A NON-STRICTLY PSEUDOCONVEX DOMAIN FOR WHICH THE SQUEEZING FUNCTION TENDS TO ONE TOWARDS THE BOUNDARY

J. E. FORNÆSS AND E. F. WOLD

Abstract. In recent work by Zimmer it was proved that if $\Omega \subset \mathbb{C}^n$ is a bounded convex domain with C^∞-smooth boundary, then Ω is strictly pseudoconvex provided that the squeezing function approaches one as one approaches the boundary. We show that this result fails if Ω is only assumed to be C^2-smooth.

1. Introduction

We recall the definition of the squeezing function $S_\Omega(z)$ on a bounded domain $\Omega \subset \mathbb{C}^n$. If $z \in \Omega$, and $f_z : \Omega \to \mathbb{B}^n$ is an embedding with $f_z(z) = 0$, we set

$$S_{\Omega,f_z}(z) := \sup\{r > 0 : B_r(0) \subset f_z(\Omega)\}, \tag{1.1}$$

and then

$$S_\Omega(z) := \sup_{f_z}\{S_{\Omega,f_z}(z)\}. \tag{1.2}$$

A guiding question is the following: which complex analytic properties of Ω are encoded by the behaviour of S_Ω? For instance, if S_Ω is bounded away from zero, then Ω is necessarily pseudoconvex, and the Kobayashi-, Carathéodory-, Bergman- and the Kähler-Einstein metric are complete, and they are pairwise quasi-isometric (see [8]). Recently, Zimmer [9] proved that if

$$\lim_{z \to b\Omega} S_\Omega(z) = 1 \tag{1.3}$$

for a C^∞-smooth, bounded convex domain, then Ω is necessarily strictly pseudoconvex[1]. In this short note we will show that this does not hold for C^2-smooth domains.

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[1] Added in proof: Zimmer has subsequently improved his results to convex domains with $C^{2,\alpha}$-boundary.

Theorem 1.1. *There exists a bounded convex C^2-smooth domain $\Omega \subset \mathbb{C}^n$ which is not strongly pseudoconvex, but*

$$\lim_{z \to b\Omega} S_\Omega(z) = 1, \tag{1.4}$$

where $S_\Omega(z)$ denotes the squeezing function on Ω.

For further results about the squeezing function the reader may also consult the references [1], [2],[3],[4],[5],[6],[7], [8], [9]. In the last section we will post some open problems.

2. The construction

2.1. **The construction in $\mathbb{R}^n$ and curvature estimates.** We start by describing a construction of a convex domain Ω in $\mathbb{R}^n$ with a single non-strictly convex point. Afterwards we will explain how to make the construction give the conclusion of Theorem 1.1 for each $n = 2m$, when we make the identification with $\mathbb{C}^m$.

Let $x = x_1, ..., x_n$ denote the coordinates on $\mathbb{R}^n$. For any $k \in \mathbb{N}$ we let B_k denote the ball

$$B_k := \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_{n-1}^2 + (x_n - k)^2 < k^2\}. \tag{2.1}$$

On some fixed neighbourhood of the origin, each boundary bB_k may be written as a graph of a function

$$x_n = \psi_k(x') = \psi_k(x_1, ..., x_{n-1}) = k - \sqrt{k^2 - \|x'\|^2} = \frac{1}{2k}\|x'\|^2 + O(\|x\|^3). \tag{2.2}$$

Fix a smooth cut-off function $\chi(x') = \chi(|x'|)$ with compact support in $\{|x'| < 1\}$ which is one near the origin. We will create a new limit graphing function $f(x')$ by subsequently gluing the functions ψ_k and ψ_{k+1} by setting

$$g_k(x') = \psi_k(x') + \chi(\frac{x'}{\epsilon_k})(\psi_{k+1}(x') - \psi_k(x')), \tag{2.3}$$

where the sequence ϵ_k will converge rapidly to zero, and the boundary of our domain Ω will be defined (locally) as the graph Σ of the function f defined as follows: start by setting $f_k := \psi_k$ for some $k \in \mathbb{N}$. Then define f_{k+1} inductively by setting $f_{k+1} = f_k$ for $\|x'\| \geq \epsilon_k$ and then $f_{k+1} = g_k$ for $\|x'\| < \epsilon_k$. Finally we set $f = \lim_{k\to\infty} f_k$.

To show that the limit function f is C^2-smooth (if the ϵ_k's converge rapidly to zero), we need to show that the sequence $\{f_k\}$ is a Cauchy-sequence with respect to C^2-norm, *i.e.*, we need to estimate the derivatives

$$\sigma_{ij}^k(x') := \frac{\partial^2}{\partial x_i \partial x_j}(\chi(\frac{x'}{\epsilon_k})(\psi_{k+1}(x') - \psi_k(x'))). \tag{2.4}$$

Note first that

$$\psi_{k+1}(x') - \psi_k(x') = \frac{-1}{2k(k+1)}\|x'\|^2 + O(\|x\|^3). \tag{2.5}$$

We see that

$$|\sigma_{ij}^k(x')| = (\frac{1}{\epsilon_k^2}O(\|x'\|^2) + \frac{1}{\epsilon_k}O(\|x\|))\frac{1}{2k(k+1)} + \frac{1}{\epsilon_k^2}O(\|x'\|^3) + \frac{1}{\epsilon_k}O(\|x'\|^2),$$

and so for $\|x'\| < \epsilon_k$ we have that

$$|\sigma_{ij}^k(x')| \leq C \cdot \frac{1}{2k(k+1)} + O(\epsilon_k), \tag{2.6}$$

where the constants are independent of any particular choice of ϵ_k. So if ϵ_k is small enough we see that $|\sigma_{ij}^k|$ is of order of magnitude $1/k^2$, which shows that $\{f_k\}$ will be a Cauchy-sequence.

To ensure that Ω is convex we will need to estimate the curvature of Σ, and estimates of the curvature of the partial graphs $\Sigma_k = \{x, g_k(x)\}$ will be necessary to prove Theorem 1.1. Informally our goal is to show the following: *There exist $N, m \in \mathbb{N}, N > m$, such that if $k \geq N$ and if ϵ_k is sufficiently small (depending on k), then Σ_k curves, at every point and in all directions, more than bB_{k+m} and less than bB_{k-m}.*

We make this more precise. The surface Σ_k has a defining function $\rho_k(x) = g_k(x') - x_n$. If v_p is a tangent vector to Σ_k at $p = (x', g_k(x))$, the curvature of Σ_k in the direction of v_p is defined as

$$\kappa_p^{\Sigma_k}(v_p) := \frac{H\rho_k(p)(v_p)}{\|\nabla\rho_k(p)\|\|v_p\|^2}, \tag{2.7}$$

where $\nabla\rho_k$ is the gradient, and $H\rho_k$ is the Hessian of ρ_k (which is equal to the Hessian of g_k). The curvature (2.7) depends only on the direction of v_p, and the curvature of bB_k is $\frac{1}{k}$ at all points and in all directions. The precise statement of our goal stated above is

Lemma 2.1. *Let ψ_k and χ be defined as above for $k \in \mathbb{N}$. There exist $N, m \in \mathbb{N}, N > m$, such that if each ϵ_k is sufficiently small (depending on k), and $k \geq N$, then*

$$\frac{1}{k+m} \leq \kappa_p^{\Sigma_k}(v_p) \leq \frac{1}{k-m}, \tag{2.8}$$

for all v_p tangent to Σ_k.

It is now easy to see that if $\epsilon_k \searrow 0$ sufficiently fast, then Ω is convex, and strictly convex away from the origin. If we let Ω_k denote the domain whose boundary near the origin is given by the graph of f_k, we see that Ω_k is strictly convex, the Hessian being positive definite everywhere. Morover $\Omega = \cup_k\Omega_k$, and so Ω is convex.

Proof. (of Lemma 2.1) When we estimate the curvature we may assume that the functions g_k are simply

$$g_k(x') = \psi_k(x') - \chi(\frac{x'}{\epsilon_k})(\frac{1}{2k(k+1)})|x'|^2 =: \psi_k(x') + \sigma_k(x'), \tag{2.9}$$

since the higher order terms missing in this expression of g_k can be made insignificant by choosing ϵ_k small enough. Because of the $|x'|^2$-term it is easy to see that

$$dg_k(x') = d\psi_k(x') + \triangle_k(x'), \tag{2.10}$$

and

$$Hg_k(x') = H\psi_k(x') + h_k(x'), \tag{2.11}$$

where the coefficients in both $\triangle_k$ and h_k are of order of magnitude $\frac{1}{k^2}$ independently of k and of the choice of a small ϵ_k.

Fix a point x' and a vector $v \in \mathbb{R}^{n-1}$ with $\|v\| = 1$. Then a tangent vector v_p at the point $(x', g_k(x'))$ is given by

$$v_p = (v, dg_k(x')(v)) = (v, d\psi_k(x')(v) + \triangle_k(x')(v)). \tag{2.12}$$

Estimating the curvature we see that

$$\begin{aligned}
\kappa_p^{\Sigma_k}(v_p) &= \frac{(H\psi_k(x') + h_k(x'))(v_p)}{\|\nabla\rho_k(p)\|\|v_p\|^2} \\
&= \frac{(H\psi_k(x'))((v, d\psi_k(x')v) + (0', \triangle_k(x')(v)))}{\|{-\mathbf{e_n}} + \nabla\psi_k(p) + \nabla\sigma_k(x')\|\|(v, d\psi_k(x')(v)) + (0', \triangle_k(x'))\|^2} \\
&\quad + O(\frac{1}{k^2}) \\
&= \frac{(H\psi_k(x'))((v, d\psi_k(x')v))}{\| - \mathbf{e_n} + \nabla\psi_k(x')\|(1 + O(\frac{1}{k^2}))\|(v, d\psi_k(x')(v))\|^2(1 + O(\frac{1}{k^2}))^2} \\
&\quad + O(\frac{1}{k^2}) \\
&= \frac{(H\psi_k(x'))((v, d\psi_k(x')v))}{\| - \mathbf{e_n} + \nabla\psi_k(x')\|\|(v, d\psi_k(x')(v))\|^2} + O(\frac{1}{k^2}) \\
&= \frac{1}{k} + O(\frac{1}{k^2}),
\end{aligned}$$

where the term $\frac{1}{k}$ comes from the fact that the expression above is the formula for the curvature of a ball of radius k. From this it is straightforward to deduce the existence of an m such that the lemma holds. □

2.2. **The squeezing function on Ω.** We will now explain why the squeezing function goes to one uniformly as we approach $b\Omega$ provided that the ϵ_k's decrease sufficiently fast. Let N, m be as in Lemma 2.1, and start by setting $f_k = \psi_k$ for some $k > N$.

Fix some small $\delta_k > 0$. By Lemma 2.1, if ϵ_k is small enough, we can for each $p = (x', x_n) \in b\Omega_k$, $\|x'\| < \delta_k$, find a ball B of radius $k + m$ containing

Ω_k such that $p \in bB$. By the same lemma we can for each such p also find a local piece of a ball of radius $k-m$ touching p from the inside of Ω_k, and the size of the local ball is uniform. So using Lemma 3.1 we may find a $t_k > 0$ small enough such that

$$S_{\Omega_k}(x', x_n) \geq 1 - \frac{m}{(k+m)} \tag{2.13}$$

if $x_n \leq t_k$.

Next, again by Lemma 2.1, we find a $\delta_{k+1} < \delta_k$ such that if ϵ_{k+1} is small enough, then for each $p = (x', x_n) \in b\Omega_{k+1}$ with $\|x'\| < \delta_{k+1}$, we may oscillate with balls of radius $k+1-m$ and $k+1+m$ respectively. So there is a $t_{k+1} < t_k$ such that

$$S_{\Omega_{k+1}}(x', x_n) \geq 1 - \frac{m}{(k+1+m)} \tag{2.14}$$

if $x_n \leq t_{k+1}$. Furthermore, by further decreasing ϵ_{k+1} we can keep the estimate (2.13) with Ω_k replaced by Ω_{k+1}. The reason is the following. First of all, by [5] there exists a constant C_k such that

$$S_{\Omega_k}(z) \geq 1 - C_k \cdot \mathrm{dist}(z, b\Omega_k), \tag{2.15}$$

and near any compact $K \subset b\Omega_k$ away from 0, this estimate is not going to be disturbed by a small perturbation of $b\Omega_k$ near the point 0; the estimate is obtained by using oscillating balls at points of K whose boundaries will stay bounded away from 0. Furthermore, on any compact subset of Ω_k we have that $S_{\Omega_{k+1}} \to S_{\Omega_k}$ as $\epsilon_{k+1} \to 0$.

Continuing in this fashion, we obtain a decreasing sequence $0 < t_j < t_{j+1}, j = k, k+1, ...,$ and an increasing sequence of domains Ω_j, such that for each j we have that

$$S_{\Omega_j}(x', x_n) \geq 1 - \frac{m}{(k+i+m)} \tag{2.16}$$

for $t_{k+i} \leq x_n \leq t_{k+i-1}$, for $i \leq j$. The result now follows from Lemma 3.2.

3. Lemmata

Let $0 < s < 1/2, 0 < d < r < 1$, and set $B_s = B(s, 1-s)$, the ball of radius $1-s$ centred at $(s, 0')$. Furthermore we set

$$B_{s,d} = B_s \cap \{(z_1, z') \in \mathbb{B}^n : \mathcal{R}e(z_1) > d\}. \tag{3.1}$$

Lemma 3.1. *If $B_{s,d} \subset \Omega \subset \mathbb{B}^n$, and if $r > 1 - \frac{sd}{4}$, then $S_\Omega(r, 0) > 1 - s$.*

Proof. Set $\mu = 1 - s$ and $\eta = \frac{d}{2}$, and then

$$B_\eta^\mu = \{(z_1, z') \in \mathbb{C}^n : |z_1 - (1-\eta)|^2 + \frac{\eta}{\mu}|z'|^2 < \eta^2\}. \tag{3.2}$$

Then certainly $\mathcal{R}e(z_1) > d$ on B^{μ}_{η}, and we also have that $B^{\mu}_{\eta} \subset B_s$. To see the latter, we translate the two balls sending $(1, 0')$ to the origin, where they are defined by

$$\tilde{B}_s = \{(z_1, z') : 2\mu\mathcal{R}e(z_1) + |z|^2 < 0\}, \tag{3.3}$$

and

$$\tilde{B}^{\mu}_{\eta} = \{(z_1, z') : 2\eta\mathcal{R}e(z_1) + |z_1|^2 + \frac{\eta}{\mu}|z'|^2 < 0\}. \tag{3.4}$$

And

$$\begin{aligned} 2\eta\mathcal{R}e(z_1) + |z_1|^2 + \frac{\eta}{\mu}|z'|^2 < 0 &\Rightarrow 2\eta\mathcal{R}e(z_1) + \frac{\eta}{\mu}|z_1|^2 + \frac{\eta}{\mu}|z'|^2 < 0 \\ &\Leftrightarrow 2\mu\mathcal{R}e(z_1) + |z|^2 < 0. \end{aligned}$$

According to Lemma 3.5 in [5] we have that

$$S_\Omega(r, 0) \geq \sqrt{\mu}\sqrt{1 - 2(1-r)\frac{1}{\eta}} = \sqrt{(1-s)(1 - \frac{4(1-r)}{d})}, \tag{3.5}$$

from which the lemma follows easily. □

Lemma 3.2. *Let $\Omega_j \subset \Omega_{j+1}$ for $j \in \mathbb{N}$, set $\Omega = \cup_j \Omega_j$, and assume that Ω is bounded. Let $z \in \Omega$, and assume that $S_{\Omega_j}(z) > 1 - \delta$ for all j large enough so that $z \in \Omega_j$. Then $S_\Omega(z) \geq 1 - \delta$.*

Proof. Let $f_j : \Omega_j \to \mathbb{B}^n$ be an embedding such that $f_j(z) = 0$ and $B_{1-\delta}(0) \subset f_j(\Omega_j)$. By passing to a subsequence we may assume that $f_j \to f : \Omega \to \mathbb{B}^n$ u.o.c., with $f(z) = 0$. Setting $g_j = f_j^{-1} : B_{1-\delta}(0) \to \Omega$ we may also assume that $g_j \to g$ uniformly on compact sets. Then $f|_{g(B_{1-\delta}(0))} = g^{-1}$, from which the result follows. □

4. Some open problems

Problem 4.1. Does Zimmer's result hold for pseudoconvex domains of class C^∞?

Problem 4.2. How much smoothness is needed for Zimmer's result hold for pseudoconvex domains?

Problem 4.3. Let $\Omega \subset \mathbb{C}^2$ be a bounded pseudoconvex domain of class C^∞. Is $S_\Omega(z)$ bounded away from zero?

Yeung [8] showed that the answer is yes for strongly convex domains in $\mathbb{C}^n$, and Kim-Zhang [6] and Deng-Guan-Zhang [3] showed that the answer is yes for strictly pseudoconvex domains. On the other hand, Fornæss-Rong [4] showed that the answer is no for $n \geq 3$.

Quantifying the asymptotic behaviour of the squeezing function, Fornæss-Wold [5] showed that

(i) $S_\Omega(z) \geq 1 - C\mathrm{dist}(z, b\Omega)$, and

(ii) $S_\Omega(z) \geq 1 - C\sqrt{\mathrm{dist}(z, b\Omega)}$,

for strongly pseudoconvex domains of class C^4 and C^3 respectively. Diederich-Fornæss-Wold [1] showed that if the the squeezing function approaches one essentially faster than (i), then Ω is biholomorphic to the unit ball.

Problem 4.4. What is the optimal estimate for the squeezing function for strictly pseudoconvex domains of class C^k with $k < 4$?

Let $\phi : \mathbb{B}^2 \to \mathbb{C}^2$ be defined as $\phi(z_1, z_2) := (z_1, -z_2 \log(z_1 - 1))$. Then $\Omega := \phi(\mathbb{B}^2)$ is of class C^1, and $(1, 0)$ is a non-strictly pseudoconvex boundary point of Ω. So S_Ω being identically equal to one does not even imply strict pseudoconvexity in the case of C^1-smooth boundaries.

Problem 4.5. Let $\phi : \mathbb{B}^n \to \Omega$ be a biholomorphism, and assume that Ω is a bounded C^2-smooth domain. Is Ω strictly pseudoconvex?

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J. E. Fornæss: Department of Mathematical Sciences, Norwegian University of Science and Technology, 7491 Trondheim, Norway.

E. F. Wold: Department of Mathematics, University of Oslo, PO-BOX 1053 Blindern, 0316 Oslo, Norway.