterval successively (we have already shown this on the first subinterval $[0, \Delta T]$). Let $M = T/\Delta T$ be the number of subintervals. Now, observe from (5.6) and (5.7) that at the first step of the iteration a quantity of energy about $N^{-2\sigma(s)}$ is added (this is considered as an error).

To show well-posedness of (5.1), (5.2) on the entire interval $[0, T]$, we have to control the total added energy after M iterations which is $\sim MN^{-2\sigma(s)}$ by the initial energy which is $\sim N^{2(1-s)}$. Thus, we must have

$$MN^{-2\sigma(s)} = T(\Delta T)^{-1}N^{-2\sigma(s)} \lesssim N^{2(1-s)}.$$

Then the range of s where global well-posedness holds can be computed from this inequality using the explicit formula for ΔT (which depends on N) and $\sigma(s)$, which we do not discuss here.

5.2. **The I-method and almost conservation law.** We consider (5.1) for data in a slightly different spaces, $(u_0, u_1) \in H^s \times H^{s-1}$. The basic idea here is to apply a smoothing operator I of order $1-s$ to (5.1), (5.2), and then replace the conserved quantity $E(u)$, which is no longer available for $s < 1$, with a smoothed out variant $E(Iu)$. However, $E(Iu)$ is not conserved either since Iu is not a solution anymore, but one hopes that some cancellation will still occur so that the increment (error) of $E(Iu)$ can be proved to be small. This is indeed the case for certain values of $s < 1$, and one obtains in this way global well-posedness below H^1. For the sake of simplicity we set $u_1 = 0$.

The smoothing operator I is defined, for $s < 1$ and $N \gg 1$, by

$$\widehat{If}(\xi) = m(\xi)\widehat{f}(\xi), \qquad m(\xi) = \begin{cases} 1, & |\xi| < N, \\ \left(\frac{N}{|\xi|}\right)^{1-s}, & |\xi| > 2N, \end{cases} \tag{5.9}$$

where m is smooth and monotone. It can be seen that

$$\|u_0\|_{H^s} \lesssim \|Iu_0\|_{H^1} \lesssim N^{1-s}\|u_0\|_{H^s}. \tag{5.10}$$

Another ingredient in the proof of global well-posedness of the Cauchy problem is a modified local well-posedness theorem for the I-system

$$\Box Iu = I(u^3), \tag{5.11}$$

with data

$$Iu(0) = Iu_0 \in H^1, \qquad \partial_t Iu(0) = 0. \tag{5.12}$$

One can show that this I-Cauchy problem is locally well-posed with time of existence, say $\Delta T = \Delta T(\|Iu_0\|_{H^1})$, such that the solution satisfies the property

$$\|u[t]\|_{H^1} \lesssim E(Iu_0) \quad \text{for all } 0 \le t \le \Delta T. \tag{5.13}$$

The next crucial step is to use (5.10), (5.13) and some kind of cancellation property in $E(Iu)$ to prove the *almost conservation energy*

$$E(Iu(t)) = E(Iu_0) + O(N^{-\beta}) \quad \text{for all } 0 \le t \le \Delta T, \tag{5.14}$$

for some $\beta = \beta(s) > 0$.

As in the preceding subsection, let $M = T/\Delta T$ be the number of subintervals. Global well-posedness of (5.1), (5.2) will follow if we show well-posedness on $[0, T]$.

Observe from (5.14) that at the first step of iteration, i.e., for the solution on the subinterval $[0, \Delta T]$ an increment energy of size $O(N^{-\beta})$ is added to the initial energy. Therefore, to reach the final time T by iteration, we have to control the total added energy after M iteration which is $\sim MN^{-\beta(s)}$ by the energy of the initial data which is $E(Iu_0) \sim N^{2(1-s)}$; Thus, we must have

$$MN^{-\beta(s)} = T(\Delta T)^{-1} N^{-\beta(s)} \lesssim N^{2(1-s)}.$$

Then the range of s where global well-posedness holds can be computed from this inequality using the explicit formula for ΔT (which depends on N) and $\beta(s)$.

Recently, using the I-method and almost conservation laws, T. Roy [26] proved the Cauchy problem (5.1), (5.2) to be global well-posedness for data $(u_0, u_1) \in H^s \times H^{s-1}$ with $13/18 < s < 1$, which is below the energy class.

6. Summary of papers

The following papers are part of the present thesis.

6.1. **Paper I: Low regularity and local well-posedness for the 1+3 dimensional Dirac-Klein-Gordon system.**

In this paper we consider the DKG system (0.3) in 3d with Dirac matrices given by (0.4). Then given the initial data (2.1) with regularity

$$(\psi_0, \phi_0, \phi_1) \in H^s \times H^r \times H^{r-1}$$

with (s, r) in the region

$$s > 0, \quad \max\left(\frac{1}{2} + \frac{s}{3}, \frac{1}{3} + \frac{2s}{3}, s\right) < r < \min\left(\frac{1}{2} + 2s, 1 + s\right).$$

we prove that there exists a time $T > 0$ and a solution of (0.3),

$$(\psi, \phi) \in C([0, T], H^s) \times C([0, T], H^r) \cap C^1\left([0, T], H^{r-1}\right),$$

which depends continuously on the data, and moreover, the solution is unique in some subspace of

$$C([0, T], H^s) \times C([0, T], H^r) \cap C^1\left([0, T], H^{r-1}\right).$$

Further more, (s, r) such that $r = 1 + s$ if $s > 1/2$ and $r = s$ if $s > 1$, are allowed.

Our proof relies on the null structure in the system, and bilinear spacetime estimates of Klainerman-Machedon type.

6.2. **Paper II: Low regularity well-posedness for one dimensional Dirac-Klein-Gordon system.** In this paper we consider the DKG system (0.3) in 1d with the Dirac matrices given by (0.6). Then given the initial data (2.1) with regularity

$$(\psi_0, \phi_0, \phi_1) \in H^s \times H^r \times H^{r-1},$$

with (s, r) in the region

$$s > -\frac{1}{4}, \quad r > 0, \quad |s| \leq r \leq 1 + s,$$

we prove that there exists a time $T > 0$ and a solution of (0.3),

$$(\psi, \phi) \in C([0, T], H^s) \times C([0, T], H^r) \cap C^1\left([0, T], H^{r-1}\right),$$

which depends continuously on the data, and moreover, the solution is unique in some subspace of

$$C([0, T], H^s) \times C([0, T], H^r) \cap C^1\left([0, T], H^{r-1}\right).$$

Furthermore, we show that our result is best possible up to endpoint cases, if one works in Bourgain-Klainerman-Machedon spaces.

Our proof relies on the null structure in the system, and bilinear spacetime estimates of Klainerman-Machedon type.

6.3. **Paper III: Global well-posedness of the 1D Dirac-Klein-Gordon system in Sobolev spaces of negative index.** In this paper we consider the DKG system (0.1) in 1d with Dirac matrices given by (0.5). Then given the initial data (2.1) with regularity

$$(\psi_0, \phi_0, \phi_1) \in H^s \times H^r \times H^{r-1}$$

with (s, r) in the region

$$-\frac{1}{8} < s < 0, \qquad s + \sqrt{s^2 - s} < r \leq 1 + s,$$

we prove that there exists a global-in-time solution of (0.1)

$$(\psi, \phi) \in C([0, \infty), H^s) \times C([0, \infty), H^r) \cap C^1\left([0, \infty), H^{r-1}\right),$$

which depends continuously on the data, and moreover, the solution is unique in some subspace of

$$C([0, \infty), H^s) \times C([0, \infty), H^r) \cap C^1\left([0, \infty), H^{r-1}\right).$$

The main ingredient in our proof is the theory of almost conservation law and I-method introduced by Colliander, Keel, Staffilani, Takaoka and Tao. Our proof also relies on the null structure in the system, and bilinear spacetime estimates of Klainerman-Machedon type.

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Paper I

Low regularity local well-posedness for the 1+3 dimensional Dirac-Klein-Gordon system.

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LOW REGULARITY AND LOCAL WELL-POSEDNESS FOR THE 1+3 DIMENSIONAL DIRAC-KLEIN-GORDON SYSTEM

ACHENEF TESFAHUN

ABSTRACT. We prove that the Cauchy problem for the Dirac-Klein-Gordon system of equations in 1+3 dimensions is locally well-posed in a range of Sobolev spaces for the Dirac spinor and the meson field. The result contains and extends the earlier known results for the same problem. Our proof relies on the null structure in the system, and bilinear spacetime estimates of Klainerman-Machedon type.

1. INTRODUCTION

We consider the Dirac-Klein-Gordon system (DKG) in three space dimensions,

$$\begin{gathered} \big(D_t + \alpha \cdot D_x\big)\psi = -M\beta\psi + \phi\beta\psi, \quad (D_t = -i\partial_t,\ D_x = -i\nabla) \\ \square\,\phi = m^2\psi - \langle \beta\psi, \psi\rangle, \quad (\square = -\partial_t^2 + \Delta) \end{gathered} \tag{1.1}$$

with initial data

$$\psi\big|_{t=0} = \psi_0 \in H^s, \quad \phi\big|_{t=0} = \phi_0 \in H^r, \quad \partial_t\phi\big|_{t=0} = \phi_1 \in H^{r-1}, \tag{1.2}$$

where $\psi(t,x)$ is the Dirac spinor, regarded as a column vector in $\mathbb{C}^4$, and $\phi(t,x)$ is the meson field which is real-valued; both the Dirac spinor and the meson field are defined for $t \in \mathbb{R}$, $x \in \mathbb{R}^3$; $M, m \ge 0$ are constants; $\nabla = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$; $\langle u, v\rangle := \langle u, v\rangle_{\mathbb{C}^4} = v^\dagger u$ for column vectors $u, v \in \mathbb{C}^4$, where $v^\dagger$ is the complex conjugate transpose of v; $H^s = (1+\sqrt{-\Delta})^{-s} L^2(\mathbb{R}^3)$ is the standard Sobolev space of order s. The Dirac matrices are given in 2×2 block form by

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha^j = \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix},$$

where

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices. The Dirac matrices α^j, β satisfy

$$\beta^\dagger = \beta, \quad (\alpha^j)^\dagger = \alpha^j, \quad \beta^2 = (\alpha^j)^2 = I, \quad \alpha^j\beta + \beta\alpha^j = 0. \tag{1.3}$$

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For the DKG system there are many conserved quantities which are not positive definite, such as the energy, see [11]. However, there is a known positive conserved quantity, namely the charge, $\|\psi(t,.)\|_{L^2} = \text{const}$. To study questions of global regularity, a natural strategy is to study local (in time) well-posedness (LWP) for low regularity data, and then try to exploit the conserved quantities of the system. See, e.g., the global result of Chadam [8] for 1+1 dimensional DKG system. The LWP results for DKG in 1+3 dimensions are summarized in Table 1

For DKG in 1+3 dimensions the scale invariant data is (see [1])

$$(\psi_0, \phi_0, \phi_1) \in L^2 \times \dot{H}^{1/2} \times \dot{H}^{-1/2},$$

where $\dot{H}^s = (\sqrt{-\Delta})^{-s} L^2$. Heuristically, one cannot expect well-posedness below this regularity. This scaling also suggests that $r = 1/2 + s$ is the line where equation (1.1) is LWP. Concerning LWP of the DKG system in 1+3 dimensions, the best result to date is due to P. d'Ancona, D. Foschi and S. Selberg in [1] for data

$$\psi_0 \in H^{\varepsilon}, \quad \phi_0 \in H^{1/2+\varepsilon}, \quad \phi_1 \in H^{-1/2+\varepsilon},$$

where $\varepsilon > 0$ is arbitrary. This result is arbitrarily close to the minimal regularity predicted by the scaling ($\varepsilon = 0$). The key achievement in this result is that a null structure occurs not only in the Klein-Gordon part (in the nonlinearity $\langle \beta\psi, \psi\rangle$) which was known to be a null form (see [1] for references)), but also in the Dirac part (in the nonlinearity $\phi\beta\psi$) of the system, which they discover using a duality argument. This requires first to diagonalize the system by using the eigenspace projections of the Dirac operator. The same authors used their result on the null structure in $\phi\beta\psi$ to prove LWP below the charge norm of the DKG system in 1+2 dimensions (see [2]).

In the present paper we study the LWP of the DKG system in 1+3 dimensions. We prove that (1.1)–(1.2) is LWP for (s, r) in the convex region shown in Figure 1, extending to the right, which contains the union of all the results shown in Table 1 as a proper subset. In our proof, we take advantage of the null structure in the nonlinearity $\phi\beta\psi$ found in [1] besides the null structure in the nonlinearity $\langle \beta\psi, \psi\rangle$, and some bilinear spacetime estimates.

We now describe our main result.

Theorem 1.1. *Suppose $(s, r) \in \mathbb{R}^2$ belongs to the convex region described by (see Figure 1) the region*

$$s > 0, \quad \max\left(\frac{1}{2} + \frac{s}{3}, \frac{1}{3} + \frac{2s}{3}, s\right) < r < \min\left(\frac{1}{2} + 2s, 1 + s\right).$$

Then the DKG system (1.1) *is LWP for data* (1.2). *Moreover, we can allow $r = 1+s$ if $s > 1/2$, and $r = s$ if $s > 1$.*

If A, B, C, D are points in the (s, r)–plane, the symbol AB represents a line from A to B, ABC represents a triangle and $ABCD$ a quadrilateral, all of them excluding the boundaries. We use the following notation for different regions in Figure 1:

$$\begin{aligned}
R_1 &:= ACD \cup AD, \\
R_2 &:= ABD, \\
R_3 &:= D \cup F \cup CD \cup DF \cup FE \cup CDFE, \\
R_4 &:= G \cup BG \cup GF \cup BDGF, \\
R &:= BD \cup \cup_{j=1}^4 R_j.
\end{aligned} \tag{1.4}$$

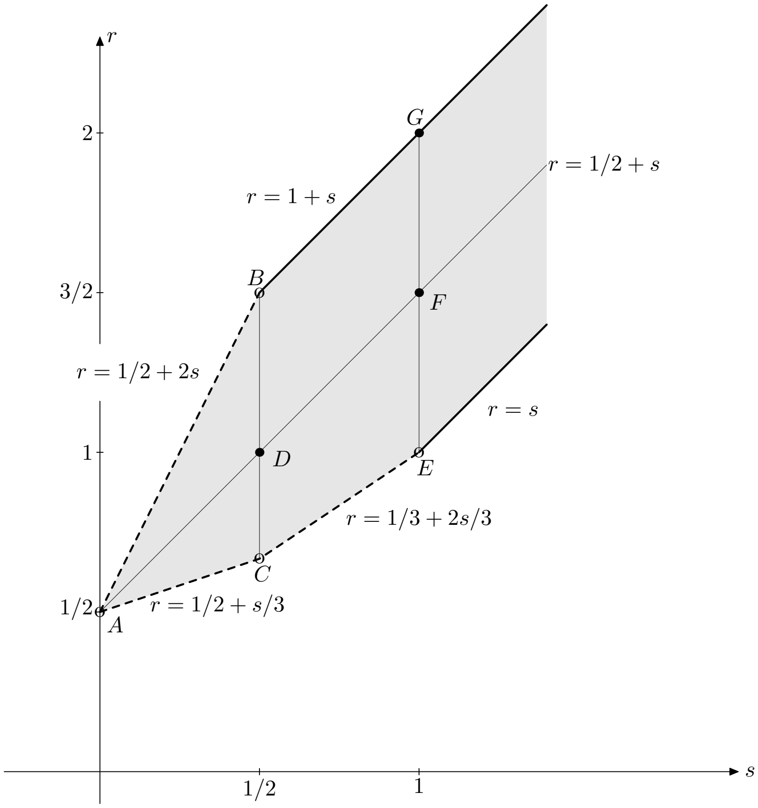

Figure 1. LWP holds in the interior of the shaded region, extending to the right. Moreover, we can allow the line $r = 1 + s$ for $s > 1/2$, and the line $r = s$ for $s > 1$. The line $r = 1/2 + s$ represents the regularity predicted by the scaling.

This paper is organized as follows. In the next section we fix some notation, state definitions and basic estimates. In addition, we shall rewrite the system (1.1) by splitting ψ as the sum $P_+(D_x)\psi + P_-(D_x)\psi$, where $P_\pm(D_x)$ are the projections onto the eigenspaces of the matrix $\alpha.D_x$. We also state the reduction of Theorem 1.1 to two $X^{s,b}$ bilinear estimates. In Section 3 we review the crucial null structure of the bilinear forms involved, and we discuss product estimates for wave-Sobolev spaces $H^{s,b}$. In Section 4 we interpolate between the product estimates from Section 3 to get a wider range of estimates. In Sections 5 and 6 we apply the estimates from Sections 3 and 4 to prove the bilinear estimates from Section 2. In Section 7 we prove that these bilinear estimates are optimal up to some endpoint cases, by constructing counterexamples.

For simplicity we set $M = m = 0$ in the rest of the paper, but the discussion can easily be modified to handle the massive case as well.

2. Notation and preliminaries

In estimates, we use the symbols $\lesssim$, $\simeq$, $\gtrsim$ to denote relations $\le$, $=$, $\ge$ up to a positive constant which may depend on s and r. Also, if $K_1 \lesssim K_2 \lesssim K_1$ we

Table 1. LWP exponents for (1.1), (1.2). That is, if the data $(\psi_0, \phi_0, \phi_1) \in H^s \times H^r \times H^{r-1}$, then there exists a time $T > 0$ and a solution of (1.1), $(\psi(t), \phi(t)) \in C([0,T], H^s) \times C([0,T], H^r)$ which depends continuously on the data. The solution is also unique in some subspace of $C\big([0,T], H^s\big) \times C\big([0,T], H^r\big)$. Here $\varepsilon > 0$ is an arbitrary parameter.

Reference	s	r
classical methods	$1+\varepsilon$	$3/2+\varepsilon$
Bachelot [3], 1984	1	3/2
Strichartz estimate [7, 15], 1993	$1/2+\varepsilon$	$1+\varepsilon$
Beals and Bezard [4], 1996	1	2
Bournaveas [7], 1999	1/2	1
Fang and Grillakis [9], 2005	$(1/4, 1/2]$	1
D'Ancona, Foschi and Selberg [1], 2005	ε	$1/2+\varepsilon$

will write $K_1 \approx K_2$. If in the inequality $\lesssim$ the multiplicative constant is much smaller than 1 then we use the symbol $\ll$; similarly, if in $\gtrsim$ the constant is much greater than 1 then we use $\gg$. Throughout we use the notation $\langle\cdot\rangle = 1 + |\cdot|$. The characteristic function of a set A is denoted by $\mathbf{1}_A$. For $a \in \mathbb{R}$, $a^\pm := a \pm \epsilon$ for sufficiently small $\epsilon > 0$. The Fourier transforms in space and space-time are defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)\,dx, \quad \widetilde{u}(\tau,\xi) = \int_{\mathbb{R}^{1+3}} e^{-i(t\tau + x\cdot\xi)} u(t,x)\,dt\,dx.$$

Then $\widetilde{D_t u} = \tau\widetilde{u}$, and $\widetilde{D_x u} = \xi\widetilde{u}$. If $\phi : \mathbb{R}^3 \to \mathbb{C}$, we define the multiplier $\phi(D)$ by

$$\widehat{\phi(D) f}(\xi) = \phi(\xi)\widehat{f}(\xi).$$

If X, Y, Z are normed function spaces, we use the notation $X \cdot Y \hookrightarrow Z$ to mean that

$$\|uv\|_Z \lesssim \|u\|_X \|v\|_Y .$$

In the study of non-linear wave equations it is standard that the following spaces of Bourgain-Klainerman-Machedon type are used. For $a, b \in \mathbb{R}$, define $X_\pm^{a,b}$, $H^{a,b}$ to be the completions of $\mathcal{S}(\mathbb{R}^{1+3})$ with respect to the norms

$$\|u\|_{X_\pm^{a,b}} = \big\|\langle\xi\rangle^a \langle\tau \pm |\xi|\rangle^b \widetilde{u}(\tau,\xi)\big\|_{L^2_{\tau,\xi}},$$
$$\|u\|_{H^{a,b}} = \big\|\langle\xi\rangle^a \langle|\tau| - |\xi|\rangle^b \widetilde{u}(\tau,\xi)\big\|_{L^2_{\tau,\xi}},$$

We also need the restrictions to a time slab $S_T = (0,T) \times \mathbb{R}^3$, since we study local in time solutions. The restriction $X_\pm^{a,b}(S_T)$ is a Banach space with norm

$$\|u\|_{X_\pm^{a,b}(S_T)} = \inf_{\tilde{u}_{|S_T} = u} \|\tilde{u}\|_{X_\pm^{a,b}} .$$

The restrictions $H^{a,b}(S_T)$ is defined in the same way. We now collect some facts about these spaces which will be needed in the later sections. It is well known that the following interpolation property holds:

$$(H^{s_0,\alpha_0}, H^{s_1,\alpha_1})_{[\theta]} = H^{s,\alpha}, \tag{2.1}$$

where $0 \le \theta \le 1$, $s = (1-\theta)s_0 + \theta s_1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$ and $(.,.)_{[\theta]}$ is the intermediate space with respect to the interpolation pair (.,.). It immediately follows from a general bilinear complex interpolation for Banach spaces (see for example [6]) that if

$$H^{a_0,\alpha_0} \cdot H^{b_0,\beta_0} \hookrightarrow H^{-c_0,-\gamma_0},$$
$$H^{a_1,\alpha_1} \cdot H^{b_1,\beta_1} \hookrightarrow H^{-c_1,-\gamma_1},$$

then

$$H^{a,\alpha} \cdot H^{b,\beta} \hookrightarrow H^{-c,-\gamma},$$

where $0 \le \theta \le 1$, $a = (1-\theta)a_0 + \theta a_1$, $b = (1-\theta)b_0 + \theta b_1$, $c = (1-\theta)c_0 + \theta c_1$, $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$, $\beta = (1-\theta)\beta_0 + \theta\beta_1$ and $\gamma = (1-\theta)\gamma_0 + \theta\gamma_1$.

We shall also need the fact that

$$X_\pm^{a,b}(S_T) \hookrightarrow H^{a,b}(S_T) \hookrightarrow C\big([0,T], H^a\big) \quad \text{provided } b > 1/2, \tag{2.2}$$
$$X_\pm^{a,b} \hookrightarrow H^{a,b} \quad \text{for all } b \ge 0. \tag{2.3}$$

The embedding (2.2) is equivalent to the estimate

$$\|u(t)\|_{H^a} \le C_1 \|u\|_{H^{a,b}(S_T)} \le C_2 \|u\|_{X_\pm^{a,b}(S_T)},$$

for all $0 \le t \le T$ and $C_1, C_2 \ge 1$. In the first inequality, C_1 will depend on b (see [1] for the proof), and the second inequality follows from the fact that $\langle |\tau| - |\xi| \rangle \le \langle \tau \pm |\xi| \rangle$ (hence $C_2 = 1$), which also implies (2.3).

Following [1], we diagonalize the system by defining the projections

$$P_\pm(\xi) = \frac{1}{2}\big(I \pm \hat{\xi} \cdot \alpha\big),$$

where $\hat{\xi} \equiv \xi/|\xi|$. Then the spinor field splits into $\psi = \psi_+ + \psi_-$, where $\psi_\pm = P_\pm(D_x)\psi$. Now applying $P_\pm(D_x)$ to the Dirac equation in (1.1), and using the identities

$$\begin{gathered} \alpha \cdot D_x = |D_x| P_+(D_x) - |D_x| P_-(D_x), \\ P_\pm^2(D_x) = P_\pm(D_x) \quad \text{and} \quad P_\pm(D_x) P_\mp(D_x) = 0, \end{gathered} \tag{2.4}$$

we obtain

$$\begin{gathered} \big(D_t + |D_x|\big)\psi_+ = P_+(D_x)(\phi\beta\psi), \\ \big(D_t - |D_x|\big)\psi_- = P_-(D_x)(\phi\beta\psi), \\ \Box\,\phi = -\langle \beta\psi, \psi \rangle, \end{gathered} \tag{2.5}$$

which is the system we shall study.

We iterate in the spaces

$$\psi_+ \in X_+^{s,\sigma}(S_T), \quad \psi_- \in X_-^{s,\sigma}(S_T), \quad (\phi, \partial_t\phi) \in H^{r,\rho} \times H^{r-1,\rho}(S_T),$$

where

$$\frac{1}{2} < \sigma, \rho < 1$$

will be chosen depending on r, s. By a standard argument (see [1] for details) Theorem 1.1 then reduces to

$$\big\| P_\pm(D_x)(\phi\beta P_{[\pm]}(D_x)\psi) \big\|_{X_\pm^{s,\sigma-1+\varepsilon}} \lesssim \|\phi\|_{H^{r,\rho}} \|\psi\|_{X_{[\pm]}^{s,\sigma}}, \tag{2.6}$$
$$\big\| \langle \beta P_{[\pm]}(D_x)\psi, P_\pm(D_x)\psi' \rangle \big\|_{H^{r-1,\rho-1+\varepsilon}} \lesssim \|\psi\|_{X_{[\pm]}^{s,\sigma}} \|\psi'\|_{X_\pm^{s,\sigma}}, \tag{2.7}$$

for all $\phi, \psi, \psi' \in \mathcal{S}(\mathbb{R}^{1+3})$, where $\pm$ and $[\pm]$ denote independent signs, and $\varepsilon > 0$ is sufficiently small.

But in [1], it was shown that (2.6) is equivalent, by duality, to an estimate similar to (2.7), namely

$$\big\|\langle\beta P_{[\pm]}(D_x)\psi, P_\pm(D_x)\psi'\rangle\big\|_{H^{-r,-\rho}} \lesssim \|\psi\|_{X^{s,\sigma}_{[\pm]}} \|\psi'\|_{X_\pm^{-s,1-\sigma-\varepsilon}}\,, \tag{2.6$'$}$$

for all $\psi, \psi' \in \mathcal{S}(\mathbb{R}^{1+3})$. Note that in this formulation, the bilinear null form $\langle\beta P_{[\pm]}(D_x)\psi, P_\pm(D_x)\psi'\rangle$, appears again. Thus, Theorem 1.1 has been reduced to proving (2.6′) and (2.7). We shall prove the following theorem, which implies Theorem 1.1.

Theorem 2.1. *Suppose*

$$s > 0, \quad \max\big(\frac{1}{2} + \frac{s}{3}, \frac{1}{3} + \frac{2s}{3}, s\big) < r < \min\big(\frac{1}{2} + 2s, 1 + s\big). \tag{2.8}$$

Then there exist $1/2 < \rho, \sigma < 1$ *and* $\varepsilon > 0$ *such that* (2.6′) *and* (2.7) *hold simultaneously for all* $\psi, \psi' \in \mathcal{S}(\mathbb{R}^{1+3})$. *Moreover, in addition to* (2.8) *we can allow* $r = 1 + s$ *if* $s > 1/2$, *and* $r = s$ *if* $s > 1$. *The parameters* ρ, σ *can be chosen as follows:*

$$\rho = 1/2 + \varepsilon, \tag{2.9}$$

$$\sigma = \begin{cases} 1/2 + s/3 & \text{if } (s,r) \in R_1, \\ 1/2 + s & \text{if } (s,r) \in R_2, \\ 5/6 - s/3 + \varepsilon & \text{if } (s,r) \in R_3, \\ 3/2 - s + 4\varepsilon & \text{if } (s,r) \in R_4, \\ 1 - \varepsilon & \text{if } (s,r) \in BD, \\ \text{any number in } (1/2, 1) & \text{otherwise}, \end{cases} \tag{2.10}$$

with $\varepsilon > 0$ *sufficiently small depending on* s, r *(see* (1.4) *to locate* (s, r) *in the case of* (2.10)*).*

3. Null structure and a product law for wave Sobolev spaces

Let us first discuss the null structure in $\langle\beta P_{[\pm]}(D_x)\psi, P_\pm(D_x)\psi'\rangle$. The discussion here follows [1]. Taking the spacetime Fourier transform on this bilinear form we get

$$\begin{aligned} &\big[\langle\beta P_{[\pm]}(D_x)\psi, P_\pm(D_x)\psi'\rangle\big]\widetilde{\ }(\tau,\xi) \\ &= \int_{\mathbb{R}^{1+3}} \langle\beta P_{[\pm]}(\eta)\widetilde{\psi}(\lambda,\eta), P_\pm(\eta-\xi)\widetilde{\psi}'(\lambda-\tau,\eta-\xi)\rangle\, d\lambda\, d\eta, \end{aligned}$$

where we have $(\lambda - \tau, \eta - \xi)$ as an argument of $\tilde{\psi}'$ instead of $(\tau - \lambda, \xi - \eta)$ because of the complex conjugation in the inner product. Since $P_\pm(\eta - \xi)^\dagger = P_\pm(\eta - \xi)$, and $P_\pm(\eta - \xi)\beta = \beta P_\mp(\eta - \xi)$, we obtain

$$\begin{aligned} &\langle\beta P_{[\pm]}(\eta)\widetilde{\psi}(\lambda,\eta), P_\pm(\eta-\xi)\widetilde{\psi}'(\lambda-\tau,\eta-\xi)\rangle \\ &= \langle P_\pm(\eta-\xi)\beta P_{[\pm]}(\eta)\widetilde{\psi}(\lambda,\eta), \widetilde{\psi}'(\lambda-\tau,\eta-\xi)\rangle \\ &= \langle \beta P_\mp(\eta-\xi) P_{[\pm]}(\eta)\widetilde{\psi}(\lambda,\eta), \widetilde{\psi}'(\lambda-\tau,\eta-\xi)\rangle. \end{aligned}$$

The matrix $\beta P_\mp(\eta - \xi)P_{[\pm]}(\eta)$ is the symbol of the bilinear operator $(\psi, \psi') \mapsto \langle\beta P_{[\pm]}(D_x)\psi, P_\pm(D_x)\psi'\rangle$. By orthogonality, $P_\mp(\eta - \xi)P_{[\pm]}(\eta)$ vanishes when the vectors $[\pm]\eta$ and $\pm(\eta-\xi)$ line up in the same direction. The following lemma, proved in [1], quantifies this cancellation. We shall use the notation $\measuredangle(\eta,\zeta)$ for the angle between vectors $\eta,\zeta\in\mathbb{R}^3$.

Lemma 3.1. $\beta P_{\mp}(\eta-\xi)P_{[\pm]}(\eta) = O\left(\measuredangle([\pm]\eta,\pm(\eta-\xi))\right)$.

As a result of this lemma, we get

$$|\langle \beta P_{[\pm]}(D_x)\psi, P_{\pm}(D_x)\psi'\rangle\widetilde{\ }(\tau,\xi)| \lesssim \int_{\mathbb{R}^{1+3}} \theta_{[\pm],\pm}|\widetilde{\psi}(\lambda,\eta)||\widetilde{\psi}'(\lambda-\tau,\eta-\xi)|\,d\lambda\,d\eta, \tag{3.1}$$

where $\theta_{[\pm],\pm}=\measuredangle\big([\pm]\eta,\pm(\eta-\xi)\big)$.

The strategy for proving Theorem 2.1 is to make use of this null form estimate, (3.1), and reduce (2.6′) and (2.7) to some well-known bilinear spacetime estimates of Klainerman-Machedon type for products of free waves. We now discuss some product laws for the wave Sobolev spaces $H^{a,\alpha}$ in the following theorems.

Theorem 3.2. *Let $d>1/2$. Then*

$$H^{a,d}\cdot H^{b,d}\hookrightarrow L^2, \tag{3.2}$$

provided that $a,b\ge 0$, and $a+b>1$.

Proof. By the same proof as in Corollary 3.3 in [5], but using the dyadic estimates in Theorem 12.1 in [10], we have, for any $\varepsilon>0$,

$$\|uv\|_{L^2(\mathbb{R}^{1+3})}\lesssim \|u_0\|_{H^{1+\varepsilon}(\mathbb{R}^3)}\|v_0\|_{L^2(\mathbb{R}^3)}.$$

It follows by the transfer principle (see [1], Lemma 4) that

$$H^{1+\varepsilon,d}\cdot H^{0,d}\hookrightarrow L^2.$$

Now, interpolation between

$$H^{1+\varepsilon,d}\cdot H^{0,d}\hookrightarrow L^2,$$
$$H^{0,d}\cdot H^{1+\varepsilon,d}\hookrightarrow L^2,$$

gives

$$H^{(1+\varepsilon)(1-\theta),d}\cdot H^{(1+\varepsilon)\theta,d}\hookrightarrow L^2,$$

for $\theta\in[0,1]$. If there exists $\theta\in[0,1]$ such that $a\ge(1+\varepsilon)(1-\theta)$ ($\Leftrightarrow \theta\ge 1-a/(1+\varepsilon)$) and $b\ge(1+\varepsilon)\theta$ ($\Leftrightarrow \theta\le b/(1+\varepsilon)$), then we have

$$H^{a,d}\cdot H^{b,d}\hookrightarrow L^2.$$

If $a,b\ge 0$ and $a+b>1$, then such $\theta\in[0,1]$ exists, if we choose $\varepsilon>0$ small enough. This proves Theorem 3.2. □

Theorem 3.3 ([10, 13, 14])**.** *Let $s_1,s_2,s_3\in\mathbb{R}$. For free waves $u(t)=e^{\pm it|D_x|}u_0$ and $v(t)=e^{[\pm]it|D_x|}v_0$ (where $\pm$ and $[\pm]$ are independent signs), we have the estimate*

$$\big\||D_x|^{-s_3}(uv)\big\|_{L^2(\mathbb{R}^{1+3})}\lesssim \|u_0\|_{\dot H^{s_1}}\|v_0\|_{\dot H^{s_2}} \tag{3.3}$$

if and only if

$$s_1+s_2+s_3=1,\quad s_1+s_2>1/2,\quad s_1,s_2<1. \tag{3.4}$$

As an application of Theorems 3.2 and 3.3 we have the following result.

Theorem 3.4. *Suppose $s_1, s_2, s_3 \in \mathbb{R}$ and $d > 1/2$. Then*

$$H^{s_1,d} \cdot H^{s_2,d} \hookrightarrow H^{-s_3,0} \tag{3.5}$$

provided s_1, s_2, s_3 satisfy

$$\begin{gathered} s_1 + s_2 + s_3 = 1, \quad s_1 + s_2 > 1/2, \\ s_1 + s_3 \geq 0, \quad s_2 + s_3 \geq 0, \\ s_1, s_2 < 1, \end{gathered} \tag{3.6}$$

or

$$\begin{gathered} s_1 + s_2 + s_3 > 1, \quad s_1 + s_2 > 1/2, \\ s_1 + s_3 \geq 0, \quad s_2 + s_3 \geq 0. \end{gathered} \tag{3.7}$$

Proof. First, let us prove (3.5) for $s_1, s_2, s_3 \in \mathbb{R}$ satisfying (3.6). By Theorem 3.3 and the transfer principle (see [1], Lemma 4), we obtain

$$H^{s_1,d} \cdot H^{s_2,d} \hookrightarrow H^{-s_3,0} \quad \text{if} \quad \begin{cases} s_1 + s_2 + s_3 = 1, \\ s_1 + s_2 > 1/2, \\ s_1, s_2, s_3 \geq 0, \; s_1, s_2 < 1. \end{cases} \tag{3.8}$$

Note that in view of (3.6) at most one of s_1, s_2, s_3 can be ≤ 0. But by the triangle inequality in Fourier space (i.e., Leibniz rule), we can always reduce the problem to the case $s_1, s_2, s_3 \geq 0$. Indeed, if $s_3 \leq 0$, then (3.5) reduces to

$$H^{s_1+s_3,b} \cdot H^{s_2,d} \hookrightarrow L^2 \quad \text{and} \quad H^{s_1,d} \cdot H^{s_2+s_3,d} \hookrightarrow L^2.$$

In view of (3.8) these estimates hold for s_1, s_2, s_3 satisfying (3.6). If $s_1 \leq 0$, then (3.5) reduces to

$$H^{0,d} \cdot H^{s_1+s_2,d} \hookrightarrow H^{-s_3,0} \quad \text{and} \quad H^{0,d} \cdot H^{s_2,d} \hookrightarrow H^{-(s_1+s_3),0},$$

and again by (3.8) these hold for s_1, s_2, s_3 satisfying (3.6). The case $s_2 \leq 0$ is symmetrical to that of $s_1 \leq 0$.

It remains to show (3.5) for s_1, s_2, s_3 satisfying (3.7). Write $s_1 + s_2 + s_3 = 1 + \varepsilon$ where $\varepsilon > 0$. We consider three cases: $s_3 \leq 0$, $0 < s_3 < 1/2$ and $s_3 \geq 1/2$.

Case 1: $s_3 \leq 0$. In this case (using $s_3 = 1 + \varepsilon - s_1 - s_2$), (3.5) reduces to

$$H^{1+\varepsilon-s_2,d} \cdot H^{s_2,d} \hookrightarrow L^2 \quad \text{and} \quad H^{s_1,d} \cdot H^{1+\varepsilon-s_1,d} \hookrightarrow L^2,$$

which hold by Theorem 3.2 (since $s_1, s_2 \geq 0$, by (3.7) and the assumption $s_3 \leq 0$).

Case 2: $0 < s_3 < 1/2$. Here we consider three subcases: $s_1 \leq 0$, $s_2 \leq 0$ and $s_1, s_2 \geq 0$. By symmetry it suffices to consider $s_1 \leq 0$ and $s_1, s_2 \geq 0$. Assume $s_1 \leq 0$; then (using $s_1 = 1 + \varepsilon - s_2 - s_3$) (3.5) reduces to

$$H^{0,d} \cdot H^{1+\varepsilon-s_3,d} \hookrightarrow H^{-s_3,0} \tag{3.9}$$

$$H^{0,d} \cdot H^{1+\varepsilon-s_1-s_3,d} \hookrightarrow H^{-(s_1+s_3),0}. \tag{3.10}$$

Since (3.6) implies (3.5), we have

$$H^{0,d} \cdot H^{1/2+\varepsilon,d} \hookrightarrow H^{-(1/2-\varepsilon),0} \hookrightarrow H^{-1/2,0}.$$

Interpolating between this and

$$H^{0,d} \cdot H^{1+\varepsilon,d} \hookrightarrow L^2,$$

with $\theta = 2s_3$, gives (3.9) (note that $\theta \in (0,1)$ by the assumption on s_3). The same interpolation, but now with $\theta = 2(s_1 + s_3)$ ($\theta \in [0,1]$ by the assumption on s_1 and s_3), gives (3.10).

Assume next $s_1, s_2 \geq 0$. Choose $0 \leq s_1' \leq s_1$, $0 \leq s_2' \leq s_2$ such that $s_1', s_2' < 1$ and $s_1'+s_2'+s_3 = 1$. Indeed, we can choose such s_1' and s_2' as follows: If $s_2+s_3 \leq 1$, take $s_1' := 1-(s_2+s_3) \in [0,1)$ and $s_2' := s_2 \in [0,1)$. If $s_2+s_3 > 1$, take $s_1' := 0$ and $s_2' := 1-s_3 \in (1/2,1)$. Then the problem reduces to

$$H^{s_1',d} \cdot H^{s_2',d} \hookrightarrow H^{-s_3,0},$$

which holds since (3.6) implies (3.5).
Case 3: $s_3 \geq 1/2$. Take $s_3' = 1/2-\delta$, where $\delta > 0$ is chosen such that $s_1+s_2+s_3' > 1$ (this is possible due to the assumption $s_1+s_2 > 1/2$ in (3.5)). Then

$$H^{-s_3',0} \hookrightarrow H^{-s_3,0},$$

so the problem reduces to case 2 for s_1, s_2 and s_3'. □

We also need the following product law for the Wave Sobolev spaces.

Theorem 3.5 ([16]). *Let $t_1, t_2, t_3 \in \mathbb{R}$. Then*

$$H^{t_1,d_1} \cdot H^{t_2,d_2} \hookrightarrow H^{-t_3,-d_3} \tag{3.11}$$

provided

$$\begin{gathered} t_1+t_2+t_3 > 3/2, \\ t_1+t_2 \geq 0, \quad t_2+t_3 \geq 0, \quad t_1+t_3 \geq 0 \\ d_1+d_2+d_3 > 1/2, \\ d_1, d_2, d_3 \geq 0. \end{gathered} \tag{3.12}$$

Moreover, we can allow $t_1+t_2+t_3 = 3/2$, provided $t_j \neq 3/2$ for $1 \leq j \leq 3$. Similarly, we may take $d_1+d_2+d_3 = 1/2$, provided $d_j \neq 1/2$ for $1 \leq j \leq 3$.

Proof. In view of (3.12), at most one of t_1, t_2, t_3 can be negative. But by the same Leibniz rule as in the proof of Theorem 3.4 this can be reduced to the case $t_1, t_2, t_3 \geq 0$, which was proved in [16, Proposition 10]. □

Theorem 3.6. *Let $\epsilon > 0$. Then*

$$H^{1/2+\epsilon,1/2^+} \cdot H^{\epsilon,1/2^+} \hookrightarrow H^{-1+\epsilon,1/2}. \tag{3.13}$$

Proof. The embedding (3.13) is equivalent to the estimate

$$I \lesssim \|u\|_{L^2(\mathbb{R}^{1+3})} \|u\|_{L^2(\mathbb{R}^{1+3})},$$

where

$$I = \Big\| \int_{\mathbb{R}^{1+3}} \frac{\langle |\tau|-|\xi|\rangle^{1/2} \widetilde{u}(\lambda,\eta)\widetilde{v}(\tau-\lambda,\xi-\eta)}{\langle\xi\rangle^{1-\epsilon}\langle\eta\rangle^{1/2+\epsilon}\langle\xi-\eta\rangle^{\epsilon}\langle|\lambda|-|\eta|\rangle^{1/2^+}\langle|\tau-\lambda|-|\xi-\eta|\rangle^{1/2^+}} d\lambda d\eta \Big\|_{L^2_{(\tau,\xi)}}.$$

By the 'hyperbolic' Leibniz rule (see [12] lemma 3.2), we reduce this to three estimates

$$H^{1/2+\varepsilon,0} \cdot H^{\varepsilon,1/2^+} \hookrightarrow H^{-1+\varepsilon,0},$$

$$H^{1/2+\varepsilon,1/2^+} \cdot H^{\varepsilon,0} \hookrightarrow H^{-1+\varepsilon,0},$$

and (using also transfer principle to one free wave estimate)

$$\big\| |D_x|^{-1+\epsilon} D_-^{1/2}(uv) \big\|_{L^2} \lesssim \big\| |D_x|^{1/2+\epsilon/2} u_0 \big\|_{L^2} \big\| |D_x|^{\epsilon/2} v_0 \big\|_{L^2},$$

where $u = e^{\pm it|D_x|}u_0$ and $v = e^{\pm it|D_x|}v_0$, and the operator D_- corresponds to the symbol $||\tau|-|\xi||$. The first two estimates hold by Theorem 3.5, and the last estimate holds by Theorem 1.1 in [10]. □

4. Interpolation results

By bilinear interpolation between special cases of Theorems 3.4 and 3.5, and at one point Theorem 3.6, we obtain a series of estimates which will be useful in the proof of Theorem 2.1. For $a, b, c, \alpha, \beta, \gamma \in \mathbb{R}$, and $\epsilon > 0$ sufficiently small, we obtain the following estimates (the proof is given below):

$$H^{a,\alpha} \cdot H^{0,1/2^+} \hookrightarrow H^{-c,0} \quad \text{if} \quad \begin{cases} a, c, \alpha \ge 0, \\ 3\min(a/2, \alpha) + c > 3/2. \end{cases} \tag{4.1}$$

$$H^{a,\alpha} \cdot H^{0,1/2^+} \hookrightarrow H^{-c,0} \quad \text{if} \quad \begin{cases} a, \alpha \ge 0,\ c \ge 1/2, \\ \min(a, \alpha) + c/2 > 3/4. \end{cases} \tag{4.2}$$

$$H^{a,\alpha} \cdot H^{0,\beta} \hookrightarrow H^{0,-\gamma} \quad \text{if} \quad \begin{cases} a > 1,\ \alpha > 0,\ \beta, \gamma \ge 0, \\ a + \min(\alpha, \beta) > 3/2, \\ \gamma + \min(\alpha, \beta) > 1/2. \end{cases} \tag{4.3}$$

$$H^{a,1/2^+} \cdot H^{b,\beta} \hookrightarrow H^{-c,0} \quad \text{if} \quad \begin{cases} c, \beta \ge 0,\ a, b > 0, \\ a + b = 1, \\ c + \beta > 1/2. \end{cases} \tag{4.4}$$

$$H^{1,1/2^+} \cdot H^{0,\beta} \hookrightarrow H^{-c,0} \quad \text{if} \quad \begin{cases} \beta \ge 0,\ c > 0, \\ c + \beta > 1/2. \end{cases} \tag{4.5}$$

$$H^{a,\alpha} \cdot H^{b,1/2^+} \hookrightarrow H^{-c,0} \quad \text{if} \quad \begin{cases} a, b, \alpha \ge 0,\ c \ge 1/2, \\ \min(a, \alpha) + 2b/3 > 1/2, \\ \min(a, \alpha) + 2c > 3/2. \end{cases} \tag{4.6}$$

$$H^{a,1/2^+} \cdot H^{b,\beta} \hookrightarrow L^2 \quad \text{if} \quad \begin{cases} b, \beta \ge 0,\ a \ge 1/2, \\ a + 2\min(b, \beta) > 3/2. \end{cases} \tag{4.7}$$

$$H^{a,1/2^+} \cdot H^{1/2,\beta} \hookrightarrow L^2 \quad \text{if} \quad \begin{cases} \beta \ge 0,\ a \ge 1/2, \\ a + \beta > 1. \end{cases} \tag{4.8}$$

$$H^{a,1/2^+} \cdot H^{\epsilon,\beta} \hookrightarrow H^{-1+\epsilon,-\gamma} \quad \text{if} \quad \begin{cases} a, \beta \ge 0,\ \gamma \ge -1/2, \\ \min(a, \beta) + \gamma/2 > 1/4. \end{cases} \tag{4.9}$$

$$H^{1/2,1/2^+} \cdot H^{0,\beta} \hookrightarrow H^{-c,0} \quad \text{if} \quad \begin{cases} \beta \ge 0,\ c > 1/2, \\ c + \beta > 1. \end{cases} \tag{4.10}$$

Proof of (4.1)–(4.10). The parameter $\varepsilon > 0$ is assumed to be sufficiently small. To prove (4.1) we interpolate between

$$H^{1+\varepsilon,1/2+\varepsilon} \cdot H^{0,1/2^+} \hookrightarrow L^2,$$
$$L^2 \cdot H^{0,1/2^+} \hookrightarrow H^{-(3/2+\varepsilon),0}.$$

This gives

$$H^{(1+\varepsilon)(1-\theta),(1/2+\varepsilon)(1-\theta)} \cdot H^{0,1/2^+} \hookrightarrow H^{-(3/2+\varepsilon)\theta,0}$$

for $\theta \in [0,1]$. Now, if there exists $\theta \in [0,1]$ such that $a \ge (1+\varepsilon)(1-\theta)$ $(\Leftrightarrow \theta \ge 1 - a/(1+\varepsilon))$, $\alpha \ge (1/2+\varepsilon)(1-\theta)$ $(\Leftrightarrow \theta \ge 1 - 2\alpha/(1+2\varepsilon))$ and $c \ge (3/2+\varepsilon)\theta$ $(\Leftrightarrow \theta \le 2c/(3+2\varepsilon))$, then we have $H^{a,\alpha} \cdot H^{0,1/2^+} \hookrightarrow H^{-c,0}$. But such a $\theta \in [0,1]$ exists if $a, \alpha, c \geq 0$, $3a + 2c \geq 3 + 5\varepsilon - 2\varepsilon(a + c) + 2\varepsilon^2$ and $2c + 6\alpha \geq 3 + 8\varepsilon - 2\varepsilon(c + \alpha) + 4\varepsilon^2$. Since $\varepsilon > 0$ is very small, it is enough to have $a, \alpha, c \geq 0$, $3a + 2c > 3$ and $2c + 6\alpha > 3$. This proves (4.1). Interpolation between

$$H^{1/2+\varepsilon,1/2+\varepsilon} \cdot H^{0,1/2^+} \hookrightarrow H^{-(1/2+\varepsilon),0},$$
$$L^2 \cdot H^{0,1/2^+} \hookrightarrow H^{-(3/2+\varepsilon),0},$$

with a similar argument as above, proves (4.2).

To prove (4.3), we interpolate between

$$H^{1+\varepsilon,1/2+\varepsilon} \cdot H^{0,1/2+\varepsilon} \hookrightarrow L^2,$$
$$H^{3/2+\varepsilon,\varepsilon} \cdot L^2 \hookrightarrow H^{0,-(1/2-\varepsilon)}.$$

This gives

$$H^{(1+\varepsilon)(1-\theta)+(3/2+\varepsilon)\theta,(1/2+\varepsilon)(1-\theta)+\varepsilon\theta} \cdot H^{0,(1/2+\varepsilon)(1-\theta)} \hookrightarrow H^{0,-(1/2-\varepsilon)\theta},$$

for $\theta \in [0,1]$. If there exists $\theta \in [0,1]$ such that $a \geq (1+\varepsilon)(1-\theta) + (3/2+\varepsilon)\theta$, $\alpha \geq (1/2+\varepsilon)(1-\theta) + \varepsilon\theta$, $\beta \geq (1/2+\varepsilon)(1-\theta)$ and $\gamma \geq (1/2-\varepsilon)\theta$, then we have

$$H^{a,\alpha} \cdot H^{0,\beta} \hookrightarrow H^{0,-\gamma}.$$

By a similar argument as in the proof of (4.1), such a $\theta \in [0,1]$ exists if $a > 1$, $\alpha > 0$, $\beta, \gamma \geq 0$, $a + \alpha > 3/2$, $a + \beta > 3/2$, $\alpha + \gamma > 1/2$ and $\beta + \gamma > 1/2$. This proves (4.3).

To prove (4.4), we interpolate between

$$H^{a,1/2^+} \cdot H^{b,1/2+\varepsilon} \hookrightarrow L^2,$$
$$H^{a,1/2^+} \cdot H^{b,0} \hookrightarrow H^{-1/2,0},$$

which both hold if $a + b = 1$, $a, b > 0$, by Theorems 3.4 and 3.5, respectively. This gives

$$H^{a,1/2^+} \cdot H^{b,(1/2+\varepsilon)(1-\theta)} \hookrightarrow H^{-\theta/2,0}$$

for $\theta \in [0,1]$. If there exists $\theta \in [0,1]$ such that $\beta \geq (1/2+\varepsilon)(1-\theta)$ and $c \geq \theta/2$, then we have

$$H^{a,1/2^+} \cdot H^{b,\beta} \hookrightarrow H^{-c,0},$$

for $a + b = 1$, $a, b > 0$. By a similar argument as before such a $\theta \in [0,1]$ exists if $\beta, c \geq 0$ and $c + \beta > 1/2$.

For (4.5)–(4.10), similar arguments as in the proof of (4.1) are used, so we only give the interpolation pairs, which give the desired estimate when interpolated.

For (4.5), we use

$$H^{1,1/2^+} \cdot H^{0,1/2+\varepsilon} \hookrightarrow H^{-\varepsilon,0},$$
$$H^{1,1/2^+} \cdot L^2 \hookrightarrow H^{-1/2,0}.$$

For (4.6), we interpolate between

$$H^{1/2+\varepsilon,1/2+\varepsilon} \cdot H^{0,1/2^+} \hookrightarrow H^{-(1/2-\varepsilon),0},$$
$$L^2 \cdot H^{3/4,1/2^+} \hookrightarrow H^{-3/4,0}.$$

For (4.7), we interpolate between

$$H^{1/2,1/2^+} \cdot H^{1/2,1/2+\varepsilon} \hookrightarrow L^2,$$

$$H^{3/2+\varepsilon,1/2^+} \cdot L^2 \hookrightarrow L^2.$$

For (4.8), we interpolate between

$$H^{1/2,1/2^+} \cdot H^{1/2,1/2+\varepsilon} \hookrightarrow L^2,$$
$$H^{1,1/2^+} \cdot H^{1/2,0} \hookrightarrow L^2.$$

For (4.9), we interpolate between

$$H^{0,1/2^+} \cdot H^{\varepsilon,0} \hookrightarrow H^{-(1-\varepsilon),-(1/2+\varepsilon)},$$
$$H^{1/2+\varepsilon,1/2^+} \cdot H^{\varepsilon,1/2+\varepsilon} \hookrightarrow H^{-(1-\varepsilon),1/2},$$

where the second embedding holds by Theorem 3.6. For (4.10), we interpolate between

$$H^{1/2,1/2^+} \cdot H^{0,1/2+\varepsilon} \hookrightarrow H^{-(1/2+\varepsilon),0},$$
$$H^{1/2,1/2^+} \cdot L^2 \hookrightarrow H^{-1,0},$$

where the first embedding does not directly follow from Theorems 3.4 and 3.5, but from interpolation between

$$H^{1/2+\varepsilon,1/2^+} \cdot H^{0,1/2+\varepsilon} \hookrightarrow H^{-(1/2-\varepsilon),0},$$
$$H^{0,1/2^+} \cdot H^{0,1/2+\varepsilon} \hookrightarrow H^{-(3/2+\varepsilon),0},$$

which gives

$$H^{(1/2+\varepsilon)(1-\theta),1/2^+} \cdot H^{0,1/2+\varepsilon} \hookrightarrow H^{-(1/2-\varepsilon)(1-\theta)-(3/2+\varepsilon)\theta,0}$$

for $\theta \in [0,1]$. Choosing $\theta = \frac{2\varepsilon}{1+2\varepsilon}$ gives the desired estimate. $\square$

In the following two sections, we shall present the proof of the bilinear estimates (2.6′) and (2.7) for all $\psi, \psi' \in \mathcal{S}(\mathbb{R}^{1+3})$ provided (r,s), ρ and σ are as in (2.8), (2.9) and (2.10) respectively. These will imply Theorem 2.1. First we prove (2.7), and then (2.6′). Note that using (2.3) we can reduce $X^{s,b}$ type estimates to $H^{s,b}$ type estimates, which we shall do in the following two sections.

5. Proof of (2.7)

Without loss of generality we take $[\pm] = +$. Assume $\psi, \psi' \in \mathcal{S}(\mathbb{R}^{1+3})$. Using (3.1), we can reduce (2.7) (write $\rho = 1/2 + \varepsilon$, as in (2.9)) to

$$I^{\pm} \lesssim \|\psi\|_{X_+^{s,\sigma}} \|\psi'\|_{X_\pm^{s,\sigma}} ,$$

where

$$I^{\pm} = \Big\| \int_{\mathbb{R}^{1+3}} \frac{\theta_\pm}{\langle \xi \rangle^{1-r} \langle |\tau| - |\xi| \rangle^{1/2-2\varepsilon}} |\widetilde{\psi}(\lambda,\eta)| |\widetilde{\psi}'(\lambda-\tau,\eta-\xi)| \, d\lambda \, d\eta \Big\|_{L^2_{\tau,\xi}},$$

and $\theta_\pm = \measuredangle\big(\eta, \pm(\eta-\xi)\big)$. The low frequency case, where $\min(|\eta|, |\eta-\xi|) \le 1$ in $I^\pm$, follows from a similar argument as in [2], and hence we do not consider this question here. From now on we assume that in $I^\pm$,

$$|\eta|, |\eta - \xi| \ge 1. \tag{5.1}$$

We shall use the following notation in order to make expressions manageable:

$$F(\lambda,\eta) = \langle \eta \rangle^s \langle \lambda + |\eta| \rangle^\sigma |\widetilde{\psi}(\lambda,\eta)|, \quad G_\pm(\lambda,\eta) = \langle \eta \rangle^s \langle \lambda \pm |\eta| \rangle^\sigma |\widetilde{\psi}'(\lambda,\eta)|,$$
$$\Gamma = |\tau| - |\xi|, \quad \Theta = \lambda + |\eta|, \quad \Sigma_\pm = \lambda - \tau \pm |\eta - \xi|,$$

$$\kappa_+ = |\xi| - \big||\eta| - |\eta-\xi|\big|, \quad \kappa_- = |\eta| + |\eta-\xi| - |\xi|.$$

We shall need the estimates (see [1]):

$$\theta_+^2 \sim \frac{|\xi|\kappa_+}{|\eta||\eta-\xi|}, \quad \theta_-^2 \sim \frac{(|\eta|+|\eta-\xi|)\kappa_-}{|\eta||\eta-\xi|} \sim \frac{\kappa_-}{\min(|\eta|,|\eta-\xi|)}. \tag{5.2}$$

$$\kappa_\pm \le 2\min(|\eta|,|\eta-\xi|), \tag{5.3}$$

$$\kappa_\pm \le |\Gamma| + |\Theta| + |\Sigma_\pm|. \tag{5.4}$$

5.1. **Estimate for I^+.** By (5.2), and using (5.1)

$$I^+ \lesssim \Big\| \int_{\mathbb{R}^{1+3}} \frac{\kappa_+^{1/2} F(\lambda,\eta) G_+(\lambda-\tau,\eta-\xi)}{\langle\xi\rangle^{1/2-r}\langle\eta\rangle^{1/2+s}\langle\eta-\xi\rangle^{1/2+s}\langle\Gamma\rangle^{1/2-2\varepsilon}\langle\Theta\rangle^\sigma\langle\Sigma_+\rangle^\sigma}\, d\lambda\, d\eta \Big\|_{L^2_{\tau,\xi}}.$$

By (5.3) and (5.4)

$$\kappa_+^{1/2} \lesssim |\Gamma|^{1/2-2\varepsilon}\min(|\eta|,|\eta-\xi|)^{2\varepsilon} + |\Theta|^{1/2} + |\Sigma_+|^{1/2}.$$

Moreover, by symmetry we may assume $|\eta| \ge |\eta-\xi|$ in I^+. By (2.8), $r > 1/2$, so we have by the triangle inequality

$$\langle\xi\rangle^{r-1/2} \lesssim \langle\eta\rangle^{r-1/2} + \langle\eta-\xi\rangle^{r-1/2} \lesssim \langle\eta\rangle^{r-1/2}. \tag{5.5}$$

Hence the estimate reduces to

$$I_j^+ \lesssim \|F\|_{L^2}\,\|G_+\|_{L^2}\,, \quad j = 1,2,3,$$

where

$$I_1^+ = \Big\| \int_{\mathbb{R}^{1+3}} \frac{F(\lambda,\eta) G_+(\lambda-\tau,\eta-\xi)}{\langle\eta\rangle^{1+s-r}\langle\eta-\xi\rangle^{1/2+s-2\varepsilon}\langle\Theta\rangle^\sigma\langle\Sigma_+\rangle^\sigma}\, d\lambda\, d\eta \Big\|_{L^2_{\tau,\xi}},$$

$$I_2^+ = \Big\| \int_{\mathbb{R}^{1+3}} \frac{F(\lambda,\eta) G_+(\lambda-\tau,\eta-\xi)}{\langle\eta\rangle^{1+s-r}\langle\eta-\xi\rangle^{1/2+s}\langle\Gamma\rangle^{1/2-2\varepsilon}\langle\Theta\rangle^{\sigma-1/2}\langle\Sigma_+\rangle^\sigma}\, d\lambda\, d\eta \Big\|_{L^2_{\tau,\xi}},$$

$$I_3^+ = \Big\| \int_{\mathbb{R}^{1+3}} \frac{F(\lambda,\eta) G_+(\lambda-\tau,\eta-\xi)}{\langle\eta\rangle^{1+s-r}\langle\eta-\xi\rangle^{1/2+s}\langle\Gamma\rangle^{1/2-2\varepsilon}\langle\Theta\rangle^{\sigma}\langle\Sigma_+\rangle^{\sigma-1/2}}\, d\lambda\, d\eta \Big\|_{L^2_{\tau,\xi}}.$$

5.1.1. *Estimate for I_1^+.* The problem reduces to

$$H^{1+s-r,\sigma} \cdot H^{s+1/2-2\varepsilon,\sigma} \hookrightarrow L^2,$$

which holds by Theorem 3.4 for all $1/2 < \sigma < 1$ provided the conditions

$$s > -1/2 \quad r < 1/2 + 2s \quad \text{and} \quad r \le 1 + s$$

are satisfied, which they are by (2.8), and provided also that $\varepsilon > 0$ is sufficiently small, which is tacitly assumed in the following discussion.

5.1.2. *Estimate for I_2^+.* We assume that $|\Gamma| \lesssim \min(|\eta|,|\eta-\xi|) = |\eta-\xi|$, since otherwise I^+ reduces to I_1^+ in view of (5.3). Giving up the weight $\langle\Theta\rangle^{-\sigma+1/2}$ in the integral, we get

$$I_2^+ \lesssim \Big\| \int_{\mathbb{R}^{1+3}} \frac{F(\lambda,\eta) G_+(\lambda-\tau,\eta-\xi)}{\langle\eta\rangle^{1+s-r}\langle\eta-\xi\rangle^{1/2+s-3\varepsilon}\langle\Sigma_+\rangle^\sigma\langle\Gamma\rangle^{1/2+\varepsilon}}\, d\lambda\, d\eta \Big\|_{L^2_{\tau,\xi}}.$$

Then the problem reduces to

$$H^{1+s-r,0} \cdot H^{1/2+s-3\varepsilon,\sigma} \hookrightarrow H^{0,-1/2-\varepsilon}.$$

But by duality this is equivalent to the embedding

$$H^{0,1/2+\varepsilon} \cdot H^{1/2+s-3\varepsilon,\sigma} \hookrightarrow H^{-1-s+r,0},$$

which holds by Theorem 3.4 for all $1/2 < \sigma < 1$ provided

$$s > 0, \quad r < 1/2 + 2s \quad \text{and} \quad r \le 1 + s,$$

which are true by (2.8).

5.1.3. *Estimate for* I_3^+. As in the argument as for I_2^+, we assume that $|\Gamma| \lesssim \min(|\eta|, |\eta - \xi|) = |\eta - \xi|$. Then

$$I_3^+ \lesssim \Big\| \int_{\mathbb{R}^{1+3}} \frac{F(\lambda,\eta)G_+(\lambda-\tau,\eta-\xi)}{\langle\eta\rangle^{1+s-r}\langle\eta-\xi\rangle^{1/2+s-3\varepsilon}\langle\Gamma\rangle^{1/2+\varepsilon}\langle\Theta\rangle^{\sigma}\langle\Sigma_+\rangle^{\sigma-1/2}}\, d\lambda\, d\eta \Big\|_{L^2_{\tau,\xi}}.$$

Hence the problem reduces to proving

$$H^{1+s-r,\sigma} \cdot H^{1/2+s-3\varepsilon,\sigma-1/2} \hookrightarrow H^{0,-1/2-\varepsilon}. \tag{5.6}$$

By duality this is equivalent to the embedding

$$H^{1+s-r,\sigma} \cdot H^{0,1/2+\varepsilon} \hookrightarrow H^{-1/2-s+3\varepsilon,-\sigma+1/2}, \tag{5.7}$$

which holds by Theorem 3.4 if $s > -1/2$ and $r < \min(1/2+2s, 1/2+s)$. But $s > 0$ by (2.8), so (5.7) holds for $r < 1/2 + s$ and all $1/2 < \sigma < 1$.

If $s > 1$ and $r \le 1 + s$ (see figure 1), then (5.7) reduces to

$$H^{0,\sigma} \cdot H^{0,1/2+\varepsilon} \hookrightarrow H^{-1/2-s+3\varepsilon,-\sigma+1/2},$$

which is true by Theorem 3.5 for all $1/2 < \sigma < 1$.

If $s > 1/2$ and $r = 1/2 + s$ (this includes $(s,r) \in DF \cup F$, see figure 1), then (5.7) becomes

$$H^{1/2,\sigma} \cdot H^{0,1/2+\varepsilon} \hookrightarrow H^{-1/2-s+3\varepsilon,-\sigma+1/2},$$

which is true by Theorem 3.5 for all $1/2 < \sigma < 1$.

It remains to prove (5.7) for (see figure 1)

$$(s,r) \in D \cup AD \cup BD \cup R_2 \cup R_4.$$

To do this, we need special choices of σ which will depend on s and r as in (2.10). We shall consider five cases based on these regions. In the rest of the paper, $\theta \in [0,1]$ is an interpolation parameter, $\varrho > 0$ depends on s and r, and $\varepsilon, \delta > 0$ will be chosen sufficiently small, depending on ϱ. We may also assume that $\varrho \gg \delta \gg \varepsilon$.

Case 1: $(s,r) \in R_2$. Then according to (2.10) we choose $\sigma = 1/2 + s$ (note that $1/2 < \sigma < 1$, since $0 < s < 1/2$ in this region). Write $r = 1/2 + 2s - \varrho$; Then (5.7) becomes

$$H^{1/2-s+\varrho,1/2+s} \cdot H^{0,1/2+\varepsilon} \hookrightarrow H^{-1/2-s+3\varepsilon,-s}. \tag{5.8}$$

At $s = \delta$, (5.8) becomes

$$H^{1/2-\delta+\varrho,1/2+\delta} \cdot H^{0,1/2+\varepsilon} \hookrightarrow H^{-1/2-\delta+3\varepsilon,-\delta}, \tag{5.9}$$

which holds by Theorem 3.4. At $s = 1/2 - \delta$, (5.8) becomes

$$H^{\delta+\varrho,1-\delta} \cdot H^{0,1/2+\varepsilon} \hookrightarrow H^{-1+\delta+3\varepsilon,-1/2+\delta}. \tag{5.10}$$

By duality this equivalent to

$$H^{1-\delta-3\varepsilon,1/2-\delta} \cdot H^{0,1/2+\varepsilon} \hookrightarrow H^{-\delta-\varrho,-1+\delta},$$

which is true by (4.1). Now, interpolation between (5.9) and (5.10) with $\theta = \frac{2(s-\delta)}{1-4\delta}$ (note that $0 \le \theta \le 1$ whenever $\delta \le s \le 1/2 - \delta$) gives (5.8).

Case 2: $(s,r) \in AD$. Here $0 < s < 1/2$, $r = 1/2 + s$. According to (2.10) we choose $\sigma = 1/2 + s/3$. Then (5.7) becomes

$$H^{1/2,1/2+s/3} \cdot H^{0,1/2+\varepsilon} \hookrightarrow H^{-1/2-s+3\varepsilon,-s/3},$$

which holds by (4.2) for $s \geq \delta$.
Case 3: $(s,r) \in R_4$. By (2.10), we choose $\sigma = 3/2 - s + 4\varepsilon$. Since $r \geq 1 + s$, (5.7) reduces to (using also duality)

$$H^{1/2+s-3\varepsilon,1-s+4\varepsilon} \cdot H^{0,1/2+\varepsilon} \hookrightarrow H^{0,-3/2+s-4\varepsilon},$$

which holds by (4.3) for $1/2 < s \leq 1$.
Case 4: $(s,r) \in BD$. Here $s = 1/2$ and $1 < r < 3/2$. According to (2.10), we choose $\sigma = 1 - \varepsilon$. Then (5.7) after duality becomes

$$H^{1-3\varepsilon,1/2-\varepsilon} \cdot H^{0,1/2+\varepsilon} \hookrightarrow H^{-3/2+r,-1+\varepsilon}$$

which holds by (4.1).
Case 5: $(s,r) \in D$ (i.e, $(s,r) = (1/2,1)$). Then by (2.10) we have $\sigma = 2/3 + \varepsilon$. Hence (5.7) becomes

$$H^{1/2,2/3+\varepsilon} \cdot H^{0,1/2+\varepsilon} \hookrightarrow H^{-1+3\varepsilon,-1/6-\varepsilon},$$

which is true by (4.2).

5.2. **Estimate for I^-.** Assume first $|\eta| \ll |\eta - \xi|$. Then $|\xi| \sim |\eta - \xi|$, so by (5.2),

$$\theta_-^2 \sim \frac{|\xi|\kappa_-}{|\eta||\eta-\xi|},$$

and hence we have the same estimate for θ_- as for θ_+. Moreover, by (5.3) and (5.4) we have

$$\kappa_-^{1/2} \lesssim \langle\Gamma\rangle^{1/2-2\varepsilon} \min(|\eta|,|\eta-\xi|)^{2\varepsilon} + \langle\Theta\rangle^{1/2} + \langle\Sigma_-\rangle^{1/2}, \tag{5.11}$$

so the analysis of I^+ in the previous subsection applies also to I^-. The same is true if $|\eta| \gg |\eta - \xi|$ or $|\xi| \sim |\eta| \sim |\eta - \xi|$. Hence we assume from now on that

$$|\xi| \ll |\eta| \sim |\eta - \xi|, \tag{5.12}$$

in I^-. By (5.1) and (5.2), we have

$$I^- \lesssim \Big\| \int_{\mathbb{R}^{1+3}} \frac{\kappa_-^{1/2} F(\lambda,\eta) G_-(\lambda-\tau,\eta-\xi)}{\langle\xi\rangle^{1-r}\langle\eta\rangle^s\langle\eta-\xi\rangle^{1/2+s}\langle\Gamma\rangle^{1/2-2\varepsilon}\langle\Theta\rangle^\sigma\langle\Sigma_-\rangle^\sigma}\, d\lambda\, d\eta \Big\|_{L^2_{\tau,\xi}}.$$

By (5.11), the estimate reduces to

$$I_j^- \lesssim \|F\|_{L^2} \|G_-\|_{L^2}, \quad j = 1,2,3,$$

where

$$I_1^- = \Big\| \int_{\mathbb{R}^{1+3}} \frac{F(\lambda,\eta) G_-(\lambda-\tau,\eta-\xi)}{\langle\xi\rangle^{1-r}\langle\eta-\xi\rangle^{1/2+2s-2\varepsilon}\langle\Theta\rangle^\sigma\langle\Sigma_-\rangle^\sigma}\, d\lambda\, d\eta \Big\|_{L^2_{\tau,\xi}},$$

$$I_2^- = \Big\| \int_{\mathbb{R}^{1+3}} \frac{F(\lambda,\eta) G_-(\lambda-\tau,\eta-\xi)}{\langle\xi\rangle^{1-r}\langle\eta-\xi\rangle^{1/2+2s}\langle\Gamma\rangle^{1/2-2\varepsilon}\langle\Theta\rangle^{\sigma-1/2}\langle\Sigma_-\rangle^\sigma}\, d\lambda\, d\eta \Big\|_{L^2_{\tau,\xi}},$$

$$I_3^- = \Big\| \int_{\mathbb{R}^{1+3}} \frac{F(\lambda,\eta) G_-(\lambda-\tau,\eta-\xi)}{\langle\xi\rangle^{1-r}\langle\eta\rangle^{1/2+2s}\langle\Gamma\rangle^{1/2-2\varepsilon}\langle\Theta\rangle^\sigma\langle\Sigma_-\rangle^{\sigma-1/2}}\, d\lambda\, d\eta \Big\|_{L^2_{\tau,\xi}}.$$

By symmetry it suffices to consider I_1^- and I_2^-.

5.2.1. *Estimate for I_1^-.* Here the problem reduces to

$$H^{0,\sigma}\cdot H^{1/2+2s-2\varepsilon,\sigma}\hookrightarrow H^{-1+r,0},$$

which holds by Theorem 3.4 provided

$$r\le 1,\quad s>0,\quad r<1/2+2s,$$

and $\sigma>1/2$. Now assuming $r\ge 1$, which implies $\langle\xi\rangle^{r-1}\lesssim\langle\eta\rangle^{r-1}+\langle\eta-\xi\rangle^{r-1}\sim\langle\eta-\xi\rangle^{r-1}$, the problem reduces to

$$H^{0,\sigma}\cdot H^{3/2+2s-r-2\varepsilon,\sigma}\hookrightarrow L^2,$$

which is true by Theorem 3.4 provided $r<1/2+2s$ and $\sigma>1/2$. Thus, the estimate for I_1^- holds in the desired region described in figure 1.

5.2.2. *Estimate for I_2^-.* We may assume $|\Gamma|\lesssim\min(|\eta|,|\eta-\xi|)\sim|\eta-\xi|$, since otherwise I^- reduces to I_1^-. Giving up the weight $\langle\Theta\rangle$, the problem reduces to

$$L^2\cdot H^{1/2+2s-3\varepsilon,\sigma}\hookrightarrow H^{-1+r,-1/2-\varepsilon}.$$

By duality this is equivalent to the embedding

$$H^{1-r,1/2+\varepsilon}\cdot H^{1/2+2s-3\varepsilon,\sigma}\hookrightarrow L^2,$$

which holds by Theorem 3.4 if

$$r<1,\quad s>-1/4,\quad r<1/2+2s,$$

and $\sigma>1/2$. For $r\ge 1$, using the triangle inequality as in the previous subsection, the problem reduces to

$$H^{0,1/2+\varepsilon}\cdot H^{3/2+2s-r-3\varepsilon,\sigma}\hookrightarrow L^2,$$

which is true by Theorem 3.4 if $r<1/2+2s$ and $\sigma>1/2$. Thus, the estimate for I_2^- holds in the desired region described in figure 1.

6. Proof of (2.6′)

Without loss of generality we take $[\pm]=+$. Assume $\psi,\psi'\in\mathcal{S}(\mathbb{R}^{1+3})$. In view of the null form estimate (3.1), we can reduce (2.6) (write $\rho=1/2+\varepsilon$, as in (2.9)) to

$$J^\pm\lesssim\|\psi\|_{X_+^{s,\sigma}}\|\psi'\|_{X_\pm^{-s,1-\sigma-\varepsilon}}\,, \tag{6.1}$$

where now

$$J^\pm=\Big\|\int_{\mathbb{R}^{1+3}}\frac{\theta_\pm}{\langle\xi\rangle^r\langle|\tau|-|\xi|\rangle^{1/2+\varepsilon}}|\widetilde{\psi}(\lambda,\eta)||\widetilde{\psi}'(\lambda-\tau,\eta-\xi)|\,d\lambda\,d\eta\Big\|_{L^2_{\tau,\xi}},$$

and $\theta_\pm=\measuredangle\big(\eta,\pm(\eta-\xi)\big)$ as before. We use the same notation as in the previous section, except that now

$$G_\pm(\lambda,\eta)=\langle\eta\rangle^{-s}\langle\lambda\pm|\eta|\rangle^{1-\sigma-\varepsilon}|\widetilde{\psi}'(\lambda,\eta)|.$$

The low frequency case, $\min(|\eta|,|\eta-\xi|)\le 1$ in $J^\pm$, follows from a similar argument as in [2], and hence we do not consider this question here. From now on we therefore assume that in $J^\pm$,

$$|\eta|,|\eta-\xi|\ge 1. \tag{6.2}$$

6.1. **Estimate for J^+.** By (5.2) and (6.2),

$$J^+ \lesssim \Big\| \int_{\mathbb{R}^{1+3}} \frac{\kappa_+^{1/2} F(\lambda,\eta) G_+(\lambda-\tau,\eta-\xi)}{\langle\xi\rangle^{r-1/2}\langle\eta\rangle^{1/2+s}\langle\eta-\xi\rangle^{1/2-s}\langle\Gamma\rangle^{1/2+\varepsilon}\langle\Theta\rangle^{\sigma}\langle\Sigma_+\rangle^{1-\sigma-\varepsilon}}\, d\lambda\, d\eta \Big\|_{L^2_{\tau,\xi}}.$$

By (5.3) and (5.4),

$$\kappa_+^{1/2} \lesssim |\Gamma|^{1/2} + |\Theta|^{1/2} + |\Sigma_+|^{1-\sigma-\varepsilon}|\eta-\xi|^{\sigma-1/2+\varepsilon}.$$

Hence the estimate reduces to

$$J_j^+ \lesssim \|F\|_{L^2}\, \|G_+\|_{L^2}\,, \quad j=1,2,3,$$

where

$$J_1^+ = \Big\| \int_{\mathbb{R}^{1+3}} \frac{F(\lambda,\eta) G_+(\lambda-\tau,\eta-\xi)}{\langle\xi\rangle^{r-1/2}\langle\eta\rangle^{1/2+s}\langle\eta-\xi\rangle^{1/2-s}\langle\Theta\rangle^{\sigma}\langle\Sigma_+\rangle^{1-\sigma-\varepsilon}}\, d\lambda\, d\eta \Big\|_{L^2_{\tau,\xi}},$$

$$J_2^+ = \Big\| \int_{\mathbb{R}^{1+3}} \frac{F(\lambda,\eta) G_+(\lambda-\tau,\eta-\xi)}{\langle\xi\rangle^{r-1/2}\langle\eta\rangle^{1/2+s}\langle\eta-\xi\rangle^{1/2-s}\langle\Gamma\rangle^{1/2+\varepsilon}\langle\Theta\rangle^{\sigma-1/2}\langle\Sigma_+\rangle^{1-\sigma-\varepsilon}}\, d\lambda\, d\eta \Big\|_{L^2_{\tau,\xi}},$$

$$J_3^+ = \Big\| \int_{\mathbb{R}^{1+3}} \frac{F(\lambda,\eta) G_+(\lambda-\tau,\eta-\xi)}{\langle\xi\rangle^{r-1/2}\langle\eta\rangle^{1/2+s}\langle\eta-\xi\rangle^{1-s-\sigma-\varepsilon}\langle\Gamma\rangle^{1/2+\varepsilon}\langle\Theta\rangle^{\sigma}}\, d\lambda\, d\eta \Big\|_{L^2_{\tau,\xi}},$$

6.1.1. *Estimate for J_1^+.* The problem reduces to

$$H^{1/2+s,\sigma}\cdot H^{1/2-s,1-\sigma-\varepsilon} \hookrightarrow H^{1/2-r,0}. \tag{6.3}$$

If $s>1$ and $r\geq s$, then (6.3) reduces to

$$H^{1/2+s,\sigma}\cdot H^{1/2-s,1-\sigma-\varepsilon} \hookrightarrow H^{1/2-s,0},$$

which is true by Theorem 3.5, for all $1/2<\sigma<1$.

It remains to prove (6.3) in the region R (see figure 1). We split this into the following five cases:

Case 1: $(s,r)\in R_1$. Then according to (2.10), we choose $\sigma=1/2+s/3$. Write $r=1/2+s/3+\varrho$; (6.3) becomes

$$H^{1/2+s,1/2+s/3}\cdot H^{1/2-s,1/2-s/3-\varepsilon} \hookrightarrow H^{-s/3-\varrho,0},$$

which holds by (4.4) for $0<s<1/2$.

Case 2: $(s,r)\in R_2$. Then by (2.10), we choose $\sigma=1/2+s$. Write $r=1/2+s+\varrho$; (6.3) becomes

$$H^{1/2+s,1/2+s}\cdot H^{1/2-s,1/2-s-\varepsilon} \hookrightarrow H^{-s-\varrho,0},$$

which holds by (4.4) for $0<s<1/2$.

Case 3: $(s,r)\in R_3$. Then according to (2.10), we choose $\sigma=5/6-s/3+\varepsilon$. Writing $r=1/3+2s/3+\varrho$, (6.3) becomes

$$H^{1/2+s,5/6-s/3+\varepsilon}\cdot H^{1/2-s,1/6+s/3-2\varepsilon} \hookrightarrow H^{1/6-2s/3-\varrho,0}. \tag{6.4}$$

At $s=1/2$, (6.4) becomes

$$H^{1,2/3+\varepsilon}\cdot H^{0,1/3-2\varepsilon} \hookrightarrow H^{-1/6-\varrho,0} \tag{6.5}$$

which holds by (4.5). At $s=1$, (6.4) becomes

$$H^{3/2,1/2+\varepsilon}\cdot H^{-1/2,1/2-2\varepsilon} \hookrightarrow H^{-1/2-\varrho,0}, \tag{6.6}$$

which is true by Theorem 3.5. Hence we get (6.4) by interpolating between (6.5) and (6.6) with $\theta=-1+2s$.

Case 4: $(s,r) \in BD$. Here $s = 1/2$ and $1 < r < 3/2$. Then we choose $\sigma = 1 - \varepsilon$ in view of (2.10). Hence (6.3) becomes

$$H^{1,1-\varepsilon} \cdot L^2 \hookrightarrow H^{1/2-r,0},$$

which holds by Theorem 3.5.
Case 5: $(s,r) \in BD$. Then in view of (2.10), we choose $\sigma = 3/2 - s + 4\varepsilon$. Writing $r = 1/2 + s + \varrho$, (6.3) reduces to

$$H^{1/2+s,3/2-s+4\varepsilon} \cdot H^{1/2-s,-1/2+s-5\varepsilon} \hookrightarrow H^{-s-\varrho,0},$$

which is true by Theorem 3.5 for $1/2 < s \le 1$.

6.1.2. *Estimate for J_2^+.* By duality the problem reduces to

$$H^{-1/2+r,1/2+\varepsilon} \cdot H^{1/2-s,1-\sigma-\varepsilon} \hookrightarrow H^{-1/2-s,1/2-\sigma}. \tag{6.7}$$

Assume $s > 1$ and $r \ge s$. Then (6.7) reduces to proving

$$H^{-1/2+s,1/2+\varepsilon} \cdot H^{1/2-s,1-\sigma-\varepsilon} \hookrightarrow H^{-1/2-s,1/2-\sigma},$$

which holds by Theorem 3.5 for all $1/2 < \sigma < 1$.

To prove (6.7) for $(s,r) \in R$, we consider the following five cases.
Case 1: $(s,r) \in R_1$. Then by (2.10), we choose $\sigma = 1/2 + s/3$. Writing $r = 1/2 + s/3 + \varrho$, (6.7) becomes

$$H^{s/3+\varrho,1/2+\varepsilon} \cdot H^{1/2-s,1/2-s/3-\varepsilon} \hookrightarrow H^{-1/2-s,-s/3}.$$

At $s = \delta$, this holds by (4.6), and at $s = 1/2 - \delta$ by (4.9); interpolation implies the intermediate cases.
Case 2: $(s,r) \in R_2$. By (2.10), we choose $\sigma = 1/2 + s$. Then writing $r = 1/2 + s + \varrho$, (6.7) becomes

$$H^{s+\varrho,1/2+\varepsilon} \cdot H^{1/2-s,1/2-s-\varepsilon} \hookrightarrow H^{-1/2-s,-s}.$$

At $s = \delta$, this holds by (4.6), and at $s = 1/2 - \delta$ by Theorem 3.5; the intermediate cases follows by interpolation.
Case 3: $(s,r) \in R_3$. Then according to (2.10), we choose $\sigma = 5/6 - s/3 + \varepsilon$. Write $r = 1/3 + 2s/3 + \varrho$; (6.7) becomes

$$H^{-1/6+2s/3+\varrho,1/2+\varepsilon} \cdot H^{1/2-s,1/6+s/3-2\varepsilon} \hookrightarrow H^{-1/2-s,-1/3+s/3-\varepsilon}. \tag{6.8}$$

At $s = 1/2$, (6.8) reduces to

$$H^{1/6+\varrho,1/2+\varepsilon} \cdot H^{0,1/3-2\varepsilon} \hookrightarrow H^{-1,-1/6}. \tag{6.9}$$

Using the triangle inequality $\langle \eta - \xi \rangle \lesssim \langle \xi \rangle + \langle \eta \rangle$, (6.8) can be reduced to

$$H^{1/6+\varrho-\delta,1/2+\varepsilon} \cdot H^{\delta,1/3-2\varepsilon} \hookrightarrow H^{-1,-1/6}$$

and

$$H^{1/6+\varrho,1/2+\varepsilon} \cdot H^{\delta,1/3-2\varepsilon} \hookrightarrow H^{-1+\delta,-1/6},$$

which both hold by (4.9). At $s = 1$, (6.8) becomes

$$H^{1/2+\varrho,1/2+\varepsilon} \cdot H^{-1/2,1/2-2\varepsilon} \hookrightarrow H^{-3/2,-\varepsilon}, \tag{6.10}$$

which holds by Theorem 3.5. Interpolation between (6.9) and (6.10) with $\theta = 2s - 1$, gives (6.8).

Case 4: $(s,r) \in BD$. We choose $\sigma = 1-\varepsilon$, by (2.10). Then (6.7) becomes

$$H^{-1/2+r,1/2+\varepsilon} \cdot L^2 \hookrightarrow H^{-1,-1/2-\varepsilon},$$

which is true by Theorem 3.5.
Case 5: $(s,r) \in R_4$. Then by (2.10), we choose $\sigma = 3/2-s+4\varepsilon$. Write $r = 1/2+s+\varrho$; (6.7) reduces to

$$H^{s+\varrho,1/2+\varepsilon} \cdot H^{1/2-s,-1/2+s-5\varepsilon} \hookrightarrow H^{-1/2-s,-1+s-4\varepsilon},$$

which holds by Theorem 3.5 for $s > 1/2$.

6.1.3. *Estimate for J_3^+.* By duality the problem reduces to

$$H^{1/2+s,\sigma} \cdot H^{-1/2+r,1/2+\varepsilon} \hookrightarrow H^{-1+s+\sigma+\varepsilon,0}. \tag{6.11}$$

Assume $s > 1$ and $r \geq s$. Then (6.11) reduces to

$$H^{1/2+s,\sigma} \cdot H^{-1/2+s,1/2+\varepsilon} \hookrightarrow H^{-1+s+\sigma+\varepsilon,0},$$

which holds by Theorem 3.4 for all $1/2 < \sigma < 1$.

Next, we prove that (6.11) holds for $(s,r) \in R$.
Case 1: $(s,r) \in R_1$. Then by (2.10), we choose $\sigma = 1/2 + s/3$. Write $r = 1/2 + s/3 + \varrho$; (6.11) becomes

$$H^{1/2+s,1/2+s/3} \cdot H^{s/3+\varrho,1/2+\varepsilon} \hookrightarrow H^{-1/2+4s/3+\varepsilon,0},$$

which is true by Theorem 3.4 for $0 < s < 1/2$.
Case 2: $(s,r) \in R_2$. We choose $\sigma = 1/2+s$, by (2.10). Then writing $r = 1/2+s+\varrho$, (6.11) becomes

$$H^{1/2+s,1/2+s} \cdot H^{s+\varrho,1/2+\varepsilon} \hookrightarrow H^{-1/2+2s+\varepsilon,0}.$$

which holds by Theorem 3.4 for $0 < s < 1/2$.
Case 3: $(s,r) \in R_3$. . Then according to (2.10), we choose $\sigma = 5/6 - s/3 + \varepsilon$. Write $r = 1/3 + 2s/3 + \varrho$; (6.11) becomes

$$H^{1/2+s,5/6-s/3+\varepsilon} \cdot H^{-1/6+2s/3+\varrho,1/2+\varepsilon} \hookrightarrow H^{-1/6+2s/3+2\varepsilon,0},$$

which is true by Theorem 3.4 for $1/2 \leq s \leq 1$.
Case 4: $(s,r) \in BD$. . Here, $s = 1/2$ and $1 < r < 3/2$). By (2.10), we choose $\sigma = 1-\varepsilon$. Then (6.11) becomes

$$H^{1,1-\varepsilon} \cdot H^{-1/2+r,1/2+\varepsilon} \hookrightarrow H^{1/2,0},$$

which holds by Theorem 3.4.
Case 5: $(s,r) \in R_4$. Then by (2.10), we choose $\sigma = 3/2-s+4\varepsilon$. Write $r = 1/2+s+\varrho$; (6.11) becomes

$$H^{1/2+s,3/2-s+4\varepsilon} \cdot H^{s+\varrho,1/2+\varepsilon} \hookrightarrow H^{1/2+5\varepsilon,0},$$

which is true by Theorem 3.4 for $1/2 < s \leq 1$.

6.2. **Estimate for J^-.** By the same argument as in subsection 5.2, we may assume

$$|\xi| \ll |\eta| \sim |\eta - \xi|.$$

Combining this with (5.2) and (6.2), we get

$$J^- \lesssim \Big\| \int_{\mathbb{R}^{1+3}} \frac{\kappa_-^{1/2} F(\lambda,\eta) G_-(\lambda-\tau,\eta-\xi)}{\langle\xi\rangle^r \langle\eta\rangle^{1/4} \langle\eta-\xi\rangle^{1/4} \langle\Gamma\rangle^{1/2+\varepsilon} \langle\Theta\rangle^{\sigma} \langle\Sigma_-\rangle^{1-\sigma-\varepsilon}}\, d\lambda\, d\eta \Big\|_{L^2_{\tau,\xi}}.$$

By (5.3) and (5.4), we get $\kappa_-^{1/2} \lesssim |\Gamma|^{1/2} + |\Theta|^{1/2} + |\Sigma_-|^{1-\sigma-\varepsilon} |\eta-\xi|^{\sigma-1/2+\varepsilon}$. Hence the estimate reduces to

$$J_j^- \lesssim \|F\|_{L^2} \|G_-\|_{L^2}, \quad j = 1,2,3,$$

where

$$J_1^- = \Big\| \int_{\mathbb{R}^{1+3}} \frac{F(\lambda,\eta) G_(\lambda-\tau,\eta-\xi)}{\langle\xi\rangle^r \langle\eta\rangle^{1/2} \langle\Theta\rangle^{\sigma} \langle\Sigma_-\rangle^{1-\sigma-\varepsilon}}\, d\lambda\, d\eta \Big\|_{L^2_{\tau,\xi}},$$
$$J_2^- = \Big\| \int_{\mathbb{R}^{1+3}} \frac{F(\lambda,\eta) G_-(\lambda-\tau,\eta-\xi)}{\langle\xi\rangle^r \langle\eta-\xi\rangle^{1/2} \langle\Gamma\rangle^{1/2+\varepsilon} \langle\Theta\rangle^{\sigma-1/2} \langle\Sigma_-\rangle^{1-\sigma-\varepsilon}}\, d\lambda\, d\eta \Big\|_{L^2_{\tau,\xi}},$$
$$J_3^- = \Big\| \int_{\mathbb{R}^{1+3}} \frac{F(\lambda,\eta) G_-(\lambda-\tau,\eta-\xi)}{\langle\xi\rangle^r \langle\eta\rangle^{1-\sigma-\varepsilon} \langle\Theta\rangle^{\sigma} \langle\Gamma\rangle^{1/2+\varepsilon}}\, d\lambda\, d\eta \Big\|_{L^2_{\tau,\xi}}.$$

6.2.1. *Estimate for J_1^-.* The problem reduces to the estimate

$$H^{1/2,\sigma} \cdot H^{0,1-\sigma-\varepsilon} \hookrightarrow H^{-r,0}. \tag{6.12}$$

If $s > 1$ and $r \geq s$, then (6.12) reduces to

$$H^{1/2,\sigma} \cdot H^{0,1-\sigma-\varepsilon} \hookrightarrow H^{-s,0},$$

which holds by Theorem 3.5 for all $1/2 < \sigma < 1$.

We now prove (6.12) for $(s,r) \in R$.

Case 1: $(s,r) \in R_1$. Then by (2.10), we choose $\sigma = 1/2 + s/3$. Write $r = 1/2 + s/3 + \varrho$; (6.12) becomes

$$H^{1/2,1/2+s/3} \cdot H^{0,1/2-s/3-\varepsilon} \hookrightarrow H^{-1/2-s/3-\varrho,0},$$

which holds by (4.10) for $0 < s < 1/2$.

Case 2: $(s,r) \in R_2$. Then we choose $\sigma = 1/2+s$, by (2.10). Writing $r = 1/2+s+\varrho$, (6.12) becomes

$$H^{1/2,1/2+s} \cdot H^{0,1/2-s-\varepsilon} \hookrightarrow H^{-1/2-s-\varrho,0},$$

which is true by (4.10) for $0 < s < 1/2$.

Case 3: $(s,r) \in R_3$. Then according to (2.10), we choose $\sigma = 5/6 - s/3 + \varepsilon$. Write $r = 1/3 + 2s/3 + \varrho$; (6.12) becomes

$$H^{1/2,5/6-s/3+\varepsilon} \cdot H^{0,1/6+s/3-2\varepsilon} \hookrightarrow H^{-1/3-2s/3-\varrho,0},$$

which holds by (4.10) for $1/2 \leq s \leq 1$.

Case 4: $(s,r) \in BD$. Here $s = 1/2$ and $1 < r < 3/2$. We choose $\sigma = 1-\varepsilon$ by (2.10). Then (6.12) becomes

$$H^{1/2,1-\varepsilon} \cdot L^2 \hookrightarrow H^{-r,0},$$

which is true by Theorem 3.5.

Case 5: $(s,r) \in R_4$. Then by (2.10), we choose $\sigma = 3/2-s+4\varepsilon$. Write $r = 1/2+s+\varrho$; (6.12) becomes

$$H^{1/2,3/2-s+4\varepsilon} \cdot H^{0,-1/2+s-5\varepsilon} \hookrightarrow H^{-1/2-s-\varrho,0},$$

which is true by Theorem 3.5 for $1/2 < s \leq 1$.

6.2.2. *Estimate for J_2^-*. Giving up the weight $\langle\Theta\rangle^{\sigma-1/2}$ and keep duality, the problem reduces to

$$H^{r,1/2+\varepsilon} \cdot H^{1/2,1-\sigma-\varepsilon} \hookrightarrow L^2. \tag{6.13}$$

Assume $s > 1$ and $r \geq s$. Then (6.13) reduces to proving

$$H^{s,1/2+\varepsilon} \cdot H^{1/2,1-\sigma-\varepsilon} \hookrightarrow L^2,$$

which holds by Theorem 3.5 for all $1/2 < \sigma < 1$.

It remains to prove (6.13) for $(s,r) \in R$, which we shall do in the following five cases.

Case 1: $(s,r) \in R_1$. Then by (2.10), we choose $\sigma = 1/2 + s/3$. Write $r = 1/2 + s/3 + \varrho$; (6.13) becomes

$$H^{1/2+s/3+\varrho,1/2+\varepsilon} \cdot H^{1/2,1/2-s/3-\varepsilon} \hookrightarrow L^2.$$

At $s = \delta$, this holds by (4.7), and at $s = 1/2 - \delta$, by (4.8); the intermediate cases follows by interpolation.

Case 2: $(s,r) \in R_2$. We choose $\sigma = 1/2+s$, by (2.10). Then writing $r = 1/2+s+\varrho$, (6.13) becomes

$$H^{1/2+s+\varrho,1/2+\varepsilon} \cdot H^{1/2,1/2-s-\varepsilon} \hookrightarrow L^2.$$

At $s = \delta$, this holds by (4.7), and at $s = 1/2 - \delta$ by Theorem 3.5; interpolation implies the intermediate cases.

Case 3: $(s,r) \in R_3$. Then according to (2.10), we choose $\sigma = 5/6 - s/3 + \varepsilon$. Write $r = 1/3 + 2s/3 + \varrho$; (6.13) becomes

$$H^{1/3+2s/3+\varrho,1/2+\varepsilon} \cdot H^{1/2,1/6+s/3-2\varepsilon} \hookrightarrow L^2,$$

which holds by (4.8) for $1/2 \leq s \leq 1$.

Case 4: $(s,r) \in BD$. We choose $\sigma = 1 - \varepsilon$, by (2.10). Then (6.13) becomes

$$H^{r,1/2+\varepsilon} \cdot H^{1/2,0} \hookrightarrow L^2,$$

which is true by Theorem 3.5.

Case 5: $(s,r) \in R_4$. Then by (2.10), we choose $\sigma = 3/2-s+4\varepsilon$. Write $r = 1/2+s+\varrho$; (6.13) becomes

$$H^{1/2+s+\varrho,1/2+\varepsilon} \cdot H^{1/2,-1/2+s-5\varepsilon} \hookrightarrow L^2.$$

which holds by Theorem 3.5 for $1/2 < s \leq 1$.

6.2.3. *Estimate for J_3^-*. By duality, the problem reduces to

$$H^{r,1/2+\varepsilon} \cdot H^{1-\sigma-\varepsilon,\sigma} \hookrightarrow L^2. \tag{6.14}$$

If $s > 1$ and $r \geq s$, then (6.14) reduces to

$$H^{s,1/2+\varepsilon} \cdot H^{1-\sigma-\varepsilon,1/2+\varepsilon} \hookrightarrow L^2,$$

which holds by Theorem 3.4 for all $1/2 < \sigma < 1$.

We next prove (6.14) for $(s,r) \in R$.

Case 1: $(s,r) \in R_1$. Then by (2.10), we choose $\sigma = 1/2 + s/3$. Write $r = 1/2 + s/3 + \varrho$; (6.14) becomes

$$H^{1/2+s/3+\varrho,1/2+\varepsilon} \cdot H^{1/2-s/3-\varepsilon,1/2+s/3} \hookrightarrow L^2.$$

which holds by Theorem 3.4 for $0 < s < 3/2$.
Case 2: $(s,r) \in R_2$. We choose $\sigma = 1/2+s$, by (2.10). Then writing $r = 1/2+s+\varrho$, (6.14) becomes

$$H^{1/2+s+\varrho,1/2+\varepsilon} \cdot H^{1/2-s-\varepsilon,1/2+s} \hookrightarrow L^2,$$

which holds by Theorem 3.4 for $0 < s < 1/2$.
Case 3: $(s,r) \in R_3$. Then according to (2.10), we choose $\sigma = 5/6 - s/3 + \varepsilon$. Write $r = 1/3 + 2s/3 + \varrho$; (6.14) becomes

$$H^{1/3+2s/3+\varrho,1/2+\varepsilon} \cdot H^{1/6+s/3-2\varepsilon,1/2+\varepsilon} \hookrightarrow L^2.$$

which holds by Theorem 3.4 for $s \geq 1/2$.
Case 4: $(s,r) \in BD$. Then by (2.10), we choose $\sigma = 1 - \varepsilon$. Hence (6.14) becomes

$$H^{r,1/2+\varepsilon} \cdot H^{0,1/2+\varepsilon} \hookrightarrow L^2,$$

which is true by Theorem 3.4.
Case 5: $(s,r) \in R_1$. Then by (2.10), we choose $\sigma = 3/2-s+4\varepsilon$. Write $r = 1/2+s+\varrho$; (6.14) becomes

$$H^{1/2+s+\varrho,1/2+\varepsilon} \cdot H^{-1/2+s-5\varepsilon,1/2+\varepsilon} \hookrightarrow L^2.$$

which holds by Theorem 3.4 for $s > 1/2$.

7. Counterexamples

Here we prove optimality conditions on s and r in Theorem 1.1, as far as iteration in the spaces $X_\pm^{s,\sigma}$, $H^{r,\rho}$ is concerned. To be precise, we prove:

Theorem 7.1. *If $s \leq 0$ or $r \leq \frac{1}{2}$ or $r < s$ or $r > 1+s$ or $r > \frac{1}{2} + 2s$, then for all $\sigma, \rho \in \mathbb{R}$ and $\varepsilon > 0$, at least one of the estimates (2.6′) or (2.7) fails.*

More generally, we prove:

Theorem 7.2. *Let $a_1, a_2, a_3, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$. If the 4-spinor estimate*

$$\|\langle \beta P_+(D_x)\psi, P_\pm(D_x)\psi'\rangle\|_{H^{-a_3,-\alpha_3}} \lesssim \|\psi\|_{X_+^{a_1,\alpha_1}} \|\psi'\|_{X_\pm^{a_2,\alpha_2}},$$

holds for all $\psi, \psi' \in \mathcal{S}(\mathbb{R}^{1+3})$, then:

$$a_1 + a_2 + a_3 \geq \frac{1}{2}, \tag{7.1}$$

$$\frac{a_1+\alpha_1}{2} + a_2 + a_3 \geq \frac{3}{4} \tag{7.2}$$

$$a_1 + \frac{a_2+\alpha_2}{2} + a_3 \geq \frac{3}{4}, \tag{7.3}$$

$$a_1 + a_3 \geq 0. \tag{7.4}$$

$$a_2 + a_3 \geq 0. \tag{7.5}$$

$$a_1 + a_2 + \alpha_3 \geq 0. \tag{7.6}$$

7.1. **Proof of Theorem 7.1.** Applying (7.1) and (7.5) in Theorem 7.2 to (2.6′), with $(a_1, a_2, a_3, \alpha_1, \alpha_2, \alpha_3) = (s, -s, r, \sigma, 1-\sigma-\varepsilon, \rho)$, we see that the conditions $r \geq 1/2$ and $r \geq s$ are necessary. Similarly, we apply (7.1) and (7.5) in Theorem 7.2 to (2.7), with $(a_1, a_2, a_3, \alpha_1, \alpha_2, \alpha_3) = (s, s, 1-r, \sigma, \sigma, 1-\rho-\varepsilon)$, to obtain the necessary conditions $r \leq 1/2+2s$ and $r \leq 1+s$. We further apply the summation of (7.2) and (7.3) to (2.6′) to obtain the necessary condition $r > 1/2$ ($r \geq 1/2 + \varepsilon/4$), which is stronger than $r \geq 1/2$. Finally, we combine the necessary conditions $r > 1/2$ and $r \leq 1/2 + 2s$ to conclude that $s > 0$ is also a necessary condition.

7.2. **Proof of Theorem 7.2.** The following counterexamples are directly adapted from those for the 2d case in [2], and depend on a large, positive parameter L going to infinity. We choose $A, B, C \subset \mathbb{R}^3$, depending on L and concentrated along the ξ_1-direction, with the property

$$\eta \in A,\ \xi \in C \implies \eta - \xi \in B. \tag{7.7}$$

Using these sets, we then construct ψ and ψ' depending on L, such that

$$\frac{\|\langle \beta P_+(D_x)\psi, P_\pm(D_x)\psi'\rangle\|_{H^{-a_3,-\alpha_3}}}{\|\psi\|_{X_+^{a_1,\alpha_1}} \|\psi'\|_{X_\pm^{a_2,\alpha_2}}} \gtrsim \frac{1}{L^\delta}, \tag{7.8}$$

for some $\delta = \delta(a_1, a_2, a_3, \alpha_1, \alpha_2, \alpha_3)$. This inequality will lead to the necessary condition $\delta \geq 0$.

Let us take the plus sign in (7.8) for the moment. Later, we will also use the minus sign. Assuming A, B, C have been chosen, we set

$$\widetilde{\psi}(\lambda, \eta) = \mathbf{1}_{\lambda+\eta_1=O(1)} \mathbf{1}_{\eta \in A} v_+(\eta), \tag{7.9}$$

$$\widetilde{\psi}'(\lambda-\tau, \eta-\xi) = \mathbf{1}_{\lambda-\tau+\eta_1-\xi_1=O(1)} \mathbf{1}_{\eta-\xi \in B} v_+(\eta-\xi), \tag{7.10}$$

where

$$v_+(\xi) = \left[1, 0, \hat{\xi}_3, \hat{\xi}_1 + i\hat{\xi}_2\right]^T \tag{7.11}$$

is an eigenvector of $P_+(\xi)$, and $\hat{\xi} \equiv \frac{\xi}{|\xi|}$.

Observe that

$$\langle \beta v_+(\eta), v_+(\zeta)\rangle = 1 - \hat{\eta}\cdot\hat{\zeta} + i\hat{\eta}' \wedge \hat{\zeta}', \tag{7.12}$$

where $\hat{\eta}' \wedge \hat{\zeta}' = \hat{\eta}_1\hat{\zeta}_2 - \hat{\eta}_2\hat{\zeta}_1$ and $\xi' = (\xi_1, \xi_2)$. Hence

$$\operatorname{Im}\langle \beta v_+(\eta), v_+(\eta-\xi)\rangle = \pm \sin\theta_+ \sim \pm\theta_+, \tag{7.13}$$

where the sign in front of $\sin\theta_+$ depends on the orientation of $(\eta', \eta'-\xi')$. But the sets A, B, C will be chosen so that the orientation of the pair $(\eta', \eta'-\xi')$ is fixed; hence we conclude (see [2]) that

$$\|\langle \beta P_+(D)\psi, P_+(D)\psi'\rangle\|_{H^{-a_3,-\alpha_3}} \geq K^+, \tag{7.14}$$

where

$$K^+ = \Big\| \int_{\mathbb{R}^{1+3}} \frac{\theta_+}{\langle\xi\rangle^{a_3}\langle|\tau|-|\xi|\rangle^{\alpha_3}} \mathbf{1}_{\{\eta\in A,\ \lambda+\eta_1=O(1)\}} \mathbf{1}_{\{\xi\in C,\ \tau+\xi_1=O(1)\}}\, d\lambda\, d\eta \Big\|_{L^2_{\tau,\xi}}.$$

We now construct the counterexamples, by choosing the sets A, B, C. Note that in K^+,

$$\begin{gathered} \eta \in A, \quad \xi \in C, \quad \eta - \xi \in B, \\ \lambda + \eta_1 = O(1), \quad \tau + \xi_1 = O(1), \quad \lambda - \tau + \eta_1 - \xi_1 = O(1). \end{gathered} \tag{7.15}$$

7.2.1. *Necessity of* (7.1). We consider high-high frequency interaction giving out put at high frequency. Set

$$\begin{aligned} A &= \Big\{\xi \in \mathbb{R}^3 : |\xi_1 - L| \le L/4,\ |\xi_2 - L^{1/2}| \le L^{1/2}/4,\ \ |\xi_3 - L^{1/2}| \le L^{1/2}/4\Big\}, \\ B &= \Big\{\xi \in \mathbb{R}^3 : |\xi_1 - 2L| \le L/2,\ |\xi_2| \le L^{1/2}/2,\ \ |\xi_3| \le L^{1/2}/2\Big\}, \\ C &= \Big\{\xi \in \mathbb{R}^3 : |\xi_1 + L| \le L/4,\ |\xi_2 - L^{1/2}| \le L^{1/2}/4,\ \ |\xi_2 - L^{1/2}| \le L^{1/2}/4\Big\}. \end{aligned}$$

Then (7.7) holds. By (7.15), we have

$$\theta_+ = \measuredangle(\eta', \eta' - \xi') \sim \frac{1}{L^{1/2}}, \quad |\xi|, |\eta|, |\eta - \xi| \sim L,$$

and

$$\lambda + |\eta| = \lambda + \eta_1 + |\eta| - \eta_1 = \lambda + \eta_1 + \frac{\eta_2^2 + \eta_3^2}{|\eta| + \eta_1} = O(1). \tag{7.16}$$

Similarly,

$$\lambda - \tau + |\eta - \xi| = O(1), \quad \big||\tau| - |\xi|\big| = \big|\tau - |\xi|\big| \le \tau + \xi_1 = O(1). \tag{7.17}$$

Let $|A|$ denote the volume of A. Then

$$K^+ \sim \frac{|A||C|^{1/2}}{L^{1/2+a_3}} \quad \text{and} \quad \|\psi\|_{X_+^{a_1,\alpha_1}} \sim L^{a_1}|A|^{1/2}, \quad \|\psi'\|_{X_+^{a_2,\alpha_2}} \sim L^{a_2}|B|^{1/2}$$

Since $|A| = |C| \sim L^2$, we conclude that (7.8) holds with $\delta(a_1, a_2, a_3, \alpha_1, \alpha_2, \alpha_3) = a_1 + a_2 + a_3 - 1/2$, proving the necessity of $a_1 + a_2 + a_3 \ge 1/2$.

7.2.2. *Necessity of* (7.2) *and* (7.3). We consider high-low frequency interaction with output at high frequency.

$$\begin{aligned} A &= \Big\{\xi \in \mathbb{R}^3 : |\xi_1| \le L^{1/2}/2,\ |\xi_2 - 1| \le L^{1/2}/2,\ |\xi_3 - 1| \le L^{1/2}/2\Big\}, \\ B &= \Big\{\xi \in \mathbb{R}^3 : |\xi_1 - L| \le L^{1/2},\ |\xi_2| \le L^{1/2},\ |\xi_3| \le L^{1/2}\Big\}, \\ C &= \Big\{\xi \in \mathbb{R}^3 : |\xi_1 + L| \le L^{1/2}/2,\ |\xi_2 - 1| \le L^{1/2}/2,\ |\xi_3 - 1| \le L^{1/2}/2\Big\}. \end{aligned}$$

Then $\theta_+ = \measuredangle(\eta', \eta' - \xi') \sim 1$, $|\eta| \sim L^{1/2}$ and $|\xi|, |\eta - \xi| \sim L$. Further, (7.17) still holds, whereas the calculation in (7.16) shows that $\lambda + |\eta| \sim L^{1/2}$, since $|\eta| + \eta_1 \ge \eta_2 - \eta_1 \ge L^{1/2}/2$. Thus,

$$K^+ \sim \frac{|A||C|^{1/2}}{L^{a_3}}, \quad \|\psi\|_{X_+^{a_1,\alpha_1}} \sim L^{a_1/2+\alpha_1/2}|A|^{1/2}, \quad \|\psi'\|_{X_+^{a_2,\alpha_2}} \sim L^{s}|B|^{1/2}.$$

But $|A|, |B|, |C| \sim L^{3/2}$, hence (7.8) holds with $\delta(a_1, a_2, a_3, \alpha_1, \alpha_2, \alpha_3) = \frac{a_1+\alpha_1}{2} + a_2 + a_3 - 3/4$, proving the necessity of (7.2).

To show the necessity of (7.3), we only need to modify A and B such that in A, we set $|\xi_1 + L| \le L^{1/2}/2$ instead of $|\xi_1| \le L^{1/2}/2$, and in B we set $|\xi_1| \le L^{1/2}$ instead of $|\xi_1 - L| \le L^{1/2}/2$. Otherwise, the same argument as above shows the necessity of (7.3).

7.2.3. *Necessity of* (7.4) *and* (7.5). The configuration is the same as in the previous subsection, except that the squares A, B, C now have side length ~ 1. We set

$$A = \left\{\xi \in \mathbb{R}^3 : |\xi_1| \le 1/2,\ |\xi_2 - 1| \le 1/2,\ |\xi_3 - 1| \le 1/2\right\},$$
$$B = \left\{\xi \in \mathbb{R}^3 : |\xi_1 - L| \le 1,\ |\xi_2| \le 1,\ |\xi_3 - 1| \le 1/2\right\},$$
$$C = \left\{\xi \in \mathbb{R}^3 : |\xi_1 + L| \le 1/2,\ |\xi_2 - 1| \le 1/2,\ |\xi_3 - 1| \le 1/2\right\}.$$

Then $\theta_+ \sim 1$, $|\eta| \sim 1$, $|\xi|, |\eta - \xi| \sim L$, and (7.16) holds. Since (7.17) also holds, we conclude:

$$K^+ \sim \frac{|A||C|^{1/2}}{L^c}, \quad \|\psi\|_{X_+^{a,\alpha}} \sim |A|^{1/2}, \quad \|\psi'\|_{X_+^{b,\beta}} \sim L^b |B|^{1/2}.$$

But $|A|, |B|, |C| \sim 1$, so (7.8) holds with $\delta(a_1, a_2, a_3, \alpha_1, \alpha_2, \alpha_3) = a_1 + a_2$, proving necessity of (7.5). By symmetry (7.4) is also necessary.

7.3. **Necessity of** (7.6)**.** Here we consider high-high frequency interaction with output at low frequency, and we choose the minus sign in (7.8).

$$A = \left\{\xi \in \mathbb{R}^3 : |\xi_1 - L| \le 1/4,\ |\xi_2 - 1| \le 1/4,\ |\xi_3 - 1| \le 1/4\right\},$$
$$B = \left\{\xi \in \mathbb{R}^3 : |\xi_1 - L| \le 1/2,\ |\xi_2| \le 1/2,\ |\xi_3| \le 1/2\right\},$$
$$C = \left\{\xi \in \mathbb{R}^3 : |\xi_1| \le 1/4,\ |\xi_2 - 1| \le 1/4,\ |\xi_3| \le 1/2\right\}.$$

We now restrict the integration to

$$\eta \in A, \quad \lambda + |\eta| = O(1), \quad \xi \in C, \quad \tau + 2L = O(1),$$

which implies

$$\eta - \xi \in B, \quad \lambda - \tau - |\eta - \xi| = \lambda + |\eta| - \tau - 2L + L - |\eta| + L - |\eta - \xi| = O(1),$$

since $L - |\eta| = L - \eta_1 - (\eta_2^2 + \eta_3^2)/(|\eta| + \eta_1) = O(1)$ and, similarly, $L - |\eta - \xi| = O(1)$. Now set

$$\widetilde{\psi}(\lambda, \eta) = \mathbf{1}_{\lambda + |\eta| = O(1)} \mathbf{1}_{\eta \in A} v_+(\eta),$$
$$\widetilde{\psi}'(\lambda - \tau, \eta - \xi) = \mathbf{1}_{\lambda - \tau - |\eta - \xi| = O(1)} \mathbf{1}_{\eta - \xi \in B} v_-(\eta - \xi),$$

where $v_-(\xi) = v_+(-\xi)$ and $v_+(\xi)$ is given by (7.11). Thus, $v_-(\xi)$ is an eigenvector of $P_-(\xi) = P_+(-\xi)$. Since $\theta_- = \measuredangle(\eta', \xi' - \eta') \sim 1$, we then get, arguing as in (7.14), and using (7.12),

$$\|\langle \beta P_+(D)\psi, P_-(D)\psi' \rangle\|_{H^{-a_3, -\alpha_3}} \ge K^-,$$

where

$$K^- = \Big\| \int_{\mathbb{R}^{1+3}} \frac{1}{\langle \xi \rangle^{a_3} \langle |\tau| - |\xi| \rangle^{\alpha_3}} \mathbf{1}_{\{\eta \in A,\ \lambda + |\eta| = O(1)\}} \mathbf{1}_{\{\xi \in C,\ \tau + 2L = O(1)\}} \, d\lambda \, d\eta \Big\|_{L^2_{\tau,\xi}}.$$

Since $|\xi| \sim 1$, $|\eta|, |\eta - \xi| \sim L$ and $|\tau| - |\xi| \sim |\tau| \sim L$, we see that

$$K^- \sim \frac{|A||C|^{1/2}}{L^{\alpha_3}}, \quad \|\psi\|_{X_+^{a_1,\alpha_1}} \sim L^{a_1} |A|^{1/2}, \quad \|\psi'\|_{X_-^{b,\beta}} \sim L^{a_2} |B|^{1/2}.$$

But $|A|, |B|, |C| \sim 1$, hence (7.8) holds with $\delta(a_1, a_2, a_3, \alpha_1, \alpha_2, \alpha_3) = a_1 + a_2 + \alpha_3$, proving necessity of (7.6) .

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Paper II

Low regularity well-posedness for the one-dimensional Dirac-Klein-Gordon system.

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LOW REGULARITY WELL-POSEDNESS FOR THE ONE DIMENSIONAL DIRAC-KLEIN-GORDON SYSTEM

SIGMUND SELBERG AND ACHENEF TESFAHUN

ABSTRACT. We extend recent results of S. Machihara and H. Pecher on low regularity well-posedness of the Dirac-Klein-Gordon (DKG) system in one dimension. Our proof, like that of Pecher, relies on the null structure of DKG, recently completed by D'Ancona, Foschi and Selberg, but we show that in 1d the argument can be simplified by modifying the choice of projections for the Dirac operator. We also show that the result is best possible up to endpoint cases, if one works in Bourgain-Klainerman-Machedon spaces.

1. INTRODUCTION

We consider the Dirac-Klein-Gordon system (DKG) in one space dimension,

$$
(1) \qquad \begin{cases} D_t\psi + \alpha D_x\psi + M\psi = \phi\beta\psi, & (D_t = -i\partial_t, \ D_x = -i\partial_x) \\ -\Box\phi + m^2\phi = \langle \beta\psi, \psi \rangle_{\mathbb{C}^2}, & (\Box = -\partial_t^2 + \partial_x^2) \end{cases}
$$

with initial data

$$
(2) \qquad \psi|_{t=0} = \psi_0 \in H^s, \qquad \phi|_{t=0} = \phi_0 \in H^r, \qquad \partial_t\phi|_{t=0} = \phi_1 \in H^{r-1},
$$

where $\phi(t,x)$ is real-valued and $\psi(t,x) \in \mathbb{C}^2$ is the Dirac spinor, regarded as a column vector with components ψ_1, ψ_2; $M, m \geq 0$ are constants. The 2×2 matrices α, β should be hermitian and satisfy $\beta^2 = \alpha^2 = I$, $\alpha\beta + \beta\alpha = 0$. A particular representation is

$$
\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

Global well-posedness for DKG in 1d was proved by Chadam [4], for data (2) with $(r,s) = (1,1)$. Several authors have improved Chadam's result, in the sense that the required regularity (r,s) has been lowered; see Table 1 for an overview.

The global results are obtained by first proving local well-posedness and then using the conservation of the charge norm $\|\psi(t)\|_{L^2}$ together with a suitable a priori estimate for $\phi(t)$, to show that the solution extends globally.

Thus, the main step is to prove local well-posedness, and the best such results to date are due to Machihara [10] and Pecher [11], who worked independently of each other. Machihara proved local well posedness of (1) for data (2) with (s,r) in the region

$$
-\frac{1}{4} < s \leq 0, \qquad 2\,|s| \leq r, \qquad r \leq 1 + 2s.
$$

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Table 1. Global well-posedness for (1), (2)

	s	r
Chadam [4], 1973	1	1
Bournaveas [2], 2000	0	1
Fang [7], 2004	0	(1/2,1]
Bournaveas and Gibbeson [3], 2006	0	[1/4,1]
Machihara [10], Pecher [11], 2006	0	(0,1]

Pecher obtained the region

$$s > -\frac{1}{4}, \qquad r > 0, \qquad |s| \le r, \qquad r < 1 + 2s, \qquad r \le 1 + s.$$

To compare the two results, note that in Pecher's region, intersected with the strip $-1/4 < s \le 0$, the lower bound for r is $|s|$, which is better than Machihara's lower bound $|2s|$, but on the other hand, Pecher has $r < 1 + 2s$ in this strip, whereas Machihara has $r \le 1 + 2s$.

Here we prove local well-posedness in a strictly larger region of the (s, r)-plane, which contains the union of the Pecher's and Machihara's regions. In fact, we show that in the strip $-1/4 < s \le 0$, the bound $r \le 1 + 2s$ can be relaxed to $r \le 1 + s$.

Theorem 1. *The DKG system* (1) *is locally well posed for data* (2) *with* (s, r) *in the region*

$$s > -\frac{1}{4}, \qquad r > 0, \qquad |s| \le r \le 1 + s.$$

Moreover, we show that this result is best possible, except possibly for the endpoint $(s, r) = (0, 0)$, if one uses iteration in Bourgain-Klainerman-Machedon spaces; see Section 4.

Our proof of Theorem 1, like Pecher's original proof, relies on the null structure of DKG, which was completed recently by D'Ancona, Foschi and Selberg [6]. To see the null structure, one starts by decomposing the spinor into eigenvectors of the Dirac operator. This approach was used by Beals and Bezard [1] to show that $\langle \beta\psi, \psi \rangle$ is a null form.[1] The new idea introduced in [6] is that this null form then appears again in the Dirac equation, after a duality argument. The null structure was used in [6] to prove almost optimal local well-posedness of the 3d DKG system, and in [5] to treat the 2d case. Pecher's proof for the 1d case follows closely the argument in [6], but here we show that in 1d the argument can be simplified by choosing the Dirac projections in a different way.

This paper is organized as follows: In the next section we reduce Theorem 1 to two bilinear estimates, and introduce the main tools needed for their proofs, which are given in Section 3. In Section 4 we prove the optimality of our result, by constructing explicit counterexamples for the iterative estimates. In Section 5 we prove a product law for Wave-Sobolev spaces (see Theorem 2) which is needed for the proof of Theorem 1.

[1] The fact that this expression is a null form was proved even earlier by Klainerman and Machedon [8], but they used a different, more indirect method.

Let us fix some notation. We use $\lesssim$ to mean $\le$ up to multiplication by a positive constant C which may depend on s and r. If a, b are nonnegative quantities, $a \sim b$ means $b \lesssim a \lesssim b$. The Fourier transforms in space and space-time are defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\,dx,$$

$$\widetilde{u}(\tau,\xi) = \int_{\mathbb{R}^{1+1}} e^{-i(t\tau + x\xi)} u(t,x)\,dt\,dx,$$

so $\widetilde{D_x u} = \xi \widetilde{u}$, $\widetilde{D_t u} = \tau \widetilde{u}$. $H^s = H^s(\mathbb{R})$ is the Sobolev space with norm

$$\|f\|_{H^s} = \left\| \langle \xi \rangle^s \widehat{f}(\xi) \right\|_{L^2_\xi}.$$

Here $\langle \cdot \rangle = 1 + |\cdot|$. For $a, \alpha \in \mathbb{R}$, let $X_\pm^{a,\alpha}$ and $H^{a,\alpha}$ be the completions of $\mathcal{S}(\mathbb{R}^{1+1})$ with respect to

$$\|u\|_{X_\pm^{a,\alpha}} = \left\| \langle \xi \rangle^a \langle \tau \pm \xi \rangle^\alpha \widetilde{u}(\tau,\xi) \right\|_{L^2_{\tau,\xi}},$$

$$\|u\|_{H^{a,\alpha}} = \left\| \langle \xi \rangle^a \langle |\tau| - |\xi| \rangle^\alpha \widetilde{u}(\tau,\xi) \right\|_{L^2_{\tau,\xi}}.$$

See [6] for more details about these spaces. Finally, if X, Y, Z are normed function spaces, we use the notation

$$X \cdot Y \hookrightarrow Z$$

to mean that $\|uv\|_Z \lesssim \|u\|_X \|v\|_Y$.

2. Preliminaries

The Dirac operator αD_x has Fourier symbol $\alpha\xi$, whose eigenvalues are $\pm\xi$. The eigenspace projections are

$$P_\pm = \frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}.$$

Following [6], Pecher used instead the ordering $\pm|\xi|$ of the eigenvalues, yielding nonconstant projections (with our choice of α, β)

$$\pi_\pm(\xi) = \frac{1}{2}\begin{pmatrix} 1 & \pm \operatorname{sgn}\xi \\ \pm \operatorname{sgn}\xi & 1 \end{pmatrix}.$$

The fact that our projections are constant simplifies the argument considerably.

We now write $\psi = \psi_+ + \psi_-$, where

$$\psi_+ = P_+\psi = \frac{1}{2}\begin{pmatrix} \psi_1 + \psi_2 \\ \psi_1 + \psi_2 \end{pmatrix}, \quad \psi_- = P_-\psi = \frac{1}{2}\begin{pmatrix} \psi_1 - \psi_2 \\ \psi_2 - \psi_1 \end{pmatrix}.$$

Applying $P_\pm$ on both sides of the first equation in (1), and using the identities $\alpha = P_+ - P_-$, $P_\pm^2 = P_\pm$ and $P_\pm P_\mp = 0$, (1) is rewritten as

$$\begin{cases} (D_t + D_x)\psi_+ = P_+(\phi\beta\psi), \\ (D_t - D_x)\psi_- = P_-(\phi\beta\psi), \\ \Box\phi = -\langle \beta\psi, \psi \rangle_{\mathbb{C}^2}. \end{cases} \tag{3}$$

We iterate in the spaces

$$\psi_+ \in X_+^{s,\sigma}, \quad \psi_- \in X_-^{s,\sigma}, \quad (\phi, \partial_t\phi) \in H^{r,\rho} \times H^{r-1,\rho},$$

where

$$\frac{1}{2} < \sigma, \rho \le 1$$

will be chosen depending on r, s. By a standard argument (see [6] for details) Theorem 1 then reduces to

$$\left\| P_\pm(\phi\beta P_{[\pm]}\psi) \right\|_{X_\pm^{s,\sigma-1+\varepsilon}} \lesssim \|\phi\|_{H^{r,\rho}} \|\psi\|_{X_{[\pm]}^{s,\sigma}}, \tag{4}$$

$$\left\| \left\langle \beta P_{[\pm]}\psi, P_\pm\psi' \right\rangle_{\mathbb{C}^2} \right\|_{H^{r-1,\rho-1+\varepsilon}} \lesssim \|\psi\|_{X_{[\pm]}^{s,\sigma}} \|\psi'\|_{X_\pm^{s,\sigma}}, \tag{5}$$

where $\pm$ and $[\pm]$ denote independent signs and $\varepsilon > 0$ is sufficiently small; the introduction of the parameter ε is a technical detail needed in the time localized linear estimates (see [6, Lemmas 5 and 6]).

But by a duality argument introduced in [6], estimate (4) is in fact equivalent to

$$\left\| \left\langle \beta P_{[\pm]}\psi, P_\pm\psi' \right\rangle_{\mathbb{C}^2} \right\|_{H^{-r,-\rho}} \lesssim \|\psi\|_{X_{[\pm]}^{s,\sigma}} \|\psi'\|_{X_\pm^{-s,1-\sigma-\varepsilon}}. \tag{4'}$$

The advantage of this formulation is that, like (5), it contains the bilinear form $\left\langle \beta P_{[\pm]}\psi, P_\pm\psi' \right\rangle$, which turns out be a null form: With our choice of projections, this comes out very easily, since by the self-adjointness, idempotency and orthogonality of the $P_\pm$, as well as the identity $P_\pm\beta = \beta P_\mp$, we see that

$$\left\langle \beta P_+\psi, P_+\psi' \right\rangle_{\mathbb{C}^2} = \left\langle \beta P_-\psi, P_-\psi' \right\rangle_{\mathbb{C}^2} = 0.$$

As a result, (4′) and (5) can be reduced to

$$\|u\bar{v}\|_{H^{-r,-\rho}} \lesssim \|u\|_{X_+^{s,\sigma}} \|v\|_{X_-^{-s,1-\sigma-\varepsilon}}. \tag{6}$$

$$\|u\bar{v}\|_{H^{r-1,\rho-1+\varepsilon}} \lesssim \|u\|_{X_+^{s,\sigma}} \|v\|_{X_-^{s,\sigma}}, \tag{7}$$

where u, v are $\mathbb{C}$-valued and $\bar{v}$ denotes the complex conjugate. The crucial point to note here is the difference in signs on the right, due to the null structure; if we had two equal signs, then the estimates would fail at the regularity prescribed in Theorem 1 (cf. the conditions in Theorem 2 below). There are two key reasons why things are better when the signs are different: The first reason is the algebraic constraint given in Lemma 1 below, which is the analogue, in the current setting, of Lemma 7 in [6]; the second reason is the bilinear estimate given in Lemma 2 below.

Lemma 1. *Define, for $\tau, \lambda, \xi, \eta \in \mathbb{R}$,*

$$\Gamma = |\tau| - |\xi|, \quad \Theta_+ = \lambda + \eta, \quad \Sigma_- = \lambda - \tau - (\eta - \xi).$$

Then

$$\min(|\eta|, |\eta - \xi|) \le \frac{3}{2} \max(|\Gamma|, |\Theta_+|, |\Sigma_-|).$$

Proof. We have

$$\Gamma = \begin{cases} \Theta_+ - \Sigma_- - (2\eta - \xi + |\xi|) & \text{if } \tau \ge 0, \\ -\Theta_+ + \Sigma_- + (2\eta - \xi - |\xi|) & \text{if } \tau \le 0. \end{cases}$$

and the terms in parentheses equal 2η or $2(\eta - \xi)$, depending on the sign of ξ. Therefore, $2\min(|\eta|, |\eta - \xi|) \le |\Gamma| + |\Theta_+| + |\Sigma_-|$. □

This lemma is applied in tandem with the following product law for the Wave-Sobolev spaces $H^{a,\alpha}$. The sufficiency of the conditions (9) and (10) in the following theorem can easily be deduced from [9, Proposition A.1], but here we also prove sufficiency, up to endpoints; see Section 5.

Theorem 2. *Suppose* $a, b, c \in \mathbb{R}$, $\alpha, \beta, \gamma \geq 0$ *and* $\alpha + \beta + \gamma > \frac{1}{2}$. *Then*

$$H^{a,\alpha} \cdot H^{b,\beta} \hookrightarrow H^{-c,-\gamma}, \tag{8}$$

provided that

$$a + b + c > \frac{1}{2}, \tag{9}$$

$$a + b \geq 0, \quad a + c \geq 0, \quad b + c \geq 0. \tag{10}$$

Furthermore, these conditions are sharp up to equality, in the sense that if (8) *holds, then* (10) *must hold, and* (9) *must hold with* $\geq$.

Remark 1. The above product law is analogous to the one for the standard Sobolev spaces, which in 1d reads $\|fg\|_{H^{-c}} \lesssim \|f\|_{H^a} \|g\|_{H^b}$, with the same conditions on a, b, c as in the above theorem.

The algebraic constraint (Lemma 1) and the product law for Wave-Sobolev spaces are enough to prove the result of Pecher, but to improve on that result, we use also the following bilinear space-time estimate for 1d free waves, where again the different signs are crucial.

Lemma 2. *Suppose* u, v *solve*

$$\begin{aligned} (D_t + D_x)u &= 0, \qquad u(0,x) = f(x), \\ (D_t - D_x)v &= 0, \qquad v(0,x) = g(x), \end{aligned}$$

where $f, g \in L^2(\mathbb{R})$. *Then*

$$\|uv\|_{L^2(\mathbb{R}^{1+1})} \leq \sqrt{2} \|f\|_{L^2} \|g\|_{L^2}.$$

Proof. We have $\widetilde{u}(\tau,\xi) = \delta(\tau+\xi)\widehat{f}(\xi)$ and $\widetilde{v}(\tau,\xi) = \delta(\tau-\xi)\widehat{g}(\xi)$, so

$$\begin{aligned} \widetilde{uv}(\tau,\xi) &= \int_{\mathbb{R}^{1+1}} \widetilde{u}(\lambda,\eta)\widetilde{v}(\tau-\lambda,\xi-\eta)\, d\lambda\, d\eta \\ &= \int \delta(\tau + 2\eta - \xi)\widehat{f}(\eta)\widehat{g}(\xi-\eta)\, d\eta \\ &= \widehat{f}\left(\frac{\xi-\tau}{2}\right)\widehat{g}\left(\frac{\xi+\tau}{2}\right). \end{aligned}$$

The claimed estimate now follows from Plancherel's theorem and an obvious change of variables. □

By the transfer principle (see [6, Lemma 4]), Lemma 2 implies:

Corollary 1. *For any* $\alpha > 1/2$,

$$X_+^{0,\alpha} \cdot X_-^{0,\alpha} \hookrightarrow L^2.$$

Again, this would fail if we had equal signs in the left hand side.

We now have all the tools needed to finish the proof of the main estimates.

3. Proof of Theorem 1

3.1. **Proof of** (6)**.** With notation as in Lemma 1, the estimate is equivalent to, using Plancherel's theorem,

$$\left\| \int_{\mathbb{R}^2} \frac{F(\lambda,\eta)G(\lambda-\tau,\eta-\xi)d\lambda\, d\eta}{\langle\xi\rangle^r\langle\eta\rangle^s\langle\eta-\xi\rangle^{-s}\langle\Gamma\rangle^\rho\langle\Theta_+\rangle^\sigma\langle\Sigma_-\rangle^{1-\sigma-\varepsilon}} \right\|_{L^2_{\tau,\xi}} \lesssim \|F\|_{L^2}\|G\|_{L^2},$$

for arbitrary $F, G \in L^2(\mathbb{R}^2)$. In view of Lemma 1 we can add either ρ, σ or $1-\sigma-\varepsilon$ to the exponent of either the $\langle\eta\rangle$ weight or the $\langle\eta-\xi\rangle$ weight, at the expense of giving up one of the "hyperbolic" weights $\langle\Gamma\rangle$, $\langle\Theta_+\rangle$ or $\langle\Sigma_-\rangle$. Then we apply Theorem 2. In fact, since (recall $\rho,\sigma > 1/2$)

$$\min(\rho,\sigma,1-\sigma-\varepsilon) = 1-\sigma-\varepsilon,$$

we can reduce to Theorem 2 with a, b, c as in the first two rows of Table 2. The conditions on a, b, c in Theorem 2 impose the following restrictions:

$$r > \sigma - \frac{1}{2} + \varepsilon, \tag{11}$$

$$r \geq |s|, \tag{12}$$

$$\sigma \leq 1-\varepsilon. \tag{13}$$

Finally, we mention that the hypotheses on (α,β,γ) in Theorem 2 are indeed satisfied in this situation, as follows from (13) and the fact that we require

$$\frac{1}{2} < \rho, \sigma \leq 1. \tag{14}$$

So we conclude that (6) holds provided (11)–(14) are verified.

3.2. **Proof of** (7)**.** This reduces to

$$I := \left\| \int_{\mathbb{R}^2} \frac{F(\lambda,\eta)G(\lambda-\tau,\eta-\xi)d\lambda\, d\eta}{\langle\xi\rangle^{1-r}\langle\eta\rangle^s\langle\eta-\xi\rangle^s\langle\Gamma\rangle^{1-\rho-\varepsilon}\langle\Theta_+\rangle^\sigma\langle\Sigma_-\rangle^\sigma} \right\|_{L^2_{\tau,\xi}} \lesssim \|F\|_{L^2}\|G\|_{L^2}.$$

We consider two cases, with notation as in Lemma 1:

3.2.1. *Case 1:* $\max(|\Gamma|, |\Theta_+|, |\Sigma_-|) \sim |\Gamma|$. By symmetry we may assume $|\eta| \leq |\eta-\xi|$ in I. Then either $|\eta| \sim |\eta-\xi|$, or $|\eta| \ll |\eta-\xi| \sim |\xi|$, hence, using Lemma 1,

$$I \lesssim I_1 + I_2$$

where

$$I_1 = \left\| \int_{|\eta|\sim|\eta-\xi|} \frac{F(\lambda,\eta)G(\lambda-\tau,\eta-\xi)d\lambda\, d\eta}{\langle\xi\rangle^{1-r}\langle\eta\rangle^{2s+1-\rho-\varepsilon}\langle\Theta_+\rangle^\sigma\langle\Sigma_-\rangle^\sigma} \right\|_{L^2_{\tau,\xi}},$$

$$I_2 = \left\| \int \frac{F(\lambda,\eta)G(\lambda-\tau,\eta-\xi)d\lambda\, d\eta}{\langle\eta\rangle^{s+1-\rho-\varepsilon}\langle\eta-\xi\rangle^{s+1-r}\langle\Theta_+\rangle^\sigma\langle\Sigma_-\rangle^\sigma} \right\|_{L^2_{\tau,\xi}}.$$

Moreover, if $r > 1$, then we can use $\langle\xi\rangle^{r-1} \lesssim \langle\eta\rangle^{r-1} + \langle\eta-\xi\rangle^{r-1}$ to further reduce I_1 to

$$I_{1,r>1} = \left\| \int_{|\eta|\sim|\eta-\xi|} \frac{F(\lambda,\eta)G(\lambda-\tau,\eta-\xi)d\lambda\, d\eta}{\langle\eta\rangle^{2s+1-\rho-\varepsilon+1-r}\langle\Theta_+\rangle^\sigma\langle\Sigma_-\rangle^\sigma} \right\|_{L^2_{\tau,\xi}}.$$

TABLE 2. Exponents used in Theorem 2

a	b	c
s	$-s+1-\sigma-\varepsilon$	r
$s+1-\sigma-\varepsilon$	$-s$	r
$s+\sigma$	s	$1-r$

Applying Corollary 1, we then see that $I_i \lesssim \|F\|_{L^2} \|G\|_{L^2}$ $(i = 1, 2)$, provided

$$r \le 1 + s, \tag{15}$$

$$s \ge -\frac{1}{2} + \frac{\rho+\varepsilon}{2} \tag{16}$$

$$s \ge -1 + \rho + \varepsilon \tag{17}$$

$$r \le 1 + 2s + 1 - \rho - \varepsilon. \tag{18}$$

Remark 2. If we had applied Theorem 2 here, the last condition would have been replaced by (due to the requirement $a + b + c > 1/2$ in Theorem 2)

$$r < \frac{1}{2} + 2s + 1 - \rho - \varepsilon,$$

which is still sufficient to obtain the result of Pecher. So it is exactly at this point that we gain something more.

3.2.2. *Case 2:* $\max(|\Gamma|, |\Theta_+|, |\Sigma_-|) \sim |\Theta_+|$ *or* $|\Sigma_-|$. Then by Lemma 1 we reduce to Theorem 2 with a, b, c as in the last row of Table 2, and $(\alpha, \beta, \gamma) = (0, \sigma, 1-\rho-\varepsilon)$ or $(\sigma, 0, 1-\rho-\varepsilon)$. The conditions on $a, b, c, \alpha, \beta, \gamma$ in Theorem 2 yield the restrictions

$$r < \frac{1}{2} + \sigma + 2s, \tag{19}$$

$$r \le 1 + s, \tag{20}$$

$$s \ge -\frac{\sigma}{2} \tag{21}$$

$$\rho \le 1 - \varepsilon. \tag{22}$$

Note that (20) is the same as (15).

We conclude that (7) holds if (15)–(22) are satisfied.

3.3. **Conlusion of the proof.** It only remains, given (s, r) satisfying the hypotheses

$$s > -\frac{1}{4}, \qquad r > 0, \qquad |s| \le r \le 1 + s \tag{23}$$

of Theorem 1, to choose $\rho, \sigma, \varepsilon$ in such a way that the constraints (11)–(22) are all satisfied. We shall need the fact that (23) implies

$$r < 3/2 + 2s. \tag{24}$$

Clearly, we get the best results by choosing ρ and ε as small as possible, so let us set

$$\rho = \frac{1}{2} + \varepsilon,$$

where $\varepsilon > 0$ will be chosen sufficiently small. Note that (22) is satisfied provided $\varepsilon \le 1/4$. Condition (16) becomes

$$s \ge -\frac{1}{4} + \varepsilon,$$

which is compatible with the assumption $s > -1/4$ in Theorem 1; (17) and (21) are weaker than (16), so they are also satisfied. Condition (18) becomes

$$r \le \frac{3}{2} + 2s - 2\varepsilon,$$

in accordance with (24).

The only remaining conditions are (11), (13) and (19) (as well as (14), which requires $\sigma > 1/2$), and these conditions can be summed up as follows:

$$\frac{1}{2} < \sigma \le 1 - \varepsilon,$$
$$\sigma - \frac{1}{2} + \varepsilon < r < \sigma + \frac{1}{2} + 2s.$$

Since $0 < r < 3/2 + 2s$, by (23) and (24), it is clear that we can find σ and $\varepsilon > 0$ such that the last two conditions are satisfied. This completes the proof of Theorem 1.

4. Counterexamples

Here we prove the optimality, except for the endpoint $(s, r) = (0, 0)$, of the conditions on s and r in Theorem 1, as far as iteration in the Bourgain-Klainerman-Machedon spaces $X_\pm^{s,\sigma}$, $H^{r,\rho}$ is concernced. To be precise, we prove:

Theorem 3.

(a) *The estimate* (5) *fails (for every choice of* $1/2 < \sigma, \rho \le 1$ *and* $\varepsilon > 0$*) if* $s \le -1/4$ *or* $r > 1 + s$.

(b) *The estimate* (4′)*, hence also* (4)*, fails (for every choice of* $1/2 < \sigma, \rho \le 1$ *and* $\varepsilon > 0$*) if* $r < |s|$.

More generally, we prove:

Theorem 4. *Let* $a, b, c, \alpha, \beta, \gamma \in \mathbb{R}$. *If the 2-spinor estimate*

$$\| \langle \beta P_+ \psi, P_- \psi' \rangle_{\mathbb{C}^2} \|_{H^{-c,-\gamma}} \lesssim \|\psi\|_{X_+^{a,\alpha}} \|\psi'\|_{X_-^{b,\beta}} ,$$

holds, then:

$$a + b + \min(\alpha, \beta, \gamma) \ge 0, \tag{25}$$

$$a + b + c + \min(\alpha, \beta) \ge \frac{1}{2} \tag{26}$$

$$a + b + c + \gamma \ge 0, \tag{27}$$

$$\min(a, b) + c \ge 0. \tag{28}$$

4.1. **Proof of Theorem 3.** We apply Theorem 4. For part (a) we take $(a,b,c) = (s,s,1-r)$ and $(\alpha,\beta,\gamma) = (\sigma,\sigma,1-\rho-\varepsilon)$. Then (25) gives the necessary condition $2s+1-\rho-\varepsilon \geq 0$, i.e., $s \geq -1/2 + (\rho+\varepsilon)/2 > -1/4$, where the last inequality holds since $\rho > 1/2$. Moreover, (28) gives the necessary condition $s+1-r \geq 0$. This proves part (a).

To prove part (b), take $(a,b,c) = (s,-s,r)$ and $(\alpha,\beta,\gamma) = (\sigma, 1-\sigma-\varepsilon, \rho)$. Then (28) implies $-|s| + r \geq 0$.

Note that we only used (25) and (28) to prove Theorem 3.

4.2. **Proof of Theorem 4.** The following counterexamples are adapted from those for the 2d case in [5], and depend on a large, positive parameter L going to infinity. We choose intervals $A, B, C \subset \mathbb{R}$, depending on L, with the property

$$\eta \in A, \ \xi \in C \implies \eta - \xi \in B. \tag{29}$$

We shall denote by $|A|$ the length of the interval A.

We shall set

$$\psi(t,x) = u(t,x)\begin{pmatrix}1\\1\end{pmatrix}, \quad \psi'(t,x) = v(t,x)\begin{pmatrix}1\\-1\end{pmatrix}, \tag{30}$$

where $u, v : \mathbb{R}^{1+1} \to \mathbb{C}$ are defined on the Fourier transform side by

$$\widetilde{u}(\lambda,\eta) = \mathbf{1}_{\lambda+\eta=O(1)}\mathbf{1}_{\eta\in A}, \quad \widetilde{v}(\lambda-\tau,\eta-\xi) = \mathbf{1}_{\lambda-\tau+\eta-\xi=O(1)}\mathbf{1}_{\eta-\xi\in B}, \tag{31}$$

and A, B remain to be chosen. Here $\mathbf{1}_{(\cdot)}$ stands for the indicator function of the set determined by the condition in the subscript. Then

$$\langle\, \beta P_+\psi, P_-\psi' \,\rangle_{\mathbb{C}^2} = \langle\, \beta\psi, \psi' \,\rangle_{\mathbb{C}^2} = 2u\bar{v}, \tag{32}$$

so in fact it suffices to find counterexamples to

$$\|u\bar{v}\|_{H^{-c,-\gamma}} \lesssim \|u\|_{X_+^{a,\alpha}} \|v\|_{X_-^{b,\beta}}. \tag{33}$$

Each counterexample will be of the form

$$\frac{\|u\bar{v}\|_{H^{-c,-\gamma}}}{\|u\|_{X_+^{a,\alpha}} \|v\|_{X_-^{b,\beta}}} \gtrsim \frac{1}{L^{\delta(a,b,c,\alpha,\beta,\gamma)}}, \tag{34}$$

which leads to the necessary condition $\delta(a,b,c,\alpha,\beta,\gamma) \geq 0$.

Observe that

$$\begin{aligned}
&\|u\bar{v}\|_{H^{-c,-\gamma}} \\
&= \left\| \int_{\mathbb{R}^{1+1}} \frac{1}{\langle\xi\rangle^c \langle |\tau|-|\xi| \rangle^\gamma} \mathbf{1}_{\left\{\substack{\eta\in A\\ \lambda+\eta=O(1)}\right\}} \mathbf{1}_{\left\{\substack{\eta-\xi\in B\\ \lambda-\tau+\eta-\xi=O(1)}\right\}} \, d\lambda\, d\eta \right\|_{L^2_{\tau,\xi}} \\
&\geq I := \left\| \int_{\mathbb{R}^{1+1}} \frac{1}{\langle\xi\rangle^c \langle |\tau|-|\xi| \rangle^\gamma} \mathbf{1}_{\left\{\substack{\eta\in A\\ \lambda+\eta=O(1)}\right\}} \mathbf{1}_{\left\{\substack{\xi\in C\\ \tau+\xi=O(1)}\right\}} \, d\lambda\, d\eta \right\|_{L^2_{\tau,\xi}},
\end{aligned}$$

where to get the last inequality we restrict the L^2 norm to $\tau+\xi = O(1)$, $\xi \in C$, make use of (29), and note that

$$\lambda+\eta = O(1), \ \tau+\xi = O(1) \implies \lambda-\tau+\eta-\xi = O(1).$$

Note also that $\big||\tau|-|\xi|\big| \leq |\tau+\xi| = O(1)$. (So to get counterexamples involving γ, we shall later have to modify I).

4.2.1. *Necessity of* (25) *when* $\min(\alpha,\beta,\gamma)=\alpha$ *or* β. Define

$$A=[L-1/2,L+1/2],\quad B=[L-1,L+1],\quad C=[-1/2,1/2].$$

Then $|\xi|=O(1)$, $|\eta|\sim L$, $|\eta-\xi|\sim L$ and

$$\lambda-\tau-(\eta-\xi)=\lambda-\tau+(\eta-\xi)-2(\eta-\xi)\sim L,$$

hence

$$I\sim|A|\,|C|^{1/2},\quad \|u\|_{X_+^{a,\alpha}}\sim L^a\,|A|^{1/2},\quad \|v\|_{X_-^{b,\beta}}\sim L^{b+\beta}\,|B|^{1/2}.$$

But $|A|,|B|,|C|\sim 1$, so (34) holds with $\delta(a,b,c,\alpha,\beta,\gamma)=a+b+\beta$, which gives the necessary condition $a+b+\beta\geq 0$. By symmetry, we must also have $a+b+\alpha\geq 0$.

4.2.2. *Necessity of* (26). Set

$$A=[L/4,L/2],\quad B=[L/2,3L/2],\quad C=[-L,-L/2].$$

Then $|\eta|,|\xi|,|\eta-\xi|\sim L$ and (as above) $\lambda-\tau-(\eta-\xi)\sim L$, so

$$I\sim\frac{|A|\,|C|^{1/2}}{L^c},\quad \|u\|_{X_+^{a,\alpha}}\sim L^a\,|A|^{1/2},\quad \|v\|_{X_-^{b,\beta}}\sim L^{b+\beta}\,|B|^{1/2}.$$

Since $|A|,|B|,|C|\sim L$, we conclude that (34) holds with $\delta(a,b,c,\alpha,\beta,\gamma)=a+b+c+\beta-1/2$, proving the necessity of $a+b+c+\beta\geq 1/2$. By symmetry, we also need $a+b+c+\alpha\geq 1/2$

4.2.3. *Necessity of* (28). Here we set

$$A=C=[L-1/2,L+1/2],\quad B=[-1,1].$$

Then $|\xi|\sim L$, $|\eta|\sim L$, $|\eta-\xi|=O(1)$ and

$$\lambda-\tau-(\eta-\xi)=\lambda-\tau+(\eta-\xi)-2(\eta-\xi)=O(1),$$

so

$$I\sim\frac{|A|\,|C|^{1/2}}{L^c},\qquad \|u\|_{X_+^{a,\alpha}}\sim L^a\,|A|^{1/2},\qquad \|v\|_{X_-^{b,\beta}}\sim|B|^{1/2}.$$

But $|A|,|B|,|C|\sim 1$, hence (34) holds with $\delta(a,b,c,\alpha,\beta,\gamma)=a+c$, proving necessity of $a+c\geq 0$. By symmetry, $a+c\geq 0$ is also necessary.

4.2.4. *Necessity of* (25) *when* $\min(\alpha,\beta,\gamma)=\gamma$. Set

$$A=[L-1,L+1],\quad B=[L-2,L+2],\quad C=[-1,1].$$

Again we use (30), with u as in (31), but we change v to:

$$\widetilde{v}(\lambda-\tau,\eta-\xi)=\mathbf{1}_{\lambda-\tau-(\eta-\xi)=O(1)}\mathbf{1}_{\eta-\xi\in B}.$$

Since (32) is unchanged, it suffices to disprove (33), but now, in view of the modification of v,

$$\begin{aligned}
&\|u\bar{v}\|_{H^{-c,-\gamma}}\\
&=\left\|\int_{\mathbb{R}^{1+1}}\frac{1}{\langle\xi\rangle^c\langle|\tau|-|\xi|\rangle^\gamma}\mathbf{1}_{\left\{\substack{\eta\in A\\ \lambda+\eta=O(1)}\right\}}\mathbf{1}_{\left\{\substack{\eta-\xi\in B\\ \lambda-\tau-(\eta-\xi)=O(1)}\right\}}\,d\lambda\,d\eta\right\|_{L^2_{\tau,\xi}}\\
&\geq I:=\left\|\int_{\mathbb{R}^{1+1}}\frac{1}{\langle\xi\rangle^c\langle|\tau|-|\xi|\rangle^\gamma}\mathbf{1}_{\left\{\substack{\eta\in A\\ \lambda+\eta=O(1)}\right\}}\mathbf{1}_{\left\{\substack{\xi\in C\\ \tau+2L=O(1)}\right\}}\,d\lambda\,d\eta\right\|_{L^2_{\tau,\xi}},
\end{aligned}$$

where in the last step we restrict the L^2 norm to the region $\tau + 2L = O(1)$, $\xi \in C$, make use of (29), and note that

$$\lambda - \tau - (\eta - \xi) = (\lambda + \eta) + 2(L - \eta) - (\tau + 2L) + \xi = O(1),$$

since each term is $O(1)$. So now $|\xi| = O(1)$, $|\eta| \sim L$, $|\eta - \xi| \sim L$, and $\big||\tau| - |\xi|\big| \sim L$, hence

$$I \sim \frac{|A| \, |C|^{1/2}}{L^\gamma}, \quad \|u\|_{X_+^{a,\alpha}} \sim L^a \, |A|^{1/2}, \quad \|v\|_{X_-^{b,\beta}} \sim L^b \, |B|^{1/2}.$$

Since $|A|, |B|, |C| \sim 1$, (34) holds with $\delta(a, b, c, \alpha, \beta, \gamma) = a + b + \gamma$.

4.2.5. *Necessity of* (27). Here we use the same u, v as in subsection 4.2.4. Set

$$A = [L - 1, L + 1], \quad B = [2L - 2, 2L + 2], \quad C = [-L - 1, -L + 1].$$

Then as in subsection 4.2.4, we have $\|u\bar{v}\|_{H^{-c,-\gamma}} \geq I$, with the only difference that the condition $\tau + 2L = O(1)$ in I has been replaced by $\tau + 3L = O(1)$, for then we can write

$$\lambda - \tau - (\eta - \xi) = (\lambda + \eta) + 2(L - \eta) - (\tau + 3L) + (\xi + L) = O(1),$$

each term being $O(1)$. So $|\xi|, |\eta|, |\eta - \xi| \sim L$, and $\big||\tau| - |\xi|\big| \sim L$, hence

$$I \sim \frac{|A| \, |C|^{1/2}}{L^{c+\gamma}}, \quad \|u\|_{X_+^{a,\alpha}} \sim L^a \, |A|^{1/2}, \quad \|v\|_{X_-^{b,\beta}} \sim L^b \, |B|^{1/2}.$$

Since $|A|, |B|, |C| \sim 1$, (34) holds with $\delta(a, b, c, \alpha, \beta, \gamma) = a + b + c + \gamma$.

5. Proof of Theorem 2

In this section we fix $\alpha, \beta, \gamma \geq 0$ satisfying $\alpha + \beta + \gamma > 1/2$. We shall say that a triple (a, b, c) of real numbers is *admissible* if the embedding (8) holds, i.e., if the bilinear estimate

$$\|uv\|_{H^{-c,-\gamma}} \lesssim \|u\|_{H^{a,\alpha}} \|v\|_{H^{b,\beta}} \tag{35}$$

holds.

First, assume that conditions (9) and (10) are satisfied. If $a, b, c \geq 0$, then (a, b, c) is admissible, as proved in [9, Proposition A.1]. It remains to consider the case where (a, b, c) contains a negative number. But in view of (10), at most one of the numbers a, b, c can be negative, and by symmetry it suffices to consider the case $a < 0$, say. In that case we can write $\langle \xi \rangle^{-a} \lesssim \langle \eta \rangle^{-a} + \langle \eta + \xi \rangle^{-a}$, thus reducing to the triples $(0, a + b, c)$ or $(0, b, a + c)$, which contain no negative numbers, hence are admissible, as noted above.

It remains to prove necessity of (9) (up to equality) and (10). But in fact, the counterexample constructed in subsection 4.2.2 gives, with u, v as in (31)

$$\frac{\|u\bar{v}\|_{H^{-c,-\gamma}}}{\|u\|_{H^{a,\alpha}} \|v\|_{H^{b,\beta}}} \gtrsim \frac{1}{L^{\delta(a,b,c,\alpha,\beta,\gamma)}}, \tag{36}$$

with $\delta(a, b, c, \alpha, \beta, \gamma) = a + b + c - 1/2$, proving necessity of $a + b + c \geq 1/2$, which is (9) with $\geq$. The fact that we have a conjugate here, but not in (35), is irrelevant, since $\|v\|_{H^{b,\beta}} = \|\bar{v}\|_{H^{b,\beta}}$.

Similarly, the counterexample from 4.2.3 gives (36) with $\delta(a, b, c, \alpha, \beta, \gamma) = a + c$, proving necessity of $a + c \geq 0$. By duality and symmetry in (35), we then get also $a + b \geq 0$ and $b + c \geq 0$, so we have proved necessity of (10).

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Paper III

Global Well-posedness of the 1D Dirac-Klein-Gordon system in Sobolev spaces of negative index.

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GLOBAL WELL-POSEDNESS OF THE 1D DIRAC-KLEIN-GORDON SYSTEM IN SOBOLEV SPACES OF NEGATIVE INDEX

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Abstract. We prove that the Cauchy problem for the Dirac-Klein-Gordon system of equations in 1D is globally well-posed in a range of Sobolev spaces of negative index for the Dirac spinor and positive index for the scalar field. The main ingredient in the proof is the theory of "almost conservation law" and "I-method" introduced by Colliander, Keel, Staffilani, Takaoka and Tao. Our proof also relies on the null structure in the system, and bilinear spacetime estimates of Klainerman-Machedon type.

1. Introduction

We consider the Dirac-Klein-Gordon system (DKG) in one space dimension,

$$\begin{cases} \left(-i(\gamma^0\partial_t + \gamma^1\partial_x) + M\right)\psi = \phi\psi, \\ (-\Box + m^2)\phi = \left\langle \gamma^0\psi, \psi \right\rangle_{\mathbb{C}^2}, \qquad (\Box = -\partial_t^2 + \partial_x^2) \end{cases} \tag{1}$$

with initial data

$$\psi|_{t=0} = \psi_0 \in H^s, \qquad \phi|_{t=0} = \phi_0 \in H^r, \qquad \partial_t\phi|_{t=0} = \phi_1 \in H^{r-1}. \tag{2}$$

Here $(t,x) \in \mathbb{R}^{1+1}$, $\psi = \psi(t,x) \in \mathbb{C}^2$ is the Dirac spinor and $\phi = \phi(t,x)$ is the scalar field which is real-valued; $M, m > 0$ are constants. Further, $\langle w, z \rangle_{\mathbb{C}^2} = z^* w$ for column vectors $w, z \in \mathbb{C}^2$, where z^* is the complex conjugate transpose of z; $H^s = (1-\partial_x^2)^{-s/2}L^2(\mathbb{R})$ is the standard Sobolev space of order s, and γ^0 and γ^1 are the Dirac matrices given by

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We remark that with this choice the general requirements for Dirac matrices are verified:

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I, \qquad (\gamma^0)^* = \gamma^0, \qquad (\gamma^1)^* = -\gamma^1$$

for $\mu, \nu = 0, 1$, where $(g^{\mu\nu}) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

We are interested in studying low regularity global solutions of the DKG system (1) given the initial data (2). Global well-posedness (GWP) of DKG in 1d was first proved by Chadam [4] for data

$$(\psi_0, \phi_0, \phi_1) \in H^1 \times H^1 \times L^2.$$

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Table 1. GWP for DKG in 1d for data $(\psi_0, \phi_0, \phi_1) \in H^s \times H^r \times H^{r-1}$.

	s	r
Chadam [4], 1973	1	1
Bournaveas [2], 2000	0	1
Fang [9], 2001	0	$(1/2, 1]$
Bournaveas and Gibbeson [3], 2006	0	$(1/4, 1]$
Machihara [11], Pecher [13], 2006	0	$(0, 1]$
Selberg [15], 2007	$(-1/8, 0)$	$(-s + \sqrt{s^2 - s}, 1 + s]$

This result has been improved over the years in the sense that the regularity requirements on the initial data which ensure global-in-time solutions can be lowered. The earlier known GWP results for DKG in 1d are summarized in Table 1.

It is well known that when $s \geq 0$, the question of GWP of (1), (2) reduces to the corresponding local question essentially due to the conservation of charge:

$$\|\psi(t,.)\|_{L^2} = \|\psi_0\|_{L^2}.$$

However, when $s < 0$ there is no applicable conservation law. So even if we have a local well-posedness (LWP) result for $s_0 < s < 0$ for some s_0, it seems that we are stuck when trying to extend this to a global-in-time solution.

The first breakthrough for resolving such problems came from Bourgain [1] who considered the cubic, defocusing nonlinear Schrödinger (NLS) equation in 2d, and proved GWP of NLS below the (conserved) energy norm, i.e., below H^1. The idea behind this method for a PDE is to split the rough initial data (data whose regularity is below the conserved norm; say the L^2 norm from now on) into low and high frequency parts, using a Fourier truncation operator. Consequently, one splits the PDE into two, corresponding to the initial data with low and high frequencies. The data with low frequency becomes smoother, in fact it is in L^2 , so by global well-posedness its evolution remains in L^2 for all time.

On the other hand, the difference between the original solution and the evolution of the low frequency data satisfies a modified nonlinear equation evolving the high-frequency part of the initial data. The homogeneous part of this evolution is of course no smoother than the initial data (so it may not be in L^2), but the inhomogeneous part may be better due to nonlinear smoothing effects. If the nonlinear smoothing brings the inhomogeneous part into L^2, then at the end of the time interval of existence this part can be added to the evolution of the low-frequency data, and the whole process can be iterated. Assuming that sufficiently good a priori estimates are available, this iteration allows one to reach an arbitrarily large existence time, by adjusting the frequency cut-off point of the original initial data. Several authors used Bourgain's method to prove GWP of dispersive and wave equations with rough data.

Recently, Selberg [15] used Bourgain's method to prove GWP of 1d-DKG below the charge norm, obtaining the following result (for a comparison with earlier results, see Table 1):

Theorem 1. *The DKG system* (1) *is GWP for data* (2) *provided*

$$-\frac{1}{8} < s < 0, \quad -s + \sqrt{s^2 - s} < r \le 1 + s.$$

Concerning LWP of 1d-DKG the best result so far, which we state in the next theorem, is due to S. Selberg and the present author [16], building on earlier results by several authors; see [4], [2], [9], [3], [11] and [13].

Theorem 2. *The DKG system* (1) *is LWP for data* (2) *if*

$$s > -\frac{1}{4}, \quad r > 0, \quad |s| \le r \le 1 + s.$$

As mentioned earlier, when $s \ge 0$ this LWP result can be extended to GWP result essentially due to the presence of conservation of charge. So in view of Theorem 2, we have the following (see also Table 1):

Theorem 3. *The DKG system* (1) *is GWP for data* (2) *provided*

$$s \ge 0, \quad r > 0, \quad |s| \le r \le 1 + s.$$

However, in view of Theorems 2, 3 and 1, there is still a gap left between the local and global results known so far. In the present paper, we shall relax the lower bound of r in Theorem 1. In particular, we fill the following gap left by Theorem 1 (see Figure 1):

$$-\frac{1}{8} < s < 0, \quad s + \sqrt{s^2 - s} < r \le -s + \sqrt{s^2 - s}$$

We now state our Main theorem.

Theorem 4. *The DKG system* (1) *is GWP for data* (2) *if (see Figure 1)*

$$-\frac{1}{8} < s < 0, \qquad s + \sqrt{s^2 - s} < r \le 1 + s.$$

The technique used here is the theory of "almost conservation law" and "I-method" which was developed by Colliander, Keel, Staffilani, Takaoka and Tao in a series of papers; See for instance [5], [6], [7]. The idea here is to apply a smoothing operator I to the solution of the PDE. The operator I is chosen so that it is the identity for low frequencies and an integration operator for high frequencies. The next step is to prove an "almost conservation law" for the smoothed out solution as time passes. Then one hopes that a modified version of LWP Theorem (after I is introduced) together with the "almost conservation law" will give a GWP result of the PDE for rough data.

In the DKG system, however, there is no conservation law for the field ϕ, only for the spinor ψ. Hence, we will not have "almost conservation law" for the ϕ field, which makes the problem harder. To fix this problem we use a product estimate for the Sobolev spaces for the inhomogeneous part of ϕ, the "almost conservation law" for the spinor ψ, together with an additional idea used by Selberg [15] of making use of induction argument involving a cascade of free waves.

This paper is organized as follows. In the next section we fix some notation, state definitions, and recall the derivation of the conservation of charge. In Section 3 we shall state some basic linear and bilinear estimates, and prove some null form estimates. In Section 4 we discuss the I-method, state a modified LWP theorem when we introduce the I operator, state a key Lemma concerning smoothing estimate, and show that a combination of these imply an "almost conservation law" for the

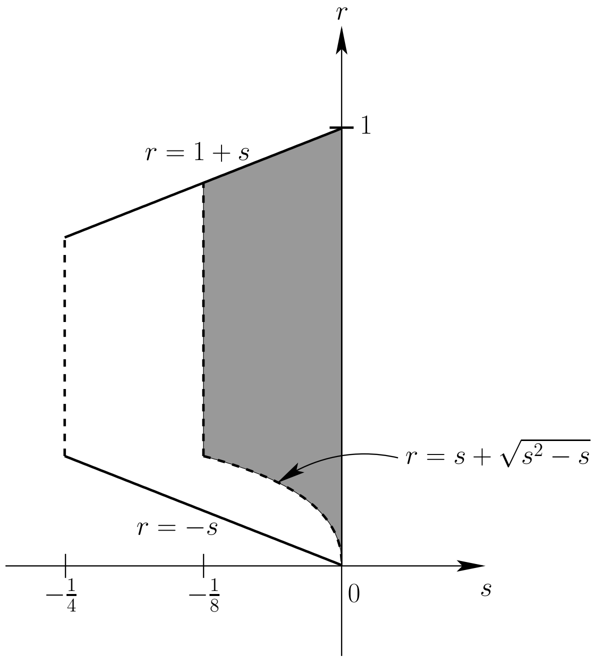

Figure 1. Global well-posedness of DKG holds in the interior of the shaded region. Moreover, we can allow the line $r = 1 + s$ for $-1/8 < s < 0$. The larger region which is contained in the strip $-1/4 < s < 0$ is where Local well-posedness of DKG holds.

charge. Here, we also state another key Lemma which is used to control the growth of solution of the Klein-Gordon part of DKG, ϕ. In Section 5 we put everything from section 4 together and prove our main theorem. In Sections 6 and 7 we prove the two key lemmas stated in section 4. In section 8 we prove the modified LWP theorem.

2. Preliminaries

2.1. **Notation and Definitions.** In estimates, C denotes a positive constant which can vary from line to line and may depend on the Sobolev exponents s and r in (2). We use the shorthand $X \lesssim Y$ for $X \le CY$, and if $C \ll 1$ we use the symbol $\ll$ instead of $\lesssim$. We use the shorthand $X \approx Y$ for $Y \lesssim X \lesssim Y$. Throughout the paper ε is considered to be a sufficiently small positive number in the sense that $0 < \varepsilon \ll 1$. We also use the notation

$$\langle\cdot\rangle = \sqrt{1 + |\cdot|^2}.$$

The Fourier transforms in space and space-time are defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx, \qquad \widetilde{u}(\tau,\xi) = \int_{\mathbb{R}^{1+1}} e^{-i(t\tau + x\xi)} u(t,x)\, dt\, dx.$$

We denote $D = -i\partial_x$, so $\widehat{Du}(\xi) = \xi \widehat{u}(\xi)$. We also write $D_+ := \partial_t + \partial_x$ and $D_- := \partial_t - \partial_x$, hence $\Box = -D_+D_-$.

We use the following spaces of Bourgain-Klainerman-Machedon type: For $a, b \in \mathbb{R}$, define $X_\pm^{a,b}$, $H^{a,b}$ and $\mathcal{H}^{a,b}$ to be the completions of $\mathcal{S}(\mathbb{R}^{1+1})$ with respect to the norms

$$\|u\|_{X_\pm^{a,b}} = \left\| \langle\xi\rangle^a \langle\tau \pm \xi\rangle^b \widetilde{u}(\tau,\xi) \right\|_{L^2_{\tau,\xi}},$$
$$\|u\|_{H^{a,b}} = \left\| \langle\xi\rangle^a \langle|\tau| - |\xi|\rangle^b \widetilde{u}(\tau,\xi) \right\|_{L^2_{\tau,\xi}},$$
$$\|u\|_{\mathcal{H}^{a,b}} = \|u\|_{H^{a,b}} + \|\partial_t u\|_{H^{a-1,b}}.$$

We also need the restrictions to a time slab $S_T = (0,T) \times \mathbb{R}$. The restriction $X_\pm^{a,b}(S_T)$ is a Banach space with norm

$$\|u\|_{X_\pm^{a,b}(S_T)} = \inf_{\tilde{u}_{|S_T} = u} \|\tilde{u}\|_{X_\pm^{a,b}}.$$

The restrictions $H^{a,b}(S_T)$ and $\mathcal{H}^{a,b}(S_T)$ are defined in the same way. See [8] for more details about these spaces.

2.2. **Rewriting DKG and Conservation of charge.** To see the symmetry in the DKG system, we shall rewrite (1) as follows: Let

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix}$$

for $u, v \in \mathbb{C}$. Then we calculate

$$(\gamma^0 \partial_t + \gamma^1 \partial_x)\psi = \begin{pmatrix} v_t - v_x \\ u_t + u_x \end{pmatrix}$$

and

$$\left\langle \gamma^0 \psi, \psi \right\rangle_{\mathbb{C}^2} = \overline{u}v + u\overline{v} = 2\,\mathrm{Re}(u\overline{v}).$$

Using this information, we rewrite (1) as

$$\begin{cases} i(u_t + u_x) = Mv - \phi v, \\ i(v_t - v_x) = Mu - \phi u, \\ \Box\phi = m^2\phi - 2\,\mathrm{Re}(u\overline{v}), \end{cases} \tag{3}$$

with the initial data (2) transformed to

$$\begin{cases} u(0) = u_0 \in H^s, & v(0) = v_0 \in H^s, \\ \phi(0) = \phi_0 \in H^r, & \partial_t\phi(0) = \phi_1 \in H^{r-1}. \end{cases} \tag{4}$$

We shall then work with the Cauchy problem (3), (4) in the rest of the paper.

To motivate the derivation of the "almost conservation law", we first recall the proof of the conservation of L^2-norm of the solution to the Dirac part of the equation (3), using integration by parts. To do this we first assume u, v to be smooth functions that decay at spatial infinity. For general well posed solutions of (3) where $s \geq 0$, the conservation of charge will follow by a density argument.

Multiplying the first and second equations in (3) by $-i\overline{u}$ and $-i\overline{v}$, respectively, we get

$$\begin{cases} \overline{u}u_t + \overline{u}u_x = -iM\overline{u}v + i\phi\overline{u}v, \\ \overline{v}v_t - \overline{v}v_x = -iMu\overline{v} + i\phi u\overline{v}. \end{cases}$$

Adding these two we obtain

$$\overline{u}u_t + \overline{v}v_t + \overline{u}u_x - \overline{v}v_x = 2i(-M + \phi)\,\mathrm{Re}(u\overline{v}).$$

We now take the real part of this equation to get

$$\mathrm{Re}(\overline{u}u_t) + \mathrm{Re}(\overline{v}v_t) + \mathrm{Re}(\overline{u}u_x) - \mathrm{Re}(\overline{v}v_x) = 0.$$

Using the identity $(\overline{u}u)_t = \overline{u}u_t + \overline{u}_t u = 2\,\mathrm{Re}(\overline{u}u_t)$ (and the same identity if we take partial derivative in x), we have

$$(|u|^2)_t + (|v|^2)_t + (|u|^2)_x - (|v|^2)_x = 0.$$

We get after integrating in x

$$\frac{d}{dt}\left(\|u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2\right) = 0,$$

which implies the conservation charge:

$$\|u(t)\|_{L^2}^2 + \|v(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 . \tag{5}$$

3. Linear and bilinear estimates

The representation formula in Fourier space for the inhomogeneous nonmassive Dirac Cauchy problem

$$\begin{cases} iD_\pm w_\pm = F_\pm(t,x), \\ w_\pm(0,x) = f_\pm(x), \end{cases} \tag{6}$$

is given by

$$\widehat{w_\pm(t)}(\xi) = e^{\mp i\xi)t}\hat{f}_\pm(\xi) + \int_0^t e^{\mp i\xi)(t-t')}\widehat{F}_\pm(t',\xi)dt'. \tag{7}$$

Similarly, the representation formula in Fourier space for the inhomogeneous massive Klein-Gordon Cauchy problem

$$\begin{cases} \Box z = m^2 z + F(t,x), \\ z(0,x) = f(x), \quad \partial_t z(0,x) = g(x), \end{cases} \tag{8}$$

is given by

$$\widehat{z(t)}(\xi) = \cos(t\langle\xi\rangle_m)\widehat{f}(\xi) + \frac{\sin(t\langle\xi\rangle_m)}{\langle\xi\rangle_m}\widehat{g}(\xi) + \int_0^t \frac{\sin\left((t-t')\langle\xi\rangle_m\right)}{\langle\xi\rangle_m}\widehat{F(t')}(\xi)dt', \tag{9}$$

where $\langle\xi\rangle_m = \sqrt{m^2 + |\xi|^2}$.

3.1. Linear estimates.

Throughout the paper, we use the notation

$$\|z[t]\|_{H^a} \equiv \|z(t)\|_{H^a} + \|\partial_t z(t)\|_{H^{a-1}} .$$

From the solution formulas (7) and (9) we deduce the following energy estimates for the solution of Cauchy problems (6) and (8), respectively:

$$\|w_\pm(t)\|_{H^a} \le \|f_\pm\|_{H^a} + \int_0^t \|F_\pm(t')\|_{H^a}\,dt', \tag{10}$$

$$\|z[t]\|_{H^a} \le C\left(\|f\|_{H^a} + \|g\|_{H^{a-1}} + \int_0^t \|F(t')\|_{H^{a-1}}\,dt'\right), \tag{11}$$

[1] for some $C > 0$ and for all $t > 0$.

The estimates we present in the following two lemmas are a priori estimates for the solutions of the nonmassive Dirac and massive Klein-Gordon Cauchy problems,

[1] If we set $m = 0$ in (8), then the constant C in the energy estimate (11) will depend on t.

and they are crucial for the reduction of the local existence problem to bilinear estimates. The following Lemma is proved in [8, Lemma 5]:

Lemma 1. *Let $1/2 < b \le 1$, $a \in \mathbb{R}$, $0 < T \le 1$ and $0 \le \delta \le 1-b$. Then for all data $F_\pm \in X_\pm^{a,b-1+\delta}(S_T)$ and $f_\pm \in H^a$, we have the following estimate for the solution (7) of the Dirac Cauchy problem (6):*

$$\|w_\pm\|_{X_\pm^{a,b}(S_T)} \le C\Big(\|f_\pm\|_{H^a} + T^\delta \|F_\pm\|_{X_\pm^{a,b-1+\delta}(S_T)}\Big), \tag{12}$$

where C depends only on b.

The estimate in the following Lemma is a variant of the estimate in [8, Lemma 6], i.e., when $m = 0$.

Lemma 2. *Let $1/2 < b \le 1$, $a \in \mathbb{R}$, $0 < T \le 1$ and $0 \le \delta \le 1-b$. Then for all data $F \in H^{a-1,b-1+\delta}(S_T)$, $f \in H^a$ and $g \in H^{a-1}$, we have the following estimate for the solution (9) of the Klein-Gordon Cauchy problem (8):*

$$\|z\|_{\mathcal{H}^{a,b}(S_T)} \le C\Big(\|f\|_{H^a} + \|g\|_{H^{a-1}} + T^{\delta/2} \|F\|_{H^{a-1,b-1+\delta}(S_T)}\Big), \tag{13}$$

where C depends only on b.

Proof. Define the space $H_m^{s,\theta}$ with a norm

$$\|u\|_{H_m^{s,\theta}} = \left\| \langle \xi \rangle^s \langle |\tau| - \langle \xi \rangle_m \rangle^\theta \widetilde{u}(\tau,\xi) \right\|_{L^2_{\tau,\xi}},$$

and the space $\mathcal{H}_m^{s,\theta}$ with a norm

$$\|u\|_{\mathcal{H}_m^{s,\theta}} = \|u\|_{H_m^{s,\theta}} + \|\partial_t u\|_{H_m^{s-1,\theta}}.$$

So, in view of [14, Theorem 12] the estimate (13) holds if we replace the spaces $H^{s,\theta}$ and $\mathcal{H}^{s,\theta}$ by $H_m^{s,\theta}$ and $\mathcal{H}_m^{s,\theta}$, respectively. Hence, to complete the proof it suffices to show

$$H^{s,\theta} = H_m^{s,\theta}.$$

This reduces to proving

$$\langle |\tau| - |\xi| \rangle \approx \langle |\tau| - \langle \xi \rangle_m \rangle. \tag{14}$$

Assume $\tau \ge 0$. Then

$$\begin{aligned}\langle -\tau + |\xi| \rangle \approx 1 + |-\tau + |\xi|| &\le 1 + |-\tau + \langle \xi \rangle_m| + \langle \xi \rangle_m - |\xi| \\ &\le 1 + m + |-\tau + \langle \xi \rangle_m| \\ &\lesssim \langle -\tau + \langle \xi \rangle_m \rangle.\end{aligned}$$

Conversely,

$$\begin{aligned}\langle -\tau + \langle \xi \rangle_m \rangle &\approx 1 + |-\tau + \langle \xi \rangle_m| \\ &\le 1 + |-\tau + |\xi|| + \langle \xi \rangle_m - |\xi| \\ &= 1 + m + |-\tau + |\xi|| \lesssim \langle -\tau + |\xi| \rangle.\end{aligned}$$

Similarly, it can be shown that the estimate (14) holds true for $\tau < 0$. This completes the proof of the Theorem. ∎

We shall need the fact that if $b > 1/2$, then

$$\|u(t)\|_{H^a} \le C \|u\|_{H^{a,b}(S_T)} \le C \|u\|_{X_\pm^{a,b}(S_T)} \qquad \text{for } 0 \le t \le T, \tag{15}$$

where C depends only on b. The following estimate will also be needed in the last section (see [12] for the proof):

$$\|u\|_{X_\pm^{a,\varepsilon}(S_T)} \le CT^{1/2-2\varepsilon} \|u\|_{X_\pm^{a,1/2-\varepsilon}(S_T)}, \tag{16}$$

for all $\varepsilon > 0$ sufficiently small, and $0 < T \le 1$.

Lemma 3. *Let $2 \le q \le \infty$ and $\varepsilon > 0$ be sufficiently small. Then*

$$\|u\|_{H^{0,-1/2+1/q-\varepsilon}} \lesssim \|u\|_{L_t^{q'} L_x^2}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$.

Remark 1. This Lemma also holds if we replace $H^{0,-1/2+1/q-\varepsilon}$ by $X_\pm^{0,-1/2+1/q-\varepsilon}$, simply because $H^{0,\alpha} \hookrightarrow X_\pm^{0,\alpha}$ for any $\alpha \le 0$.

Proof of lemma 3. By duality, the estimate is equivalent to

$$\|u\|_{L_t^q L_x^2} \lesssim \|u\|_{H^{0,1/2-1/q+\varepsilon}}. \tag{17}$$

By Sobolev embedding in t

$$\|u\|_{L_t^\infty L_x^2} \lesssim \|u\|_{H^{0,1/2+\varepsilon}}.$$

Interpolating this with

$$\|u\|_{L_t^2 L_x^2} = \|u\|_{L_t^2 L_x^2}$$

gives

$$\|u\|_{L_t^q L_x^2} \lesssim \|u\|_{H^{0,b}}$$

where

$$\frac{1}{q} = \frac{\theta}{\infty} + \frac{1-\theta}{2}, \qquad b = \theta(1/2+\varepsilon)$$

for $\theta \in [0,1]$. Thus $b = \frac{1}{2} - \frac{1}{q} + \varepsilon(1 - \frac{2}{q}) < \frac{1}{2} - \frac{1}{q} + \varepsilon$, and hence (17) follows. This concludes the proof of the Lemma. $\square$

3.2. **Bilinear estimates.** We shall need the standard product estimate for the Sobolev spaces H^s, which reads as follows:

Lemma 4. *Suppose $a_1, a_2, a_3 \in \mathbb{R}$. Then*

$$\|fg\|_{H^{-a_3}} \lesssim \|f\|_{H^{a_1}} \|g\|_{H^{a_2}}. \tag{18}$$

provided

$$\begin{aligned} &a_1 + a_2 + a_3 > 1/2, \\ &a_1 + a_2 \ge 0, \quad a_1 + a_3 \ge 0, \quad a_2 + a_3 \ge 0. \end{aligned} \tag{19}$$

The following estimate is just the analogue of Lemma 4 for the wave-Sobolev space $H^{s,b}$.

Lemma 5. [14, 16]. *Suppose $a_1, a_2, a_3 \in \mathbb{R}$ satisfy (19). Let $\alpha, \beta, \gamma \ge 0$ and $\alpha + \beta + \gamma > \frac{1}{2}$. Then*

$$\|wz\|_{H^{-a_3,-\gamma}} \lesssim \|w\|_{H^{a_1,\alpha}} \|z\|_{H^{a_2,\beta}}. \tag{20}$$

The following comparison estimate between elliptic and hyperbolic weights proved in [16] will be needed in the proof of Lemma 7 below. This estimate is used to identify null structure in bilinear estimates.

Lemma 6. *Denote*

$$\Gamma = |\tau| - |\xi|, \qquad \Theta_+ = \lambda + \eta, \qquad \Sigma_- = \tau - \lambda - (\xi - \eta).$$

Then

$$\min(|\eta|, |\xi - \eta|) \lesssim \max(|\Gamma|, |\Theta_+|, |\Sigma_-|).$$

We now prove the following null form estimates. We remark that the null structure of DKG in 1d is reflected in the difference of signs in the r.h.s. of the estimate (21), and the difference of signs in the r.h.s. and l.h.s. of the estimates (22) and (23) below; for equal signs the estimates would fail.

Lemma 7. *Let $b = \frac{1}{2} + \varepsilon$ for sufficiently small $\varepsilon > 0$. The bilinear estimates*

$$\|wz\|_{H^{-s_1,b-1}} \lesssim \|w\|_{X_+^{s_2,b}} \|z\|_{X_-^{s_3,b}}, \tag{21}$$

$$\|wz\|_{X_-^{-s_3,b-1}} \lesssim \|w\|_{H^{s_1,b}} \|z\|_{X_+^{s_2,b}}, \tag{22}$$

$$\|wz\|_{X_+^{-s_3,b-1}} \lesssim \|w\|_{H^{s_1,b}} \|z\|_{X_-^{s_2,b}} \tag{23}$$

hold provided

$$\begin{aligned} &s_1 + s_2 + s_3 > \varepsilon, \\ &s_2 + s_3 \geq -1/2 + \varepsilon, \\ &s_1 + s_2 \geq 0, \quad s_1 + s_3 \geq 0. \end{aligned} \tag{24}$$

Remark 2. The bilinear estimates (21)–(23) will still hold if we replace z in the l.h.s. of the inequalities in these estimates by $\overline{z}$. We also note that these bilinear estimates will imply the corresponding estimates where the spaces are restricted in time (refer [8] for the detail).

Proof of Lemma 7. We only prove (21) and (22), since (23) will follow from (22) by symmetry. We first prove (21).

Set

$$F_+(\lambda, \eta) = \langle \eta \rangle^{s_2} \langle \lambda + \eta \rangle^b |\widetilde{w}(\lambda, \eta)|,$$
$$G_-(\lambda, \eta) = \langle \eta \rangle^{s_3} \langle \lambda - \eta \rangle^b |\widetilde{z}(\lambda, \eta)|.$$

Then (21) is equivalent to

$$J \lesssim \|F_+\|_{L^2} \|G_-\|_{L^2}$$

where

$$J := \left\| \int_{\mathbb{R}^{1+1}} \frac{F_+(\lambda, \eta) G_-(\tau - \lambda, \xi - \eta)\, d\lambda\, d\eta}{\langle \xi \rangle^{s_1} \langle \eta \rangle^{s_2} \langle \xi - \eta \rangle^{s_3} \langle \Gamma \rangle^{1-b} \langle \Theta_+ \rangle^b \langle \Sigma_- \rangle^b} \right\|_{L^2_{\tau,\xi}},$$

where Γ, Θ_+ and Σ_- are defined as in Lemma 6.

By symmetry, we may assume $|\eta| \leq |\xi - \eta|$. If $\max(|\Gamma|, |\Theta_+|, |\Sigma_-|) = |\Gamma|$, then in view of Lemma 6 the estimate for J reduces to (20) with exponents $(a_1, a_2, a_3) = (s_2 + 1 - b, s_3, s_1)$, $(\alpha, \beta, \gamma) = (b, b, 0)$. If $\max(|\Gamma|, |\Theta_+|, |\Sigma_-|) = |\Theta_+|$ or $|\Sigma_-|$, then the estimate for J reduces to (20) with exponents $(a_1, a_2, a_3) = (s_2 + b, s_3, s_1)$, $(\alpha, \beta, \gamma) = (0, b, 1 - b)$ or $(b, 0, 1 - b)$.

Then the conditions on (a_1, a_2, a_3), (19), will be satisfied (for all the cases above) as long as (24) holds.

Next, we prove (22). By duality, proving the estimate (22) is equivalent to proving

$$\|wz\|_{H^{-s_1,-b}} \lesssim \|w\|_{X_+^{s_2,b}} \|z\|_{X_-^{s_3,1-b}}, \tag{25}$$

where $w,\ z$ are $\mathbb{C}$-valued functions. Define F_+ as before, and redefine G_- as

$$G_-(\lambda,\eta) = \langle \eta \rangle^{s_3} \langle \lambda - \eta \rangle^{1-b} \left| \widetilde{z}(\lambda,\eta) \right|.$$

Then (25) is equivalent to

$$L \lesssim \left\| F_+ \right\|_{L^2} \left\| G_- \right\|_{L^2}$$

where

$$L := \left\| \int_{\mathbb{R}^{1+1}} \frac{F_+(\lambda,\eta) G_-(\tau-\lambda,\xi-\eta)\, d\lambda\, d\eta}{\langle \xi \rangle^{s_1} \langle \eta \rangle^{s_2} \langle \xi-\eta \rangle^{s_3} \langle \Gamma \rangle^{b} \langle \Theta_+ \rangle^{b} \langle \Sigma_- \rangle^{1-b}} \right\|_{L^2_{\tau,\xi}}.$$

We use the same argument as in the estimate for J above. In view of Lemma 6 we can add $1-b$ to the exponent of either the weight $\langle \eta \rangle$ or $\langle \xi-\eta \rangle$, at the cost of giving up one of the hyperbolic weights $\langle \Gamma \rangle$, $\langle \Theta_+ \rangle$ or $\langle \Sigma_- \rangle$. Then we apply Lemma 5. In fact, we can reduce the estimate for L to (20) with exponents $(a_1,a_2,a_3) = (s_2+1-b, s_3, s_1)$ or (s_2, s_3+1-b, s_1). Then the condition (19) is satisfied, since we assume (24). □

4. I-Method and Almost Conservation Law

Let $s<0$ and $N \gg 1$ be fixed. Define the Fourier multiplier operator

$$\widehat{If}(\xi) = q(\xi)\widehat{f}(\xi), \qquad q(\xi) = \begin{cases} 1, & |\xi|<N, \\ N^{-s}|\xi|^s, & |\xi|>2N, \end{cases} \tag{26}$$

with q even, smooth and monotone.

Observe that on low frequencies $\{\xi : |\xi|<N\}$, I is the identity operator. The operator I commutes with differential operators. We also have the following properties: For $a,b \in \mathbb{R}$,

$$\left\| If \right\|_{H^a} \lesssim \left\| f \right\|_{H^a}, \quad \left\| Iw \right\|_{H^{a,b}} \lesssim \left\| w \right\|_{H^{a,b}}, \tag{27}$$

$$\left\| f \right\|_{H^s} \lesssim \left\| If \right\|_{L^2} \lesssim N^{-s} \left\| f \right\|_{H^s}, \tag{28}$$

$$\left\| f \right\|_{H^a} \lesssim \left\| I^2 f \right\|_{H^{a-2s}} \lesssim N^{-2s} \left\| f \right\|_{H^a}, \tag{29}$$

and if $\operatorname{supp} \hat{z}(t,\cdot) \subset \{\xi : |\xi| \gtrsim N\}$, we have

$$\left\| I^{-1} z \right\|_{H^{a,b}} \lesssim N^s \left\| z \right\|_{H^{a-s,b}},$$

which in turn implies

$$\left\| Iz \right\|_{H^{a,b}} \lesssim N^s \left\| I^2 z \right\|_{H^{a-s,b}}. \tag{30}$$

Let (s,r) be such that $-\frac{1}{6}<s<0$ and $-s \le r < \frac{1}{2}+2s$. Then from the modified LWP theorem which we state in the next section, there exists a $\Delta T>0$ depending on

$$\left\| Iu_0 \right\|_{L^2} + \left\| Iv_0 \right\|_{L^2} + \left\| I^2 \phi_0 \right\|_{H^{r-2s}} + \left\| I^2 \phi_1 \right\|_{H^{r-2s-1}},$$

such that (3), (4) has solution for times $0 \le t \le \Delta T$. Of course, (3), (4) has solution for (s,r) in a larger region as in Theorem 2, but now we reprove the Theorem in the above restricted region with a different time of existence of solution.

Now, we observe using the Fundamental Theorem of Calculus that

$$\left\| Iu(\Delta T) \right\|_{L^2}^2 + \left\| Iv(\Delta T) \right\|_{L^2}^2 = \left\| Iu_0 \right\|_{L^2}^2 + \left\| Iv_0 \right\|_{L^2}^2 + R_1(\Delta T) + R_2(\Delta T),$$

where

$$R_1(\Delta T) = \int_0^{\Delta T} \frac{d}{d\tau}(Iu(\tau), Iu(\tau))d\tau,$$
$$R_2(\Delta T) = \int_0^{\Delta T} \frac{d}{d\tau}(Iv(\tau), Iv(\tau))d\tau,$$

and $(.,.)$ denotes the scalar product in L^2. By the first equation in (3),

$$\begin{aligned} R_1(\Delta T) &= \int_0^{\Delta T} \frac{d}{d\tau}(Iu(\tau), Iu(\tau))d\tau \\ &= 2\,\mathrm{Re}\int_0^{\Delta T} (I\dot{u}(\tau), Iu(\tau))d\tau \\ &= 2\,\mathrm{Re}\int_0^{\Delta T} (I\left[-iMv + i\phi v - u_x\right](\tau), Iu(\tau))d\tau \\ &= 2\,\mathrm{Re}\int_0^{\Delta T} (-iMIv(\tau), Iu(\tau))d\tau + 2\,\mathrm{Re}\int_0^{\Delta T} (iI(\phi v)(\tau), Iu(\tau))d\tau \\ &\quad + 2\,\mathrm{Re}\int_0^{\Delta T} (-Iu_x(\tau), Iu(\tau))d\tau. \end{aligned}$$

But the third term is zero. Indeed,

$$\begin{aligned} 2\,\mathrm{Re}\int_0^{\Delta T} (-Iu_x(\tau), Iu(\tau))d\tau &= -2\,\mathrm{Re}\int_0^{\Delta T}\int_{\mathbb{R}} \overline{Iu_x}(\tau) Iu(\tau)dxd\tau \\ &= -\int_0^{\Delta T}\int_{\mathbb{R}} \left(\overline{Iu}(\tau)Iu(\tau)\right)_x dxd\tau = 0. \end{aligned}$$

Hence

$$R_1(\Delta T) = 2\,\mathrm{Re}\int_0^{\Delta T}\int_{\mathbb{R}} -iMIv(\tau)\overline{Iu(\tau)}dxd\tau + 2\,\mathrm{Re}\int_0^{\Delta T}\int_{\mathbb{R}} iI(\phi v)(\tau)\overline{Iu(\tau)}dxd\tau.$$

Similarly, by the second equation in (3)

$$R_2(\Delta T) = 2\,\mathrm{Re}\int_0^{\Delta T}\int_{\mathbb{R}} -iMIu(\tau)\overline{Iv(\tau)}dxd\tau + 2\,\mathrm{Re}\int_0^{\Delta T}\int_{\mathbb{R}} iI(\phi u)(\tau)\overline{Iv(\tau)}dxd\tau.$$

We therefore get

$$\begin{aligned} R(\Delta T) :&= R_1(\Delta T) + R_2(\Delta T) \\ &= 2\,\mathrm{Re}\int_0^{\Delta T}\int_{\mathbb{R}} -2Mi\,\mathrm{Re}(Iu(\tau)\overline{Iv(\tau)})dxd\tau \\ &\quad + 2\,\mathrm{Re}\int_0^{\Delta T}\int_{\mathbb{R}} iI(\phi u)(\tau)\overline{Iv(\tau)}dxd\tau + 2\,\mathrm{Re}\int_0^{\Delta T}\int_{\mathbb{R}} iI(\phi v)(\tau)\overline{Iu(\tau)}dxd\tau \\ &= 2\,\mathrm{Re}\int_0^{\Delta T}\int_{\mathbb{R}} iI(\phi u)(\tau)\overline{Iv(\tau)}dxd\tau + 2\,\mathrm{Re}\int_0^{\Delta T}\int_{\mathbb{R}} iI(\phi v)(\tau)\overline{Iu(\tau)}dxd\tau. \end{aligned}$$

Now, observe that

$$-iI\phi Iu\overline{Iv} - iI\phi Iv\overline{Iu} = -2iI\phi\,\mathrm{Re}(Iu\overline{Iv}).$$

Using this identity and the fact that $I\phi$ is real-valued (recall that the multiplier q is assumed to be even), we obtain

$$2\,\mathrm{Re}\left[\int_0^{\Delta T}\int_{\mathbb{R}} -iI\phi(\tau)Iu(\tau)\overline{Iv}(\tau)dxd\tau + \int_0^{\Delta T}\int_{\mathbb{R}} -iI\phi(\tau)Iv(\tau)\overline{Iu}(\tau)dxd\tau\right] = 0.$$

We can therefore add this term to $R(\Delta T)$ for free. We remark that adding this term to $R(\Delta T)$ gives us a cancellation on the dangerous interaction in frequencies, and this makes it possible for proving some smoothing estimates. This in turn enables us to get the desired *almost conservation law* (see below for the details). We can now write

$$\begin{aligned} R(\Delta T) = 2\,\mathrm{Re}\int_0^{\Delta T}\int_{\mathbb{R}} i\{I(\phi u) - I\phi Iu\}(\tau)\overline{Iv(\tau)}dxd\tau \\ + 2\,\mathrm{Re}\int_0^{\Delta T}\int_{\mathbb{R}} i\{I(\phi v) - I\phi Iv\}(\tau)\overline{Iu(\tau)}dxd\tau. \end{aligned}$$

We therefore conclude

$$\|Iu(\Delta T)\|_{L^2}^2 + \|Iv(\Delta T)\|_{L^2}^2 = \|Iu_0\|_{L^2}^2 + \|Iv_0\|_{L^2}^2 + R(\Delta T). \tag{31}$$

The quantity that could make $\|Iu(\Delta T)\|_{L^2}^2 + \|Iv(\Delta T)\|_{L^2}^2$ too large in the future is $R(\Delta T)$. The idea is then to use bilinear estimates to show that locally in time $R(\Delta T)$ is small. By Plancherel and Cauchy-Schwarz, we obtain

$$\begin{aligned} |R(\Delta T)| \lesssim \|I(\phi u) - I\phi Iu\|_{X_-^{0,-b}(S_{\Delta T})}\|Iv\|_{X_-^{0,b}(S_{\Delta T})} \\ + \|I(\phi v) - I\phi Iv\|_{X_+^{0,-b}(S_{\Delta T})}\|Iu\|_{X_+^{0,b}(S_{\Delta T})}, \end{aligned} \tag{32}$$

for $b \in \mathbb{R}$.

We denote

$$Q_I(f,g) = I(fg) - If \cdot Ig.$$

Lemma 8. *(Smoothing estimate). Suppose*

$$-1/3 < s < 0, \qquad -s < r \le 1 + 2s. \tag{33}$$

Let $b = \frac{1}{2} + \varepsilon$ for sufficiently small $\varepsilon > 0$ depending on s, r. Then

$$\|Q_I(\phi,u)\|_{X_-^{0,-b}(S_{\Delta T})} \le CN^{-r+2s+2\varepsilon}\left\|I^2\phi\right\|_{H^{r-2s,b}(S_{\Delta T})}\|Iu\|_{X_+^{0,b}(S_{\Delta T})}, \tag{34}$$

$$\|Q_I(\phi,v)\|_{X_+^{0,-b}(S_{\Delta T})} \le CN^{-r+2s+2\varepsilon}\left\|I^2\phi\right\|_{H^{r-2s,b}(S_{\Delta T})}\|Iv\|_{X_-^{0,b}(S_{\Delta T})}, \tag{35}$$

where C depends on s, r, ε, but not N or ΔT .

In order to apply the I-method, we need a variant of Theorem 2, which we call a modified LWP Theorem for the I-modified equation

$$\begin{cases} iD_+(Iu) = MIv - I(\phi v), \\ iD_-(Iv) = MIu - I(\phi u), \\ \square(I^2\phi) = m^2 I^2\phi - 2I^2(\mathrm{Re}(u\overline{v})), \end{cases} \tag{36}$$

which is obtained from (3) by applying I. The corresponding I-initial data obtained from (4) are

$$\begin{cases} Iu(0) = Iu_0 \in L^2, \qquad Iv(0) = Iv_0 \in L^2, \\ I^2\phi(0) = I^2\phi_0 \in H^{r-2s}, \qquad \partial_t I^2\phi(0) = I^2\phi_1 \in H^{r-2s-1}. \end{cases} \tag{37}$$

Combining (31), (32), (34) and (35) we obtain, for s, r and ε as in Lemma 8 ,

$$\begin{aligned} &\|Iu(\Delta T)\|_{L^2}^2 + \|Iv(\Delta T)\|_{L^2}^2 \\ &\le \|Iu_0\|_{L^2}^2 + \|Iv_0\|_{L^2}^2 \\ &\quad + CN^{-r+2s+2\varepsilon} \left\|I^2\phi\right\|_{H^{r-2s,b}(S_{\Delta T})} \|Iu\|_{X_+^{0,b}(S_{\Delta T})} \|Iv\|_{X_-^{0,b}(S_{\Delta T})}, \end{aligned} \tag{38}$$

where C depends on s, r and ε, but not N or ΔT.

In view of (28) and (29), we have

$$\begin{aligned} &\|Iu_0\|_{L^2} + \|Iv_0\|_{L^2} \le AN^{-s}, \\ &\left\|I^2\phi_0\right\|_{H^{r-2s}} + \left\|I^2\phi_1\right\|_{H^{r-2s-1}} \le BN^{-2s}, \end{aligned} \tag{39}$$

for some $A, B > 0$. Here, A depends on $\|u_0\|_{L^2} + \|v_0\|_{L^2}$ whereas B depends on $\|\phi_0\|_{H^r} + \|\phi_1\|_{H^{r-1}}$.

We now state the modified LWP theorem which will be proved in the last section.

Theorem 5. *Suppose*

$$-\frac{1}{6} < s < 0, \quad -s \le r < \frac{1}{2} + 2s, \tag{40}$$

Let $b = \frac{1}{2} + \varepsilon$ for sufficiently small $\varepsilon > 0$ depending on s, r. Assume also that A and B in (39) are such that

$$C(B + A^2)(N^{-2\varepsilon} + N^{-r+2\varepsilon}) \le 1. \tag{41}$$

Then there exists

$$\Delta T \approx N^{(s-\varepsilon)/(r-2s-2\varepsilon)} \tag{42}$$

such that (3), (4) has a unique solution

$$(u, v, \phi) \in X_+^{s,b}(S_{\Delta T}) \times X_-^{s,b}(S_{\Delta T}) \times \mathcal{H}^{r,b}(S_{\Delta T})$$

on the time interval $0 \le t \le \Delta T$. Moreover,

$$\|Iu\|_{X_+^{0,b}(S_{\Delta T})} + \|Iv\|_{X_-^{0,b}(S_{\Delta T})} \le CAN^{-s}, \tag{43}$$

$$\left\|I^2\phi\right\|_{\mathcal{H}^{r-2s,b}(S_{\Delta T})} \le C(B + A^2)N^{-2s}, \tag{44}$$

where C depends on s, r and ε, but not N or ΔT.

Combining (38), (43) and (44) we conclude the following *almost conservation law*:

Corollary 1. *Let $s, r, \Delta T, \varepsilon, A, B, u$ and v be as in Theorem 5. Then*

$$\|Iu(\Delta T)\|_{L^2}^2 + \|Iv(\Delta T)\|_{L^2}^2 \le \|Iu_0\|_{L^2}^2 + \|Iv_0\|_{L^2}^2 + C(B + A^2)A^2N^{-r-2s+2\varepsilon}. \tag{45}$$

As a consequence of this Corollary and (39), we obtain

$$\|Iu(\Delta T)\|_{L^2}^2 + \|Iv(\Delta T)\|_{L^2}^2 \le A^2N^{-2s} + C(B + A^2)A^2N^{-r-2s+2\varepsilon}. \tag{46}$$

We also need to control the growth of $I^2\phi$. To do so, we first split ϕ into its homogeneous and inhomogeneous parts. Let $\phi^{(0)}$ be solution of the homogenous Klein-Gordon Cauchy problem

$$\begin{cases} \left(\Box - m^2\right)\phi^{(0)} = 0 \\ \phi^{(0)}(0) = \phi_0, \quad \partial_t\phi^{(0)}(0) = \phi_1. \end{cases} \tag{47}$$

Then we write

$$\phi = \phi^{(0)} + \Phi$$

where

$$\Phi = \left(\Box - m^2\right)^{-1}(-2(\mathrm{Re}(u\overline{v}))). \tag{48}$$

Here $\left(\Box - m^2\right)^{-1} F$ denotes the solution of $\left(\Box - m^2\right) w = F$ with vanishing initial data.

The solution of the homogeneous Cauchy problem (47) in Fourier space is given by

$$\widehat{\phi^{(0)}(t)}(\xi) = \cos(t\langle\xi\rangle_m)\widehat{\phi_0}(\xi) + \frac{\sin(t\langle\xi\rangle_m)}{\langle\xi\rangle_m}\widehat{\phi_1}(\xi). \tag{49}$$

Then by the energy estimate we have

$$\left\|I^2\phi^{(0)}[t]\right\|_{H^{r-2s}} \le C(\left\|I^2\phi_0\right\|_{H^{r-2s}} + \left\|I^2\phi_1\right\|_{H^{r-2s-1}}), \tag{50}$$

for some $C > 0$ and for all $t \ge 0$.

Now, consider the inhomogeneous part, (48). Since the multiplier q is assumed to be even, we obtain

$$I^2\,\mathrm{Re}(u\overline{v}) = \mathrm{Re}(I^2(u\overline{v})) = \mathrm{Re}(I(Iu \cdot I\overline{v})) + \mathrm{Re}(IQ_I(u,\overline{v})).$$

Using this identity, we write

$$I^2\Phi = \left(\Box - m^2\right)^{-1}(-2\,\mathrm{Re}(I(Iu\cdot I\overline{v}))) + (\Box - m^2)^{-1}(-2\,\mathrm{Re}(IQ_I(u,\overline{v}))). \tag{51}$$

We then prove the following:

Lemma 9. *Suppose*

$$-1/4 < s < 0, \qquad 0 < r < 1/2 + 2s. \tag{52}$$

Let $b = \frac{1}{2} + \varepsilon$ for sufficiently small $\varepsilon > 0$ depending on s, r, and ΔT be as in Theorem 5. Then

$$\begin{aligned}\left\|I^2\Phi[\Delta T]\right\|_{H^{r-2s}} &\le C\Delta T(\|Iu_0\|_{L^2}^2 + \|Iv_0\|_{L^2}^2)\\ &\quad + C\Delta T N^{-r+2s+2\varepsilon}\left\|I^2\phi\right\|_{H^{r-2s,b}(S_{\Delta T})}\|Iu\|_{X_+^{0,b}(S_{\Delta T})}\|Iv\|_{X_-^{0,b}(S_{\Delta T})}\\ &\quad + CN^{-1/2+2\varepsilon}\|Iu\|_{X_+^{0,b}(S_{\Delta T})}\|Iv\|_{X_-^{0,b}(S_{\Delta T})},\end{aligned} \tag{53}$$

where C depends on s, r, and ε, but not N or ΔT.

Then, by (39), (43), (44) and (53) we conclude

Corollary 2. *Let A, B, ΔT be as in Theorem 5 and s, r, ε be as in Lemma 9. Then*

$$\left\|I^2\Phi[\Delta T]\right\|_{H^{r-2s}} \le CA^2\left(\Delta T N^{-2s} + (B + A^2)\Delta T N^{-r-2s+2\varepsilon} + N^{-1/2-2s+2\varepsilon}\right). \tag{54}$$

By (39) and (50), we also have

$$\left\|I^2\phi^{(0)}[t]\right\|_{H^{r-2s}} \le CBN^{-2s}, \tag{55}$$

for some $C > 0$ and for all $t \ge 0$.

5. Proof of Theorem 4

We first remark that by propagation of higher regularity (see Remark 1.4 in [15] for the detail on this argument), it suffices to prove Theorem 4 for $r < 1/2 + 2s$. We therefore fix s and r satisfying

$$-\frac{1}{8} < s < 0, \quad s + \sqrt{s^2 - s} < r < \frac{1}{2} + 2s. \tag{56}$$

Observe that this region is contained in the intersection of the regions in (33), (40) and (52), so the statements made in Theorem 5, Lemmas 8 and 9, Corollaries 1 and 2, (46) and (54) hold true for s, r satisfying (56).

Global well-posedness of (3), (4) will follow if we show well-posedness on $[0, T]$ for arbitrary $0 < T < \infty$. We have already shown in Theorem 5 that (3), (4) is well-posed on $[0, \Delta T]$, where the size of ΔT is given by (42). Now, we divide the interval $[0, T]$ into subintervals of length ΔT. Let K be the number of subintervals, so

$$K = \frac{T}{\Delta T} \approx N^{(-s+\varepsilon)/(r-2s-2\varepsilon)}. \tag{57}$$

To reach the given time T, we need to advance the solution from ΔT to $2\Delta T$ etc. up to $K\Delta T$, successively.

We shall use induction argument to show well-posedness of (3), (4) up to time T. We denote the solution of (3), (4) on the n-th subinterval $[(n-1)\Delta T, n\Delta T]$, where $1 \le n \le K$, by (u_n, v_n, ϕ_n). Now, consider the DKG system

$$\begin{cases} iD_+ u_n = M v_n - \phi_n v_n, \\ iD_- v_n = M u_n - \phi_n u_n, \\ \Box \phi_n = m^2 \phi_n - 2\,\mathrm{Re}(u_n \overline{v}_n). \end{cases} \tag{58}$$

The initial data for this system at $t = (n-1)\Delta T$ is specified by the induction scheme

$$\begin{cases} u_n((n-1)\Delta T) = u_{n-1}((n-1)\Delta T) \in H^s, \\ v_n((n-1)\Delta T) = v_{n-1}((n-1)\Delta T) \in H^s, \\ \phi_n((n-1)\Delta T) = \phi_{n-1}((n-1)\Delta T) \in H^r, \\ \partial_t \phi_n((n-1)\Delta T) = \partial_t \phi_{n-1}((n-1)\Delta T) \in H^{r-1}. \end{cases} \tag{59}$$

The corresponding I-system will be

$$\begin{cases} iD_+(Iu_n) = MIv_n - I(\phi_n v_n), \\ iD_-(Iv_n) = MIu_n - I(\phi_n u_n), \\ \Box(I^2\phi_n) = m^2 I^2 \phi_n - 2I^2(\mathrm{Re}(u_n \overline{v}_n)), \end{cases} \tag{60}$$

with the I-initial data at $t = (n-1)\Delta T$:

$$\begin{cases} Iu_n((n-1)\Delta T) = Iu_{n-1}((n-1)\Delta T) \in L^2, \\ Iv_n((n-1)\Delta T) = Iv_{n-1}((n-1)\Delta T) \in L^2, \\ I^2\phi_n((n-1)\Delta T) = I^2\phi_{n-1}((n-1)\Delta T) \in H^{r-2s}, \\ \partial_t I^2\phi_n((n-1)\Delta T) = \partial_t I^2\phi_{n-1}((n-1)\Delta T) \in H^{r-2s-1}. \end{cases} \tag{61}$$

Note that for $n = 1$, this I-initial value problem corresponds to (36), (37).

In the following estimates and the rest of this section we shall use the notation

$$S_{n\Delta T} = [(n-1)\Delta T, n\Delta T] \times \mathbb{R}.$$

Recall that (u_n, v_n, ϕ_n) is a solution of DKG on the n-th subinterval $[(n-1)\Delta T, n\Delta T]$ for given data at $t = (n-1)\Delta T$. Then in view of (38) we have

$$\begin{aligned} &\|Iu_n(n\Delta T)\|_{L^2}^2 + \|Iv_n(n\Delta T)\|_{L^2}^2 \\ &\le \|Iu_n((n-1)\Delta T)\|_{L^2}^2 + \|Iv_n((n-1)\Delta T)\|_{L^2}^2 \\ &\quad + CN^{-r+2s+2\varepsilon} \left\|I^2\phi_n\right\|_{H^{r-2s,b}(S_{n\Delta T})} \|Iu_n\|_{X_+^{0,b}(S_{n\Delta T})} \|Iv_n\|_{X_-^{0,b}(S_{n\Delta T})} . \end{aligned} \tag{62}$$

On the other hand, splitting ϕ_n into its homogeneous and inhomogeneous parts, $\phi_n = \phi_n^{(0)} + \Phi_n$, we have in view of (50) and (53)

$$\begin{aligned} &\left\|I^2\Phi_n[n\Delta T]\right\|_{H^{r-2s}} \\ &\le C\Delta T(\|Iu_n((n-1)\Delta T)\|_{L^2}^2 + \|Iv_n((n-1)\Delta T)\|_{L^2}^2) \\ &\quad + C\Delta T N^{-r+2s+2\varepsilon} \left\|I^2\phi_n\right\|_{H^{r-2s,b}(S_{n\Delta T})} \|Iu_n\|_{X_+^{0,b}(S_{n\Delta T})} \|Iv_n\|_{X_-^{0,b}(S_{n\Delta T})} \\ &\quad + CN^{-1/2+2\varepsilon} \|Iu_n\|_{X_+^{0,b}(S_{n\Delta T})} \|Iv_n\|_{X_-^{0,b}(S_{n\Delta T})} , \end{aligned} \tag{63}$$

and

$$\sup_{0\le t\le T} \left\|I^2\phi_n^{(0)}[t]\right\|_{H^{r-2s}} \le C \left\|I^2\phi_n[(n-1)\Delta T]\right\|_{H^{r-2s}} . \tag{64}$$

Our induction hypotheses will be

$$\|Iu_n((n-1)\Delta T)\|_{L^2} + \|Iv_n((n-1)\Delta T)\|_{L^2} \le A_n N^{-s}, \tag{65}$$

$$\left\|I^2\phi_n[(n-1)\Delta T]\right\|_{H^{r-2s}} \le B_n N^{-2s}, \tag{66}$$

for some $1 \le n < K$, where A_n and B_n are independent of N. Again, at the first induction step, $n = 1$, (65) and (66) hold by (39). Now, by Theorem 5 we know that (u_n, v_n, ϕ_n) solves (58), (59) on the n-th subinterval $[(n-1)\Delta T, n\Delta T]$, where the size of ΔT is given by (42), provided that the boot-strap condition

$$C(B_n + A_n^2)(N^{-2\varepsilon} + N^{-r+2\varepsilon}) \le 1 \tag{67}$$

is satisfied. Moreover, these solutions satisfy the bound

$$\|Iu_n\|_{X_+^{0,b}(S_{n\Delta T})} + \|Iv_n\|_{X_-^{0,b}(S_{n\Delta T})} \le CA_n N^{-s}, \tag{68}$$

$$\left\|I^2\phi_n\right\|_{\mathcal{H}^{r-2s,b}(S_{n\Delta T})} \le C(B_n + A_n^2)N^{-2s}. \tag{69}$$

So, if we can prove that A_n and B_n stay bounded for all $1 \le n \le K$, then (67) will be satisfied for all $1 \le n \le K$, choosing ε small enough and N large enough (recall $r > 0$). We can therefore apply Theorem 5 K times, and hence prove well-posedness on $[0, T]$.

By (68), (69) and the induction hypotheses (65) and (66), the estimates (62) and (63) imply

$$\|Iu_n(n\Delta T)\|_{L^2}^2 + \|Iv_n(n\Delta T)\|_{L^2}^2 \le A_n^2 N^{-2s} + C(B_n + A_n^2)A_n^2 N^{-r-2s+2\varepsilon}, \tag{70}$$

$$\left\|I^2\Phi_n[n\Delta T]\right\|_{H^{r-2s}} \le CA_n^2 \left(\Delta T N^{-2s} + (B_n + A_n^2)\Delta T N^{-r-2s+2\varepsilon} + N^{-1/2-2s+2\varepsilon}\right), \tag{71}$$

whereas (64) and (66) imply

$$\sup_{0\le t\le T} \left\|I^2\phi_n^{(0)}[t]\right\|_{H^{r-2s}} \le CB_n N^{-2s}. \tag{72}$$

By (61) and (70) we obtain

$$\begin{aligned}\|Iu_{n+1}(n\Delta T)\|_{L^2}^2 + \|Iv_{n+1}(n\Delta T)\|_{L^2}^2 &= \|Iu_n(n\Delta T)\|_{L^2}^2 + \|Iv_n(n\Delta T)\|_{L^2}^2 \\ &\le A_n^2 N^{-2s} + C(B_n + A_n^2)A_n^2 N^{-r-2s+2\varepsilon}.\end{aligned}$$

We therefore have

$$A_{n+1}^2 \le A_n^2 + C(B_n + A_n^2)A_n^2 N^{-r+2\varepsilon}. \tag{73}$$

On the other hand, by (61), (71) and (72) we get

$$\begin{aligned}\left\|I^2\phi_{n+1}[n\Delta T]\right\|_{H^{r-2s}} &= \left\|I^2\phi_n[n\Delta T]\right\|_{H^{r-2s}} \\ &\le \left\|I^2\phi_n^{(0)}[n\Delta T]\right\|_{H^{r-2s}} + \left\|I^2\Phi_n[n\Delta T]\right\|_{H^{r-2s}} \\ &\le CB_nN^{-2s} + CA_n^2\left(\Delta T N^{-2s} + (B_n + A_n^2)\Delta T N^{-r-2s+2\varepsilon} + N^{-1/2-2s+2\varepsilon}\right)\end{aligned}$$

Therefore,

$$B_{n+1} \le CB_n + CA_n^2\Delta T + C(B_n + A_n^2)A_n^2\Delta T N^{-r+2\varepsilon} + CA_n^2 N^{-1/2+2\varepsilon}. \tag{74}$$

However, the presence of a constant C in front of B_n in the first term of the r.h.s. of this inequality is bad, since then B_n will grow exponentially in n; after n induction steps, $B_n \approx C^n$. To fix this problem, we follow [15] to write $\phi_n^{(0)}$ as a cascade of free waves:

$$\phi_{n+1}^{(0)} = \phi_1^{(0)} + \tilde\phi_2^{(0)} + \cdots + \tilde\phi_n^{(0)} + \tilde\phi_{n+1}^{(0)},$$

for $n \ge 1$, where

$$\begin{cases}\left(\Box - m^2\right)\tilde\phi_{n+1}^{(0)} = 0 \\ \tilde\phi_{n+1}^{(0)}(n\Delta T) = \Phi_n(n\Delta T), \\ \partial_t\tilde\phi_{n+1}^{(0)}(n\Delta T) = \partial_t\Phi_n(n\Delta T).\end{cases} \tag{75}$$

Now, by energy inequality and (71) we have

$$\left\|I^2\tilde\phi_{n+1}^{(0)}[t]\right\|_{H^{r-2s}} \le CA_n^2\left(\Delta T N^{-2s} + (B_n + A_n^2)\Delta T N^{-r-2s+2\varepsilon} + N^{-1/2-2s+2\varepsilon}\right), \tag{76}$$

in the entire time interval $0 \le t \le T$.

We now replace the induction hypothesis (66) by the stronger condition

$$\sup_{0\le t\le T}\left\|I^2\phi_n^{(0)}[t]\right\|_{H^{r-2s}} \le B_n N^{-2s}. \tag{77}$$

Since $\phi_{n+1}^{(0)} = \phi_n^{(0)} + \tilde\phi_{n+1}^{(0)}$, we have

$$\left\|I^2\phi_{n+1}^{(0)}[t]\right\|_{H^{r-2s}} \le \left\|I^2\phi_n^{(0)}[t]\right\|_{H^{r-2s}} + \left\|I^2\tilde\phi_{n+1}^{(0)}[t]\right\|_{H^{r-2s}},$$

for all $0 \le t \le T$. Then using (77) and (76), we conclude

$$B_{n+1} \le B_n + CA_n^2\Delta T + C(B_n + A_n^2)A_n^2\Delta T N^{-r+2\varepsilon} + CA_n^2 N^{-1/2+2\varepsilon}. \tag{78}$$

This estimate will be a replacement for the "bad" estimate (74).

Now, we claim that if $\varepsilon > 0$ is chosen small enough, and then N large enough, depending on ε, then for $1 \le n \le K$,

$$A_n \le \rho \equiv 2A_1, \qquad B_n \le \sigma \equiv 2B_1 + 4CTA_1^2. \tag{79}$$

We proceed by induction. Assume that (79) holds for $1 \le n < k$, for some $k \le K$. Then (67) reduces to

$$C(\sigma+\rho^2)(N^{-2\varepsilon}+N^{-r+2\varepsilon}) \le 1, \tag{80}$$

for $n < k$. Since $r > 0$, we can choose ε very small and N very large to ensure that (80) is satisfied. So by (73) and (78), and the assumption that (79) holds for $n < k$, we get (for $n < k$)

$$\begin{aligned} A_{n+1}^2 &\le A_1^2 + nC\sigma\rho^2 N^{-r+2\varepsilon} \\ B_{n+1} &\le B_1 + n\left[C\rho^2\Delta T + C\sigma\rho^2\Delta T N^{-r+2\varepsilon} + C\rho^2 N^{-1/2+2\varepsilon}\right]. \end{aligned}$$

Furthermore, (79) will be satisfied for A_k and B_k provided that

$$\begin{aligned} &(k-1)C\sigma\rho^2 N^{-r+2\varepsilon} \le 3A_1^2 \\ &(k-1)\left(C\rho^2\Delta T + C\sigma\rho^2\Delta T N^{-r+2\varepsilon} + C\rho^2 N^{-1/2+2\varepsilon}\right) \le B_1 + 4CTA_1^2. \end{aligned}$$

Now, since $k \le K = T/(\Delta T) \le CN^{(-s+\varepsilon)/(r-2s-2\varepsilon)}$, by (42), it suffices to have

$$C\sigma\rho^2 N^{(-s+\varepsilon)/(r-2s-2\varepsilon)-r+2\varepsilon} \le 3A_1^2, \tag{81}$$

$$CT\sigma\rho^2 N^{-r+2\varepsilon} \le B_1/2, \tag{82}$$

$$C\rho^2 N^{(-s+\varepsilon)/(r-2s-2\varepsilon)-1/2+2\varepsilon} \le B_1/2, \tag{83}$$

$$CT\rho^2 \le 4CTA_1^2. \tag{84}$$

Here, to get the l.h.s. of (84) we used the fact that $(k-1)\Delta T \le K\Delta T = T$; In fact, (84) holds with equality, since $\rho = 2A$. Since $r > 0$, (82) will be satisfied by choosing first ε small enough and then N sufficiently large. To satisfy (81) and (83), it suffices to have

$$\frac{-s+\varepsilon}{r-2s-2\varepsilon} - r + 2\varepsilon < 0, \qquad \frac{-s+\varepsilon}{r-2s-2\varepsilon} - 1/2 + 2\varepsilon < 0. \tag{85}$$

The first condition is equivalent to $r^2 - 2sr + s > \varepsilon(4(r-s)+1-4\varepsilon)$. Choosing $\varepsilon > 0$ very small, this reduces to $r^2 - 2sr + s > 0$, i.e., $r > s + \sqrt{s^2 - s}$, which holds by assumption (56). The second condition in (85) is weaker than the first condition since by assumption (56), $r < 1/2 + 2s$ and $s < 0$.

Thus, (79) holds for $n = 1, \cdots, K$, and hence the proof is complete.

6. Proof of Lemma 8

Taking the Fourier transform in space, we get

$$[Q_I(f,g)]\widehat{\ }(\xi) = \int [q(\xi) - q(\eta)q(\xi-\eta)]\hat{f}(\eta)\hat{g}(\xi-\eta)d\eta. \tag{86}$$

Recall that the symbol $q(\zeta) = 1$ for $|\zeta| < N$.

We now write $u = u_l + u_h$, $v = v_l + v_h$, $\phi = \phi_l + \phi_h$ with $\widehat{u_l}$, $\widehat{v_l}$, $\widehat{\phi_l}$ supported on $\{\xi : |\xi| \ll N\}$ and $\widehat{u_h}$, $\widehat{v_h}$, $\widehat{\phi_h}$ supported on $\{\xi : |\xi| \gtrsim N\}$. Since we are considering (weighted) L^2 norms, we can replace $\widehat{u}$, $\widehat{v}$ and $\widehat{\phi}$ by $|\widehat{u}|$, $|\widehat{v}|$ and $|\widehat{\phi}|$. Assume therefore that $\widehat{u}, \widehat{v}, \widehat{\phi} \ge 0$.

We only prove (34) since the proof for (35) is quite similar. The only difference is that to prove (34), we use the product estimate (22), but to prove (35), we use (23). We prove (34) for all possible interactions. As a matter of convenience we skip the time restriction in this section.

6.1. **Low/low interaction.** Recalling (86), we have

$$[Q_I(\phi_l, u_l)]\widehat{\ }(\xi) = \int [q(\xi) - q(\eta)q(\xi-\eta)]\hat{\phi}_l(\eta)\hat{u}_l(\xi-\eta)d\eta.$$

But since $|\eta|, |\xi-\eta| \ll N$, which in turn implies $|\xi| < N$, the expression inside the square bracket in the above integral vanishes.

6.2. **Low/high interaction.** Then

$$[Q_I(\phi_l, u_h)]\widehat{\ }(\xi) = \int [q(\xi) - q(\xi-\eta)]\hat{\phi}_l(\eta)\hat{u}_h(\xi-\eta)d\eta,$$

because $q(\eta) = 1$ on the support of $\hat{\phi}_l$. By the mean value theorem,

$$|q(\xi) - q(\xi-\eta)| \le |q'(\zeta)| |\eta|,$$

where ζ lies between ξ and $\xi - \eta$.

Now, assume $|\xi-\eta| \gg N$. Then $|\eta| \ll |\xi-\eta|$, and this implies

$$|\xi| \approx |\xi-\eta| \approx |\zeta|.$$

Hence

$$|q'(\zeta)| = N^{-s}\left|s|\zeta|^{s-1}\right| \approx N^{-s}\left|s|\xi-\eta|^{s-1}\right|$$

Next, assume $|\xi-\eta| \approx N$. If $|\zeta| < N$, then $q'(\zeta) = 0$. If $|\zeta| > 2N$, then

$$|q'(\zeta)| = N^{-s}\left|s|\zeta|^{s-1}\right| \lesssim N^{-s}|\xi-\eta|^{s-1}.$$

Finally, assume $N \le |\zeta| \le 2N$. In this case, we define $q(\xi) = \chi(\xi/N)$ where χ is a smooth, even and monotone function defined by

$$\chi(\sigma) = \begin{cases} 1 & \text{if } 0 \le \sigma < 1, \\ \sigma^s & \text{if } \sigma > 2. \end{cases}$$

Then

$$|q'(\zeta)| \lesssim N^{-1} \lesssim N^{s}|\xi-\eta|^{s-1}.$$

We therefore conclude

$$|q(\xi) - q(\xi-\eta)| \lesssim N^{-s}|\xi-\eta|^{s-1}|\eta|.$$

Interpolating this with the trivial estimate

$$|q(\xi) - q(\xi-\eta)| \lesssim N^{-s}|\xi-\eta|^{s}$$

we get

$$|q(\xi) - q(\xi-\eta)| \lesssim N^{-s}|\xi-\eta|^{s}|\xi-\eta|^{-\theta}|\eta|^{\theta},$$

for $\theta \in [0, 1]$.

Then

$$\begin{aligned} |[Q_I(\phi_l, u_h)]\widehat{\ }(\xi)| &\lesssim \int |\eta|^{\theta}\hat{\phi}_l(\eta)|\xi-\eta|^{-\theta}N^{-s}|\xi-\eta|^{s}\hat{u}_h(\xi-\eta)d\eta \\ &\lesssim [D^{\theta}\phi_l \cdot D^{-\theta}Iu_h]\widehat{\ }(\xi). \end{aligned} \tag{87}$$

Now, choosing $\theta = r - 2s$ and applying the product estimate (22), we get

$$\begin{aligned} \|Q_I(\phi_l, u_h)\|_{X_-^{0,-b}} &\lesssim \left\|D^{r-2s}\phi_l \cdot D^{-r+2s}Iu_h\right\|_{X_-^{0,-b}} \\ &\lesssim \left\|D^{r-2s}\phi_l\right\|_{H^{0,b}}\left\|D^{-r+2s}Iu_h\right\|_{X_+^{2\varepsilon,b}} \\ &\lesssim N^{-r+2s+2\varepsilon}\|\phi_l\|_{H^{r-2s,b}}\|Iu_h\|_{X_+^{0,b}}. \end{aligned}$$

6.3. **High/low interaction.** A calculation similar to the preceding low/high interaction estimate gives

$$|[Q_I(\phi_h, u_l)]^\frown(\xi)| \lesssim [D^{-\theta} I\phi_h \cdot D^\theta u_l]^\frown(\xi).$$

Take $\theta = 0$. Applying the product estimate (22) and (30), we get

$$\begin{aligned}
\|Q_I(\phi_h, u_l)\|_{X_-^{0,-b}} &\lesssim \|I\phi_h \cdot u_l\|_{X_-^{0,-b}} \\
&\lesssim \|I\phi_h\|_{H^{2\varepsilon,b}} \|u_l\|_{X_+^{0,b}} \\
&\lesssim N^{-r+s+2\varepsilon} \|I\phi_h\|_{H^{r-s,b}} \|u_l\|_{X_+^{0,b}}, \\
&\lesssim N^{-r+2s+2\varepsilon} \left\|I^2\phi_h\right\|_{H^{r-2s,b}} \|u_l\|_{X_+^{0,b}}.
\end{aligned}$$

6.4. **High/high interaction.** Here, we do not take advantage of any cancellation. We instead use the triangle inequality to get

$$\|Q_I(\phi_h, u_h)\|_{X_-^{0,-b}} \leq \|I(\phi_h u_h)\|_{X_-^{0,-b}} + \|I\phi_h \cdot Iu_h\|_{X_-^{0,-b}}.$$

By (27), the product estimate (22), and (30), we get

$$\begin{aligned}
\|I(\phi_h u_h)\|_{X_-^{0,-b}} &\lesssim \|\phi_h u_h\|_{X_-^{0,-b}} \\
&\lesssim \|\phi_h\|_{H^{-s+2\varepsilon,b}} \|u_h\|_{X_+^{s,b}} \\
&= \|\phi_h\|_{H^{r-r-s+2\varepsilon,b}} N^s N^{-s} \|u_h\|_{X_+^{s,b}} \\
&\lesssim N^{-r-s+2\varepsilon} \|\phi_h\|_{H^{r-s+s,b}} N^s \|Iu_h\|_{X_+^{0,b}} \\
&= N^{-r+s+2\varepsilon} \|I\phi_h\|_{H^{r-s,b}} \|Iu_h\|_{X_+^{0,b}}, \\
&\lesssim N^{-r+2s+2\varepsilon} \left\|I^2\phi_h\right\|_{H^{r-2s,b}} \|Iu_h\|_{X_+^{0,b}},
\end{aligned}$$

and

$$\begin{aligned}
\|I\phi_h \cdot Iu_h\|_{X_-^{0,-b}} &\lesssim \|I\phi_h\|_{H^{2\varepsilon,b}} \|Iu_h\|_{X_+^{0,b}} \\
&\lesssim N^{-r+s+2\varepsilon} \|I\phi_h\|_{H^{r-s,b}} \|Iu_h\|_{X_+^{0,b}} \\
&\lesssim N^{-r+2s+2\varepsilon} \left\|I^2\phi_h\right\|_{H^{r-2s,b}} \|Iu_h\|_{X_+^{0,b}}.
\end{aligned}$$

7. Proof of Lemma 9

First, we estimate the first term in the right hand side of (51). By energy inequality, (27), Lemma 4 and (38) we get (recall that $r < 1/2 + 2s$)

$$\begin{aligned}
&\left\| \left(\Box - m^2\right)^{-1} 2\,\mathrm{Re}(I(Iu \cdot I\overline{v}))[\Delta T] \right\|_{H^{r-2s}} \\
&\le C \int_0^{\Delta T} \|\mathrm{Re}(I(Iu(t) \cdot I\overline{v}(t)))\|_{H^{r-2s-1}}\, dt \\
&\le C \int_0^{\Delta T} \|Iu(t) \cdot I\overline{v}(t)\|_{H^{r-2s-1}}\, dt \\
&\le C \int_0^{\Delta T} \|Iu(t)\|_{L^2} \|Iv(t)\|_{L^2}\, dt \\
&\le C \int_0^{\Delta T} \|Iu(t)\|_{L^2}^2 + \|Iv(t)\|_{L^2}^2\, dt \\
&\le C\Delta T \left(\|Iu_0\|_{L^2}^2 + \|Iv_0\|_{L^2}^2 \right) \\
&\quad + C\Delta T N^{-r+2s+2\varepsilon} \left\| I^2 \phi \right\|_{H^{r-2s,b}(S_{\Delta T})} \|Iu\|_{X_+^{0,b}(S_{\Delta T})} \|Iv\|_{X_-^{0,b}(S_{\Delta T})} .
\end{aligned} \tag{88}$$

Now, we estimate the second term in the right hand side of (51). We claim that

$$\begin{aligned}
&\left\| \left(\Box - m^2\right)^{-1} 2\,\mathrm{Re}(IQ_I(u, \overline{v}))[\Delta T] \right\|_{H^{r-2s}} \\
&\le CN^{-1/2+2\varepsilon} \|Iu\|_{X_+^{0,b}(S_{\Delta T})} \|Iv\|_{X_-^{0,b}(S_{\Delta T})} .
\end{aligned} \tag{89}$$

Assume for the moment that this claim is true. Then a combination of the estimates (51), (88) and (89) proves the Lemma.

It remains to prove the claim, (89). By (15), Lemma 2 and (27)

$$\begin{aligned}
&\left\| \left(\Box - m^2\right)^{-1} \mathrm{Re}(IQ_I(u, \overline{v}))[\Delta T] \right\|_{H^{r-2s}} \\
&\le C \left\| \left(\Box - m^2\right)^{-1} \mathrm{Re}(IQ_I(u, \overline{v})) \right\|_{H^{r-2s,b}(S_{\Delta T})} \\
&\le C \|IQ_I(u, \overline{v})\|_{H^{r-2s-1,b-1}(S_{\Delta T})} \\
&\le C \|Q_I(u, \overline{v})\|_{H^{r-2s-1,b-1}(S_{\Delta T})} .
\end{aligned}$$

Then to estimate $\|Q_I(u, \overline{v})\|_{H^{r-2s-1,b-1}(S_{\Delta T})}$ we follow a similar argument as in the preceding subsection. As a matter of convenience we skip the time restriction in the rest of the section. The contribution from the low/low frequency interaction, $Q_I(u_l, v_l)$, vanishes by the same argument as in the low/low frequency case in the preceding section. For the low/high frequency case we use (87) with $\theta = 0$ (the high/low frequency case is similar) to get

$$\left| [Q_I(u_l, \overline{v_h})]^\frown(\xi) \right| \lesssim \left| [u_l \cdot I\overline{v_h}]^\frown(\xi) \right| .$$

Then by (21),

$$\begin{aligned}
\|u_l \cdot I\overline{v_h}\|_{H^{r-2s-1,b-1}} &\lesssim \|u_l\|_{X_+^{0,b}} \|Iv_h\|_{X_-^{-1/2+2\varepsilon,b}} \\
&\lesssim N^{-1/2+2\varepsilon} \|u_l\|_{X_+^{0,b}} \|Iv_h\|_{X_-^{0,b}} .
\end{aligned}$$

To estimate the contribution from high/high interaction, we first use the triangle inequality to get

$$\|Q_I(u_h, \overline{v_h})\|_{H^{r-2s-1,b-1}} \le \|I(u_h\overline{v_h})\|_{H^{r-2s-1,b-1}} + \|Iu_h \cdot I\overline{v}_h\|_{H^{r-2s-1,b-1}} .$$

Then applying (21), we obtain

$$\begin{aligned}\|I(u_h\overline{v_h})\|_{H^{r-2s-1,b-1}} &\lesssim \|u_h\overline{v_h}\|_{H^{r-2s-1,b-1}} \\ &\lesssim \|u_h\|_{X_+^{-1/4,b}} \|v_h\|_{X_-^{-1/4+2\varepsilon,b}} \\ &\lesssim N^{-1/4-s} \|u_h\|_{X_+^{s,b}} N^{-1/4-s+2\varepsilon} \|v_h\|_{X_-^{s,b}} \\ &\lesssim N^{-1/2+2\varepsilon} \|Iu_h\|_{X_+^{0,b}} \|Iv_h\|_{X_-^{0,b}} ,\end{aligned}$$

and

$$\begin{aligned}\|Iu_h \cdot I\overline{v_h}\|_{H^{r-2s-1,b-1}} &\lesssim \|Iu_h\|_{X_+^{-1/4,b}} \|Iv_h\|_{X_-^{-1/4+2\varepsilon,b}} \\ &\lesssim N^{-1/2+2\varepsilon} \|Iu_h\|_{X_+^{0,b}} \|Iv_h\|_{X_-^{0,b}} .\end{aligned}$$

8. Proof of Theorem 5

Assume $0 < \Delta T < 1$. Define

$$\begin{aligned}\|Iw\|_{X^{0,b}(S_{\Delta T})} &= \|Iu\|_{X_+^{0,b}(S_{\Delta T})} + \|Iv\|_{X_-^{0,b}(S_{\Delta T})} , \\ \|Iw_0\|_{L^2} &= \|Iu_0\|_{L^2} + \|Iv_0\|_{L^2} .\end{aligned}$$

Applying Lemma 1 to the first two equations and Lemma 2 to the third equation of the I-system (36), we get

$$\begin{aligned}\|Iu\|_{X_+^{0,b}(S_{\Delta T})} &\le C\left\{\|Iu_0\|_{L^2} + \|Iv\|_{X_+^{0,b-1}(S_{\Delta T})} + \|I(\phi v)\|_{X_+^{0,b-1}(S_{\Delta T})}\right\}, \\ \|Iv\|_{X_-^{0,b}(S_{\Delta T})} &\le C\left\{\|Iv_0\|_{L^2} + \|Iu\|_{X_-^{0,b-1}(S_{\Delta T})} + \|I(\phi u)\|_{X_-^{0,b-1}(S_{\Delta T})}\right\}, \\ \left\|I^2\phi\right\|_{\mathcal{H}^{r-2s,b}(S_{\Delta T})} &\le C\left\{\left\|I^2\phi[0]\right\|_{H^{r-2s}} + \left\|I^2(u\overline{v})\right\|_{H^{r-2s-1,b-1}(S_{\Delta T})}\right\}.\end{aligned}$$

Using estimate (16), we have

$$\begin{aligned}\|Iv\|_{X_+^{0,b-1}(S_{\Delta T})} &\le \|Iv\|_{X_-^{0,\varepsilon}(S_{\Delta T})} \lesssim (\Delta T)^{1/2-2\varepsilon} \|Iv\|_{X_-^{0,b}(S_{\Delta T})} , \\ \|Iu\|_{X_-^{0,b-1}(S_{\Delta T})} &\le \|Iu\|_{X_+^{0,\varepsilon}(S_{\Delta T})} \lesssim (\Delta T)^{1/2-2\varepsilon} \|Iu\|_{X_+^{0,b}(S_{\Delta T})} .\end{aligned} \tag{90}$$

Now, we claim the following:

$$\|I(\phi u)\|_{X_-^{0,b-1}(S_{\Delta T})} \le C\Gamma_1 \left\|I^2\phi\right\|_{H^{r-2s,b}(S_{\Delta T})} \|Iu\|_{X_+^{0,b}(S_{\Delta T})} , \tag{91}$$

$$\|I(\phi v)\|_{X_+^{0,b-1}(S_{\Delta T})} \le C\Gamma_1 \left\|I^2\phi\right\|_{H^{r-2s,b}(S_{\Delta T})} \|Iv\|_{X_-^{0,b}(S_{\Delta T})} , \tag{92}$$

$$\left\|I^2(u\overline{v})\right\|_{H^{r-2s-1,b-1}(S_{\Delta T})} \le C\Gamma_2 \|Iu\|_{X_+^{0,b}(S_{\Delta T})} \|Iv\|_{X_-^{0,b}(S_{\Delta T})} , \tag{93}$$

where

$$\begin{aligned}\Gamma_1 = \Gamma_1(N,\Delta T) &:= (\Delta T)^{2r-4s-4\varepsilon} + N^{-r+2s+2\varepsilon}, \\ \Gamma_2 = \Gamma_2(N,\Delta T) &:= (\Delta T)^{1-4\varepsilon} + N^{-1/2+2\varepsilon}.\end{aligned}$$

Assume for the moment that the claim is true. Then using (90)–(93),

$$\begin{aligned}\|Iw\|_{X^{0,b}(S_{\Delta T})} \le C\|Iw_0\|_{L^2} &+ C(\Delta T)^{1/2-2\varepsilon} \|Iw\|_{X^{0,b}(S_{\Delta T})} \\ &+ C\Gamma_1 \left\|I^2\phi\right\|_{H^{r-2s,b}(S_{\Delta T})} \|Iw\|_{X^{0,b}(S_{\Delta T})} ,\end{aligned} \tag{94}$$

$$\left\|I^2\phi\right\|_{\mathcal{H}^{r-2s,b}(S_{\Delta T})} \le C\left\|I^2\phi[0]\right\|_{H^{r-2s}} + C\Gamma_2 \|Iw\|^2_{X^{0,b}(S_{\Delta T})}. \tag{95}$$

In (94), the term $\|Iw\|_{X^{0,b}(S_{\Delta T})}$ can be moved to the left hand side provided

$$C(\Delta T)^{1/2-2\varepsilon} < 1. \tag{96}$$

Thus, using (95), the estimate (94) reduces to

$$\begin{aligned}\|Iw\|_{X^{0,b}(S_{\Delta T})} &\le C\|Iw_0\|_{L^2} + C\Gamma_1\left\|I^2\phi[0]\right\|_{H^{r-2s}}\|Iw\|_{X^{0,b}(S_{\Delta T})} + C\Gamma_1\Gamma_2\|Iw\|^3_{X^{0,b}(S_{\Delta T})}\\ &\le CAN^{-s} + CBN^{-2s}\Gamma_1\|Iw\|_{X^{0,b}(S_{\Delta T})} + C\Gamma_1\Gamma_2\|Iw\|^3_{X^{0,b}(S_{\Delta T})}.\end{aligned} \tag{97}$$

So if

$$CBN^{-2s}\Gamma_1(2CAN^{-s}) + C\Gamma_1\Gamma_2(2CAN^{-s})^3 \le CAN^{-s}, \tag{98}$$

then it follows by a boot-strap argument (see the Remark below for the detail on this argument) that

$$\|Iw\|_{X^{0,b}(S_{\Delta T})} \le 2CAN^{-s}. \tag{99}$$

Now, if we choose

$$\Delta T \approx N^{(s-\varepsilon)/(r-2s-2\varepsilon)}, \tag{100}$$

the boot-strap condition (98) reduces to (modifying C)

$$C(B + A^2)\left(N^{-2\varepsilon} + N^{-r+2\varepsilon}\right) \le 1. \tag{101}$$

Note that the choice of ΔT in (100) also satisfies condition (96) since N is assumed to.

On the other hand, by (95) we get (modifying C)

$$\left\|I^2\phi\right\|_{\mathcal{H}^{r-2s,b}(S_{\Delta T})} \le CBN^{-2s} + 4CA^2N^{-2s}\left(N^{(s-\varepsilon)(1-4\varepsilon)/(r-2s-2\varepsilon)} + N^{-1/2+2\varepsilon}\right).$$

The second term in the r.h.s. of this inequality can be bounded by $C(B+A^2)N^{-2s}$ since the quantity in the bracket is very small. So, we obtain

$$\left\|I^2\phi\right\|_{\mathcal{H}^{r-2s,b}(S_{\Delta T})} \le 2C(B + A^2)N^{-2s}. \tag{102}$$

Remark 3. The above estimates imply LWP of (3), (4) with time of existence up to $\Delta T > 0$ given by (100) provided that the condition (101) is satisfied. The boot-strap argument mentioned above can be shown using the standard iteration argument: Set $u^{(-1)} = v^{(-1)} = 0$, and define for $n \ge -1$ inductively

$$\begin{cases} iD_+(Iu^{(n+1)}) = MIv^{(n)} - I(\phi^{(n)}v^{(n)}),\\ iD_-(Iv^{(n+1)}) = MIu^{(n)} - I(\phi^{(n)}u^{(n)}),\\ Iu^{(n+1)}(0) = Iu_0 \in L^2, \quad Iv^{(n+1)}(0) = Iv_0 \in L^2,\end{cases} \tag{103}$$

where

$$\Box\phi^{(n)} = m^2\phi^{(n)} - 2\,\mathrm{Re}(u^{(n)}\overline{v}^{(n)}),$$

with the same data as for ϕ.

Then, defining $y_n = \|Iw^n\|_{X^{0,b}}$ for $n \ge 0$, (97) becomes

$$y_{n+1} \le CAN^{-s} + CBN^{-2s}\Gamma_1 y_n + C\Gamma_1\Gamma_2 y_n^3.$$

By (12) and (39), $y_0 \le 2CAN^{-s}$. Now, if (101) holds, we conclude by induction that $y_n \le 2CAN^{-s}$ for all $n \ge 0$. On the other hand, we know from [16] that $(u^{(n)}, v^{(n)}) \to (u, v) \in X_+^{s,b} \times X_-^{s,b}$ as $n \to \infty$, which implies $Iw^{(n)} \to Iw \in X^{0,b}$ as $n \to \infty$, and hence (99) follows.

It remain to prove the claim; i.e., (91)–(93). The estimates (91) and (92) are symmetrical. Hence we only prove (91) and (93). As in Section 6, we decompose u, v, ϕ into low and high frequencies, and prove the bilinear estimates for all possible interactions.

8.1. **Proof of (91).** We recall that $s > -1/6$, $-s \le r < 1/2+2s$, $b = 1/2+\varepsilon$, and the operator I is the identity for low frequencies. Note also that low-low interaction yields low frequency output. Then for the low/low interaction, we have by (27) and the product estimate (22)

$$\begin{aligned} \|I(\phi_l u_l)\|_{X_-^{0,b-1}(S_{\Delta T})} &\le C \|\phi_l u_l\|_{X_-^{0,b-1}(S_{\Delta T})} \\ &\le C \|\phi_l\|_{H^{2\varepsilon,b}(S_{\Delta T})} \|u_l\|_{X_+^{0,b}(S_{\Delta T})}. \end{aligned} \tag{104}$$

On the other hand, by (27), Lemma 3, Hölder in t, (18) and (15), we have

$$\begin{aligned} \|I(\phi_l u_l)\|_{X_-^{0,b-1}(S_{\Delta T})} &\le C \|\phi_l u_l\|_{X_-^{0,b-1}(S_{\Delta T})} \\ &\le C \|\phi_l u_l\|_{L_t^{\frac{1}{1-2\varepsilon}} L_x^2(S_{\Delta T})} \\ &\le C(\Delta T)^{1-2\varepsilon} \|\phi_l u_l\|_{L_t^\infty L_x^2(S_{\Delta T})} \\ &\le C(\Delta T)^{1-2\varepsilon} \|\phi_l\|_{L_t^\infty H_x^{1/2+\varepsilon}(S_{\Delta T})} \|u_l\|_{L_t^\infty L_x^2(S_{\Delta T})} \\ &\le C(\Delta T)^{1-2\varepsilon} \|\phi_l\|_{H^{1/2+\varepsilon,b}(S_{\Delta T})} \|u_l\|_{X_+^{0,b}(S_{\Delta T})}. \end{aligned} \tag{105}$$

Then interpolation between (104) and (105), for $0 \le \theta \le 1$, gives

$$\begin{aligned} &\|I(\phi_l u_l)\|_{X_-^{0,b-1}(S_{\Delta T})} \\ &\le C(\Delta T)^{(1-2\varepsilon)\theta} \|\phi_l\|_{H^{2\varepsilon(1-\theta)+(1/2+\varepsilon)\theta,b}(S_{\Delta T})} \|u_l\|_{X_+^{0,b}(S_{\Delta T})}. \end{aligned}$$

We take $\theta = \frac{2r-4s-4\varepsilon}{1-2\varepsilon}$ (by the hypothesis made on s, r, we then have $\theta \in [0,1]$). This implies $2\varepsilon(1-\theta)+(1/2+\varepsilon)\theta = r-2s$ and $(1-2\varepsilon)\theta = 2r-4s-4\varepsilon$. Consequently,

$$\|I(\phi_l u_l)\|_{X_-^{0,b-1}(S_{\Delta T})} \le C(\Delta T)^{2r-4s-4\varepsilon} \|\phi_l\|_{H^{r-2s,b}(S_{\Delta T})} \|u_l\|_{X_+^{0,b}(S_{\Delta T})}. \tag{106}$$

The contribution from low/high can be estimated using (27) and the product estimate (22) as

$$\begin{aligned} \|I(\phi_l u_h)\|_{X_-^{0,b-1}(S_{\Delta T})} &\le C \|\phi_l u_h\|_{X_-^{0,b-1}(S_{\Delta T})} \\ &\le C \|\phi_l\|_{H^{r-2s,b}(S_{\Delta T})} \|u_h\|_{X_+^{2s-r+2\varepsilon,b}(S_{\Delta T})} \\ &= C \|\phi_l\|_{H^{r-2s,b}(S_{\Delta T})} N^s N^{-s} \|u_h\|_{X_+^{s+s-r+2\varepsilon,b}(S_{\Delta T})} \\ &\le CN^{-r+2s+2\varepsilon} \|\phi_l\|_{H^{r-2s,b}(S_{\Delta T})} \|Iu_h\|_{X_+^{0,b}(S_{\Delta T})}. \end{aligned} \tag{107}$$

The contribution from high/low can be estimated using (27), the product estimate (22), and (30) as

$$\begin{aligned} \|I(\phi_h u_l)\|_{X_-^{0,b-1}(S_{\Delta T})} &\le C \|\phi_h u_l\|_{X_-^{0,b-1}(S_{\Delta T})} \\ &\le C \|\phi_h\|_{H^{2\varepsilon,b}(S_{\Delta T})} \|u_l\|_{X_+^{0,b}(S_{\Delta T})} \\ &\lesssim CN^{-r+s-s+2\varepsilon} \|\phi_h\|_{H^{r-s+s,b}(S_{\Delta T})} \|u_l\|_{X_+^{0,b}(S_{\Delta T})} \\ &= CN^{-r+s+2\varepsilon} \|I\phi_h\|_{H^{r-s,b}(S_{\Delta T})} \|u_l\|_{X_+^{0,b}(S_{\Delta T})} \\ &\le CN^{-r+2s+2\varepsilon} \left\|I^2\phi_h\right\|_{H^{r-2s,b}(S_{\Delta T})} \|u_l\|_{X_+^{0,b}(S_{\Delta T})}. \end{aligned} \tag{108}$$

Similarly, we estimate the high/high interaction using (27), the product estimate (22), and (30) as

$$\begin{aligned}\|I(\phi_h u_h)\|_{X_-^{0,b-1}(S_{\Delta T})} &\le C\,\|\phi_h u_l\|_{X_-^{0,b-1}(S_{\Delta T})}\\ &\le C\,\|\phi_h\|_{H^{-s+2\varepsilon,b}(S_{\Delta T})}\,\|u_h\|_{X_+^{s,b}(S_{\Delta T})}\\ &= C\,\|\phi_h\|_{H^{r-s-r+2\varepsilon,b}(S_{\Delta T})}\,N^sN^{-s}\,\|u_h\|_{X_+^{s,b}(S_{\Delta T})}\\ &\le CN^{-r+s+2\varepsilon}\,\|I\phi_h\|_{H^{r-s,b}(S_{\Delta T})}\,\|Iu_h\|_{X_+^{0,b}(S_{\Delta T})}\\ &\le CN^{-r+2s+2\varepsilon}\,\left\|I^2\phi_h\right\|_{H^{r-2s,b}(S_{\Delta T})}\,\|Iu_h\|_{X_+^{0,b}(S_{\Delta T})}.\end{aligned} \tag{109}$$

Therefore, (91) follows from the estimates (106)–(109).

8.2. **Proof of** (93)**.** We recall that $s > -1/6$, $-s \le r < 1/2 + 2s$ and $b = 1/2 + \varepsilon$. Noting that I is the identity for low frequencies, we have by (27), Lemma 5 and (16)

$$\begin{aligned}\left\|I^2(u_l\overline{v_l})\right\|_{H^{r-2s-1,b-1}(S_{\Delta T})} &\le C\,\|u_l\overline{v_l}\|_{H^{r-2s-1,-1/2+\varepsilon}(S_{\Delta T})}\\ &\le C\,\|u_l\|_{X_+^{0,\varepsilon}(S_{\Delta T})}\,\|v_l\|_{X_-^{0,\varepsilon}(S_{\Delta T})}\\ &\le C(\Delta T)^{1-4\varepsilon}\,\|u_l\|_{X_+^{0,1/2-\varepsilon}(S_{\Delta T})}\,\|v_l\|_{X_-^{0,1/2-\varepsilon}(S_{\Delta T})}\\ &\le C(\Delta T)^{1-4\varepsilon}\,\|u_l\|_{X_+^{0,b}(S_{\Delta T})}\,\|v_l\|_{X_-^{0,b}(S_{\Delta T})}.\end{aligned} \tag{110}$$

The contribution from low/high interaction is estimated using (27) and the product estimate (21) as

$$\begin{aligned}\left\|I^2(u_l\overline{v_h})\right\|_{H^{r-2s-1,b-1}(S_{\Delta T})} &\le C\,\|u_l\overline{v_h}\|_{H^{-1/2,b-1}(S_{\Delta T})}\\ &\le C\,\|u_l\|_{X_+^{0,b}(S_{\Delta T})}\,\|v_h\|_{X_-^{-1/2+2\varepsilon,b}(S_{\Delta T})}\\ &= C\,\|u_l\|_{X_+^{0,b}(S_{\Delta T})}\,\|v_h\|_{X_-^{-1/2-s+2\varepsilon+s,b}(S_{\Delta T})}\\ &\le C\,\|u_l\|_{X_+^{0,b}(S_{\Delta T})}\,N^{-1/2-s+2\varepsilon}\,\|v_h\|_{X_-^{s,b}(S_{\Delta T})}\\ &= CN^{-1/2+2\varepsilon}\,\|u_l\|_{X_+^{0,b}(S_{\Delta T})}\,\|Iv_h\|_{X_-^{0,b}(S_{\Delta T})}.\end{aligned} \tag{111}$$

By symmetry

$$\left\|I^2(u_h\overline{v_l})\right\|_{H^{r-2s-1,b-1}(S_{\Delta T})} \le CN^{-1/2+2\varepsilon}\,\|Iu_h\|_{X_+^{0,b}(S_{\Delta T})}\,\|v_l\|_{X_-^{0,b}(S_{\Delta T})}. \tag{112}$$

Finally, for the high/high interaction we obtain using (27) and the product estimate (21)

$$\begin{aligned}\left\|I^2(u_h\overline{v_h})\right\|_{H^{r-2s-1,b-1}(S_{\Delta T})} &\le C\,\|u_h\overline{v_h}\|_{H^{-1/2,b-1}(S_{\Delta T})}\\ &\le C\,\|u_h\|_{X_+^{-1/4+2\varepsilon,b}(S_{\Delta T})}\,\|v_h\|_{X_-^{-1/4,b}(S_{\Delta T})}\\ &\le CN^{-1/4-s+2\varepsilon}\,\|u_h\|_{X_+^{s,b}(S_{\Delta T})}\,N^{-1/4-s}\,\|v_h\|_{X_-^{s,b}(S_{\Delta T})}\\ &= CN^{-1/2+2\varepsilon}\,\|Iu_h\|_{X_+^{0,b}(S_{\Delta T})}\,\|Iv_h\|_{X_-^{0,b}(S_{\Delta T})}.\end{aligned} \tag{113}$$

We therefore conclude that (93) follows from the estimates (110)–(113).

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