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On a Traffic Flow Model

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PREFACE

This thesis is submitted for the degree philosophiae doctor (PhD) at the Norwegian University of Science and Technology (NTNU) in Trondheim, Norway. The work has been financed by and performed at the Department of Mathematical Sciences, NTNU. The thesis is based on three research papers, and my supervisor has been Harald Hanche-Olsen, NTNU.

The topic of this PhD project has been modeling of vehicular traffic flow by nonlinear partial differential equations and in particular hyperbolic conservation laws. However, it turned out that all the research contained in this thesis concerns about the same macroscopic traffic flow model.

I am grateful for the opportunities I have been given during the last years to study and do mathematics, learn about how to do research, meet great mathematicians and take part at conferences. First of all I would like to thank my supervisor Harald Hanche-Olsen for his valuable support. He has taught me a lot about mathematics and showed me alternative ways to attack and solve problems. I would also like to thank Helge Holden for his constant encouragement.

It has been great to be part of the differential equation group. In particular I would like to thank Hilde for our discussions and travels. My colleagues and friends at the department have all contributed to making these years, which have been filled with hard work, joyful and happy.

Finally I would like to express my gratitude to my friends, my family and Eskil for being patient with me during these years.

Marte Godvik,
Trondheim, September 2008

CONTENTS

INTRODUCTION

Macroscopic models for traffic flow, which are based on nonlinear hyperbolic conservation laws, have received a lot of attention and been investigated intensively during the last years. In these models cars are considered as small particles and the main quantity to model is the density of the cars on the roadway. The topic of this thesis is one such model, namely the so-called Aw–Rascle model proposed by Aw and Rascle in 2000.

Macroscopic models of traffic flow: an overview

Modeling of traffic flow by macroscopic models started back in the 1950s by Lighthill and Whitham [36] and independently Richards [42] proposing to apply fluid dynamics concepts to traffic flow. The Lighthill–Whitham–Richards (LWR) model describes car traffic on a one-dimensional unidirectional single roadway. It is a so-called first order model, i.e. a scalar conservation law, expressing conservation of mass,

$$\rho_t + \big(\rho V(\rho)\big)_x = 0,$$

where $x \in \mathbb{R}$ is the position on the roadway, $t \in \mathbb{R}^+$ denotes time and $\rho = \rho(x, y)$ is the density of cars on the roadway. The velocity of the cars, $V(\rho)$, is nonincreasing and nonnegative for $\rho \in [0, \rho_{\max}]$, where $\rho_{\max}$ denotes some maximal density. Despite the simplicity of the LWR model it correctly models important features like formation, propagation and dissolution of traffic flow. The model still receives a lot of attention, both from a mathematical modeling, an analytical and a numerical point of view.

Many different features have been included in the model giving rise to analytical studies. In [30] the model is extended to include merging and diverging traffic, and also a mixed type of vehicles is taken into account. This is done by including a time independent inhomogeneity factor $a(x)$ in the flux function. A study of the model with inhomogeneous road conditions is given in [38]. Furthermore, the model is generalized to multi-class flow, see [5]. More precisely, the density of each class i is described by the LWR model, and the equations are coupled through the velocity functions V_i, which depends on the density of all classes. In [11] and [12] multilane flow is modeled by n inhomogeneous scalar LWR equations, one for each lane, and the equations are coupled through the source terms, which describe the intra-lane dynamics. Furthermore, in [14] the LWR model is studied with a variable unilateral constraint modeling for instance a toll gate at the roadway. Also, as considered in [15], the phenomena of phase transition is included in the model. The velocity $V(\rho)$ consists of two disconnected functions, which are defined on disjoint intervals representing the free and congested phase, respectively.

The model is also extended to describe traffic flow on a network, that is a collection of unidirectional roads connected by junctions. This was first done by Holden and Risebro in [29]. On each single road the traffic is given by the LWR model. The system is underdetermined at the junctions, and so-called coupling conditions for the junctions have to be given. There are different approaches for treating the traffic distributions at the junctions, and another set of coupling conditions is given in [7]. A different point of view for junctions is presented in [37], which takes into account the interactions of the cars at a junctions. The networks are extended to include traffic lights, bottlenecks and traffic circles, see Piccoli and Garavello [19]. Furthermore, there is also a network model which includes sources and destinations, that are areas from which the cars start and end their travels. This model is actually a multi-class model on a road network and is discussed in [17] and [24]. In [26] highway networks with junctions are described as merging and dispersing roadways.

The models on networks are used in numerical simulation, see for instance [6]. The goal is modeling of large highways and complex road networks in big cities. This may help predicting the traffic behavior so that road networks can be planned to for instance avoid congestion and maximize traffic flow.

In the LWR model the velocity of cars depends on the density only, which is not always a realistic model. Trying to overcome this drawback and include other effects such as the time lag of the response of the drivers and the cars gave rise to the so-called second order models, that is models of two equations. The prototype of these models was until recently the Payne–Whitham (PW) model from the early 1970s, see [40] and [46]. This model is inspired by fluid flow and includes, in addition to the density, also the velocity $v = v(x,t)$ of cars on the roadway. In non conservative form the model is

$$\begin{aligned} \rho_t + \big(\rho v\big)_x &= 0, \\ v_t + v v_x + \frac{1}{\rho} p(\rho)_x &= \frac{1}{\epsilon}\big(V(\rho) - v\big), \end{aligned}$$

where $p(\rho)$ is a so-called pressure function, $V(\rho)$ denotes an equilibrium velocity and ϵ denotes the relaxation time.

However, there are some differences between gas and traffic flow, as pointed out by Daganzo in [16]. For instance, in contrast to gas, traffic flow does not respond to stimuli from behind, so vehicles should not influence the behavior of the vehicles in front of them. Daganzo shows that the second-order models give some unphysical effects and concludes by rejecting these models. The drawbacks of these models are clearly explained in [2]. First, under some conditions the models predict negative flows and negative speed, that is the cars drive backwards. Next, the characteristic speed of the second wave family is larger than the velocity of the cars, so some information travels faster than the cars, which is clearly unsatisfactory. Furthermore, unlike gas, a driver with dense and fast traffic in front of him or her will accelerate instead of braking.

In 2000 Aw and Rascle [2] proposed a new second order model for traffic flow. Motivated by the drawbacks of the PW model, they simply replaced the time derivative in the second equation in the homogeneous model by a convective derivative. It turned out that by this modification they resolved the inconsistencies of the PW model. In [47] Zhang independently proposed the same model. However, in this thesis the model is considered in the framework by Aw and Rascle. The model is given as

$$\left.\begin{aligned} \rho_t + (\rho v)_x &= 0 \\ (\rho w)_t + (\rho w v)_x &= 0 \end{aligned}\right\} \qquad \text{where } w = v + p(\rho). \tag{1}$$

The function w can be viewed as the preferred velocity of the drivers, while the "density" function p slows down the traffic. We will assume that the function $p(\rho)$ is smooth, strictly increasing and Lipschitz continuous in its argument and satisfies $p(0) = 0$ and $p'(0) = 0$. The prototype of this function is $p(\rho) \propto \rho^\gamma$ where $\gamma > 1$. The eigenvalues of the model are $\lambda_1 = v - \rho p'(\rho)$ and $\lambda_2 = v$, so the waves do not travel faster than the traffic, and vehicles will not be influenced by what happens behind. Furthermore, as discussed in [2] the model predicts instabilities near vacuum, which might be reasonable in traffic with very few drivers.

An improved version of the model includes a relaxation term in the second equation, which accounts for drivers' attempt to drive at some ideal speed and the time lag in the response of the driver and the car. The model is

$$\left.\begin{aligned} \rho_t + (\rho v)_x &= 0 \\ (\rho w)_t + (\rho w v)_x &= \frac{1}{\epsilon}\rho\big(V(\rho) - v\big) \end{aligned}\right\} \qquad \text{where } w = v + p(\rho), \tag{2}$$

and the relaxation time ϵ corresponds to the average acceleration time. The equilibrium velocity $V(\rho)$ is smooth and satisfies the subcharacteristic condition, that is $-p'(\rho) \leq V'(\rho) \leq 0$. This model is discussed in for instance [41].

A two lane extension of the Aw–Rascle model is proposed in [23] . It gives (ρ, v), taken over both lanes, on a multilane roadway by incorporating two equilibrium functions $V_1(\rho)$ and $V_2(\rho)$ and a switching mechanism in the model. In [3] the model is extended to include a source term that models a highway entry. For this model, in order to achieve a meaningful invariant domain, the pressure function has to be negative, for instance $p(\rho) = C \ln \frac{\rho}{\rho_{\max}}$. Furthermore, in [44] another source term to the second equation of the model is introduced, motivated by experimental data. Thus, the system yields an unstable regime for intermediate densities, as observed in real traffic dynamics and discussed in [32]. Another approach for modeling this instability is given in [20], where the Aw–Rascle model and the LWR model are coupled together as a phase transition model. Like the LWR model, the Aw–Rascle model is also extended to networks. On each road in the network the traffic flow is modeled by this second order model. Different sets of coupling conditions are given in [18], [27] and [28]. In [35] the Zhang-version of the model (2) is extended to include time dependent effects such as weather conditions, traffic jams and traffic congestion managements by allowing the functions corresponding to p and V to depend on time.

In Lagrangian coordinates the Aw–Rascle system is

$$\left.\begin{aligned} \tau_t - v_y &= 0 \\ w_t &= R(\tau, w) \end{aligned}\right\} \qquad \text{where } w = v + Q(\tau), \tag{3}$$

$\tau(y,t)$ is the inverse density of cars on the roadway, and $y \in \mathbb{R}$ and $t \in \mathbb{R}^+$ are the Lagrangian mass and time variable, respectively. Note that we abuse the notation such that v and w depend on (y,t) or (x,t) depending on whether we consider the Lagrangian or Eulerian form of the system. The function $Q(\tau) = p(\rho)$ and the assumptions on Q corresponds to the assumptions on p. Furthermore, the function R denotes the relaxation term. For further discussions of the model and different choices of R, see [21] and [22]. Due to the results by Wagner in [45] the Eulerian and Lagrangian form of the Aw–Rascle system are equivalent.

In [4] the homogenoues model is used to describe multi-class traffic flow by introducing a class variable a. The behavior of different types of vehicles or drivers is accounted for in the model by setting $w = v + \tilde{Q}(\tau, a)$ where, for instance, $\tilde{Q}(a, \tau) = aQ(\tau)$. Since the model is in Lagrangian form, a is constant in time and thus given by $a_t = 0$. An extension of this multi-class model to road networks is proposed in [25].

The Aw–Rascle model has been rigorously derived from a microscopic car-following model,

$$\left.\begin{aligned} \dot{\tau}_k^\delta(t) &= \frac{1}{\delta}\left(v_{k+1}^\delta(t) - v_k^\delta(t)\right) \\ \dot{w}_k^\delta(t) &= R\left(\tau_k^\delta(t), w_k^\delta(t)\right) \end{aligned}\right\} \qquad \text{where } w_k^\delta = v_k^\delta + Q(\tau_k^\delta). \tag{4}$$

The functions $\tau_k^\delta(t)$ and $v_k^\delta(t)$ are the inverse density (the distance to car $k+1$) and velocity of car k, respectively. The Lagrangian position of car k on the roadway is $y = k\delta$ for some $\delta > 0$. This semi-discrete car-following model was proposed by Aw, Klar, Materne and Rascle in [1], and they show a connection between this microscopic model and a semi-discretization of the macroscopic Aw–Rascle model. Furthermore, for the homogeneous model without vacuum they establish that the semi-discretization of the macroscopic model is the limit of the time discretization of the microscopic model. In **Paper II** the Aw–Rascle model is derived directly from this microscopic model.

Recently a fully discrete hybrid model, which combines a macroscopic description based on the Aw–Rascle model away from obstacles as for instance traffic lights and a microscopic view near the obstacles, is proposed in [39].

Inspired by the Aw–Rascle model, Colombo proposed in [8] another new second order model. This model predicts the density of cars and an auxiliary variable motivated by the linear momentum in gas dynamics, and the velocity of the cars is given as a function of these two variables. In [3] the model is extended to include entries and exists and changes in the traffic speed due to inhomogeneities of the roadway. All these features are introduced in the model as different choices of source terms. Furthermore, in order to describe phase transitions in traffic flow the model is combined with the LWR-model, see [9] and [15]. In [13] this phase transition model is extended to comprehend junctions.

Weak solutions of the Aw–Rascle model

In this thesis we study global weak solutions of the Aw–Rascle model and address questions concerning existence, entropy admissibility, uniqueness and stability of such solutions. The main difficulty is the appearance of vacuum in the initial data and in the solutions.

The Aw–Rascle model, as given in (1), is a system of hyperbolic conservation laws. The eigenvalues of the model is,

$$\lambda_1 = v - \rho p'(\rho), \qquad \lambda_2 = v.$$

Away from vacuum the system is strictly hyperbolic, and, under the assumption that $\rho p''(\rho) + 2p'(\rho) > 0$, the first wave family is genuinely nonlinear and the second wave family is linearly degenerate. Furthermore, the system is of Temple class, that is the wave curves of the shock and rarefaction waves coincide. The Riemann invariants are w and v, and combining the above properties yields that the wave curves are given by $w = \text{const.}$ and $v = \text{const.}$ We will denote the conservative variables $(\rho, \rho w)$ by u.

Weak solutions away from vacuum. In general, a solution of the Riemann problem without vacuum is given by a rarefaction wave or a shock connecting the left state to some middle state and then the middle state connects to the right state by a contact discontinuity. Vacuum appear in the solution of a Riemann problem if $w_{\mathrm{L}} < v_{\mathrm{R}}$, where L and R denote the left and right state, respectively. Furthermore, as discussed in **Paper I**, for initial data consisting of several Riemann problems vacuum may appear in the weak solution even though it is not present in the initial data or appears immediately. Thus, in order to stay away from vacuum invariant domains for the Riemann problem are of the form

$$\mathcal{D} = \{u \in \mathbb{R}^2 : w_{\min} \leq w(u) \leq w_{\max}, 0 \leq v_{\min} \leq v(u) \leq v_{\max}\}, \tag{5}$$

where $w_{\min} > v_{\max}$. The domain $\mathcal{D}$ and a general solution of the Riemann problem in the (w, v)-plane are shown in figure 1(a). Note however that these invariant domains are severely restricted.

Existence of weak solutions $u \in \mathrm{BV}(\mathbb{R})^2$ for system (1) with initial data $\bar{u}$ in $\mathcal{D}$ and initial Riemann invariants $(\bar{w}, \bar{v}) \in \mathrm{BV}(\mathbb{R})^2$ follows by the Glimm scheme, see for instance [43, Theorem 5.4.1]. Since the wave curves are given by the Riemann invariants, the approximate solution defined by the Glimm scheme has Riemann invariants with nonincreasing total variation. Away from vacuum $(p^{-1})'$ is bounded, which yields a bound on the total variation of the conservative variables of the Glimm approximation. Convergence of a subsequence of approximate solutions follows by Helly's theorem. The limit is a weak solution which satisfies the following entropy inequality,

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}} \big(\eta(u)\phi_t + q(u)\,\phi_x\big)\,dx\,dt + \int_{\mathbb{R}} \eta(\bar{u})\phi(x,0)\,dx \geq 0, \tag{6}$$

for all nonnegative test functions $\phi(x,t)$ in $C_0^\infty(\mathbb{R}\times\mathbb{R}^+)$ and all strictly convex entropies $\eta(u)$ with flux $q(u)$.

This existence result is extended to the Aw–Rascle model with source terms, like the system with relaxation (2), by operator splitting assuming the domain $\mathcal{D}$ is invariant for the system of ordinary differential equations generated by the right hand side, see **Paper I**. Furthermore, these inhomogeneous systems, with initial data in $\mathcal{D}$, often satisfy the assumptions made by Colombo and Corli in [10]. They show well-posedness for strictly hyperbolic Temple systems with source, assuming the eigenvalues are separated on every compact subset of this domain. In particular, away from vacuum this last assumption is satisfied for the case $p(\rho)=\rho^\gamma$ where $\gamma>1$, which yields $\max_{u\in\mathcal{U}}\lambda_1 < 0 < \min_{u\in\mathcal{U}}\lambda_2$. Furthermore, in [3] well-posedness is proved with $p(\rho) = P\ln\frac{\rho}{\rho_{\max}}$, where P is some positive constant, and the source terms modeling highway entries.

The Zhang version of the Aw–Rascle model with relaxation is studied in [33] and [34]. Weak entropy solutions are constructed and uniqueness is proved by means of a finite difference approximation.

In **Paper II** we construct a weak solution by considering the Lagrangian form of the Aw–Rascle system (3). Our approach is to carefully study the semi-discrete car-following model given by (4). For initial data of bounded variation taking values in $\mathcal{D}$ the solution (τ^δ, w^δ) of the semi-discrete car-following model converges in $L^1_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}^+)^2$ to a weak solution of (3). Furthermore, the solution satisfies an entropy inequality corresponding to the one given in (6). In particular, this approach yields a direct derivation of the Aw–Rascle model with relaxation.

For the homogeneous system, that is for $R=0$, we show monotonicity of $\tau^\delta(t)$ with respect to the initial data, that is if

$$\bar{\tau}_k^1 \leq \bar{\tau}_k^2,\;\; w_{k+1}^1 - w_k^1 \leq w_{k+1}^2 - w_k^2 \quad \text{for all } \; k,$$

then

$$\tau_k^1(t) \leq \tau_k^2(t) \quad \text{for all } \; k.$$

By using this monotonicity property, we prove that the constructed solution $\tau(y,t)$ is stable with respect to the initial data,

$$\|\tau^1(t) - \tau^2(s)\| \leq \left\|\bar{\tau}^1 - \bar{\tau}^2\right\| + (t\wedge s)\,\mathrm{TV}(w^1-w^2) + C|t-s|, \tag{7}$$

where $C = \mathrm{TV}(\bar{v}^1)\wedge\mathrm{TV}(\bar{v}^2)$.

In [31] Karlsen, Risebro and Towers give, as a special case, uniqueness criteria for scalar hyperbolic equations with discontinuous flux. Since w is constant in time, the Lagrangian form of the homogeneous Aw–Rascle model rewrites as

$$\tau_t + \big(Q(\tau) - w\big)_y = 0.$$

By the results in **Paper III** we obtain that the weak entropy solution constructed satisfies the conditions needed for uniqueness, and in particular a Kružkov-type

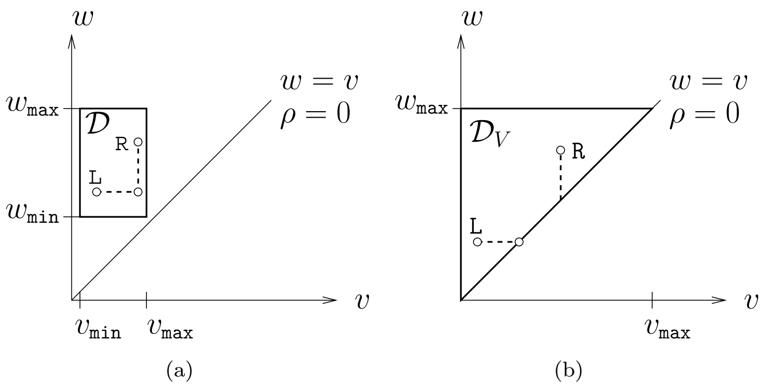

FIGURE 1. The invariant domains (a) $\mathcal{D}$ and (b) $\mathcal{D}_V$ and general solutions of the Riemann problem.

entropy condition. Furthermore, the Kružkov-type entropy condition is translated into Eulerian coordinates and we show that it is satisfied for the Eulerian solution constructed in **Paper I**. Thus, by Wagner [45], the two constructed solutions correspond and are unique.

Weak solutions with vacuum. At vacuum the eigenvalues coincide so the hyperbolic Aw–Rascle system is not strictly hyperbolic. A Riemann problem which yields vacuum consists of a rarefaction wave connecting the left state to vacuum, and then vacuum is connected to the right state by a discontinuity with speed v_{R}. We describe the vacuum as the state $w = v = w_{\mathrm{L}}$. This definition of vacuum and solution of the Riemann problem is entropy admissible as shown in **Paper III**. Even though entropy inequalities as (6) do put restrictions on the values of w at a vacuum, these are not strong enough to specify w. However, in conservative variables $(\rho, \rho w)$ these solutions are equal.

When vacuum is included in the solution, invariant domains for the Riemann problem are of the form

$$\mathcal{D}_V = \left\{ u \in \mathbb{R}^2 : 0 \leq v(u) \leq w(u) \leq v_{\max} \right\}.$$

We can now include any nonnegative value of the density ρ and the velocity v by just increasing the choice of the maximal velocity $v_{\max}$. The domain $\mathcal{D}_V$ and a general solution of the Riemann problem including vacuum in the (w, v)-plane are shown in figure 1(b).

In the solution of the Riemann problem, both w and v changes over the wave connecting vacuum to some right state. However, the total variation of the Riemann invariants is still nonincreasing. But the derivative of the inverse of $p(\rho)$ is unbounded so we do not get a bound on the total variation of the conservative variables from $\mathrm{TV}(w, v)$. Thus, we can not use the Glimm scheme to obtain existence of weak solutions as is the case when we stay away from vacuum.

In **Paper I** we consider the Eulerian form of the model with initial data in $\mathcal{D}_V$ and initial Riemann invariants $(\bar{w}, \bar{v}) \in \mathrm{BV}(\mathbb{R})^2$ and show existence of weak entropy solutions in $L^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^+)^2$. Our strategy is to consider slightly modified systems for which we can control the total variation of the conservative variables and thus get existence of weak entropy solutions by the Glimm scheme. By introducing a cut-off function and using a compactness argument we achieve convergence of a subsequence of approximate weak solutions. The limit turns out to be a weak solution satisfying an entropy inequality as given in (6), for all nonnegative test functions $\phi(x,t)$ in $C_0^\infty(\mathbb{R} \times \mathbb{R}^+)$ and all entropies $\eta(u)$, which is strictly convex in ρ, with corresponding flux $q(u)$. When vacuum is included there exists no strictly convex entropy $\eta(u)$. However, it turns out that there exists an entropy as introduced above and, away from vacuum, this entropy function makes the entropy inequality fail for inadmissible discontinuities. Operator splitting and the same technique as used to obtain convergence of the approximate solutions extends the existence result to the inhomogeneous model (2).

For the Lagrangian form of the homogeneous system the construction of weak solutions, which are stable in the L^1-norm, as the limit of the semi-discrete car-following model is extended to Cauchy problems with initial data in $\mathcal{D}_V$, see **Paper II**. However, when vacuum is included in the model, the inverse density $\tau(y,t)$ does not belong to $L^\infty(\mathbb{R} \times \mathbb{R}^+)$, so convergence of τ^δ has to be obtained by a different approach. The idea is now to study the Eulerian space variable $x = x(y,t)$ as given by

$$\tau = \frac{\partial x}{\partial y}, \quad v = \frac{\partial x}{\partial t} \quad \text{a.e.}$$

Consider the semi-discrete car-following model (4). For a fixed t the piecewise linear $x^\delta(y,t)$ is strictly increasing and bounded and it is Lipschitz continuous in time as a function into $L^1_{\mathrm{loc}}(\mathbb{R})$. Furthermore, if there is no vacuum in the initial data it is also continuous. So, by Helly's theorem $x^\delta \to x^\delta$ in $L^1_{\mathrm{loc}}(\mathbb{R} \times \mathbb{R}^+)$, which implies weak convergence of τ^δ to a limit $\hat{\tau}$.

However, the problem is to show that the relation $w = v + Q(\tau)$ holds for the limit and to handle vacuum in the initial data. We restrict to functions w in $\mathrm{BV}(\mathbb{R})$ that have positive jumps only in a discrete set V and satisfy a one-sided Lipschitz condition limiting the rate of growth between the points of V. Furthermore, the solution τ^δ is monotonicity preserving, that is if

$$\bar{\tau}_k \le \bar{\tau}_{k+1} \text{ and } \quad w_{k+1} - w_k \le w_{k+2} - w_{k+1} \text{ for all } k,$$

then

$$\tau_k(t) \le \tau_{k+1}(t) \quad \text{for all} \quad k.$$

It follows, by using this property, that vacuum only appears at the set V. Thus, in the solution a vacuum is represented as a stationary delta-shock,

$$d\hat{\tau} = h(y)\, d\delta(y),$$

where $h(y) \ge 0$ is the size of the vacuum, that is the length of an empty road section.

We show that the strictly increasing initial $\bar{x}$ with jumps in V can be approximated by a strictly increasing function $\bar{x}^\delta$ which converges to $\bar{x}$ in $L^1_{\text{loc}}(\mathbb{R})$. For fixed t the limit $x(y,t)$ is monotone and piecewise continuous, so $\hat{\tau}$ is a locally bounded measure of the form

$$d\hat{\tau} = \tau(y,t)\,dy + \sum_{y_m \in V} h(y,t)\,d\delta(y - y_m), \tag{8}$$

where $\tau(y,t) \in L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^+)$. It follows that $\tau^\delta \to \tau$ in $L^1_{\text{loc}}(\mathbb{R} \setminus V \times \mathbb{R}^+)$, and we show $Q(\tau^\delta) \rightharpoonup Q(\tau)$ and thus $\hat{\tau}$ is a weak solution. Furthermore, $\hat{\tau}$ satisfies an entropy inequality of the form in (6) with a flux function given as the measure

$$d\hat{\eta} = \eta\,dy + d\eta_s,$$

where $\eta\,dy$ and $d\eta_s$ is the absolutely continuous and singular part of the measure, respectively. The function $\eta(\tau, w)$ corresponds to the Eulerian entropy function and is strictly convex in τ away from vacuum.

By the same approach as without vacuum it follows that the constructed solution is stable with respect to the initial data, that is $\hat{\tau}$ satisfies the estimate given for τ in (7).

In **Paper III** we consider uniqueness of weak solutions. The Lagrangian form of the Aw-Rascle model rewrites as a scalar conservation law with a singular source,

$$\tau_t + Q(\tau)_y = w_y.$$

Vacuum will only appear at points at which w makes positive jumps. We make the same assumptions on w as we made in **Paper II**, and assume in addition that all the jumps in w, which are the singularities of the source, are located in the discrete set V. Clearly the inverse density does not belong to L^∞ when vacuum is included. Furthermore, it follows by an entropy condition that vacuum only will appear in the discrete set V. We consider weak solutions of the form

$$d\hat{\tau} = \tau\,dy + h\,d\Delta,$$

where $(\tau, w) \in \mathcal{D}_V$, $\tau \in L^1_{loc}(\mathbb{R}^+ \times \mathbb{R})$ and $d\Delta$ is the counting measure over V.

In order to obtain a uniqueness result we follow the method by Karlsen, Risebro and Towers in [31], where uniqueness is obtained from a Kružkov-type entropy condition. Our entropy condition has an additional term due to the delta-shocks, and is given as

$$\begin{aligned} \int_{\mathbb{R}^+}\int_{\mathbb{R}} \Big[\Big(|\tau - c| + \sum_{y_m \in V} h(y,t)\delta(y-y_m)\Big)\phi_t - |Q(\tau) - Q(c)|\,\phi_y\Big]\,dy\,dt& \\ + \int_{\mathbb{R}^+}\int_{\mathbb{R}\setminus V} \operatorname{sign}(\tau - c) w'(y)\,\phi(y,t)\,dt\,dy& \\ + \int_{\mathbb{R}^+}\sum_{y_m \in V} |w(y_m^+) - w(y_m^-)|\phi(y_m,t)\,dt \geq 0,& \end{aligned} \tag{9}$$

for all c in $\mathbb{R}$ and all test functions $0 \leq \phi \in C_0^\infty(\mathbb{R} \times \mathbb{R}^+)$. The above entropy condition allows discontinuities directly to vacuum, and so does the entropy condition in (6) with entropy fluxes convex in ρ. By physical considerations a solution should approach vacuum continuously from the left. For instance, if a driver has no cars in front, he or she will speed up. So, in order to avoid unphysical vacuum states, we require explicitly that for $t > 0$ and $y_m \in V$ such that $h(y_m, t) > 0$,

$$\lim_{y \to y_m^-} \tau(y, t) = \tau(y_m, t) = \infty. \tag{10}$$

We show that a weak solution of the Cauchy problem with initial data given by (8) satisfying the entropy conditions (9) and (10) is unique. Furthermore, by considering the semi-discrete model we show that the solution constructed in **Paper II** satisfies both the Kružkov-type entropy inequality and approaches vacuum continuously from the left. Hence it is unique and the Cauchy problem for the Lagrangian form of the Aw–Rascle model is well-posed.

By Wagner we know that the weak solution of the system in Lagrangian form corresponds to a weak solution of the Eulerian form of the system. The Kružkov-type entropy condition (9) is translated into Eulerian coordinates and we show that the weak solution constructed in **Paper I** satisfies this condition. As for the Lagrangian solution, the entropy inequality (6) and the Eulerian form of (9) both allow some weak solutions which connect to vacuum by a discontinuity. The uniqueness of the Eulerian solution constructed is an open question since we do not know whether the solution always approaches vacuum continuously from the left or sometimes makes jumps directly to vacuum. However, the Cauchy problem for the Eulerian system with vacuum is not well-posed. In [2] several examples of Riemann problems are given in which small perturbations of the initial conservative variables $(\rho, \rho w)$ totally change to solution, and hence, weak solutions do not depend continuously on the initial data when vacuum is included.

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Paper I

Existence of Solutions for the Aw–Rascle Traffic Flow Model with Vacuum

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EXISTENCE OF SOLUTIONS FOR THE AW–RASCLE TRAFFIC FLOW MODEL WITH VACUUM

MARTE GODVIK AND HARALD HANCHE-OLSEN

Abstract. We consider the macroscopic model for traffic flow proposed by Aw and Rascle in 2000. The model is a 2×2 system of hyperbolic conservation laws, or, when the model includes a relaxation term, a 2×2 system of hyperbolic balance laws. The main difficulty is the presence of vacuum, which makes control of the total variation of the conservative variables impossible. We allow vacuum to appear and prove existence of a weak entropy solution to the Cauchy problem.

1. Introduction

In [2] Aw and Rascle introduced a new macroscopic model for traffic flow,

$$\left.\begin{aligned} \rho_t + (\rho v)_x &= 0 \\ (\rho w)_t + (\rho w v)_x &= 0 \end{aligned}\right\} \qquad \text{where } w = v + p(\rho), \tag{1}$$

the functions $\rho(x,t)$ and $v(x,t)$ are the density and the velocity of cars on the roadway and $x \in \mathbb{R}$ and $t \in \mathbb{R}^+$ are the Eulerian space and time variable, respectively. For simplicity we write the system as

$$u_t + f(u)_x = 0,$$

where $u = (\rho, y) = (\rho, \rho w) \in \mathcal{U} \subset \mathbb{R}^2$. The function $p(\rho)$ is smooth and strictly increasing and it satisfies

$$p(0) = 0, \quad \lim_{\rho \to 0} \rho p'(\rho) = 0 \quad \text{and } \rho\, p''(\rho) + 2\, p'(\rho) > 0 \text{ for } \rho > 0. \tag{2}$$

The last assumption ensures strict hyperbolicity for $\rho > 0$. The prototype of the function $p(\rho)$ is

$$p(\rho) \propto \rho^\gamma, \qquad \gamma > 0. \tag{3}$$

The eigenvalues of the system are

$$\lambda_1 = v - \rho p'(\rho) \text{ and } \lambda_2 = v.$$

For $\rho > 0$ the first wave family is genuinely nonlinear and the second family is linearly degenerate. Moreover, for $\rho = 0$ the eigenvalues coincide and the system is only hyperbolic. In [2] Aw and Rascle solve the Riemann problem for this

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model and they include the vacuum state. For a discussion of the model, see also [1], [5], [6], [8], [9], [10], [11], [12] and [15].

The model is of Temple class, i.e., the shock and rarefaction curves coincide. The Riemann invariants are w and v, and the wave curves are given by $w = \text{const.}$ and $v = \text{const.}$, respectively. A solution of the Riemann problem that does not include vacuum consists of at most two waves, one of each family. Thus, any subdomain $\mathcal{D} \subset \mathcal{U}$ defined by

$$\mathcal{D} = \{u \in \mathcal{U} : w_- \le w(u) \le w_+, 0 \le v_- \le v(u) \le v_+\}, \tag{4}$$

where $w_- > v_+$, is invariant for the Riemann problem. Further, for a solution of the Riemann problem with initial data in $\mathcal{D}$, the total variation of the Riemann invariants is nonincreasing. The domain $\mathcal{D}$ is shown in figure 1(a). If we pick initial values $u_0(x)$ in $\mathcal{D}$, vacuum will not appear in the solution of the Cauchy problem. Thus the function $(p^{-1})'$ is bounded, and it is possible to obtain a bound on the total variation of (ρ, y) from the total variation of the Riemann invariants. By using this property and the Glimm scheme [7], it can be shown that the Cauchy problem, with $u(x, 0) = u_0(x) \in \mathrm{BV}(\mathbb{R})^2$, has a weak entropy solution. This argument is given in Serre [16, Chapter 5].

As long as we must exclude the vacuum state, our choice of an invariant domain is severely limited. If we, for example, want to increase the maximal velocity v_+, we also have to increase w_- in order to stay away from vacuum. When including the vacuum state, the available invariant domains are given by (4) requiring $w(u) \ge v(u)$ instead of $w_- > v_+$. All these regions are subdomains of

$$\mathcal{D}_V = \{u \in \mathcal{U} : 0 \le v(u) \le w(u) \le v_+\}. \tag{5}$$

The domain $\mathcal{D}_V$ is depicted in figure 1(b)–1(c). Note that we now can include any nonnegative value of the car density ρ_0 and velocity v_0 in $\mathcal{D}_V$ by choosing a larger value of v_+.

A Riemann problem with left state u_L and right state u_R produces a vacuum state at time $t = 0^+$ if and only if

$$v_R \ge w_L. \tag{6}$$

When the Riemann data satisfies this condition, the solution consists of a rarefaction wave that connects u_L to a vacuum state given by $w = v = w_L$, and a contact wave that connects the vacuum state given by $v = w = v_R$ to the right state u_R. Hence the total variation of the Riemann invariants is still nonincreasing. So, if $p'(\rho) > \epsilon > 0$, it is in general possible to obtain a bound on the total variation of (ρ, y) from the total variation of the Riemann invariants, and the Glimm scheme yields existence of a weak entropy solution.

In [19] it is shown for the p-system that, unless vacuum is present initially or appears immediately, the solution will not reach a vacuum state in finite time. This is not the case for the Aw–Rascle model. In order to show this, assume initial Cauchy data consisting of three constant states denoted by u_L, u_M and u_R such that $v_L = v_M$, $w_M = w_R$ and $w_L \le v_R$. Further, assume that the Cauchy data does not include vacuum. Thus, at time $t = 0^+$ the solution consists of

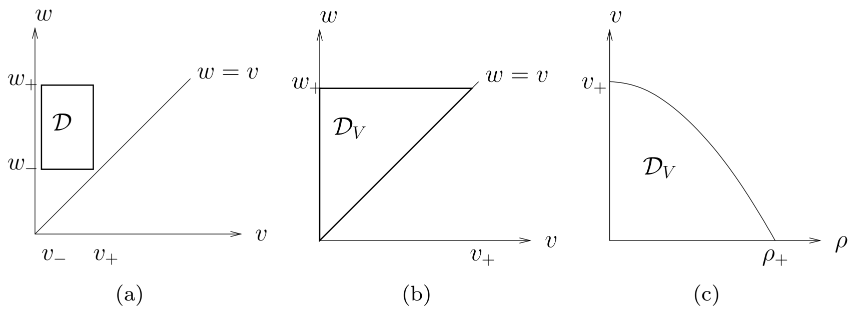

FIGURE 1. The domains (a) $\mathcal{D}$ and (b)-(c) $\mathcal{D}_V$.

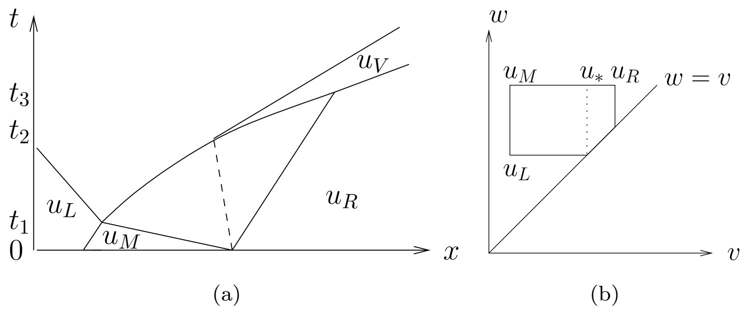

FIGURE 2. A contact discontinuity and a rarefaction wave collide and a vacuum state u_V appears in the solution. The state along the dashed characteristic in figure (a) is the state denoted by u_* in figure (b).

a contact discontinuity separating the leftmost state u_L and the middle state u_M, and a rarefaction wave connecting u_M and the rightmost state u_R. The solution, as a function of time and space, is shown in figure 2(a), and figure 2(b) gives the states in the (w, v)-plane. The contact travels faster than the rarefaction, and the two waves will collide at time t_1. Since $w_L \leq v_R$, the resultant state once the incoming waves have passed through each other includes vacuum. Across the incoming rarefaction wave the velocity v increases. Thus, as the contact discontinuity transverses the rarefaction, the velocity of the contact increases. Since the velocity of the rightmost part of the incoming rarefaction is $\lambda_1(u_R) < \lambda_2(u_R)$, the contact travels through the rarefaction in finite time and leaves the rarefaction at time $t = t_3$. Thus, vacuum will appear in the solution at some time t_2. Further, several waves can collide at the same time. Then vacuum appears if and only if (6) is satisfied for L denoting the leftmost wave and R the rightmost wave.

We want to show existence of a weak entropy solution of the Cauchy problem for system (1), with initial data $u_0(x)$ taking values in $\mathcal{D}_V$. Assume

$$p'(0) = 0, \quad p'(\rho) > 0 \text{ for } \rho > 0, \quad \text{and } |p(\rho_1) - p(\rho_2)| \leq L\,|\rho_1 - \rho_2|\,, \tag{7}$$

for some constant L, which is satisfied for the prototype function given by (3) with $\gamma > 1$. Since we allow vacuum states and $p'(0) = 0$, we are unable to control the total variation of the conservative variables, and thus we can not use the Glimm scheme to show existence. However, we consider slightly modified systems for which we can control the total variation and show existence of weak entropy solutions. The Riemann invariants have nonincreasing total variation and they are Lipschitz continuous in time as functions into $L^1_{loc}(\mathbb{R})$. Thus, by introducing a cut-off function and using a compactness argument, we show convergence of a sequence of weak entropy solutions of the slightly modified systems. It turns out that the limit is a weak solution of the original system. However, when we include vacuum the system has no strictly convex entropy $\eta(\rho, y)$. We relax the assumptions on the entropy function and assume only that $\eta(\rho, y)$ is strictly convex in ρ. For this choice of entropy functions we show that admissible, in the sense of Lax, discontinuous solutions of the Riemann problem for (1) satisfy the entropy inequality, and that it fails for inadmissible discontinuities. Finally, we show that the weak solution obtained for the Cauchy problem for (1) satisfies the entropy inequality. In Section 2 we prove the following theorem:

Theorem 1. *Let the initial Riemann invariants* $(w_0(x), v_0(x))$ *be in* $\mathrm{BV}(\mathbb{R})^2$ *and take values in* $\mathcal{D}_V$. *Assume* $p(\rho)$ *satisfies* (2) *and* (7). *Then there exists a weak entropy solution* $u(x,t)$ *in* $L^1_{loc}(\mathbb{R} \times \mathbb{R}^+)^2$ *of system* (1) *with initial data* $u_0(x)$. *Further, the total variation of the Riemann invariants* $w(x,t)$ *and* $v(x,t)$ *is nonincreasing.*

We expect our proof technique to work also for the network model [5], [12], provided that the junctions do not cause the total variation of v and w to blow up. It should also prove useful in overcoming any difficulty arising from loss of control of the total variation of ρ near vacuum for related traffic models.

An improved version of the model (1) includes a relaxation term in the second equation,

$$\begin{aligned} \rho_t + (\rho v)_x &= 0 \\ y_t + (yv)_x &= \rho R(\rho, y). \end{aligned} \tag{8}$$

We assume the relaxation term satisfies

$$R(\rho, y) \begin{cases} \geq 0 & \text{for } v = 0, \;\; w \leq w_+, \\ \leq 0, & \text{for } v \leq v_+, \;\; w = w_+, \end{cases} \tag{9}$$

and is Lipschitz in v and w,

$$|R(\rho_1, y_1) - R(\rho_2, y_2)| < L\left(|v_1 - v_2| + |w_1 - w_2|\right). \tag{10}$$

By using operator splitting and Theorem 1 we obtain approximate solutions of the above system. The approximate Riemann invariants have bounded total variation and are Lipschitz continuous in time. Thus, by using the same technique as used to achieve Theorem 1, we prove, see Section 3, the following theorem:

Theorem 2. *Assume the conditions of Theorem 1 are satisfied. Furthermore, assume $R(\rho, y)$ satisfies (9)–(10). Then there exists a weak entropy solution $u(x,t)$ in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)^2$ of system (8) with initial data $u_0(x)$. Further, the total variation of the Riemann invariants $w(x,t)$ and $v(x,t)$ is bounded.*

Remark. The operator splitting technique, as used in the proof of Theorem 2, also yields existence of a weak entropy solution of the Cauchy problem for strictly hyperbolic Temple systems with a source term. Consider the strictly hyperbolic Temple system $u_t + f(u)_x = g(u)$, where u, f and g are vectors in $\mathbb{R}^n$ and $(x,t) \in \mathbb{R}\times\mathbb{R}^+$. Assume the initial Riemann invariants, $r_0(x) \in \mathbb{R}^n$, have bounded total variation. We split the system into a homogenoues hyperbolic part, $u_t + f(u)_x = 0$, and a system of ordinary differential equations, $u_t = g(u)$. Consider the homogeneous hyperbolic system. From the theory of Temple systems there exists a bounded invariant domain $\mathcal{D}$, and the Riemann invariants, $r \in \mathbb{R}^n$, have nonincreasing total variation and are Lipschitz continuous in time as function into $L^1_{loc}(\mathbb{R})^n$. If the map $r \mapsto u(r)$ is a diffeomorphism, there exists a weak entropy solution of the Cauchy problem for the system. Consider the system of ordinary differential equations, and assume that the domain $\mathcal{D}$ is an invariant domain for the system. Denote $r = R(u)$ and $u = U(r)$. Thus, by a change of variables the system transforms to $r_t = dR(U(r))g(U(r))$. If g, U and dR are Lipschitz, Grönwall's inequality yields $\mathrm{TV}(r(t)) \le e^{Ct}\,\mathrm{TV}(r_0)$, for a constant C, and Lipschitz continuity in time of r. Finally, the same operator splitting technique as used to prove Theorem 2, yields existence of a weak entropy solution $u(x,t)$ of the inhomogeneous hyperbolic system, and the solution has bounded total variation. In [4] Colombo and Corli prove well-posedness for a class of strictly hyperbolic Temple systems with a source, assuming the eigenvalues of the system are separated on every compact subset of $\mathcal{D}$. The above proof holds for more general systems, but we only obtain existence of a weak entropy solution. In particular, the eigenvalues of the Aw–Rascle model are not separated on $\mathcal{D}_V$ (5), so [4] is not directly applicable.

A specific choice of the relaxation term, as given in [15], is

$$R(\rho, v) = \frac{1}{\tau}\left(V(\rho) - v\right) = \tilde{R}(w, v) = \frac{1}{\tau}\left(V \circ p^{-1}(w - v) - v\right), \tag{11}$$

where the constant τ is the relaxation time and the smooth function $V(\rho)$ is an equilibrium velocity. Further, assume the subcharacteristic condition, see [15],

$$-p'(\rho) \le V'(\rho) \le 0 \tag{12}$$

is satisfied.

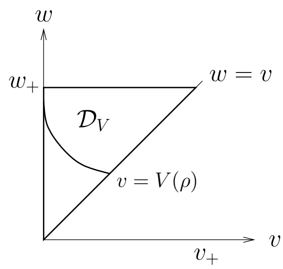

FIGURE 3. The curve $v = V(\rho)$ and the domain $\mathcal{D}_V$.

We want to show that (9)–(10) are satisfied with this particular choice of $R(\rho, v)$ under these extra conditions: There is some density $\rho_0 > 0$ so that

$$w_+ \geq p(\rho_0) \quad \text{and} \quad V(\rho) \begin{cases} > 0, & 0 \leq \rho < \rho_0, \\ = 0, & \rho > \rho_0. \end{cases} \tag{13}$$

The top half of (9) is trivially satisfied. As to the bottom half, note that (12) implies $\tilde{R}_v \leq 0$ and $w_+ \geq p(\rho_0)$ implies $p^{-1}(w_+) \geq \rho_0$. Thus $\tilde{R}(v, w_+) \leq \tilde{R}(0, w_+) = \tau^{-1} V \circ p^{-1}(w_+) = 0$. The domain $\mathcal{D}_V$ and a curve $V(\rho)$ are shown in figure 3.

The Lipschitz condition (10) follows too, since (12) implies $-\tau^{-1} \leq \tilde{R}_v \leq 0$ and $-\tau^{-1} \leq \tilde{R}_w \leq 0$, and (10) is satisfied with $L = \tau^{-1}$. In conclusion, if we assume (12)–(13), the assumptions in Theorem 2 hold for $R(\rho, v)$ given by (11). Thus there exists a weak entropy solution including vacuum for system (8) with this specific choice of relaxation term.

In [17] Siebel and Mauser introduce another source term $\rho R(\rho, v)$ to the second equation in the Aw–Rascle model. Their function $R(\rho, v)$ satisfies (9)–(10), and by Theorem 2 there exits a weak entropy solution of the system. The source term is motivated by experimental data and the system gives an unstable regime for intermediate densities. For the unstable region the equilibrium density curves in the fundamental diagram is shifted towards an inverse-λ shape. This feature is observed in traffic dynamics, as discussed in [14]. Another approach for modeling this instability using the Aw–Rascle model is given in [8].

Before proving the theorems we introduce some notation. Let Ω denote any bounded subset of $\mathbb{R}$. The purpose is to compute in $L^1(\Omega)$ and tacitly draw conclusions about $L^1_{loc}(\mathbb{R})$. The norm on $L^1(\Omega)$ is denoted by $\|\cdot\|$. Further, we will usually omit subscripts on subsequences and let any subsequence of the sequence u^δ be denoted by u^δ. In particular, whenever speaking of convergence of u^δ, we really mean convergence of some subsequence. For simplicity, we write $u(t)$ instead of $u(\cdot, t)$. Finally, let $a \vee b$ denote $\max(a, b)$.

2. Proof of Theorem 1.

In order to prove Theorem 1, we first define an appropriate approximation of (1). For $\delta > 0$, consider

$$\begin{aligned} \rho_t + (\rho v)_x &= 0 \\ \left[\rho\left(v + p^\delta(\rho)\right)\right]_t + \left[\rho v\left(\rho + p^\delta(\rho)\right)\right]_x &= 0, \end{aligned} \tag{14}$$

where

$$p^\delta(\rho) = \begin{cases} \dfrac{p(\delta)}{\delta}\,\rho, & \rho \le \delta \\ p(\rho), & \rho > \delta, \end{cases} \tag{15}$$

and $w = v + p^\delta(\rho)$. The initial Riemann invariants $v_0(x)$ and $w_0(x)$ are in $BV(\mathbb{R})$ and take values in $\mathcal{D}_V$. Further, we take them to be independent of δ. The initial data $u_0^\delta(x)$ depends on δ since $w_0 = v_0 + p^\delta(\rho^\delta)$. Assume $\delta < \rho_+$, where $\rho_+ = (p^\delta)^{-1}(w_+) = p^{-1}(w_+)$ is the maximal density. We denote the solution of the conservative problem by (ρ^δ, y^δ) and the Riemann invariants by (v^δ, w^δ). The modified system is also of Temple class. As discussed in the previous section, the total variation of the Riemann invariants, even when including the vacuum state, is nonincreasing,

$$\mathrm{TV}\left(w^\delta(t), v^\delta(t)\right) \le \mathrm{TV}\left(w_0, v_0\right) \le C, \tag{16}$$

where C is a constant. For $\delta > 0$ the function $p^\delta(\rho)$ satisfies

$$C_\delta\,\left|p^\delta(\rho_1) - p^\delta(\rho_2)\right| \ge \left|\rho_1 - \rho_2\right|, \tag{17}$$

where $C_\delta = \left\|1/(p^\delta)'\right\|_\infty$. >From $p^\delta(\rho^\delta) = w^\delta - v^\delta$ we find

$$\mathrm{TV}\left(\rho^\delta(t)\right) \le C_\delta\ \mathrm{TV}\left(w^\delta(t), v^\delta(t)\right) \le C_\delta\, C, \tag{18}$$

which implies

$$\begin{aligned} \mathrm{TV}\left(y^\delta(t)\right) &\le \rho_+\,\mathrm{TV}\left(w^\delta(t)\right) + w_+\,\mathrm{TV}\left(\rho^\delta(t)\right) \\ &\le \rho_+ C + w_+ C_\delta\, C. \end{aligned}$$

Thus, for some constant M_δ depending on δ, we have

$$\mathrm{TV}\left(\rho^\delta(t), y^\delta(t)\right) \le M_\delta. \tag{19}$$

Notice that M_δ increases without bound as $\delta \to 0$, because the same is true for C_δ.

For initial Riemann data taking values in $\mathcal{D}_V$, there exists a solution to the Riemann problem for (14)–(15) with $\delta > 0$, see [2]. The domain $\mathcal{D}_V$ is an invariant region in the sense that if the initial data lies in $\mathcal{D}_V$, then so does the solution. Consider the Cauchy problem for (14)–(15) with initial Riemann invariants $(v_0(x), w_0(x))$ in $BV(\mathbb{R})^2$ taking values in $\mathcal{D}_V$ and initial Cauchy data $u_0^\delta(x)$. Since $\mathcal{D}_V$ is bounded, the Glimm approximate solutions can be defined for all times t, and they are bounded. Further, the total variation of the Glimm approximations are bounded. Thus, by Serre [16, Theorem 5.4.1], the Glimm scheme yields existence of a weak entropy solution $u^\delta(x,t)$ of the Cauchy problem.

The solution is bounded in $L^\infty(\mathbb{R}\times[0,T])^2$ and the total variation is given by (19). Further, the solution is Lipschitz continuous in time,

$$\|u^\delta(t)-u^\delta(s)\| \le 2\ \mathrm{TV}\left(u^\delta(t)\right)|t-s| \le M_\delta|t-s|,$$

for $s\le t\le T$.

In order to show that the Riemann invariants are Lipschitz in time independently of δ, we consider the Glimm scheme as given in [16, Chapter 5]. We denote by $u_h^\delta(t)$ and $\left(w_h^\delta(t), v_h^\delta(t)\right)$ the approximate solution and the approximate Riemann invariants of (14) given by the Glimm scheme at time t. Let $h=\Delta x = c\,\Delta t$, where Δx and Δt are the space and time step, respectively, and c is some constant. For a fixed t in $[0,T]$, it is shown that $u_h^\delta(t)$ converges to $u^\delta(t)$ in $L^1_{loc}(\mathbb{R})^2$ as $h\to 0$. Further, since the Riemann invariants are bounded in $L^\infty(\mathbb{R})$ and have bounded total variation independently of h, Helly's theorem yields the existence of a subsequence converging to some limit $(w^\delta(t), v^\delta(t))$ in $L^1_{loc}(\mathbb{R})^2$. The limits should satisfy $w^\delta = v^\delta + p^\delta(\rho^\delta)$ and $y^\delta=\rho^\delta w^\delta$. Since p^δ is Lipschitz in its argument and $w_h^\delta = v_h^\delta + p^\delta(\rho_h^\delta)$, the first equality is satisfied. Further, since $y_h^\delta=\rho_h^\delta w_h^\delta$, the second equality is satisfied.

A diagonal argument gives us a subsequence $(w_h^\delta(t), v_h^\delta(t))$ converging for all t in $\mathcal{S}$, where $\mathcal{S}$ is a countable and dense subset of $[0,T]$. By the same technique as used in [16, Chapter 5.4] when proving that $u_h^\delta(t)$ is Lipschitz continuous in time, it can be shown, for any $s\le t$ in $[0,T]$, that

$$\begin{aligned}\left\|w_h^\delta(t)-w_h^\delta(s)\right\| + \left\|v_h^\delta(t)-v_h^\delta(s)\right\| &\le 2\ \mathrm{TV}\big(w_h^\delta(t), v_h^\delta(t)\big)(|t-s|+h)\\ &\le C\,(|t-s|+h).\end{aligned} \tag{20}$$

Notice by (16) that the constant C is independent of δ. Assume $t\in[0,T]$ and $t\notin\mathcal{S}$. Let t_k be a sequence in $\mathcal{S}$ such that $t_k\to t$ as $k\to\infty$. Finally,

$$\begin{aligned}\left\|v_{h_m}^\delta(t)-v_{h_n}^\delta(t)\right\| \le \left\|v_{h_m}^\delta(t)-v_{h_m}^\delta(t_k)\right\| + \left\|v_{h_m}^\delta(t_k)-v_{h_n}^\delta(t_k)\right\|&\\ + \left\|v_{h_n}^\delta(t_k)-v_{h_n}^\delta(t)\right\|&.\end{aligned}$$

By the Lipschitz continuity of v_h^δ in time, the first and last terms can be made arbitrary small by choosing k large. Fixing such a k, if m and n are large, the middle term is small. Thus, the sequence v_h^δ is Cauchy in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$. Further,

$$\left\|v^\delta(t)-v^\delta(s)\right\| \le \left\|v^\delta(t)-v_h^\delta(t)\right\| + \left\|v_h^\delta(t)-v_h^\delta(s)\right\| + \left\|v_h^\delta(s)-v^\delta(s)\right\|.$$

Apply (20) to the middle term and let $h\to 0$. The same arguments hold for w^δ and therefore the Riemann invariants are Lipschitz in time,

$$\left\|w^\delta(t)-w^\delta(s)\right\| + \left\|v^\delta(t)-v^\delta(s)\right\| \le C\,|t-s|. \tag{21}$$

Since the Riemann invariants $w^\delta(t)$ and $v^\delta(t)$ are bounded in $L^\infty(\mathbb{R})$ and have bounded total variation independently of δ, Helly's theorem yields the existence of a subsequence (w^δ, v^δ) converging to some limit in $L^1_{loc}(\mathbb{R})^2$. Using the same argument as for the convergence when $h\to 0$, we end the above discussion by the following conclusion:

Lemma 1. *(w^δ, v^δ) converges to some limit (w, v) in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)^2$ as $\delta \to 0$ and the limit (v, w) satisfies*

$$\mathrm{TV}\big(w(t), v(t)\big) \le \mathrm{TV}\left(w_0, v_0\right) \le C, \tag{22}$$

and

$$\|w(t) - w(s)\| + \|v(t) - v(s)\| \le C\,|t - s|. \tag{23}$$

The above properties of the limit (w, v) follows directly from (16) and (21).

We are going to prove convergence of a subsequence $u^\delta(t)$ to some limit in $L^1_{loc}(\mathbb{R})$ by using the following result:

Lemma 2. *Let K be a set of nonnegative functions uniformly bounded in $L^\infty(\mathbb{R})$. If, for any $\epsilon > 0$, there is a constant M_ϵ so that $\mathrm{TV}(u \vee \epsilon) \le M_\epsilon$ for all $u \in K$, then the set K is precompact in $L^1_{loc}(\mathbb{R})$.*

Proof. For a fixed $\epsilon > 0$, let $K_\epsilon = \{u \vee \epsilon : u \in K\}$. We have $u \vee \epsilon$ bounded in $L^\infty(\mathbb{R})$ and $\mathrm{TV}(u\vee\epsilon) \le M_\epsilon$. Thus, by Helly's theorem, the set K_ϵ is precompact in $L^1_{loc}(\mathbb{R})$. Since u is bounded in $L^\infty(\mathbb{R})$, we have $K \subset L^1_{loc}(\mathbb{R})$. Further, since $\|u - u \vee \epsilon\| < \epsilon\,|\Omega|$ for all $u \in K$, we find $\mathrm{dist}(u, K_\epsilon) < \epsilon\,|\Omega|$ for all u in K. By [13, Lemma A.4], K is precompact in $L^1_{loc}(\mathbb{R})$. □

Fix $t \in [0, T]$ and define the set $K = \{\rho^\delta(t)\}_{\delta>0}$. The solution $\rho^\delta(t)$ takes values in $\mathcal{D}_V$, and hence $K \subset L^\infty(\mathbb{R})$. Consider the total variation of $\rho^\delta \vee \epsilon$. Since

$$\frac{1}{\left(p^\delta\right)'\left(\rho \vee \epsilon\right)} \le \begin{cases} 1/p'(\epsilon), & \delta \le \epsilon \\ 1/\left(p^\epsilon\right)'(\epsilon), & \delta > \epsilon, \end{cases} \tag{24}$$

where p^ϵ is defined by (15) replacing δ by ϵ, the constant C_δ given in inequality (17) now depends on ϵ instead of δ. Thus, the same argument as used to obtain (18) yields

$$\mathrm{TV}(\rho^\delta \vee \epsilon) \le M_\epsilon, \qquad \text{for all } \delta \ge 0. \tag{25}$$

By Lemma 2 the set K is precompact in $L^1_{loc}(\mathbb{R})$. Thus $\rho^\delta(t)$ converges in $L^1_{loc}(\mathbb{R})$ to some limit as $\delta \to 0$. A diagonal argument gives us a subsequence $\rho^\delta(t)$ converging for all $t \in \mathcal{S}$.

Lemma 3. *(ρ^δ, y^δ) converges to some limit (ρ, y) in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$ as $\delta \to 0$.*

Proof. It remains to show convergence of $u^\delta(t)$ for all t in $[0, T]$. Let

$$\rho^\delta_\epsilon = \epsilon \vee \rho^\delta, \quad y^\delta_\epsilon = \left(\epsilon \vee \rho^\delta\right) w^\delta, \qquad \text{for all } \epsilon > 0. \tag{26}$$

Denote $u^\delta_\epsilon = (\rho^\delta_\epsilon, y^\delta_\epsilon)$. Notice that u^δ_ϵ converges uniformly to u^δ as $\epsilon \to 0$. From the previous paragraph we have $\rho^\delta_\epsilon(t)$ converging to some limit $\rho_\epsilon(t)$ for all t in $\mathcal{S}$. Then, $y^\delta_\epsilon \to y_\epsilon = \rho_\epsilon w$, as the factors converge in L^1_{loc} and are uniformly bounded in L^∞. Hence we can conclude that the sequence $u^\delta_\epsilon(t)$ converges to some limit $u_\epsilon(t)$ in $L^1_{loc}(\mathbb{R})^2$ for all $t \in \mathcal{S}$. The Lipschitz continuity in time of w^δ and v^δ yields Lipschitz continuity in time for $p^\delta(\rho^\delta) = w^\delta - v^\delta$,

$$\left\|p^\delta\left(\rho^\delta(t)\right) - p^\delta\left(\rho^\delta(s)\right)\right\| \le C\,|t - s|. \tag{27}$$

By using (17), (24) and the above inequality we can show that $\rho_\epsilon^\delta(t)$ is Lipschitz continuous in time for all t, s in $[0,T]$,

$$\begin{aligned}\left\|\rho_\epsilon^\delta(t)-\rho_\epsilon^\delta(s)\right\| &\leq \left\|\frac{1}{(p^\delta)'\left(\rho_\epsilon^\delta\right)}\right\|_\infty \left\|p^\delta\left(\rho_\epsilon^\delta(t)\right)-p^\delta\left(\rho_\epsilon^\delta(s)\right)\right\| \\ &\leq C_\epsilon\,|t-s|,\end{aligned} \tag{28}$$

where the constant C_ϵ depends on ϵ. Also, $y_\epsilon^\delta(t)$ is Lipschitz continuous in time,

$$\begin{aligned}\left\|y_\epsilon^\delta(t)-y_\epsilon^\delta(s)\right\| &\leq \left\|\rho_\epsilon^\delta(t)\left(w^\delta(t)-w^\delta(s)\right)\right\| + \left\|\left(\rho^\delta(t)-\rho_\epsilon^\delta(s)\right)w^\delta(s)\right\| \\ &\leq (\rho_+ C + w_+ C_\epsilon)\,|t-s|.\end{aligned} \tag{29}$$

Now, assume $t\in[0,T]$ and $t\notin\mathcal{S}$ and let t_k be a sequence in $\mathcal{S}$ such that $t_k\to t$ as $k\to\infty$. Consider

$$\begin{aligned}\left\|u_\epsilon^{\delta_m}(t)-u_\epsilon^{\delta_n}(t)\right\| \leq \left\|u_\epsilon^{\delta_m}(t)-u_\epsilon^{\delta_m}(t_k)\right\| &+ \left\|u_\epsilon^{\delta_m}(t_k)-u_\epsilon^{\delta_n}(t_k)\right\| \\ &+ \left\|u_\epsilon^{\delta_n}(t_k)-u_\epsilon^{\delta_n}(t)\right\|.\end{aligned} \tag{30}$$

Since u_ϵ^δ is Lipschitz continuous in time with a Lipschitz constant independent of δ, the first and third term can be made small by choosing k large, i.e. the term $|t-t_k|$ is small. For a fixed k, the middle term is small for m and n large. Hence the sequence is Cauchy in $L^1_{loc}(\mathbb{R})^2$, and $u_\epsilon^\delta(t)$ converges to $u_\epsilon(t)$ for all $t\in[0,T]$. Finally, consider the sequence $u^\delta(t)$ and some t in $[0,T]$. We have

$$\begin{aligned}\left\|u^{\delta_m}(t)-u^{\delta_n}(t)\right\| \leq \left\|u^{\delta_m}(t)-u_\epsilon^{\delta_m}(t)\right\| &+ \left\|u_\epsilon^{\delta_m}(t)-u_\epsilon^{\delta_n}(t)\right\| \\ &+ \left\|u_\epsilon^{\delta_n}(t)-u^{\delta_n}(t)\right\|.\end{aligned} \tag{31}$$

Since ϵ is arbitrary, the first and third term can be made small by choosing ϵ small. Then, by choosing m and n large, the middle term is small. Hence the sequence is Cauchy in $L^1_{loc}(\mathbb{R})^2$, and there exists a sequence $u^\delta(t)$ converging to some limit $u(t)$ in $L^1_{loc}(\mathbb{R})^2$ as $\delta\to 0$. □

The function $u^\delta(x,t)$ is a weak solution of the approximate problem (14) with initial data $u_0^\delta(x)$. Thus, for all test functions $\phi(x,t)\in C_0^\infty(\mathbb{R}\times\mathbb{R}^+)$,

$$\int_{\mathbb{R}}\int_{\mathbb{R}^+}\left(u^\delta\phi_t + f^\delta(u^\delta)\,\phi_x\right)\,dt\,dx + \int_{\mathbb{R}} u_0^\delta\,\phi(x,0)\,dx = 0$$

where

$$f^\delta(u) = \left(y-\rho p^\delta(\rho),\; y(w-p^\delta(\rho))\right).$$

The function f^δ converges uniformly to f. Since u^δ and (w^δ, v^δ) are bounded, $f^\delta(u^\delta)$ converges in L^1_{loc} to $f(u)$ as $\delta\to 0$. Thus Lebesgue's dominated convergence theorem yields

$$\int_{\mathbb{R}}\int_{\mathbb{R}^+}\left(u\,\phi_t + f(u)\,\phi_x\right)\,dt\,dx + \int_{\mathbb{R}} u_0(x)\phi(x,0)\,dx = 0.$$

For all $t\in[0,T]$, the weak solution $u^\delta(t)$ of system (14) converges in $L^1_{loc}(\mathbb{R})$ to a weak solution $u(t)$ of system (1). Further, the convergence is in $C([0,T]; L^1_{loc}(\mathbb{R})^2)$.

The limits satisfy $w = v + p(\rho)$ and $y = \rho w$ and thus $w(x,t)$ and $v(x,t)$ are the Riemann invariants of system (1).

It remains to prove that the limit $u(x,t)$ is a weak entropy solution of (1) with initial data u_0. An entropy/entropy flux pair (η, q) for a hyperbolic system is an entropy $\eta : \mathcal{U} \to \mathbb{R}$ and a flux $q : \mathcal{U} \to \mathbb{R}$ satisfying

$$\nabla_u q = \nabla_u \eta \, df,$$

where df is the Jacobian matrix of $f(u)$. By [16] we know that u^δ is a weak entropy solution of (14),

$$\int_{\mathbb{R}} \int_{\mathbb{R}^+} \left(\eta^\delta(u^\delta)\phi_t + q^\delta(u^\delta)\,\phi_x\right) dt\,dx + \int_{\mathbb{R}} \eta^\delta(u_0^\delta)\phi(x,0)\,dx \geq 0, \tag{32}$$

for all nonnegative test functions $\phi(x,t) \in C_0^\infty(\mathbb{R}\times\mathbb{R})$ and all convex entropies η^δ with corresponding flux q^δ. Consider system (1). Using the transformation given by Wagner [18, Theorem 1],

$$\frac{\partial X}{\partial x} = \rho, \quad \frac{\partial y}{\partial t} = -\rho v,$$

where $X(x,t)$ is the Lagrangian mass coordinate, we rewrite system (1) as

$$\begin{aligned} \tau_t + (\tilde{p}(\tau) - w)_X &= 0 \\ w_t &= 0, \end{aligned} \tag{33}$$

where $\tau = 1/\rho$ and $\tilde{p}(\tau) = p(\rho)$. For $0 < \rho \leq \rho_+$, system (1) is equivalent to the above system and the relation between the entropy/entropy flux pairs (η, q), in Eulerian coordinates, and $(\tilde{\eta}, \tilde{q})$, in Lagrangian coordinates, is

$$\begin{aligned} \tilde{\eta}(\tau, w) &= \frac{1}{\rho}\eta(\rho, y) \\ \tilde{q}(\tau, w) &= q(\rho, y) - \rho v\,\tilde{\eta}(\tau, w). \end{aligned} \tag{34}$$

Moreover, the entropy η is convex if and only if $\tilde{\eta}$ is convex.

Any entropy/entropy flux pair (η, q) of (33) must satisfy

$$\begin{aligned} \tilde{q}_\tau - \tilde{p}'(\tau)\,\tilde{\eta}_\tau &= 0 \\ \tilde{\eta}_\tau + \tilde{q}_w &= 0. \end{aligned}$$

Solving this system yields

$$\begin{aligned} \tilde{q}(\tau, w) &= g\left(v\right), \\ \tilde{\eta}(\tau, w) &= h(w) - \int_{\tau_0}^{\tau} g'\left(w - \tilde{p}(\xi)\right) d\xi, \end{aligned}$$

where τ_0 is a constant value and $h = h(w)$ and $g = g(v)$ are smooth functions in $L^\infty(\mathbb{R})$ having continuous second and third order derivatives, respectively. In order to obtain strict convexity of η we compute

$$\begin{aligned}\tilde{\eta}_{\tau\tau} &= g''(v)\,\tilde{p}'(\tau),\\ \tilde{\eta}_{\tau w} &= \tilde{\eta}_{w\tau} = -g''(v),\\ \tilde{\eta}_{ww} &= h''(w) - \int_{\tau_0}^{\tau} g'''\big(w-\tilde{p}(\xi)\big)\,d\xi.\end{aligned}$$

The Hessian matrix of $\tilde{\eta}$ is positive definite if $\tilde{\eta}_{\tau\tau} > 0$ and $\tilde{\eta}_{\tau\tau}\tilde{\eta}_{ww} > (\tilde{\eta}_{\tau w})^2$. Since $(\tilde{p})' < 0$, we require $g'' < 0$ and

$$\tilde{p}'(\tau)\left(h''(w) - \int_{\tau_0}^{\tau} g'''\big(w-\tilde{p}(\xi)\big)\,d\xi\right) < g''(v)\,. \tag{35}$$

For fixed τ and w, the above inequality is satisfied for h'' big enough. However, when w is fixed and $\tau \to \infty$ the left hand side of inequality (35) is asymptotically equal to $-\tilde{p}'(\tau)\tau g'''(w)$, which goes to 0 as $\tau \to \infty$. Thus we have to require $g'' = 0$, and we conclude that there are no strictly convex entropies when vacuum is included.

We now define a *semiconvex* entropy η with corresponding flux q as a an entropy satisfying $\eta_{\rho\rho} > 0$ for $\rho > 0$. Further, an entropy solution is a weak solution which satisfies an entropy inequality for all such entropy/entropy flux pairs. The relation between η and $\tilde{\eta}$ yields $\eta_{\rho\rho} = \tau^3\tilde{\eta}_{\tau\tau}$. Thus, for $\rho > 0$ we have $\eta_{\rho\rho} > 0$ if and only if $\tilde{\eta}_{\tau\tau} > 0$. Since $h(w)_t = h'(w)w_t = 0$ in Lagrangian coordinates and we now have a weaker assumption on η, we can, for simplicity, choose $h(w) = 0$. Thus the semiconvex entropy/entropy flux pairs of system (33) are

$$\tilde{q}(\tau,w) = g\left(w-\tilde{p}(\tau)\right), \quad \tilde{\eta}(\tau,w) = -\int_{\tau_0}^{\tau} g'\big(w-\tilde{p}(\xi)\big)\,d\xi, \tag{36}$$

where $g(v)$ is a smooth function such that $g''(v) < 0$. A calculation of the above entropy/entropy flux pairs is also done in [3].

Lemma 4. *The admissible discontinuities satisfy the entropy inequality for all semiconvex entropies with corresponding entropy fluxes and the entropy inequality fails for the inadmissible discontinuities.*

Proof. Discontinuous solutions u satisfy the Rankine–Hugionot condition,

$$s(u_R - u_L) = f(u_R) - f(u_L),$$

where s is the speed of the discontinuity and L and R denote the left and right state, respectively. Further, by Lax' admissibility condition a shock of the first family is admissible if

$$\lambda_1(u_R) \le s \le \lambda_1(u_L).$$

Consider the Riemann problem for system (33) in Lagrangian coordinates. The eigenvalues are

$$\tilde{\lambda}_1 = \tilde{p}'(\tau) \text{ and } \tilde{\lambda}_2 = 0.$$

For $\tau < \infty$ the first family is genuinely nonlinear and the second family is linearly degenerate. Moreover, at the vacuum state $\tilde{\lambda}_1 = \tilde{\lambda}_2 = 0$. The wave curves are given by $w =$ const. and $v =$ const., respectively. Then, consider a point on an isolated discontinuity and a test function $\phi(x,t)$ whose support lies entirely inside a small neighborhood of this point. By Green's theorem an entropy inequality as given in (32) is satisfied for $\tilde{u}$ if

$$s\big(\tilde{\eta}(\tilde{u}_R) - \eta(\tilde{u}_L)\big) \geq \tilde{q}(\tilde{u}_R) - \tilde{q}(\tilde{u}_L).$$

Across contact discontinuities of the second wave family the velocity v is preserved and s equals 0. Thus, for contact discontinuities the above inequality is satisfied with equality. Consider admissible shock solutions. Since w is constant across a shock, both τ and v should decrease across the jump, i.e. $v_L > v_R$ and $\tau_L > \tau_R$. In order to show that the admissible shocks do satisfy the entropy inequality, we insert the expressions of $\tilde{\eta}$ and $\tilde{q}$ into the above inequality, with the result,

$$s \int_{\tau_R}^{\tau_L} g'\big(w - \tilde{p}(\xi)\big)\, d\xi \geq g(v_R) - g(v_L),$$

where $w = w_L = w_R$. Further, inserting the expressions of s given by the Rankine–Hugionot condition into the above inequality yields

$$\frac{1}{\tau_L - \tau_R} \int_{\tau_R}^{\tau_L} g'\big(w - \tilde{p}(\xi)\big)\, d\xi \begin{cases} \leq \dfrac{g(v_R) - g(v_L)}{v_R - v_L}, & v_L > v_R, \\ \geq \dfrac{g(v_R) - g(v_L)}{v_R - v_L}, & v_L < v_R, \end{cases}$$

where $w = w_L = w_R$. Consider the left hand side in the above inequality. By the substitution $\xi = t\tau_L + (1-t)\tau_R$ and using the strict convexity of $\tilde{p}$ and the fact that g' is decreasing, we get

$$\begin{aligned} \frac{1}{\tau_L - \tau_R} \int_{\tau_R}^{\tau_L} g'\big(w - \tilde{p}(\xi)\big)\, d\xi &= \int_0^1 g'\big(w - \tilde{p}\left(t\tau_L + (1-t)\tau_R\right)\big)\, dt \\ &< \int_0^1 g'\big(w - t\tilde{p}(\tau_L) + (1-t)\tilde{p}(\tau_R)\big)\, dt \\ &= \int_0^1 g'\left(tv_L + (1-t)v_R\right)\, dt \\ &= \frac{g(v_R) - g(v_L)}{v_R - v_L}. \end{aligned}$$

Thus, the admissible shocks will satisfy, and the inadmissible shocks will violate, the entropy inequality (32) for all semiconvex entropies $\tilde{\eta}$ and corresponding entropy fluxes $\tilde{q}$. □

By the previous lemma the solution of the Riemann problem for (14) satisfies (32) for all semiconvex entropy/entropy flux pairs (η^δ, q^δ), and then so does the solution $u^\delta(x,t)$ of the Cauchy problem, see [16, Chapter 5.4].

By equation (34) and (36) the entropy/entropy flux pair (η^δ, q^δ) of system (14) for $\rho > 0$ is

$$\begin{aligned}\eta^\delta(\rho, y) &= \frac{1}{\tau}\tilde{\eta}^\delta(\tau, w) = \frac{1}{\tau}\int_{\tau_0}^{\tau} g'\left(w - \tilde{p}^\delta(\xi)\right)\, d\xi,\\ q^\delta(\rho, y) &= \tilde{q}^\delta(\tau, w) + \frac{w - \tilde{p}^\delta(\tau)}{\tau}\,\tilde{\eta}^\delta(\tau, w)\\ &= g\left(w - \tilde{p}^\delta(\tau)\right) - \left(w - \tilde{p}^\delta(\tau)\right)\frac{1}{\tau}\int_{\tau_0}^{\tau} g'\left(w - \tilde{p}^\delta(\xi)\right)\, d\xi.\end{aligned}$$

Thus, $\eta^\delta(\rho, y) \to g'(w)$ and $q^\delta(\rho, y) \to g(w) - wg'(w)$ as $\rho \to 0$. The Lipschitz continuity of $g(v)$ and $g'(v)$ with respect to v yields uniform convergence of (η^δ, q^δ) to some limit (η, q) as $\delta \to 0$. Further, since $df^\delta \to df$, the limit (η, q) satisfies $\nabla q = df \cdot \nabla\eta$.

Consider inequality (32). Let $\delta \to 0$ and use Lebesgue's dominated convergence theorem and the Lipschitz properties. Thus, we finally conclude that for all nonnegative test functions $\phi(x,t)$ in $C_0^\infty(\mathbb{R} \times \mathbb{R}^+)$ and all semiconvex entropies η with corresponding entropy fluxes q,

$$\int_{\mathbb{R}}\int_{\mathbb{R}^+} \left(\eta(u)\phi_t + q(u)\,\phi_x\right) dt\, dx + \int_{\mathbb{R}} \eta(u_0)\phi(x, 0)\, dx \geq 0,$$

and $u(x,t)$ is a weak entropy solution of (1).

3. Proof of Theorem 2.

Consider system (8) with initial Riemann invariants $(v_0(x), w_0(x))$ in $BV(\mathbb{R})^2$ taking values in $\mathcal{D}_V$ and initial Cauchy data $u_0(x)$. In order to show existence of a weak entropy solution, we split the system into a hyperbolic part, given by (1), and a pair of ordinary differential equations in time, $\rho_t = 0$ and $\left[\rho\left(v + p(\rho)\right)\right]_t = \rho R(\rho, v)$. More conveniently we write

$$\begin{aligned}\rho_t &= 0\\ w_t &= v_t = R(\rho, v),\end{aligned} \tag{37}$$

where $v_t = w_t$ follows from $\rho(x,t)$ being constant in time. The hyperbolic system is treated in the previous section. For initial data given by $u_0(x)$, we denote a weak entropy solution of the hyperbolic part at time t as $H(t)u_0(x)$. The domain $\mathcal{D}_V$ is an invariant domain for the system. Further, the total variation of the Riemann invariants is nonincreasing and they are Lipschitz continuous in time as functions into $L^1_{loc}(\mathbb{R})$. Moreover, by Theorem 1 there exists a weak entropy solution in $L^1_{loc}(\mathbb{R} \times \mathbb{R}^+)^2$.

Consider the system of ordinary differential equations (37) with initial data $u_0(x)$. Denote the solution at time t by $S(t)u_0(x)$. For simplicity, we abuse this notation and write the variables w, v and y at time t as $S(t)w_0(x)$, $S(t)v_0(x)$ and $S(t)y_0(x)$, respectively. However, bear in mind that $S(t)w_0$ in fact depends on v_0 as well as w_0, and similarly for the other variables.

First, we want to show that if the initial data lies in the domain $\mathcal{D}_V$ given by (5), then so does the solution $S(t)u_0(x)$. Consider the boundary of the domain $\mathcal{D}_V$. When $w = w_+$ we want $w_t \leq 0$, and thus we require $R(\rho, y) \leq 0$. On the part of the boundary where $v = 0$ we want $v_t \geq 0$, which is satisfied if $R(\rho, y) \geq 0$. For $w = v$ we have $S(t)w = S(t)v$, and the solution is still the vacuum state. Thus, requiring (9) yields invariance of $\mathcal{D}_V$ with respect to the system of ordinary differential equations (37).

We now want to consider the total variation of the solution. From equation (37) we have

$$\big(S(t)v_1 - S(t)v_2\big)_t = R\big(S(t)\rho_1, S(t)v_1\big) - R\big(S(t)\rho_2, S(t)v_2\big).$$

Multiplying the equality by $\operatorname{sign}(S(t)v_1 - S(t)v_2)$ yields

$$\begin{aligned}\big|S(t)v_1 - S(t)v_2\big|_t &\leq \big|R\big(S(t)\rho_1, S(t)v_1\big) - R\big(S(t)\rho_2, S(t)v_2\big)\big| \\ &\leq L\Big(\big|S(t)w_1 - S(t)w_2\big| + \big|S(t)v_1 - S(t)v_2\big|\Big).\end{aligned}$$

The same argument is true for $S(t)w$. Thus, by Grönwall's inequality,

$$|S(t)w_1 - S(t)w_2| + |S(t)v_1 - S(t)v_2| \leq e^{Lt}\left(|w_1 - w_2| + |v_1 - v_2|\right),$$

and this implies

$$\operatorname{TV}\big(S(t)w, S(t)v\big) \leq e^{Lt}\operatorname{TV}(w, v). \tag{38}$$

Since $R(\rho, v)$ is bounded, $S(t)w$ and $S(t)v$ are Lipschitz continuous in time,

$$\int_\Omega |S(t)w - S(s)w|\;dx + \int_\Omega |S(t)v - S(s)v|\;dx \leq \tilde{C}|t - s|, \tag{39}$$

where $\tilde{C} = 2\|R\|_\infty |\Omega|$.

We define an approximate solution of system (8) at time $t_n = n\Delta t$, where $\Delta t = T/N$, by

$$u^n = [H(\Delta t)S(\Delta t)]^n u_0(x).$$

Since $\mathcal{D}_V$ is invariant for both the hyperbolic part and the system of differential equations, the approximate solution u^n takes values in $\mathcal{D}_V$. Further, by (22), (38) and induction the total variation of w^n and v^n is bounded,

$$\operatorname{TV}(w^n, v^n) \leq Ce^{LT}.$$

By (23) and (39) we obtain Lipschitz continuity in time for v^n,

$$\begin{aligned}\|v^n - v^{n+m}\| &\leq \sum_{i=n}^{m-1} \|v^i - v^{i+1}\| \\ &\leq \sum_{i=n}^{m-1} \left(\|v^i - S(\Delta t)v^i\| + \|S(\Delta t)v^i - H(\Delta t)S(\Delta t)v^i\|\right) \qquad (40)\\ &\leq \sum_{i=n}^{m-1} (\tilde{C} + Ce^{LT})\Delta t = \hat{C}|t_n - t_{n+m}|.\end{aligned}$$

The same argument holds for w^n.

We now consider an approximate solution that is defined for all times t in $[0,T]$,

$$u_{\Delta t}(x,t) = \begin{cases} H\big(2(t-t_n)\big)u^n(x), & t\in[t_n,t_{n+1/2}) \\ S\big(2(t-t_{n+1/2})\big)u^{n+1/2}(x), & t\in[t_{n+1/2},t_{n+1}), \end{cases}$$

where $u^{n+1/2}(x) = H(\Delta t)u^n$. >From the above results, $u_{\Delta t}$ takes values in the invariant domain $\mathcal{D}_V$. Further, $w_{\Delta t}$ and $v_{\Delta t}$ have bounded total variation,

$$\mathrm{TV}\,(w_{\Delta t}, v_{\Delta t}) \le Ce^{LT}, \tag{41}$$

and they are Lipschitz continuous in time for t_n, n in $\mathbb{N}$. By (23), (39) and (40), $w_{\Delta t}$ and $v_{\Delta t}$ are Lipschitz continuous in time,

$$\|w_{\Delta t}(s) - w_{\Delta t}(t)\| + \|v_{\Delta t}(s) - v_{\Delta t}(t)\| \le \bar{C}|s-t|, \tag{42}$$

for all s and t in $[0,T]$ and a constant $\bar{C}$ independent of Δt.

In order to prove convergence of a subsequence $u_{\Delta t}(x,t)$ to some limit in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)$, we use Lemma 2 and the same technique as we used to prove convergence of $u^\delta(x,t)$ in the previous section. Since $p'(\rho_{\Delta t}\vee\epsilon)$ is bounded, equation (41) and $w_{\Delta t} = v_{\Delta t} + p(\rho_{\Delta t})$ give

$$\mathrm{TV}(\rho_{\Delta t}\vee\epsilon) \le M_\epsilon \qquad \text{for all } \Delta t \ge 0.$$

Thus, replacing u^δ by $u_{\Delta t}$ in equation (24)–(25) and (26)–(31) yields convergence of $u_{\Delta t}(x,t)$ to some limit $u(x,t)$ in $L^1_{loc}(\mathbb{R}\times\mathbb{R}^+)^2$ as $\Delta t\to 0$. Further, the limits w and v also have bounded total variation (41) and they are Lipschitz continuous in time (42).

It remains to show that the limit $u(x,t)$ is a weak solution of system (8). For simplicity, introduce the vector $r(u) = (0, R(\rho,v))$. Since $u_{\Delta t}(x,t)$ is a weak solution of the hyperbolic system for $t\in[t_n,t_{n+1/2})$, we have for all test functions $\phi(x,t)\in C_0^\infty(\mathbb{R}\times\mathbb{R}^+)$,

$$\begin{aligned}\int_{\mathbb{R}}\int_{t_n}^{t_{n+1/2}} &\big(u_{\Delta t}\phi_t + 2f(u_{\Delta t})\phi_x\big)\,dt\,dx \\ &- \int_{\mathbb{R}} u_{\Delta t}(x,t_{n+1/2})\phi(x,t_{n+1/2})\,dx + \int_{\mathbb{R}} u_{\Delta t}(x,t_n)\phi(x,t_n)\,dx = 0.\end{aligned}$$

Further, $u_{\Delta t}(x,t)$ is a solution of the system of ordinary differential equations for $t\in[t_{n+1/2},t_{n+1})$. After multiplying with a test function ϕ and partial integration,

$$\begin{aligned}\int_{\mathbb{R}}\int_{t_{n+1/2}}^{t_{n+1}} &u_{\Delta t}\phi_t\,dt\,dx + \int_{\mathbb{R}}\int_{t_{n+1/2}}^{t_{n+1}} 2r(u_{\Delta t})\phi\,dt\,dx \\ &- \int_{\mathbb{R}} u_{\Delta t}(x,t_{n+1})\phi(x,t_{n+1})\,dx + \int_{\mathbb{R}} u_{\Delta t}(x,t_{n+1/2})\phi(x,t_{n+1/2})\,dx = 0.\end{aligned}$$

Adding the two equations and summing over $n = 0, 1, ..., N-1$ yields

$$\begin{aligned}\int_{\mathbb{R}}\int_0^T u_{\Delta t}\phi_t \, dx\, dt + 2\chi_{\Delta t}\int_{\mathbb{R}}\int_0^T f(u_{\Delta t})\phi_x \, dt\, dx \\ + 2\tilde{\chi}_{\Delta t}\int_{\mathbb{R}}\int_0^T r(u_{\Delta t})\phi \, dt\, dx + \int_{\mathbb{R}} u_{\Delta t}(x,t_0)\phi(x,t_0)\, dx \\ - \int_{\mathbb{R}} u_{\Delta t}(x,T)\phi(x,T)\, dx = 0,\end{aligned}$$

where $\chi_{\Delta t} = \chi_{\cup[t_n, t_{n+1/2})}$ and $\tilde{\chi}_{\Delta t} = \chi_{\cup[t_{n+1/2}, t_{n+1})}$. Notice that $\chi_{\Delta t}$ converges weakly to $\frac{1}{2}$ as $\Delta t \to 0$. Remember that $u_{\Delta t}$ is bounded and Lipschitz continuous in time. Thus, letting $\Delta t \to 0$ in the above equation, Lebesgue's dominated convergence theorem yields

$$\begin{aligned}\int_{\mathbb{R}}\int_0^T \left(u(x,t)\phi_t(x,t) + f(u(x,t)\phi_x\right) \, dt\, dx + \int_{\mathbb{R}} u_0(x)\phi(x,0)\, dx \\ - \int_{\mathbb{R}}\int_0^T r(u(x,t))\, dt\phi(x,t)\, dx = 0 \quad (43)\end{aligned}$$

i.e. the limit $u(x,t)$ is a weak solution of system (8).

In the proof of Theorem 1 we show existence of semiconvex entropy/entropy flux pairs (η, q) of the flux f, where we assume that $\eta_{\rho\rho} > 0$. By definition, $u_{\Delta t}$ is a weak entropy solution of the hyperbolic part for t in $[t_n, t_{n+1/2})$. Further, we multiply system (37) with $\nabla_u \eta\phi$ and integrate over t in $[t_{n+1/2}, t_{n+1})$. Thus, the same arguments as used to achieve (43) yields

$$\begin{aligned}\int_{\mathbb{R}}\int_0^T \big(\eta(u)\phi_t + q(u)\phi_x\big)\, dx\, dt + \int_{\mathbb{R}} u_0(x)\phi(x,0)\, dx \\ + \int_{\mathbb{R}}\int_0^T \nabla_u\eta(u)r(u)\,\phi\, dt\, dx \geq 0.\end{aligned}$$

For all $t \in [0,T]$, $u_{\Delta t}(t)$ converges in $L^1_{loc}(\mathbb{R})^2$ to a weak entropy solution $u(t)$ of system (8) and the convergence is in $C([0,T]; L^1_{loc}(\mathbb{R})^2)$. Further, the limits $w(x,t)$ and $v(x,t)$ are the Riemann invariants of system (8).

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ERRATA AND COMMENTS TO PAPER I

In this note we point out some errors in **Paper I**, *Existence of Solutions for the Aw–Rascle Traffic Flow Model with Vacuum*, and add a few comments.

(i) The last assumption in (2) ensures genuinely nonlinearity of the first wave family and not strict hyperbolicity as stated in the paper.

(ii) In equation (24) we have assumed that $p(\rho)$ is convex for small densities $\rho > 0$. This assumption should have been stated in the paper.

(iii) Lemma 4 does only hold away from vacuum. It should have been shown that it also holds for a discontinuity connecting vacuum to a right state, as shown in **Paper III**.

(iv) An admissble weak solution should approach vacuum continuously from the left, that is $\lim_{x\to\bar{x}^-} \rho(x,t) = \rho(\bar{x},t) = 0$ if there is a vacuum at $x = \bar{x}$ at time t. In **Paper III** it is shown that the entropy inequality considered in Lemma 4 holds for such a vacuum. However, in addition it also holds for some inadmissible vacuum, more specific vacuum which are the right state for a contact discontinuity.

(v) In this paper approximate weak solutions are constructed by the Glimm scheme, and a description of the general solution of a Riemann problem which yields vacuum is given. We do not show that this weak solution and the definition of the vacuum state is entropy admissible. However, this is discussed in **Paper III**.

(vi) If there is a vacuum in the initial Riemann data we would have to give $v_0 = w_0$ at vacuum the value of w to the left of the vacuum state. Otherwise our solutions of the Riemann problem will not make sense and our solution machinery will not work. However, since the conservative variables $(\rho, \rho w) = (0,0)$ at vacuum, the values of v and w at vacuum are artificial and do not change the conservative weak solutions. Furthermore, when $\rho = 0$ the velocity is not physically defined.

Paper II

Car-Following and the Macroscopic Aw–Rascle Traffic Flow Model

M. Godvik and H. Hanche-Olsen

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Paper III

Well-Posedness for the Aw–Rascle Traffic Flow Model with Vacuum

M. Godvik and H. Hanche-Olsen

Preprint.

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