Model Reference Adaptive Control of 2×2 Coupled Linear Hyperbolic PDEs

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***Abstract*—We solve a model reference adaptive control problem for a class of linear 2×2 hyperbolic partial differential equations (PDEs) with uncertain system parameters subject to harmonic disturbances, from a single boundary measurement anti-collocated with the actuation. This is done by transforming the system into a canonical form, from which filters are designed so that the states can be expressed as linear combinations of the filters and uncertain parameters, a representation facilitating for the design of adaptive laws. A stabilizing controller is then combined with the adaptive laws to make the measured signal asymptotically track the output of a reference model. The reference model is taken as a simple transport partial differential equation. Moreover, pointwise boundedness of all variables in the closed loop is proved, provided the reference signal is bounded. The theory is demonstrated in a simulation.**

I. Introduction

In this paper, we investigate a model reference adaptive control problem for a class of 2×2 coupled linear hyperbolic partial differential equations (PDEs) with uncertain system parameters and influenced by harmonic disturbances. Linear hyperbolic PDEs have attracted considerable attention due to the vast amount of different physical systems that can be modeled by them, ranging from open channel flows [1] and oil wells [2] to road traffic [3] and predator-pray systems [4]. Linear hyperbolic PDEs therefore give rise to important estimation and control problems, for which early results can be found in [5], [6], [7] and more recently in [8].

Infinite-dimensional backstepping for distributed systems, originally presented in [9], has in the last couple of decades emerged as a general framework for stabilization of PDEs. When using backstepping for PDE control synthesis, one designs a Volterra transformation and a control law that map the system of interest into an auxiliary "target" system designed with some desirable stability properties. It is then proved that the transformation is boundedly invertible, so that the two systems' stability properties are the same. While the infinite-dimensional backstepping method was originally developed for parabolic equations, it has later been extended to 1-D PDEs of hyperbolic type in [10], and to linear 2×2 coupled hyperbolic PDEs of the type investigated in this paper in [11]. Extensions to higher dimensions have also been made in [12] ($n+1$ systems) and [13] ($n+m$ systems). A slight modification of the method from [13] resulted in [14] to a controller and observer for $n+m$ systems converging in minimum time. A controller and observer for hyperbolic PIDE systems with non-strict feedback terms in the form of Fredholm integrals was presented in [15], while a Luenberger-type observer was designed in [16] for PIDE systems with time-varying coefficients.

Adaptive control of PDEs, where one or several of the system parameters are unknown, is a well-established field when it comes to PDEs of parabolic type [17], [18], [19], [20], and has recently also started to emerge for hyperbolic PDEs. The first result on the latter was presented in [21], where a single hyperbolic partial-integro differential equation (PIDE) was mapped to a target system using an invertible backstepping transformation, before a filter-based control law was designed. Backstepping was then used once more to establish closed-loop stability and convergence to zero. A full-state feedback stabilizing controller for a subclass of the 1-D hyperbolic PDEs investigated in [21] was offered in [22], while adaptive full-state feedback controllers for coupled 2×2 systems with uncertain in-domain parameters were given in [23], using identifier-based design, and [24], using the swapping filter-based design originally presented for hyperbolic systems in [25]. The output-feedback solution from [21] has also in [26] been extended to 2×2 hyperbolic PDEs of the type investigated in the present paper, offering an adaptive stabilizing controller using a single boundary measurement only. A similar problem was solved in [27], allowing non-local source terms to be present but limited to the case of having constant and equal transport speeds, and also requiring sensing to be taken at both boundaries.

A disturbance rejection problem was investigated for 2×2 systems in [28]. In that paper, a disturbance entered at one boundary, while sensing and actuation were limited to the opposite boundary. The disturbance was modeled as a linear autonomous ordinary differential equation (ODE), particularly aimed at modeling periodic disturbances with a bias, as in the present paper. The point of rejection was the boundary where the disturbance entered, a limitation later relaxed in [29] where the point of rejection could be anywhere in the domain. Extensions to $n+1$ systems were done in [30], assuming sensing at both boundaries, and in [31] where sensing was restricted to the boundary of actuation. The general $n+m$ case was covered in [32]. Common for all of these methods is that they apply the separation principle of linear systems, combining state feedback laws with state observers, and also assume all system parameters to be known. An adaptive disturbance rejection scheme was recently developed in [33], where the disturbance's bias, frequencies, amplitudes and phases were all unknown, and in [34], where the system's parameters were allowed to be uncertain.

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A tracking problem for the same type of systems considered in the present paper was solved in [35]. In that paper, a reference trajectory was generated by "inversely" using backstepping on a very simple reference model, before a standard PI controller was used to drive the measured output to the generated reference signal. A similar problem was solved in [13] for $n+m$ systems of coupled PDEs with known, constant coefficients. In [36] a tracking problem for $n+m$ systems with known, spatially varying coefficients was solved. The convergence time was lower than in [13], since the control design was based on the minimum-time controller derived in [14]. A related problem was investigated in [37], where both a disturbance rejection problem and tracking problem for the same type of systems investigated in the present paper were solved simultaneously. However, in all of these papers, all system parameters were assumed known.

In the problem to be investigated in this paper - formally stated in Section II - actuation is on one boundary only, while the measurement is restricted to the opposite boundary. Both of these are allowed to be scaled by uncertain nonzero constants, and affected by the disturbance. The goal is to make the measured signal track the output of a reference model. The reference model is essentially a transport delay, with the delay corresponding to the propagation time from the actuated to the measured boundary, an unavoidable restriction that only can be overcome if some sort of prediction can be made on the reference signal. The only assumption made on the system, is that the total transport delays in each direction, the disturbances' frequencies and the sign of the product of the actuation and measurement scaling constants are known. All other parameters are unknown. We believe the work presented here to be the first result on model reference adaptive control for coupled linear hyperbolic PDEs. A subproblem only involving disturbance rejection was solved in [34], but the objective was rather restrictive, and the actuation and measurement were not allowed to be scaled. We offer here an extension to the general disturbance rejection case, and also solve a model reference adaptive control problem. The method we use is an extension of the filter-based stabilizing controller derived for 2×2-systems in [26].

In this paper, we only consider variables that are real. For a variable $z(x,t)$ defined for $0 \leq x \leq 1$, $t \geq 0$, we will in subsequent sections denote by $||z||$ the L_2-norm

$$||z(t)|| = \sqrt{\int_0^1 z^2(x,t)dx}. \tag{1}$$

For a time-varying, real signal $f(t)$,

$$f \in \mathcal{L}_p([a,b]) \Leftrightarrow \left(\int_a^b |f(t)|^p dt\right)^{\frac{1}{p}} < \infty \tag{2}$$

for $p \geq 1$ with the particular case

$$f \in \mathcal{L}_\infty([a,b]) \Leftrightarrow \sup_{a \leq t \leq b} |f(t)| < \infty. \tag{3}$$

The argumentless shorthand notations

$$\mathcal{L}_p = \mathcal{L}_p([0,\infty]), \qquad \mathcal{L}_\infty = \mathcal{L}_\infty([0,\infty]) \tag{4}$$

will also be used. Moreover, we will in subsequent sections often omit writing the argument in time, so that e.g. $||u|| = ||u(t)||$ and $z(x) = z(x,t)$.

II. Problem statement

In this paper, we investigate systems on the form

$$u_t(x,t) + \lambda(x)u_x(x,t) = c_1(x)v(x,t) + d_1(x,t) \tag{5a}$$
$$v_t(x,t) - \mu(x)v_x(x,t) = c_2(x)u(x,t) + d_2(x,t) \tag{5b}$$
$$u(0,t) = qv(0,t) + d_3(t) \tag{5c}$$
$$v(1,t) = k_1U(t) + d_4(t) \tag{5d}$$
$$y(t) = k_2v(0,t) + d_5(t) \tag{5e}$$

where $u(x,t)$, $v(x,t)$ are the system states, and $d_1(x,t)$, $d_2(x,t)$, $d_3(t)$, $d_4(t)$, $d_5(t)$ are biased, harmonic disturbances containing a known number of known frequencies, but with unknown amplitudes, phases and biases. The parameters μ, λ, c_1, c_2, q, k_1, k_2 are unknown, but assumed to satisfy

$$\mu, \lambda \in C^1([0,1]),\ \mu(x), \lambda(x) > 0\ \ \forall x \in [0,1] \tag{6a}$$
$$c_1, c_2 \in C^0([0,1]) \tag{6b}$$
$$q, k_1, k_2 \in \mathbb{R}\backslash\{0\}. \tag{6c}$$

Although the exact profiles of λ and μ are not needed, we assume the total transport delays in each direction are known, that is

$$d_\alpha = \bar{\lambda}^{-1} = \int_0^1 \frac{d\gamma}{\lambda(\gamma)}, \qquad d_\beta = \bar{\mu}^{-1} = \int_0^1 \frac{d\gamma}{\mu(\gamma)} \tag{7}$$

are known quantities, also the sign of the product k_1k_2, that is

$$\mathrm{sign}\,(k_1k_2)\,, \tag{8}$$

is assumed known. The initial conditions $u(x,0) = u_0(x)$, $v(x,0) = v_0(x)$ are assumed to satisfy

$$u_0,\ v_0 \in \mathcal{B}, \tag{9}$$

where the space $\mathcal{B}$ is the set of functions defined over $[0,1]$, so that

$$f \in \mathcal{B} \Leftrightarrow |f(x)| < \infty,\ \forall x \in [0,1] \tag{10}$$

with norm

$$||f||_\infty = \sup_{x \in [0,1]} |f(x)|. \tag{11}$$

Remark 1: Under the above conditions, (5) has a unique weak solution for all $(x,t) \in [0,1] \times [0,T]$ for any $T > 0$. The solution can be constructed by transforming (5) to an integral equation and applying successive approximations (see [38] for details). In fact, it can be shown [38] that

$$\sup_{t \in [0,T]} (||u||_\infty + ||v||_\infty) \tag{12}$$

is bounded, with bound depending on T, $\sup_{t \in [0,T]} |U(t)|$, $||u_0||_\infty$, $||v_0||_\infty$, as well as bounds on d_i, $i = 1, \ldots, 5$. That is, $u, v \in \mathcal{B}$ for all $t \geq 0$. The significance of these facts is that even though the Lyapunov analysis that follows in this paper is carried out in the L_2-norm, pointwise evaluation of the distributed states makes sense.

The goal is to design an adaptive control law $U(t)$ in (5d) so that system (5) is adaptively stabilized, in the sense that the objective

$$\lim_{t\to\infty}\int_t^{t+T}(y(s)-y_r(s))^2ds = 0 \tag{13}$$

is obtained for some bounded constant $T > 0$, where the reference signal y_r is generated from the reference model

$$b_t(x,t) - \bar{\mu}b_x(x,t) = 0 \tag{14a}$$
$$b(1,t) = r(t) \tag{14b}$$
$$y_r(t) = b(0,t) \tag{14c}$$

with some initial condition

$$b(x,0) = b_0(x) \tag{15}$$

of choice, provided $b_0 \in \mathcal{B}$. The signal $r(t)$ is a bounded signal of choice. The goal (13) should be achieved from using the sensing (5e) only. Moreover, all additional variables in the closed loop system should be bounded pointwise in space.

The disturbance rejection problem solved in [34] is a subproblem of (13), corresponding to $k_1 = k_2 = 1$, $d_1 = d_2 = d_4 = d_5 \equiv 0$ and $r \equiv 0$.

III. Transform to canonical form

A. Disturbance parametrization

In the transformations to follow, we will need a parametrization of the disturbance terms d_1, d_2, d_3, d_4, d_5. As they are all assumed to be biased, harmonic disturbances with a known number n of distinct frequencies, they can all be represented as outputs of an autonomous linear system. Hence, we parameterize the disturbances as follows

$$d_1(x,t) = g_1^T(x)X(t), \qquad d_2(x,t) = g_2^T(x)X(t) \tag{16a}$$
$$d_3(t) = g_3^TX(t), \qquad d_4(t) = g_4^TX(t) \tag{16b}$$
$$d_5(t) = g_5^TX(t), \qquad \dot{X}(t) = AX(t), \tag{16c}$$

where the matrix $A \in \mathbb{R}^{(2n+1)\times(2n+1)}$ is known and has the form

$$A = \text{diag}\,\{0, A_1, A_2, \dots, A_n\} \tag{17}$$

where

$$A_i = \begin{bmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{bmatrix} \tag{18}$$

for $i = 1 \dots n$. The vectors g_1, g_2, g_3, g_4, g_5 and the disturbance model's initial condition $X(0) = X_0$, however, are unknown.

B. Decoupling

Lemma 2: System (5) is through an invertible backstepping transformation, which is characterized in the proof, equivalent to the system

$$\check{\alpha}_t(x) + \lambda(x)\check{\alpha}_x(x) = 0 \tag{19a}$$
$$\check{\beta}_t(x) - \mu(x)\check{\beta}_x(x) = 0 \tag{19b}$$
$$\check{\alpha}(0) = q\check{\beta}(0) \tag{19c}$$
$$\check{\beta}(1) = k_1U - \int_0^1 m_1(\xi)\check{\alpha}(\xi)d\xi - \int_0^1 m_2(\xi)\check{\beta}(\xi)d\xi - m_3^TX \tag{19d}$$
$$y = k_2\check{\beta}(0) \tag{19e}$$

for some (continuous) functions m_1, m_2, m_3 of the unknown parameters $\mu, \lambda, c_1, c_2, q$.

Proof: We will prove that the systems (5) and (19) are connected through an invertible backstepping transformation. To ease the derivations to follow, we write system (5) on vector form as follows

$$\zeta_t(x) + \Lambda(x)\zeta_x(x) = \Pi(x)\zeta(x) + G(x)X \tag{20a}$$
$$\zeta(0) = Q_0\zeta(0) + G_3X \tag{20b}$$
$$\zeta(1) = R_1\zeta(1) + k_1\bar{U} + G_4X \tag{20c}$$

where

$$\zeta(x) = \begin{bmatrix} u(x) \\ v(x) \end{bmatrix}, \qquad \Lambda(x) = \begin{bmatrix} \lambda(x) & 0 \\ 0 & -\mu(x) \end{bmatrix} \tag{21a}$$
$$\Pi(x) = \begin{bmatrix} 0 & c_1(x) \\ c_2(x) & 0 \end{bmatrix}, \quad G(x) = \begin{bmatrix} g_1^T(x) \\ g_2^T(x) \end{bmatrix} \tag{21b}$$
$$Q_0 = \begin{bmatrix} 0 & q \\ 0 & 1 \end{bmatrix}, \qquad R_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{21c}$$
$$\bar{U} = \begin{bmatrix} 0 \\ U \end{bmatrix}, \qquad G_3 = \begin{bmatrix} g_3^T \\ 0 \end{bmatrix} \tag{21d}$$
$$G_4 = \begin{bmatrix} 0 \\ g_4^T \end{bmatrix}. \tag{21e}$$

Consider the backstepping transformation

$$\gamma(x) = \zeta(x) - \int_0^x K(x,\xi)\zeta(\xi)d\xi - F(x)X \tag{22}$$

where

$$\gamma(x) = \begin{bmatrix} \check{\alpha}(x) \\ \check{\beta}(x) \end{bmatrix} \tag{23}$$

contains the new set of variables, and

$$K(x,\xi) = \begin{bmatrix} K^{uu}(x,\xi) & K^{uv}(x,\xi) \\ K^{vu}(x,\xi) & K^{vv}(x,\xi) \end{bmatrix} \tag{24a}$$
$$F(x) = \begin{bmatrix} f_1^T(x) \\ f_2^T(x) \end{bmatrix}. \tag{24b}$$

Differentiating (22) with respect to time, inserting the dynamics (20a) and (16c) and integration by parts, we find

$$\begin{aligned}\zeta_t(x) &= \gamma_t(x) - K(x,x)\Lambda(x)\zeta(x) + K(x,0)\Lambda(0)\zeta(0) \\ &\quad + \int_0^x \Big[K_\xi(x,\xi)\Lambda(\xi) + K(x,\xi)\Lambda'(\xi) \\ &\qquad + K(x,\xi)\Pi(\xi)\Big]\zeta(\xi)d\xi \\ &\quad + \int_0^x K(x,\xi)G(\xi)Xd\xi + F(x)AX. \end{aligned} \tag{25}$$

Equivalently, differentiating (22) with respect to space, we find

$$\begin{aligned}\zeta_x(x) &= \gamma_x(x) + K(x,x)\zeta(x) \\ &\quad + \int_0^x K_x(x,\xi)\zeta(\xi)d\xi + F'(x)X. \end{aligned} \tag{26}$$

Inserting (25) and (26) into (20a), we find

$$\begin{aligned}&\gamma_t(x)+\Lambda(x)\gamma_x(x)+K(x,0)\Lambda(0)Q_0\zeta(0)\\&+\left[\Lambda(x)K(x,x)-K(x,x)\Lambda(x)-\Pi(x)\right]\zeta(x)\\&+\int_0^x\left[\Lambda(x)K_x+K_\xi\Lambda(\xi)+K\Pi(\xi)+K\Lambda'(\xi)\right]\zeta(\xi)d\xi\\&+\Bigg[\Lambda(x)F'(x)-G(x)+F(x)A+\int_0^x K(x,\xi)G(\xi)d\xi\\&\quad+K(x,0)\Lambda(0)G_3\Bigg]X=0. \end{aligned}\tag{27}$$

If K and F satisfy the following equations

$$0=\Lambda(x)K_x+K_\xi\Lambda(\xi)+K\Pi(\xi)+K\Lambda'(\xi) \tag{28a}$$
$$0=\Lambda(x)K(x,x)-K(x,x)\Lambda(x)-\Pi(x) \tag{28b}$$
$$0=K(x,0)\Lambda(0)Q_0 \tag{28c}$$
$$\begin{aligned}0&=\Lambda(x)F'(x)-G(x)+F(x)A\\&\quad+\int_0^x K(x,\xi)G(\xi)d\xi+K(x,0)\Lambda(0)G_3,\end{aligned} \tag{28d}$$

we obtain the target system equations (19a)–(19b). Inserting the transformation (22) into the boundary condition (5c) and the measurement (5e), we obtain

$$\check{\alpha}(0)+f_1^T(0)X=q\check{\beta}(0)+qf_2^T(0)X+g_3^TX \tag{29a}$$
$$y=k_2\check{\beta}(0)+k_2f_2^TX+g_5^TX. \tag{29b}$$

Choosing

$$f_1^T(0)=-\frac{q}{k_2}g_5^T+g_3^T \tag{30a}$$
$$f_2^T(0)=-\frac{1}{k_2}g_5^T \tag{30b}$$

we obtain (19c) and (19e). The equations consisting of (28a)–(28c) have a unique, continuous solution K according to [11]. The equations consisting of (28d) and (30) is a standard matrix ODE which can be explicitly solved for F. The inverse of (22) is given as

$$\zeta(x)=\gamma(x)+\int_0^x L(x,\xi)\gamma(\xi)d\xi+R(x)X \tag{31}$$

where

$$L(x,\xi)=\begin{bmatrix}L^{\alpha\alpha}(x,\xi) & L^{\alpha\beta}(x,\xi)\\ L^{\beta\alpha}(x,\xi) & L^{\beta\beta}(x,\xi)\end{bmatrix} \tag{32a}$$
$$R(x)=\begin{bmatrix}r_1^T(x)\\ r_2^T(x)\end{bmatrix}. \tag{32b}$$

Inserting the backstepping transformation (22) into (31) and changing the order of integration in the double integral, we find

$$\begin{aligned}0=&\int_0^x\left[L(x,\xi)-K(x,\xi)-\int_\xi^x L(x,s)K(s,\xi)ds\right]\zeta(\xi)d\xi\\&+\left[R(x)-F(x)-\int_0^x L(x,\xi)F(\xi)d\xi\right]X\end{aligned} \tag{33}$$

Hence, the gains L, R in the inverse transformation are given from the backstepping gains as

$$L(x,\xi)=K(x,\xi)+\int_\xi^x L(x,s)K(s,\xi)ds \tag{34a}$$
$$R(x)=F(x)+\int_0^x L(x,\xi)F(\xi)d\xi. \tag{34b}$$

From inserting $x=1$ into (31), we obtain (19d), where

$$m_1(\xi)=L^{\beta\alpha}(1,\xi) \tag{35a}$$
$$m_2(\xi)=L^{\beta\beta}(1,\xi) \tag{35b}$$
$$m_3^T=r_2^T(1)-g_4^T. \tag{35c}$$

■

C. Scaling and mapping to constant transport speeds

We now use a transformation to get rid of the spatially varying transport speeds in (19), and also scale the variables to ease subsequent analysis.

Lemma 3: System (19) is equivalent to the system

$$\alpha_t(x)+\bar{\lambda}\alpha_x(x)=0 \tag{36a}$$
$$\beta_t(x)-\bar{\mu}\beta_x(x)=0 \tag{36b}$$
$$\alpha(0)=\beta(0) \tag{36c}$$
$$\begin{aligned}\beta(1)=&\rho U-\int_0^1\kappa(\xi)\alpha(\xi)d\xi\\&-\int_0^1\sigma(\xi)\beta(\xi)d\xi-m_4^TX\end{aligned} \tag{36d}$$
$$y=\beta(0) \tag{36e}$$

where ρ,κ,σ,m_4 are continuous functions of m_1,m_2,m_3,k_1 and k_2.

Proof: Consider the invertible mapping

$$\alpha(x)=\frac{k_2}{q}\check{\alpha}(h_\alpha^{-1}(x)) \tag{37a}$$
$$\beta(x)=k_2\check{\beta}(h_\beta^{-1}(x)) \tag{37b}$$

where

$$h_\alpha(x)=\bar{\lambda}\int_0^x\frac{d\gamma}{\lambda(\gamma)} \tag{38a}$$
$$h_\beta(x)=\bar{\mu}\int_0^x\frac{d\gamma}{\mu(\gamma)} \tag{38b}$$

with $\bar{\lambda},\bar{\mu}$ defined in (7), are strictly increasing and hence invertible functions. The invertiblility of the transform (37) therefore follows. The rest of the proof follows immediately from insertion and noting that

$$h'_\alpha(x)=\frac{\bar{\lambda}}{\lambda(x)},\qquad h'_\beta(x)=\frac{\bar{\mu}}{\mu(x)} \tag{39a}$$
$$h_\alpha(0)=h_\beta(0)=0,\qquad h_\alpha(1)=h_\beta(1)=1 \tag{39b}$$

and is therefore omitted. The new parameters are given as

$$\rho=k_1k_2 \tag{40a}$$
$$\kappa(x)=qd_\alpha\lambda(h_\alpha^{-1}(x))m_1(h_\alpha^{-1}(x)) \tag{40b}$$
$$\sigma(x)\ =d_\beta\mu(h_\beta^{-1}(x))m_2(h_\beta^{-1}(x)) \tag{40c}$$
$$m_4=k_2m_3^T. \tag{40d}$$

■

D. Extension of reference model and error dynamics

In view of the structure of system (36), we extend the reference model (14) with an additional variable a as follows

$$a_t(x) + \bar{\lambda} a_x(x) = 0 \quad (41a)$$
$$b_t(x) - \bar{\mu} b_x(x) = 0 \quad (41b)$$
$$a(0) = b(0) \quad (41c)$$
$$b(1) = r \quad (41d)$$

with initial conditions (15) and

$$a(x,0) = a_0(x) \quad (42)$$

with $a_0 \in \mathcal{B}$.

Lemma 4: Consider the system (36) and the extended reference model (41). The error variables

$$w(x) = \alpha(x) - a(x) \quad (43a)$$
$$\check{z}(x) = \beta(x) - b(x) \quad (43b)$$

satisfy the dynamics

$$w_t(x) + \bar{\lambda} w_x(x) = 0 \quad (44a)$$
$$\check{z}_t(x) - \bar{\mu} \check{z}_x(x) = 0 \quad (44b)$$
$$w(0) = \check{z}(0) \quad (44c)$$
$$\check{z}(1) = \rho U - r + \int_0^1 \kappa(\xi)(w(\xi) + a(\xi))d\xi + \int_0^1 \sigma(\xi)(\check{z}(\xi) + b(\xi))d\xi + m_4^T X \quad (44d)$$

with the measurement (36e) becoming

$$y = \check{z}(0) + b(0). \quad (45)$$

Proof: The proof is straight forward, and therefore omitted. ■

E. Canonical form

Lemma 5: System (44) is equivalent to the system

$$w_t(x) + \bar{\lambda} w_x(x) = 0 \quad (46a)$$
$$z_t(x) - \bar{\mu} z_x(x) = \bar{\mu}\theta(x) z(0) \quad (46b)$$
$$w(0) = z(0) \quad (46c)$$
$$z(1) = \rho U - r + \int_0^1 \kappa(\xi)(w(\xi) + a(\xi))d\xi + \int_0^1 \theta(\xi) b(1-\xi)d\xi + m_4^T X \quad (46d)$$

with measurement

$$y = v(0) = z(0) + b(0) \quad (47)$$

where

$$\theta(x) = \sigma(1-x). \quad (48)$$

Proof: Consider the transformation

$$z(x) = \check{z}(x) - \int_0^x \sigma(1 - x + \xi)\check{z}(\xi)d\xi, \quad (49)$$

which is invertible with inverse

$$\check{z}(x) = z(x) + \int_0^x \omega(x-\xi) z(\xi) d\xi \quad (50)$$

where ω satisfies the Volterra equation

$$\omega(x) = \int_0^x \omega(x-\xi)\sigma(1-\xi)d\xi - \sigma(1-x). \quad (51)$$

This can be verified from insertion and changing the order of integration in the double integral. Differentiating (49) with respect to time and space, respectively, we find

$$\check{z}_t(x) = z_t(x) + \bar{\mu}\sigma(1)\check{z}(x) - \bar{\mu}\sigma(1-x)\check{z}(0) - \int_0^x \bar{\mu}\sigma'(1-x+\xi)\check{z}(\xi)d\xi \quad (52)$$

and

$$\check{z}_x(x) = z(x) + \sigma(1)\check{z}(x) - \int_0^x \sigma'(1-x+\xi)\check{z}(\xi)d\xi. \quad (53)$$

Inserting (52) and (53) into (44b), we obtain

$$\check{z}_t(x) - \bar{\mu}\check{z}_x(x) = z_t(x) - \bar{\mu} z(x) - \bar{\mu}\sigma(1-x)\check{z}(0) = 0 \quad (54)$$

which gives (46b) with θ defined in (48), since

$$\check{z}(0) = z(0). \quad (55)$$

Lastly using (49) and (44d), we have

$$\begin{aligned} z(1) &= \rho U - r + \int_0^1 \kappa(\xi)(w(\xi) + a(\xi))d\xi \\ &\quad + \int_0^1 \sigma(\xi)(\check{z}(\xi) + b(\xi))d\xi \\ &\quad - \int_0^1 \sigma(1 - 1 + \xi)\check{z}(\xi)d\xi + m_4^T X \\ &= \rho U - r + \int_0^1 \kappa(\xi)(w(\xi) + a(\xi))d\xi \\ &\quad + \int_0^1 \sigma(\xi) b(\xi)d\xi + m_4^T X \end{aligned} \quad (56)$$

which gives (46d), in view of the identity

$$\int_0^1 \sigma(\xi) b(\xi) d\xi = \int_0^1 \theta(1-\xi) b(\xi)d\xi = \int_0^1 \theta(\xi) b(1-\xi)d\xi. \quad (57)$$

■

Remark 6: It is important to notice that the formulas expressing how θ, ρ, κ and m_4 relate to the original system parameters of (5), even though specified in Lemmas 2–5, are not needed in the control design that follows. The adaptive parameter update laws are designed for the parameters of the system in canonical form (46).

IV. Adaptive control

We have thus shown that stabilizing (46) is equivalent to stabilizing the original system (5), because the reference system (41) itself is stable for any bounded r. Moreover, the objective (13) can be stated in terms of z as

$$\lim_{t\to\infty} \int_t^{t+T} z^2(0,s)ds = 0. \quad (58)$$

The goal is to design a control law U so that z and w converge in $\mathcal{B}$ at least asymptotically to zero, while at the same time ensuring pointwise boundedness of all variables and convergence of $z(0)$ to zero in the sense of (58).

A. Reparametrization of X

We reparameterize the disturbance term $m_4^T X$ as follows

$$m_4^T X = \chi^T \nu \tag{59}$$

where

$$\begin{aligned}\chi^T &= \chi^T(t)\\ &= \begin{bmatrix}1 & \sin(\omega_1 t) & \cos(\omega_1 t) & \dots & \sin(\omega_n t) & \cos(\omega_n t)\end{bmatrix}\end{aligned} \tag{60}$$

contains known components, while

$$\nu = \begin{bmatrix}a_0 & a_1 & b_1 & \dots & a_n & b_n\end{bmatrix}^T \tag{61}$$

contains the unknown amplitudes and bias.

B. Filter design

We introduce the following filters

$$\psi_t(x) - \bar{\mu}\psi_x(x) = 0, \qquad \psi(1) = U \tag{62a}$$
$$\phi_t(x) - \bar{\mu}\phi_x(x) = 0, \qquad \phi(1) = y - b(0) \tag{62b}$$
$$\vartheta_t(x) - \bar{\mu}\vartheta_x(x) = 0, \qquad \vartheta(1) = \chi \tag{62c}$$
$$P_t(x,\xi) + \bar{\lambda}P_\xi(x,\xi) = 0, \qquad P(x,0) = \phi(x) \tag{62d}$$

and the derived filters

$$p_0(x) = P(0,x) \tag{63a}$$
$$p_1(x) = P(1,x) \tag{63b}$$

and also the filtered reference variables

$$M_t(x,\xi) - \bar{\mu}M_x(x,\xi) = 0, \quad M(1,\xi) = a(\xi) \tag{64a}$$
$$N_t(x,\xi) - \bar{\mu}N_x(x,\xi) = 0, \quad N(1,\xi) = b(1-\xi) \tag{64b}$$

with the derived filtered reference variables

$$n_0(x) = N(0,x) \tag{65a}$$
$$m_0(x) = M(0,x) \tag{65b}$$

for some initial conditions $\psi(x,0) = \psi_0(x)$, $\phi(x,0) = \phi_0(x)$, $\vartheta(x,0) = \vartheta_0(x)$, $P(x,\xi,0) = P_0(x,\xi)$, $M(x,\xi,0) = M_0(x,\xi)$, $N(x,\xi,0) = N_0(x,\xi)$ satisfying

$$\psi_0, \phi_0, \vartheta_0 \in \mathcal{B} \tag{66a}$$
$$\begin{aligned}&P_0(x,\cdot), P_0(\cdot,\xi), M_0(x,\cdot), M_0(\cdot,\xi)\\ &\qquad N_0(x,\cdot), N_0(\cdot,\xi) \in \mathcal{B},\ \forall x,\xi \in [0,1].\end{aligned} \tag{66b}$$

One can now construct non-adaptive estimates of the variables w and z as

$$\bar{w}(x) = p_1(x) \tag{67a}$$
$$\begin{aligned}\bar{z}(x) &= \rho\psi(x) - b(x) + \int_x^1 \theta(\xi)\phi(1-(\xi-x))d\xi\\ &\quad + \int_0^1 \kappa(\xi)\left[P(x,\xi) + M(x,\xi)\right]d\xi\\ &\quad + \int_0^1 \theta(\xi)N(x,\xi)d\xi + \vartheta^T(x)\nu.\end{aligned} \tag{67b}$$

Lemma 7: Consider the system (46) and state estimates (67) generated using the filters (62) and (63). After a finite time t_F given as

$$t_F = d_\alpha + d_\beta \tag{68}$$

we will have

$$\bar{w} \equiv w \qquad \text{and} \qquad \bar{z} \equiv z. \tag{69}$$

Proof: Consider the non-adaptive estimation errors

$$e(x) = w(x) - \bar{w}(x) \tag{70a}$$
$$\epsilon(x) = z(x) - \bar{z}(x). \tag{70b}$$

Then the dynamics can straight forwardly be shown to satisfy

$$e_t(x) + \bar{\lambda}e_x(x) = 0 \tag{71a}$$
$$\epsilon_t(x) - \bar{\mu}\epsilon_x(x) = \int_0^1 \kappa(\xi)\left[\bar{\mu}P_x(x,\xi) - P_t(x,\xi)\right]d\xi \tag{71b}$$
$$e(0) = 0 \tag{71c}$$
$$\epsilon(1) = \int_0^1 \kappa(\xi)e(\xi)d\xi. \tag{71d}$$

It can be shown using the boundary condition $P(x,0) = \phi(x)$ in (62d) and the dynamics of ϕ in (62b), that $P_t(x,\xi) = \bar{\mu}P_x(x,\xi)$ for $t \geq d_\alpha$. Moreover, from (71a) and (71c), it is observed that $e \equiv 0$ for $t \geq d_\alpha$, and therefore (71b) and (71d) imply that $\epsilon \equiv 0$ for $t \geq t_F$ where t_F is given by (68). ■

C. Adaptive laws

We start by assuming the following:

Assumption 8: Bounds on ρ, θ, κ, ν are known. That is, we are in knowledge of some constants $\underline{\rho}, \bar{\rho}, \underline{\theta}, \bar{\theta}, \underline{\kappa}, \bar{\kappa}, \underline{\nu}_i, \bar{\nu}_i$, $i = 1\dots(2n+1)$ so that

$$\underline{\rho} \leq \rho \leq \bar{\rho} \tag{72a}$$
$$\underline{\theta} \leq \theta(x) \leq \bar{\theta}, \quad \forall x \in [0,1] \tag{72b}$$
$$\underline{\kappa} \leq \kappa(x) \leq \bar{\kappa}, \quad \forall x \in [0,1] \tag{72c}$$
$$\underline{\nu}_i \leq \nu_i \leq \bar{\nu}_i, \quad i = 1\dots(2n+1) \tag{72d}$$

for all $x \in [0,1]$, where

$$\nu = \begin{bmatrix}\nu_1 & \nu_2 & \dots & \nu_{2n+1}\end{bmatrix}^T \tag{73a}$$
$$\underline{\nu} = \begin{bmatrix}\underline{\nu}_1 & \underline{\nu}_2 & \dots & \underline{\nu}_{2n+1}\end{bmatrix}^T \tag{73b}$$
$$\bar{\nu} = \begin{bmatrix}\bar{\nu}_1 & \bar{\nu}_2 & \dots & \bar{\nu}_{2n+1}\end{bmatrix}^T \tag{73c}$$

and with

$$0 \notin [\underline{\rho}, \bar{\rho}]. \tag{74}$$

The assumption (74) is equivalent to knowing the sign of the product k_1k_2. The remaining assumptions should not be a limitation, since the bounds can be made arbitrary large.

Motivated by the parametrization (67), we generate an estimate of z from

$$\begin{aligned}\hat{z}(x) &= \hat{\rho}\psi(x) - b(x) + \int_x^1 \hat{\theta}(\xi)\phi(1-(\xi-x))d\xi\\ &\quad + \int_0^1 \hat{\kappa}(\xi)\left[P(x,\xi) + M(x,\xi)\right]d\xi\end{aligned}$$

$$+\int_0^1 \hat{\theta}(\xi)N(x,\xi)d\xi + \vartheta^T(x)\hat{\nu} \tag{75}$$

and the corresponding prediction error as

$$\hat{\epsilon}(x) = z(x) - \hat{z}(x). \tag{76}$$

The dynamics of (75) is

$$\begin{aligned}\hat{z}_t(x) - \bar{\mu}\hat{z}_x(x) &= \bar{\mu}\hat{\theta}(x)z(0)\\ &+ \int_0^1 \hat{\kappa}(\xi)\left[P_t(x,\xi) - \bar{\mu}P_x(x,\xi)\right]d\xi\\ &+ \dot{\hat{\rho}}\psi(x)\\ &+ \int_x^1 \hat{\theta}_t(\xi)\phi(1-(\xi-x))d\xi\\ &+ \int_0^1 \hat{\kappa}_t(\xi)[P(x,\xi)+M(x,\xi)]d\xi\\ &+ \int_0^1 \hat{\theta}_t(\xi)N(x,\xi)d\xi + \vartheta^T(x)\dot{\hat{\nu}} \end{aligned}\tag{77a}$$

$$\begin{aligned}\hat{z}(1) = \hat{\rho}U - r &+ \int_0^1 \hat{\kappa}(\xi)(p_1(\xi)+a(\xi))d\xi\\ &+ \int_0^1 \hat{\theta}(\xi)b(1-\xi)d\xi\end{aligned}\tag{77b}$$

where the term in the first integral of (77a) will be zero in a finite time d_α. Moreover, we have

$$\begin{aligned}y &= z(0) + b(0)\\ &= \rho\psi(0) + \int_0^1 \theta(\xi)\left[\phi(1-\xi)+n_0(\xi)\right]d\xi\\ &\quad+ \int_0^1 \kappa(\xi)\left[p_0(\xi)+m_0(\xi)\right]d\xi\\ &\quad+ \vartheta^T(0)\nu + \epsilon(0)\end{aligned}\tag{78}$$

where the error term $\epsilon(0)$ converges to zero in a finite time $t_F = d_\alpha + d_\beta$. From (78), we propose the adaptive laws

$$\dot{\hat{\rho}} = \begin{cases} 0 & \text{for } t < t_F\\ \text{proj}_{\underline{\rho},\bar{\rho}}\left\{\tau_1, \hat{\rho}\right\} & \text{for } t \geq t_F\end{cases}\tag{79a}$$

$$\hat{\theta}_t(x) = \begin{cases} 0 & \text{for } t < t_F\\ \text{proj}_{\underline{\theta},\bar{\theta}}\left\{\tau_2(x), \hat{\theta}(x)\right\} & \text{for } t \geq t_F\end{cases}\tag{79b}$$

$$\hat{\kappa}_t(x) = \begin{cases} 0 & \text{for } t < t_F\\ \text{proj}_{\underline{\kappa},\bar{\kappa}}\left\{\tau_3(x, \hat{\kappa}(x)\right\} & \text{for } t \geq t_F\end{cases}\tag{79c}$$

$$\dot{\hat{\nu}} = \begin{cases} 0 & \text{for } t < t_F\\ \text{proj}_{\underline{\nu},\bar{\nu}}\left\{\tau_4, \hat{\nu}\right\} & \text{for } t \geq t_F\end{cases}\tag{79d}$$

where

$$\tau_1 = \gamma_1\frac{\hat{\epsilon}(0)\psi(0)}{1+f^2}\tag{80a}$$

$$\tau_2(x) = \gamma_2(x)\frac{\hat{\epsilon}(0)(\phi(1-x)+n_0(x))}{1+f^2}\tag{80b}$$

$$\tau_3(x) = \gamma_3(x)\frac{\hat{\epsilon}(0)(p_0(x)+m_0(x))}{1+f^2}\tag{80c}$$

$$\tau_4 = \Gamma_4\frac{\hat{\epsilon}(0)\vartheta(0)}{1+f^2}\tag{80d}$$

with

$$\hat{\epsilon}(0) = z(0) - \hat{z}(0) = y - b(0) - \hat{z}(0)\tag{81}$$

and

$$\begin{aligned}f^2 &= \psi^2(0) + ||\phi||^2 + ||p_0||^2\\ &\quad+ ||m_0||^2 + ||n_0||^2 + |\vartheta(0)|^2\end{aligned}\tag{82}$$

with $\gamma_1 > 0$, $\gamma_2(x), \gamma_3(x) > 0$ for all $x \in [0,1]$ and $\Gamma_4 > 0$ being some bounded design gains, the initial guesses are chosen inside the feasible domain

$$\underline{\rho} \leq \hat{\rho}(0) \leq \bar{\rho}\tag{83a}$$

$$\underline{\theta} \leq \hat{\theta}(x,0) \leq \bar{\theta}, \quad \forall x \in [0,1]\tag{83b}$$

$$\underline{\kappa} \leq \hat{\kappa}(x,0) \leq \bar{\kappa}, \quad \forall x \in [0,1]\tag{83c}$$

$$\underline{\nu}_i \leq \hat{\nu}_i(0) \leq \bar{\nu}_i, \quad i = 1\ldots(2n+1)\tag{83d}$$

and the projection operator is given as

$$\text{proj}_{a,b}(\tau,\omega) = \begin{cases} 0 & \text{if } \omega = a \text{ and } \tau \leq 0\\ 0 & \text{if } \omega = b \text{ and } \tau \geq 0\\ \tau & \text{otherwise.}\end{cases}\tag{84}$$

We note that

$$|\vartheta(0)|^2 = n+1\tag{85}$$

for $t \geq d_\beta$.

Lemma 9: The adaptive laws (79) with initial conditions (83) have the following properties

$$\underline{\rho} \leq \hat{\rho} \leq \bar{\rho}, t \geq 0\tag{86a}$$

$$\underline{\theta} \leq \hat{\theta}(x) \leq \bar{\theta}, \forall x \in [0,1], t \geq 0\tag{86b}$$

$$\underline{\kappa} \leq \hat{\kappa}(x) \leq \bar{\kappa}, \forall x \in [0,1], t \geq 0\tag{86c}$$

$$\underline{\nu}_i \leq \hat{\nu}_i \leq \bar{\nu}_i, i = 1\ldots(2n+1), t \geq 0\tag{86d}$$

$$\frac{|\hat{\epsilon}(0,\cdot)|}{\sqrt{1+f^2}} \in \mathcal{L}_\infty([t_F,\infty]) \cap \mathcal{L}_2([t_F,\infty])\tag{86e}$$

$$|\dot{\hat{\rho}}|, ||\hat{\theta}_t||, ||\hat{\kappa}_t||, |\dot{\hat{\nu}}| \in \mathcal{L}_\infty \cap \mathcal{L}_2\tag{86f}$$

where $\tilde{\rho} = \rho - \hat{\rho}$, $\tilde{\theta} = \theta - \hat{\theta}$, $\tilde{\kappa} = \kappa - \hat{\kappa}$, $\tilde{\nu} = \nu - \hat{\nu}$, with f^2 given in (82).

Proof of Lemma 9: The properties (86a)–(86d) follow from the projection operator used in (79) and the initial conditions (83). Consider the Lyapunov function candidate

$$\begin{aligned}V &= \frac{1}{2\gamma_1}\tilde{\rho}^2 + \frac{1}{2}\int_0^1 \gamma_2^{-1}(x)\tilde{\theta}^2(x)dx\\ &\quad+ \frac{1}{2}\int_0^1 \gamma_3^{-1}(x)\tilde{\kappa}^2(x)dx + \frac{1}{2}\tilde{\nu}^T\Gamma_4^{-1}\tilde{\nu}.\end{aligned}\tag{87}$$

Differentiating with respect to time, inserting the adaptive laws and using the property $-\tilde{\nu}^T\text{proj}_{\underline{\nu},\bar{\nu}}(\tau,\hat{\nu}) \leq -\tilde{\nu}^T\tau$ ([39, Lemma E.1]), and similarly for $\hat{\rho}$, $\hat{\theta}$ and $\hat{\kappa}$, we get

$$\dot{V} = 0 \text{ for } t < t_F\tag{88}$$

and

$$\begin{aligned}\dot{V} \leq -\frac{\hat{\epsilon}(0)}{1+f^2}\Bigg[\tilde{\rho}\psi(0) &+ \int_0^1 \Big(\tilde{\theta}(x)(\phi(1-x)+n_0(x))\\ &+ \tilde{\kappa}(x)(p_0(x)+m_0(x))\Big)dx + \vartheta^T(0)\tilde{\nu}\Bigg], \text{ for } t \geq t_F\end{aligned}\tag{89}$$

We note that

$$\hat{\epsilon}(0) = \epsilon(0) + \tilde{\rho}\psi(0) + \int_0^1 \tilde{\theta}(\xi)(\phi(1-\xi) + n_0(\xi))d\xi + \int_0^1 \tilde{\kappa}(\xi)(p_0(\xi) + m_0(\xi))d\xi + \vartheta^T(0)\tilde{\nu}, \tag{90}$$

where $\epsilon(0) = 0$ for $t \geq d_\alpha + d_\beta = t_F$, and inserting this into (89), we obtain

$$\dot{V} \leq \begin{cases} 0 & \text{for } t < t_F \\ -\dfrac{\hat{\epsilon}^2(0)}{1+f^2} & \text{for } t \geq t_F \end{cases}. \tag{91}$$

This proves that V is bounded and nonincreasing, and hence has a limit as $t \to \infty$. Integrating (91) from zero to infinity gives

$$\frac{|\hat{\epsilon}(0,\cdot)|}{\sqrt{1+f^2}} \in \mathcal{L}_2([t_F, \infty]) \tag{92}$$

Using (90), we obtain, for $t \geq t_F$

$$\begin{aligned} \frac{|\hat{\epsilon}(0,\cdot)|}{\sqrt{1+f^2}} &\leq |\tilde{\rho}|\frac{|\psi(0)|}{\sqrt{1+f^2}} + ||\tilde{\theta}||\frac{||\phi|| + ||m_0||}{\sqrt{1+f^2}} \\ &\quad + ||\tilde{\kappa}||\frac{||p_0|| + ||n_0||}{\sqrt{1+f^2}} + |\tilde{\nu}|\frac{|\vartheta(0)|}{\sqrt{1+f^2}} \\ &\leq |\tilde{\rho}| + ||\tilde{\theta}|| + ||\tilde{\kappa}|| + |\tilde{\nu}| \end{aligned} \tag{93}$$

which gives

$$\frac{|\hat{\epsilon}(0,\cdot)|}{\sqrt{1+f^2}} \in \mathcal{L}_\infty([t_F, \infty]). \tag{94}$$

From the adaptation laws (79), we have, for $t \geq t_F$

$$|\dot{\hat{\rho}}| \leq \gamma_1 \frac{|\hat{\epsilon}(0)|}{\sqrt{1+f^2}}\frac{|\psi(0)|}{\sqrt{1+f^2}} \leq \gamma_1\frac{|\hat{\epsilon}(0)|}{\sqrt{1+f^2}} \tag{95a}$$

$$\begin{aligned} ||\hat{\theta}_t|| &\leq ||\gamma_2||\frac{|\hat{\epsilon}(0)|}{\sqrt{1+f^2}}\frac{||\phi|| + ||n_0||}{\sqrt{1+f^2}} \\ &\leq ||\gamma_2||\frac{|\hat{\epsilon}(0)|}{\sqrt{1+f^2}} \end{aligned} \tag{95b}$$

$$\begin{aligned} ||\hat{\kappa}_t|| &\leq ||\gamma_3||\frac{|\hat{\epsilon}(0)|}{\sqrt{1+f^2}}\frac{||p_0|| + ||m_0||}{\sqrt{1+f^2}} \\ &\leq ||\gamma_3||\frac{|\hat{\epsilon}(0)|}{\sqrt{1+f^2}} \end{aligned} \tag{95c}$$

$$|\dot{\hat{\nu}}| \leq |\Gamma_4|\frac{|\hat{\epsilon}(0)|}{\sqrt{1+f^2}}\frac{|\vartheta(0)|}{\sqrt{1+f^2}} \leq |\Gamma_4|\frac{|\hat{\epsilon}(0)|}{\sqrt{1+f^2}} \tag{95d}$$

which, along with (86e), gives (86f). ■

D. Main theorem

Theorem 10: Consider the system (5), the filters (62) and (63b), the reference model (41), and the adaptive laws (79). Suppose r is bounded. Then the control law

$$U = \frac{1}{\hat{\rho}}\left(r + \int_0^1 \hat{g}(1-\xi)\hat{z}(\xi)d\xi - \int_0^1 \hat{\kappa}(\xi)(p_1(\xi) + a(\xi))d\xi - \int_0^1 \hat{\theta}(\xi)b(1-\xi)d\xi - \chi^T\hat{\nu}\right) \tag{96}$$

where $\hat{z}$ is generated using (75), and $\hat{g}$ is the on-line solution to the Volterra equation

$$\hat{g}(x) = \int_0^x \hat{g}(x-\xi)\hat{\theta}(\xi)d\xi - \hat{\theta}(x), \tag{97}$$

with $\hat{\rho}$, $\hat{\theta}$, $\hat{\kappa}$ and $\hat{\nu}$ generated from the adaptive laws (79) guarantees (13). Moreover, all additional variables in the closed loop system are bounded for $t \geq t_F$.

This Theorem is proved in Section IV-F, but first, we introduce a backstepping transformation which facilitates a Lyapunov analysis, and also establish some useful properties.

E. Backstepping

Consider the transformation

$$\eta(x) = \hat{z}(x) - \int_0^x \hat{g}(x-\xi)\hat{z}(\xi)d\xi = T[\hat{z}](x) \tag{98}$$

where g is the solution to

$$\hat{g}(x) = -T[\hat{\theta}](x) = \int_0^x \hat{g}(x-\xi)\hat{\theta}(\xi)d\xi - \hat{\theta}(x). \tag{99}$$

The transformation (98) is invertible, with inverse

$$\hat{z}(x) = \eta(x) - \int_0^x \hat{\theta}(x-\xi)\eta(\xi)d\xi = T^{-1}[\eta](x), \tag{100}$$

which is easily verified from inserting (100), (98), changing the order of integration in the double integral and using (97).

Lemma 11: The transformation (98) with inverse (100) and controller (96) maps the system (77) into

$$\begin{aligned} \eta_t(x) - \bar{\mu}\eta_x(x) = &-\bar{\mu}\hat{g}(x)\hat{\epsilon}(0) \\ &+ T\left[\int_0^1 \hat{\kappa}(\xi)\left[P_t(x,\xi) - \bar{\mu}P_x(x,\xi)\right]d\xi\right] \\ &+ T\left[\dot{\hat{\rho}}\psi(x)\right] \\ &+ T\left[\int_x^1 \hat{\theta}_t(\xi)\phi(1-(\xi-x))d\xi\right] \\ &+ T\left[\int_0^1 \hat{\kappa}_t(\xi)\left[P(x,\xi) + M(x,\xi)\right]d\xi\right] \\ &+ T\left[\int_0^1 \hat{\theta}_t(\xi)N(x,\xi)d\xi\right] + T\left[\vartheta^T(x)\right]\dot{\hat{\nu}} \\ &- \int_0^x \hat{g}_t(x-\xi)T^{-1}[\eta](\xi)d\xi \end{aligned} \tag{101a}$$

$$\eta(1) = 0. \tag{101b}$$

Moreover

$$||\hat{g}|| \leq M_g, \qquad ||\hat{g}_t|| \leq M_1||\hat{\theta}_t|| \tag{102}$$

for some positive constants M_g and M_1.

The proof of this Lemma is given in Appendix A.

F. Proof of the main theorem

Proof of Theorem 10: First off, we note that, since $r, \chi \in \mathcal{L}_\infty$, we have

$$a(x,\cdot), b(x,\cdot), m_0(x,\cdot), n_0(x,\cdot) \in \mathcal{L}_\infty \tag{103a}$$

$$M(x,\xi,\cdot), N(x,\xi,\cdot) \in \mathcal{L}_\infty \tag{103b}$$

$$||a||, ||b||, ||M||, ||N||, ||m_0||, ||n_0|| \in \mathcal{L}_\infty \tag{103c}$$

$$\vartheta(x,\cdot) \in \mathcal{L}_\infty \tag{103d}$$

$$||\vartheta|| \in \mathcal{L}_\infty \tag{103e}$$

for all $x, \xi \in [0,1]$. Consider the Lyapunov function candidates

$$V_1 = \bar{\mu}^{-1}\int_0^1 (1+x)\eta^2(x)dx \tag{104a}$$

$$V_2 = \bar{\mu}^{-1}\int_0^1 (1+x)\phi^2(x)dx \tag{104b}$$

$$V_3 = \bar{\lambda}^{-1}\int_0^1\int_0^1 (2-\xi)P^2(x,\xi)d\xi dx \tag{104c}$$

$$V_4 = \bar{\lambda}^{-1}\int_0^1 (2-x)p_0^2(x)dx \tag{104d}$$

$$V_5 = \bar{\lambda}^{-1}\int_0^1 (2-x)p_1^2(x)dx \tag{104e}$$

$$V_6 = \bar{\mu}^{-1}\int_0^1 (1+x)\psi^2(x)dx. \tag{104f}$$

It is possible to show that these satisfy, for $t \geq d_\alpha$ (see Appendix B for details)

$$\begin{aligned}\dot{V}_1 \leq& -\eta^2(0) - \frac{\bar{\mu}}{4}V_1 + h_1\frac{\hat{\epsilon}^2(0)}{1+f^2}\psi^2(0)\\ &+ l_1V_1 + l_2V_2 + l_3V_3 + l_4V_4 + l_5V_6 + l_6\end{aligned} \tag{105a}$$

$$\begin{aligned}\dot{V}_2 \leq& -\phi^2(0) + 4\eta^2(0) - \frac{\bar{\mu}}{2}V_2 + 4\frac{\hat{\epsilon}^2(0)}{1+f^2}\psi^2(0)\\ &+ l_7V_2 + l_8V_4 + l_9\end{aligned} \tag{105b}$$

$$\dot{V}_3 \leq -\frac{1}{2}\bar{\lambda}V_3 + 2\bar{\mu}V_2 \tag{105c}$$

$$\dot{V}_4 \leq 2\phi^2(0) - \frac{\bar{\lambda}}{2}V_4 \tag{105d}$$

$$\begin{aligned}\dot{V}_5 \leq& 4\eta^2(0) - \frac{\bar{\lambda}}{2}V_5 + 4\frac{\hat{\epsilon}^2(0)}{1+f^2}\psi^2(0)\\ &+ l_7V_2 + l_8V_4 + l_9\end{aligned} \tag{105e}$$

$$\begin{aligned}\dot{V}_6 \leq& -\psi^2(0) - \frac{\bar{\mu}}{2}V_6 + h_2r^2 + h_3V_1 + h_4V_5\\ &+ h_5||a||^2 + h_6||b||^2 + h_7||\chi||^2.\end{aligned} \tag{105f}$$

for some bounded, integrable functions $l_1 \dots l_9$, and positive constants $h_1 \dots h_7$. Forming

$$V_7 = 64V_1 + 8V_2 + V_3 + 4V_4 + 8V_5 + 2k_1V_6 \tag{106}$$

where

$$k_1 = \min\{\bar{\mu}h_3^{-1}, \bar{\lambda}h_4^{-1}\}, \tag{107}$$

we obtain

$$\begin{aligned}\dot{V}_7 \leq& -cV_7 + l_{10}V_7 + l_{11}\\ &- \left(2k_1 - 64(1+h_1)\frac{\hat{\epsilon}^2(0)}{1+f^2}\right)\psi^2(0) + 2k_1h_2r^2\\ &+ 2k_1h_5||a||^2 + 2k_1h_6||b||^2 + 2k_1h_7||\chi||^2\end{aligned} \tag{108}$$

for some positive constant c and integrable functions l_9 and l_{10}. The terms in $r, ||a||, ||b||$ and $||\chi||$ are all bounded, and hence for V_7 to be unbounded, the term in the brackets needs to be negative on a set whose measure increases unboundedly as $t \to \infty$. This is the persistence of excitation (PE) requirement of V in (91), meaning that V converges exponentially to zero, and hence $|\tilde{\rho}|$, $||\tilde{\theta}||$, $||\tilde{\kappa}||$, $|\tilde{\nu}|$ can be made as small as one pleases. However, by (93), this means that the fraction $\frac{\hat{\epsilon}^2(0)}{1+f^2}$ can also be made as small as desired, and eventually, we will have $\frac{\hat{\epsilon}^2(0)}{1+f^2} < \frac{k_1}{32(1+h_1)}$, contradicting the initial assumption. Hence $V_7 \in \mathcal{L}_\infty$ and

$$||\eta||, ||\phi||, ||P||, ||p_0||, ||p_1||, ||\psi|| \in \mathcal{L}_\infty. \tag{109}$$

and from the transform (100), we will also have

$$||\hat{z}|| \in \mathcal{L}_\infty. \tag{110}$$

From the definition of the filter ψ in (62a) and the control law U in (96), we will then have $U \in \mathcal{L}_\infty$, an

$$\psi(x,\cdot) \in \mathcal{L}_\infty \tag{111}$$

and particularly, $\psi(0,\cdot) \in \mathcal{L}_\infty$. Now forming

$$V_8 = 64V_1 + 8V_2 + V_3 + 4V_4 + 8V_5 \tag{112}$$

we obtain in a similar way

$$\dot{V}_8 \leq -\bar{c}V_8 + l_{12}V_8 + l_{13} + 64(1+h_1)\frac{\hat{\epsilon}^2(0)}{1+f^2}\psi^2(0) \tag{113}$$

for some positive constant $\bar{c}$ and integrable functions l_{12} and l_{13}. Since $\frac{\hat{\epsilon}^2(0)}{1+f^2} \in \mathcal{L}_1([d_\alpha,\infty])$ and $\psi(0) \in \mathcal{L}_\infty$, the latter term is integrable, and hence

$$\dot{V}_8 \leq -\bar{c}V_8 + l_{12}V_8 + l_{14} \tag{114}$$

for an integrable function l_{14}. It then follows from [40, Lemma B.6] that

$$V_8 \in \mathcal{L}_1 \cap \mathcal{L}_\infty, \tag{115}$$

and hence

$$||\eta||, ||\phi||, ||P||, ||p_0||, ||p_1|| \in \mathcal{L}_\infty \cap \mathcal{L}_2. \tag{116}$$

From (100), it then follows that

$$||\hat{z}|| \in \mathcal{L}_\infty \cap \mathcal{L}_2 \tag{117}$$

while from (75), we have

$$||\psi|| \in \mathcal{L}_\infty. \tag{118}$$

From the invertibility of the transforms, and the fact that $||a||$ and $||b||$ are bounded, we obtain

$$||u||, ||v|| \in \mathcal{L}_\infty. \tag{119}$$

We proceed by proving pointwise boundedness. From (67b), (70b), (75) and (76) we have

$$\begin{aligned}\hat{\epsilon}(x) =& \epsilon(x) - \tilde{\rho}\psi(x) - \int_x^1 \tilde{\theta}(\xi)\phi(1-(\xi-x))d\xi\\ &- \int_0^1 \tilde{\kappa}(\xi)[P(x,\xi) + M(x,\xi)]d\xi\\ &- \int_0^1 \tilde{\theta}(\xi)N(x,\xi)d\xi\end{aligned} \tag{120}$$

and

$$z(x) = \hat{\rho}\psi(x) - b(x) + \int_x^1 \hat{\theta}(\xi)\phi(1-(\xi-x))d\xi$$

$$+\int_0^1 \hat{\kappa}(\xi)[P(x,\xi)+M(x,\xi)]d\xi$$
$$+\int_0^1 \hat{\theta}(\xi)N(x,\xi)d\xi + \hat{\epsilon}(x) \tag{121}$$

with ϵ converging to zero in a finite time d_β. From this, we find

$$\hat{\epsilon}(x,\cdot) \in \mathcal{L}_\infty([d_\beta,\infty]) \tag{122}$$

and

$$z(x,\cdot) \in \mathcal{L}_\infty([d_\beta,\infty]). \tag{123}$$

From $z(0,\cdot) \in \mathcal{L}_\infty([d_\beta,\infty])$, we then obtain

$$\phi(x,\cdot), P(x,\xi,\cdot), p_0(x,\xi,\cdot), p_1(x,\xi,\cdot) \in \mathcal{L}_\infty([d_\beta,\infty]). \tag{124}$$

From (67a) and (70a), we get

$$w(x,\cdot) \in \mathcal{L}_\infty([t_F,\infty]). \tag{125}$$

Since a and b are also pointwise bounded, we obtain

$$\alpha(x,\cdot), \beta(x,\cdot) \in \mathcal{L}_\infty([t_F,\infty]). \tag{126}$$

From the invertibility of the transforms, we finally get

$$u(x,\cdot), v(x,\cdot) \in \mathcal{L}_\infty([t_F,\infty]). \tag{127}$$

Lastly, we prove that the tracking goal (13) is achieved. Using Lemma 12 in Appendix C on (114) with $g = V_8$ and $f = l_{12}V_8 + l_{14} \in \mathcal{L}_1$, yields

$$V_8 \to 0 \tag{128}$$

and hence

$$||\eta||, ||\phi||, ||P||, ||p_0||, ||p_1|| \to 0. \tag{129}$$

By solving (62b), we find

$$\phi(x,t) = \phi(1, t - d_\beta(1-x)) = z(0, t - d_\beta(1-x)) \tag{130}$$

for $t \geq d_\beta(1-x)$. Moreover, we have

$$||\phi||^2 = \int_0^1 \phi^2(x,t)dx = \int_0^1 z^2(0, t - d_\beta(1-x))dx \to 0 \tag{131}$$

for $t \geq d_\beta$. Which proves that

$$\int_t^{t+T} z^2(0,s)ds \to 0 \tag{132}$$

for any $T > 0$, and from the definition of $z(0,t)$ in (47), this implies that

$$\int_t^{t+T} (y(s) - y_r(s))^2 ds \to 0 \tag{133}$$

for any $T > 0$. ■

V. Simulation

The system (5), the reference model (41) and the filters (62)–(65) were implemented in MATLAB along with the adaptive laws (79) and the controller of Theorem 10. The system parameters were set to

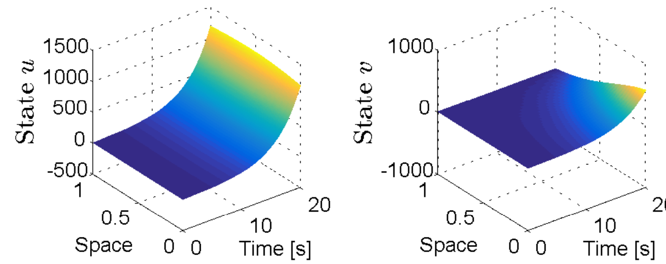

Fig. 1: System states in the open loop case.

$$\lambda(x) = 1 + x, \qquad \mu(x) = e^x \tag{134a}$$
$$c_1(x) = 1 + x, \qquad c_2(x) = \frac{1}{2}(1 + \sin(x)) \tag{134b}$$
$$q = 2 \tag{134c}$$

with the disturbance terms being

$$d_1(x,t) = \frac{1}{2}x \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \chi(t) \tag{135a}$$
$$d_2(x,t) = \frac{1}{20}e^x \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \chi(t) \tag{135b}$$
$$d_3(t) = \frac{1}{4} \begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \chi(t) \tag{135c}$$
$$d_4(t) = \frac{1}{4} \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} \chi(t) \tag{135d}$$
$$d_5(t) = \frac{1}{4} \begin{bmatrix} -1 & -1 & 2 \end{bmatrix} \chi(t) \tag{135e}$$

where

$$\chi(t) = \begin{bmatrix} 1 & \sin(t) & \cos(t) \end{bmatrix}^T. \tag{136}$$

The reference signal r was set to

$$r(t) = 1 + \sin\left(\frac{\pi}{10}t\right) + 2\sin\left(\frac{\sqrt{2}}{2}t\right), \tag{137}$$

while the initial conditions of the system were set to

$$u_0(x) = x, \qquad v_0(x) = \sin(2\pi x). \tag{138}$$

All initial conditions for the filters and parameter estimates were set to zero, except

$$\hat{\rho}(0) = 1. \tag{139}$$

The adaptation gains were set to

$$\gamma_1 = 5 \tag{140a}$$
$$\gamma_2(x) = \gamma_3(x) = 5, \quad \forall x \in [0,1] \tag{140b}$$
$$\Gamma_4 = 5I_{3\times3} \tag{140c}$$

with the bounds on ρ, θ, κ and ν set to

$$\underline{\rho} = 0.1, \quad \bar{\rho} = 100 \tag{141a}$$
$$\underline{\theta} = \underline{\kappa} = \underline{\nu}_i = -100 \tag{141b}$$
$$\bar{\theta} = \bar{\kappa} = \bar{\nu}_i = 100 \tag{141c}$$

for $i = 1 \ldots 3$. The integral equation (97) was solved using successive approximations. System (5) with parameters (134) is unstable, as seen from the open loop ($U \equiv 0$) simulation

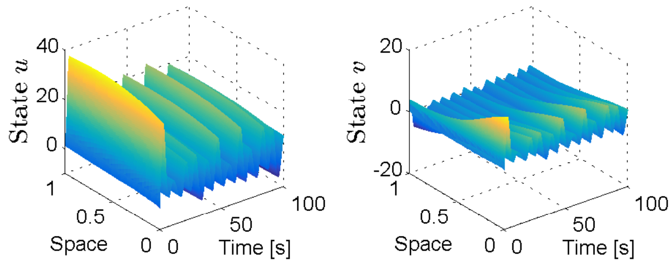

Fig. 2: System states in the adaptive tracking case.

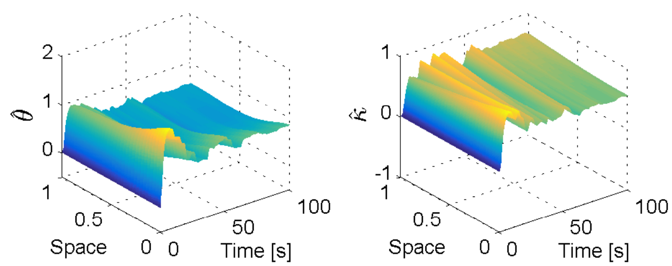

Fig. 3: Estimated system parameters.

displayed in Figure 1. With the controller active, it is noted from Figures 2 and 5 that the states are bounded, and that the measured output y tracks the signal y_r after approximately 60 seconds. It is also noted from Figures 3 and 4 that the estimates $\hat{\rho}$, $\hat{\theta}$, $\hat{\kappa}$ and $\hat{\nu}$ do not stagnate, but continuously adapt. The reason for this may be that the values of θ and κ for which the goal is achieved are not unique.

VI. Conclusions

We have solved a model reference adaptive control problem for a class of linear 2×2 hyperbolic partial differential equations with uncertain boundary parameters and harmonic disturbances, with sensing restricted to the boundary anti-collocated with the actuation. This was achieved using a series of transformations that mapped the system into a canonical form, before a filter-based control law was designed that ensured pointwise boundedness of all variables in the system, and also asymptotic tracking of the measured signal. The theory was verified in a simulation.

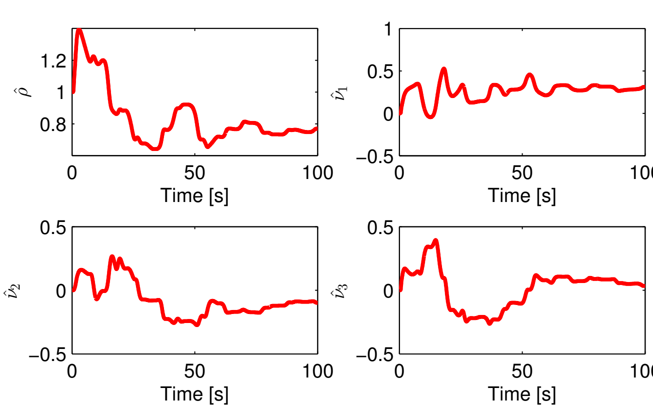

Fig. 4: Estimated ρ and ν.

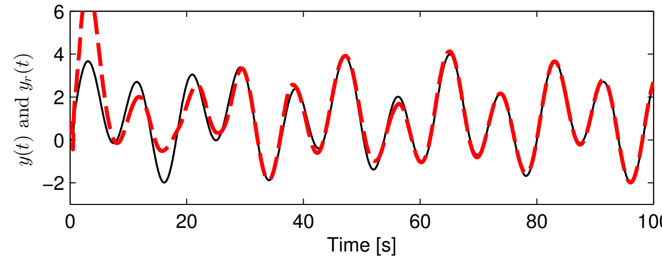

Fig. 5: Objective: measured signal y (dashed) and reference y_r (solid).

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APPENDIX

A. Proof of Lemma 11

Proof of Lemma 11: Differentiating (98) with respect to time inserting the dynamics (77a) and integration by parts, we obtain

$$\begin{aligned}
\hat{z}_t(x) &= \eta_t(x) + \hat{g}(0)\bar{\mu}\hat{z}(x) - \hat{g}(x)\bar{\mu}\hat{z}(0) \\
&\quad + \int_0^x \hat{g}_x(x-\xi)\bar{\mu}\hat{z}(\xi)d\xi + \int_0^x \hat{g}(x-\xi)\bar{\mu}\hat{\theta}(\xi)z(0)d\xi \\
&\quad + \int_0^x \hat{g}(x-\xi)\int_0^1 \kappa(s)\left[P_t(\xi,s) - \bar{\mu}P_x(\xi,s)\right]dsd\xi \\
&\quad + \int_0^x \hat{g}(x-\xi)\dot{\hat{\rho}}\psi(\xi)d\xi \\
&\quad + \int_0^x \hat{g}(x-\xi)\int_\xi^1 \hat{\theta}_t(s)\phi(1-(s-\xi))dsd\xi \\
&\quad + \int_0^x \hat{g}(x-\xi)\int_0^1 \hat{\kappa}_t(s)[P(\xi,s) + M(\xi,s)]dsd\xi \\
&\quad + \int_0^x \hat{g}(x-\xi)\int_0^1 \hat{\theta}_t(s)N(\xi,s)dsd\xi \\
&\quad + \int_0^x \hat{g}(x-\xi)\vartheta^T(\xi)\dot{\hat{\nu}}d\xi \\
&\quad + \int_0^x \hat{g}_t(x-\xi)\hat{z}(\xi)d\xi. \qquad (142)
\end{aligned}$$

Equivalently, differentiating (98) with respect to space, we find

$$\hat{z}_x(x) = \eta_x(x) + \hat{g}(0)\hat{z}(x) + \int_0^x \hat{g}_x(x-\xi)\hat{z}(\xi)d\xi. \qquad (143)$$

Inserting the results into (77a), yields

$$\begin{aligned}
&\eta_t(x) - \bar{\mu}\eta_x(x) - \left[\bar{\mu}\hat{\theta}(x) - \int_0^x \hat{g}(x-\xi)\bar{\mu}\hat{\theta}(\xi)d\xi\right]\hat{\epsilon}(0) \\
&- \int_0^1 \kappa(\xi)\left[P_t(x,\xi) - \bar{\mu}P_x(x,\xi)\right]d\xi \\
&+ \int_0^x \hat{g}(x-\xi)\int_0^1 \kappa(s)\left[P_t(\xi,s) - \bar{\mu}P_x(\xi,s)\right]dsd\xi \\
&- \dot{\hat{\rho}}\psi(x) + \int_0^x \hat{g}(x-\xi)\dot{\hat{\rho}}\psi(\xi)d\xi \\
&- \int_x^1 \hat{\theta}_t(\xi)\phi(1-(\xi-x))d\xi \\
&+ \int_0^x \hat{g}(x-\xi)\int_\xi^1 \hat{\theta}_t(s)\phi(1-(s-\xi))dsd\xi \\
&- \int_0^1 \hat{\kappa}_t(\xi)[P(x,\xi) + M(x,\xi)]d\xi \\
&+ \int_0^x \hat{g}(x-\xi)\int_0^1 \hat{\kappa}_t(s)[P(\xi,s) + M(\xi,s)]dsd\xi \\
&- \int_0^1 \hat{\theta}_t(\xi)N(x,\xi)d\xi + \int_0^x \hat{g}(x-\xi)\int_0^1 \hat{\theta}_t(s)N(\xi,s)dsd\xi \\
&- \vartheta^T(x)\dot{\hat{\nu}} + \int_0^x \hat{g}(x-\xi)\vartheta^T(\xi)\dot{\hat{\nu}}d\xi \\
&+ \int_0^x \hat{g}_t(x-\xi)\hat{z}(\xi)d\xi = 0 \qquad (144)
\end{aligned}$$

which can we rewritten as (101a). The boundary condition (101b) follows from inserting $x = 1$ into (98), and using (77b) and (96).

Note that, from invertibility of the transforms (98) and (100) and the fact that the estimate $\hat{\theta}$ and hence also $\hat{g}$ are bounded by projection, we have the inequalities

$$||T[u]|| \leq G_1||u||, \qquad ||T^{-1}[u]|| \leq G_2||u|| \tag{145}$$

for some positive constants G_1 and G_2. This means, from (99), that

$$||\hat{g}|| \leq G_1||\hat{\theta}|| \leq G_1 M_\theta \tag{146}$$

where $M_\theta = \max\{|\underline{\theta}|, |\bar{\theta}|\}$. Taking $M_g = G_1 M_\theta$ proves the first part of (102). Differentiating (97) with respect to time, we find

$$\begin{aligned}\hat{g}_t(x) - &\int_0^x \hat{\theta}(x-\xi)\hat{g}_t(\xi)d\xi \\ &= \int_0^x \hat{g}(x-\xi)\hat{\theta}_t(\xi)d\xi - \hat{\theta}_t(x),\end{aligned} \tag{147}$$

which, using (98) and (100) can be written

$$T^{-1}[\hat{g}_t](x) = -T[\hat{\theta}_t](x), \tag{148}$$

and hence $\hat{g}_t(x) = -T[T[\hat{\theta}_t]](x)$. Taking $M_1 = G_1^2$ proves the last part of (102). ■

B. Details regarding Theorem 10

1) Bounds on V_1: From differentiating V_1 in (104a) with respect to time and inserting the dynamics (101a), we find, for $t \geq d_\alpha$

$$\begin{aligned}\dot{V}_1 = &\, 2\int_0^1 (1+x)\eta(x)\eta_x(x)dx \\ &- 2\int_0^1 (1+x)\eta(x)g(x)dx\hat{\epsilon}(0) \\ &+ \frac{2}{\bar{\mu}}\int_0^1 (1+x)\eta(x)T\left[\dot{\hat{\rho}}\psi(x)\right]dx \\ &+ \frac{2}{\bar{\mu}}\int_0^1 (1+x)\eta(x)T\left[\int_x^1 \hat{\theta}_t(\xi)\phi(1-(\xi-x))d\xi\right]dx \\ &+ \frac{2}{\bar{\mu}}\int_0^1 (1+x)\eta(x)T\left[\int_0^1 \hat{\kappa}_t(\xi)P(x,\xi)d\xi\right]dx \\ &+ \frac{2}{\bar{\mu}}\int_0^1 (1+x)\eta(x)T\left[\int_0^1 \hat{\kappa}_t(\xi)M(x,\xi)d\xi\right] \\ &+ \frac{2}{\bar{\mu}}\int_0^1 (1+x)\eta(x)T\left[\int_0^1 \hat{\theta}_t(\xi)N(x,\xi)d\xi\right]dx \\ &+ \frac{2}{\bar{\mu}}\int_0^1 (1+x)\eta(x)T\left[\vartheta^T\dot{\hat{\nu}}\right](x)dx \\ &- \frac{2}{\bar{\mu}}\int_0^1 (1+x)\eta(x)\int_0^x g_t(x-\xi)T^{-1}[\eta](\xi)d\xi dx\end{aligned} \tag{149}$$

where we have utilized that $P_t - \bar{\mu}P_x$ is zero for $t \geq d_\alpha$. Using integration by parts and Cauchy-Schwartz' inequality on the cross terms, we find the following upper bounds

$$\begin{aligned}\dot{V}_1 \leq &-\eta^2(0) \\ &- \bar{\mu}\left[\frac{1}{2} - \rho_1 - \rho_2 - \rho_3 - \rho_4 - \rho_5 - \rho_6 - \rho_7 - \rho_8\right]V_1 \\ &+ \frac{1}{\rho_1\bar{\mu}^2}\int_0^1 (1+x)T\left[\dot{\hat{\rho}}\psi(x)\right]^2 dx \\ &+ \frac{1}{\rho_2\bar{\mu}^2}\int_0^1 (1+x)T\left[\int_x^1 \hat{\theta}_t(\xi)\phi(1-(\xi-x))d\xi\right]^2 dx \\ &+ \frac{1}{\rho_3}\int_0^1 (1+x)g^2(x)dx\hat{\epsilon}^2(0) \\ &+ \frac{1}{\rho_4\bar{\mu}^2}\int_0^1 (1+x)T\left[\int_0^1 \hat{\kappa}_t(\xi)P(x,\xi)d\xi\right]^2 dx \\ &+ \frac{1}{\rho_5\bar{\mu}^2}\int_0^1 (1+x)T\left[\int_0^1 \hat{\kappa}_t(\xi)M(x,\xi)d\xi\right]^2 dx \\ &+ \frac{1}{\rho_6\bar{\mu}^2}\int_0^1 (1+x)T\left[\int_0^1 \hat{\theta}_t(\xi)N(x,\xi)d\xi\right]^2 dx \\ &+ \frac{1}{\rho_7\bar{\mu}^2}\int_0^1 (1+x)T[\vartheta^T(x)\dot{\hat{\nu}}]^2 dx \\ &+ \frac{1}{\rho_8\bar{\mu}^2}\int_0^1 (1+x)\left[\int_0^x g_t(x-\xi)T^{-1}[\eta](\xi)d\xi\right]^2 dx,\end{aligned} \tag{150}$$

for some arbitrary positive constants ρ_i, $i = 1\ldots 8$. $\dot{V}_1$ can be upper bounded by

$$\begin{aligned}\dot{V}_1 \leq &-\eta^2(0) \\ &- \bar{\mu}\left[\frac{1}{2} - \rho_1 - \rho_2 - \rho_3 - \rho_4 - \rho_5 - \rho_6 - \rho_7 - \rho_8\right]V_1 \\ &+ \frac{2}{\rho_1\bar{\mu}^2}G_1^2|\dot{\hat{\rho}}|^2||\psi||^2 + \frac{2}{\rho_2\bar{\mu}^2}G_1^2||\theta_t||^2||\phi||^2 \\ &+ \frac{2\bar{g}^2}{\rho_3}\hat{\epsilon}^2(0) + \frac{2}{\rho_4\bar{\mu}^2}G_1^2||\hat{\kappa}_t||^2||P||^2 \\ &+ \frac{2}{\rho_5\bar{\mu}^2}G_1^2||\hat{\kappa}_t||^2||M||^2 + \frac{2}{\rho_6\bar{\mu}^2}G_1^2||\hat{\theta}_t||^2||N||^2 \\ &+ \frac{2}{\rho_7\bar{\mu}^2}G_1^2|\dot{\hat{\nu}}|^2||\vartheta||^2 + \frac{2}{\rho_8\bar{\mu}^2}G_2^2||g_t||^2||\eta||^2.\end{aligned} \tag{151}$$

Let

$$\rho_i = \frac{1}{32}, \quad i = 1\ldots 8, \tag{152}$$

then

$$\begin{aligned}\dot{V}_1 \leq &-\eta^2(0) - \frac{\bar{\mu}}{4}V_1 + \frac{64}{\bar{\mu}^2}G_1^2|\dot{\hat{\rho}}|^2||\psi||^2 + \frac{64}{\bar{\mu}^2}G_1^2||\theta_t||^2||\phi||^2 \\ &+ 64M_g^2\frac{\hat{\epsilon}^2(0)}{1+f^2} + 64M_g^2\frac{\hat{\epsilon}^2(0)}{1+f^2}\psi^2(0) \\ &+ 64M_g^2\frac{\hat{\epsilon}^2(0)}{1+f^2}||\phi||^2 + 64M_g^2\frac{\hat{\epsilon}^2(0)}{1+f^2}||p_0||^2 \\ &+ 64M_g^2\frac{\hat{\epsilon}^2(0)}{1+f^2}||m_0||^2 + 64M_g^2\frac{\hat{\epsilon}^2(0)}{1+f^2}||n_0||^2 \\ &+ 64M_g^2\frac{\hat{\epsilon}^2(0)}{1+f^2}|\vartheta(0)|^2 + \frac{64}{\bar{\mu}^2}G_1^2||\hat{\kappa}_t||^2||P||^2 \\ &+ \frac{64}{\bar{\mu}^2}G_1^2||\hat{\kappa}_t||^2||M||^2 + \frac{64}{\bar{\mu}^2}G_1^2||\hat{\theta}_t||^2||N||^2 \\ &+ \frac{64}{\bar{\mu}^2}G_1^2|\dot{\hat{\nu}}|^2||\vartheta||^2 + \frac{64}{\bar{\mu}^2}G_2^2||g_t||^2||\eta||^2.\end{aligned} \tag{153}$$

Define the bounded, integrable functions

$$l_1 = \frac{64}{\bar{\mu}}G_2^2||g_t||^2 \tag{154a}$$

$$l_2 = \frac{64}{\bar{\mu}}G_1^2||\theta_t||^2 + 64\bar{\mu}M_g^2\frac{\hat{\epsilon}^2(0)}{1+f^2} \tag{154b}$$

$$l_3 = \frac{64\bar{\lambda}}{\bar{\mu}^2} G_1^2 ||\hat{\kappa}_t||^2 \tag{154c}$$

$$l_4 = 64\bar{\lambda} M_g^2 \frac{\hat{\epsilon}^2(0)}{1+f^2} \tag{154d}$$

$$l_5 = \frac{64}{\bar{\mu}} G_1^2 |\dot{\hat{\rho}}|^2 \tag{154e}$$

$$\begin{aligned} l_6 = {} & 64M_g^2 \frac{\hat{\epsilon}^2(0)}{1+f^2} + 64M_g^2 \frac{\hat{\epsilon}^2(0)}{1+f^2} ||m_0||^2 \\ & + 64M_g^2 \frac{\hat{\epsilon}^2(0)}{1+f^2} ||n_0||^2 + 64M_g^2 \frac{\hat{\epsilon}^2(0)}{1+f^2} |\vartheta(0)|^2 \\ & + \frac{64}{\bar{\mu}^2} G_1^2 ||\hat{\kappa}_t||^2 ||M||^2 + \frac{64}{\bar{\mu}^2} G_1^2 ||\hat{\theta}_t||^2 ||N||^2 \\ & + \frac{64}{\bar{\mu}^2} G_1^2 |\dot{\hat{\nu}}|^2 ||\vartheta||^2 \end{aligned} \tag{154f}$$

and the positive constant

$$h_1 = 64M_g^2 \tag{155}$$

then (153) can be written as

$$\begin{aligned} \dot{V}_1 \le {} & -\eta^2(0) - \frac{\bar{\mu}}{4} V_1 + h_1 \frac{\hat{\epsilon}^2(0)}{1+f^2} \psi^2(0) \\ & + l_1 V_1 + l_2 V_2 + l_3 V_3 + l_4 V_4 + l_5 V_6 + l_6. \end{aligned} \tag{156}$$

2) Bounds on V_2: Similarly, differentiating V_2 in (104b) with respect to time, inserting the dynamics (62b), and integration by parts, we find

$$\begin{aligned} \dot{V}_2 &= 2\int_0^1 (1+x)\phi(x)\phi_x(x)dx \\ &= 2\phi^2(1) - \phi^2(0) - \int_0^1 \phi^2(x)dx \\ &\le -\phi^2(0) + 4\eta^2(0) - \frac{1}{2}\bar{\mu} V_2 + 4\hat{\epsilon}^2(0) \end{aligned} \tag{157}$$

where we have inserted the boundary condition in (62b). Inequality (157) can be written as

$$\begin{aligned} \dot{V}_2 \le {} & -\phi^2(0) + 4\eta^2(0) - \frac{\bar{\mu}}{2} V_2 + 4\frac{\hat{\epsilon}^2(0)}{1+f^2}\Big(1 + \psi^2(0) \\ & + ||\phi||^2 + ||p_0||^2 + ||m_0||^2 + ||n_0||^2 + |\vartheta(0)|^2\Big). \end{aligned} \tag{158}$$

Defining the functions

$$l_7 = 4\bar{\mu} \frac{\hat{\epsilon}^2(0)}{1+f^2} \tag{159a}$$

$$l_8 = 4\bar{\lambda} \frac{\hat{\epsilon}^2(0)}{1+f^2} \tag{159b}$$

$$l_9 = 4\frac{\hat{\epsilon}^2(0)}{1+f^2}(1 + ||m_0||^2 + ||n_0||^2 + |\vartheta(0)|^2) \tag{159c}$$

which from (86e) are bounded and integrable, we obtain

$$\begin{aligned} \dot{V}_2 \le {} & -\phi^2(0) + 4\eta^2(0) - \frac{\bar{\mu}}{2} V_2 + 4\frac{\hat{\epsilon}^2(0)}{1+f^2}\psi^2(0) \\ & + l_7 V_2 + l_8 V_4 + l_9. \end{aligned} \tag{160}$$

3) Bounds on V_3: Differentiating V_3 in (104c) with respect to time and inserting the dynamics (62d), we find

$$\begin{aligned} \dot{V}_3 &= -2\int_0^1 \int_0^1 (2-\xi) P(x,\xi) P_\xi(x,\xi) d\xi dx \\ &= -\int_0^1 P^2(x,1)dx + 2\int_0^1 P^2(x,0)dx \\ &\quad - \int_0^1 \int_0^1 P^2(x,\xi) d\xi dx. \end{aligned} \tag{161}$$

Inserting the boundary condition in (62d), we obtain

$$\dot{V}_3 \le -\frac{1}{2}\bar{\lambda} V_3 + 2\bar{\mu} V_2 \tag{162}$$

4) Bounds on V_4: From differentiating V_4 in (104d) with respect to time and inserting p_0's dynamics derived from the relationship given in (63a), we find

$$\begin{aligned} \dot{V}_4 &= -2\int_0^1 (2-x) p_0(x) \partial_x p_0(x) dx \\ &= -p_0^2(1) + 2p_0^2(0) - \frac{\bar{\lambda}}{2} V_4. \end{aligned} \tag{163}$$

Using (63a) and (62d) yields

$$\dot{V}_4 \le 2\phi^2(0) - \frac{\bar{\lambda}}{2} V_4. \tag{164}$$

5) Bounds on V_5: Similarly, differentiating V_5 in (104e) with respect to time and, we find

$$\begin{aligned} \dot{V}_5 &= -2\int_0^1 (2-x) p_1(x) \partial_x p_1(x) dx \\ &= -p_1^2(1) + 2p_1^2(0) - \frac{\bar{\lambda}}{2} V_5. \end{aligned} \tag{165}$$

Using (63a) and (62d) yields

$$\dot{V}_5 \le 2\phi^2(1) - \frac{\bar{\lambda}}{2} V_5 \tag{166}$$

$$\begin{aligned} &\le 4\eta^2(0) - \frac{\bar{\lambda}}{2} V_5 + 4\frac{\hat{\epsilon}^2(0)}{1+f^2}\Big(1 + \psi^2(0) + ||\phi||^2 \\ &\quad + ||p_0||^2 + ||m_0||^2 + ||n_0||^2 + |\vartheta(0)|^2\Big) \end{aligned} \tag{167}$$

which can be written as

$$\begin{aligned} \dot{V}_5 \le {} & 4\eta^2(0) - \frac{\bar{\lambda}}{2} V_5 + 4\frac{\hat{\epsilon}^2(0)}{1+f^2}\psi^2(0) \\ & + l_7 V_2 + l_8 V_4 + l_9, \end{aligned} \tag{168}$$

for the integrable functions defined in (159a).

6) Bounds on V_6: Lastly, from differentiating V_6 in (104f) with respect to time and the dynamics (62a), we find

$$\begin{aligned} \dot{V}_6 &= 2\int_0^1 (1+x)\psi(x)\psi_x(x)dx \\ &= 2\psi^2(1) - \psi^2(0) - \frac{\bar{\mu}}{2} V_6. \end{aligned} \tag{169}$$

Inserting the boundary condition (63a) and the control law (96), we can bound this as

$$\begin{aligned} \dot{V}_6 \le {} & -\psi^2(0) - \frac{\bar{\mu}}{2} V_6 + 12M_\rho^2 r^2 \\ & + 12M_\rho^2 \int_0^1 \hat{g}^2(1-\xi)\hat{z}^2(\xi)d\xi \\ & + 12M_\rho^2 \int_0^1 \hat{\kappa}^2(\xi)p_1^2(\xi)d\xi + 12M_\rho^2 \int_0^1 \hat{\kappa}^2(\xi)a^2(\xi)d\xi \\ & + 12M_\rho^2 \int_0^1 \hat{\theta}^2(\xi)b^2(1-\xi)d\xi + 12M_\rho^2(\chi^T\hat{\nu})^2 \end{aligned} \tag{170}$$

where

$$M_\rho = \frac{1}{\min\{|\underline{\rho}|, |\bar{\rho}|\}}. \tag{171}$$

Inequality (170) can be bounded as

$$\begin{aligned}\dot{V}_6 \leq& -\psi^2(0) - \frac{\bar{\mu}}{2}V_6 + 12M_\rho^2 r^2 + 12M_\rho^2 M_g^2 G_2^2||\eta||^2 \\ &+ 12M_\rho^2 M_\kappa^2||p_1||^2 + 12M_\rho^2 M_\kappa^2||a||^2 \\ &+ 12M_\rho^2 M_\theta^2||b||^2 + 12(2n+1)M_\rho^2 M_\nu^2||\chi||^2 \end{aligned} \tag{172}$$

where

$$M_\kappa = \max\{|\underline{\kappa}|, |\bar{\kappa}|\} \tag{173a}$$

$$M_\nu = \max_{i\in[1\dots(2n+1)]}\{|\underline{\nu}_i|, |\bar{\nu}_i|\} \tag{173b}$$

Defining the positive constants

$$h_2 = 12M_\rho^2 \tag{174a}$$

$$h_3 = 12M_\rho^2 M_g^2 G_2^2\bar{\mu} \tag{174b}$$

$$h_4 = 12M_\rho^2 M_\kappa^2\bar{\lambda} \tag{174c}$$

$$h_5 = 12M_\rho^2 M_\kappa^2 \tag{174d}$$

$$h_6 = 12M_\rho^2 M_\theta^2 \tag{174e}$$

$$h_7 = 12(2n+1)M_\rho^2 M_\nu^2 \tag{174f}$$

then (172) can be written as

$$\begin{aligned}\dot{V}_6 \leq& -\psi^2(0) - \frac{\bar{\mu}}{2}V_6 + h_2 r^2 + h_3 V_1 + h_4 V_5 \\ &+ h_5||a||^2 + h_6||b||^2 + h_7||\chi||^2. \end{aligned} \tag{175}$$

C. *Lemma 2.17 from [41]*

Lemma 12 (Lemma 2.17 from [41]): Consider a signal g satisfying

$$\dot{g}(t) = -ag(t) + bh(t) \tag{176}$$

for a signal $h \in \mathcal{L}_1$ and some constants $a > 0, b > 0$. Then

$$g \in \mathcal{L}_\infty \tag{177}$$

and

$$\lim_{t\to\infty} g(t) = 0. \tag{178}$$

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