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Quasi-linear functionals Theory and Applications

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Introduction

The theory of bounded operators on Hilbert spaces has been a subject of great interest for the last six decades, in particular norm closed, self-adjoint algebras of operators. These were shown by the Gelfand-Naimark-Segal construction to be the *C*-algebras*. A number of fruitful problems concerning C*-algebras have been posed and answered during the years. One of the earlier questions was posed by R. V. Kadison in [3], namely are there non-linear states on C*-algebras. A state is here assumed to be a complex valued function on a C*-algebra which is a positive functional on subalgebras generated by a single self-adjoint element. If the C*-algebra is unital a state is assumed to map the unit to $1 \in \mathbb{R}$. The answer came as late as 1990, by Johan F. Aarnes in [1], where non-linear states were constructed explicitly on a unital commutative C*-algebra. The commutative C*-algebras are classified by the Gelfand theory. That is, a unital commutative C*-algebra is isomorphic to the continuous functions on some compact Hausdorff space. For the non-unital commutative case the C*-algebra is isomorphic to the continuous functions "vanishing at infinity" on some locally compact Hausdorff space.

The construction in [1] was done by realizing that a slight generalization of regular Borel measures on compact Hausdorff spaces would represent any state (linear or not) through an integral construction. These generalized measures were refered to as quasi-measures, we include their definition:

Definition 1 *Let X be a compact Hausdorff space. Let $\mathcal{O}$ and $\mathcal{C}$ denote the open and closed subsets of X respectively. Furthermore, put $\mathcal{A} = \mathcal{O} \cup \mathcal{C}$. A positive set function $\mu : \mathcal{A} \to \mathbb{R}^+$ is a quasi-measure if*

1. $\mu(\biguplus_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu A_i$ *(where $\biguplus$ indicates disjoint union, and all A_i and $\biguplus_{i=1}^{\infty} A_i$ are assumed to be in $\mathcal{A}$)*
2. $\mu U = \sup\{\mu C : C \subset U, C \in \mathcal{C}\}$ *for all $U \in \mathcal{O}$.*

Notice that their definition is identical to regular Borel measures except for the domain of definition. The integral is now given by a transformation of integrals (see e.g. the first article in this thesis for the construction of an integral with respect to a quasi-measure).

Since their initial introduction the quasi-measures have gotten a life of their own. They are no longer confined to the C*-algebra setting, and are objects of interest in their own right.

The six articles in this thesis are devoted to developping the theory of quasi-measures, and to provide quasi-measures with substantial applications. The articles can basically be divided into two categories, namely theory and applications. The first and last article develops the theory of quasi-measures, whereas the other four bring the quasi-measures into other fields of mathematics.

Article 1: Unbounded quasi-integrals.

The first article goes back to the origin of quasi-measures and represents states in non-unital commutative C*-algebras. Uniform continuity of states in the unital case was initially proven by E. Christensen (c.f. [1]). We prove that states are also uniformly continuous in the non-unital case. Moreover, unbounded quasi-integrals are presented and constructed from unbounded quasi-measures. The Riesz representation theorem is proved for unbounded quasi-integrals and quasi-measures, presenting a proper generalization of the Riesz representation theorem as it is stated in [4]. This of course is beyond the setting of C*-algebras and examplifies that the quasi-measures have evolved into an integration theory in its own right.

Article 2: Quasi-measures and probability -a new interpretation.

Once defined the quasi-measures are still difficult to construct. This article presents a new and general construction technique: the q-functions. The q-functions demonstrate the existence of continuously varying quasi-measures similar to Lebesgue measure. In particular, they give the construction of a quasi-measure modelling a statistical problem. This was the first concrete example where quasi-measures proved efficient for modelling purposes.

Article 3: Quasi-measures, Image transformations and self-similar sets.

This article brings quasi-measures into the theory of Chaos and Fractals, more precisely iterated function systems and self-similar sets. Iterated systems of continuous functions are not adequate for obtaining self-similar sets from quasi-measures. The concept of an image transformation is introduced, generalizing continuous maps. The classical theory for iterated function systems is generalized to iterated image transformation systems. Astonishing examples of quasi-measures modelling image structures impossible by ordinary Borel measures are presented. Theory expanding these examples are presented, the reader is encouraged to see the article for the "whole story".

Article 4: Quasi-measures with image transformations as generalized variables.

A more general image transformation than above is presented and shown to generalize measurable maps. Their various mathematical properties are given through a series of theorems. In probability theory the measurable maps are the variables. This was the starting point of this paper. The last section presents how basic concepts like the median and sample median are modelled by quasi-measures in the general setting of a probability space. However, the sample median is not a variable (as is expected), but turns out to be an image transformation.

Article 5: The multidimensional median and sample median defined as quasi-probabilities.

This article is a natural continuation of the work in the preceeding article. The results are aimed at probability theorists, and accordingly address the probabilistic properties of the median and sample median, whereas the previous article focused on the mathematical properties of the image transformations. Numerous results and examples are given, we will not present them here. Still, we should point out that the solid variables are introduced. They generalize continuous monotone maps on the real line to the generality of topological spaces. They are shown to be median preserving transformations. Moreover, any concept of a median being preserved under solid variables and equal to the ordinary median on the real line is shown to be a quasi-measure. It should be noted that this is the first time the median and sample median have been defined for the general setting of probability spaces, although multidimensional constructions have been sought throughout the last century.

Article 6: Construction and properties of quasi-linear functionals.

The last article brings us back to the C*-algebra setting. Here we study quasi-linear functionals on unital commutative C*-algebras, i.e. bounded maps on the C*-algebra which are functionals on each C*-subalgebra generated by the unit and a self-adjoint element. They were represented by signed quasi-measures in [2], and their integration theory was presented there. However, most of the basic problems remained open, such as continuity, decomposition and construction. These are all answered affirmatively in this paper.

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Unbounded quasi-integrals

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Abstract

Let X be a locally compact Hausdorff space. We define a quasi-measure in X, a quasi-integral on $C_0(X)$, and a quasi-integral on $C_c(X)$. We show that all quasi-integrals on $C_0(X)$ are bounded, continuity properties of the quasi-integral on $C_c(X)$, representation of quasi-integrals on $C_c(X)$ in terms of quasi-measures, and unique extension of quasi-integrals on $C_c(X)$ to $C_0(X)$.

1. Introduction

The notion of a quasi-measure was introduced in [1] by J. F Aarnes. In [1], physical states on commutative unital C*-algebras where represented by quasi-measures. The quasi-measure in [1] was defined as a regular, finitely additive set function on open and closed subsets of a compact Hausdorff space X. The quasi-integral (physical state) with respect to a quasi-measure was constructed on the space of continuous functions on X (denoted $C(X)$). The quasi-integrals were shown to be the maps linear on each uniformly closed, singly generated subalgebra of $C(X)$.

Recent results (c.f. [2], [4] and [6]) indicate that the quasi-measures are interesting as a generalization of regular Borel measures. The restriction of a quasi-measure to a compact Hausdorff space is therefore unfortunate. Accordingly, the work presented here aims to extend the theory in [1] to X being a locally compact Hausdorff space.

In the sequel we let X denote a locally compact Hausdorff space. A set is called bounded if it's closure is compact. $\mathcal{F}$ and $\mathcal{O}$ denotes respectively the class of closed and the class of open subsets of X. Similarly, $\mathcal{C}$ and $\mathcal{O}^*$ denotes respectively the class of compact and the class of open bounded subsets. Furthermore we put $\mathcal{A} = \mathcal{F} \cup \mathcal{O}$ and $\mathcal{A}^* = \mathcal{C} \cup \mathcal{O}^*$.

Definition 1.1. *A quasi-measure in X is a function $\mu : \mathcal{A} \to [0, \infty]$ satisfying the following conditions:*

1. $\mu(A) < \infty$ *if* $A \in \mathcal{A}^*$
2. *For any finite, disjoint collection* $\{A_i\}_{i=1}^n \subset \mathcal{C} \cup \mathcal{O}$ *with* $\bigcup_{i=1}^n A_i \in \mathcal{C} \cup \mathcal{O}$, *then*
$$\mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$$
3. $\mu(U) = \sup\{\mu(K) : K \subset U, K \in \mathcal{C}\}, U \in \mathcal{O}$
4. $\mu(F) = \inf\{\mu(U) : F \subset U, U \in \mathcal{O}\}, F \in \mathcal{F}$

Our quasi-measure corresponds to the quasi-measure in [3], and the reader will find numerous properties of the quasi-measure there. The notion of a quasi-measure in a locally compact Hausdorff space can also be found in [7]. The definition in [7] is more restrictive than ours and does not produce the quasi-integrals given below.

Let $C_0(X)$ denote the real-valued continuous functions on X vanishing at infinity and let $C_c(X)$ be the functions in $C_0(X)$ with compact support. The support of a function $a \in C_0(X)$ will be denoted by $\operatorname{supp} a$ and the range of a in $\mathbf{R}$ by $\operatorname{sp} a$. If $a \in C_0(X)$, let $\mathbf{A}_0(a)$ denote the smallest uniformly closed subalgebra of $C_0(X)$ containing a. For a subsets $K, O \subset X$ we will let $K \prec a$ and $a \prec O$ denote that $a \in C_c(X), \operatorname{sp} a \subset [0,1]$ and respectively that $x \in K \Rightarrow a(x) = 1$ and $\operatorname{supp} a \subset O$.

Definition 1.2. *A real valued function ρ on $C_0(X)$ is called* a quasi-integral *if the following conditions are satisfied:*

1. $b \geq 0 \Rightarrow \rho(b) \geq 0$ *whenever* $b \in C_0(X)$
2. ρ *is linear on* $\mathbf{A}_0(a)$ *for each* $a \in C_0(X)$

When $\sup\{\rho(a) : a \prec X\} < \infty$ *we say that* ρ *is bounded. If in addition* $\sup\{\rho(a) : a \prec X\} = 1$ *we say that* ρ *is a* quasi-state.

In the C*-algebra setting this corresponds to the commutative, nonunital case, where ρ is characterized by linearity on closed subalgebras generated by self-adjoint elements.

If $a \in C_c(X)$ then we have $\mathbf{A}_0(a) \subset C_c(X)$. Hence we may define a quasi-integral on $C_c(X)$ similarly as above:

Definition 1.3. *A real-valued function* ρ *on* $C_c(X)$ *is called a* quasi-integral *if:*

1. $f \geq 0 \Rightarrow \rho(f) \geq 0$ *whenever* $f \in C_c(X)$.
2. ρ *is linear on* $\mathbf{A}_0(f)$ *for each* $f \in C_c(X)$.

If in addition $\sup\{\rho(f) : f \prec X\} < \infty$, *then* ρ *is bounded and we put* $\|\rho\| = \sup\{\rho(f) : f \prec X\}$.

The only difference between the definition above and Definition 1.2 is that ρ is now restricted to $C_c(X)$. However we will show that if ρ is bounded these two definitions coincide (Corollary 3.10). The key results in this article are boundedness of quasi-integrals on $C_0(X)$ and a representation theorem between the quasi-measures in X and the quasi-integrals on $C_c(X)$. The representation is a generalization of the Riesz Representation Theorem in [5].

The section below presents some preparatory results on the quasi-measures and quasi-integrals on $C_0(X)$. The section ends with the boundedness theorem for quasi-integrals on $C_0(X)$. The next and last section presents construction of the quasi-integral on $C_c(X)$ with respect to a quasi-measure. Monotonicity and continuity properties of the quasi-integral is given. The section highlights with the representation theorem for quasi-measures and quasi-integrals on $C_c(X)$. Finally, unique extension to $C_0(X)$ of quasi-integrals on $C_c(X)$ is given.

2. Quasi-integrals on $C_0(X)$

Throughout this article we will assume that X is a locally compact Hausdorff space. The results in the following proposition were given in [3]. We will only give a brief outline of the proofs here.

Proposition 2.1. *Let* μ *be a quasi-measure in* X.

1. $\mu(\emptyset) = 0$

2. $A \subset B \Rightarrow \mu(A) \leq \mu(B), A, B \in \mathcal{A}$

3. *If* $K \in \mathcal{C}, F \in \mathcal{F}$ *are disjoint, then* $\mu(F \cup K) = \mu(F) + \mu(K)$

4. μ *is countably additive on open sets.*

5. *Let* μ *be a quasi-measure in* X. *For any increasing family of open sets* $\{V_\lambda\}$, *if* $V_\lambda \nearrow V$ *(i.e.* $\bigcup V_\lambda = V$*) then* $\mu(V_\lambda) \nearrow \mu(V)$.

Proof. With $A_1 = A_2 = \emptyset$ in Definition 1.1.2 we get $\mu(\emptyset) = 0$. The monotonicity follows from regularity (1.1.3 and 1.1.4). The third statement follows from regularity and a Urysohn's lemma argument. The fifth statement follows from regularity (1.1.3). The fifth statement and finite additivity (1.1.2) implies the fourth statement.

Proposition 2.2. *A set function* $\mu : A \to [0, \infty]$ *satisfying 1., 3. and 4. of Definition 1.1 is a quasi-measure if and only if*

1. *If* $O_1, O_2 \in O$ *are disjoint, then* $\mu(O_1 \cup O_2) = \mu(O_1) + \mu(O_2)$.

2. *If* $K \subset O \in O$ *with* K *compact, then* $\mu(O) = \mu(O \backslash K) + \mu(K)$.

Proof. The proof of the third statement in Proposition 2.1 holds for μ. Hence by induction μ is finitely additive on $\mathcal{C}$. Similarly, μ is finitely additive on $\mathcal{O}$ by assumption 2.2.1. Let $\{A_i\}_{i=1}^n \subset \mathcal{C} \cup \mathcal{O}$ with disjoint union $A = \bigcup_{i=1}^n A_i \in \mathcal{C} \cup \mathcal{O}$. We may split the union to a disjoint union of a compact and an open set by $A = (\bigcup_{A_i \in \mathcal{C}} A_i) \cup (\bigcup_{A_i \notin \mathcal{C}} A_i)$. If A is open then $\mu(A) = \mu(\bigcup_{A_i \in \mathcal{C}} A_i) + \mu(\bigcup_{A_i \notin \mathcal{C}} A_i)$ by assumption 2.2.2. With μ finitely additive on $\mathcal{C}$ and on $\mathcal{O}$ we obtain $\mu(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i)$. If A is compact we may use a similar argument. Hence it suffices to show that if O is open and $O \subset K \in \mathcal{C}$, then $\mu(K) = \mu(K \backslash O) + \mu(O)$. Let $K' \subset O$ be compact. Then since μ is monotone $\mu(K) \geq \mu(K \backslash O) + \mu(K')$. Taking supremum of all $K' \subset O$, regularity yields $\mu(K) \geq \mu(K \backslash O) + \mu(O)$. Conversely, given $\epsilon > 0$ pick an open set $U \supset K \backslash O$ with $\mu(U) < \mu(K \backslash O) + \epsilon$. Observing that $K \backslash U \subset O$ yields

$$\begin{aligned} \mu(K) &\leq \mu(U \cup O) = \mu(U) + \mu(K \backslash U) \\ &< \mu(K \backslash O) + \mu(O) + \epsilon. \end{aligned}$$

Equality follows. We have shown finite additivity for μ on $\mathcal{C} \cup \mathcal{O}$ which completes the proof.

Lemma 2.3. *A quasi-integral ρ on $C_0(X)$ is bounded on $\mathbf{A}_0(a)$ for each $a \in C_0(X)$.*

Proof. Suppose $\sup\{\rho(f) : 0 \leq f \leq 1, f \in \mathbf{A}_0(a)\} = \infty$ for some $a \in C_0(X)$. Choose ϕ_i with $\phi_i \circ a \in \mathbf{A}_0(a)$, $\rho(\phi_i(a)) > 2^{2i}$ and $0 \leq \phi_i(a) \leq 1$ for $i = 1, 2, ...$, then with $\phi = \sum_{i=1}^{\infty} 2^{-i}\phi_i$ we have $\phi \circ a \in \mathbf{A}_0(a)$, $0 \leq \phi(a) \leq 1$ and $\rho(\phi(a)) = \infty$, which is a contradiction. Hence we must have ρ bounded on $\mathbf{A}_0(a)$.

Remark 1. *Note that ρ is a linear functional on $\mathbf{A}_0(a)$ and thus boundedness implies that ρ is continuous on $\mathbf{A}_0(a)$ for each $a \in C_0(X)$. Hence*

$$\sup\{\rho(a) : a \prec X\} = \sup\{\rho(a) : 0 \leq a \leq 1, a \in C_0(X)\}$$

for all quasi-integrals ρ on $C_0(X)$. Moreover, the complexification of $\mathbf{A}_0(a)$ is a C^*-algebra so Lemma 2.3 is not a new result. We included it for completeness and the readers convenience.

Lemma 2.4. *Suppose $a \in C_c(X)$ with $0 \leq a \leq 1$. Then there is a function $f \in C_c(X)$ with $\operatorname{supp} a \prec f$. Moreover, $\operatorname{supp} a \prec f \prec X$ implies that $a, f \in \mathbf{A}_0(a+f)$ and $\rho(a) \leq \rho(f)$.*

Proof. If $a \in C_c(X)$ then $\operatorname{supp} a = K$ is compact. There is an open bounded set V containing K. By Urysohn's lemma there is a function f with $K \prec f \prec V$ which implies that $f \in C_c(X)$. Define ϕ_1 and ϕ_2 by

$$\phi_1(x) = \begin{cases} 1 & , x \geq 1 \\ x & , x < 1 \end{cases} \quad \text{and } \phi_2 = \begin{cases} x-1 & , x \geq 1 \\ 0 & , x < 1 \end{cases}.$$

Then $\phi_1(a+f) = f$ and $\phi_2(a+f) = a$ thus $a, f \in \mathbf{A}_0(a+f)$ and we get $\rho(a) \leq \rho(f)$.

Theorem 2.5 (Boundedness of quasi-integrals). *All quasi-integrals on $C_0(X)$ are bounded.*

Proof. Let ρ be a quasi-integral on $C_0(X)$ and suppose $\sup\{\rho(a) : a \prec X\} = \infty$. By lemma 2.4 construct recursively a sequence $\{a_i\}_{i=1}^{\infty}$ where $\rho(a_i) \geq 2^{2i}$ and $\operatorname{supp} a_i \prec a_{i+1} \prec X$ for each i. Let $f = \sum_{i=1}^{\infty} 2^{-i}a_i$ then $f \in C_0(X)$ since $C_0(X)$ is complete. Define ϕ_i for $i = 1, 2, ...$ by

$$\phi_i(x) = \begin{cases} 1 & , x \geq 2^{-i+1} \\ 2^i(x - 2^{-i}) & , 2^{-i} \leq x \leq 2^{-i+1} \\ 0 & , x \leq 2^{-i} \end{cases}$$

we have $\phi_i \in C(\text{sp} f)$, $\phi_i(0) = 0$ and $\phi_i(f) = a_i$ for each i and thus $\{a_i\}_{i=1}^{\infty} \subset \mathrm{A}_0(f)$. Finally $f \geq 2^{-i}a_i$ implies $\rho(f) \geq 2^{-i}\rho(a_i) \geq 2^i$ for $i = 1, 2, \ldots$ which in its turn implies that $\rho(f) = \infty$. This is a contradiction since ρ is supposed to be a quasi-integral on $C_0(X)$ we may conclude that $\sup\{\rho(a) : a \prec X\} < \infty$ so ρ is bounded.

Remark 2. *Theorem 2.5 shows that the local linearity of the quasi-integrals impose strong restrictions on their global behavior. This suggests that unbounded quasi-integrals on $C_c(X)$ may exhibit nice properties. Indeed, this is what we will devote the next and last section to.*

3. Quasi-integrals on $C_c(X)$

Proposition 3.1. *Suppose that μ is a quasi-measure in X and $f \in C_c(X)$, then there is a unique bounded regular Borel measure μ_f on $\mathbf{R}\backslash\{0\}$ with $\mu_f(O) = \mu(f^{-1}(O))$ for all open sets $O \subset \mathbf{R}\backslash\{0\}$.*

Proof. Let $\check{f}(x) = \mu(f^{-1}(-\infty, x)\backslash\{0\})$ which implies that $\check{f}$ is increasing. Since $f \in C_c(X)$ we have that $f(\text{supp} f)$ is compact. Hence $\check{f}$ is constant outside an interval $[a, b]$ for some $a, b \in \mathbf{R}$. By proposition 2.1 $\check{f}(x^-) = \check{f}(x)$ for each $x \in \mathbf{R}$, so $\check{f}$ is continuous from the left. Thus $\mu_f[x, y) = \check{f}(y) - \check{f}(x)$ uniquely determines a regular Borel measure in $\mathbf{R}$ and by regularity $\mu_f(x, y) = \check{f}(y) - \check{f}(x^+)$. If $x_\lambda \in (x, y)$ then $\check{f}(y) \geq \mu(f^{-1}[(x_\lambda, y)\backslash\{0\}]) + \check{f}(x_\lambda)$ since μ is monotone, hence $\check{f}(y) - \check{f}(x^+) \geq \mu(f^{-1}[(x, y)\backslash\{0\}])$. Conversely finite additivity and monotonicity of μ yields

$$\begin{aligned}\check{f}(y) &= \mu(f^{-1}[(x, y)\backslash\{0\}]) + \mu(f^{-1}(\{x\}\backslash\{0\}) + \check{f}(x) \\ &\leq \mu(f^{-1}[(x, y)\backslash\{0\}]) + \check{f}(x_\lambda),\end{aligned}$$

so $\check{f}(y) - \check{f}(x^+) \leq \mu(f^{-1}[(x, y)\backslash\{0\}])$.We have $\mu(f^{-1}[(x, y)\backslash\{0\}]) = \check{f}(y) - \check{f}(x^+) = \mu_f(x, y)$, and since both μ and μ_f are countably additive on open sets the proof is complete.

Remark 3. *We will call μ_f the measure corresponding to μ and f. Notice that Proposition 3.1 only is stated for open sets not containing $\{0\}$, whereas the proof produces a measure on $\mathbf{R}$ with $\mu_f(\{0\}) = 0$. This is convenient when the quasi-measure is an extended real-valued function. In fact, Lemma 3.3 is not valid unless zero is omitted.*

Definition 3.2. *A map $f \mapsto \mu_f$ from $C_c(X)$ into the regular Borel measures in $\mathbf{R}\backslash\{0\}$ is consistent if $\mu_{\phi\circ f} = \mu_f \circ \phi^{-1}$ for each $f \in C_c(X)$ and $\phi \in C(\operatorname{sp} f), \phi(0) = 0$.*

Lemma 3.3. *Let μ be a quasi-measure in X. Let μ_f denote the measure corresponding to μ and $f \in C_c(X)$, then the map $f \mapsto \mu_f$ is consistent.*

Proof. Let $f \in C_c(X)$, $\phi \circ f \in \mathbf{A}_0(f)$ and $K \subset \mathbf{R}\backslash\{0\}$ be compact. Now $0 \notin \phi^{-1}(K)$ implies:

$$\begin{aligned} \mu_{\phi\circ f}(K) &= \mu((\phi \circ f)^{-1}(K)) \\ &= \mu(f^{-1}(\phi^{-1}(K))) \\ &= \mu_f(\phi^{-1}(K)). \end{aligned}$$

Note that since K is compact in $\mathbf{R}\backslash\{0\}$, then K is compact in $\mathbf{R}$ by the identity map. So $f^{-1}(\phi^{-1}(K))$ is a closed subset of $\operatorname{supp} f$, and thus compact. The result now follows from the regularity of μ_f.

In the sequel we will assume that the measure corresponding to a quasi-measure μ and a function $f \in C_c(X)$ is extended to $\mathbf{R}$ by $\mu_f\{0\} = 0$.

Proposition 3.4. *Let ρ be a quasi-integral on $C_c(X)$. If $f, g \in C_c(X)$ and $f \leq g$ then $\rho(f) \leq \rho(g)$.*

Proof. Given $\delta > 0$, suppose $f \geq 0$ and $g(x) \geq \delta + f(x)$ when $x \in \operatorname{supp} f$. Pick a natural number n such that $n\delta > \max g$ and define $\phi_i \in C(\operatorname{sp} f)$, $1 \leq i \leq n$ by:

$$\phi_i(x) = \begin{cases} 0 & , x \leq (i-1)\delta \\ x - (i-1) & , (i-1)\delta < x < i\delta \\ \delta & , x \geq i\delta. \end{cases}$$

Then $x \in \operatorname{supp} \phi_i(f) \Rightarrow \phi_i(g(x)) = \delta$, thus $\phi_i(f), \phi_i(g) \in \mathbf{A}_0(\phi_i(f) + \phi_i(g))$ which imply $\rho(\phi_i(f)) \leq \rho(\phi_i(g))$. Now $\sum \phi_i(f) = f, \sum \phi_i(g) = g$ implies $\rho(f) \leq \rho(g)$. Given $\epsilon > 0$, suppose now that $0 \leq f \leq g$, choose h and $\delta > 0$ with $\operatorname{supp} g \prec h \in C_c(X)$ and $\rho(\delta h) < \epsilon$. We have $\rho(f) \leq \rho(g + \delta h) < \rho(g) + \epsilon$. Let $f \leq g \in C_c(X)$ be arbitrary, then $f^+, f^- \in A(f)$ and $f^+ \leq g^+, f^- \geq g^-$. We have $\rho(f) = \rho(f^+) - \rho(f^-) \leq \rho(g^+) - \rho(g^-) = \rho(g)$ by the previous argument. The proof is complete.

Corollary 3.5. *Let ρ be a quasi-integral on $C_c(X)$ and let K be an arbitrary compact subset of X. Then there is a $k \in \mathbf{R}$ such that whenever $\operatorname{supp} f_i \subset K, f_i \in C_c(X)$ for $i = 1, 2$, we have:*

$$|\rho(f_1) - \rho(f_2)| \leq k \|f_1 - f_2\|.$$

Proof. Pick an $g \succ K$ and let $\rho(g) = k$. Then $f_1 \leq f_2 + g \|f_1 - f_2\|$ which implies that $\rho(f_1) - \rho(f_2) \leq \rho(g) \|f_1 - f_2\|$ and conversely $\rho(f_2) - \rho(f_1) \leq \rho(g) \|f_1 - f_2\|$. But then we must have $|\rho(f_1) - \rho(f_2)| \leq k \|f_1 - f_2\|$.

Remark 4. *In general ρ is not uniformly continuous (since it is a generalization of regular Borel measures). However, ρ is continuous with respect to the topology of uniform convergence on compacta. Hence this is a sharp result, we can not expect stronger continuity properties.*

Corollary 3.6. *Let ρ be a bounded quasi-integral on $C_c(X)$. Then for each pair $f_1, f_2 \in C_c(X)$ we have*

$$|\rho(f_1) - \rho(f_2)| \leq \|\rho\| \|f_1 - f_2\|.$$

Proof. Pick a function $g \succ \operatorname{supp} f_1 \cup \operatorname{supp} f_2$. Then $\rho(g) \leq \|\rho\|$ and the result follows from Corollary 3.5.

Proposition 3.7. *Let μ be a quasi-measure in X. Define*

$$\rho(f) = \int i \; d\mu_f \text{ for each } f \in C_c(X),$$

where μ_f is the measure corresponding to μ and f and i is the identity map on $\boldsymbol{R}$. Then ρ is a quasi-integral on $C_c(X)$.

Proof. By the transformation theorem for integrals and Lemma 3.3, the result follows.

Lemma 3.8. *Let μ be a quasi-measure in X and let ρ be the corresponding quasi-integral. Then for each open set $O \subset X$ we have:*

$$\mu(O) = \sup\{\rho(f) : f \prec O\}.$$

Moreover, if $\mu(X) < \infty$ then ρ is bounded and $\|\rho\| = \mu(X)$.

Proof. First suppose $\mu(O) < \infty$. Choose a compact set $K \subset O$ with $\mu(K) > \mu(O) - \epsilon$, and a function f with $K \prec f \prec O$. We have:

$$\begin{aligned} \rho(f) &= \int_{\mathrm{sp}\, f} i \; d\mu_f = \int_{\{1\}} d\mu_f + \int_{(0,1)} i \; d\mu_f \\ &\geq \int_{\{1\}} d\mu_f = \mu_f(\{1\}) = \mu(f^{-1}\{1\}) \\ &\geq \mu(K) \text{ since } K \subset f^{-1}\{1\}. \end{aligned}$$

On the other hand we have:

$$\begin{array}{rll} \rho(f) & \leq \int_{\mathrm{sp}\, f} d\mu_f & ;\ \mathrm{sp}\, f \subset [0,1]. \\ & = \mu_f(\mathrm{sp}\, f\backslash\{0\}) & \\ & = \mu(f^{-1}(0,\infty)) & ;\ f^{-1}(0,\infty) = f^{-1}(\mathrm{sp}\, f\backslash\{0\}). \\ & \leq \mu(O) & ;\ f \prec O \Rightarrow f^{-1}(0,\infty) \subset O. \end{array}$$

These together imply $\mu(O) = \sup\{\rho(f) : f \prec O\}$. If $\mu(O) = \infty$, then there is for every natural number n, a compact set $K \subset O$ with $\mu(K) > n$. By the previous argument there is then a function f with $K \prec f \prec O$ and $\rho(f) > n$. Hence $\mu(O) = \sup\{\rho(f) : f \prec O\} = \infty$. If $\mu(X) < \infty$ put $O = X$ in the previous argument. Then $\mu(X) = \sup\{\rho(f) : f \prec X\} = \|\rho\| < \infty$. The proof is complete.

Theorem 3.9 (The representation theorem). *Let X be a locally compact Hausdorff space.*

1. *To each quasi-measure μ in X there is a unique quasi-integral ρ on $C_c(X)$ such that for any $f \in C_c(X)$ we have*

$$\rho(\phi(f)) = \int \phi(i)\ d\mu_f$$

for all $\phi \in \{\phi \in C(\mathrm{sp}\, f) : \phi(0) = 0\}$. Here μ_f is the regular Borel measure in $\mathbf{R}$ corresponding to μ and f.

2. *Conversely, for any quasi-integral ρ on $C_c(X)$ there is a unique quasi-measure μ in X such that ρ is the quasi-integral corresponding to μ. Specifically we have, for any open set $O \subset X$:*

$$\mu(O) = \sup\{\rho(f) : f \prec O\}. \tag{3.1}$$

Proof. The first part of the theorem follows from Proposition 3.7. Suppose ρ is a quasi-integral on $C_c(X)$. Define a set function $\mu : \mathcal{O} \to \mathbf{R} \cup \{\infty\}$ by (3.1). Extend μ to the closed subsets F of X by $\mu(F) = \inf\{\mu(O) : F \subset O, O \text{ is open}\}$. Notice that this implies $\mu(K) = \inf\{\rho(f) : f \succ K\}$ when K is compact by Urysohn's lemma and the monotonicity of ρ. We will show that μ is a quasi-measure in X. Note that $\mu(A) < \infty$ when $A \in \mathcal{A}^*$ by Urysohn's lemma and Corollary 3.5. Suppose that O_1 and O_2 are open disjoint subsets of X. Pick f_i with $f_i \prec O_i$ and $\rho(f_i) > \mu(O_i) - \epsilon$ for $i = 1, 2$. We have $f_1 f_2 = 0$ which implies $f_1, f_2 \in \mathbf{A}(f_1 - f_2)$ and thus

$$\begin{array}{rl} \mu(O_1 \cup O_2) & \geq \rho(f_1 + f_2) = \rho(f_1) + \rho(f_2) \\ & \geq \mu(O_1) + \mu(O_2) + 2\epsilon. \end{array}$$

Conversely if $f \prec O_1 \cup O_2$, the opposite equality follows from observing that $f = f_1 + f_2$ where $f_i(x) = f(x)$ if $x \in O_i$ and elsewhere zero.
Let $K \subset O \subset X$ where K is compact and O is open. By Urysohn's lemma there is an open bounded set U and functions f_K, f_U such that $K \subset U \subset \bar{U} \subset O$, $K \prec f_K \prec O$ and $\bar{U} \prec f_U \prec O$ with $\rho(f_U) > \mu(O) - \epsilon$. Then $f_K, f_U \in \mathbf{A}(f_K + f_U)$ and $f_U - f_K \prec O \backslash K$ thus

$$\begin{aligned} \mu(O\backslash K) &\geq \rho(f_U - f_K) = \rho(f_U) - \rho(f_K) \\ &> \mu(O) - \mu(K) - \epsilon. \end{aligned}$$

Which yields $\mu(O) \leq \mu(O\backslash K) + \mu(K)$ when $\mu(O) < \infty$ and equality when $\mu(O) = \infty$. Conversely, if $f \prec O\backslash K$ with $\rho(f) > \mu(O\backslash K) - \epsilon$, then $K' = \operatorname{supp} f \subset O\backslash K$, so $(X\backslash K') \cap U$ is an open set containing K. Pick f_K such that $K \prec f_K \prec (X\backslash K') \cap U$, then $f f_K = 0$. We have

$$\begin{aligned} \mu(O) &\geq \rho(f_K + f) = \rho(f_K) + \rho(f) \\ &> \mu(K) + \mu(O\backslash K) - \epsilon. \end{aligned}$$

We have shown that μ is a quasi-measure in X. The uniqueness of μ follows from Lemma 3.8. Let ρ_μ denote the quasi-integral corresponding to μ, it remains to prove that ρ_μ is equal to ρ. Let $f \in C_c(X)$ be arbitrary. Then $\rho_f : \phi \to \rho(\phi(f))$ is a functional on $\{\phi : \phi \in C(\operatorname{sp} f), \phi(0) = 0\}$. Extend ρ_f to a functional F on $C(\operatorname{sp} f)$ by $F(\phi) = \rho_f(\phi - \phi(0)) + \phi(0)$, $\phi \in C(\operatorname{sp} f)$. Now $\operatorname{sp} f$ is compact and thus F determines a unique regular Borel measure ν_f on $\operatorname{sp} f$ such that

$$F(\phi) = \rho(\phi(f)) = \int \phi(i)\, d\nu_f \text{ when } \phi \in C(\operatorname{sp} f) \text{ and } \phi(0) = 0.$$

So by regularity it suffices to show that $\mu_f = \nu_f$ on the open subsets of $E = \operatorname{sp} f\backslash\{0\}$. Suppose $\epsilon > 0$ and $U \subset E$ open are arbitrary. Pick a compact set $K \subset U$ such that $\nu_f(K) > \nu_f(U) - \epsilon$. Choose an Urysohn function $K \prec \phi \prec U$, then $\nu_f(U) - \epsilon < \nu_f(K) \leq \rho(\phi(f)) \leq \mu(f^{-1}(U)) = \mu_f(U)$ since $\phi \circ f \prec f^{-1}(U)$. Conversely, pick a compact set $K \subset f^{-1}(U)$ such that $\mu(K) > \mu(f^{-1}(U)) - \epsilon$ and a function ϕ with $f(K) \prec \phi \prec U$. Then since $\phi \circ f \succ f^{-1}(K)$ we have

$$\mu_f(U) - \epsilon = \mu(f^{-1}(U)) - \epsilon < \mu(K) \leq \rho(\phi \circ f) = \rho_f(\phi) \leq \nu_f(U).$$

The proof is complete.

Corollary 3.10. *Let ρ be a quasi-integral on $C_c(X)$. If ρ is bounded then ρ has a unique extension to a quasi-integral on $C_0(X)$.*

Proof. By Corollary 3.6 ρ is uniformly continuous. Extend ρ by continuity to a function $\rho_0 : C_0(X) \to \mathbf{R}$, for example by the functions ϕ_ϵ defined by

$$\phi_\epsilon(x) = \begin{cases} 0 & ,x < \epsilon \\ 2x - 2\epsilon & ,\epsilon \leq x \leq 2\epsilon \\ x & ,x > 2\epsilon \end{cases}$$

Obviously $\rho_0(\alpha f) = \alpha\rho_0(f)$ for all $\alpha \in \mathbf{R}, f \in C_0(X)$. Suppose $f \in C_0(X)$ and $\phi_1(f), \phi_2(f) \in \mathbf{A}_0(f)$. Then $\phi_i(\phi_\epsilon(f)) \in \mathbf{A}_0(\phi_\epsilon(f))$ for all $\epsilon > 0$ and $i = 1, 2$. Note that $\phi_i(\phi_\epsilon(f))$ converges uniformly to $\phi_i(f)$ when ϵ tends to zero. Hence by continuity

$$\begin{aligned} \rho(\phi_1(f) + \phi_2(f)) &= \lim_{\epsilon \to 0} \rho(\phi_1(\phi_\epsilon(f)) + \phi_2(\phi_\epsilon(f))) \\ &= \lim_{\epsilon \to 0} [\rho(\phi_1(\phi_\epsilon(f))) + \rho(\phi_2(\phi_\epsilon(f)))] \\ &= \rho(\phi_1(f)) + \rho(\phi_2(f)). \end{aligned}$$

We have shown that ρ_0 is a quasi-integral on $C_0(X)$. The uniqueness of the extension is immediate from the continuity of ρ_0. The proof is complete.

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Probability and quasi-measures

-a new interpretation

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1. Introduction

Let X be a compact Hausdorff space, and let $\mathcal{C}$ (respectively $\mathcal{O}$) denote the collection of closed (respectively open) subsets of X. Let $\mathcal{A} = \mathcal{C} \cup \mathcal{O}$. A *quasi-measure* in X is a function $\mu : \mathcal{A} \to \mathbb{R}^+$ which is monotone, additive and regular. More precisely we have:

(*i*) $\mu(A_1) \leq \mu(A_2)$ if $A_1 \subset A_2$

(*ii*) $\mu(\biguplus_{i=1}^n A_i) = \sum_{i=1}^n \mu A_i$ ($\biguplus$ indicates disjoint union, and we assume all A_i and $\biguplus_{i=1}^n A_i$ in $\mathcal{A}$)

(*iii*) $\mu U = \sup\{\mu C : C \subset U; C \in \mathcal{C}\}$ for all U in $\mathcal{O}$.

One may show that a quasi-measure has a (necessarily unique) extension to a regular Borel measure in X if and only if it is *subadditive* on $\mathcal{C}$, i.e. if it satisfies

(iv) $\mu(C_1 \cup C_2) \leq \mu C_1 + \mu C_2$ for all C_1, C_2 in $\mathcal{C}$.

The whole point here is that quasi-measures that *do not* satisfy condition (iv), exist. Their basic construction has been given in [1], [2] and [3]. The main construction result ([1], Theorem 5.1) assumes that a set function μ initially is given on a fundamental family of sets $\mathcal{A}_s$, called the *solid sets,* and extended to all of $\mathcal{A}$ (a set $A \in \mathcal{A}$ is solid if A and its compliment are both connected). A function $\mu : \mathcal{A}_s \rightarrow \mathbb{R}^+$ is a *solid set-function* if it satisfies

(A) $\sum_{i=1}^{n} \mu C_i \leq \mu C$ whenever $\biguplus_{i=1}^{n} C_i \subset C$, $C_i, C \in \mathcal{C}_s$; $i = 1, 2, ..., n$

(B) $\mu U = \sup\{\mu C : C \subset U; C \in \mathcal{C}_s\}$ for all $U \in \mathcal{O}_s$

(C) $\mu A + \mu(X \backslash A) = \mu X$

Here $\mathcal{C}_s$ (respectively $\mathcal{O}_s$) denotes the family of closed (respectively open) solid sets in X. We assume here that X is connected and locally connected, and for simplicity we will also require that it has genus $g = 0$ (for details see [1]). These conditions will be met in standard spaces like balls and spheres, for instance. The main construction theorem now states that each solid set function has a unique extension to a quasi-measure in X. By this result, the construction problem is reduced to that of obtaining the solid set-functions. It is the purpose of this note to describe how to do this by means of applying certain functions to existing (Borel or quasi-) measures. By this process we also obtain a new interpretation in probability theory, presented in the final section in this paper.

2. Functions composed with measures

We first introduce the class of functions we are going to consider.

Definition 2.1. *A function* $f : [0, 1] \rightarrow [0, 1]$ *is called* a q-function *if it is continuous from the right and satisfies*

1. $f(0) = 0$, $f(x-) + f(1 - x) = 1$
2. $\sum_{i=1}^{n} f(x_i) \leq f(\sum_{i=1}^{n} x_i)$ *whenever* $x_1, x_2, ..., x_n \in [0, 1]$ *and* $\sum_{i=1}^{n} x_i < 1$.

Let ν be a normalized Borel (or quasi-) measure in X, i.e. $\nu(X) = 1$. We say that ν is *non-splitting* if there is no disjoint pair $C_1, C_2 \in \mathcal{C}_s$ such that $\nu C_1 > 0$, $\nu C_2 > 0$ and $\nu C_1 + \nu C_2 = 1$. For instance, Lebesgue-measure on the unit disk, or the unit sphere (normalized) is non-splitting.

Proposition 2.2. *Let f be a q-function, and let ν be a normalized regular Borel (or quasi-) measure in X. Define μ on $\mathcal{A}_s$ by: $\mu C = f(\nu C)$; $C \in \mathcal{C}_s$ and $\mu U = 1 - \mu(X \backslash U)$; $U \in \mathcal{O}_s$. If either ν is non-splitting or f is continuous, then μ is a solid set-function.*

Proof. We first verify that f is non-decreasing: Let $0 \leq x < y < 1$. By 2.1.2 : $f(x) \leq f(x) + f(y-x) \leq f(x + (y-x)) = f(y)$. Also, by 2.1.1 : $f(1-) + f(0) = 1 \Rightarrow f(1-) = 1$. Putting $x = 0$ in 2.1.1 yields $f(0) + f(1) = 1 \Rightarrow f(1) = 1$. Now let $C_1, ..., C_n, C \in \mathcal{C}_s$ and suppose $\biguplus_{i=1}^n C_i \subset C$. If $\sum_{i=1}^n \mu C_i < 1$ then by 2.1.2

$$\sum \mu C_i = \sum f(\nu C_i) \leq f(\sum \nu C_i) = f(\nu(\biguplus C_i)) \leq f(\nu C) = \mu C$$

since f is non-decreasing. If ν is non-splitting then $\sum \nu C_i = 1$ is impossible unless $n = 1$. This follows from lemma 3.3 in [1]. So, if $\sum \nu C_i = 1$ we may assume that f is continuous. Condition 2.1.1 then yields $f(x) + f(1-x) = 1$ for all $x \in [0,1]$ so that $\sum \mu C_i = 1 = \mu C$. Next, to show that μ is regular we first note that if $C \subset U$; $C \in \mathcal{C}_s, U \in \mathcal{O}_s$, then $\mu C \leq \mu U$. For if $C' = X \backslash U$ then $C \cap C' = \emptyset$ and it follows from the argument above that $\mu C + \mu C' \leq 1 \Rightarrow \mu C \leq 1 - \mu C' = \mu U$. Now let $U \in \mathcal{O}_s$ be arbitrary. Since ν is (inner) regular we get

$$\begin{aligned} \mu C &= 1 - \mu(X \backslash U) = 1 - f(\nu(X \backslash U)) \\ &= 1 - f(1 - \nu U) = f(\nu U -) \\ &= \sup_{C \subset U} f(\nu C) = \sup_{C \subset U} \mu C \end{aligned}$$

We have now verified conditions (A) and (B) for a solid set-function, and (C) is true by definition. ■

We next turn to the question of determining what functions that are q-functions.

Lemma 2.3. *Let $f : [0,1] \to [0,1]$ be a function satisfying:*

1. *$f(0) = 0$, $f(x) + f(1-x) = 1$*
2. *f is convex on $[0, \frac{1}{2}]$*

Then f is non-decreasing and satisfies

$$\sum_{i=1}^{n} f(x_i) \leq f(\sum_{i=1}^{n} x_i) \text{ whenever } \sum_{i=1}^{n} x_i \leq 1, \; x_1, ..., x_n \in [0,1]; n \in \mathbb{Z}^+. \quad (*)$$

Proof. Let $0 < c < \frac{1}{2}$ and let $l = l(x)$ be the straight line through the origin and the point $(c, f(c))$. By the convexity of f we have

$$\begin{aligned} f(x) &\leq l(x) \quad \text{if } 0 \leq x \leq c \\ f(x) &\geq l(x) \quad \text{if } c \leq x \leq \tfrac{1}{2} \end{aligned}$$

We first prove (*) when $n = 2$.

i) Let $0 < x_1 \leq x_2 \; ; x_1 + x_2 \leq \frac{1}{2}$. Taking $c = x_2$ we get $f(x_1) + f(x_2) \leq l(x_1) + l(x_2) = l(x_1 + x_2) \leq f(x_1 + x_2)$.

ii) $0 < x_1 \leq x_2 \leq \frac{1}{2} \; ; x_1 + x_2 \geq \frac{1}{2}$. By 2.3.1 we get $f(x) \leq x$ if $x \leq \frac{1}{2}$ and $f(x) \geq x$ if $x \geq \frac{1}{2}$. Hence $f(x_1) + f(x_2) \leq x_1 + x_2 \leq f(x_1 + x_2)$.

iii) $0 < x_1 < \frac{1}{2} < x_2 < 1 \; ; x_1 + x_2 \leq 1$. We have $0 \leq 1 - x_1 - x_2 \leq \frac{1}{2}$, and $(1 - x_1 - x_2) + x_1 = 1 - x_2 < \frac{1}{2}$,so by 2.3.1 and case *i)* above we get

$$\begin{aligned} f(1 - x_1 - x_2) + f(x_1) &\leq f(1 - x_2), \text{ i. e.} \\ 1 - f(x_2 + x_1) + f(x_1) &\leq 1 - f(x_2) \\ \Rightarrow f(x_1) + f(x_2) &\leq f(x_1 + x_2) \end{aligned}$$

This establishes (*) when $n = 2$. An easy induction argument shows that it is true in general. ∎

Corollary 2.4. *Let $f : [0,1] \to [0,1]$ be a function satisfying 2.3.1 and 2.3.2 of the lemma above , and which is continuous at $x = \frac{1}{2}$. Then f is a continuous q-function.*

Proof. Since f is non-decreasing and convex on $[0, \frac{1}{2})$ it is continuous on $[0, \frac{1}{2})$. By 2.3.1 and continuity at $x = \frac{1}{2}$ it follows that f is continuous on $[0, 1]$. ∎

Corollary 2.5. *Let $f : [0,1] \to [0,1]$ be continuous, convex on $[0,1]$ and satisfy $f(0) = 0 \; ; f(x) + f(1-x) = 1$. Then $\mu A = f(\nu A) \; ; A \in \mathcal{A}_s$ is a solid set-function for each normalized quasi-measure in X.*

Remark 1. *As described in the introduction, each solid set-function extends to a unique quasi-measure in X. In general the quasi-measures obtained by q-functions will be proper quasi-measures, they are not subadditive. In fact, even if the initial set-function ν is a regular Borel measure, the only ordinary measure obtained from the process is ν itself, coming from $f(x) = x$. To illustrate this, let $X = D$ =unit disk, and let ν be normalized Lebesgue measure. If f is a continuous, convex q-function which is not the identity function, we must have $f(\frac{1}{4}) < \frac{1}{4}$. Let C_1 and C_2 be disks in X with area $\nu(C_1) = \nu(C_2) = \frac{1}{4}$. Assume $C_1 \cap C_2 \neq \emptyset$ and $C_1 \cup C_2 \in \mathcal{C}_s$. We have $\mu C_1 + \mu C_2 = f(\nu C_1) + f(\nu C_2) = 2f(\frac{1}{4}) < \frac{1}{2}$. On the other hand, by making $\nu(C_1 \cap C_2)$ small we can have $\nu(C_1 \cup C_1) \to \frac{1}{2}$, and then by the continuity of f we can get $\mu(C_1 \cup C_2) = f(\nu(C_1 \cup C_2)) > \mu C_1 + \mu C_2$. Hence μ is not subadditive.*

We conclude this section with some examples.

Example 2.6. *Let $n \in \mathbb{Z}^+$ be arbitrary, and let*

$$\begin{aligned} I_k &= [\tfrac{k}{n+1}, \tfrac{k+1}{n+1}) \quad ; k = 0, 1, ..., n-1 \\ I_n &= [\tfrac{n}{n+1}, 1] \end{aligned}$$

Define $f(x) = \frac{k}{n}$ on I_k ; $k = 0, ..., n$. Then f is a q-function. If ν is a non-splitting regular Borel measure we therefore obtain a quasi-measure μ. If $n = 1$ μ is simple, i.e. it only takes the values 0 and 1. For general n one may show that μ is an extreme point in the set $Q(X)$ of all normalized quasi-measures in X.

Example 2.7. $f(x) = \sin^2(\frac{\pi}{2}x) = \frac{1}{2}(1 - \cos \pi x)$ *is a continuous q-function which is convex on $[0, \frac{1}{2}]$.*

Example 2.8. $p(x) = 3x^2 - 2x^3$ *and* $q(x) = 2x^2 + 2x^3 - 5x^4 + 2x^5$ *are polynomials of the type above.*

3. Quasi-measures and probability

A normalized quasi-measure in X is called a *quasi-probability*. The preceding section has shown how one may construct quasi-probabilities from q-functions and a given probability measure ν in X. We formalize this procedure. Let $Q(X)$ denote the set of quasi-probabilities in X and let $q[0,1]$ denote the set of q-functions. Both these sets are convex. If ν is a fixed non-splitting element of

$Q(X)$ we obtain a map $F_\nu : q[0,1] \to Q(X)$ by defining $\mu = F_\nu(f)$ where μ is given by Proposition 2.2 on $\mathcal{A}_s$ and then extended to all of $\mathcal{A}$ (Theorem 5.1 in [1]). Since μ is uniquely determined by its values on $\mathcal{A}_s$ it follows that F_ν is an *affine map* so the range of F_ν is a convex subset of $Q(X)$ which we will denote by $Q_\nu(X)$, and consists of the quasi-probabilities that are *associated* with ν. Note that if ν is simple then $Q_\nu(X)$ only consists of ν, i. e. $F_\nu(f) = \nu$ for all $f \in q[0,1]$. Our goal here is to give a probabilistic interpretation of the quasi-probabilities that are associated with an ordinary probability ν. This will be done by considering concrete examples. A more general approach to quasi-probabilities may be found in [4].

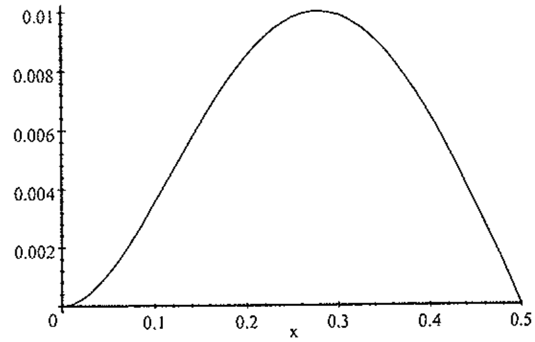
Example 3.1 (Quasi-probability). *Let D be the unit disk, and let ν be the normalized Lebesgue measure in D. Then $(D, \mathcal{B}(D), \nu)$ where $\mathcal{B}(D)$ is the Borel sets in D is a probability space. Further let $X_i = id_D$ (identity map) for $i = 1, 2, 3$ be independent random variables on D. Then $P(\bigcap_{i=1}^3 X_i \in A_i) = P_1(X_1 \in A_1) \cdot P_2(X_2 \in A_2) \cdot P_3(X_3 \in A_3)$ for any triple $\{A_i\}_{i=1}^3 \subset \mathcal{B}(D)$, where of course P_i is given by $P_i(X_i \in A_i) = \nu(X_i^{-1}(A_i)) = \nu(A_i)$; $i = 1, 2, 3$. Note that P is the product measure ν^3 in D^3. We define a set function $\mu : \mathcal{A}_s \to [0,1]$ on the solid subsets of D by*

$$\mu(A) = \{\text{The probability of at least two } X_i\text{'s being in } A\}. \tag{3.1}$$

μ may be calculated combinatorically considering D^3 with the X_i occuring respectively in the three disks. We then obtain $\mu(A) = \nu(A)^3 + \binom{3}{2}\nu(A)^2\nu(D \backslash A) = 3\nu(A)^2 - 2\nu(A)^3$. Notice that $\mu \in Q_\nu(X)$ by the q-function $p(x)$ in Example 2.8 and hence determines a unique quasi-measure in D. One might be tempted to think that μ determines a new probability measure on D but this is not so (see Remark 1). It is not difficult to imagine a situation where the set-function μ is interesting: Imagine an airdrop of three objects where in order for the drop to be successful you need to find two objects and the ground you cover searching is a solid set.

Remark 2. *Although the specific problem above can be solved with ordinary probability theory one should bear in mind that this is a very simple example to illustrate that the quasi-measures arises naturally in probability theory. Note that the construction above could analogously be done on the sphere. The resulting quasi-measure μ would then be translation invariant on solid sets and yet still not a measure. The "quasi" behavior of μ appears when the sets get larger.*

Example 3.2. *In the example above the observations are made in triples where we can split the triple into three independent variables. However experimental statisticians often face the problem of choosing a model for dependent observations. In the example above this can be illustrated by the three points being charged particles dropped simultaneously onto the disk. The standard approach is then to try to determine how the observations are dependent, an approach often without success (we might have incompatible experiments, for details see [4]). Again consider observations on the disk in triples. After a series of experiments one might find an estimate for μ in 3.1. For instance, $F_\nu(f)$ where ν is the Lebesgue measure and f is the function in Example 2.7 could be a suitable model. A statistical interpretation of this model would be that the three points are more likable to be further apart from each other than if they were independent. This can easily be seen by looking at the function $p(x) - f(x) = 3x^2 - 2x^3 - \sin^2(\frac{\pi}{2}x)$:*



Example 3.3 (Quasi-variable). *Given any continuous function $f : X \to [0,1]$ and quasi-measure $\mu \in Q(X)$ one obtain a probability measure μ_f on $[0,1]$ by $\mu_f(A) = \mu(f^{-1}(A))$ for all open or closed sets $A \subset [0,1]$. Let $X = D$ =unit disk, let μ be the quasi-probability in Example 3.1 and define $f : D \to [0,1]$ by $re^{i\theta} \mapsto r$. Then $\mu_f[0,r]$ is the probability of at least two points being within a radius r of the origin. Using the formula 3.1 with D_x being the disk with radius x we find that*

$$F_X(x) = \mu_f[0,x) = 3x^4 - 2x^6 \quad , x \in [0,1],$$

where $F_X(x)$ is the cumulative distribution function of a random variable X on $[0,1]$. Differentiating we find the Radon-Nikodym derivative $f_X(x) = 12x^3(1 - x^2)$ of μ_f with respect to the Lebesgue measure on $[0,1]$ (of course $f_X(x)$ is the probability distribution function of X). Now we may calculate the expectation of

X:

$$E(X) = \int_0^1 x f_X(x) dx = \frac{24}{35}.$$

But what is X? The variable X interpreted in this example is the radius of the second point from the origin. Of course this problem can be solved by ordinary probability theory considering the radius of each point as an independent variable and then calculate the distribution of the second largest one.

Remark 3. *The example above enables us to investigate the quasi-variable using ordinary variables since the problem splits into three independent variables. However one might verify that the calculations above can be done with μ as in Example 3.2, one of the essential properties being that any continuous map $f : D \to [0,1]$ maps μ onto a probability measure in $[0,1]$.*

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Quasi-measures, Image transformations and Self-similar Sets

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Abstract

This paper introduces a novel idea: the concept of an image transformation. We also introduce the closely related concept of a quasi-homomorphism, and study the properties of these mathematical objects, and give several examples. In particular we investigate iterated systems of image transformations, which we believe give a more realistic approach to the study of so called self-similar structures in nature than what is obtained by iterated function systems.

1 Introduction

The theory of chaos and fractals has been a subject of great interest for the last two decades. Hutchinson has given one of the major contributions with his work in [9]. This contribution has also been applied to commercial software for image compression, but the difficulties in finding effective general algorithms are evident. Our aim for the work presented here is to generalize the theory of iterated function systems.

Advocates of chaos and fractals have argued that self-similarity is frequently observed in a variety of objects. In particular we may observe such structures in plants, animals and scenery (e.g. clouds, coastlines, mountains). The mathematical model of the object is then given by a self-similar set. This model is however not completely correct, as the structures will not continue to repeat themselves forever, at some scale they break down and new structures may be formed (e.g. we cannot expect to find the same patterns in an actual fern if we zoom in indefinitely). This is also the case for any image, since the structures in any image of a fern will be limited by the resolution of the media used. The random iteration algorithm is often used to generate self-similar sets. More precisely, it generates the support of a Borel measure. Our approach will be similar, but we will allow the fixed point measure to be a quasi-measure (see the definition below).

Throughout the text we will assume that X is a compact Hausdorff space. We will let $\mathcal{O}(X)$ and $\mathcal{C}(X)$ denote the open and closed subsets of X respectively. Furthermore we put $\mathcal{A}(X) = \mathcal{O}(X) \cup \mathcal{C}(X)$. When there is no confusion concerning the space in question, we will omit the space from the notation. A positive set function $\mu : \mathcal{A}(X) \longrightarrow \mathbb{R}^+$ is a quasi-measure if

(i) $\mu(\biguplus_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu A_i$ (where $\biguplus$ indicates disjoint union, and all A_i and $\cup A_i$ are assumed to be in $\mathcal{A}$)

(ii) $\mu U = \sup\{\mu C : C \subset U, C \in \mathcal{C}\}$ for all $U \in \mathcal{O}$.

We denote the set of all quasi-measures in X by $Q(X)$. The quasi-measures originated in [1]. It was proved in [6] that quasi-measures are countably additive. Their definition only differs from that of regular Borel measures by their domain of definition. However, the main difference is that they constitute a vastly larger class of set functions with a rich mathematical structure.

We want our transformations to preserve quasi-measures, so ordinary measure preserving transformations will not suffice. This leads to the concept of an *image transformation* (for definition, see section 3). The image transformations turn out to provide a great variety of examples, where two classes are presented here. We present iterated image transformation systems as a generalization of iterated function systems. They will have a quasi-measure as a unique and attracting fix point. The most basic class of examples are the "collapsing" quasi-measures. They behave like ordinary Borel measures on large sets but vanish on small sets. The idea is that when a set is too small, we do not have any information. The second class of examples are by "volume measures". They are quasi-measures measuring the volume or area of an object depending on the resolution level chosen. The idea here is that an object might have one structure at one level but then a totally different structure on a smaller level (e.g. an object might look quite different in a microscope).

2 Basic results

An integration theory has been developed with respect to a quasi-measure. The integral is defined for all continuous functions $f \in C(X)$. Let $\mu \in Q(X)$, $f \in C(X)$; then defining $\mu_f(A) = \mu(f^{-1}(A)), A \in \mathcal{A}(\mathrm{sp}(f))$ yields a regular Borel measure μ_f in $\mathbb{R}$. In general, such combination with a continuous function maps quasi-measures to quasi-measures. However, on the real line all quasi-measures extend uniquely to Borel measures.

Definition 1 *Let $\mu \in Q(X)$ and $f \in C(X)$. Then we define*

$$\mu(f) = \int x d\mu_f(x)$$

where μ_f is the Borel measure given by $\mu_f(A) = \mu(f^{-1}(A)), A \in \mathcal{A}(X)$.

In the study of quasi-measures and integrals the singly generated subalgebras of $C(X)$ play a crucial part. For $f \in C(X)$ let A_f denote the uniformly closed subalgebra generated by f and the constant functions. By the spectral theorem this algebra is isomorphic to the continuous functions on the range of f.

Definition 2 *If a function $\rho : C(X) \to \mathbb{R}$ is a positive linear functional on A_f for every $f \in C(X)$ we call ρ a* quasi-integral.

The quasi-integrals were shown in [1] to be exactly the integrals with respect to quasi-measures. Furthermore, the Riesz representation theorem holds, so that the measures and integrals are in one-to-one correspondence. Quasi-integrals are also shown to be monotone i.e. $f \leq g$ implies $\rho(f) \leq \rho(g)$ and hence also uniformly continuous (c.f. [1])

The family of normalized quasi-measures is a convex set. Its extreme points are, however much more complex than the dirac measures. They are not in general $\{0,1\}$-valued.

Definition 3 *The* representable quasi-measures *is the convex closure of the $\{0,1\}$-valued quasi-measures, and will be denoted by $Q_r(X)$. The collection of $\{0,1\}$-valued quasi-measures will be denoted by X^*.*

Notation 4 *We will denote the Borel probability measures in X with $P(X)$. Its extreme points, the dirac measures will be denoted by $P_e(X)$. Moreover, the normalized quasi-measures in X (i.e. the quasi-measures where $\mu(X) = 1$) will be denoted by $Q_1(X)$.*

The quasi-integrals, and hence measures, were shown in [1] to be uniformly continuous. That is, for $\mu \in Q(X)$ we have $\|\mu\| = \mu(X)$, and $|\mu(f) - \mu(g)| \leq \|\mu\| \, \|f - g\|_\infty$ for all $f, g \in C(X)$. In [2] a weak topology for $Q_1(X)$ was introduced: Any function $f \in C(X)$ may be represented as a functional $\hat{f}$ on $Q(X)$ by $\hat{f}(\mu) = \mu(f)$. The topology on $Q(X)$ is defined to be the topology induced by the separating space of functionals $\{\hat{f} : f \in C(X)\}$. This turns $Q_1(X)$, $Q_r(X)$ and X^* into compact Hausdorff spaces. Moreover, the sets $V^* = \{\mu \in X^* : \mu(V) = 1\}$; $(V \in \mathcal{O}(X))$ is a subbasis for the topology on X^*.

Definition 5 *If a set $A \in \mathcal{A}$ and its complement are both connected we will call the set* solid. *A restriction to solid sets will be denoted with a subscript s (e.g. $\mathcal{C}_s$ will denote the compact solid sets).*

It was shown in [5] that $\{V^* : V \in \mathcal{O}_s(X)\}$ actually is a subbasis for the topology on X^*. The solid sets play an important role in the theory of quasi-measures. They constitute a small and manageable family of sets that totally determines a quasi-measure. This is illustrated by the solid set-functions, they were introduced in [3] and their properties were investigated there. In particular they are invaluable tools for constructing quasi-measures. We include their definition here for the convenience of the reader.

Definition 6 *We say that X has genus zero if $X = \biguplus_{i=1}^{n} A_i$; $(A_i \in \mathcal{A}_s(X)$; $i = 1, 2, ..., n)$ implies that $n \leq 2$.*

Remark 7 *The genus requirement was treated in [3] and [11]. When X has genus zero, then X can at most be the disjoint union of two solid sets. This property is shared by a large class of spaces (e.g. when X is simply connected).*

Definition 8 *If X is a connected, locally connected, compact Hausdorff space with genus zero, we call X a* q-space.

Let X be a q-space. Then a function $\mu : \mathcal{A}_s \to \mathbb{R}^+$ is a *solid set-function* if it satisfies

(A) $\sum_{i=1}^{n} \mu C_i \leq \mu C$ whenever $\biguplus_{i=1}^{n} C_i \subset C; C_i, C \in \mathcal{C}_s$ for $i = 1, 2, ..., n$

(B) $\mu U = \sup\{\mu C : C \subset U; C \in \mathcal{C}_s\}$ for all $U \in \mathcal{O}_s$

(C) $\mu A + \mu(X \backslash A) = \mu X$

Theorem 9 *If X is a q-space and μ is a solid set function, then μ extends uniquely to a quasi-measure in X. Conversely, the restriction of a quasi-measure in X to the solid sets is a solid set function.*

We have the following definition and Proposition from [12]:

Definition 10 *Let X_1 and X_2 be compact Hausdorff spaces. A map $f : X_1 \to X_2$ will be called a* solid variable *if f is continuous and $f^{-1}(\mathcal{A}_s(X_2)) \subset \mathcal{A}_s(X_1)$.*

Proposition 11 (Urysohn's lemma for solid variables) *Let X be any connected and locally connected compact Hausdorff space. If $C \in \mathcal{C}_s(X)$ and $F \in \mathcal{C}(X)$ are disjoint and non-empty, there is a solid variable $f : X \to [0,1]$ such that $f|_C \equiv 0$ and $f|_F \equiv 1$. If in addition X is a metric space we may assume that $f^{-1}(0) = C$.*

For our constructions the concept of non-splitting quasi-measures will be important:

Definition 12 *We say that a quasi-measure μ on X is splitting if there exists disjoint sets $C_1, C_2 \in \mathcal{C}_s(X)$ such that $\mu(C_1)+\mu(C_2) = 1$ with $\mu(C_1), \mu(C_2) > 0$. If no such pair exists we call μ* non-splitting.

Note that Lebesgue measure in the unit square is non-splitting.

3 Image transformations and quasi-homomorphisms

Definition 13 *Let X and Y be compact Hausdorff spaces. An image transformation is a map $q : \mathcal{A}(X) \to \mathcal{A}(Y)$ with $q(\mathcal{O}(X)) \subset \mathcal{O}(Y)$ satisfying*

a) $A \cap B = \emptyset \Rightarrow qA \uplus qB = q(A \cup B)$

b) $qX = Y$

c) $U_\lambda \nearrow U \Rightarrow qU_\lambda \nearrow qU$ whenever U_λ, $U \in \mathcal{O}(X)$ and $\lambda \in \Lambda$ where Λ is a directed set.

Note that the composition of two image transformations (when defined) is again an image transformation.

We include some immediate properties of image transformations below.

Proposition 14 *Let $q : \mathcal{A}(X) \to \mathcal{A}(Y)$ be an image transformation. The following hold:*

1. $q(\mathcal{C}(X)) \subset \mathcal{C}(Y)$
2. $A \subset B \Rightarrow q(A) \subset q(B)$; $A, B \in \mathcal{A}(X)$
3. *If $A_1, A_2, ..., A_n$ are mutually disjoint sets in $\mathcal{A}(X)$ whose union also belongs to $\mathcal{A}(X)$, then*

$$q(\bigcup_{i=1}^{n} A_i) = \bigcup_{i=1}^{n} q(A_i)$$

4. *If $K \subset q(U)$; $K \in \mathcal{C}(Y)$, $U \in \mathcal{O}(X)$ there is $C \subset U$; $C \in \mathcal{C}(X)$ such that $K \subset q(C) \subset q(U)$*
5. *If $q(C) \subset V$; $C \in \mathcal{C}(X)$, $V \in \mathcal{O}(Y)$ there is $U \supset C$; $U \in \mathcal{O}(X)$ such that $q(C) \subset q(U) \subset V$*
6. *If $C_\lambda \searrow C$ in $\mathcal{C}(X)$ then $q(C_\lambda) \searrow q(C)$ in $\mathcal{C}(Y)$*

Example 15 *Let $w : Y \to X$ be a continuous map and define*

$$q(A) = w^{-1}(A) = \{y \in Y : w(y) \in A\} \text{ for all } A \in \mathcal{A}(X)$$

Then q is an image transformation, and we say that q is derived from the function w.

Example 16 *Let $\Psi^* : \mathcal{A}(X) \to \mathcal{A}(X^*)$ be defined by $\Psi^*(A) = A^*$, where*

$$A^* = \{\mu \in X^* : \mu(A) = 1\}$$

Then Ψ^* is an image transformation. Moreover, if $X \neq X^*$ then Ψ^* is not derived from a function.

Definition 17 *Let $q : \mathcal{A}(X) \to \mathcal{A}(Y)$ be an image transformation. The* adjoint map *$q^* : Q(Y) \to Q(X)$ is given by $(q^*\mu)(A) = \mu(qA)$ for all $A \in \mathcal{A}(X)$, $\mu \in Q(X)$.*

It is straight forward to show that $q^*(\mu)$ is a quasi-measure. We leave this verification to the reader.

Example 18 *Let $\mu \in X^*$, and define $q : \mathcal{A}(X) \to \mathcal{A}(X)$ by*

$$q(A) = \begin{cases} \emptyset, & \mu(A) = 0 \\ X, & \mu(A) = 1 \end{cases} \quad ; \ A \in \mathcal{A}(X).$$

Then q is an image transformation. The adjoint q^* is the constant map $Q(X) \mapsto \{\mu\}$.

Let $\iota_Y : Y \to Y^*$ be the map which assigns to each point $y \in Y$ the corresponding Dirac measure μ_y. We may then formulate the *structure theorem for image transformations*:

Theorem 19 *There is a one-to-one correspondence between image transformations $q : \mathcal{A}(X) \to \mathcal{A}(Y)$ and continuous functions $w : Y \to X^*$ such that $q = w^{-1} \circ \Psi^*$, moreover $w = q^* \circ \iota_Y$.*

Proof. Let $q : \mathcal{A}(X) \to \mathcal{A}(Y)$ be an image transformation. It is immediate from the definition that q^* maps Y^* into X^*. Fix an arbitrary point $\mu_0 \in Y^*$ and let W be an open neighborhood of $q^*\mu_0$ in X^*. Then there are open sets $V_1, V_2, ,..., V_n$ in X such that $q^*\mu_0 \in V_1^* \cap V_2^* \cap \cdots \cap V_n^* \subset W$. Let $U_i = q(V_i)$, $i = 1, 2, ..., n$, and suppose $\mu \in U_1^* \cap U_2^* \cap \cdots \cap U_n^*$. Then $(q^*\mu)(V_i) = \mu(q(V_i)) = \mu(U_i) = 1 \Rightarrow q^*\mu \in W$, which establishes the desired continuity. It follows that $w = q^* \circ \iota_Y : Y \to X^*$ is continuous. Let $C \in \mathcal{C}(X)$ be arbitrary. Then $y \in q(C) \Leftrightarrow \mu_y(q(C)) = 1 \Leftrightarrow q^*\mu_y \in C^* \Leftrightarrow w(y) \in C^* \Leftrightarrow y \in w^{-1}(C^*) = w^{-1}(\Psi^*(C))$. Hence $q = w^{-1} \circ \Psi^*$. Then q is an image transformation of $\mathcal{A}(X)$ into $\mathcal{A}(Y)$. Let $y \in Y$ and $A \in \mathcal{A}(X)$ be arbitrary. We have $(q^*\mu_y)(A) = \mu_y(q(A)) = \mu_y(w^{-1}(A^*)) = 1 \Leftrightarrow y \in w^{-1}(A^*) \Leftrightarrow w(y) \in A^* \Leftrightarrow w(y)(A) = 1$. It follows that $q^* \circ \iota_Y = w$, and the proof is complete. ■

Corollary 20 *Let $q : \mathcal{A}(X) \to \mathcal{A}(Y)$ be an image transformation. the following are equivalent:*

(i) *q is derived from a continuous function $u : Y \to X$ (i.e. $q = u^{-1}$).*

(ii) $Y = Y_1 = \bigcup_{x \in X} q(\{x\})$

(iii) $q^*(P_e(Y)) \subset P_e(X)$

Proof. The implication (i)⇒(ii) is obvious. Suppose $y \in q(\{x\})$ for some $x \in X$ if $C \in \mathcal{C}(X)$ then $x \in C \Leftrightarrow q(\{x\}) \subset q(C)$, so $(q^*\mu_y)(C) = \mu_y(q(C)) = \mu_x(C)$. Hence $q^*\mu_y = \mu_x$, and (ii)⇒(iii). Now assume (iii) to hold. Then $u = \iota_X^{-1} \circ q^* \circ \iota_Y$ is a continuous function on Y into X, and $u^{-1} = w^{-1} \circ \iota_X$ in the notation Theorem 19. Since the range of w is contained in $P_e(X)$, we must have $w^{-1}(C^* \cap \iota_X(X)) = w^{-1}(\iota_X(C))$. It follows that $q = u^{-1}$. ■

The structure theorem provides the following useful construction:

Proposition 21 *Let X be a q-space. A map $q : \mathcal{A}_s(X) \to \mathcal{A}(Y)$ extends uniquely to an image transformation $\tilde{q} : \mathcal{A}(X) \to \mathcal{A}(Y)$ if and only if the following hold:*

a') If $C, C_i \in \mathcal{C}_s; i = 1, 2, ..., n$ and $\biguplus C_i \subset C$, then $\biguplus q(C_i) \subset qC$.

b') If $U, U_\lambda \in \mathcal{O}_s$ and $U_\lambda \nearrow U$, then $q(U_\lambda) \nearrow qU$ with $\lambda \in \Lambda$ where Λ is a directed set.

c') For any $A \in \mathcal{A}_s$ we have $q(A) \biguplus q(X \backslash A) = Y$.

Proof. If $\tilde{q}$ is an image transformation, then obviously a'), b') and c') hold. Conversely, for any $\mu \in Q(Y)$, define $q^*\mu : \mathcal{A}_s(X) \to \mathbb{R}$ by $(q^*\mu)(A) = \mu(q(A))$. Then $q^*\mu$ is a solid set function and hence extends uniquely to a quasi-measure on X. By Theorem 19 it suffices to show that $w = q^* \circ \iota_Y$ is continuous (any two image transformations agreeing on solid sets must be identical by uniqueness of extension of solid set functions). For $V \in \mathcal{O}_s(X)$ we have $q(V) = w^{-1}(V^*) \in \mathcal{O}(Y)$. Since $\{V^*\}_{V \in \mathcal{O}_s(X)}$ is a subbasis for the topology of X^*, it follows that w is continuous. ■

An image transformation may be applied to level sets of functions. This leads to the concept of a quasi-homomorphism:

Definition 22 *A function $r : C(X) \to C(Y)$ is a* quasi-homomorphism *if r is an algebra homomorphism of A_f onto $A_{r(f)}$ for each $f \in C(X)$.*

Specifically this means that $r(1_X) = 1_Y$, $r(f) \geq 0$ if $f \geq 0$ and that r is multiplicative and linear on each singly generated subalgebra A_f ; $f \in C(X)$.

We recall the following result from [2] (Proposition 4.1):

Proposition 23 *Let $\Psi : C(X) \to C(X^*)$ be given by $\Psi(f) = \hat{f}$, where $\hat{f}(\mu) = \mu(f)$; $\mu \in X^*$. Then Ψ is an algebra isomorphism of A_f onto $A_{\hat{f}}$ for each $f \in C(X)$. Moreover, Ψ is an order-preserving isometry with closed range $\mathbf{B} \subset C(X^*)$.*

Remark 24 *In particular, Ψ is an injective quasi-homomorphism.*

Definition 25 *Let $r : C(X) \to C(Y)$ be a quasi-homomorphism. The* adjoint *of r is the map $r^* : Q(Y) \to Q(X)$ given by $(r^*\mu)(f) = \mu(r(f))$; $(f \in C(X), \mu \in Q(Y))$.*

Theorem 26 *There is a one-to-one correspondence between quasi-homomorphisms $r : C(X) \to C(Y)$ and algebra homomorphisms $h : C(X^*) \to C(Y)$ such that $r = h \circ \Psi$ and $h(f) = f \circ r^* \circ \iota_Y$; $(f \in C(X))$.*

Proof. Let $r : C(X) \to C(Y)$ be a quasi-homomorphism. By the definition of the topology on quasi-measures, r^* is continuous on $Q(Y)$ into $Q(X)$. Since r is multiplicative on each singly generated subalgebra $A(f)$, it follows that r^* maps Y^* into X^*. Hence $u = r^* \circ \iota_Y$ is a continuous map of Y into X^*. Composition $g \mapsto g \circ u$; $(g \in C(X^*))$ then defines an algebra homomorphism $h : C(X^*) \to C(Y)$. Let $f \in C(X)$. Then $r(f)(y) = \mu_y(r(f)) = (r^*\mu_y)(f) = u(y)(f) = \hat{f}(u(y))$ for each $y \in Y$. Hence $r(f) = \hat{f} \circ u = h(\hat{f}) = h(\Psi(f))$, i.e. $r = h \circ \Psi$, which proves the assertion one way. Conversely, let $h : C(X^*) \to C(Y)$ be an algebra homomorphism, and define $r = h \circ \Psi$. Then r is a quasi-homomorphism of $C(X)$ into $C(Y)$. There is a unique continuous function $u : Y \to X^*$ such that $h(g) = g \circ u$; $(g \in C(X^*))$. Hence $r(f) = \hat{f} \circ u$; $(f \in C(X))$. Let $y \in Y$ be arbitrary. Then $u(y)(f) = \hat{f}(u(y)) = (\hat{f} \circ u)(y) = r(f)(y) = \mu_y(r(f)) = (r^*\mu_y)(f)$ which implies that $u(y) = (r^* \circ \iota_Y)(y)$ for all $y \in Y$, and consequently that $u = r^* \circ \iota_Y$. The assertion now follows. ■

On the basis of the two structure theorems 19 and 26 it is clear that there is a one-to-one correspondence between image transformations $q : \mathcal{A}(X) \to \mathcal{A}(Y)$ and quasi-homomorphisms $r : C(X) \to C(Y)$, since they are both uniquely determined by the set of continuous functions from Y to X^*. We make this correspondence more explicit below:

Proposition 27 *There is a one-to-one correspondence between image transformations $q : \mathcal{A}(X) \to \mathcal{A}(Y)$ and quasi-homomorphisms $r : C(X) \to C(Y)$ such that if $q \leftrightarrow r$, then*

1. $r(f)(y) = (q^*\mu_y)(f)$; $(f \in C(X), y \in Y)$
2. $q(C) = \bigcap_{f \succ C}\{y \in Y : r(f)(y) = 1\}$; $(C \in \mathcal{C}(X))$
3. $q(f^{-1}(D)) = (r(f))^{-1}(D)$; $(f \in C(X)$, *D open or closed in* $\mathbb{R}\}$

Moreover, $r^* = q^*$

Proof. Let $w : Y \to X^*$ be a continuous function. Define $q = w^{-1} \circ \Psi^*$ and $r = h \circ \Psi$, where $h(g) = g \circ w$; $(g \in C(X^*))$. Then $q \leftrightarrow r$, and (1.) follows from Theorem 19 and Theorem 26 which yields $w = q^* \circ \iota_Y = r^* \circ \iota_Y$. Next, let $C \in \mathcal{C}(X)$ be arbitrary. By definition $q(C) = w^{-1}(C^*) = w^{-1}(\bigcap_{f \succ C}\{\mu \in X^* : \mu(f) = 1\}) = \bigcap_{f \succ C} w^{-1}(\{\mu \in X^* : \mu(f) = 1\}) = \bigcap_{f \succ C} w^{-1}(\{y \in Y : w(y)(f) = 1\})$. Since $r(f) = \hat{f} \circ w$, the relation (2.) follows. To establish (3.), let D be an arbitrary open or closed set in $\mathbb{R}$, f an arbitrary element in $C(X)$. Then we have $(f^{-1}(D))^* = \hat{f}^{-1}(D)$. Hence $q(f^{-1}(D)) =$

$w^{-1}((f^{-1}(D))^*) = w^{-1}(\hat{f}^{-1}(D)) = (\hat{f} \circ w)^{-1}(D) = (r(f))^{-1}(D)$ as desired. It remains to verify that $r^* = q^*$. It suffices to show that $(r^*\mu)(f) = (q^*\mu)(f)$; $(f \in C(X), \mu \in Y^*)$ or explicitly $\int x d\mu_{r(f)}(x) = \int x d(q^*\mu)_f(x)$. For any set $D \in \mathcal{A}(\mathbb{R})$ we have by (3.): $(q^*\mu)_f(D) = (q^*\mu)(f^{-1}(D)) = \mu(q(f^{-1}(D))) = \mu((r(f)^{-1})(D)) = \mu_{r(f)}(D)$ and the assertion follows. ■

Lemma 28 *Let $r : C(X) \to C(Y)$ be a quasi-homomorphism. If $f, g \in C(X)$ then*

1. $f \leq g \Rightarrow r(f) \leq r(g)$
2. $\|r(f) - r(g)\|_\infty \leq \|f - g\|_\infty$

Proof. For $y \in Y$, $f \leq g$, we get $r(g)(y) - r(f)(y) = (r^*\mu_y)(g) - (r^*\mu_y)(f) \geq 0$, so (1.) follows. For arbitrary $f, g \in C(X)$ we have $|r(f)(y) - r(g)(y)|$ $|(r^*\mu_y)(f) - (r^*\mu_y)(g)| \leq \|r^*\mu_y\| \, \|f - g\|_\infty = \|f - g\|_\infty$, and (2.) follows. ■

Proposition 29 *Let $r : C(X) \to C(Y)$ be a quasi-homomorphism, and let q be the corresponding image transformation. The following are equivalent:*

1. *$q(\{x\}) \neq \emptyset$ for all $x \in X$.*
2. *q is one-to-one.*
3. *r is an isometry.*
4. *r is one-to-one.*

Proof. Assume (1.) and let $A \neq B$; $(A, B \in \mathcal{A}(X))$. There is an $x \in A \backslash B$ and since q is monotone we have $\emptyset \neq q(\{x\}) \subset q(A) \backslash q(B)$. Hence $q(A) \neq q(B)$ and (2.) follows. Obviously (2.)$\Rightarrow$(1.), so we get (1.)$\Leftrightarrow$(2.) We next prove (2.)$\Rightarrow$(3.). Let $f, g \in C(X)$ and choose $x \in X$ so that $\|f - g\|_\infty = |f(x) - g(x)|$. Assuming (1.), there is an element $y \in q(\{x\})$ so we get $\mu_x = q^*\mu_y = r^*\mu_y$. Hence

$$\|f - g\|_\infty = |(r^*\mu_y)(f) - (r^*\mu_y)(g)| = |r(f)(y) - r(g)(y)| \leq \|r(f) - r(g)\|_\infty .$$

Combined with the preceding lemma this yields (3.) The implication (3.)$\Rightarrow$(4.) is trivial, so it remains to show that (4.)$\Rightarrow$(1.). Suppose that for some $x \in X$ we have $q(\{x\}) = \emptyset$. Let $\{V_\lambda\}_{\lambda \in \Lambda}$ be a basis for neighborhoods around x. We may assume Λ to be a directed index set and that $\bigcap_{\lambda \in \Lambda} \bar{V}_\lambda = \{x\}$, i.e. $\bar{V}_\lambda \searrow \{x\}$. Then $q(\bar{V}_\lambda) \searrow q(\{x\}) = \emptyset$, so by compactness there is a V_λ

such that $q(\bar{V}_\lambda) = \emptyset \Rightarrow q(V_\lambda) = \emptyset$. Let $f \in C(X)$satisfy $\{x\} \prec f \prec V_\lambda$. Then $f > 0$ so by assumption $r(f) > 0$, i.e. there is an $\alpha > 0$ such that $\emptyset \neq (r(f))^{-1}(\alpha, \infty) = q(f^{-1}(\alpha, \infty)) \subset q(V_\lambda) = \emptyset$ so we have a contradiction. Hence $q(\{x\}) \neq \emptyset$ for all $x \in X$ and the proof is complete. ■

Proposition 30 *Let $r : C(X) \to C(Y)$ be a quasi-homomorphism, and let q be the corresponding image transformation. The following are equivalent:*

1. *r is linear.*
2. *r is an algebra homomorphism.*
3. *$r^*(P_e(Y)) \subset P_e(X)$.*
4. *q is derived from a continuous function $u : Y \to X$ (i.e. $q = u^{-1}$).*

Proof. (1.)⇒(2.): Let $f, g \in C(X)$. We have $fg = \frac{1}{4}[(f+g)^2 - (f-g)^2]$. If r is a linear quasi-homomorphism it now follows by simple algebra that $r(fg) = r(f)r(g)$, i.e. r is an algebra homomorphism.

(2.)⇒(3.) is classical, and (3.)⇒(1.) is trivial. (3.)⇔(4.) follows by Corollary 20 since $r^* = q^*$. ■

Proposition 31 *Let $r : C(X) \to C(Y)$ be a bijective quasi-homomorphism. Then r is an algebra homomorphism.*

Proof. By the preceding result we only have to verify that $r^*(P_e(Y)) \subset P_e(X)$. Since r is bijective it has an inverse $s = r^{-1} : C(Y) \to C(X)$ which is a quasi-homomorphism. It is easily verified that $s^*r^* = \mathrm{id}_{Q(Y)}$ and $r^*s^* = \mathrm{id}_{Q(X)}$. Let $y \in Y$ and suppose $r^*\mu_y = \sigma \in X^*$, i.e. $s^*\sigma = \mu_y$. Let p be the image transformation corresponding to s. Then $\sigma(p(\{y\})) = (p^*\sigma)(\{y\}) = (s^*\sigma)(\{y\}) = \mu_y(\{y\}) = 1 \Rightarrow p(\{y\}) \neq \emptyset$. But then, is $x \in p(\{y\})$ we have $s^*\mu_x = p^*\mu_x = \mu_y$. Since s^* is one-to-one we must have $\sigma = \mu_x \in P_e(X)$ which proves the assertion. ■

4 Iterated image transformation systems

To study iterated image transformation systems more generally we need a proper framework. The adjoint operation on image transformations is a contravariant functor (the image transformation mimics the inverse image of a map). We will study the dynamics of the image transformation by investigating the adjoint.

In this section we will assume that X is a compact metric space, equipped with a metric d. This enables us to introduce the *Hutchinson metric* d_H on $Q_1(X)$, analogous to the metric on $P(X)$ as given in [9]:

Notation 32 *Let $d : X \times X \to \mathbb{R}^+$ be a metric on X. For any $r \geq 0$; $(r \in \mathbb{R})$ we put $L_r = \{f \in C(X) : |f(x) - f(y)| \leq rd(x,y)\ \forall x, y \in X\}$.*

Definition 33 *The Hutchinson metric on $Q_1(X)$ is defined by*

$$d_H(\mu, \mu') = \sup_{f \in L_1} \{\mu(f) - \mu'(f)\}$$

For the convenience of the reader we include a proof of the following:

Proposition 34 *d_H is a metric on $Q_1(X)$, and the topology it induces coincides with the weak topology.*

Proof. The only non-trivial property to verify, for d_H to be a metric, is that $\mu \neq \mu' \Rightarrow d_H(\mu, \mu') > 0$. If $\mu \neq \mu'$ we may assume that $\mu'(C) - \mu(C) = \alpha > 0$ for some set $C \in \mathcal{C}(X)$. There is an open set $U \supset C$ such that $\mu(U) < \mu(C) + \alpha$. We claim that there is a real number $r > 0$ and a function $f \in L_r$ such that $C \prec f \prec U$. Assuming this to be true, suppose $d_H(\mu, \mu') = 0$. Then $\mu(g) = \mu'(g)$ for all $g \in L_1$, and consequently μ and μ' also agree on L_r, so that $\mu(f) = \mu'(f)$. But then $\mu'(C) \leq \mu'(f) = \mu(f) \leq \mu(U) < \mu(C) + \alpha$ which is a contradiction. To establish the claim, let $K = X \backslash U$ and define $g(x) = d(x, K) = \inf_{y \in K}\{d(x,y)\}$. Then $g \in L_1$, and $g(x) > 0$ on C. Let $m = \min_{x \in C}\{g(x)\}$ and put $f = \frac{1}{m} g \wedge 1$. Then $0 \leq f \leq 1$, $f = 0$ on K, $f = 1$ on C and $f \in L_r$ with $r = \frac{1}{m}$. This verifies the claim, and shows that d_H is a metric in $Q_1(X)$.

It remains to show that the topology induced by the metric coincides with the weak topology on $Q_1(X)$. It suffices to show that the identity map id: $(Q_1(X), w) \to (Q_1(X), d_H)$ is continuous, since $Q_1(X)$ is weakly compact. By definition L_1 is equicontinuous and therefore by the Ascoli-theorem totally bounded in the uniform topology. Fix an arbitrary element $\mu_0 \in Q_1(X)$ and let $\varepsilon > 0$ be given. There are elements $f_1, f_2, ..., f_n$ in L_1 such that $L_1 \subset \bigcup_{i=1}^{n} N(f_i, \frac{\varepsilon}{3})$ where $N(g, \delta) = \{f \in C(X) : \|g - f\|_\infty \geq \delta\}$. Let $W = \{\mu \in Q_1(X) : |\mu(f_i) - \mu_0(f_i)| < \frac{\varepsilon}{3}; i = 1, 2, ..., n\}$. Suppose $\mu \in W$ and let $f \in L_1$ be arbitrary. We have $f \in N(f_i, \frac{\varepsilon}{3})$ for some $i \in \{1, 2, ..., n\}$ and consequently:

$$\begin{aligned} |\mu(f) - \mu_0(f)| &\leq |\mu(f) - \mu(f_i)| + |\mu(f_i) - \mu_0(f_i)| + |\mu_0(f_i) - \mu_0(f)| \\ &\leq \|f - f_i\|_\infty + \tfrac{\varepsilon}{3} < \varepsilon \end{aligned}$$

where we have used the uniform continuity of the quasi-integral. It follows that $d_H(\mu, \mu_0) \leq \varepsilon$ which concludes the argument. ■

Remark 35 *The metric d_H is the usual Hutchinson metric on $P(X)$ extended to $Q_1(X)$. With the proposition below this allows us to exploit the already well developed theory for iterated function systems.*

For non-void sets $C_1, C_2 \in \mathcal{C}(X)$ we define $\delta(C_1, C_2) = \min\{d(x,y) : x \in C_1, y \in C_2\}$. Note that δ is not a metric, but $\delta(C_1, C_2) > 0$ if and only if $C_1 \cap C_2 = \emptyset$.

Proposition 36 *Let $q : \mathcal{A}(X) \to \mathcal{A}(X)$ be an image transformation, let $w = q^* \circ \iota_X$, and let $s \geq 0$ be a real number. The following are equivalent:*

1. *$\delta(C_1, C_2) \leq s\delta(q(C_1), q(C_2))$ whenever $C_1, C_2 \in \mathcal{C}(X)$ and $q(C_1), q(C_2)$ are both non-void.*

2. *$q(L_1) \subset L_s$.*

3. *$d_H(w(x), w(y)) \leq sd(x,y)$ whenever $x, y \in X$.*

4. *$d_H(q^*\mu, q^*\nu) \leq s\, d_H(\mu, \nu)$ for all $\mu, \nu \in Q(X)$*
   The smallest s such that all conditions hold is called the contractivity factor of q.

Proof. (1.)$\Rightarrow$(2.): Suppose (1.) holds, let $f \in L_1$ and $y_1, y_2 \in X$ be arbitrary. Suppose $(qf)(y_i) = \alpha_i$, $i = 1, 2$. By Proposition 27.3 $q(f^{-1}(\{\alpha_i\}) = (qf)(\{\alpha_i\}) \neq \emptyset$, so that by (1.)

$$\delta(f^{-1}(\{\alpha_1\}), f^{-1}(\{\alpha_2\})) \leq s\delta(q(f^{-1}(\{\alpha_1\})), q(f^{-1}(\{\alpha_2\}))) \leq sd(y_1, y_2).$$

Since d is continuous on $X \times X$ there are points $x_i \in f^{-1}(\{\alpha_i\})$; $i = 1, 2$, such that $d(x_1, x_2) \leq sd(y_1, y_2)$, which proves (2.).

(2.)$\Rightarrow$(4.): Assume (2.) to hold, let $\lambda, \mu \in Q(X)$, $f \in L_1$ be given. Suppose first that $s > 0$. Then $s^{-1}q(f) \in L_1$ and consequently:

$$\begin{aligned}(q^*\lambda)(f) - (q^*\mu)(f) &= \lambda(q(f)) - \mu(q(f)) = s(\lambda(s^{-1}qf) - \mu(s^{-1}qf)) \\ &\leq sd_H(\lambda, \mu).\end{aligned}$$

If $s = 0$ the $q(f) = k$ -a constant function, so $(q^*\lambda)(f) - (q^*\mu)(f) = \lambda(k) - \mu(k) = 0$.

(4.)$\Rightarrow$(3.): We first observe that $d_H(\mu_x, \mu_y) = d(x,y)$ whenever $x, y \in X$. Indeed, if $f \in L_1$ then $\mu_x(f) - \mu_y(f) = f(x) - f(y) \leq d(x,y)$, showing that $d_H(\mu_x, \mu_y) \leq d(x,y)$. Let $g(x') = d(x', y)$; $x' \in X$. Then $g \in L_1$ and $\mu_x(g) - \mu_y(g) = d(x,y)$, so $d_H(\mu_x, \mu_y) = d(x,y)$.

(3.)⇒(1.): Let δ_H denote the distance between sets in $\mathcal{C}(X^*)$ with respect to the metric d_H, i.e. $\delta_H(F,G) = \inf_{\lambda \in F, \mu \in G}\{d_H(\lambda,\mu)\}$; $(F, G \in \mathcal{C}(X^*))$. We claim that $\delta_H(C^*, K^*) = \delta(C, K)$; $(C, K \in \mathcal{C}(X^*))$:

Since $d_H(\mu_x, \mu_y) = d(x,y)$; $(x, y \in X)$ we must have $\delta_H(C^*, K^*) \leq \delta(C, K)$. To show the opposite inequality we may assume that $C \cap K = \emptyset$, otherwise there is nothing to prove. Let $f(x) = d(x, C)$, so $f \in L_1$. Moreover $f = 0$ on C and $0 < \alpha = \delta(K, C) \leq f(x)$ for $x \in K$. Hence $K^f_\alpha = \{x : f(x) \leq \alpha\} \supset K$. Let $\lambda \in C^*$ and $\mu \in K^*$ be arbitrary. Then $\mu(K) = 1 \Rightarrow \mu(K^f_\alpha) = 1 \Rightarrow \mu(f) \geq \alpha$. For any $\beta > 0$ we have $K^f_\beta \cap C = \emptyset$, which implies that $\lambda(K^f_\beta) = 0$ for all $\beta > 0$. But then $\lambda(f) = 0$. Therefore $\mu(f) - \lambda(f) \geq \alpha = \delta(C, K) \Rightarrow d_H(\lambda, \mu) \geq \delta(C, K) \Rightarrow \delta_H(C^*, K^*) \geq \delta(C, K)$ since $\lambda \in C^*, \mu \in K^*$ were arbitrary. This establishes the claim.

Now assume (3.) to hold, let $C_1, C_2 \in \mathcal{C}(X)$ and suppose $q(C_1), q(C_2)$ to be non-void. Pick $y_i \in C_i$; $i = 1, 2$. By (3.) we get $d_H(w(y_1), w(y_2)) \leq sd(y_1, y_2)$. Now $w(y_i)(C_i) = 1$, so $w(y_i) \in C_i^*$; $i = 1, 2$, and consequently $\delta_H(C_1^*, C_2^*) \leq sd(y_1, y_2)$. Combining this with the previous claim we get $\delta(C_1, C_2) \leq sd(y_1, y_2)$. Since y_1, y_2 were arbitrary, (1.) follows. ∎

Remark 37 *Since image transformations are often given by their values on solid sets, one might hope it would suffice to verify (1.) for solid sets only. This is not generally sufficient, as example 48 will show. However, for many metric q-spaces it is sufficient to check (1.) for solid sets. We will prove this for q-spaces whose metric is given by the length of a (geodesic) path between the points, which includes spheres as well as convex subsets of* $\mathbb{R}^n$.

Definition 38 *If X is a metric q-space where the metric is given by the length of a (geodesic) path between points, we will call X a* convex q-space.

Theorem 39 *Let X be a metric q-space.*

If an image transformation $q : \mathcal{A}(X) \to \mathcal{A}(X)$ satisfies 36.1 for C_1, C_2 connected, then q has contractivity factor s.

If X is a convex q-space, then it is sufficient that 36.1 is satisfied for C_1, C_2 solid.

Proof. In the following, let C_1, C_2 be disjoint compact sets with $q(C_1)$, $q(C_2)$ nonempty.

First, assume that 36.1 holds for solid compact sets, and that X is a convex q-space. Let C_1, C_2 be connected. Let O_1 be the (solid open) component of $X \setminus C_1$ containing C_2, and similarly for O_2. Then $H_1 = X \setminus O_1$ and $H_2 = X \setminus O_2$ are disjoint solid compacts containing C_1 and C_2, respectively.

Let $x_1 \in H_1$, $x_2 \in H_2$ be such that $d(x_1, x_2) = \delta(H_1, H_2)$, and let $p : [0,1] \to X$ be a geodesic path from x_1 to x_2. Then $p((0,1))$ is disjoint from both H_1 and H_2, and therefore x_1 is contained in the boundary of H_1, which is itself a subset of the boundary of C_1. We therefore have that $x_1 \in C_1$, and similarly, $x_2 \in C_2$. From this we get $\delta(H_1, H_2) = \delta(C_1, C_2)$.

Since H_1, H_2 are solid, we have (by assumption)

$$\delta(C_1, C_2) = \delta(H_1, H_2) \leq s\delta(q(H_1), q(H_2)) \leq s\delta(q(C_1), q(C_2)).$$

Secondly, we no longer assume that X is convex, but we instead assume that 36.1 holds for any connected compact sets. Let C_1, C_2 have a finite number of components $C_1^1, C_1^2, \ldots, C_1^{n_1}$ and $C_2^1, C_2^2, \ldots, C_2^{n_2}$ respectively, $n_1, n_2 = 1, 2, \ldots$. Then

$$\begin{aligned}
\delta(C_1, C_2) &= \min\{\delta(C_1^i, C_2^j) | i = 1, 2, \ldots, n_1, \, j = 1, 2, \ldots, n_2\} \\
&\leq \min\{s\delta(q(C_1^i), q(C_2^j)) | i = 1, 2, \ldots, n_1, \, j = 1, 2, \ldots, n_2\} \\
&= s\delta(q(C_1), q(C_2)).
\end{aligned}$$

Finally, assume that 36.1 holds for compact sets with a finite number of components, and let C_1, C_2 be arbitrary compact.

For each $n = 1, 2, \ldots$, $i = 1, 2$ let $O_i^n = \{x \in X | d(x, C_i) < 1/n\}$. For each $n = 1, 2, \ldots$, choose C_1^n, C_2^n to be compact sets with a finite number of components, $O_i^{n+1} \subseteq C_i^n \subseteq O_i^n$. This is always possible since a finite set of components of O_i^n must cover the compact subset $\overline{O_i^{n+1}}$.

Since $C_i^n \searrow C_i$, we have $q(C_i^n) \searrow q(C_i)$, and so

$$\delta(C_1, C_2) = \lim_{n \to \infty} \delta(C_1^n, C_2^n) \leq \lim_{n \to \infty} s\delta(q(C_1^n), q(C_2^n)) = s\delta(q(C_1), q(C_2)).$$

■

Definition 40 *Let $q_1, q_2, \ldots, q_n$ be image transformations each with contractivity factor smaller than or equal to $s < 1$. An image transformation system is a map $M : Q_1(X) \to Q_1(X)$ defined by $M(\mu) = \sum_{i=1}^n \alpha_i q_i^*(\mu)$ where $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$ and $\sum_{i=1}^n \alpha_i = 1$.*

Proposition 41 *An image transformation system M is a contraction on $(Q_1(X), d_H)$ with contractivity factor s and hence has a unique (attracting) fix point μ_M in $Q_1(X)$.*

Proof. Let $\lambda, \mu \in Q_1(X)$, $f \in L_1$ be arbitrary. Then

$$\begin{aligned}
|(M\lambda)(f) - (M\mu)(f)| &= |\textstyle\sum_{i=1}^n \alpha_i (q_i^*\lambda)(f) - \sum_{i=1}^n \alpha_i (q_i^*\mu)(f)| \\
&\leq \textstyle\sum_{i=1}^n \alpha_i |\lambda(q_i f) - \mu(q_i f)|.
\end{aligned}$$

By virtue of Proposition 36 $q_i f \in L_s$. If $s = 0$ this implies that the expression to the right above is equal to zero. If $s > 0$ then $s^{-1}(q_i f) \in L_1$; $(i = 1, 2, \ldots, n)$, the above is $\leq s \sum_{i=1}^n \alpha_i d_H(\lambda, \mu) = s d_H(\lambda, \mu)$. The assertion now follows. ■

Remark 42 *The approach chosen here is similar to the random iteration algorithm, obtaining a fixed quasi-measure as the attractor. It is possible to draw this analogy further since* $\{q_1^*, q_2^*, ..., q_n^*; X^*\}$ *is an iterated function system with a corresponding probability measure* $\mu \in P(X^*)$*. However, this leads to an attractor in* $C(X^*)$*. For modelling purposes this is not favorable since* X^* *is an abstract object. Still interpreting a quasi-measure as an image is not straight forward. The support of a quasi-measure is not appropriate for our purposes (see [7] for a treatment of the support).*

Proposition 43 μ_M *is a representable quasi-measure.*

Proof. An image transformation necessarily maps X^* into X^*. In addition image transformations preserve convex combinations and are continuous. With the representable quasi-measures being a compact they must be invariant under the image transformations. Hence iterating any representable measure by M must converge to a representable measure, which must be μ_M. ■

Remark 44 *In view of Proposition 43, the representable quasi-measures are our object of interest. Even more so the proposition below ensures that we may approximate any representable quasi-measure by image transformation systems.*

Proposition 45 *Whenever* $\mu \in Q_r(X)$ *there is a net of image transformation systems whose invariant quasi-measures converge weakly to* μ*.*

Proof. Since μ is representable there is by definition a net of quasi-measures $\{\mu_\lambda\}$ converging weakly to μ where each μ_λ is a finite convex combination of elements in X^*. It suffices to construct an iterated image transformation system whose invariant quasi-measure is μ_λ. Assume $\mu_\lambda = \sum_{i=1}^n \alpha_i \mu_i$ with $\mu_i \in X^*, \alpha_i \geq 0$ for $i = 1, 2, ..., n$ and $\sum_{i=1}^n \alpha_i = 1$. By Theorem 19 there are constant image transformations $q_i^* : M(X) \longmapsto \mu_i$ (these image transformations were given explicitly in Example 18). We have the desired image transformation system $M_\lambda = \sum_{i=1}^n \alpha_i q_i^*$ whose contraction factor is zero. ■

Example 46 *Let* $X = [0,1] \times [0,1]$*, and let* $r > 0$*. For each* $x \in X$ *let* D_x *denote the disk with radius* r *around* x*. Let* C_x *be the boundary of* $D_x \cap X$ *in* $\mathbb{R}^2$*.* C_x *is a simple closed curve in* X*, and we let* σ_x *denote the "circle-measure" associated with* C_x*, i.e. the quasi-measure determined by*

$$\sigma_x(C) = \begin{cases} 1 & \text{if } C_x \subset C \\ 1 & \text{if } x \in C \text{ and } C_x \cap C \neq \emptyset \\ 0 & \text{elsewhere} \end{cases} \; ; (C \in C_s(X))$$

We may note that if $x \in \partial X$ then σ_x coincides with the pointmass at x. The map $w : x \mapsto \sigma_x$ of X into X^* is continuous, so we may define an image transformation by $q = w^{-1} \circ \Psi^*$. First observe that if $C \in \mathcal{C}_s(X)$, $C \cap \partial X = \emptyset$ and $\operatorname{diam}(C) = \sup\{d(x,x') : x, x' \in C\} < r$, then $\sigma_x(C) = 0$. Now any set $K \subset X \backslash \partial X$ with $\operatorname{diam}(K) < \frac{r}{2}$ is contained in a circular disk D with diameter $< r$. It follows that $\sigma_x(K) = 0$ for all such sets and all $x \in X$. Consequently we have $q(K) = \emptyset$ for such sets. On the other hand, if K is connected and $\operatorname{diam}(K) \geq r$, there is an $x \in X$ such that $\sigma_x(K) = 1$, which implies that $q(K) \neq \emptyset$. This of course just says that connected sets that are disjoint from the boundary of X must have a certain size to be recognized by q. Similarly, the ability of q to distinguish between two sets depends on their separation. For suppose C_1 and C_2 are connected and compact, both have diameter $\geq 2r$ and $\mathrm{h}(C_1, C_2) > r$ (h denotes the Hausdorff metric in $\mathcal{C}(X)$). Then there is a point $x \in C_1$ (or vice versa) such that $\mathrm{d}(x, C_2) > r$. Now C_1 is connected and $\operatorname{diam}(C_1) \geq 2r$ so there is a point x' in C_1 such that $\mathrm{d}(x,x') = r$. Then $x' \notin C_2$, and it follows that there is an element x such that $\sigma_x(C_1) = 1$ and $\sigma_x(C_2) = 0$. But then $q(C_1) \neq q(C_2)$.

The quasi-measures are not in general subadditive, so quasi-measures may vanish on small sets. The example above suggests that it is possible to model the collapse of a self similar structure by image transformations. What seems to be a fundamental image transformation to achieve this is given by the following lemma:

Lemma 47 *Let X be a q-space, let $0 \leq \varepsilon \leq \frac{1}{2}$ and let m be a non-splitting Borel probability measure on X. Now define $q_\varepsilon : \mathcal{C}_s(X) \to \mathcal{C}_s(X)$ by*

$$q_\varepsilon(C) = \begin{cases} \emptyset \; ; & m(C) < \varepsilon \\ C \; ; & \varepsilon \leq m(C) < 1 - \varepsilon \\ X \; ; & m(C) \geq 1 - \varepsilon \end{cases}$$

Then q_ε extends uniquely to an image transformation $q_\varepsilon : \mathcal{A}(X) \to \mathcal{A}(X)$.

Moreover, if X is a convex q-space, then q_ε has contractivity factor $= 1$.

Proof. By Proposition 21 it suffices to check a'), b') and c') for solid sets. Clearly q_ε is monotone on $\mathcal{C}_s(X)$. Extending q to open solid sets by complement we trivially get c'). Let $\biguplus C_j \subset C \in \mathcal{C}_s(X)$ be a finite union of compact solid sets. If $mC < \varepsilon$ then $\biguplus q_i(C_j) = \emptyset \subset q_i(C)$. When $mC_i < 1 - \varepsilon$ for all indices and $mC \geq \varepsilon$ we get $\biguplus q_i(C_j) \subset \biguplus C_j \subset q_i(C)$. If $m(C_i) \geq 1 - \varepsilon$ for one i, then $m(C) \geq 1 - \varepsilon$ and $q(C) = X$ which concludes a'). Now let $U_i \nearrow U$ be open solid sets. The regularity of m assures that if

$m(U) > 1-\varepsilon$ then $m(U_i) > 1-\varepsilon$ for some i (i.e. $q(U_i) = X$) and hence for all following indices. Accordingly $q(U_i) \nearrow X = q(U)$. Finally if $m(U) \leq \varepsilon$ then $q(U_i) = \emptyset = q(U)$ for all i. We have shown that q satisfies the requirements and hence extends uniquely to an image transformation by Proposition 21.

It is easily seen that q_ε has contractivity factor $s = 1$ on solid compact sets. If X is a convex q-space then theorem 39 applies. ∎

Example 48 *Let X be the compact metric subspace of the complex numbers given by*

$$X = \{re^{i\theta} | 1 \leq r \leq \sqrt{3},\, 0 \leq \theta \leq 6\}.$$

This space is homeomorphic to a disk and so is a q-space, but it is not convex.

If ε is sufficiently small, then the compact sets

$$\begin{aligned} C_1 &= \{re^{i\theta} | 1 \leq r \leq \sqrt{3},\, \varepsilon/2 \leq \theta \leq 3\varepsilon/2\}, \\ q_\varepsilon(C_1) &= \{re^{i\theta} | 1 \leq r \leq \sqrt{3},\, 0 \leq \theta \leq 3\varepsilon/2\}, \\ C_2 &= \{re^{i\theta} | 1 \leq r \leq \sqrt{3},\, 6 - 3\varepsilon/2 \leq \theta \leq 6 - \varepsilon/2\}, \\ q_\varepsilon(C_2) &= \{re^{i\theta} | 1 \leq r \leq \sqrt{3},\, 6 - 3\varepsilon/2 \leq \theta \leq 6\} \end{aligned}$$

will be counterexamples for the contractivity of q_ε constructed from Lebesgue measure on this space.

We give a general example of iterated image transformation systems in the proposition below, providing a large class of quasi-measures. The systems all have the property that the unique invariant quasi-measure vanishes on small sets.

Proposition 49 *Let $\{f_1, f_2, \ldots, f_n; (X, d)\}$ be an iterated function system with contractivity factor $s < 1$. Assume that X is a convex q-space with a non-splitting Borel probability measure* m. *Now define $q_i : C_s(X) \longrightarrow \mathbb{R}^+$ by*

$$q_i(C) = \begin{cases} \emptyset & ; \quad m(C) < \varepsilon \\ f_i^{-1}(C) & ; \quad \varepsilon \leq m(C) < 1 - \varepsilon \\ X & ; \quad m(C) \geq 1 - \varepsilon \end{cases}$$

then $\sum_{i=1}^n \alpha_i q_i^*$ whenever $\alpha_1, \alpha_2, \ldots, \alpha_n \geq 0$ and $\sum_{i=1}^n \alpha_i = 1$ is an image transformation system. Moreover the image transformation system has contractivity factor $r \leq s$

Proof. Notice that $q_i^* = q_\varepsilon^* \circ f_{i*}$ since $q_i = f_i^{-1} \circ q_\varepsilon$ and hence they are image transformations by Lemma 47. Clearly each q_i has contractivity factor smaller than or equal to s. Since they are all image transformations the assertion follows. ∎

Remark 50 *It is easy to determine the values of μ_M in this particular class of examples. In addition, the random iteration algorithm applies in illustrating the measure.*

Remark 51 *Note that a similar approach as above applies to Example 46: Let X and q be as in Example 46, i.e. $q = w^{-1} \circ \Psi^*$, where $w(x) = \sigma_x; (x \in X)$. Let $\{f_1, ..., f_n; (X, d)\}$ be an iterated function system with contractivity factor $s < 1$. Define $w_i = w \circ f_i$, and let $q_i = w_i^{-1} \circ \Psi^*; i = 1, ..., n$. One may show that w is an isometry of (X, d) into (X^*, d_H), so by virtue of Proposition 36 any convex combination $\sum \alpha_i q_i^*$ $(0 < \alpha_i < 1$ for $i = 1, ..., n)$ is an image transformation system with contractivity factor s.*

Example 52 (Structure within structure - the volume measure) *When we want to draw a picture of an object, the resolution chosen is of fundamental importance. What seems like a smooth surface on a macroscopic level, may look more irregular and complex on a microscopic level. For instance, a sandstone looks solid and compact at a distance, but if we get up close we will notice pores of millimeter size. Without such pores the world would have considerably less oil. When looking at a sandstone in an electron microscope we will be certain to discover yet another structure. This kind of behavior cannot be modelled by ordinary measures, whereas we shall see that the representable quasi-measures apply. We will construct a general example with discrete transitions between structures below.*

Proposition 53 *Let μ be a non-splitting Borel probability measure in a q-space X. Let $\{A_i\}_{i=1}^{\infty}$ be a family of disjoint Borel sets in X. Puy $A_0 = X \setminus \cup_{i=1}^{\infty} A_i$, let $t_0 = 0$ and $\{t_i\}_{i=1}^{\infty} \subset [0, \frac{1}{2}]$ and let $\{q_{t_i}\}_{i=1}^{\infty}$ be the coprresponding image transformations in X, as defined by Lemma 47, with respect to μ. Finally, let $\mu_{A_i} = \mu|_{A_i}$ for $i = 0, 1, 2, ...$ Then the series*

$$\sum_{i=1}^{\infty} q_{t_i}^*(\mu_{A_i})(f) \tag{1}$$

converges for each $f \in C(X)$ and defines an element $\nu \in Q_r(X)$. Moreover, the convergence of the series is uniform on the unit ball of $C(X)$ (viz: $\{f \in C(X) : \|f\|_\infty \leq 1\}$).

Proof. Let r_{t_i} be the quasi-homomorphism in $C(X)$ corresponding to q_{t_i} (Proposition 27). Then

$$\begin{aligned}\sum_{i=0}^{\infty} \left| q_{t_i}^*(\mu_{A_i})(f) \right| &= \sum_{i=0}^{\infty} \left| r_{t_i}^*(\mu_{A_i})(f) \right| = \sum_{i=0}^{\infty} \left| \mu_{A_i}(r_{t_i}(f)) \right| \leq \\ \sum_{i=0}^{\infty} \|\mu_{A_i}\| \, \|f\|_\infty &= \sum_{i=0}^{\infty} \mu(A_i) \, \|f\|_\infty = \mu(X) \, \|f\|_\infty\end{aligned}$$

where we have used that each r_{t_i} is norm-reducing. Hence the series 1 is absolutely convergent. It is straight forward to check that 1 defines a quasi-state ν on $C(X)$. Since each summand is a representable quasi-measure it follows that ν is representable. The uniform convergence on the unit ball is also clear. ■

5 Integration with image transformations

In this section we will study integration of quasi-measures and image transformations, the latter then gives rise to affine transformations of quasi-measures. The non-uniqueness of representations of quasi-measures gives both the space of quasi-measures and the space of affine transformations a more subtle structure than in the analogous case with Borel measures. We will start by presenting the classical constructions.

Notation 54 *Let $L_1^+(C(X))$ denote the normalized, positive linear maps from $C(X)$ (here we also consider the complex-valued functions) into $C(X)$. That is, for $T \in L_1^+(C(X))$ we have: $f \geq 0 \Rightarrow T(f) \geq 0$ (positivity) and $T(1_X) = 1_X$ -the function identical to one on X (normalized). Furthermore, we let $\mathrm{Aff}_{w^*}(P(X))$ denote the weak*-weak* continuous, affine maps from $P(X)$ to $P(X)$.*

Recall that a map $T \in L_1^+(C(X))$ is representable by a function $\tau \in C(X,X)$ (continuous function from X to X) such that $T(f) = f \circ \tau$ if and only if T is a $*$-homomorphism. We need one unpublished theorem by Ionescu-Tulcea, a proof of which may be found in e.g. [13], Corollary 3.6:

Theorem 55 *$T \in \partial_e L_1^+(C(X))$ (the extreme boundary) if and only if T is a $*$-homomorphism.*

Proposition 56 *$M \in \partial_e \mathrm{Aff}_{w^*}(P(X))$ if and only if $M = \tau_*$ (i.e. $M(\mu) = \mu \circ \tau^{-1}$) where $\tau \in C(X,X)$.*

Proof. The proof is by constructing an affine one-to-one correspondance between $L_1^+(C(X))$ and $\mathrm{Aff}_{w^*}(P(X))$, and then applying Theorem 55. Define $\Phi : L_1^+(C(X)) \to \mathrm{Aff}_{w^*}(P(X))$ by $[\Phi(T)(\mu)](f) = \int T(f)d\mu$; $T \in L_1^+(C(X))$, $\mu \in P(X)$, $f \in C(X)$. By the Riesz representation theorem $\Phi(T)$ takes $P(X)$ into $P(X)$. It is straight forward to verify that Φ is affine, and that $\Phi(T)$ is affine and weak*-weak*continuous. Now define $\Psi : \mathrm{Aff}_{w^*}(P(X)) \to L_1^+(C(X))$ by $[\Psi(M)(f)](x) = (M(\delta_x))(f)$; $M \in \mathrm{Aff}_{w^*}(P(X))$, $f \in C(X)$, $x \in X$. Again, it is straight forward to check that

$\Psi(M) \in L_1^+(C(X))$ and that Ψ is affine. Moreover, $\Phi \circ \Psi$ is the identity map on $\mathrm{Aff}_{w^*}(P(X))$ and $\Psi \circ \Phi$ is the identity on $L_1^+(C(X))$. Hence Φ is an affine one-to-one correspondance between $L_1^+(C(X))$ and $\mathrm{Aff}_{w^*}(P(X))$, and must accordingly give a one-to-one correspondance between the extreme boundaries of the two spaces. ∎

The Krein-Milman theorem does not apply since our space is not compact, and changing the topology of $P(X)$ may introduce new extreme points in $\mathrm{Aff}_{w^*}(P(X))$.

Proposition 57 $\partial_e Q_r(X) = X^*$

Proof. By definition we have $X^* \subset \partial_e Q_r(X)$. Let μ be an extreme point of $Q_r(X)$. Let $K_\mu = \{\lambda \in P(X^*) : \mu = \lambda \circ \Psi^*$. K_μ is a convex subset of $P(X^*)$, for $\lambda_1, \lambda_2 \in K_\mu; 0 < \alpha < 1$, $A \in \mathcal{A}(X)$ we have

$$\begin{aligned}(\alpha\lambda_1 + (1-\alpha)\lambda_2)(\Psi^*(A)) &= \alpha\lambda_1(A^*) + (1-\alpha)\lambda_2(A^*) = \\ \alpha\mu(A^*) + (1-\alpha)\mu(A^*) &= \mu(A^*)\end{aligned}$$

K_μ is closed in $P(X^*)$ with respect to the weak*-topology. Let λ_0 be an extreme point of K_μ. We claim that λ_0 is an extreme point of $P(X^*)$. For suppose $\lambda_0 = \alpha\lambda_1 + (1-\alpha)\lambda_2$ with $\lambda_1, \lambda_2 \in P(X^*)$. Then $\lambda_1, \lambda_2 \notin K_\mu$ so $\mu_1 = \lambda_1 \circ \Psi^*$ and $\mu_2 = \lambda_2 \circ \Psi^*$ are different from μ, but $\mu = \alpha\mu_1 + (1-\alpha)\mu_2$ is a contradiction since μ is extreme. ∎

Proposition 58 $M \in \partial_e \mathrm{Aff}_{w^*}(Q_r(X))$ *if* $M = q^*$ *where* $q : \mathcal{C}(X) \to \mathcal{C}(X)$ *is an image transformation.*

Proof. Assume that q is an image transformation and that $q^* = \alpha_1 M_1 + \alpha_2 M_2$ ($M_1 \neq M_2 \in \mathrm{Aff}_{w^*}(Q_r(X))$, $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 = 1$). Then there must be some $\mu \in X^*$ such that $M_1(\mu) \neq M_2(\mu)$. Since $q^*(\mu) \in X^*$ we have $\alpha_1 M_1(\mu) + \alpha_2 M_2(\mu) \in X^*$ and hence $\alpha_1 = 0$ or $\alpha_2 = 0$. ∎

It is still an open problem whether the image transformations are all the extreme points. This is connected with the fact that representations by convex combinations of $\{0,1\}$-valued quasi-measures are not unique, and the relations are not well understood.

Examples of identical convex combinations are easily obtained: Take X to be the unit square $[0,1] \times [0,1]$, and let $p_1 = (0,0)$, $p_2 = (0,1)$, $p_3 = (1,1)$ and $p_4 = (1,0)$. We may define the "three point quasi-measure" $\sigma_{\{x_1,x_2,x_3\}}$ for any three fixed points $\{x_1, x_2, x_3\} \subset X$ by:

$$\sigma_{\{x_1,x_2,x_3\}}(A) = \begin{cases} 0 \text{ if } |A \cap \{x_1,x_2,x_3\}| < 2 \\ 1 \text{ if } |A \cap \{x_1,x_2,x_3\}| \geq 2 \end{cases} \quad \text{for all } A \in \mathcal{A}_s(X)$$

Clearly $\sigma_{\{x_1,x_2,x_3\}}$extends to an extreme quasi-measure. With σ_i as the three point quasi-measure corresponding to $\{p_j\}_{j\neq i}$ we have

$$\sigma_0 + \sigma_2 = \sigma_1 + \sigma_3$$

In order to create examples with continuously varying transitions, we wish to define integrals of functions with values that are quasi-measures or image transformations.

We have already defined the weak*-topology of quasi-measures by pointwise application as integrals to single functions. This principle allows us to define measurability as well as integration:

Definition 59 *Let $\mathcal{B}$ be the σ-algebra of sets in $Q_1(X)$ generated by sets of the form $\{\mu|\mu(f) \in [a, b)\}$, $f \in C(X)$, $a, b \in \mathbb{R}$. Then a function $g : S \to Q_1(X)$ from a measurable space $(S, \mathcal{B}')$ into $Q_1(X)$ is* measurable *if $g^{-1}(M) \in \mathcal{B}'$ for each $M \in \mathcal{B}$.*

Equivalently, $g : S \to Q_1(X)$ is measurable if $g_f : s \mapsto g(s)(f)$ is Borel measurable for each $f \in C(X)$.

Let $(S, \nu, \mathcal{B})$ be a finite measure space and let $\mu : S \to Q_1(X)$ be a measurable function. Then we define $\int \mu(s)\ d\nu(s) \in Q_1(X)$ by

$$\left(\int \mu(s)\ d\nu(s)\right)(f) = \int \mu(s)(f)\ d\nu(s)$$

for each $f \in C(X)$.

The definitions for image transformations are similarly based on pointwise application to measures, except that image transformations do not by themselves form a convex set, but (when considered via the adjoint as acting on measures) just part of the convex set of affine maps from $Q_1(X)$ to $Q_1(Y)$, which also includes the combined transformations given by iterated image transformation systems.

Definition 60 *We define the topology of the set $\mathrm{Aff}(Q_1(X), Q_1(Y))$ of affine maps from $Q_1(X)$ to $Q_1(Y)$ by pointwise convergence on each quasi-measure in $Q_1(X)$.*

We define measurability of a function $g : S \to \mathrm{Aff}(Q_1(X), Q_1(Y))$ (into this set) by measurability with respect to the σ-algebra generated by sets of the form $\{T \in \mathrm{Aff}(Q_1(X), Q_1(Y))|T(\mu)(f) \in [a, b)\}$ for $\mu \in Q_1(X)$, $f \in C(Y)$, $a, b \in \mathbb{R}$. This is again equivalent to the measurability of the functions $g_\mu : s \mapsto g(s)(\mu)$, $\mu \in Q_1(X)$.

Again, let $(S, \nu, \mathcal{B})$ be a finite measure space and this time let $g : S \longrightarrow \mathrm{Aff}(Q_1(X), Q_1(Y))$ be a measurable function. Then we define $\int g(s)\, d\nu(s) \in \mathrm{Aff}(Q_1(X), Q_1(Y))$ by

$$(\int g(s)\ d\nu(s))(\mu) = \int g(s)(\mu)\ d\nu(s)$$

for each $\mu \in Q_1(X)$.

6 Volume measures with continuous transitions

We will recall some results on uniformly closed subalgebras of $C(X)$. When A is a uniformly closed subalgebra of $C(X)$ containing the constants, then A is isomorphic to $C(\hat{A})$, where $\hat{A}$ is the compact Hausdorff space of characters on A. Moreover, the isomorphism is established by the continuous surjection $i^* : X \longrightarrow \hat{A}$, $i^*(x) : a \mapsto a(x)$. Precisely,

$$f \in C(\hat{A}) \mapsto f \circ i^* \in A.$$

Moreover, a function $f \in C(X)$ is in A if and only if it takes equal values at points identified by i^*. Since i^* maps a compact space onto a Hausdorff space, the topology of $\hat{A}$ must be identical to the quotient topology.

Definition 61 *An* analytic subalgebra *A of $C(X)$ is a uniformly closed subalgebra containing the constants and with the property that whenever $f^2 \in A$, $f \in C(X)$ then $f \in A$.*

Remark 62 *The concept of an analytic subalgebra may be found in [4] and [10].*

Proposition 63 *Let A be a uniformly closed subalgebra of $C(X)$ containing the constants, and let $i^* : X \longrightarrow \hat{A}$ be the corresponding surjection onto its character space. Then the following are equivalent:*

1. *A is analytic.*
2. *For each $\hat{p} \in \hat{A}$, $i^{*-1}(\hat{p})$ is connected.*
3. *For each connected subset $\hat{M} \subseteq \hat{A}$, $i^{*-1}(\hat{M})$ is connected.*

Proof. For (1) $\Rightarrow$ (2), assume to the contrary that $i^{*-1}(\hat{p})$ is not connected for a point $\hat{p} \in \hat{A}$. Find disjoint open sets O_1, $O_2 \subset X$ separating $i^{*-1}(\hat{p})$, and let $O = O_1 \cup O_2$. Then $i^*(X \setminus O) \subset \hat{A}$ is a compact set not containing $\hat{p}$. By Urysohn's lemma let $\hat{g} : \hat{A} \to [0,1]$ be a continuous function with $\hat{g}(\hat{p}) = 1$, $\hat{g}(i^*(X \setminus O)) = 0$. Then $g = \hat{g} \circ i^* \in A$. Define $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} g(x), & x \in O_1 \\ -g(x), & x \in O_2 \\ 0, & x \in X \setminus O. \end{cases}$$

Then f is a continuous function, $f^2 = g^2$, and so $f \in A$. But f takes different values at different elements of $i^{*-1}(\hat{p})$, a contradiction.

For (3) $\Rightarrow$ (1), assume to the contrary that $f \in C(X) \setminus A$, $f^2 \in A$. Let $x_1, x_2 \in X$ be two points such that $i^*(x_1) = i^*(x_2) = \hat{x}$, while $f(x_1) \neq f(x_2)$. Since $f^2 \in A$, $|f(x)| = f(x_1)$ for $i^*(x) = \hat{x}$. But then $i^{*-1}(\hat{x})$ is separated by the closed sets $\{x \in X | f(x) = f(x_1)\}$ and $\{x \in X | f(x) = f(x_2) = -f(x_1)\}$, a contradiction.

For (2) $\Rightarrow$ (3), assume to the contrary that $\hat{M} \subseteq \hat{A}$ is connected with $i^{*-1}(\hat{M})$ disconnected. Let $i^{*-1}(\hat{M})$ be divided into the (compact set separated) sets $M_1 \subset C_1$, $M_2 \subset C_2$. $i^*(M_1)$ and $i^*(M_2)$ cannot be disjoint, as otherwise $i^*(C_1)$ and $i^*(C_2)$ would separate $\hat{M}$. Let $\hat{p}$ be a common point of $i^*(M_1)$ and $i^*(M_2)$; then $i^{*-1}(\hat{p})$ is itself separated by C_1 and C_2, a contradiction. ∎

Remark 64 *Proposition 63 conveniently expresses some facts about analytic subalgebras in terms of the corresponding surjection i^*. The equivalence (1.)⇔(2.) is essentially Theorem 16.30 and Lemma 16.31 in [4]. (2.) is the definition of a monotone mapping, and the equivalence (2.)⇔(3.) holds generally for closed maps of topological spaces.*

Corollary 65 *If A is analytic, then:*

1. *If $\hat{M}$ is a solid set in $\hat{A}$, then $i^{*-1}(\hat{M})$ is a solid set in X.*
2. *If $\hat{f}$ is a solid function on $\hat{A}$, then $\hat{f} \circ i^*$ is a solid function on X.*
3. *If X is connected, then so is $\hat{A}$.*
4. *If X is locally connected, then so is $\hat{A}$.*
5. *If X has genus 0, then so does $\hat{A}$.*
6. *If X is simply connected, then so is $\hat{A}$.*

Proof. (1) is obvious from 63.3. (2) is obvious from (1). (3) and (6) are obvious given just the continuity of i^*. (5) is obvious from (1) and the definition of genus by existence of non-trivial solid partitions.

For (4), assume now that X is locally connected. Let $\hat{O}$ be an open subset of $\hat{A}$, and let $\hat{M}$ be one of its components. Then $i^{*-1}(\hat{M})$ is connected, and is a component of the open set $i^{*-1}(\hat{O})$. Since X is locally connected, $i^{*-1}(\hat{M})$ is open, and so $\hat{M}$ is open. Since the components of open sets in $\hat{A}$ are open, $\hat{A}$ is locally connected. ∎

The smallest analytic subalgebra containing a given function $f \in C(X)$, will be denoted by $A(f)$. The following theorem explains the main importance of analytic subalgebras for quasi-measure theory:

Theorem 66 *Every quasi-integral is linear on each $A(f)$.*

Remark 67 *For a proof, (generalized to quasi-linear functionals) see [8].*

In the remainder of this section, X is also assumed to be connected and locally connected.

Theorem 68 (Urysohn's lemma for sums of solid functions) *Let C_0 and C_1 be disjoint compact subsets of X. Then there exists a function $u_{C_1}^{C_0} : X \to [0,1]$, which is a finite sum of solid functions, and such that $u_{C_1}^{C_0}(C_0) = 0$, $u_{C_1}^{C_0}(C_1) = 1$.*

Proof. We will reduce the problem step by step to the case where one of the sets is solid, for which the result is known (with a single solid function.)

First, separate C_0 and C_1 by open sets O_0 and O_1. A finite number of components of O_1 suffices to cover C_1. By replacing C_1 by the union of the closures of these components, we may assume that C_1 has a finite number of (compact) connected components.

Now, separate the components M_i, $i = 1, \dots, n$ of C_1 by open sets O_i disjoint from C_0. By replacing C_1 by each M_i and C_0 by the corresponding $X \setminus O_i$, we reduce the problem to a number of sub-cases with C_1 connected. The solution to the original problem is then built from the sub-cases as

$$u_{C_1}^{C_0} = \sum_{i=1}^{n} u_{M_i}^{X \setminus O_i}.$$

Assuming now that C_1 is connected, its complement is a disjoint union of open solid components, a finite number of which suffices to cover C_0. Let $O_1^1, \dots, O_1^n$ be such a finite collection of components. By replacing C_1 by each $X \setminus O_i$ and C_0 by $C_0 \cap O_i$, we reduce the problem to a number of sub-cases with C_1 solid. The solution to the original problem is then built from the sub-cases as

$$u_{C_1}^{C_0} = \left(\sum_{i=1}^{n} u_{X\setminus O_i}^{C_0 \cap O_i} \right) - (n-1).$$

■

Corollary 69 *Any function $f \in C(X)$ may be uniformly approximated by sums of solid functions contained in $A(f)$, the analytic closed subalgebra generated by f.*

Proof. Let $A = A(f)$. Since $f \in A$, we have $f = \hat{f} \circ i^*$, $\hat{f} \in C(\hat{A})$. Assume $\hat{f}(\hat{A}) = f(X) = [a, b]$, $a < b$ (The case $a = b$ is trivial.) Let $\epsilon > 0$ be given, and let $n > (b-a)/\epsilon$. Then in the notation of theorem 68,

$$\hat{g} = a + \frac{1}{n} \sum_{i=0}^{n-1} u_{\{x \in \hat{A} | \hat{f} \geq a+(i+1)/n\}}^{\{x \in \hat{A} | \hat{f} \leq a+i/n\}}$$

is an ϵ-uniform approximation of $\hat{f}$ by sums of solid functions in $C(\hat{A})$. It follows from 65.2 that $g = \hat{g} \circ i^*$ is a sum of solid functions in $A(f)$ approximating f ϵ-uniformly. ■

Remark 70 *It is not possible in general to write a continuous function exactly as a finite sum of solid functions. E.g. for $X = [0, 1]$, solid functions are monotone, and so a sum of solid functions must be of bounded variation. Therefore, no function of unbounded variation in $C([0, 1])$*

$$(e.g.\ f(x) = \begin{cases} 0, & x = 0 \\ x \sin \frac{1}{x}, & 0 < x \leq 1 \end{cases})$$

may be such a sum.

We are going to conclude this paper by an explicit calculation of an integral of the type introduced in Definition 60. To this end some preliminaries are necessary.

Let μ be a normalized quasi-measure in X and let μ_f denote the corresponding regular Borel measure in $\mathrm{Sp}(f)$ ($f \in C(X)$, $\mathrm{Sp}(f) = \{f(x) : x \in X\}$), i.e. $\mu_f(A) = \mu(f^{-1}(A))$; $A \in \mathcal{A}(\mathrm{Sp}(f))$. Then

$$\mu(f) = \int_{\mathrm{Sp}(f)} s \, d\mu_f(s)$$

It is useful to recognize that the integral is nothing but an ordinary Riemann-Stieltjes integral with respect to the left-continuous function $\tilde{f}_\mu(s) = \mu(f^{-1}(-\infty, s))$; $s \in \mathbb{R}$. Indeed, if $\mathrm{Sp}(f) = [\lambda_1, \lambda_2]$ we get $\int_{\mathrm{Sp}(f)} s\ d\mu_f(s) = \int_{\lambda_1}^{\lambda_2} s\ d\tilde{f}_\mu(s) + \lambda_2 \cdot \mu(f^{-1}\{\lambda_2\}) = [s \cdot \tilde{f}_\mu(s)]_{\lambda_1}^{\lambda_2} - \int_{\lambda_1}^{\lambda_2} \tilde{f}_\mu(s)\ ds$, i.e.

$$\mu(f) = \lambda_2 - \int_{\lambda_1}^{\lambda_2} \tilde{f}_\mu(s)\ ds$$

Now let q_t be the "cut-off" image transformation given by Lemma 47, with $\varepsilon = t \in [0, \frac{1}{2}]$, viz.:

$$q_t(U) = \begin{cases} \emptyset & \text{if} \quad m(U) \leq t \\ U & \text{if} \quad t < m(U) \leq 1 - t \\ X & \text{if} \quad 1 - t < m(U) \leq 1 \end{cases}$$

Here m is a fixed non-splitting probability measure in X and U is an arbitrary open solid set in X.

Proposition 71 *$t \mapsto q_t^* \in \mathrm{Aff}(Q(X), Q(X))$ is a continuous (and hence measurable) function on $[0, 1/2]$.*

Proof. We need to show that the map $t \mapsto q_t^*(\mu)(f)$ is continuous for each $\mu \in Q(X)$ and each $f \in C(X)$. Suppose this is true for each solid function f. Let $\varepsilon > 0$ be arbitrary, and let $g \in C(X)$. By Corrolary 68 there is a finite set of solid functions $\{f_1, f_2, ..., f_n\} \subset A(g)$ such that $\|g - \sum_{i=1}^n f_i\|_\infty < \varepsilon$. For any $\mu \in Q(X)$, $t \in [0, \frac{1}{2}]$ we have, by linearity on $A(g)$:

$$\left| q_t^*(\mu)(g) - q_t^*(\mu)(\sum_{i=1}^n f_i) \right| < \varepsilon$$

Now, if continuity holds for solid functions we obtain, for $|t - t'|$ sufficiently small:

$$|q_t^*(\mu)(\sum_{i=1}^n f_i) - q_{t'}^*(\mu)(\sum_{i=1}^n f_i)| \leq \sum_{i=1}^n |q_t^*(\mu)(f_i) - q_{t'}^*(\mu)(f_i)| < \epsilon$$

By a standard 3ε-argument it follows that $|q_t^*(\mu)(g) - q_{t'}^*(\mu)(g)| < 3\varepsilon$. We are left with the task of showing that $t \mapsto q_t^*(\mu)(f)$ is continuous while f is assumed to be a solid function. However, in this case an explicit calculation can be made. We have

$$q_t^*(\mu)(f) = (\mu \circ q_t)(f) = \lambda_2 - \int_{\lambda_1}^{\lambda_2} \tilde{f}_{\mu \circ q_t}(s)\ ds \text{ , where } \mathrm{Sp}(f) = [\lambda_1, \lambda_2]$$

In order to compute $\tilde{f}_{\mu o q_t}(s) = \mu(q_t(f^{-1}(-\infty, s)))$ we must look at $m(f^{-1}(-\infty, s)) = \tilde{f}_m(s)$. Let

$$\begin{aligned} s_1(t) &= \sup\{s : \tilde{f}_m(s) \leq t\} \\ s_2(t) &= \sup\{s : \tilde{f}_m(s) \leq 1 - t\} \end{aligned}$$

We have $\lambda_1 \leq s_1(t) \leq s_2(t) \leq \lambda_2$ for $t \in [0, \frac{1}{2}]$, and by left continuity of $\tilde{f}_m$ we get

$$\begin{aligned} s \leq s_1(t) \Rightarrow m(f^{-1}(-\infty, s)) \leq t \Rightarrow q_t(f^{-1}(-\infty, s)) = \emptyset \\ s_1(t) < s \leq s_2(t) \;\Rightarrow\; t < m(f^{-1}(-\infty, s)) \leq 1 - t \\ \Rightarrow\; q_t(f^{-1}(-\infty, s)) = f^{-1}(-\infty, s) \\ s_2(t) < s \Rightarrow m(f^{-1}(-\infty, s)) > 1 - t \Rightarrow q_t(f^{-1}(-\infty, s)) = X \end{aligned}$$

Consequently, for $s \in [\lambda_1, \lambda_2]$

$$\tilde{f}_{\mu o q_t}(s) = \begin{cases} 0 & \text{if} \quad s \leq s_1(t) \\ \tilde{f}_\mu(s) & \text{if} \quad s_1(t) < s \leq s_2(t) \\ 1 & \text{if} \quad s > s_2(t) \end{cases}$$

Hence, we obtain

$$q_t^*(\mu)(f) = \lambda_2 - \int_{s_1(t)}^{s_2(t)} \tilde{f}_\mu(s)\, ds + \int_{s_2(t)}^{\lambda_2} -ds = s_2(t) - \int_{s_1(t)}^{s_2(t)} \tilde{f}_\mu(s)\, ds \qquad (2)$$

Now observe that because m is non-splitting the function $\tilde{f}_m$ is strictly increasing on $[\lambda_1, \lambda_2]$, possibly with jumps. For if $\alpha < \beta$ and $\tilde{f}_m(\alpha) = \tilde{f}_m(\beta)$ and we put $C_1 = f^{-1}(-\infty, \alpha]$, $C_2 = f^{-1}[\beta, \infty)$, then $C_1 \cap C_2 = \emptyset$ and $m(C_1) + m(C_2) = 1$; $C_1, C_2 \neq \emptyset$ a contradiction. If $\tilde{f}_m$ is continuous at a point α and $\tilde{f}_m(\alpha) = t$ we have $s_1(t) = \alpha$. If $\tilde{f}_m$ has a jump from t_1 to $t_2 \leq \frac{1}{2}$ in α, then the function s_1 is constant with value α on $[t_1, t_2]$. s_1 is therefore a continuous functionon $[0, \frac{1}{2}]$ which satisfies $s_1(\tilde{f}_m(s)) = s$ for $s \in [\lambda_1, \lambda_0]$, where $\lambda_0 = s_1(\frac{1}{2})$. Next, let $u = 1 - t$, $u \in [\frac{1}{2}, 1]$ and put $h(u) = s_2(1 - u) = \sup\{s : \tilde{f}_m(s) \leq u\}$. Then h is a continuous function on $[\frac{1}{2}, 1]$ which satisfies $h(\tilde{f}_m(s)) = s$ for $s \in [\lambda_0, \lambda_2]$. Hence $s_2(1 - \tilde{f}_m(s)) = s$ for $s \in [\lambda_0, \lambda_2]$. By the continuity of s_1 and s_2 we obtain from 2 that $q_t^*(\mu)(f)$ is a continuous function of t when f is a solid function. This completes the proof. ∎

Proposition 72 *With notation as above we have* $(\mu \in Q(X),\ f \in C_s(X))$

$$\begin{aligned} \int_0^{\frac{1}{2}} q_t^*(\mu)(f)\, dt &= -\frac{\lambda_2}{2} + \lambda_0 - \int_{\lambda_1}^{\lambda_0} \tilde{f}_\mu(s)\tilde{f}_m(s)\, ds \\ &\quad + \int_{\lambda_0}^{\lambda_2} [\tilde{f}_m(s) - \tilde{f}_\mu(s) + \tilde{f}_\mu(s)\tilde{f}_m(s)]\, ds \end{aligned}$$

where $\lambda_0 = s_1(\frac{1}{2}) = s_2(\frac{1}{2})$

Proof. From 2 we get

$$I = \int_0^{\frac{1}{2}} q_t^*(\mu)(f)\ dt = \int_0^{\frac{1}{2}} s_2(t)\ dt - \int_0^{\frac{1}{2}} (\int_{s_1(t)}^{s_2(t)} \tilde{f}_\mu(s)\ ds)\ dt$$

Changing the order of integration in the last integral we get

$$\int_0^{\frac{1}{2}} (\int_{s_1(t)}^{s_2(t)} \tilde{f}_\mu(s)\ ds)\ dt = \int_{\lambda_1}^{\lambda_0} \tilde{f}_\mu(s)(\int_0^{\tilde{f}_m(s)} dt)\ ds + \int_{\lambda_0}^{\lambda_2} \tilde{f}_\mu(s)(\int_0^{1-\tilde{f}_m(s)} dt)\ ds$$

where we have utilized the relation $s_1(\tilde{f}_m(s)) = s$ if $s \in [\lambda_1, \lambda_0]$ and $s_2(1 - \tilde{f}_m(s)) = s$ if $s \in [\lambda_0, \lambda_2]$. Calculating the expression above we obtain

$$\int_{\lambda_1}^{\lambda_0} \tilde{f}_\mu(s)\tilde{f}_m(s)\ ds + \int_{\lambda_0}^{\lambda_2} \tilde{f}_\mu(s)(1 - \tilde{f}_m(s))\ ds$$

Since $s_2(t)$ is the "inverse" of $1 - \tilde{f}_m(s)$ for $s \in [\lambda_0, \lambda_2]$, $t \in [0, \frac{1}{2}]$ we must have

$$\int_0^{\frac{1}{2}} s_2(t)\ dt + \int_{\lambda_0}^{\lambda_2} (1 - \tilde{f}_m(s))\ ds = \frac{1}{2}\lambda_2$$

Hence $I = \frac{\lambda_2}{2} - \int_{\lambda_0}^{\lambda_2}(1 - \tilde{f}_m(s))\ ds - \int_{\lambda_1}^{\lambda_0} \tilde{f}_\mu(s)\tilde{f}_m(s)\ ds - \int_{\lambda_0}^{\lambda_2} \tilde{f}_\mu(s)(1 - \tilde{f}_m(s))$ ds which yields the desired formula. ∎

Example 73 *Take* $X = [0,1] \times [0,1]$, *let* m *be normalized Lebesgue measure in* X *and let* q_t *be defined as above with respect to* m. *Now let* $\mu = \delta_{(x_0,y_0)}$ *(=point-measure at* $(x_0, y_0) \in X$*). We are going to compute the value of the quasi-state*

$$\rho = \int_0^{\frac{1}{2}} q_t^*(\mu)\ dt$$

applied to each of the three solid functions $f(x,y) = x$, $g(x,y) = y$, $h(x,y) = x + y$. *Straight forward calculations yield*

$$\begin{aligned}
\tilde{f}_m(s) &= \tilde{g}_m(s) = s\ ;\ s \in [0,1] \\
\tilde{h}_m(s) &= \begin{cases} \frac{1}{2}s^2 & \textit{if}\ \ 0 \le s \le 1 \\ -\frac{1}{2}s^2 + 2s - 1 & \textit{if}\ \ 1 \le s \le 2 \end{cases} \\
\tilde{f}_\mu(s) &= \chi_{(x_0,1]}(s)\ ;\ s \in [0,1] \\
\tilde{g}_\mu(s) &= \chi_{(y_0,1]}(s)\ ;\ s \in [0,1] \\
\tilde{h}_\mu(s) &= \chi_{(x_0+y_0,2]}(s)\ ;\ s \in [0,2]
\end{aligned}$$

Also, for f and g we get $\lambda_0 = \frac{1}{2}$, and for h we have $\lambda_0 = 1$. If $x_0 \le \frac{1}{2}$ we therefore get from Proposition 72:

$$\rho(f) = -\int_{x_0}^{\frac{1}{2}} s\ ds + \int_{\frac{1}{2}}^{1} (s - 1 + s)\ ds = \frac{1}{8} + \frac{1}{2}x_0^2$$

If $x_0 \ge \frac{1}{2}$ we get

$$\rho(f) = \int_{\frac{1}{2}}^{1} s\ ds - \int_{x_0}^{1} (1-s)\ ds = -\frac{1}{8} - \frac{1}{2}x_0^2 + x_0$$

Similarily

$$\rho(g) = \frac{1}{8} + \frac{1}{2}y_0^2 \text{ if } y_0 \le \frac{1}{2},\ \text{and } \rho(g) = -\frac{1}{8} - \frac{1}{2}y_0^2 + y_0 \text{ if } y_0 \ge \frac{1}{2}$$

For h we have $\lambda_1 = 0$, $\lambda_0 = 1$, $\lambda_2 = 2$. If $x_0 + y_0 \le 1$ we get

$$\begin{aligned}\rho(h) &= \int_{1}^{2} (-\frac{1}{2}s^2 + 2s - 1)\ ds - \int_{x_0+y_0}^{2} [1 - (-\frac{1}{2}s^2 + 2s - 1)]\ ds \\ &= -\frac{1}{2} + \frac{1}{6}(x_0 + y_0)^3 - (x_0 + y_0)^2 + 2(x_0 + y_0)\end{aligned}$$

For instance, if $x_0 = y_0 = \frac{1}{2}$, we obtain that $\rho(f) = \rho(g) = \frac{1}{4}$, while $\rho(h) = \rho(f+g) = \frac{2}{3}$ which shows that ρ is non-additive.

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Quasi-measures with Image transformations as generalized variables

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Abstract

We introduce an image transformation as a generalization of measurable maps. Structure properties of the image transformation is given. Unique extension of image transformations from solid sets. The multidimensional medain and sample median is presented as image transformations, providing a construction suitable for probability theorists.

Keywords: Quasi-measure, image transformation, generalized variable, q-function, quasi-probability

1 Introduction

The quasi-measure originated in [1] as a solution to the problem of finding non-linear states on C*-algebras formulated by R. Kadison (c.f. [10]). The quasi-measures are topological measures in the sense that they are only defined on closed and open sets. Let X be a compact Hausdorff space. We will let $\mathcal{C}(X)$ and $\mathcal{O}(X)$ respectively denote the closed and open subsets of X. In addition we put $\mathcal{A}(X) = \mathcal{C}(X) \cup \mathcal{O}(X)$. When there is no confusion concerning the space in question, we will omit the space in the notation. With a *quasi-measure* in X we mean a set function $\mu : \mathcal{A} \to \mathbb{R}^+$ such that the following hold:

(i) $\mu(\biguplus_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i)$ ($\biguplus$ indicates disjoint union, and we assume all A_i and $\biguplus A_i$ in $\mathcal{A}$)

(ii) $\mu U = \sup\{\mu C : C \subset U; C \in \mathcal{C}\}$ for all U in $\mathcal{O}$

Note that one immediate consequence of (i) and (ii) is the monotonicity of the quasi-probability. That is $A_1 \subset A_2$ implies $\mu A_1 \leq \mu A_2$ whenever $A_1, A_2 \in \mathcal{A}$. The quasi-measures are in fact countably additive (c.f. [9]), but being defined only on closed and open sets they are a vastly larger class of set functions than Borel measures. Perhaps the most characteristic difference is that they are not in general subadditive.

With respect to a quasi-measure, an integration theory has been developed, where the quasi-integral differs from the usual integral in not being linear. However, the quasi-integral is linear on different classes of functions such as singly generated algebras of continuous functions.

In section 2 we introduce a generalized image transformation. The image transformations map sets to sets and resemble the inverse images of maps. The notion of an image transformation was first given in [4] as a generalization of continuous functions. We show that our image transformations naturally induce a mapping of measures into quasi-measures. In section 3 we present a structure theorem. The result states that our image transformations generalize measurable maps. From the structure theorem we deduce that image transformations defined on solid sets extend uniquely to image transformations on all open or closed sets (a set is called solid if both the set and its compliment is connected). Next we establish a composition of the image transformation with the set map into the closed and open sets of the Stone space of the measurable space in question. This enables us to conclude that our image transformations correspond to continuous image transformations. The continuity of the corresponding image transformation implies that the image transformations map measures to representable quasi-measures (i.e. those spanned by $\{0,1\}$-valued quasi-measures). We conclude section 3 by showing that an image transformation is naturally lifted to a quasi-homomorphism of conitnuous functions into measurable functions (quasi-homomorphism is defined in section 3)

The last section was the motivation for developing the theory presented in this paper. The work in [6] presented the quasi-measure as a modelling tool for statistical problems. In addition a simple and general construction of the quasi-measures by $q - functions$ (see [6] for details) was given. The examples in [6] suggested that there was a connection between the quasi-measure and the statistical median. A literature search indicated that the median had no basis in probability theory. Still a median in spaces other than $\mathbb{R}$ has been sought for throughout the twentieth century (many have been suggested but none suitable for measure theory). This encouraged us to investigate whether the quasi-measure could serve as a model for the median. Indeed, it turned out that basic desired properties of the median forces us to consider a quasi-measure. We will not go into detail here, for a broader exposition of the median the reader is referred to [13] ([13] is a preprint for this article).

The $q-functions$ mentioned above were inadequate for constructing a median in general. The image transformation on the other hand proved to be an efficient tool. In the last section we present definitions and constructions of the median and sample median. Preservation (equivariance in statistical literature) properties of the medians are given.

2 Quasi-measures and Image transformations

The letter X will denote a compact Hausdorff space and $(Y, \mathcal{B})$ will denote a measurable space in the sequel. The measurable spaces will be the domain of our variables and hence we will impose no restrictions on the space. In our construction we will however require some properties of the image space of the variables. We formalize these properties in the definition below.

Definition 1 *If X is locally connected, connected and has genus equal zero ($g(X) = 0$) we will call X a* q-space.

Remark 2 *These properties are shared by a large class of spaces such as closed intervals and disks in addition to balls and spheres in $\mathbb{R}^n, n \geq 3$. The genus requirement is treated (and defined) in [3] and [11], we will not elaborate on that issue here. The reader may settle with the fact that simply connected spaces have $g = 0$.*

The restriction to compact spaces is unfortunate. The restriction was made because the theory of quasi-measures is well established there. However, we hope that the results here may be generalized to the locally compact setting. To this extent the integration theory for quasi-measures in locally compact spaces has been developed and may be found in [12].

If a set $A \in \mathcal{A}$ and its compliment both are connected we will call the set solid. The solid sets play an important role in the theory of quasi-measures.. They constitute a small and manageable family of sets that totally determines a quasi-measure. This is illustrated by the solid set-functions, they were introduced in [3] and their properties were investigated there. In particular they are invaluable tools for constructing quasi-measures. We include their definition here for the convenience of the reader. The restriction to the solid sets will be denoted with a subscript s (e.g. $\mathcal{C}_s$ will denote the compact solid sets). Let X be a *q-space*. Then a function $\mu : \mathcal{A}_s \to \mathbb{R}^+$ is a solid set-function if it satisfies

(A) $\sum_{i=1}^{n} \mu C_i \leq \mu C$ whenever $\biguplus_{i=1}^{n} C_i \subset C; C_i, C \in \mathcal{C}_s$ for $i = 1, 2, ..., n$

(B) $\mu U = \sup \{\mu C : C \subset U; C \in \mathcal{C}_s\}$ for all $U \in \mathcal{O}_s$

(C) $\mu A + \mu(X \backslash A) = \mu X$

Remark 3 *For our purposes we will assume that $\mu(X) = 1$, accordingly we will call the quasi-measure a quasi-probability. The basic construction of quasi-probabilities has been given in [1], [3] and [11]. The main construction result ([3], Theorem 5.1) states that a solid set-function uniquely extends to a quasi-probability on $\mathcal{A}$.*

Image transformations were introduced in [4] as a generalization of continuous maps. In measure theory the measurable maps rather than the continuous maps are the natural variables and hence we will need a more general tool. This is provided by the definition below.

Definition 4 *We define an image transformation to be a map $q : \mathcal{A}(X) \to \mathcal{B}(Y)$ from the closed subsets of a metric Hausdorff space X into the sigma algebra of a measurable space $(Y, \mathcal{B})$, such that the following is satisfied*

1. $A_1 \cap B_2 = \emptyset \Rightarrow qA_1 \uplus qB_2 = q(A_1 \cup B_2)$
2. $qX = Y$
3. $U_i \nearrow U \Rightarrow qU_i \nearrow qU \;; U_i, U \in \mathcal{O}(X)$ *for* $i = 1, 2, ...$

If in addition Y is a compact Hausdorff space and $q(\mathcal{O}(X)) \subset \mathcal{O}(Y)$ we will call q a continuous image transformation.

Remark 5 *This definition generalizes the image transformations in [4] by its image being measurable subsets of a measurable space rather than compact subsets of a compact Hausdorff space. In addition we restrict ourselves to the metric situation for the space X, as we will see in Proposition 10 this is connected to property 3 of Definition 4.*

Example 6 *Let $T : Y \to X$ be a measurable map with respect to the Borel sets in X. Then the map $T^{-1} : \mathcal{A}(X) \to \mathcal{B}(Y)$ is an image transformation. In this case we say that the image transformation is derived from the function T. The image transformations derived from functions are trivial in the sense that their adjoint will map measures to measures.*

The example above is very important as the image transformation is really a generalization of measurable maps. However, an image transformation is not in general the inverse image of a map.

The following proposition is a routine application of Definition 4 and included for the reader's convenience. The proofs are straight forward from the definition of the image transformations.

Proposition 7 *If $q : \mathcal{A}(X) \to \mathcal{B}(Y)$ is an image transformation the following hold*

1. $A \subset B \Rightarrow qA \subset qB$ for any $A, B \in \mathcal{A}(X)$
2. $q(\biguplus_{i=1}^n A_i) = \biguplus_{i=1}^n qA_i; A_i, \biguplus_{i=1}^n A_i \in \mathcal{A}$ for $i = 1, 2, ..., n$
3. $C_i \searrow C \Rightarrow qC_i \searrow qC \,; C_i, C \in \mathcal{C}(X)$ for $i = 1, 2, ...$

Remark 8 *Notice that by compliment Proposition 7.3 is equivalent with Definition 4.3 under the assumption of 4.1 and 4.2.*

Lemma 9 *Let μ be a monotone set function on $\mathcal{A}$ satisfying the additivity (i) of a quasi-probability. Assume that X is a metric compact Hausdorff space. If μ satisfies $\mu U_i \nearrow \mu U$ whenever U_i is an increasing sequence of open sets with $\bigcup U_i = U$, then μ is a quasi-probability.*

Proof. We will use the fact that metrizability is equivalent with second countability (i.e. countable basis for the topology) for compact Hausdorff spaces. First we show that if Λ is a directed set and $U_\lambda \nearrow U$; $U_\lambda, U \in \mathcal{O}$ and $\lambda \in \Lambda$ then $\mu U_\lambda \nearrow \mu U$. Let $\{O_n\}$ be a basis for the topology τ of X and pick an arbitrary $\lambda_0 \in \Lambda$. Then recursively pick $\lambda_{i+1} \leq \lambda_i$ such that $\bigcup_{\{n:O_n \subset U_{\lambda_i}\}} O_n \nearrow U$ as $i \to \infty$. By the assumption of the Lemma $\mu U_{\lambda_i} \nearrow \mu U$. From monotonicity of μ it is clear that we must have $\mu U_\lambda \nearrow \mu U$. Let $U \subset X$ be an arbitrary open set. Order all open sets $U_\lambda \subset U$ with $\bar{U}_\lambda \subset U$ by inclusion. By Urysohn's lemma we have $U_\lambda \nearrow U$, so $\mu U_\lambda \nearrow \mu U$ by the previous argument. Now monotonicity twice yields $\mu \bar{U}_\lambda \nearrow \mu U$ and then $\mu U = \{\mu C : C \subset U, C \in \mathcal{C}\}$. We have proven the regularity of μ, the additivity was assumed. Hence, μ is a quasi-probability. ∎

Proposition 10 *If $(Y, \mathcal{B}, P)$ is a probability space and $q : \mathcal{A}(X) \to \mathcal{B}(Y)$ is an image transformation, then $q^*(P)$ defined by $(q^*P)A = P(qA)$ for all $A \in \mathcal{A}(X)$ is a quasi-probability in X.*

Proof. The additivity requirement (i) follows from Proposition 7.2. Assume that $U_i \nearrow U$ is an increasing sequence of open sets. Then by Definition 4.3 $q(U_i) \nearrow qU$ is an increasing sequence of measurable sets. Monotone convergence of P implies $P(qU_i) \nearrow P(qU)$, so $(q^*P)(U_i) \nearrow (q^*P)(U)$. By Lemma 9 q^*P is regular (ii) and hence a quasi-probability. ∎

Notation 11 *We will denote the probability measures of a measurable space $(Y, \mathcal{B})$ by $\mathcal{M}(Y)$, and the quasi-probabilities of a compact Hausdorff space X by $Q(X)$.*

The map $q^* : \mathcal{M}(Y) \to Q(X)$ will be called the adjoint of q. Note that if q is continuous and Y is compact Hausdorff we are in the situation in [4] where q^* can be extended to $Q(Y)$. If q is derived from a measurable map we of course get the well known situation of transformations of measures. However, image transformations are in general not derived from maps. The last section will provide examples of such in terms of the median.

If $q : \mathcal{A}(X) \to \mathcal{B}(Y)$ is an image transformation, we may restrict it to the solid sets. By Proposition 7 it is easy to verify that

(A') If $C, C_i \in \mathcal{C}_s; i = 1, 2, ..., n$ and $\biguplus C_i \subset C$, then $\biguplus q(C_i) \subset qC$.

(B') If $U, U_i \in \mathcal{O}_s; i = 1, 2, ...$ and $U_i \nearrow U$, then $q(U_i) \nearrow qU$.

(C') For any $A \in \mathcal{A}_s$ we have $q(A) \uplus q(X \backslash A) = Y$.

Proposition 12 *Let X be a q-space. If $(Y, \mathcal{B}, \mu)$ is a probability space and $q : \mathcal{A}_s(X) \to \mathcal{B}(Y)$ satisfies (A'), (B') and (C'), then $q^*(\mu)$ defined by $(q^*\mu)A = \mu(qA)$ for all $A \in \mathcal{A}_s(X)$ extends uniquely to a quasi-measure in X.*

Proof. This is a consequence of $q^*\mu$ being a solid set-function. Property (A) and (C) is immediate from (A') and (C'). The regularity can be shown from (B') and Lemma 3.3 in [3]. The lemma states that given any compact solid set contained in any open set U, it is possible to find an open solid set between them with closure in U. ∎

Remark 13 *In the last section of this paper (Proposition 21) we will show in that the image transformations corresponds 1-1 with solid set maps satisfying (A'), (B') and (C'). We will not distinguish between the two concepts, and refer to both as image transformations.*

3 Image transformations and structure

Throughout this section we will let $(Y, \mathcal{B}, P)$ denote a probability space where the sigma algebra $\mathcal{B}$ contains the singleton sets (i.e. $\{y\} \in \mathcal{B}$ for all $y \in Y$). Then we can define $\iota_Y : Y \to \mathcal{M}(Y)$ which denotes the map assigning each point $y \in Y$ to the corresponding pointmass δ_y in $\mathcal{M}(Y)$.

For the arguments to come we will need a topology on the space of quasi-probabilities in a compact Hausdorff space X. The Riesz representation theorem holds for quasi-probabilities and quasi-integrals (c.f. [1]). Hence we can define a weak*-topology on $Q(X)$ by identifying the quasi-probabilities with the quasi-integrals. That is, we define the topology on $Q(X)$ to be topology of pointwise convergence on $C(X)$ (the continuous functions on X). This turns $Q(X)$ into a compact Hausdorff space.

In the set of quasi-probabilities the $\{0,1\}$-valued quasi-probabilities need not be pointmasses (e.g. the median in section 4). The collection of $\{0,1\}$-valued quasi-probabilities in a compact Hausdorff space X will be denoted X^*. Moreover we will refer to them as the simple quasi-probabilities. The reason for not referring to them as extreme is due to the fact that they generally (in contrast to measures) is a proper subset of the extreme quasi-probabilities (see for instance [2]). The space X^* is given the topology relative to $Q(X)$.

Example 14 *For any compact Hausdorff space X we may define a map Ψ^*_α : $\mathcal{O}(X) \to \mathcal{O}(Q(X))$ by $\Psi^*_\alpha(U) = U^*_\alpha$, where $U^*_\alpha = \{\mu \in Q(X) : \mu(U) > \alpha\}$ with $\alpha \in \mathbb{R}$ fixed. The sets $\{U^*_\alpha : U \in \mathcal{O}(X), \alpha \in \mathbb{R}\}$ can be shown to be a subbasis for the topology of $Q(X)$ (c.f. [8]), furthermore the map Ψ^*_α extends to a continuous image transformation.*

The result of the following proposition is known (c.f. [2]). However, we will explicitly use the construction in our proof of Proposition 15, and therefore we include the result.

Proposition 15 *If X is a metric compact Hausdorff space, then $Q(X)$ is metric. In particular X^* is metric.*

Proof. Since X is second countable, there is a countable basis $\{O_i\}$ for the topology of X. For any finite subset $S \subset \mathbb{N}$ put $O_S = \bigcup_{i \in S} O_i$. Then the set $E = \{O_S : S \subset \mathbb{N} \text{ is finite}\}$ is countable. In particular the set

$$\tau_{0=} \{\Psi^*_r(O_S) : S \subset \mathbb{N} \text{ is finite, } r \in \mathbb{Q}\} \leftrightarrow E \times \mathbb{Q}$$

is countable. It suffices to show that this set is a subbasis for the topology of $Q(X)$. Let $U \in \mathcal{O}(X)$ and $\alpha \in \mathbb{R}$ be arbitrary. Assume $\mu \in U^*_\alpha$, then pick an increasing sequence $\{O_{S_i}\} \subset E$ such that $O_{S_i} \nearrow U$. Then $\mu(O_{S_i}) \nearrow \mu U$ and so there is an $O_S \in E$ with $O_S \subset U$ and $\mu(O_S) > \alpha$. Pick a rational number r with $\alpha \leq r < \mu(O_S)$, then $\mu \in \Psi^*_r(O_S) \subset U^*_\alpha$ which shows that τ_0 is a subbasis for the topology of $Q(X)$. ■

Remark 16 *In addition to proving the result, the proof also gives an explicit way of constructing the subbasis through the image transformation and finite unions of basis open sets. For the subspace X^* the measures only takes the values zero and one hence it suffices to fix α in Ψ^*_α equal to zero.*

Corollary 17 *If X is a metric compact Hausdorff space, there is a countable family of open sets $\{O_i\}$ in X such that $\{\Psi^*_0(O_i)\}$ is a subbasis for the topology of X^*.*

Example 18 *Let X be any metric q-space, and suppose $q : \mathcal{A}(X) \to \mathcal{B}$ is an image transformation. Then we can define a map $w : Y \to X^*$ by $w = q^* \circ \iota_Y$. If Y is compact Hausdorff and q is a continuous image transformation, then w is a continuous function (c.f. [4, Theorem 3.5]). This property is the reason* *for us labelling them continuous. We clarify this by our generalization of the structure theorem for image transformations. In the structure theorem below we shall see that our image transformations correspond to the measurable maps from Y to X^*.*

Theorem 19 (The structure theorem for image transformations) *Let X be any metric q-space. Then there is a one-to-one correspondence between image transformations $q : \mathcal{A}(X) \to \mathcal{B}$ and measurable maps $w : Y \to X^*$ such that the following diagram commutes*

$$\begin{array}{cccc} & \mathcal{A}(X^*) & & w^{-1} \\ \Psi_0^* & \uparrow & \searrow & \\ & \mathcal{A}(X) & \longrightarrow & \mathcal{B} \\ & & q & \end{array} \tag{1}$$

where the measurable map w is given by $w = q^* \circ \iota_Y$, and the σ-algebra of X^* is the Borel sets.

Proof. We start by showing that the diagram commutes with $w = q^* \circ \iota_Y$. For any $U \in \mathcal{O}(X)$ we have $y \in q(U) \Leftrightarrow \delta_y(q(U)) = 1 \Leftrightarrow q^*\delta_y \in U^* \Leftrightarrow w(y) \in U^* \Leftrightarrow y \in w^{-1}(\Psi_0^*(U))$. The closed sets follow by compliment. Hence the diagram commutes. Next we show that w is measurable. Since the diagram is commutative all sets of the form $w^{-1}(\Psi_0^*(U)), U \in \mathcal{O}(X)$ will be measurable. By Corollary 17 a countable collection of such sets will be a subbasis for the topology of X^*. Since X^* is second countable we can get any open subset O of X^* by finite intersections and countable unions of such sets. Hence $w^{-1}(O) \in \mathcal{B}$ for any $O \in \mathcal{O}(X^*)$, accordingly w is measurable with respect to the Borel sets in X^*. Conversely, given a measurable map $w : Y \to X^*$define $q = w^{-1} \circ \Psi_0^*$. Then q is an image transformation, so it suffices to show that $w = q^* \circ \iota_Y$. For any $y \in Y$ and $U \in \mathcal{O}(X)$ we have $(q^*\delta_y)(U) = \delta_y(q(U)) = \delta_y(w^{-1}(U^*)) = 1 \Leftrightarrow w(y) \in U^* \Leftrightarrow w(y)(U) = 1$. The proof is complete. ■'

Remark 20 *The adjoint q^* of a continuous image transformation is defined on all of $Q(Y)$. In [4] it was shown that the adjoint of a continuous image transformation is actually continuous on $Q(Y)$.*

Recall that any compact Hausdorff space has a neighborhoodbase around each of its points consisting of compact connected sets.

Proposition 21 *If $q : \mathcal{A}_s(X) \to \mathcal{B}(Y)$ satisfies (A'), (B') and (C'), then q extends uniquely to an image transformation $\tilde{q} : \mathcal{A}(X) \to \mathcal{B}(Y)$.*

Proof. The adjoint of q is well defined, and we denote it by q^*. Define $w : Y \to X^*$ by $w = q^* \circ \iota_Y$. Analogous to the argument in the structure theorem we may verify that $qA = w^{-1}(A^*)$ for all $A \in \mathcal{A}_s$. We must now show that w is measurable. By [3, Lemma 3.2] the compliment of any compact connected set K is a disjoint union of open solid sets. Since X is second countable this union must be countable. Let $\biguplus O_i; O_i \in \mathcal{O}_s(X)$ be a disjoint union, and let $\mu \in X^*$ with $\mu(\biguplus O_i) = 1$. Then by regularity and additivity $\mu(O_i) = 1$ for exactly one i. This means that $\Psi_0^*(\biguplus O_i) = \biguplus O_i^*$, implying that $w^{-1}[\Psi_0^*(\biguplus O_i)] \in \mathcal{B}$. Hence, by compliment $w^{-1}(K^*) \in \mathcal{B}$. In particular, any finite union of compact connected sets can be written as a finite disjoint union of compact connected sets (if two sets intersect their union is connected). The argument above with additivity yields $\Psi_0^*(\biguplus_{i=1}^n K_i) = \biguplus_{i=1}^n K_i^*$. Hence $w^{-1}[\Psi_0^*(\bigcup K_i)] \in \mathcal{B}$ for any finite union of compact connected sets K_i. Finally, assume that $O \in \mathcal{O}$ is any open set. Then by Lemma 21 it is a union of compact connected neighborhoods of each of its points, $O = \bigcup_{x \in O} K_x$. By picking open neighborhoods inside each K_x, and using second countability we may assume that the union is countable (i.e. $O = \bigcup K_i = \bigcup O_i$ with $O_i \subset K_i$) . The proof of Proposition 15 now applies since $\bigcup_{i=1}^n O_i \subset \bigcup_{i=1}^n K_i \nearrow O$, so for any $\mu \in O^*$ we can find a finite union of compact connected sets $\bigcup_{i=1}^n K_i \subset O$ such that $[\bigcup_{i=1}^n K_i]^* \subset O^*$ is a neighborhood of μ in X^*. Hence w^{-1} maps a subbasis for $\mathcal{O}(X^*)$ into $\mathcal{B}$. Again by second countability of X^* we must have $w^{-1}[\mathcal{O}(X^*)] \subset \mathcal{B}$. We have shown that $w : Y \to X^*$ is measurable. By the structure theorem for image transformations w corresponds to a unique image transformation $\tilde{q}$ such that the diagram 1 commutes. By the commutativity of the diagram we must have $q(A) = \tilde{q}(A)$ for all $A \in \mathcal{A}_s(X)$. Accordingly $\tilde{q}$ is an extension of q from the solid sets. By the structure theorem this extension must be unique since w is unique. The proof is complete. ∎

Recall that any σ-algebra $\mathcal{B}$ is naturally a Boolean algebra under union and intersection. The Stone space K of $\mathcal{B}$ can then be constructed from a subset of $\mathbb{Z}_1^{\mathcal{B}}$ endowed with product topology (see for instance [14] for details). It can then be shown that the Stone space is an extremely disconnected compact Hausdorff space. Further more there is a canonical map $T : \mathcal{B} \to \mathcal{O}(K) \cap \mathcal{C}(K)$ where T is a Boolean algebra isomorphism from the σ-algebra

$\mathcal{B}$ to the Boolean algebra of closed and open subsets of K.

Lemma 22 *If X is a metric compact Hausdorff space and $q : \mathcal{A}(X) \to \mathcal{B}(Y)$ is an image transformation, then $T \circ q : \mathcal{A}(X) \to \mathcal{A}(K)$ is a continuous image transformation.*

The proof is a routine verification of the requirements of an image transformation exploiting the fact that T is a Boolean algebra isomorphism.

In particular, T induces a bijection of measures $T^* : \mathcal{M}(K) \to \mathcal{M}(Y)$ (c.f. [16]) canonically.

As mentioned earlier the $\{0,1\}$-valued quasi-probabilities are not the only extreme quasi-probabilities. We denote the closed convex span of X^* with $R(X)$. The convex space $R(X)$ is referred to as the representable quasi-probabilities and is in general different from $Q(X)$.

Proposition 23 *Let X be metric compact Hausdorff space, and let $q : \mathcal{A}(X) \to \mathcal{B}$ be any image transformation. Then $q^*(\mathcal{M}(Y)) \subset R(X)$.*

Proof. The Stone space K of $\mathcal{B}$ is extremely disconnected, hence $Q(K) = \mathcal{M}(K)$ =Borel probability measures in K (this was actually proven first in [7] for quasi-integrals when the underlying space K was totally disconnected). Now the adjoints of image transformations are affine maps (i.e. preserves convex combinations). Hence $[(T \circ q)^* \circ \iota_K](K) \subset R(X)$ implies $(T \circ q)^*$ maps the convex span of the Dirac measures into $R(X)$. By continuity of w in Lemma 22 this extends to the closed convex span and hence to $\mathcal{M}(K) = Q(K)$. The result is now due to T^* being a bijection of measures such that $(T \circ q)^*[\mathcal{M}(K)] = q^*(\mathcal{M}(Y))$. ■

It is well known that the map T into the closed and open sets of the Stone space naturally induces an isometric algebra isomorphism $T^{**} : l_b(Y) \to C(K)$ of the measurable bounded functions on Y onto the continuous functions on K. Both algebras are endowed with supremum norm. In the theory of quasi-measures we can only expect linear behavior on the singly generated subalgebras of functions. With X and Y compact Hausdorff the image transformation in [4] was lifted to a map $q^{**} : C(X) \to C(Y)$ of the continuous functions. The map was shown to be an algebra-homomorphism on singly generated subalgebras, and named quasi-homomorphism. Inspired by the quasi-homomorphisms in [4] we give the definition below.

For a normed unital algebra $\mathfrak{A}$ we will denote the closed subalgebra generated by $1_{\mathfrak{A}}$ and an element $f \in \mathfrak{A}$ with A_f.

Definition 24 *Let $\mathfrak{A}$ and $\mathfrak{B}$ be normed unital algebras. A function $T : \mathfrak{A} \to \mathfrak{B}$ is a quasi-homomorphism if T is an algebra-homomorphism of A_f onto $A_{T(f)}$ for each self-adjoint $f \in \mathfrak{A}$*

Hence a quasi-homomorphism $q^{**} : C(X) \to l_b(Y)$ is defined to be a map which is an algebra homomorphism on each closed singly generated subalgebra $A_f; f \in C(X)$ of $C(X)$ onto each subalgebra of $l_b(Y)$ generated by $q^{**}(f)$. Note that our interest is real valued functions here. One may complexify linearly however, and formulate the definition for C*-algebras.

Proposition 25 *An image transformation $q : \mathcal{A}(X) \to \mathcal{B}(Y)$ lifts naturally to a quasi-homomorphism $q^{**} : C(X) \to l_b(Y)$.*

Proof. The map $T \circ q$ is a continuous image transformation, and hence corresponds to a unique quasi-homomorphism $(T \circ q)^{**} : C(X) \to C(K)$ by [4, Theorem 4.3]. The desired quasi-homomorphism is then obtained by $(T \circ q)^{**} \circ (T^{**})^{-1}$. ■

Remark 26 *The corresponding quasi-homomorphism in [4] may be obtained by moving level sets of the functions with the image transformations. Essentially this is the same construction as in Banach-Stones Theorem. Accordingly, by composition of T^{**} and q^{**} one may show that the quasi-homomorphism above is obtained by moving level sets of the functions. We will not do so here.*

4 The median and the sample median

Throughout this section we will omit proofs. A more complete treatment of the median is beside the scope of this treatment. At this point we want to illustrate that the image transformations serve to generalize variables.

We will denote the cardinality of a finite set S with $|S|$. In addition, we will put $I_n = \{i\}_{i=1}^n$.

Definition 27 *Let $(Y, \mathcal{B}, P)$ be a probability space and X be a metric q-space. If $\{T_i : Y \to X\}_{i=1}^{2n+1}$ is an odd numbered collection of measurable maps (i.e. random variables) with respect to the Borel sets in X we define the sample median of $\{T_i\}$ to be a set function $\mu : \mathcal{A}_s(X) \to \mathbb{R}$ by $\mu C = P(|T_i \in C| > n)$, i.e. the probability of over half of the variables being in C.*

Notice that our definition is with respect to any collection of variables regardless of dependencies between them. This generality is particularly amenable in situations where independence of observations can not be assumed, as often is the case in experimental statistics. Moreover, the definition is topological where the geometry of the space is replaced by the concept of solid sets. The definition is even independent on the choice of metric for X. The simplicity of the definition should also make it easily accessible to undergraduate students.

Theorem 28 *The sample median extends uniquely to a quasi-probability in X. The construction is given by an image transformation $q : \mathcal{A}_s(X) \to \mathcal{B}$ with*

$$qA = \bigcup_{\{S \subset I_{2n+1} : |S| > n\}} [\bigcap_S T_i^{-1}(A)] \qquad \text{for all } A \in \mathcal{A}_s(X)$$

Remark 29 *The crucial part of the proof is realizing that the sample median is given by the image transformation. Even for the most basic examples this image transformation is not the inverse image of a measurable map. Hence we claim that the sample median should not be thought of as a variable, but in terms of its inverse images -the image transformation. We will denote the sample median of $\{T_i\}_{i \in I}$ with μ_I. Similarly the image transformation q in the proof depends on the measurable maps and will be denoted $M_{\{T_i\}}$. In view of Theorem 28 the sample median will be assumed to be a quasi-probability defined on all open or closed sets.*

The median (and sample median) in $\mathbb{R}$ is preserved under monotone maps. For our general setting we will need a more general concept than monotone maps. This is provided below with the solid variables.

Definition 30 *Let X_1 and X_2 be compact Hausdorff spaces. A map $f : X_1 \to X_2$ will be called a* solid variable *if f is continuous and $f^{-1}(\mathcal{A}_s(X_2)) \subset \mathcal{A}_s(X_1)$. Similarly a continuous image transformation $q : \mathcal{A}(X_2) \to \mathcal{A}(X_1)$ will be called solid if $q(\mathcal{A}_s(X_2)) \subset \mathcal{A}_s(X_1)$.*

Theorem 31 *Let X_1 and X_2 be metric q-spaces. Given a measurable space $(Y, \mathcal{B})$ and measurable maps $T_i : Y \to X_1$ for $i = 1, 2, ..., n$. Then for any solid variable $f : X_1 \to X_2$ we have*

$$f^* \circ M^*_{\{T_i\}} = M^*_{\{f \circ T_i\}}$$

on the set of probability measures in $(Y, \mathcal{B})$.

Remark 32 *This result is the* raison d'être *for our median allowing us to preserve the sample median under an abundance of transformations. The preservation property is usually referred to as equivariance in statistical terminology. The theorem shows that the sample median is exactly the set function on $\mathcal{A}_s(X)$ that corresponds to the transformation of the (ordinary one dimensional) sample median under the solid variables. Hence the equivariance properties forces us to consider a quasi-probability.*

In the limiting case letting the number of variables tend to infinity we should have the notion of a median. Hence, since we are dealing with a limit of measures, the natural median should be a measure rather than points. This is in contrast to the approach by statisticians where a point or even a set of points is sought. For the construction we will need the notion of *splitting* measures (c.f.. [3]). We say that a quasi-probability P in a compact Hausdorff space X is splitting if there exists disjoint sets $C_1, C_2 \in \mathcal{C}_s(X)$ such that $P(C_1) + P(C_2) = 1$ with $P(C_1), P(C_2) > 0$. If no such pair exists we call P *non-splitting*. How to construct the median of splitting measures is given in the preprint [13], we will not present that here.

Definition 33 *Let $(X, \mathcal{B}, P)$ be a probability space where X is a q-space, $\mathcal{B}$ consists of the Borel sets in X, and P is a non-splitting probability measure. The median of P is defined to be a set function $P_m : \mathcal{C}_s \to \{0, 1\}$ by*

$$P_m(C) = \begin{cases} 0 & ; \quad P(C) < \frac{1}{2} \\ 1 & ; \quad P(C) \geq \frac{1}{2} \end{cases}$$

Proposition 34 *The median uniquely extends to a quasi-probability in X.*

Remark 35 *Our construction differs fundamentally with the classical notion of a median. We claim that the natural median is a set function, namely a quasi-probability rather than being a set of points. In $\mathbb{R}$ the median will be a point mass, where the point is the ordinary median in one dimension.*

Note that when the median P_m can be constructed from an image transformation $M_P : \mathcal{C}_s(X) \to \mathcal{C}_s(X)$ by

$$M_P C = \begin{cases} \emptyset & ; P(C) < \frac{1}{2} \\ X & ; P(C) \geq \frac{1}{2} \end{cases}$$

Where the extension to open solid sets is by complement. Hence we have a map $M^* : Q(X) \to Q(X)$ where $P \mapsto M_P^*(P) = P_m$ which is just sending the measure to the median in terms of image transformations.

Theorem 36 *If X_1, X_2 are q-spaces and $q : \mathcal{A}(X_2) \to \mathcal{A}(X_1)$ is a solid image transformation, then the following diagram is commutative*

$$\begin{array}{ccc} Q(X_1) & \overset{q^*}{\longrightarrow} & Q(X_2) \\ M^* \uparrow & & \uparrow M^* \\ Q(X_1) & \underset{q^*}{\longrightarrow} & Q(X_2) \end{array}$$

Remark 37 *Notice that this theorem is the medians version of Theorem 31. However, this statement is more general involving solid image transformations. Still an important class of examples is when the image transformation is derived from a solid variable.*

We have only presented the definitions and equivariance properties of the median. However, it should be noted that the median has a non-linear behavior in multidimensional spaces (in contrast to the mean). This is well known, but has not been well understood. Introducing the quasi-measure and quasi-integral we have a solution to the linearity problem. The linearity of the median is reduced to determining whether the corresponding quasi-integral is linear. In [17] it was shown that quasi-measures are restrictions of regular Borel measures in one dimensional spaces. Accordingly the median exhibits linear behavior in the one dimensional setting.

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The multidimensional median and sample median defined as quasi-probabilities

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Abstract

The objective of the paper is to give a theoretic and uniting generalization of the median and sample median for multidimensional problems. The sample median is defined with respect to a finite collection of variables. The median is defined for regular Borel measures. Both concepts are shown to yield quasi-measures. The construction is done by a generalized image transformation. Numerous examples are provided throughout the paper.

Keywords: Multidimensional median, quasi-measure, quasi-probability, image transformation.

1 Introduction

The concept of median in spaces of more than one dimension dates back to the turn of the century. One of the earlier works was done by J. Hayford and may be found in [9]. The problem was to find an estimator for the center of the population of the United States. Hayford proposed to use the vector of medians of the coordinates, although recognizing that this concept is dependent on the choice of axis. Since then, several attempts have been made to give a natural definition of a higher dimensional median, but no particular concept has prevailed. For a survey of different multidimensional medians at hand the reader is referred to [14]. A main theme is the search of a symmetry center, where various definitions of symmetry have been applied (i.e. finding points which satisfies certain symmetry requirements, see [14] for details). Researchers are left with no general and uniting concept. This can be illustrated by the fact that the distributional analogue to the median (i.e. the median of a distribution rather than a sample) is missing, not well defined or not in general existing. In this paper we concentrate on the theoretic aspects of the multidimensional median, and we provide a measure theoretic field of mathematics for handling the concept. We define the sample median of any finite collection of measurable maps on a probability space with image space being metric compact spaces. The median will be defined for any Borel measure on compact spaces with some restrictions to the space.

Our median and sample median will be constructed as a quasi-measure. The quasi-measure or quasi-probability originated in [1] as a solution to the problem of finding non-linear states on C*-algebras formulated by R. Kadison (c.f. [10]). With respect to a quasi-probability measure, an integration theory has been developed, where the quasi-integral differs from the usual integral in not being linear. However, the quasi-integral is linear on certain classes of functions. In [15] it is shown that any quasi-measure on a one dimensional space is necessarily a measure which goes to show that the sample median and median is well behaved in one dimension.

Equivariance (i.e. preserving the median under transformations) has been a major issue for the median. As mentioned initially the median in [9] was dependent on choice of axis which is very unfortunate considering the problem it was designed to solve. Different approaches have been made with defining symmetry properties. However, the different symmetry properties are not in general applicable. Perhaps the most natural equivariance property is with respect to the different coordinates, that is, the transformation taking the projection down to an axis should preserve the median. None of the proposed medians have this property, which is not strange since this property naturally leads to a quasi-measure (c.f. section 4). On the real line the class of maps for which the median is equivariant is the monotone maps. Hence there are some limitations as for how large a class of maps we can expect equivariance. In this respect we introduce solid variables as a multidimensional analogue to the monotone maps.

Finally, we present how non trivial properties of the quasi-integral yields linearity and continuity properties for the median and sample median. In particular, expectations are linear on singly generated subalgebras of variables and uniformly continuous in general.

2 Basic properties of the sample median

We will start with identifying the basic properties of the sample median in $\mathbb{R}$ leading to our definition. Let $(X, \mathcal{B}, P)$ be a probability space, and let $\{T_i : X \to \mathbb{R}\}_{i=1}^{2n-1}$ be a collection of Borel measurable variables. Then the median of the variables is well defined by the n'th order statistic $T_{(n)}$ and its distribution is given by a probability measure μ in $\mathbb{R}$. The standard approach for constructing a measure in $\mathbb{R}$ is to start with sets of the type $\{(-\infty, a), (-\infty, a]\}_{a\in\mathbb{R}}$ or their compliments. Apart from being open or closed these sets have a particular topological property defined below.

Definition 1 *An open or closed set is called solid if the set and its compliment are both connected.*

Notice that the sets given above are the only solid sets in $\mathbb{R}$. For a solid set $A \subset \mathbb{R}$ we have $T_{(n)} \in A$ if and only if $T_i \in A$ for at least n values of the index (that is, at least half the variables are in A). Accordingly we will define the probability measure of the sample median. First some notation. We denote the cardinality of a finite set S with $|S|$. Let $\mathcal{C}(X)$ and $\mathcal{O}(X)$ respectively denote the closed and open subsets of a space X. In addition we put $\mathcal{A}(X) = \mathcal{C}(X) \cup \mathcal{O}(X)$. When there is no confusion concerning the space in question, we will omit the space in the notation. Similarly we let the subscript s denote the solid sets (e.g. $\mathcal{C}_s$ are the closed solid sets).

Definition 2 *Let the probability measure μ of the sample median be defined by $\mu A = P(|T_i \in A| \geq n)$; $A \in \mathcal{A}_s(\mathbb{R})$, i.e. the probability of over half of the variables being in A.*

Of course, the set function μ is a probability measure for the variable $T_{(n)}$. It is also clear that the solid sets will determine μ completely. None of these statements are clear when the image space of the variables is higher dimensional. With the intent of generalizing the sample median this definition has one obvious advantage. It is purely topological with no concerns to the ordering of the variables. The necessary geometry of the spaces is described by connectedness.

Notice that this definition may easily be communicated to undergraduate students. Even in a setting where $\mathbb{R}$ is replaced by a more general space (e.g. $\mathbb{R}^n$). However, with an intuitive understanding of topology.

If we apply our definition to a more general space than $\mathbb{R}$ we would like to know what kind of set function our μ is. Also we would like to be able to describe the sample median in terms of a variable. To do this we need two concepts given below.

The letter X will denote a compact Hausdorff space and $(Y, \mathcal{B})$ will denote a measurable space in the sequel. The quasi-probabilities are topological measures in the sense that they are only defined on closed and open sets. With a *quasi-probability* in X we will mean a set function $\mu : \mathcal{A} \rightarrow \mathbb{R}^+$ such that the following hold:

(i) $\mu(\biguplus_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ ($\biguplus$ indicates disjoint union, and we assume all A_i and $\biguplus_{i=1}^{\infty} A_i$ in $\mathcal{A}$)

(ii) $\mu U = \sup \{\mu C : C \subset U; C \in \mathcal{C}\}$ for all U in $\mathcal{O}$

(iii) $\mu(X) = 1$

Remark 1 *Notice that one immediate consequence of (i) and (ii) is the monotonicity of the quasi-probability. That is* $A_1 \subset A_2$ *implies* $\mu A_1 \leq \mu A_2$ *whenever* $A_1, A_2 \in \mathcal{A}$.

The definition of the quasi-probability differs from that of a probability measure only by its domain. We only define it for open and closed sets. The quasi-probabilities are however a vastly larger class of set maps than the Borel measures. Perhaps their most distinct difference is that they are not in general subadditive.

Our vehicle for constructing the multidimensional median will be image transformations. Our image transformation was introduced in [13] as a generalization of measurable maps. It describes inverse images rather than values of the variable.

Definition 3 *We define an image transformation to be a map* $q : \mathcal{A}(X) \rightarrow \mathcal{B}(Y)$ *from the closed subsets of a metric (compact) space* X *into the sigma algebra of a measurable space* $(Y, \mathcal{B})$*, such that the following is satisfied*

1. $A_1 \cap B_2 = \emptyset \Rightarrow qA_1 \uplus qB_2 = q(A_1 \cup B_2)$

2. $qX = Y$

3. $U_i \nearrow U \Rightarrow qU_i \nearrow qU \; ; U_i, U \in \mathcal{O}(X)$ *for* $i = 1, 2, ...$

If in addition Y is a compact Hausdorff space and $q(\mathcal{O}(X)) \subset \mathcal{O}(Y)$ we will call q a continuous image transformation.

Remark 2 *We restrict ourselves to the metric situation for the space* X*, this is connected to property 3 of Definition 3 (details may be found in [13].The loss in generality by imposing metrizability on the image space is not crucial since it enables us to define the median in* $\mathbb{R}^n$ *for arbitrary* $n \in \mathbb{N}$ *(among other spaces).*

Example 1 *Let* $T : Y \rightarrow X$ *be a measurable map with respect to the Borel sets in* X*. Then the map* $T^{-1} : \mathcal{A}(X) \rightarrow \mathcal{B}(Y)$ *is an image transformation. In this case we say that the image transformation is derived from the function* T*. The image transformations derived from functions are trivial in the sense that their adjoint map measures to measures.*

3 Quasi-probabilities and Image transformations

Now that we have defined the quasi-probability and the image transformation it is natural to ask what type of space is X (the image space of the variables). This leads to the somewhat awkward definition below. The restrictions are closely related to the problem of constructing quasi-probabilities.

Definition 4 *If* X *is locally connected, connected and has genus equal zero (* $g(X) = 0$ *) we will call* X *a* q-space.

Remark 3 *These properties are shared by a large class of spaces such as closed intervals and disks in addition to balls and spheres in* $\mathbb{R}^n, n \geq 3$*. The genus requirement is treated (and defined) in [3] and [11], we will not elaborate on that issue here. The reader may settle with the fact that simply connected spaces have* $g = 0$*.*

The solid sets play an important role in the theory of quasi-probabilities. They constitute a small and manageable family of sets that totally determines a quasi-probability. This is illustrated by the solid set-functions, they were introduced in [3] and their properties were investigated there. In particular they are invaluable tools for constructing quasi-probabilities. We recall their definition. Let X be a *q-space*. Then a function $\mu : \mathcal{A}_s \to \mathbb{R}^+$ is a solid set-function if it satisfies

(A) $\sum_{i=1}^{n} \mu C_i \leq \mu C$ whenever $\biguplus_{i=1}^{n} C_i \subset C; C_i, C \in \mathcal{C}_s$ for $i = 1, 2, ..., n$

(B) $\mu U = \sup\{\mu C : C \subset U; C \in \mathcal{C}_s\}$ for all $U \in \mathcal{O}_s$

(C) $\mu A + \mu(X \backslash A) = \mu X$

Remark 4 *Again we will only consider the case $\mu X = 1$. The basic construction of quasi-probabilities has been given in [1], [3] and [11]. The main construction result ([3], Theorem 5.1) states that a solid set-function uniquely extends to a quasi-probability on $\mathcal{A}$.*

The following propositions give the basic properties of the image transformations. We include them for the readers convenience.

Proposition 1 *If $q : \mathcal{A}(X) \to \mathcal{B}(Y)$ is an image transformation the following hold*

1. $A \subset B \Rightarrow qA \subset qB$ for any $A, B \in \mathcal{A}(X)$
2. $q(\biguplus_{i=1}^{n} A_i) = \biguplus_{i=1}^{n} qA_i; A_i, \biguplus_{i=1}^{n} A_i \in \mathcal{A}$ for $i = 1, 2, ..., n$
3. $C_i \searrow C \Rightarrow qC_i \searrow qC \,; C_i, C \in \mathcal{C}(X)$ for $i = 1, 2, ...$

Proposition 2 (Transformation of variables for image transformations) *If $(Y, \mathcal{B}, P)$ is a probability space and $q : \mathcal{A}(X) \to \mathcal{B}(Y)$ is an image transformation, then $q^*(P)$ defined by $(q^*P)A = P(qA)$ for all $A \in \mathcal{A}(X)$ is a quasi-probability in X.*

Notation We will denote the probability measures of a measurable space $(Y, \mathcal{B})$ by $\mathcal{M}(Y)$, and the quasi-probabilities of a compact Hausdorff space X by $Q(X)$.

Remark 5 *The map $q^* : \mathcal{M}(Y) \to Q(X)$ will be called the adjoint of q. If q is derived from a measurable map we of course get the well known situation of transformations of measures. However, as we shall see examples of (both the median and sample median may be interpreted as image transformations) this is not the case in general.*

If $q : \mathcal{A}(X) \to \mathcal{B}(Y)$ is an image transformation, we may restrict it to the solid sets. By Proposition 1 it is easy to verify that

(**A'**) If $C, C_i \in \mathcal{C}_s; i = 1, 2, ..., n$ and $\biguplus C_i \subset C$, then $\biguplus q(C_i) \subset qC$.

(**B'**) If $U, U_i \in \mathcal{O}_s; i = 1, 2, ...$ and $U_i \nearrow U$, then $q(U_i) \nearrow qU$.

(**C'**) For any $A \in \mathcal{A}_s$ we have $q(A) \biguplus q(X \backslash A) = Y$.

Proposition 3 *Let X be a q-space. If $(Y, \mathcal{B}, \mu)$ is a probability space and $q : \mathcal{A}_s(X) \to \mathcal{B}(Y)$ satisfies (A'), (B') and (C'), then $q^*(\mu)$ defined by $(q^*\mu)A = \mu(qA)$ for all $A \in \mathcal{A}_s(X)$ extends uniquely to a quasi-measure in X.*

Remark 6 *It is shown in [13] that any solid set map satisfying (A'), (B') and (C') extends uniquely to an image transformation. Hence we will not distinguish between the two concepts and refer to both as image transformations.*

4 The sample median

For a finite collection of measurable maps $\{T_i : Y \to X\}$ we will define the sample median of $\{T_i\}$ when X is a metric q-space. In particular this will include any closed ball in $\mathbb{R}^n$ and hence by inclusion any compact subset of $\mathbb{R}^n$. More explicitly, if the natural image space does not satisfy the requirements of a q-space we may embed it into a q-space. Hence a natural sample median is relative to an imbedding $\Phi : X \hookrightarrow K$ where K is a q-space. We will therefore assume that the measurable maps $\{T_i\}$ have a q-space as image space rather than considering the composite maps $\{\Phi \circ T_i\}$. Although the study of different imbeddings Φ relative to the median is of interest in itself, we will not pursue that issue here.

Note that we are only considering the compact situation. The theory of quasi-measures in compact Hausdorff spaces is well established through several articles. We conjecture however that our concept of a (sample) median may be generalized to locally compact Hausdorff spaces. The quasi-measures in locally compact spaces are presented in [4] and their integration theory was developed in [12].

Notation We put $I_n = \{i\}_{i=1}^n$.

Definition 5 *Let $(Y, \mathcal{B}, P)$ be a probability space and X be a metric q-space. If $\{T_i : Y \to X\}_{i=1}^{2n-1}$ is an odd numbered collection of measurable maps (i.e. random variables) with respect to the Borel sets in X we define the sample median of $\{T_i\}$ to be a set function $\mu : \mathcal{A}_s(X) \to \mathbb{R}$ by $\mu C = P(|T_i \in C| \geq n)$, i.e. the probability of over half of the variables being in C.*

Remark 7 *Notice that our definition is with respect to any collection of variables regardless of dependencies between them. This generality is particularly amenable in situations where independence of observations can not be assumed, as often is the case in experimental statistics.*

Example 2 *Imagine a dart player. He has three arrows to throw, and the first arrow will help him to improve his aim. Hence the three throws cannot be assumed to be independent. However, different series of throws are more likely to be independent. Accordingly we may estimate the median for any particular fixed solid set in the dart board (here we regard the dart board as a closed disk) by using a series of throws.*

Theorem 1 *The sample median extends uniquely to a quasi-probability in X.*

Proof. Consider the set map $q : \mathcal{A}_s(X) \to \mathcal{B}$ defined by

$$qA = \bigcup_{\{S \subset I_{2n-1} : |S| > n\}} [\bigcap_S T_i^{-1}(A)]$$

We claim that q is an image transformation. The regularity (B') requirement is preserved by finite intersection and finite unions. The surjectivity requirement ($qX = Y$) is trivially true since the intersection will be of Y with itself. For the remaining claims notice that $y \in qC \Leftrightarrow |T_i y \in C| \geq n$. To show (C') we need to consider the case $A \biguplus B = X$ in which case it is obvious that for each $y \in Y$ we must have over half of $\{T_i y\}$ contained in either A or B. Accordingly we have $qA \biguplus qB = Y$. Finally, suppose

$C, C_i \in \mathcal{C}_s; i = 1, 2, ..., m$ with $\biguplus C_i \subset C$. If $y \in q(C_{i'})$for some i' then since the C_i's are disjoint we can not have $|T_i y \in C_j| \geq n$ for any $j \neq i'$. Hence the sets $q(C_i); i = 1, 2, ..., m$ are disjoint, and obviously if $y \in q(C_i)$ for some i, then $y \in qC$. The proof is complete. ■

Remark 8 *We will denote the sample median of $\{T_i\}_{i \in I}$ with μ_I when there is no confusion about the set of measurable maps in question. Similarly the image transformation q in the proof depends on the measurable maps and will be denoted $M_{\{T_i\}}$. In view of Theorem 1 the sample median will be assumed to be a quasi-probability defined on all open or closed sets.*

Definition 6 *For an even numbered collection $\{T_i : Y \to X\}_{i=1}^{2n}$ of measurable maps (analogous situation as in Definition 5), we define the sample median to be the quasi-probability*

$$\mu = \frac{1}{2n} \sum_E \mu_S \; ; E = \{S \subset I_{2n} : |S| = 2n - 1\}$$

Remark 9 *The concept here is that the even numbered sample median is a linear combination of sample medians rather than a transformation of the variables (as in the one dimensional case where the mean value of the two midpoints are taken). This definition is a suggestion, we realize that other proposals might be more suitable.*

Notice that given the measurable maps $\{T_i\}$ the sample median is a map from the probability measures on Y, $\mathcal{M}(Y)$ into the quasi-probabilities in a q-space X. We will denote this map with $M^*_{\{T_i\}}$ (corresponding to the notation of the adjoint of an image transformation) when there is no confusion concerning the probability space. In the odd numbered case the median can be interpreted as an image transformation (c.f. proof of Theorem 1). This image transformation will be denoted $M_{\{T_i\}}$.

Example 3 *Let X be any metric q-space, and let $x = (x_1, x_2, x_3, x_4)$ be any element of X^4. Further, let $T_i : X^4 \to X$ be the i'th projection map (i.e. $T_i(x) = x_i$). Then X^4 endowed with the Dirac measure δ_x in x, is a probability space where $\delta_x = \delta_{x_1} \times \delta_{x_2} \times \delta_{x_3} \times \delta_{x_4}$ is the product measure of the Dirac measures in the coordinates. Hence the variables $\{T_i\}$ are independent, but they are not identically distributed. Let μ_i denote the median of $\{T_j\}_{j \neq i}$. For any solid set $A \in \mathcal{A}_s(X)$ we have $\mu_i(A) = P(|T_j \in A| \geq 2)$ where $j \in$*

$I_4 \setminus \{i\}$. *Which gives us that* $\mu_i(A) = 1$ *if and only if* $\left|A \cap \{x_j\}_{j \in (I_4 \setminus \{i\})}\right| \geq 2$, *i.e. if and only if at least two of the three coordinates with index different from* i *is in* A. *We may now calculate the median of* $\{T_i\}_{i=1}^4$ *as* $\frac{1}{4}\sum \mu_i$, *and verify that for any* $A \in \mathcal{A}_s(X)$ *we have*

$$[M^*_{\{T_i\}}(\delta_x)](A) = \begin{cases} 1 & \text{if } \left|A \cap \{x_i\}_{i \in I_4}\right| > 2 \\ \frac{1}{2} & \text{if } \left|A \cap \{x_i\}_{i \in I_4}\right| = 2 \\ 0 & \text{if } \left|A \cap \{x_i\}_{i \in I_4}\right| < 2 \end{cases}$$

In the one dimensional case ($X = [a, b]$) we will denote the n'th order statistic of a sample $\{T_i : Y \to [a, b]\}_{i \in I}$ by $T_{(n)}$. That is $T_{(n)} : Y \to [a, b]$ is the function that assigns each $y \in Y$ to the n'th largest value of $\{T_i y\}$.

Proposition 4 *In the one dimensional case, i.e.* $\{T_i : Y \to [a, b]\}_{i \in I}$, *the sample median of an odd numbered collection* $I = I_{2n+1}$ *will the probability distribution of the* n'*th order statistic* $T_{(n)}$. *In the even numbered* $I = I_{2n}$ *case the sample median will be* $\mu = \frac{\mu_1 + \mu_2}{2}$ *where* μ_1 *and* μ_2 *respectively is the probability distribution of the* n'*th and the* $(n+1)$'*th order statistic of* $\{T_i\}_{i=1}^{2n}$, $T_{(n)}$ *and* $T_{(n+1)}$.

Proof. First assume that $I = I_{2n-1}$ is odd numbered where I is the index set of the variables. A solid subset of $[a, b]$ is of the form $[a, r]$, $[a, s)$ or a complement of these. In any case $\mu(A)$ is the probability of over a half of the variables being in A. Note that since we are in a one dimensional space μ is actually a probability measure. Then for an open interval $(r, s) \subset [a, b]$ we have:

$$\begin{aligned} \mu(r, s) &= \mu[a, s) - \mu[a, r] = P(T_{(n)} < s) - P(T_{(n)} \leq r) \\ &= P(T_{(n)} \in (r, s)) \end{aligned}$$

Now for $I = I_{2n}$ the situation is somewhat more complicated. Put $E_y = \{T_i y\}_{i \in I}$; $y \in Y$, then for any closed or half open interval $J \in \{[a, r], [a, r)\}_{r \in [a,b]}$ consider the set $J \cap E_y$. If and only if $|J \cap E_y| = n$ we will have $y \in q_S J$ for exactly half of the sets in $\{S \subset I : |S| = 2n - 1\}$, put $E_J^n = \{y \in Y : |J \cap E_y| = n\}$ (notice that E_J^n is the event that exactly n of the variables is in J). Hence the sets $\{q_S J \cap E_J^n\}$ will constitute n copies of E_J^n , which can be done effectively by taking intersections, difference sets and unions. The second consideration is the case where $|J \cap E_y| > n$, so put $E_J = \{y \in Y : |J \cap E_y| > n\}$. Then observe that $y \in E_J$ if and only if $y \in q_S J$ for all the sets in $\{S \subset I : |S| = 2n-1\}$. Now we can calculate the sample median μ in terms of order statistics:

$$\begin{aligned}
\mu J &= \tfrac{1}{2n}\sum \mu_S J = \tfrac{1}{2n}\sum P(q_S J) = \tfrac{1}{2n}\sum P[(q_S J \cap E_J^n) \uplus (q_S J \cap E_J)] \\
&= \tfrac{1}{2n}\sum [P(q_S J \cap E_J^n) + P(q_S J \cap E_J)] \\
&= \tfrac{1}{2n}\sum P(q_S J \cap E_J^n) + \tfrac{1}{2n}\sum P(q_S J \cap E_J) \\
&= \tfrac{1}{2n}\sum nP(E_J^n) + \tfrac{1}{2n}\sum 2nP(E_J) = \tfrac{1}{2}P(E_J^n) + P(E_J) \\
&= \tfrac{1}{2}P[(T_{(n)} \in J) \cap (T_{(n+1)} \notin J)] + P(T_{(n+1)} \in J) \\
&= \tfrac{1}{2}P(T_{(n)} \in J) + \tfrac{1}{2}P(T_{(n+1)} \in J)
\end{aligned}$$

Then for an open interval $(r, s) \subset [a, b]$ we obtain

$$\begin{aligned}
\mu(r, s) &= \mu[a, s) - \mu[a, r] \\
&= \tfrac{1}{2}[P(T_{(n)} < s) - P(T_{(n)} \le r)] + \tfrac{1}{2}[P(T_{(n+1)} < s) - P(T_{(n+1)} \le r)] \\
&= \tfrac{1}{2}P(T_{(n)} \in (r, s)) + \tfrac{1}{2}P(T_{(n+1)} \in (r, s))
\end{aligned}$$

■

Remark 10 *The even numbered median may be interpreted to be that any of the n sample medians are equally likeable to represent the median of the sample. Of course, one might question whether to weigh each of the n sample medians equally, a Bayesian approach using any a priori knowledge about the variables at hand might suggest another convex combination of the quasi-probabilities* μ_S. *Actually any convex combination* $\sum \alpha_S \mu_S; \alpha_S \ge 0, \sum \alpha_S = 1$ *will still give a meaningful sample median. In elementary courses in statistics the median of an even numbered sample is defined to be the mean value of the two midpoints, i.e. the variable* $\frac{T_{(n)}+T_{(n+1)}}{2}$. *Notice that this variable has the same expectation as our distribution* $\frac{\mu_1+\mu_2}{2}$, *but it does not have the same distribution. Accordingly* $\frac{T_{(n)}+T_{(n+1)}}{2}$ *is an unbiased estimator for our even numbered median.*

Example 4 (Independent Identically Distributed Variables) *Consider a probability space* $(X, \mathcal{B}, P)$ *with* X *any q-space,* $\mathcal{B}$ *the Borel sets in* X, *and* P *a regular Borel measure in* X. *Let* $(X^n, \mathcal{B}^n, P^n)$ *be the n'th product space. Put* T_i *equal the i'th projection map on* X^n, *that is* $T_i : X^n \to Y$ *by* $(x_1, x_2, ..., x_n) \mapsto x_i$ *for* $i = 1, 2, ..., n$. *Then* $\{T_i\}$ *are independent identically distributed random variables in* X *with probability measure* P. *The sample median* μ *of an odd numbered collection* $\{T_i\}_{i=1}^{2n-1}$ *on the solid sets*

$A \in \mathcal{A}_s(X)$ can be calculated binomially in terms of $P(A)$ and $1 - P(A)$. That is, calculating the probability of at least half of the variables being in A we have

$$P(|T_i \in A| \geq n) = \sum_{i \geq n} \binom{2n-1}{i} P(A)^i (1 - P(A))^{2n-1-i}$$

An example with X being the unit disk, three variables and P the normalized Lebesgue measure in the unit disk is outlined in [5, Example 3.1]. However, the construction in [5] is done by q-functions *(see [5] for details). Also it is noted in [5] that only in very special situations will this construction give a measure. The even numbered median is a construction of odd numbered ones. It turns out that the median for $\{T_i\}_{i=1}^{2n}$ is equal to the median of $\{T_i\}_{i=1}^{2n-1}$. This property relies on both independentness and equality of distributions.*

Remark 11 *Note that the projection maps in the example above have the property that $T_i^{-1}(\mathcal{A}_s(X)) \subset \mathcal{A}_s(X^n)$. This is an important property which is shared by a large class of continuous maps. We formalize this property in the definition below.*

Definition 7 *Let X_1 and X_2 be compact Hausdorff spaces. A map $f : X_1 \to X_2$ will be called a* solid variable *if f is continuous and $f^{-1}(\mathcal{A}_s(X_2)) \subset \mathcal{A}_s(X_1)$. Similarly a continuous image transformation $q : \mathcal{A}(X_2) \to \mathcal{A}(X_1)$ will be called solid if $q(\mathcal{A}_s(X_2)) \subset \mathcal{A}_s(X_1)$.*

Theorem 2 *Let X_1 and X_2 be metric q-spaces. Given a measurable space $(Y, \mathcal{B})$ and measurable maps $T_i : Y \to X_1$ for $i = 1, 2, ..., n$. Then for any solid variable $f : X_1 \to X_2$ we have*

$$f^* \circ M^*_{\{T_i\}} = M^*_{\{f \circ T_i\}}$$

on the set of probability measures in $(Y, \mathcal{B})$.

Proof. Let μ be any probability measure in $(Y, \mathcal{B})$ and let $A \in \mathcal{A}(X_2)$ be arbitrary. Recall that f^{-1} defines the image transformation derived from f where $f^* : Q(X_1) \to Q(X_2)$ is the corresponding map of measures. Hence for n odd it suffices to show that $(M_{\{T_i\}} \circ f^{-1})(A) = M_{\{f \circ T_i\}}(A)$. Now $y \in (M_{\{T_i\}} \circ f^{-1})(A)$ if and only if $|T_i y \in f^{-1}(A)| > \frac{n}{2}$ which is equivalent to $|f(T_i(y)) \in A| > \frac{n}{2}$ which means that $y \in M_{\{f \circ T_i\}}(A)$, so the image transformations coincide, accordingly the mappings of measures coincide. If n is even the same argument applies to all of the collections $\{T_i\}_{i\in I}$ with $I \subset \{1, 2, ..., n\}$ and $|I| = n - 1$. Assume that $(Y, \mathcal{B})$ is given a probability measure P. Let μ_i denote the sample median of $\{T_j\}_{j\neq i}$, $f^*\mu_i = \mu_{f_i}$ and $\mu_f = M^*_{\{f\circ T_i\}}(P)$. Then by the previous argument, and by definition, we have $\mu_f = \frac{1}{n}\sum \mu_{f_i}$. Moreover, $(f^* \circ M^*_{\{T_i\}})(P) = f^*(\frac{1}{n}\sum \mu_i)$. Finally, for any set $A \in \mathcal{A}(X_2)$ we have

$$\begin{aligned} [f^*(\tfrac{1}{n}\sum \mu_i)](A) &= (\tfrac{1}{n}\sum \mu_i)[f^{-1}(A)] = \tfrac{1}{n}\sum \mu_i[f^{-1}(A)] \\ &= \tfrac{1}{n}\sum \mu_{f_i}(A) \end{aligned}$$

Which completes the proof. ∎

Remark 12 *This result is very important allowing us to preserve the sample median under an abundance of transformations. The preservation property is usually referred to as equivariance. Some of the transformations for which the sample median is equivariant will be outlined below, an attempt to give a complete description of the transformationclass at hand is beyond the scope of this treatment. Note that the solid variables are neither a vector space nor a convex space. However, the composition of two solid maps is solid.*

Example 5 *Homeomorphisms $\tau : X_1 \to X_2$ are solid variables. In particular the sample median is independent of choice of axis. That is, any linear transformation by invertible matrix followed by a translation of q-spaces preserve the sample median.*

Example 6 *On the real line the solid variables will be exactly the monotone continuous maps. This is however typical for one dimension. If we again consider an n-dimensional ball $B^n_r = \{x \in \mathbb{R}^n : \|x\| \leq r\}$ with $n \geq 2$ then the norm itself is a solid variable. That is, the function $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x) = \|x\|$ is solid. Hence by composition also any monotone continuous map of the norm, e.g. $h(x) = g(\|x\|)$ with g being a continuous monotone real valued function, is solid. More generally, with appropriate choice of q-space we have the unimodal variables being solid. Which is in sharp contrast to the monotone maps in the one dimensional setting.*

Example 7 *Consider the closed n-dimensional ball with radius $r \in \mathbb{R}$, that is $B^n_r = \{x \in \mathbb{R}^n : \|x\| \leq r\}$ with euclidean norm. Let l be any straight line in $\mathbb{R}^n$, i.e. $l = \{\alpha x + x_0 : \alpha \in \mathbb{R}\}$ where $x, x_0 \in \mathbb{R}^n$. Then the orthogonal* *projection P_l on the line l is a solid variable on B_r^n. In particular the projections down to the coordinate axis are solid variables. Since any compact subset of $\mathbb{R}^n$ is contained in B_r^n for some r, this applies to any multivariate sample median (assuming bounded image for the $\mathbb{R}^n$-valued variables).*

Remark 13 *The projections will especially give us the coordinates. This equivariance property necessarily forces us to consider a quasi-probability and in itself should justify the use of quasi-measures to model the sample median. We will make this property explicit in the results below.*

Proposition 5 *Let $(Y, \mathcal{B}, P)$ be a probability space and let X be a metric q-space. Assume that $\{T_i : Y \to X\}_{i=1}^{n-1}$ is a collection of measurable maps (with respect to the Borel measures in X). Then for any solid variable $f : X \to \mathbb{R}$ we have*

$$M_{\{T_i\}}(f^{-1}(a,b)) = (f \circ T)_{(n)}^{-1}(a,b) \text{ for any } a, b \in \mathbb{R}$$

Here $(f \circ T)_{(n)}$ denotes the n'th order statistic of $\{f \circ T_i\}_{i=1}^{n-1}$.

Proof. We have

$$\begin{aligned} Y &= M_{\{T_i\}}(X) = M_{\{T_i\}}(f^{-1}(-\infty, a) \cup f^{-1}(a,b) \cup f^{-1}(b,\infty)) \\ &= M_{\{T_i\}}(f^{-1}(-\infty, a)) \cup M_{\{T_i\}}(f^{-1}(a,b)) \cup M_{\{T_i\}}(f^{-1}(b,\infty)) \end{aligned}$$

hence $M_{\{T_i\}}(f^{-1}(a,b)) = Y \setminus \left[M_{\{T_i\}}(f^{-1}(-\infty,a)) \cup M_{\{T_i\}}(f^{-1}(b,\infty))\right]$ which implies that $y \in M_{\{T_i\}}(f^{-1}(a,b))$ if and only if the median of $\{(f \circ T_i)(y)\}_{i=1}^{n-1}$ is in (a,b). ■

Proposition 6 (Urysohn's lemma for solid variables) *Let X be any q-space. If $C \in \mathcal{C}_s(X)$ and $F \in \mathcal{C}(X)$ are disjoint and nonempty, there is a solid variable $f : X \to [0,1]$ such that $f|_C \equiv 0$ and $f|_F \equiv 1$. If in addition X is metric we may assume that $f^{-1}(0) = C$.*

Proof. The standard construction in Urysohn's lemma is by an increasing family of open sets indexed and ordered by rational numbers, e.g. [16]. That is we have a family $\{U_r\}_{r\in\mathbb{Q}} \subset \mathcal{O}(X)$ where $C \subset U_r, F \cap U_r = \emptyset \; \forall r \in \mathbb{Q}$, and $r_1 < r_2 \Rightarrow \bar{U}_{r_1} \subset U_{r_2}$. By [3, Lemma 3.3] there is for any $C \in \mathcal{C}_s(X)$ and $C \subset U \in \mathcal{O}(X)$ a set $V \in \mathcal{O}_s(X)$ such that $C \subset V \subset \bar{V} \subset U$. Hence we may assume that $\{U_r\}_{r\in\mathbb{Q}} \subset \mathcal{O}_s(X)$. We then define a continuous function $f : X \to [0,1]$ by

$$f(x) = \begin{cases} 1 \text{ if } x \notin \bigcup_{r\in\mathbb{Q}} U_r \\ \inf_{r\in\mathbb{Q}} \{r : x \in U_r\} \text{ elsewhere} \end{cases}$$

We claim that f is solid. It suffices to show that $\{f^{-1}(-\infty,a), f^{-1}(-\infty,a]\}_{a\in\mathbb{R}} \subset \mathcal{A}_s(X)$. Observe that $f^{-1}(-\infty,a) = \bigcup_{r<a} U_r$ whenever $a \in (0,1]$. For other values of a we have $f^{-1}(-\infty,a) = X$ or $f^{-1}(-\infty,a) = \emptyset$ which both are solid. Now $\bigcup_{r<a} U_r$ is a union of connected sets with nonempty intersection and hence connected. For the compliment we have by DeMorgans law the intersection of continua directed by inclusion, and hence connected. For $a \in (-\infty, 1)$ notice that $f^{-1}(-\infty,a] = \bigcap_{r>a} U_r = \bigcap_{r>a} \bar{U}_r$ hence the arguments above applies and $f^{-1}(-\infty,a]$ is also solid. We have shown that f is a solid variable.

If X is metric we can construct $\{U_r\}_{r\in\mathbb{Q}}$ such that $h(C,\bar{U}_r) < r \ \forall r \in \mathbb{Q}$, where the Hausdorff distance $h : \mathcal{C}(X) \to \mathbb{R}$ is defined by $h(C_1,C_2) = \max\{\max_{x\in C_1}\{d(x,C_2)\}, \max_{y\in C_2}\{d(y,C_1)\}\}$ (see [7] for details on Hausdorff distance). Then $C = \bigcap_{r\in\mathbb{Q}} U_r$ and accordingly $f^{-1}(0) = C$. ■

Now assume that we have any general concept of a sample median, say $T_{(n)}$. Where $T_{(n)}$ is typically a variable $T_{(n)} : X^{2n-1} \to X$. Assume further that this $T_{(n)}$ coincides with the ordinary sample median in one dimension and is equivariant under the solid variables. If X is a metric q-space we will for any $A \in \mathcal{A}_s(X)$ have a solid variable f such that $f^{-1}(-\infty,0] = A$ if A is closed or $f^{-1}(-\infty,1) = A$ if A is open. By Proposition 5 we have

$$M_{\{T_i\}}(A) = T_{(n)}^{-1}(A) \text{ for any solid set } A \in \mathcal{A}_s(X).$$

Conclusively, there is no other concept of a sample median being equivariant under the solid variables.

Example 8 *Consider a probability space with one possible outcome. That is, a measure space $(Y = \{y\}, \mathcal{P}(Y), \delta_y)$, where $\mathcal{P}(Y)$ denotes all subsets of the one point set $\{y\}$ and δ_y is the one point (Dirac) measure in y. Any variable on Y will be defined by its value on y. Consider the three variables in the unit square $X = \{(x,y) : 0 \le x, y \le 1\}$ given by $T_1(y) = (\frac{1}{4}, \frac{1}{4})$, $T_2(y) = (\frac{2}{3}, \frac{1}{2})$ and $T_3(y) = (\frac{1}{2}, \frac{2}{3})$. Denote the projections on the square down to the respective axis with f_1, and f_2, i.e. $f_1 : (x,y) \mapsto x$ and $f_2 : (x,y) \mapsto y$. Then we* *have* $M^*_{\{f_1 \circ T_i\}}(\delta_y) = M^*_{\{f_2 \circ T_i\}}(\delta_y) = \delta_{\frac{1}{2}} \in \mathcal{M}[0,1]$. *Moreover* $M_{\{T_i\}}(f_1^{-1}(\frac{1}{2})) = M_{\{T_i\}}(f_2^{-1}(\frac{1}{2})) = Y$, *but* $f_1^{-1}(\frac{1}{2}) \cap f_2^{-1}(\frac{1}{2}) = (\frac{1}{2}, \frac{1}{2})$. *Hence we have*

$$Y = M_{\{T_i\}}(f_1^{-1}(\frac{1}{2})) \cap M_{\{T_i\}}(f_2^{-1}(\frac{1}{2})) \neq M_{\{T_i\}}(f_1^{-1}(\frac{1}{2}) \cap f_2^{-1}(\frac{1}{2})) = \emptyset$$

This illustrates that the image transformations, and in particular the sample median do not behave nicely under intersections in contrary to inverse images of maps.

Remark 14 *Notice that the set* $M_{\{T_i\}}(f_1^{-1}(\frac{1}{2})) \cap M_{\{T_i\}}(f_2^{-1}(\frac{1}{2}))$ *and the set* $M_{\{T_i\}}(f_1^{-1}(\frac{1}{2}) \cap f_2^{-1}(\frac{1}{2}))$ *both can be interpreted as events. The first set being the event that both of the medians of the coordinates is* $\frac{1}{2}$, *which we know has probability one. The second is the event that two of the variables is equal to* $(\frac{1}{2}, \frac{1}{2})$ *which of course none of them will be with probability one.*

5 The Median

In the limiting case for Example 4 letting the number of variables tend to infinity we should have the notion of a median. Hence, since we are dealing with a limit of measures, the natural median should be a measure rather than points. For the construction we will need the notion of *splitting* measures (c.f.. [3]). We say that a quasi-probability P in a compact Hausdorff space X is splitting if there exists disjoint sets $C_1, C_2 \in \mathcal{C}_s(X)$ such that $P(C_1)+P(C_2) = 1$ with $P(C_1), P(C_2) > 0$. If no such pair exists we call P *non-splitting.* The collection of sets that splits P such that $P(C_1) = P(C_2) = \frac{1}{2}$ will be denoted by $\mathcal{C}_{sp}(X, P)$.

Definition 8 *Let* $(X, \mathcal{B}, P)$ *be a probability space where* X *is a q-space,* $\mathcal{B}$ *consists of the Borel sets in* X, *and* P *is a probability measure. The median of* P *is defined to be a set function* $P_m : \mathcal{C}_s \to \{0, \frac{1}{2}, 1\}$ *by*

$$P_m(C) = \begin{cases} 0 & ; \quad PC < \frac{1}{2} \\ \frac{1}{2} & ; \quad PC = \frac{1}{2} \text{ and } C \in \mathcal{C}_{sp}(X,P) \\ 1 & ; \quad \text{elsewhere} \end{cases}$$

Proposition 7 *The median uniquely extends to a quasi-probability in* X.

Proof. Define the set function P_m on open solid sets by $P_m(U) = 1 - P_m(X \backslash U); U \in \mathcal{O}_s(X)$, according to the additivity of a quasi-probability. In the non-splitting case we will have $\mathcal{C}_{sp}(X, P) = \emptyset$ and so P_m will be constructed from the q-function $f(x) = \begin{cases} 0 & ; x < \frac{1}{2} \\ 1 & ; x \geq \frac{1}{2} \end{cases}, x \in [0,1]$ applied to the measure P. Hence P_m is a quasi-probability. If the measure is splitting, we have to treat the splitting sets separately. We proceed by showing the requirement (A) and (B) of a solid set function consecutively, property (C) is clear by the definition of P_m on open solid sets.

(A): Let $C \in \mathcal{C}_s$, if $P(C) < \frac{1}{2}$, then any disjoint collection of solid compact $C_i \subset C$ will have $P(C_i) < \frac{1}{2}$ and hence $P_m(C_i) = 0$ for all i. If $\frac{1}{2} \leq P(C)$ and $C \notin \mathcal{C}_{sp}(X, P)$, then at most one of the sets C_i can have $P(C_i) > \frac{1}{2}$ and so $P_m(C_i) = 1$ with $P_m(C_j) = 0$ for $j \neq i$. Two of the sets C_i can have $P(C_i) = \frac{1}{2}$ but then they are splitting and so $1 = P_m(C) = \sum P_m(C_i) = \frac{1}{2} + \frac{1}{2}$. If $C \in \mathcal{C}_{sp}(X, P)$ we only have to consider the case $P(C) = \frac{1}{2}$, then at most one of the sets C_i can have $P(C_i) = \frac{1}{2}$ and since $C_i \subset C$ it must also be splitting, so the assertion follows.

(B): Suppose $P(U) < \frac{1}{2}$, then $P(X \backslash U) > \frac{1}{2}$ and so $P_m(U) = 0$. In particular any compact solid set $C \subset U$ will have $P(C) < \frac{1}{2}$ and so $P_m(C) = 0$, which shows regularity. If $X \backslash U \in \mathcal{C}_{sp}(X, P)$, any compact solid set $C \subset U$ with $P(C) = \frac{1}{2}$ is splitting, in particular there exists such a splitting set $C \subset U$, hence the regularity holds for U. If $P(U) = \frac{1}{2}$ and $X \backslash U \notin \mathcal{C}_{sp}(X, P)$, then $P_m(U) = 0$ also any compact subset $C \subset U$ can not be splitting, hence has $P(C) < \frac{1}{2} \Rightarrow P_m(C) = 0$. Finally if $P(U) > \frac{1}{2}$, we have $P(X \backslash U) < \frac{1}{2} \Rightarrow P_m(U) = 1$, furthermore by regularity of P there is a compact solid set $C \subset U$ with $P(C) > \frac{1}{2} \Rightarrow P_m(C) = 1$.

We have shown that P_m is a solid set function, and hence it extends uniquely to a quasi-probability ■

Remark 15 *It is clear that we are really dealing with two constructions, one when the measure is splitting and one when it is not. This is illustrated in statistics with the symmetry centre (c.f. [14]), typically the symmetry centre will be a set of points rather than a single point when the measure is splitting. However, our construction differs fundamentally with the classical notion of a median. We claim that the natural median is a set function, namely a quasi-probability rather than being a set of points. One might question whether giving all the splitting sets measure $\frac{1}{2}$ each or perhaps choosing a Bayesian approach imposing different convex combinations on the different pairs of* *splitting sets. If $n > 1$ this can not be done arbitrarily because the different axis interact and may cause violation of the monotonicity of the median.*

Proposition 8 *If P is non-splitting the median coincides with the ordinary median in one dimension. If P is splitting then in one dimension we get a two point measure where the two points are the $\frac{1}{2}$ quantiles of the distribution P.*

Proof. In the non-splitting case we will have a $\{0, 1\}$ valued quasi-measure in a one dimensional space. Since the quasi-measure is a measure in one dimension, P_m is necessarily a point mass. Obviously P_m is the desired point "chopping" the distribution in half. In the splitting case we will have a three valued measure and hence a linear combination of two Dirac measures each with weight $\frac{1}{2}$. Taking solid sets downwards and upwards it is clear that the two points are the $\frac{1}{2}$ quantiles. ■

Note that when the measure P is non-splitting the median P_m can be constructed from an image transformation $M_P : \mathcal{C}_s(X) \to \mathcal{C}_s(X)$ by

$$M_P C = \begin{cases} \emptyset & ; P(C) < \frac{1}{2} \\ C & ; P(C) = \frac{1}{2}, C \in \mathcal{C}_{sp}(B_n, P) \\ X & ; P(C) \geq \frac{1}{2} \end{cases}$$

Where the extension to open solid sets is by complement. Hence we have a map $M^* : Q(X) \to Q(X)$ where $P \mapsto M_P^*(P) = P_m$ which is just sending the measure to the median in terms of image transformations.

Example 9 *Consider the unit square $E = \{(x, y) : 0 \leq x, y \leq 1\}$, fix four distinct points $\{p_i\}_{i=1}^4$ in E. Define P to be the probability measure in E assigning the probability $\frac{1}{4}$ to each of the four points. Then if a set $\mathcal{C}_s(E)$ contains two of the points, there is a compact solid set in the complement containing the two other points. Hence the set is splitting. We can now determine how the median P_m looks for any $C \in \mathcal{C}_s(E)$:*

$$P_m(C) = \begin{cases} 0 & ; \quad |C \cap \{p_i\}| \leq 1 \\ \frac{1}{2} & ; \quad |C \cap \{p_i\}| = 2 \\ 1 & ; \quad |C \cap \{p_i\}| \geq 3 \end{cases}$$

Now P_i be the probability measure obtained by giving each of the points $\{p_j\}_{j=1}^4$ probability $\frac{1}{3}$ except p_i which is given probability zero. Denote the median of

P_i *with* P_m^i. *In similar manner as above we can then determine what these medians are on compact solid sets. Since we now have odd numbered collection of points with probability greater than zero, there will be now splitting sets. The remarkable property is that we have* $P_m = \frac{1}{4}\sum_{i=1}^{4} P_m^i$. *This gives a nice analogue to the definition of the sample median of an even numbered sample.*

Theorem 3 *If* X_1, X_2 *are q-spaces and* $q : \mathcal{A}(X_2) \to \mathcal{A}(X_1)$ *is a solid image transformation, then the following diagram is commutative*

$$\begin{array}{ccc} Q(X_1) & \stackrel{q^*}{\longrightarrow} & Q(X_2) \\ M^* \uparrow & & \uparrow M^* \\ Q(X_1) & \underset{q^*}{\longrightarrow} & Q(X_2) \end{array}$$

Proof. Let $P \in Q(X_1)$ be arbitrary. If $C \in \mathcal{C}_s(X_2)\backslash\mathcal{C}_{sp}(X_2, q^*P)$ we have

$$\begin{array}{l} [M^*(q^*P)]C = 0 \Leftrightarrow (q^*P)(M_{q^*P}C) = 0 \Leftrightarrow M_{q^*P}C = \emptyset \Leftrightarrow \\ (q^*P)C < \frac{1}{2} \Leftrightarrow P(qC) < \frac{1}{2} \Leftrightarrow (M^*P)C = 0 \Leftrightarrow [q^*(M^*P)]C = 0 \end{array}$$

Since zero and one are the only possible values for the non-splitting sets this settles the problem for them. Now assume $C \in \mathcal{C}_{sp}(X_2, q^*P)$. Then there is a set $C' \in \mathcal{C}_{sp}(X_2, q^*P)$ such that $C \cap C' = \emptyset$ and $(q^*P)C = (q^*P)C' = \frac{1}{2}$. Hence $qC \in \mathcal{C}_{sp}(X_1, P)$ and so $[M^*(q^*P)]C = [q^*(M^*P)]C = \frac{1}{2}$.

Remark 16 *Notice that this theorem is the medians version of Theorem 2. However, this statement is more general involving solid image transformations. Still an important class of examples is when the image transformation is derived from a solid variable.*

Example 10 *The sphere with uniform measure (i.e. Lebesgue measure on the sphere). Any attempt to find natural points of symmetry in the sphere with this distribution would result either in the whole sphere or the empty set. Which leaves us stripped of statistical concepts to model the median. One might of course use some of the computational methods to any sample in order to find some center location. But this would be strictly computational with now clear definition of what is estimated. However, in terms of the* *coordinates on a map (as used by [9]) it does make sense to talk about sample medians also in this setting. Those are essentially projecting spheric points down to there coordinates which both are solid variables. The sphere being a metric q-space will have a median as well as a sample median for any finite number of variables in the sphere and any Borel measure.*

6 Non-linearity and continuity of expectations

The (sample) median (i.e. the median and the sample median) defined in balls is equivariant under projections. Hence we have a tool for investigating the median in terms of the different coordinates, which is perhaps the most natural variables in a multidimensional setting. In particular for the sample median we get the distribution of the middle order statistic in each coordinate. More generally we have the possibility of investigating the expectations of any transformation by a real-valued solid variable in terms of quasi-integrals. We will give a brief presentation of the integration theory below.

The integral with respect to a quasi-probability μ in a compact Hausdorff space X is defined on $C(X)$, i.e. the continuous real-valued functions on X. Given any function $f \in C(X)$, the quasi-probability μ is mapped to a quasi-probability μ_f given by $\mu_f(A) = \mu(f^{-1}(A))$ for $A \in \mathcal{A}(\mathbb{R})$. Which of course is just a transformation of μ by the variable f. Since $\mathbb{R}$ is one dimensional μ_f is a regular Borel measure. Hence we can define the integral or expectation of f with respect to μ as $E_\mu(f) = \int_{\mathbb{R}} x d\mu_f(x)$, i.e. the integral of the identity function $f(x) = x$ over $\mathbb{R}$ with respect to the measure μ_f. One of the remarkable properties of this integral is the lack of linearity. However, the integral is linear on uniformly closed singly generated subalgebras of $C(X)$. We will denote the subalgebra generated by a function $f \in C(X)$ with $A_f = \{\phi \circ f : \phi \in C(\operatorname{sp} f)\}$, where $\operatorname{sp} f = \{f(x) : x \in X\}$ is the range of f in $\mathbb{R}$.

The singly generated subalgebras are abstractly defined and so it is not always easy to decide wether two functions are contained in the same subalgebra or not. To complicate things further, it is known that the quasi-integrals are linear on even larger classes of functions, e.g. analytic subalgebras (a presentation of analytic subalgebras can be found in [8]). Exactly when quasi-integrals are linear is still not known. However, several results on explicitly determining linearity are known.

We will summarize the linearity problem in terms of medians below and accordingly give an example of non-linearity.

Proposition 9 *Let X be a metric q-space and let μ be a (sample) median in X. If $f_1, f_2 \in A_f$ for some $f \in C(X)$, then $E_\mu(f_1 + f_2) = E_\mu(f_1) + E_\mu(f_2)$. For any $\alpha \in \mathbb{R}, f \in C(X)$ we have $E_\mu(\alpha f) = \alpha E_\mu(f)$.*

Remark 17 *Some caution is necessary. The (sample) median is not equivariant under all the continuous functions in X. Hence the expectation need not make much sense in terms of the (sample) median. However, splitting a variable into a sum is often convenient in a computational setting. In that case we need not be concerned with equivariance problems.*

Example 11 *Consider the 2-simplex*

$$\Delta_r = \left\{(x, y) \in \mathbb{R}^2 : x + y \leq r \text{ and } x, y \geq 0\right\}$$

with $r > 0$ and three predetermined experiments $(0,0)$, $(0,r)$ and $(r,0)$. That is, we are considering three variables $\{T_i : \{p\} \to E\}$ by $T_1(p) = (0,0)$, $T_2(p) = (0,r)$ and $T_3(p) = (r,0)$, where $\{p\}$ is a one point space endowed with a probability measure which of course is the Dirac measure δ_p. Denote the sample median of $\{T_i\}$ with μ. Then for any solid subset $A \subset E$ we have μA equal zero if less than two of the points are contained in A and one otherwise. Let P_X and P_Y respectively be the projections down to the coordinates axis, i.e. $P_X : (x, y) \mapsto x$ and $P_Y : (x, y) \mapsto y$. Then both P_X and P_Y as well as their sum $P_X + P_Y$ are solid variables. However both μ_{P_X} and μ_{P_Y} are pointmasses in zero, whereas $\mu_{P_X + P_Y}$ is a pointmass in r. Hence we have $E_\mu(P_X + P_Y) = r$ but $E_\mu(P_X) = E_\mu(P_Y) = 0$, so obviously for any $r > 0$ we have $E_\mu(P_X + P_Y) \neq E_\mu(P_X) + E_\mu(P_Y)$. Notice that the supremum norm of the projections on the space Δ_r are both r (i.e. $\|P_X\|_\infty = \|P_Y\|_\infty = \sup\{|P_Y(x, y)| : (x, y) \in \Delta_r\} = r$). This states that the loss of linearity is as bad as the maximum of the functions involved.

Remark 18 *The example is rather disappointing. The coordinates which are the most natural variables and a combination of three pointmasses which perhaps is the simplest nontrivial sample median we can construct does not obey linearity. Also the example illustrates that we must treat the median with caution. For an applied setting where linearity is a crucial property this complicates the applicability of the median. In computational problems* *the knowledge of linearity is often employed to avoid situations of numerical instability. Proposition 9 suggests that we in some situations still can assume linearity. This is exemplified in the example below.*

Example 12 *Assume that we are doing experiments where we are observing n real valued solid variables $T_1, T_2, ..., T_n$ on a q-space X. Assume X is given a quasi-probability μ according to the sample median of a collection of variables (not the variables $\{T_i\}$) in X. Whereas we are observing $\{T_i\}_{i=1}^n$ we may be interested in estimating the expectation of a continuous variable f on some subset of $\mathbb{R}^n$, i.e. we want to estimate the expectation $E_\mu(f(T_1, T_2, ..., T_n))$. For instance in the example above the function f is addition of the projections P_X and P_Y. Suppose the variables can be assumed to be transformations of another variable $T \in C(X)$, i.e. $\{T_i\} \subset A_T$ (which is not the case for the projections). Then we are in a linear situation and we may for instance integrate the Taylor series of f term by term.*

In [1] it is shown that the quasi-integral is uniformly continuous (this is not a trivial result). We will restate that result in our setting:

Proposition 10 *The expectation of the (sample)median is norm decreasing, i.e. $|E_\mu(f) - E_\mu(g)| \leq \|f - g\|_\infty$ for any $f, g \in C(X)$.*

Remark 19 *Notice that this result ensures that a small experimental error or a small perturbation of a variable will not influence the expectation dramatically. In other words, the expectation $E_\mu : C(X) \to \mathbb{R}$ is robust.*

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Construction and properties of quasi-linear functionals

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Abstract

Quasi-linear functionals are shown to be uniformly continuous, and decomposable into a difference of two quasi-integrals. A predual space for the quasi-linear functionals inducing the weak*-topology is given. General construction of quasi-linear functionals by solid set functions and q-functions are given.

1 Introduction

The theory of quasi-measures originated in [1], and was shown there to represent quasi-integrals (for definitions see basic results section). The notion of a signed quasi-measure in a compact Hausdorff space was introduced in [6] and was shown there to represent bounded quasi-linear functionals. Basic properties such as continuity and decomposition of signed quasi-measures remained open problems. In this article we show that quasi-linear functionals are indeed uniformly continuous, countably additive and they decompose into a difference of quasi-measures in so called q-spaces. We also present a predual space for the quasi-linear functionals as a direct limit. This turns the topology of pointwise convergence into a weak*-topology.

Solid set functions have become the main tool for constructing quasi-measures, hence it would be favourable with a similar concept for signed quasi-measures. It turned out that this is possible, but the definition of a signed solid set function is more complex. The q-functions were introduced in [4] as an optional construction technique for quasi-measures, a discussion on signed q-functions is contained in the last section.

2 Basic definitions and results

Throughout the article we will let X denote a compact Hausdorff space. We will let $\mathcal{O}(X)$ and $\mathcal{C}(X)$ denote the open and closed subsets of X respectively. Furthermore we put $\mathcal{A}(X) = \mathcal{O}(X) \cup \mathcal{C}(X)$. When there is no confusion concerning the space in question, we will omit the space from the notation.

Notation 1 *Whenever $U \in \mathcal{O}$, we let $\lim_{\substack{C \subset U \\ C \in \mathcal{C}}} f(C)$ denote the limit (if any) of the net $f(C)$ with index set $\{C \in \mathcal{C} : C \subset U\}$,ordered by inclusion; conversely, whenever $C \in \mathcal{C}$, we let $\lim_{\substack{C \subset U \\ U \in \mathcal{O}}} f(U)$ denote the limit of $f(U)$ with index set $\{U \in \mathcal{O} : C \subset U\}$, ordered by reverse inclusion. (Here $f(C)$ may be any suitable expression in C.) Subfamilies of $\mathcal{C}$ and $\mathcal{O}$ may be specified to restrict the index set further.*

Definition 2 *A real valued, non-negative function μ on $\mathcal{A}$ is called a* signed quasi-measure *if the following hold:*

1. *If $\{A_i\}_{i=1}^n, \biguplus_{i=1}^n A_i \in \mathcal{A}$, then $\mu(\biguplus_{i=1}^n A_i) = \sum_{i=1}^n \mu A_i$ (where $\biguplus$ indicates disjoint union).*

2. *There is a constant $M < \infty$ such that whenever $A \in \mathcal{A}$, $|\mu A| \leq M$.*

3. *For any open set U*

$$\mu(U) = \lim_{\substack{C \subset U \\ C \in \mathcal{C}}} \mu(C).$$

Combining (1) and (2), we may also define

$$\|\mu\| = \sup\{\sum_{i=1}^{n} |\mu A_i| : \{A_i\}_{i=1}^n \subset \mathcal{A} \text{ disjoint}\}$$

Remark 3 *Our definition is different from the definition in [6], but it is straight forward to prove that the two are equivalent. Our definition is however more convenient for the results presented here.*

We denote the set of all signed quasi-measures in X by $Q_S(X)$. The collection of quasi-measures will be denoted $Q(X)$.

The quasi-measures originated in [1]. It was proved in [8] that quasi-measures are countably additive. Their definition only differs from that of regular Borel measures by their domain of definition. Still they are a vastly larger class of set functions with a rich mathematical structure.

With respect to a signed quasi-measure an integration theory has been developed. The integral is defined for all continuous functions $f \in C(X)$. Let $\mu \in Q_S(X)$, $f \in C(X)$; then defining $\mu_f(A) = \mu(f^{-1}(A)), A \in \mathcal{A}(\mathbb{R})$ yields a regular Borel measure μ_f in $\mathbb{R}$. (In general, such combination with a continuous function maps quasi-measures to quasi-measures. However, in one dimensional spaces it was shown in [7] that all signed quasi-measures extend uniquely to Borel measures.)

Definition 4 *Let $\mu \in Q_S(X)$ and $f \in C(X)$. Then we define*

$$\mu(f) = \int x d\mu_f(x)$$

where μ_f is the Borel measure given by $\mu_f(A) = \mu(f^{-1}(A)), A \in \mathcal{A}(\mathbb{R})$.

In the study of quasi-measures and integrals the singly generated subalgebras of $C(X)$ plays a crucial part. For $f \in C(X)$ let A_f denote the uniformly closed subalgebra generated by f and the constant functions. By the spectral theorem this algebra is isomorphic to the continuous functions on the range of f.

Definition 5 *A function $\rho : C(X) \to \mathbb{R}$ is called a* quasi-linear functional *if it is a linear functional on A_f for every $f \in C(X)$ and there is an $M < \infty$ such that $\rho(f) \leq M \|f\|_\infty; (f \in C(X))$. If ρ is positive (i.e. $\rho(f) \geq 0$ whenever $f \geq 0$) we call ρ a* quasi-integral.

The quasi-linear functionals and signed quasi-measures were shown to be in one-to-one correspondence through the integral (c.f. [6]). Accordingly we will not distinguish between the two and denote both with $Q_S(X)$. Similarly we will denote the quasi-integrals with $Q(X)$.

Definition 6 *If a set $A \in \mathcal{A}$ has connected complement we call A co-connected. If a set $A \in \mathcal{A}$ and its compliment are both connected we will call the set* solid. *A restriction to solid sets will be denoted with a subscript s (e.g. $\mathcal{C}_s$ will denote the compact solid sets). A restriction to connected sets will similarly be denoted by c.*

The solid sets play an important role in the theory of quasi-measures.. They constitute a small and manageable family of sets that totally determines a quasi-measure. This is illustrated by the solid set-functions, they were introduced in [3] and their properties were investigated there. In particular they are invaluable tools for constructing quasi-measures. We include some definitions and results from [3] below.

Throughout the remainder of this section we will assume that X is connected and locally connected.

Proposition 7 *The following properties hold for X:*

1. *Let $K \in \mathcal{C}$, $U \in \mathcal{O}$ and $K \subset U$. If either K or U is connected, then there is a set $V \in \mathcal{O}_c$ such that $K \subset V \subset \bar{V} \subset U$.*

2. *Let $K \in \mathcal{C}_c$. Then each connected component of $V = X \backslash K$ belongs to $\mathcal{O}_s$.*

3. *Let $K \in \mathcal{C}_s$, $U \in \mathcal{O}$ and $K \subset U$. Then there is a set $V \in \mathcal{O}_s$ such that $K \subset V \subset \bar{V} \subset U$.*

Definition 8 *A* partition *of X is a collection of mutually disjoint , non-void sets $\{A_i\}_{i \in I} \subset \mathcal{A}_s$, where at most finitely many of the sets A_i are closed, and such that $X = \bigcup_{i \in I} A_i$. The number of closed sets in a partition $\mathcal{P}$ is called the order of $\mathcal{P}$.Partitions of order 1 is called trivial.*

Definition 9 $\{A_i\}_{i \in I}$ *is* irreducible *if the following two conditions hold:*

1. *$\bigcup_{i \in I, A_i \in \mathcal{C}} A_i$ is not co-connected.*

2. *For any proper subset $I' \subset \{i \in I : A_i \in \mathcal{C}\}$, $\bigcup_{i \in I'} A_i$ is co-connected.*

Definition 10 *Let n denote the maximal order of any irreducible partition of X. If n is finite, let $g = n - 1$. If X only permits trivial partitions we call X a* q-space *and put $g = 0$.*

Remark 11 *The genus requirement was treated in [3] and [9]. When X has genus zero, then X can at most be the disjoint union of two solid sets. This property is shared by a large class of spaces (e. g. when X is simply connected).*

Proposition 12 *Let $\mathcal{F} = \{C_j\}_{j=1}^n \subset \mathcal{C}_s$ $(n \geq 1)$ be a family of disjoint sets such that $\bigcup_{j=1}^n C_j$ is not co-connected. Then $\mathcal{F}$ has a subfamily $\mathcal{F}'$ such that each connected component U_i $(i \in I)$ of $U = X \setminus \bigcup_{C_j \in \mathcal{F}'} C_j$ belongs to $\mathcal{O}_s$ and $\mathcal{F}' \cup (\bigcup_{i \in I} U_i)$ is an irreducible partition of X.*

Definition 13 *A function $\mu : \mathcal{A}_s \to \mathbb{R}^+$ is a* solid set-function *if it satisfies*

1. *For any finite collection of disjoint sets $\{C_1, ..., C_n\} \subset \mathcal{C}_s$ such that $C_i \subset C \in \mathcal{C}_s$ for $i = 1, ..., n$ we have*

$$\sum_{i=1}^{n} \mu C_i \leq \mu C.$$

2. *For all $U \in \mathcal{O}_s$ we have*

$$\mu(U) = \sup\{\mu(C) : C \subset U, C \in \mathcal{C}_s\}.$$

3. *For any trivial or irreducible partition $\{A_i\}_{i \in I}$ of X we have*

$$\mu(X) = \sum_{i \in I} \mu(A_i).$$

The main construction result (Theorem 5.1 in [3]) states that solid set functions extends uniquely to quasi-measures.

We include the following Definition and Proposition from [4] (with the generalization that we do not assume $f(1) = 1$):

Definition 14 *A function $f : [0,1] \to [0,1]$ is called a* q-function *if it is continuous from the right and satisfies*

1. *$f(0) = 0$, $f(x-) + f(1-x) = f(1)$*
2. *$\sum_{i=1}^n f(x_i) \leq f(\sum_{i=1}^n x_i)$ whenever $x_1, x_2, ..., x_n \in [0,1]$ and $\sum_{i=1}^n x_i <$ 1.*

Let ν be a normalized Borel (or quasi-) measure in X, i.e. $\nu(X) = 1$. We say that ν is *non-splitting* if there is no disjoint pair $C_1, C_2 \in \mathcal{C}_s$ such that $\nu C_1 > 0$, $\nu C_2 > 0$ and $\nu C_1 + \nu C_2 = 1$. For instance, Lebesgue-measure on the unit disk, or the unit sphere (normalized) is non-splitting.

Proposition 15 *Let X be a q-space. Let f be a q-function, and let ν be a normalized regular Borel (or quasi-) measure in X. Define μ on $\mathcal{A}_s$ by: $\mu C = f(\nu C)$; $C \in \mathcal{C}_s$ and $\mu U = f(1) - \mu(X \backslash U)$; $U \in \mathcal{O}_s$. If either ν is non-splitting or f is continuous, then μ is a solid set-function.*

3 Countable additivity

We present here a generalization to signed quasi-measures of the proof in [8] that quasi-measures are countably additive.

We recall the following lemmas, stated or implicit in [8], Section 3:

Lemma 16 (The Sierpiński Theorem) *A compact, connected Hausdorff space cannot be decomposed into a countable family of disjoint, non-empty closed sets.*

Lemma 17 *Let μ be a signed quasi-measure on X and suppose that $C \subset X$ is closed and 0-dimensional. Then the restriction of μ to the closed subsets of C extends to a signed Borel measure on C.*

Proof. The restriction gives a finitely additive signed measure on the algebra of relatively clopen sets of C, which is then extended by standard measure theory. ∎

Lemma 18 *Let $C \subset X$ be closed, and let $Y = X/\sim$ be the quotient space obtained by identifying each component of C to a point. Then Y is a Hausdorff space, and the image of C under the quotient map $\pi : X \to Y$ is 0-dimensional.*

Theorem 19 *Suppose $\{C_n : n \in \mathbb{N}\}$ is a disjoint collection of closed subsets of X with $C = \biguplus_{n\in\mathbb{N}} C_n$ closed. Then $\mu(C) = \sum_{n\in\mathbb{N}} \mu(C_n)$.*

Proof. Let $\{K_i : i \in I\}$ be the collection of connected components of C. Since $K_i = \biguplus_{n\in\mathbb{N}}(K_i \cap C_n)$, Lemma 16 gives that $K_i \subset C_{n_i}$ for some n_i. As in

Lemma 18, let $\pi(C)$ be the image of C by the quotient mapping identifying each K_i to a point. Then $\pi^{-1}(\pi(C_i)) = C_i$ and $\pi^{-1}(\pi(C)) = C$, since each C_n is a union of components of C, and also $\pi(C) = \biguplus_{n\in\mathbb{N}} \pi(C_n)$.

By Lemma 17, the restriction of the signed quasi-measure $\mu \circ \pi^{-1}$ to the closed subsets of the 0-dimensional subspace C extends to a signed Borel measure on C, and in particular it is countably additive there. So

$$\mu(C) = (\mu \circ \pi^{-1})(\pi(C)) = \sum_{n\in\mathbb{N}} (\mu \circ \pi^{-1})(\pi(C_n)) = \sum_{n\in\mathbb{N}} \mu(C_n).$$

■

Corollary 20 *Every signed quasi-measure on a compact Hausdorff space X is countably additive.*

Proof. As in [8], this follows from the above theorem together with the known countable additivity on open sets (following easily from regularity.)
■

4 Quasi-measures as a dual space

In [2] a weak topology for $Q(X)$ was introduced: Any function $f \in C(X)$ may be represented as a functional $\hat{f}$ on $Q(X)$ by $\hat{f}(\mu) = \mu(f)$. The topology on $Q(X)$ is defined to be the topology induced by the separating space of functionals $\{\hat{f} : f \in C(X)\}$. This topology might equally well be defined for $Q_S(X)$. The following theorem illustrates that this indeed gives a weak*-topology:

Theorem 21 *$Q_S(X)$ is a dual space where the induced weak*-topology is the topology of pointwise convergence on $Q_S(X)$.*

Proof. For $f \in C(X)$ let A_f denote the uniformly closed subalgebra generated by f and the constant functions. Whenever $g \in A_f$ we let $i_g^f : A_g \to A_f$ denote the inclusion map. We may now construct the algebraic direct limit

$$\mathfrak{X} = \varinjlim \{A_f, i_g^f\}$$

Defining $\|\cdot\| : \bigoplus_{f \in C(X)} A_f \to \mathbb{R}$ by $\oplus f_i \mapsto \Sigma \|f_i\|$ the quotient norm turns $\mathfrak{X}$ into a seminormed space. The map $\Phi : Q_S(X) \to \mathfrak{X}^*$ is now given canonically by $(\Phi\mu)(\oplus f_i) = \Sigma\mu(f_i)$. Note that $\Phi(\mu)$ is independent of representatives in the equivalence classes of $\mathfrak{X}$. Furthermore one may check that Φ is an invertible isometry. In particular X^* corresponds exactly to those quasi-linear functionals on $C(X)$ that are continuous at zero. Weak*– convergence is preserved both ways since pointwise convergence on single elements is equivalent with pointwise convergence on direct sums of elements. ■

Remark 22 *Several comments are in order here. Constructing a predual Banach space for $Q(X)$ was done by D. Grubb. However it is not clear what the topological properties will be for that construction. The predual space $\mathfrak{X}$ shows that the weak topology on $Q(X)$ is really a weak*-topology. Signed quasi-measures were suggested by D. Grubb in [6] as set functions representing the quasi-linear functionals on $C(X)$ continuous at zero. This suggestion is supported by our predual.*

5 Decomposition of a signed quasi-measure

In the following, μ is a signed quasi-measure on X.

Definition 23 *We define μ_+ as a real-valued set function, as follows:*

1. $\mu_+(U) = \sup\{\mu(K) | K \subseteq U, K \in \mathcal{C}(X)\}$, $U \in \mathcal{O}(X)$.
2. $\mu_+(M) = \inf\{\mu_+(U) | M \subseteq U, U \in \mathcal{O}(X)\}$, $M \in \mathcal{A}(X)$.

We define $\mu_- = (-\mu)_+$.

We note that because expression (1) is monotonic, open sets automatically fulfill (2). Thus μ_+ is well-defined.

Proposition 24 *We have the following properties for μ_+:*

1. *μ_+ is non-negative and monotonic.*
2. $\mu_+ \geq \mu$

3. $\mu_+(M) = \sup\{\mu_+(K) | K \subseteq M, K \in \mathcal{C}(X)\}$, $M \in \mathcal{A}(X)$. *(I.e. μ_+ is inner regular as well as outer regular (equation (2) of the definition.))*

4. $\mu_+(U) = \sup\{\mu(O) | O \subseteq U, O \in \mathcal{O}(X)\}$, $U \in \mathcal{O}(X)$.

5. $\mu_+(M_1 \cup M_2) = \mu_+(M_1) + \mu_+(M_2)$, *when* $M_1 \cap M_2 = \emptyset$, *and* M_1, M_2 *are both compact or both open.*

6. $\mu_+(M) \geq \sum_{i=1}^n \mu_+(M_i)$, *when* $\biguplus_{i=1}^n M_i \subseteq M$ *for* $M, M_i \in \mathcal{A}(X)$, $i = 1, \ldots, n$, $n \in \mathbb{N}$.

7. $\mu_+ - \mu_- = \mu$

8. $\mu_+(X) + \mu_-(X) = \|\mu\|$

Proof. (1) Equation (2) of the definition implies that μ_+ is monotonic. Then $\mu_+(M) \geq \mu_+(\emptyset) = \mu(\emptyset) = 0$.

(2) If U is open, $\varepsilon > 0$, then by the regularity of μ we can find $K \subseteq U$ with $\mu(K) > \mu(U) - \varepsilon$, so that $\mu_+(U) > \mu(U) - \varepsilon$. Thus $\mu_+(U) \geq \mu(U)$ for U open, and $\mu_+(K) \geq \mu(K)$ for compact sets follows by outer regularity of μ and μ_+.

(3) For closed sets, (3) is obvious from monotonicity. For open sets it follows from the definition, monotonicity and (2).

(4) Given U and $\varepsilon > 0$, select $K \subseteq U$ with $\mu(K) > \mu_+(U) - \varepsilon/2$. Then select O, $K \subseteq O \subseteq U$ with $\mu(O) > \mu(K) - \varepsilon/2 > \mu_+(U) - \varepsilon$. This gives ($\leq$) in (4), while ($\geq$) follows from monotonicity and (2).

(5) If M_1, M_2 are disjoint open sets, then any compact within their union is the disjoint union of compacts within each, and vice versa, so that (5) follows from additivity of μ. If M_1, M_2 are disjoint closed sets, (5) follows from additivity of μ_+ on disjoint open sets $O_1 \supseteq M_1$, $O_2 \supseteq M_2$.

(6) Assume that (6) does not hold. By (3), we may then replace any open M_i by a compact contained within it, without making (6) true. But if M_i are all compact, (6) follows from (5) and monotonicity, giving a contradiction.

(7) By regularity it suffices to show equality on open sets. Given U open and $\varepsilon > 0$, select $K \subseteq U$ such that $\mu(K) > \mu_+(U) - \varepsilon$, then (e.g. by (4) applied to μ_-) $\mu_-(U) \geq -\mu(U \setminus K) = \mu(K) - \mu(U) > \mu_+(U) - \mu(U) - \varepsilon$, or $\mu_+(U) - \mu_-(U) \leq \mu(U)$. Switching μ_+ and μ_- gives the opposite inequality.

(8) Let $\varepsilon > 0$. Note from (7) that if $K \in \mathcal{C}(X)$ is such that $|\mu(K) - \mu_+(X)| < \varepsilon$, then $|(-\mu(X \setminus K)) - \mu_-(X)| < \varepsilon$. Then

$$\|\mu\| \geq |\mu(K)| + |\mu(X \setminus K)| \geq \mu(K) - \mu(X \setminus K) > \mu_+(X) + \mu_-(X) - 2\varepsilon.$$

In the other direction, let $(M_i)_{i=1}^n$, with $M_i \in \mathcal{A}(X)$, $i = 1, \dots, n$, $n \in \mathbb{N}$ be a disjoint family such that $\sum_{i=1}^n |\mu(M_i)| > \|\mu\| - \varepsilon$. Let $I_+ = \{i | \mu(M_i) > 0, i = 1, \dots, n\}$ and $I_- = \{i | \mu(M_i) < 0, i = 1, \dots, n\}$. Then

$$\mu_+(X) + \mu_-(X) \geq \sum_{i \in I_+} \mu(M_i) + \sum_{i \in I_-} (-\mu(M_i)) = \sum_{i=1}^n |\mu(M_i)| > \|\mu\| - \varepsilon.$$

■

We have seen that μ_+ and μ_- share many of the properties of (non-negative) quasi-measures. In fact the only axiom missing so far is $\mu_+(K) + \mu_+(X \setminus K) = \mu_+(X)$. As the following example shows, there is no hope of getting this in general.

Example 25 *Let p_1, p_2, p_3, p_4 be four distinct points in the disk D. Define four solid set functions as follows: For M a solid set (open or closed)*

$$\nu_1(M) = \begin{cases} 1 & \text{card}(M \cap \{p_1, p_2, p_4\}) \geq 2 \\ 0 & \textit{otherwise} \end{cases}$$

$$\nu_2(M) = \begin{cases} 1 & \text{card}(M \cap \{p_1, p_3, p_4\}) \geq 2 \\ 0 & \textit{otherwise} \end{cases}$$

$$\nu_3(M) = \begin{cases} 1 & p_1 \in M \textit{ and } M \cap \{p_2, p_3, p_4\} \neq \emptyset, \\ & \textit{or } \{p_2, p_3, p_4\} \subseteq M \\ 0 & \textit{otherwise} \end{cases}$$

$$\nu_4(M) = \begin{cases} 1 & p_4 \in M \textit{ and } M \cap \{p_1, p_2, p_3\} \neq \emptyset, \\ & \textit{or } \{p_1, p_2, p_3\} \subseteq M \\ 0 & \textit{otherwise} \end{cases}$$

It is readily verified that each of the above extends to a $\{0, 1\}$-valued quasi-measure, and that $\nu_1 + \nu_2 = \nu_3 + \nu_4$. Then $\nu = \nu_1 - \nu_3 = \nu_4 - \nu_2$ is a signed quasi-measure, and furthermore we must have $\nu_+ \leq \nu_1, \nu_4$.

It follows that if ν_+ were a quasi-measure, it would be proportional to both of the extreme quasi-measures ν_1 and ν_4, and thus identically zero. Let M be a solid set containing p_2 and p_4 but not p_1 or p_3; then $\nu_+(M) \geq \nu(M) = 1$. So ν_+ cannot be a quasi-measure.

We will therefore define a suitable class of set functions containing μ_+ and μ_-.

Definition 26 *If $\mu : \mathcal{A}(X) \to [0, \infty)$ is regular and monotonic, and if it satisfies*

$$\nu(M) \geq \sum_{i=1}^{n} \nu(M_i)$$

whenever $M \supseteq \biguplus_{i=1}^{n} M_i$, with $M, M_i \in \mathcal{A}(X)$, $i = 1, \dots, n$, $n \in \mathbb{N}$, we shall call it a deficient quasi-measure.

6 Continuity and decomposition of a quasi-linear functionals

We now wish to define how to integrate continuous functions with respect to μ_+. Like in the case of quasi-measures, our first step is to use the function to move the deficient quasi-measure to the real line.

Definition 27

$$f(\mu_+)(M) = \mu_+(f^{-1}(M)), \; f \in C(X, \mathbb{R}), \; M \in \mathcal{A}(R).$$

Unlike in the case of quasi-measures, $f(\mu_+)$ cannot itself be extended to a Borel measure, as it is not additive on complements. However, it is a deficient quasi-measure, in particular monotonic and regular, which means that its restriction to leftward infinite closed intervals is extensible to some Borel measure.

Definition 28 *Given $f \in C(X, \mathbb{R})$, let $df(\mu_+)$ be the Borel measure defined by extension of*

$$df(\mu_+)((-\infty, x]) = f(\mu_+)((-\infty, x]), \; x \in \mathbb{R}$$

(equivalently, $x \mapsto f(\mu_+)((-\infty, x])$ is the cumulative distribution of $df(\mu_+)$.) We then define the integral of f with respect to μ_+ as

$$\mu_+(f) = \int_{-\infty}^{\infty} x \; df(\mu_+)(x).$$

We note that even such a simple change as using rightward infinite intervals will in general give a different integral. We also note that if we use this construction with a (possibly signed) quasi-measure ν instead of μ_+, then $df(\nu)$ becomes equal to $f(\nu)$, and so the integral is the usual quasi-integral.

Proposition 29 *We have the following properties of the integral:*

1. $\mu_+(af + b) = a\mu_+(f) + b\mu_+(X)$, $a \geq 0$, $b \in \mathbb{R}$
2. $f \leq g$ *implies* $\mu_+(f) \leq \mu_+(g)$
3. $\mu_+(f) - \mu_-(f) = \mu(f)$

Proof. (1) For $a > 0$, $x \in \mathbb{R}$ we have

$$(af + b)(\mu_+)((-\infty, ax + b]) = f(\mu_+)((-\infty, x]),$$

so that

$$\begin{aligned} \mu_+(af + b) &= \int_{-\infty}^{\infty} x \; d(af + b)(\mu_+)(x) = \int_{-\infty}^{\infty} (ax + b) \; df(\mu_+)(x) \\ &= a \int_{-\infty}^{\infty} x \; df(\mu_+)(x) + b \lim_{x \to \infty} f(\mu_+)((-\infty, x]) \\ &= a\mu_+(f) + b\mu_+(X). \end{aligned}$$

For $a = 0$, f does not affect the value of the integral, so we may assume $f = 0$; then a does not affect the value, and we may reduce to the $a > 0$ case.

(2) If $f \leq g$, then for each $x \in \mathbb{R}$ we have $f^{-1}((-\infty, x]) \supseteq g^{-1}((-\infty, x])$ and so $f(\mu_+)((-\infty, x]) \geq g(\mu_+)((-\infty, x])$.

Let $M > 0$ be such that $|f|, |g| < M$, so that both $df(\mu_+)$ and $dg(\mu_+)$ have support in $[-M, M]$. Since the identity $x \mapsto x$ on $[-M, M]$ is continuous, and both it and $x \mapsto f(\mu_+)((-\infty, x])$ are of bounded variation there, we may integrate by parts:

$$\begin{aligned}
\mu(f) &= \int_{-M}^{M} x \; df(\mu_+)(x) \\
&= (x f(\mu_+)((-\infty, x]))|_{x=-M}^{M} - \int_{-M}^{M} f(\mu_+)((-\infty, x]) \; dx \\
&= M - \int_{-M}^{M} f(\mu_+)((-\infty, x]) \; dx \\
&\leq M - \int_{-M}^{M} g(\mu_+)((-\infty, x]) \; dx \\
&= \mu(g).
\end{aligned}$$

(3) In addition to 24.8, we need merely note that for a fixed f, definitions 27 and 28 are all linear in the deficient quasi-measure. ■

Theorem 30 *Every quasi-linear functional $\mu : C(X, \mathbb{R}) \to \mathbb{R}$ is uniformly continuous; specifically, for $f, g \in C(X, \mathbb{R})$*

$$|\mu(f) - \mu(g)| \leq \|\mu\| \; \|f - g\|_\infty.$$

Proof. We have $f - \|f - g\|_\infty \leq g \leq f + \|f - g\|_\infty$, so by proposition 29

$$\begin{array}{ccccc}
\mu_+(f) - \mu_+(X) \; \|f - g\|_\infty & \leq & \mu_+(g) & \leq & \mu_+(f) + \mu_+(X) \; \|f - g\|_\infty \\
\mu_-(f) - \mu_-(X) \; \|f - g\|_\infty & \leq & \mu_-(g) & \leq & \mu_-(f) + \mu_-(X) \; \|f - g\|_\infty
\end{array}$$

and then by 24.8

$$\begin{aligned}
\mu(f) - \|\mu\| \; \|f - g\|_\infty &= \mu(f) - (\mu_+(X) + \mu_-(X)) \; \|f - g\|_\infty \\
&\leq \mu(g) \\
&\leq \mu(f) + (\mu_+(X) + \mu_-(X)) \; \|f - g\|_\infty \\
&= \mu(f) + \|\mu\| \; \|f - g\|_\infty,
\end{aligned}$$

giving the result. ■

Remark 31 *With uniform continuity the choice of definition for quasi-linear functionals seems to be the appropriate one.*

7 Decomposition into quasi-measures

We would like to investigate when it is possible to write a signed quasi-measure as the difference of two positive ones. As example 25 showed, such a decomposition need not be unique even when it exists; note that the decompositions $\nu_1 - \nu_3$ and $\nu_4 - \nu_2$ of that example both are minimal in the sense that none of the pairs (ν_1, ν_3) and (ν_2, ν_4) can bound any common positive quasi-measure other than zero, as the argument that ν_+ is not a quasi-measure would apply to it as well.

From its definition and properties, we may characterize μ_+ as follows: μ_+ is the unique smallest positive monotone regular function bounding μ from above. Clearly if $\mu = \mu_1 - \mu_2$, where μ_1, μ_2 are positive quasi-measures, then μ_1 must have the same properties. Thus we have the following proposition:

Proposition 32 *A positive quasi-measure μ' gives a decomposition of a signed quasi-measure μ into a difference of positive quasi-measures by $\mu = \mu' - (\mu - \mu')$, if and only if $\mu' \geq \mu_+$.*

In the following, let X be a q-space. We may then use the full power of the solid set function construction to construct a positive quasi-measure bounding μ_+ from above, and thus giving a decomposition of μ.

Definition 33 *Let ν be a deficient quasi-measure, and let $p \in X$ be an arbitrary point. We then define the solid set function $\nu^p : \mathcal{A}_s(X) \to \mathbb{R}$ by*

$$\nu^p(M) = \begin{cases} \nu(M), & p \notin M \\ \nu(X) - \nu(X \setminus M), & p \in M \end{cases}$$

Proposition 34 *We have the following properties for ν^p:*

1. *$\nu^p(X) = \nu(X)$.*

2. *ν^p extends to a (positive) quasi-measure on X.*

3. *$\nu^p(M) \geq \nu(M)$, $M \in \mathcal{A}(X)$.*

Proof. (1) Obvious.

(2) Since $\nu(X) - \nu(X \setminus M) \geq \nu(M)$ for any set in $\mathcal{A}(X)$, and since a set containing p cannot be a subset of a set not containing p, ν^p is monotonic on solid sets.

If U is an open solid set, $p \in U$, then we may add p to any compact set approximating U from within, and then find a solid compact subset of U containing the resulting compact set. If $p \notin U$, then any compact set contained in U cannot contain p. Thus inner regularity of ν^p follows from the regularity of each defining expression.

Additivity on complements is built into the definition.

Let M_i, $i = 1, \dots, n$, $n \in \mathbb{N}$ be disjoint solid sets, $M \supseteq \biguplus_{i=1}^n M_i$, with M a solid set. If none of the M_i contains p, then

$$\nu^p(M) \geq \nu(M) \geq \sum_{i=1}^n \nu(M_i) = \sum_{i=1}^n \nu^p(M_i).$$

If $p \in M_1$, say, then by the above

$$\begin{aligned}
\nu^p(X \setminus M_1) &\geq \sum_{i=2}^n \nu^p(M_i) + \nu^p(X \setminus M); \\
&\text{i.e.} \\
\nu^p(M) &= \nu^p(X) - \nu^p(X \setminus M) \\
&\geq \nu^p(X) - \nu^p(X \setminus M_1) + \sum_{i=2}^n \nu^p(M_i) \\
&= \sum_{i=1}^n \nu^p(M_i).
\end{aligned}$$

Thus ν^p extends to a quasi-measure.

(3) Let M be a connected compact set, and let $(M_i)_{i\in I}$ be the family of (open solid) components of its complement. If $p \in M$, then

$$\begin{aligned}
\nu^p(M) &= \nu^p(X) - \sum_{i\in I} \nu^p(M_i) \\
&= \nu(X) - \sum_{i\in I} \nu(M_i) \geq \nu(M).
\end{aligned}$$

If $p \in M_1$, say, then

$$\begin{aligned} \nu^p(M) &= \nu^p(X) - \sum_{i \in I} \nu^p(M_i) \\ &= \nu(X) - (\nu(X) - \nu(X \setminus M_1)) - \sum_{i \in I \setminus \{1\}} \nu(M_i) \\ &= \nu(X \setminus M_1) - \sum_{i \in I \setminus \{1\}} \nu(M_i) \geq \nu(M). \end{aligned}$$

Let U be an arbitrary open set; then by inner regularity $\nu^p(U)$ and $\nu(U)$ may be approximated by the values of ν^p and ν on finite disjoint unions of compact connected sets contained in U, and so satisfy the inequality. Arbitrary compact sets then follow by outer regularity. ■

Theorem 35 *Any signed quasi-measure μ on a q-space X may be written as the difference of positive quasi-measures μ_1 and μ_2. Moreover, $\|\mu\| = \mu_1(X) + \mu_2(X)$.*

Proof. Define $\mu_1 = \mu_+^p$ and $\mu_2 = \mu_-^p$. Then for any solid set M not containing p, $\mu(M) = \mu_1(M) - \mu_2(M)$, and $\mu(X) = \mu_+(X) - \mu_-(X) = \mu_1(X) - \mu_2(X)$, which by taking complements gives $\mu(M) = \mu_1(M) - \mu_2(M)$ for solid sets M containing p. By uniqueness of extension of signed solid set functions to signed quasi-measures, $\mu = \mu_1 - \mu_2$.

Finally, $\|\mu\| = \mu_+(X) + \mu_-(X) = \mu_1(X) + \mu_2(X)$. ■

8 Signed solid set functions

We wish to extend to signed quasi-measures Aarnes's construction of a quasi-measure from its values at solid sets. The extension definition is quite similar, replacing suprema by more general limits of nets. However, checking that a "signed solid set function" corresponds to a signed quasi-measure presents additional difficulties since the lack of monotonicity means that boundedness of the measure on solid sets, or even connected sets, is no longer sufficient to check boundedness of the quasi-measure; also, limits need special care since sets "squeezed" between sets close in measure need no longer themselves be close in measure.

8.1 Topological preliminaries

For the entirety of this section, X will be a connected, locally connected, compact Hausdorff space.

We start with a generalization of Proposition 7.2:

Proposition 36 *Let M be a subset of X, and let C be a component of M. Then each co-component of C contains a co-component of M. In particular, if M has a finite number of co-components, then so does C, and if M is co-connected, C is solid.*

Proof. Let D be a co-component of C, and assume that it contains no co-component of M. Since any co-component of M is a connected set contained in $X \setminus M \subset X \setminus C$, D cannot intersect it without containing it, and so $D \subset M$. But then D must be contained in a component of M other than C, and similarly that component must be contained in D. So D is a component of M.

Since C and D are distinct components of M, the closure of each does not intersect the other. But then the closure of D is a connected set disjoint from C, and containing D; since D is a co-component of C it must therefore be its own closure.

But also $D \subset X \setminus \overline{C} \subset X \setminus C$, so D is a closed component of the open set $X \setminus \overline{C}$, a contradiction since X is locally connected. ■

Definition 37 *By $\mathcal{C}_0$ we denote the family of closed sets with a finite number of components. By $\mathcal{C}_f$ we denote the subfamily of $\mathcal{C}_0$ consisting of those sets whose complements each have a finite number of co-components. By $\mathcal{O}_0$ we denote the family of open sets whose complements are in $\mathcal{C}_0$.*

Corollary 38 *Each (open) component of an $\mathcal{O}_0$-set has a finite number of co-components, each a compact solid set.*

Proof. In Proposition 36, consider first M to be the open set in question, and secondly consider M to be the complement of any of its components. ■

Proposition 39 *Let $C \in \mathcal{C}$, $U \in \mathcal{O}$, $C \subset U$. Then there exists $C' \in \mathcal{C}_f$ such that $C \subset C' \subset U$.*

Proof. Only a finite number of components of U may intersect C, so it suffices to consider the case of U connected. First by Proposition 7 let $C \subset C'' \subset U$ with C'' connected. Now we note that a finite number of co-components of $X \setminus C''$ cover $X \setminus U$; we obtain C' as the complement of their union. ■

Notation 40 *For each $A \in \mathcal{A}_0$, we write its components with indices, i.e. A_1, A_2, ..., A_i, ..., A_n. Likewise we write each (solid) co-component of A_i as A_i^j. Thus $A = \biguplus_{i=1}^{n}(X \setminus \biguplus_j A_i^j)$. This notation is unique up to obvious permutations.*

Lemma 41 *Let $C \in \mathcal{C}_c$ with $X \backslash C = \bigcup_{i \in I} C^i$ where $\{C^i\}_{i \in I} \subset \mathcal{O}_s$ is a disjoint collection. Then $C \cup (\bigcup_{i \in I_0} C^i) \in \mathcal{C}_c$ for any set of indices $I_0 \subset I$.*

Proof. Obviously $C \cup (\bigcup_{i \in I_0} C^i) = X \backslash (\bigcup_{i \notin I_0} C^i) \in \mathcal{C}$. Suppose $C' = C \cup (\bigcup_{i \in I_0} C^i)$ is not connected. Then there must be disjoint open sets U and V such that $\emptyset \neq C' \cap U, C' \cap V \in \mathcal{C}$, but C and $\{C^i\}$ are all connected, hence they are all contained in either U or V respectively. We may assume without loss of generality that $C \subset U$, then $C' \cap V = \bigcup_{C^i \subset V} C^i \in \mathcal{O}$. Now we reach a contradiction since X is connected and therefore contains no nontrivial clopen sets. ■

8.2 Definition of signed solid set functions

The following Proposition describes our general plan for how to recover the value of a signed quasi-measure from its values at solid sets.

Proposition 42 *Let μ be a signed quasi-measure. Then*

1. $\mu(C) = \sum_i \left(\mu(X) - \sum_j \mu(C_i^j)\right)$, $C \in \mathcal{C}_0$
2. $\mu(U) = \lim_{\substack{C \subset U \\ C \in \mathcal{C}_0}} \mu(C)$, $U \in \mathcal{O}$
3. $\mu(C) = \mu(X) - \mu(X \setminus C)$, $C \in \mathcal{C}$.

Proof. Equations (1) and (3) are obvious from additivity. That it suffices to consider $\mathcal{C}_0$-sets (or even $\mathcal{C}_f$-sets) in the limit follows from Proposition 39. ■

Definition 43 *A function $\mu : \mathcal{A}_s \to \mathbb{R}$ is a* signed solid set function *if it satisfies*

1. *There exists $M \geq 0$ such that for all $C \in \mathcal{C}_f$,*

$$\left| \sum_i \left(\mu(X) - \sum_j \mu(C_i^j) \right) \right| \leq M.$$

2. *For all $U \in \mathcal{O}_s$ we have*

$$\mu(U) = \lim_{\substack{C \subset U \\ C \in \mathcal{C}_s}} \mu(C).$$

3. *For any trivial or irreducible partition $\{A_i\}_{i \in I}$ of X we have*

$$\mu(X) = \sum_{i \in I} \mu(A_i).$$

Proposition 44 *Let μ be a signed solid set function, then μ may be extended to $\mathcal{A}_0$ by*

1. $\mu(C) = \sum_i \left(\mu(X) - \sum_j \mu(C_i^j) \right)$, $C \in \mathcal{C}_0$

2. $\mu(U) = \mu(X) - \mu(X \backslash U)$, $U \in \mathcal{O}_0$.

In particular, the extension of μ is bounded on $\mathcal{A}_0$.

Proof. If $C \in \mathcal{C}_0$ then for any component C_i of C, $\{\mu C_i^j\}_j$ is nonzero for at most a countable number of indices by Definition 43.1 and Lemma 41. Moreover, the sum $\sum_j \mu(C_i^j)$ is absolutely convergent. Hence μ is well-defined and real valued on $\mathcal{A}_0$. Given $\varepsilon > 0$ let $\{C_i\}_{i=1}^n$ be the components of C and for each i pick n_i such that

$$|\mu(C_i) - \left(\mu(X) - \sum_{j=1}^{n_i} \mu(C_i^j) \right)| < \frac{\varepsilon}{n}$$

and such that $C_k \subset \bigcup_{j=1}^{n_i} C_i^j$ for $k \neq i$. Then the set $C' = \bigcup_{i=1}^{n} [X \backslash (\bigcup_{j=1}^{n_i} C_i^j)]$ is in $\mathcal{C}_f$ and we get

$$|\mu(C) - \mu(C')| \leq \sum_{i=1}^{n} |\mu(C_i) - \left(\mu(X) - \sum_{j=1}^{n_i} \mu(C_i^j)\right)| < \varepsilon$$

Accordingly μ is bounded by M on $\mathcal{C}_0$. By (2) μ must be bounded by $2M$ on $\mathcal{A}_0$. One may verify that the extension is consistent with the definition of μ on $\mathcal{A}_s$. ■

Remark 45 *We will assume from now on that μ is a signed solid set function extended to $\mathcal{A}_0$ according to the proposition above.*

Corollary 46 *We have the following properties for μ.*

1. *Let $C, C' \in \mathcal{C}_0$ with $C \cap C' = \emptyset$, then $\mu(C \uplus C') = \mu(C) + \mu(C')$.*
2. *Let $C \in \mathcal{C}_0$, $U \in \mathcal{O}_0$, $C \subset U$, then $\mu(U \setminus C) = \mu(U) - \mu(C)$.*
3. *Let $C \in \mathcal{C}_c$ with co-components $\{C^j\}_{j \in J}$. Then for any subset $I \subset J$ we have*

$$\mu(\bigcup_{j \in I} C^j) = \sum_{j \in I} \mu(C^j)$$

4. *Let $(C_i)_{i \in I} \subset \mathcal{C}_0$ be a disjoint family of $\mathcal{C}_0$-sets, then $\sum_i |\mu(C_i)| \leq 2M$.*
5. *Let $(U_i)_{i \in I} \subset \mathcal{O}_s$ be a disjoint family of solid open sets, then $\sum_i |\mu(U_i)| \leq 2M$.*

Proof. (1) The family $\mathcal{C}_0$ is closed under finite unions, and the components of a (disjoint) union are the components of the individual sets.

(2) This follows from (1) by considering $C' = X \setminus U$.

(3) $C' = C \cup (\bigcup_{j \in I} C^j) \in \mathcal{C}_c$ by Lemma 41 and hence

$$\begin{aligned} \mu(\bigcup_{j \in I} C^j) &= \mu(X) - \mu(C') = \mu(X) - (\mu(X) - \sum_{j \in I} \mu(C^j)) \\ &= \sum_{j \in I} \mu(C^j) \end{aligned}$$

(4) It suffices to consider finite families. Let $I_+ = \{i|\mu(C_i) > 0\}$, and $I_- = I \setminus I_+$. Then

$$\sum_i |\mu(C_i)| = \sum_{i\in I_+} |\mu(C_i)| + \sum_{i\in I_-} |\mu(C_i)| = |\mu(\bigcup_{i\in I_+} C_i)| + |\mu(\bigcup_{i\in I_-} C_i)| \leq 2M.$$

(5) Let $0 < \varepsilon < 1$. By Definition 43.2 we may find $C_i \in \mathcal{C}_s$, $C_i \subset U_i$, with $|\mu(C_i)| \geq (1-\varepsilon)|\mu(U_i)|$, from which (4) gives the result. ∎

8.3 Regularity on $\mathcal{A}_0$

We will now embark on proving the result (Proposition 51) that the signed solid set functions are regular on $\mathcal{A}_0$. For positive quasi-measures, this essentially corresponds to condition $(Q3)_0$ of [3], section 4. Condition $(Q1)_0$ is monotonicity, which we do not have; condition $(Q2)_0$ is additivity on disjoint $\mathcal{C}_0$-sets, which is Corollary 46.1.

Lemma 47 *Given $\varepsilon > 0$ and $C \in \mathcal{C}_s$, there is a $U \in \mathcal{O}_s$ such that $K \in \mathcal{C}_0, K \subset U \backslash C$ implies $|\mu(K)| < \varepsilon$.*

Proof. Assume to the contrary that no such set $U \in \mathcal{O}_s$ exists. Then recursively there is a set $K_i \in \mathcal{C}_0, K_i \subset O_i \backslash C$ with $|\mu(K_i)| \geq \varepsilon$ and by Proposition 7.3 a set $O_{i+1} \in \mathcal{O}_s$ with $C \subset O_{i+1} \subset O_i \setminus K_i$ (O_1 may be taken to be X). Now for each finite set $I \subset \mathbb{N}$ we have $\bigcup_{i\in I} K_i \in \mathcal{C}_0$ and $\sum_{i=1}^n |\mu(K_i)| \geq \varepsilon n$ violating the boundedness of μ on $\mathcal{C}_0$. ∎

Lemma 48 *If $U \in \mathcal{O}_c$ has a finite number of co-components, which are all solid, then*

$$\mu(U) = \lim_{\substack{C\subset U \\ C\in\mathcal{C}_0}} \mu(C)$$

Proof. Let $\{K_i\}_{i=1}^n$ denote the co-components of U. Given $\varepsilon > 0$ pick sets $\{U_i\}_{i=1}^n \subset \mathcal{O}_s$ such that $K \in \mathcal{C}_0, K \subset U_i \backslash K_i$ implies $|\mu(K)| < \frac{\varepsilon}{n}$ and such that $U \in \mathcal{O}_s, K_i \subset U \subset U_i$ implies $|\mu(U) - \mu(K_i)| < \frac{\varepsilon}{n}$ (by Lemma 47 and Definition 43.2 respectively). By Proposition 7.1 there is a set $C \in \mathcal{C}_c$ with $X \backslash (\bigcup_{i\in I} U_i) \subset C \subset U$. By Lemma 41 we may assume that all co-components C^j contain exactly one K_i (replace C with $C \cup (\bigcup\{C^j : C^j \cap (\bigcup_{i=1}^n K_i) =$

$\emptyset\})$). We now claim that $F \in \mathcal{C}_0, C \subset F \subset U$ implies $|\mu(U) - \mu(F)| < 5\varepsilon$ (completing the proof).

One of the components say F' of F must contain C (since C is connected) and $F \backslash F' \in \mathcal{C}_0$ with $F \backslash F' \subset \bigcup_{i=1}^n U_i \backslash K_i$, $F \backslash F' \cap U_i \in \mathcal{C}_0$ for $i = 1, ..., n$. Hence $|\mu(F \backslash F')| < \varepsilon$.

Let $\{V_i\}_{i=1}^\infty$ (possibly finite collection) be the co-components of F' where μ is nonzero and $V_i \cap (\bigcup_{i=1}^n K_i) = \emptyset$. Pick $F_i \in \mathcal{C}_s$, $F_i \subset V_i$ with $|\mu(V_i) - \mu(F_i)| < \frac{\varepsilon}{2^i}$ for $i = 1, 2, ...$ Now $\bigcup_{i=1}^\infty F_i \subset \bigcup_{i=1}^n U_i \backslash K_i$ and $\bigcup_{i=1}^m F_i \in \mathcal{C}_0$ for $m \in \mathbb{N}$, so $|\mu(\bigcup_{i=1}^m F_i)| < \varepsilon$. For m sufficiently large we have $|\mu(\bigcup_{i=1}^\infty V_i) - \mu(\bigcup_{i=1}^m F_i)| < 2\varepsilon$, implying $|\mu(\bigcup V_i)| = |\sum \mu(V_i)| < 3\varepsilon$ (c.f. Corollary 46.3).

Finally, we have

$$|\mu(U) - \mu(F)| \leq |\mu(U) - \mu(F')| + \varepsilon \leq |\mu(U) - \mu(F' \cup (\bigcup V_i))| + 4\varepsilon < 5\varepsilon$$

where the last inequality is due to each co-component of $F' \cup (\bigcup V_i)$ being an open solid set U with $K_i \subset U \subset U_i$ for some choice of i. The proof is complete. ■

Lemma 49 *For all $U \in \mathcal{O}_0$,*

$$\lim_{\substack{C \subset U \\ C \in \mathcal{C}_0}} \mu C = \sum_i \mu U_i.$$

Proof. By Corollary 46.4 and monotone convergence we have

$$\sum_i \sup_{\substack{C \subset U_i \\ C \in \mathcal{C}_0}} |\mu C| \leq 2M.$$

Given $\varepsilon > 0$, we may therefore write $U = U' \uplus U''$, $U', U'' \in \mathcal{O}$, where $U' = \biguplus_{i=1}^n U_i'$ consists of a finite number of components of U and

$$|\sum_i \mu(U_i'')| \leq \sup_{\substack{C \subset U'' \\ C \in \mathcal{C}_0}} |\mu C| \leq \sum_i \sup_{\substack{C \subset U_i'' \\ C \in \mathcal{C}_0}} |\mu C| < \varepsilon.$$

By Lemma 48 we then select $C_i \in U_i'$ such that $F \in \mathcal{C}_0$, $C_i \subset F \subset U_i'$ implies $|\mu(U_i') - \mu(F)| < \varepsilon/n$. Let $C = \biguplus_{i=1}^n C_i$.

We then have that $F \in \mathcal{C}_0, C \subset F \subset U$ implies

$$\begin{aligned} |\sum_i \mu U_i - \mu(F)| &\leq \sum_{i=1}^n |\mu(U_i') - \mu(F \cap U_i')| + |\sum_i \mu(U_i'') - \mu(F \cap U'')| \\ &< \sum_{i=1}^n \varepsilon/n + 2\varepsilon = 3\varepsilon. \end{aligned}$$

■

Lemma 50 *Let $U \in \mathcal{O}_0$ with solid co-components. Then*

$$\mu(U) = \sum_{i \in I} \mu(U_i)$$

Proof. Let $\{C_i\}_{i=1}^n$ denote the co-components of U. We have the equivalent statement:

$$\mu(X) = \sum_{i \in I} \mu(U_i) + \sum_{i=1}^n \mu(C_i).$$

The proof is by induction on n. If $n = 0$, then $U = X$ and the statement is trivial. If $n = 1$, then $U \in \mathcal{O}_s$ and we have a trivial partition and equality follows. Assume the equation to hold up to an arbitrary $n \in \mathbb{N}$. If U is connected, then equality follows from the definition of μ on $\mathcal{O}_0$. If U is not connected, then Proposition 12 applies to the collection $\{C_i\}_{i=1}^{n+1}$. That is we have a subfamily $\{C_i\}_{i \in J}$, card $J \geq 2$ such that $X = \left(\bigcup_{i \in J} C_i\right) \cup \left(\bigcup_{i \in I'} O_i\right)$ is an irreducible partition of X (here $O = X \setminus \bigcup_{i \in J} C_i$). By Definition 43.3 we have

$$\mu(X) = \sum_{i \in I'} \mu(O_i) + \sum_{i \in J} \mu(C_i)$$

For an arbitrary choice of $i' \in I'$ we have

$$O_{i'} = \left(\bigcup_{C_i \subset O_{i'}} C_i \right) \cup \left(\bigcup_{U_i' \subset O_{i'}} U_i \right)$$

Now the induction hypothesis applies to the collection $\{C_i : C_i \subset O_{i'}\} \cup \{X \backslash O_{i'}\}$ since it consists of at most n solid compact sets. We have

$$\begin{aligned}\mu(X) &= \sum_{U_i' \subset O_{i'}} \mu(U_i) + \sum_{C_i \subset O_{i'}} \mu(C_i) + \mu(X \backslash O_{i'}) \Rightarrow \\ \mu(O_{i'}) &= \sum_{U_i' \subset O_{i'}} \mu(U_i) + \sum_{C_i \subset O_{i'}} \mu(C_i)\end{aligned}$$

Finally, we obtain

$$\mu(X) = \sum_{i \in I'} \mu(O_i) + \sum_{i \in J} \mu(C_i) = \sum_{i \in I} \mu(U_i) + \sum_{i=1}^{n+1} \mu(C_i)$$

The proof is complete. ■

Proposition 51 *For all $U \in \mathcal{O}_0$ we have*

$$\mu(U) = \lim_{\substack{C \subset U \\ C \in \mathcal{C}_0}} \mu C.$$

Proof. We will show that $\mu(U) = \sum_i \mu(U_i)$, from which the result follows by Lemma 49. Throughout the proof let $C = X \backslash U \in \mathcal{C}_0$.

We will first prove a special case. Assume that for each i there is a j such that $C_k \subset C_i^j$ for all k. Let $O = \bigcup_{j,i} \{C_i^j : C_k \cap C_i^j = \emptyset \text{ for all } k\}$.

Consider the sets $K_i = C_i \cup \left(\bigcup_{\{j : C_k \cap C_i^j = \emptyset \text{ for all } k\}} C_i^j \right)$. We have $\{K_i : i \in I\} \subset \mathcal{C}_s$, so $O' = X \backslash (\bigcup K_i)$ is an open set in $\mathcal{O}_0$ with solid co-components. By Lemma 50, $\mu(O') = \sum_i O_i'$. Then

$$\begin{aligned}\mu(U) &= \mu(X) - \sum_i (\mu(X) - \sum_j \mu(C_i^j)) \\ &= \mu(X) - \sum_i (\mu(K_i) - \sum_{C_k \cap C_i^j = \emptyset,\ \forall k} \mu(C_i^j)) \\ &= \mu(X) - \sum_i \mu(K_i) + \sum_i \mu(O_i) \\ &= \mu(O') + \sum_i \mu(O_i) = \sum_i \mu(O_i') + \sum_i \mu(O_i) \\ &= \sum_i \mu(U_i),\end{aligned}$$

completing the proof for this case.

Let $n = \text{card}\{C_i : i \in I\}$. The remaining proof will be by induction on n. For $n \leq 2$ we must be in the case already proved. Assume $\mu(U) = \sum_i \mu(U_i)$ to hold for $n = 1, 2, ..., k$, and let $C \in \mathcal{C}_0$ with card $\{C_i\} = k + 1$.

Having proved the first case, we may now assume that there are distinct i, k_1, k_2 together with j, such that $C_{k_1} \subset C_i^j$ but $C_{k_2} \cap C_i^j = \emptyset$.

Now the induction hypothesis applies to the collection $K = X \setminus O = \{C_i \cup C_i^{ij}\} \cup \{C_k : C_k \cap C_i^j = \emptyset,\ k \neq i\}$ and to the collection $K' = X \setminus O' = \{X \setminus C_i^j\} \cup \{C_k : C_k \subset C_i^j\}$. Then $K \cap K' = C$, $K \cup K' = X$ (i.e. $O \cap O' = \emptyset$) and

$$\begin{aligned}
\sum_i \mu(U_i) &= \sum_i \mu(O_i) + \sum_i \mu(O_i') = \mu(O) + \mu(O') \\
&\quad \text{(by induction)} \\
&= (\mu(X) - \mu(C_i \cup C_i^j) - \sum_{\substack{C^k \cap C_i^j = \emptyset \\ k \neq i}} \mu(C_k)) \\
&+ (\mu(X) - \mu(X \setminus C_i^j) - \sum_{C_k \subset C_i^j} \mu(C_k)) \\
&= \sum_{l \neq j} \mu(C_i^l) + \mu(C_i^j) - \sum_{k \neq i} \mu(C_k) = \mu(X) - \mu(C_i) - \sum_{k \neq i} \mu(C_k) \\
&= \mu(U).
\end{aligned}$$

■

8.4 Extension to $\mathcal{A}$

Lemma 52 *Let μ be a signed solid set function extended to $\mathcal{A}_0$. For any open set U, $\lim_{\substack{C \subset U \\ C \in \mathcal{C}_0}} \mu C$ exists.*

Proof. Let $C_1 \subset U$, $C_1 \in \mathcal{C}_0$; then recursively there is a set $V_1 \in \mathcal{O}_0$ with $C_1 \subset V_1 \subset \bar{V}_1 \subset U$ and a set $C_2 \in \mathcal{C}_0$ with $V_1 \subset C_2 \subset U$. We obtain an increasing sequence $\{C_i\}$. Now suppose to the contrary that the limit does not exist. Then there is an $\varepsilon > 0$ such that $\{C_i\}$ may be chosen with $|\mu C_i - \mu C_{i+1}| > \varepsilon$ and V_i with $|\mu V_i - \mu C_i| < \frac{\varepsilon}{2}$ for $i = 1, 2, ...$

We get $\varepsilon < |\mu C_i - \mu C_{i+1}| = |\mu C_i - \mu V_{i+1} + \mu V_{i+1} - \mu C_{i+1}| \leq |\mu C_i - \mu V_{i+1}| + |\mu V_{i+1} - \mu C_{i+1}| \leq |\mu C_i - \mu V_{i+1}| + \frac{\varepsilon}{2}$, accordingly $|\mu C_i - \mu V_{i+1}| \geq \frac{\varepsilon}{2}$. We have constructed a disjoint sequence $\{V_{2i+1}\backslash C_{2i}\} \subset \mathcal{O}_0$ with $|\mu(V_{2i+1}\backslash C_{2i})| \geq \frac{\varepsilon}{2}$; $i = 1, 2, \ldots$ Finally, pick $K_i \subset V_{2i+1}\backslash C_{2i}$, $K_i \in \mathcal{C}_0$ with $|\mu(V_{2i+1}\backslash C_{2i}) - \mu K_i| < \frac{\varepsilon}{4}$ implying $|\mu K_i| \geq \frac{\varepsilon}{4}$. We reach a contradiction with $\sum_i |\mu(K_i)| = \infty$. ■

Theorem 53 *A signed solid set function $\mu : \mathcal{A}_s \to \mathbb{R}$ extends uniquely to a signed quasi-measure in X. Conversely, the restriction of a signed quasi-measure to the solid sets is a signed solid set function.*

Proof. For an arbitrary set in $\mathcal{A}$ we define μ by

1. $\mu V = \lim_{\substack{C \subset V \\ C \in \mathcal{C}_0}} \mu C$, $V \in \mathcal{O}$;

2. $\mu C = \mu X - \mu(X\backslash C)$, $C \in \mathcal{C}$.

Clearly this is consistent with the definition on $\mathcal{A}_0$. We start by proving the regularity (Definition 2.3). Let $\varepsilon > 0$ and $U \in \mathcal{O}$ be arbitrary. Then by Lemma 52 there is a set $C \in \mathcal{C}_0$; $C \subset U$ such that for any set $C' \in \mathcal{C}_0$ with $C \subset C' \subset U$ we have $|\mu C - \mu C'| < \frac{\varepsilon}{2}$. Now let K be any closed set with $C \subset K \subset U$. It suffices to show that $|\mu C - \mu K| < \varepsilon$. To this end choose a set $K' \in \mathcal{C}_0$ such that $X\backslash U \subset K' \subset X\backslash K$ and $|\mu(X\backslash K) - \mu K'| = |\mu(X\backslash K') - \mu K| < \frac{\varepsilon}{2}$ (by Lemma 52). Again by Lemma 52 and Proposition 51 $|\mu(X\backslash K') - \mu C| \leq \frac{\varepsilon}{2}$. We have

$$|\mu C - \mu K| \leq |\mu C - \mu(X\backslash K')| + |\mu(X\backslash K') - \mu K| < \varepsilon,$$

which establishes 2.3.

For 2.2, it is clear from the definition of μ on open sets that $\sup\{|\mu U| : U \in \mathcal{O}\} \leq M$. Notice that by complement we have for any set $C \in \mathcal{C}$ that $\mu C = \lim_{\substack{C \subset U \\ U \in \mathcal{O}}} \mu U$. Hence $|\mu A| \leq M$ for any set $A \in \mathcal{A}$.

It remains to prove additivity (2.1). Let U_1 and U_2 be disjoint open sets. For any pair of closed sets $C_1, C_2 \in \mathcal{C}_0$; $C_1 \subset U_1$, $C_2 \subset U_2$ we have $C_1 \cup C_2 \in \mathcal{C}_0$; $C_1 \cup C_2 \subset U_1 \cup U_2$. Conversely, for any $C \in \mathcal{C}_0$; $C \subset U_1 \cup U_2$ we have $C \cap U_1 \in \mathcal{C}_0$ and $C \cap U_2 \in \mathcal{C}_0$. Hence by additivity of μ on $\mathcal{C}_0$ we have

$$\mu(U_1 \cup U_2) = \lim_{\substack{C \subset U_1 \cup U_2 \\ C \in \mathcal{C}_0}} \mu C = \lim_{\substack{C \subset U_1 \\ C \in \mathcal{C}_0}} \mu C + \lim_{\substack{C \subset U_2 \\ C \in \mathcal{C}_0}} \mu C = \mu U_1 + \mu U_2$$

By induction μ is finitely additive on open sets. Now let $C_1, C_2 \in \mathcal{C}$ be disjoint. Similarly as above we obtain

$$\mu(C_1 \cup C_2) = \lim_{\substack{C_1 \cup C_2 \subset U \\ U \in \mathcal{O}}} \mu U = \lim_{\substack{C_1 \subset U \\ U \in \mathcal{O}}} \mu U + \lim_{\substack{C_2 \subset U \\ U \in \mathcal{O}}} \mu U = \mu C_1 + \mu C_2$$

Finally, let $A_1, A_2, ..., A_n \in \mathcal{A}$ be a disjoint collection with $\bigcup_{i=1}^n A_i \in \mathcal{A}$. Put $A_0 = X \setminus \bigcup_{i=1}^n A_i$; by finite additivity on open and closed sets respectively, and (2) we get

$$\begin{aligned} \mu X &= \mu(\bigcup_{i=0}^{n} A_i) = \mu(\bigcup_{A_i \in \mathcal{C}} A_i) + \mu(\bigcup_{A_i \in \mathcal{O}} A_i) = \mu A_0 + \sum_{i=1}^{n} \mu A_i \\ &= \mu X - \mu(\bigcup_{i=1}^{n} A_i) + \sum_{i=1}^{n} \mu A_i, \end{aligned}$$

and hence

$$\mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu A_i.$$

The uniqueness of the extension follows from Proposition 42.

Now assume instead μ to be a signed quasi-measure, and consider its restriction to $\mathcal{A}_s$. Definition 43.1 follows by Proposition 42.1 and Definition 2.1; 43.2 follows from finite additivity, and that it suffices to consider solid sets in 43.3 follows from Proposition 7.3. ■

9 Signed q-functions

The q-functions originated in [4] and have proven to be an efficient tool for constructing quasi-measures (see e.g. [5]). It is natural to ask whether a similar construction technique might apply to signed quasi-measures. It turned out that this is possible.

Throughout this section we will let X denote a q-space. Let $Q_1(X)$ denote the normalized quasi-measures (i.e. $\mu(X) = 1$). We say that $\mu \in Q_1(X)$ is *non-splitting* if there is no disjoint pair $C_1, C_2 \in \mathcal{C}_s$ such that $\mu(C_1) > 0$, $\mu(C_2) > 0$ and $\mu(C_1) + \mu(C_2) = 1$ (Lebesgue measure on the disk or sphere are examples of non-splitting measures).

Definition 54 *For $f : [0,1] \to \mathbb{R}$, $\mu \in Q_1(X)$, $C \in \mathcal{C}_s$,define $f_*(\mu) : \mathcal{C}_s \to \mathbb{R}$ by*

$$[f_*(\mu)](C) = f(\mu(C)) \text{ for all } C \in \mathcal{C}_s$$

Then f is called a signed q-function *if $f_*(\mu)$ extends to a signed solid set function (and hence to a signed quasi-measure) in X for any q-space X, and any non-splitting $\mu \in Q_1(X)$.*

Remark 55 *If f is a signed q-function, then $f_*(\mu)$ is determined on $\mathcal{O}_s$ by*

$$(f_*\mu)(U) = (f_*\mu)(X) - (f_*\mu)(X\backslash U) = f(1) - (f_*\mu)(X\backslash U). \qquad (1)$$

Since X is a q-space it only exhibits trivial partitions. Hence (1) assures the additivity requirement (Definition 43.3) of $f_*\mu$ trivially. For any signed q-function f we must also have $f(0) = 0$ since the empty set always has measure zero.

Proposition 56 *If f is a signed q-function then*

1. *$f(x^-) + f(1-x) = f(1)$ for all $x \in (0, 1]$.*
2. *f is continuous from the right.*
3. *f has bounded variation.*

Proof. Let m be the Lebesgue measure on the unit interval. Then f_*m extends to a signed quasi-measure in $\mathbb{R}$, and hence extends uniquely to a regular signed Borel measure in $\mathbb{R}$. The Proposition now follows from elementary measure theory. ■

Example 57 *Let $f : [0,1] \to \mathbb{R}$ be given by*

$$f(x) = \begin{cases} \sqrt{x} & ; \quad x \in [0, \frac{1}{2}] \\ \sqrt{2} - \sqrt{1-x} & ; \quad x \in [\frac{1}{2}, 1] \end{cases}$$

Then $\lim_{x\to 0^+} f'(x) = \infty$, so for any M there is an $\varepsilon > 0$ and $n \in \mathbb{N}$ such that $n\varepsilon < 1$ and $nf(\varepsilon) > M$. Let m be the normalized Lebesgue measure in the unit disk D. Choose disjoint sets $\{C_i\}_{i=1}^n \subset \mathcal{C}_s$ such that $m(C_1) = \cdots = m(C_n) = \varepsilon$. Assuming that f is a signed q-function we have $(f_*m)(\bigcup C_i) = \sum (f_*m)(C_i) > M$, which is a contradiction. Apparently Proposition 56 does not provide sufficient conditions for a signed q-function.

In [4] it was shown that a convex funtion on the interval $[0, \frac{1}{2}]$ extends uniquely to a q-function on the whole unit interval. This immediately provides us with a large class of signed q-functions:

Proposition 58 *If $f : [0, \frac{1}{2}] \to \mathbb{R}$ with $f(0) = 0$ is differentiable, and f' has bounded variation, then f extends to a continuous signed q-function on $[0, 1]$.*

Proof. If f' has bounded variation, then $f' = g - h$ where g and h are monotone functions. In particular, f is the difference of two convex functions, say f_1 and f_2, on the interval $[0, \frac{1}{2}]$. Since $f(0) = f_1(0) - f_2(0) = 0$ we may assume that $f_1(0) = f_2(0) = 0$. Assuming continuity of f, Proposition 56 requires that $f(x) + f(1-x) = f(1)$; $(x \in [0,1])$. Accordingly we must have

$$f(x) = \begin{cases} f_1(x) - f_2(x) & ; \quad x \in [0, \frac{1}{2}] \\ f(1) - f_1(1-x) - f_2(1-x) & ; \quad x \in [\frac{1}{2}, 1] \end{cases}$$

implying that $f(1) = 2f_1(\frac{1}{2}) - 2f_2(\frac{1}{2})$. Now $f : [0,1] \longrightarrow \mathbb{R}$ is the difference of two q-functions k_1, k_2 given by

$$k_i(x) = \begin{cases} f_i(x) & ; \quad x \in [0, \frac{1}{2}] \\ 2f_i(\frac{1}{2}) - f_i(1-x) & ; \quad x \in [\frac{1}{2}, 1] \end{cases} \quad ; \; i = 1, 2$$

■

Example 59 *Let $f(x) = 2x^3 - 3x^2 + 2x : [0,1] \longrightarrow \mathbb{R}$, and again let m denote the normalized Lebesgue measure in the unit disk D. Then f is a signed q-function by Proposition 58. It is strictly increasing and positive. However, f splits into the difference $2x$ and $3x^2 - 2x^3$ where the latter is known to be a q-function which maps even m to non-trivial quasi-measures (i.e. quasi-measures which are not restrictions of regular Borel measures). Hence f_*m is a signed quasi-measure, but it is neither a quasi-measure nor a signed Borel measure. Indeed, let $C_1, C_2 \in \mathcal{C}_s(D)$ be disjoint with $m(C_1) = \frac{1}{3}$, $m(C_2) = \frac{2}{5}$, and choose $C \in \mathcal{C}_s(D)$ with $C_1 \cup C_2 \subset C$ and $m(C) = \frac{4}{5}$. Then $(f_*m)(C) < (f_*m)(C_1) + (f_*m)(C_2)$, and consequently f_*m is not a quasi-measure.*

Before we embark on a total classification of the signed q-functions, we will need some notation for trees:

Notation 60 *We will assume a rooted tree $T = (V, E)$ to be finite and directed, where V is the set of vertices and E is the set of edges. There is a level function $l : V \longrightarrow \mathbb{N}$ where the root has level zero, vertices adjecent to the root has level one and so forth. There are two maps $r, s : E \longrightarrow V$ determined by $e = (s(e), r(e))$ $(; e \in E)$. We call r and s the range and source map of T respectively.*

Proposition 61 *Let X be any q-space and let $C \in \mathcal{C}_f$. Then there is a finite rooted tree $T = (V, E)$ (not necessarily unique) representing C in the following sense:*

1. $\{A \in V : l(A)$ *is an odd number*$\} \subset \mathcal{C}_s$ *and* $\{A \in V : l(A)$ *is even*$\} \subset \mathcal{O}_s$.

2. *For each* $e \in E$ *we have* $\biguplus_{A \in r(e)} A \subset s(e)$.

3. *Let* $n = \max_{v \in V} \left\lceil \frac{l(v)}{2} \right\rceil$, *then*

$$C = \biguplus_{i=0}^{n-1} \left(\biguplus_{l(K)=2i+1} \left(K \setminus \left(\biguplus_{V \in r(s^{-1}(K))} V \right) \right) \right)$$

Proof. Let X be the root of T. Pick $x \in X \setminus C$ and for each i let j' be determined by $x \in C_i^{j'}$. Put $K_i = C_i \cup \left(\bigcup_{j \neq j'} C_i^j \right)$, then for $i \neq k$ we have $K_i \subset K_k$, $K_k \subset K_i$ or $K_i \cap K_k = \emptyset$. Let $r(s^{-1}(X))$ be the sets K_i ; $i = 1, 2, \ldots$ such that $K_i \setminus \left(\bigcup_{k \neq i} K_k \right) \neq \emptyset$. Now recursively for $K_i \in V \cap \mathcal{C}_s$ let $r(s^{-1}(K_i))$ be the set $\{C_i^j\}_{j \neq j'}$, and recursively for C_i^j let

$$r(s^{-1}(C_i^j)) = \left\{ K_k : K_k \subset C_i^j,\ K_i \setminus \left(\bigcup_{\substack{k \neq i \\ K_k \subset C_i^j}} K_k \right) \neq \emptyset \right\}.$$

It is now straight forward to verify the requirements of the tree T. ∎

Remark 62 *The construction also works for sets in $\mathcal{C}_0$ (which in general give infinite trees), in spaces of general genus. There is an alternative interpretation of the tree as unrooted, for which a vertex V is relabeled with the set $V \setminus \biguplus_{W \in r(s^{-1}(V))} W$, while each edge e is labeled with the original $r(e)$ if $r(e)$ is open, and with $X \setminus r(e)$ otherwise; in the second case the direction of the edge is reversed. In this case the vertices are a partition of the space X, with the source vertices corresponding to the components of C, and the edges adjacent to a vertex correspond either to its co-complements or to their complements.*

Definition 63 *Let $T = (V, E)$ be a rooted tree with root v_0, and a function $w : V \to \mathbb{R}^+$. We call T monotonic if $w(v_0) = 1$ and*

$$w(v) \geq \sum_{v \in s(e)} r(e) \text{ for all } v \in V.$$

where the inequality is strict if $l(v)$ even.

Theorem 64 *A function $f : [0,1] \to \mathbb{R}$ is a signed q-function if and only if the following hold*

1. *We have $f(0) = 0$ and*

$$f(x^-) + f(1-x) = f(1) \text{ for all } x \in (0,1].$$

2. *There is an $M \in \mathbb{R}^+$ such that for any monotonic tree (V, E, w) with root v_0 we have*

$$\left| \sum_{\{v \in V : l(v) \text{ odd}\}} f(w(v)) - \sum_{\{v \in V \setminus \{v_0\} : l(v) \text{ even}\}} f(w(v)^-) \right| \leq M$$

Proof. Suppose f is a function satisfying the requirements of the theorem. For any q-space X define $f_* : Q_1(X) \to Q(X)$ according to Definition 54. Let $\mu \in Q_1(X)$ be arbitrary, we need to show that $f_*\mu$ is a signed solid set function.

Since q-spaces only exhibit trivial partitions, the additivity requirement (43.3) is satisfyed by definition of $f_*\mu$ on $\mathcal{O}_s$. The regularity requirement (43.2) of $f_*\mu$ follows from 64.1.

Let $C \in \mathcal{C}_f$, let M be as in 64.2, and let $T = (V, E)$ be a tree representation of C according to Proposition 61, with notation as in the proof of the proposition. Then (V, E, μ) is a monotonic tree. We obtain

$$\left|\sum_i \left((f_*\mu)(X) - \sum_j (f_*\mu)(C_i^j)\right)\right|$$

$$= \left|\sum_i \left(f(1) - \sum_j (f(\mu(C_i^j)^-))\right)\right|$$

$$= \left|\sum_i \left(\left(f(\mu(K_i)) + f(\mu(C_i^{j'})^-)\right) - \sum_j (f(\mu(C_i^j)^-))\right)\right|$$

$$= \left|\sum_i \left(f(\mu(K_i)) - \sum_{j \neq j'} (f(\mu(C_i^j)^-))\right)\right| \leq M$$

By Theorem 53 and Definition 43 it is now clear that f is a signed q-function.

For the converse, assume that f is a signed q-function. By Proposition 56 it is clear that $f(x^-) + f(1-x) = f(1)$ for all $x \in (0, 1]$. Clearly, $f(0) = 0$ since the empty set is a zero set.

It remains to prove 64.2 for f. Let m denote the Lebesgue measure in the unit square $I^2 = [0,1] \times [0,1]$. Then $f_*m : \mathcal{A}_s \to \mathbb{R}$ is assumed to be a solid set function, and in particular satisfying 43.1 for some M. For any monotonic tree (V, E, w) we may construct recursively a set $C \in \mathcal{C}_f(I^2)$ such that (V, E) represents C and $w \equiv m$. The set of vertices V may even be taken to be rectangles. By the calculations above we obtain

$$\left|\sum_{\{v \in V : l(v) \text{ odd}\}} f(w(v)) - \sum_{\{v \in V \setminus \{v_0\} : l(v) \text{ even}\}} f(w(v)^-)\right|$$

$$= \left|\sum_i \left(f(m(K_i)) - \sum_{j \neq j'} f(m(C_i^j)^-)\right)\right|$$

$$= \left|\sum_i \left((f_*m)(X) - \sum_j (f_*m)(C_i^j)\right)\right| \leq M,$$

and hence f satisfyes the requirements of the theorem. The proof is complete. ■

Corollary 65 *Let m be Lebesgue measure on the unit square $[0,1] \times [0,1]$, and let $f : [0,1] \to \mathbb{R}$. Then f is a signed q-function if and only if f_*m extends to a signed quasi-measure.*

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