GI@La Trobe16:
Two classes of quadratic vector fields for which the Kahan discretization is integrable

Elena Celledoni[1], Robert I. McLachlan [2], David I. McLaren[3], Brynjulf Owren[1], G.R.W. Quispel[3],

1 Norwegian University of Science and Technology, 7491, Trondheim, Norway

2 Massey University, Palmerston North, New Zealand

3 La Trobe University, Victoria 3086, Australia

1 INTRODUCTION

Kahan's method was introduced in [1]. It was rediscovered in the context of integrable systems by Hirota and Kimura [2]. Suris and collaborators extended the applications to integrable systems significantly in a series of papers [3], [4], [5], [6], [7]. Applications to non-integrable Hamiltonian systems and the use of polarisation to discretise arbitrary degree Hamiltonian systems were studied in [8], [9] and [10].

The present paper contains an extract of the talk one of the authors (GRWQ) gave on 6th July 2016 at the 12th International Conference on Symmetries and Integrability of Difference Equations (SIDE12) in Sainte Adele, Quebec, Canada. We present two classes of 2-dimensional ODE systems with quadratic vector fields where the Kahan discretization is integrable. Both classes of systems can be cast in the form

$$\frac{dX}{dt} = \varphi(X) K \nabla H(X), \tag{1}$$

where

$$X^t = (x, y), \qquad K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In the first class the Hamiltonian function is quartic and in the second class it is sextic. These systems can be seen as generalisations of the examples of the reduced Naham equations presented in [6]. Some of the results in this paper were found independently by Petrera and Zander [11].

2 Quartic Hamiltonians in 2D

Consider the 2-dimensional ODE system (1) where $\varphi(X) = \frac{1}{ax+by}$, and the homogeneous Hamiltonian has the form $H = (ax + by)^2(cx^2 + 2dxy + ey^2)$. Then the Kahan map for this system preserves the modified Hamiltonian:

$$\widetilde{H}(X) = \frac{H}{(1+h^2D(ax+by)^2+h^2E(cx^2+2dxy+ey^2))(1+h^29D(ax+by)^2+h^2E(cx^2+2dxy+ey^2)},$$

and the measure:

$$m(x,y) = \frac{dxdy}{(ax+by)(cx^2+2dxy+ey^2)}.$$

Here, $D := ce - d^2$ and $E := 2abd - a^2e - b^2c$. It follows that the Kahan map is integrable. It also seems to preserve the genus of the level sets of H (which generically equals 1).

3 Sextic Hamiltonians in 2D

Consider the 2-dimensional ODE system (1) with $\varphi(X) = \frac{1}{(cx+dy)(ex+fy)^2}$ and with the *homogeneous* sextic Hamiltonian $H = (ax+by)(cx+dy)^2(ex+fy)^3$. Then the Kahan map for this system preserves the modified Hamiltonian:

$$\widetilde{H}(X) = \frac{H}{(1+a_5l_2^2)(1+a_3l_1^2+a_4l_3^2+a_7l_1l_3)(1+a_5l_2^2+a_6l_3^2))}$$

where

$$l_1 = ax+by, \quad l_2 = cx+dy, \quad l_3 = ex+fy,$$

and $d_{1,2} = h(ad-bc)$, $d_{2,3} = h(cf-ed)$, $d_{3,1} = h(eb-fa)$ and $a_3 = \frac{-9d_{2,3}^2}{4}$, $a_4 = \frac{-d_{1,2}^2}{4}$, $a_5 = \frac{-9d_{3,1}^2}{4}$; $a_6 = -4d_{1,2}^2$, $a_7 = \frac{3d_{1,2}d_{2,3}}{2}$. In this case the Kahan map also preserves the modified measure

$$m(x,y) = \frac{dxdy}{(ax+by)(cx+dy)(ex+fy)}.$$

Again, the Kahan map is integrable. It also seems to preserve the genus of the level sets of H (which generically equals 1).

Acknowledgment

This work was supported by the Australian Research Council and by the Research Council of Norway, and by a New Zealand Marsden Grant ??, and by the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No. 691070.

References

[1] W Kahan, *Unconventional numerical methods for trajectory calculations*, Unpublished lecture notes, 1993.

[2] K Kimura and R Hirota, *Discretization of the Lagrange top*, J. Phys. Soc. Jap. **69** (2000), 3193–3199.

[3] M Petrera and Y B Suris, *On the Hamiltonian structure of Hirota-Kimura discretization of the Euler top*, Mathematische Nachrichten **283** (11) (2010), 1654–1663.

[4] M Petrera and Y B Suris, *SV Kovalevskaya system, its generalization and discretization*, Frontiers of Mathematics in China **8** (2012), 1047–1065.

[5] M Petrera and Y B Suris, *Spherical geometry and integrable systems*, Geometriae Dedicata **169** (2014), 83–98.

[6] M Petrera, A Pfadler and Y B Suris, *On integrability of Hirota-Kimura type discretizations,* Regular and Chaotic Dynamics **16** (2011), 245–289.

[7] A Hone and M Petrera, *Three-dimensional discrete systems of Hirota-Kimura type and deformed Lie-Poisson algebras*, Journal of Geometric Mechanics **1** (2009), 55-85.

[8] E Celledoni, R I McLachlan, B Owren and G R W Quispel, *Geometric properties of Kahan's method* J Phys. A **46** (2013), 025001.

[9] E Celledoni, R I McLachlan, B Owren and G R W Quispel, *Integrability properties of Kahan's method,* J. Phys. A **47** (2014), 365202.

[10] E Celledoni, R I McLachlan, B Owren and G R W Quispel, *Discretization of polynomial vector fields by polarisation*, Proc. Roy. Soc. **A471** (2015) 20150390, 10pp. ,

[11] M Petrera and R. Zander, *New classes of quadratic vector fields admitting integral-preserving Kahan-Hirota-Kimura discretizations*, arXiv:1610.03664 [nlin.SI].