

# Analysis and design of quadratic parameter varying (QPV) control systems with polytopic attractive region

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## Abstract

This paper proposes a gain-scheduling approach for systems with a quadratic structure. Both the stability analysis and the state-feedback controller design problems are considered for quadratic parameter varying (QPV) systems. The developed approach assesses/enforces the belonging of a polytopic region of the state space to the region of attraction of the origin, and relies on a linear matrix inequality (LMI) feasibility problem. The main characteristics of the proposed approach are illustrated by means of examples, which confirm the validity of the theoretical results.

*Keywords:* Gain-scheduled control, quadratic systems, Lyapunov stability, region of attraction.

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## 1. Introduction

When dealing with systems different from linear time invariant (LTI) ones, assessing the stability of an equilibrium point might not be enough to ensure that its stability is *global*. In fact, the presence of nonlinearities or time-varyingness may result in the nonuniqueness of equilibria and the existence of limit cycles [1], which shrink the stability region of the desired equilibrium point (usually the origin) from the whole state space to the *region of attraction* (RA), i.e. the set of initial conditions from which the state converges to the equilibrium point itself.

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Knowing the RA becomes of paramount importance for control purposes, in order to ensure that the initial condition lies inside it. However, in many cases, the shape of the RA is complicated [2] and inner approximations of the RA must be obtained instead, which is a problem that has been solved using numerous methods, see e.g. [3, 4]. A common approach to obtain these approximations involves Lyapunov functions, which are first used to assess the local asymptotical stability of the equilibrium point, and later to calculate the RA estimates by considering the largest level set where the time derivative is negative. In the case of polynomial systems, i.e. systems described by the following state equation:

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \quad (1)$$

where  $f$  and  $g$  are polynomials, the Lyapunov-based approach leads to linear matrix inequalities (LMIs), for which efficient solvers are available [5, 6, 7, 8]. However, in cases where it is wished to compute the Lyapunov function such that the volume of the estimate of the RA is maximized, the above mentioned approaches either lead to non-convex optimization problems, infinite dimensional linear reformulations [9, 10] or piecewise affine approximations [11, 12], which are hardly treatable from the numerical point of view, especially in the presence of non-odd polynomial systems [13]. For this reason, recent research has been devoted to study the special case of quadratic systems, i.e. systems described by a state equation of the following type:

$$\dot{x}(t) = Ax(t) + N(x(t)) + Bu(t) + M(x(t), u(t)) \quad (2)$$

with:

$$N(x(t)) = \begin{pmatrix} x(t)^T N_1 x(t) \\ x(t)^T N_2 x(t) \\ \vdots \\ x(t)^T N_{n_x} x(t) \end{pmatrix} \quad (3)$$

$$M(x(t), u(t)) = \begin{pmatrix} x(t)^T M_1 u(t) \\ x(t)^T M_2 u(t) \\ \vdots \\ x(t)^T M_{n_x} u(t) \end{pmatrix} \quad (4)$$

which are of importance since they can explain the dynamic behavior of several phenomena in a wide range of applications [13, 14, 15]. As a matter of example, quadratic systems describe the dynamics of electric power systems [16], chemical

reactors [17], robots [18], hydrodynamic flows [19] and enzymatic reactions [20], as well as the interactions between tumors and immune systems [21].

For systems described by (2)-(4), which will be referred in the following as quadratic time invariant (QTI) systems, since the matrices  $A$ ,  $B$ ,  $N_1, \dots, N_{n_x}$ ,  $M_1, \dots, M_{n_x}$  are constant, several LMI-based algorithms have been proposed for estimating the RA. The first result in this sense was developed by [13], where a quadratic Lyapunov function was used. An approach based on polyhedral Lyapunov functions was developed by [14], with the advantage of producing less conservative results. On the other hand, [22] considered the case where a stochastic framework is used. A problem closely related to RA estimation concerns the controller design, i.e. finding a controller that guarantees the closed-loop stability of an equilibrium point (usually the origin of the state space) and enforces a given polytope to belong to the RA of that equilibrium point. The design problem has been investigated for both polynomial systems [23] and QTI systems [22, 24, 25]. It is worth recalling that several studies have addressed the problem of finding conditions guaranteeing global asymptotic stability under state feedback [26, 27]. However, as remarked by [28], these results are limited to special subclasses of quadratic systems.

A class of approaches which has received a lot of attention from the control community are the gain-scheduling ones [29]. Gain-scheduling is a broad notion that refers to changes in the controller's structure and/or parameters according to the system's operating conditions. This idea has been extensively applied to many classes of systems, e.g. nonlinear stochastic [30], Takagi-Sugeno (TS) [31, 32] and linear parameter varying (LPV) ones [33, 34]. As remarked by [29], gain scheduling enables a controller to respond to changing operating conditions, often with much less computational burden than other nonlinear approaches. Also, in the cases of TS/LPV frameworks, a well-established analysis and design theory offers the potential of both stability and performance guarantees. However, most of the gain scheduling approaches found in the literature have been developed for systems with a linear structure. In fact, the only paper that has considered the problems of local stability and stabilisation for parameter-varying quadratic systems is the recent work by Chen et al. [35].

The approach presented in [35] is appealing since it uses parameter-dependent quadratic Lyapunov functions and an S-procedure approach, and the authors show that the derived conditions are a generalisation of those presented in [24]. However, its limitation is not allowing the enforcement of a given polytopic region of the state space to belong to the RA. This problem is of interest in many practical situations, since the states of physical systems are usually constrained to take val-

ues in independent bounded intervals, in which case it is desired the equilibrium point to be attractive for every possible value of the initial condition within the polytope generated by those intervals. In this sense, the main contribution of this paper is extending the results developed by [13] to parameter-varying quadratic systems, which will be referred in the following as quadratic parameter-varying (QPV) systems, since they relate to QTI systems in the same way as LPV systems relate to LTI ones. The QPV paradigm provides a framework for characterizing some nonlinear systems found in practical applications, such as robotic manipulators [36] and inverted pendula [37], which are described typically by dynamic equations where quadratic terms depending on state variables appear due to centrifugal and Coriolis forces. Moreover, QPV models can be obtained by calculating the first and second order terms of the Taylor expansion of a nonlinear plant about a family of operating points (this approach is the quadratic equivalent of the *linearization scheduling* for LPV systems [29]).

In this paper, two different problems are considered, i.e. the stability analysis and the state-feedback controller design, both of which are solved using a quadratic Lyapunov function with a constant Lyapunov matrix. Despite the potential conservativeness introduced by using a constant Lyapunov matrix, it is worth highlighting that this choice: (i) provides fewer analysis and design conditions, which can be relevant for computational reasons; (ii) does not need any assumption about the derivative of the varying parameters, i.e. provides stability for arbitrary fast variations of the varying parameters; and (iii) in the case of LPV systems has proved to be sufficient for many practical applications, e.g. [38, 39]. More specifically, the contributions of this paper can be resumed as follows:

- the paper shows that the condition for analyzing the global stability of QPV systems, obtained by requiring the negativity of the Lyapunov function's derivative, depends on both the state vector  $x$  and the varying parameter vector  $\theta$ , which take infinite values, thus complicating its assessment. On the other hand, it is shown that constraining both the desired attractive region and the QPV system to be *polytopic* leads to a finite number of conditions, which can be assessed using available computational tools.
- the condition for designing state-feedback controllers for QPV systems, which ensure closed-loop stability of the origin with a desired polytopic region of attraction, are obtained. It is also shown that this condition can be reduced to a finite number, suitable for the application of computational tools, only if some of the matrices characterizing the QPV system (more

specifically, the ones that describe how the input vector  $u$  affects the state derivative) are constrained to be constant.

- Similarly to the case of LPV systems, detailed in [40], it is shown that the aforementioned constraint can be relaxed, such that all the matrices describing the QPV system are parameter-varying, either by pre-filtering the input vector using an LTI filter, or by relying on a gridding approach.

The paper is structured as follows. Section 2 shows how the aforementioned stability analysis can be performed. Section 3 is dedicated to the state-feedback controller design problem. The theoretical results are illustrated with an academic example and with a robotic manipulator in Section 4. Finally, Section 5 summarizes the main conclusions of this work.

*Notation:* Given a symmetric matrix  $M$ , the notation  $M \succ 0$  ( $M \prec 0$ ) means that  $M$  is positive (negative) definite. On the other hand,  $M \succeq 0$  ( $M \preceq 0$ ) denotes that  $M$  is positive (negative) semidefinite. The shorthand notation  $He\{M\}$  will be used for  $M + M^T$ .

## 2. Stability analysis for a QPV system

Let us define as autonomous (continuous-time) *quadratic parameter varying (QPV)* a system that has the following structure:

$$\dot{x}(t) = A(\theta(t))x(t) + N(\theta(t), x(t)) \quad (5)$$

where  $x \in \mathbb{R}^{n_x}$  is the system state, and  $\theta \in \Theta \subset \mathbb{R}^{n_\theta}$  is the varying parameter vector that schedules both the matrix function  $A(\theta(t))$  and the nonlinear function  $N(\theta(t), x(t))$ , defined as:

$$N(\theta(t), x(t)) = \begin{pmatrix} x(t)^T N_1(\theta(t)) x(t) \\ x(t)^T N_2(\theta(t)) x(t) \\ \vdots \\ x(t)^T N_{n_x}(\theta(t)) x(t) \end{pmatrix} \quad (6)$$

where  $N_1(\theta(t)), N_2(\theta(t)), \dots, N_{n_x}(\theta(t))$  are matrix functions of appropriate dimensions. Notice that, contrarily to QTI systems described by (2)-(4), the matrices in (5)-(6) are not constant. Also, the parameter  $\theta(t)$  is assumed to be known, which further distinguishes QPV systems from uncertain QTI systems [15].

Let us recall that for a nonlinear system, the *region of attraction of the origin* is the set:

$$\mathcal{R}_A = \left\{ x(0) \mid \lim_{t \rightarrow \infty} \phi(t, x(0)) = 0 \right\} \quad (7)$$

where  $\phi(t, x(0))$  denotes the solution that starts at initial state  $x(0)$  at time  $t = 0$ .

Let us consider the problem of assessing the *global* stability of (5)-(6), i.e. the case in which  $\mathcal{R}_A$  is given by  $\mathbb{R}^{n_x}$ , using the Lyapunov function candidate:

$$V(x(t)) = x(t)^T P x(t) \quad (8)$$

for which, taking into account the system's dynamics described by (5)-(6), the following holds:

$$\begin{aligned} \dot{V}(x(t)) &= \dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t) \\ &= x(t)^T He \left\{ PA(\theta(t)) + P \begin{pmatrix} x(t)^T N_1(\theta(t)) \\ x(t)^T N_2(\theta(t)) \\ \vdots \\ x(t)^T N_{n_x}(\theta(t)) \end{pmatrix} \right\} x(t) \end{aligned} \quad (9)$$

It follows that a global stability condition is given by:

$$He \left\{ PA(\theta) + P \begin{pmatrix} x^T N_1(\theta) \\ x^T N_2(\theta) \\ \vdots \\ x^T N_{n_x}(\theta) \end{pmatrix} \right\} \prec 0 \quad (10)$$

that must hold  $\forall x \in \mathbb{R}^{n_x}$  and  $\forall \theta \in \Theta$ . However, in general, (10) cannot be assessed  $\forall x \in \mathbb{R}^{n_x}$  using computational tools, which motivates considering a polytopic region of attraction. For this reason, we will analyze whether the polytope  $\mathcal{P} \subset \mathbb{R}^{n_x}$ , given by:

$$\mathcal{P} = Co\{x_{(1)}, x_{(2)}, \dots, x_{(p)}\} = \{x \in \mathbb{R}^{n_x} : a_k^T x \leq 1, k = 1, \dots, q\} \quad (11)$$

where  $p$  and  $q$  are suitable integer numbers,  $x_{(i)}$  denotes the  $i$ -th vertex of  $\mathcal{P}$ , and  $Co\{\cdot\}$  denotes the convex hull, belongs to  $\mathcal{R}_A$  for the QPV system (5). Notice that the vertex (V-) representation and the half-space (H-) representation in (11) are equivalent [41].

Following the reasoning applied by [13] to QTI systems, a quadratic Lyapunov function that has negative definite derivative over an invariant set that contains  $\mathcal{P}$  will be searched for in order to solve the aforementioned analysis problem.

**Theorem 1.** Let  $P \succ 0$  and  $0 < \gamma < 1$  be such that  $\forall i \in \{1, \dots, p\}, \forall k \in \{1, \dots, q\}$  and  $\forall \theta \in \Theta$ :

$$He \left\{ \gamma PA(\theta) + P \begin{pmatrix} x_{(i)}^T N_1(\theta) \\ x_{(i)}^T N_2(\theta) \\ \vdots \\ x_{(i)}^T N_{n_x}(\theta) \end{pmatrix} \right\} \prec 0 \quad (12)$$

$$x_{(i)}^T P x_{(i)} \leq 1 \quad (13)$$

$$\begin{pmatrix} 1 & \gamma a_k^T \\ \gamma a_k & P \end{pmatrix} \succeq 0 \quad (14)$$

Then,  $\mathcal{P} \subseteq \mathcal{R}_A$  for the QPV system (5)-(6).

*Proof:* Let us consider the Lyapunov function candidate (8), for which (9) is obtained, and let us define  $\tilde{\mathcal{P}}$  as an enlarged version of  $\mathcal{P}$  obtained by multiplying all the coordinates of its vertices by  $\rho = \gamma^{-1} > 1$ :

$$\tilde{\mathcal{P}} = Co \{ \rho x_{(1)}, \rho x_{(2)}, \dots, \rho x_{(p)} \} = \left\{ x \in \mathbb{R}^{n_x} : \gamma a_k^T x = \frac{a_k^T}{\rho} x \leq 1, k = 1, \dots, q \right\} \quad (15)$$

and let us notice that (12) implies:

$$He \left\{ PA(\theta) + P\rho \begin{pmatrix} x_{(i)}^T N_1(\theta) \\ x_{(i)}^T N_2(\theta) \\ \vdots \\ x_{(i)}^T N_{n_x}(\theta) \end{pmatrix} \right\} \prec 0 \quad (16)$$

which, in virtue of (11), is equivalent to:

$$He \left\{ PA(\theta) + P\rho \begin{pmatrix} x^T N_1(\theta) \\ x^T N_2(\theta) \\ \vdots \\ x^T N_{n_x}(\theta) \end{pmatrix} \right\} \prec 0 \quad \begin{matrix} \forall x \in \mathcal{P} \\ \forall \theta \in \Theta \end{matrix} \quad (17)$$

Hence, from (9) and (17), it is clear that  $\dot{V}(x(t)) < 0 \forall x \in \tilde{\mathcal{P}}$ . The remaining of the proof aims at demonstrating that  $\tilde{\mathcal{P}}$  contains a level curve of  $V(x)$  which contains  $\mathcal{P}$ .

According to [42], the ellipsoid:

$$\mathcal{E} = \{x \in \mathbb{R}^{n_x} : x^T P x \leq 1\} \quad (18)$$

contains the polytope  $\mathcal{P}$  described by (11) if (and only if) (13) holds.

On the other hand, by means of Schur complements, (14) is equivalent to:

$$\gamma a_k^T P^{-1} \gamma a_k \leq 1 \quad \forall k \in \{1, \dots, q\} \quad (19)$$

which guarantees that  $\mathcal{E} \subset \tilde{\mathcal{P}}$  [42]. Hence,  $\mathcal{E}$  is an invariant set for the QPV system (5), since its boundary corresponds to a level curve of  $V(x)$ , and  $\dot{V}(x(t)) < 0 \forall x \in \tilde{\mathcal{P}}$ . This means that  $\mathcal{E} \subseteq \mathcal{R}_{\mathcal{A}}$ , which proves that  $\mathcal{P} \subseteq \mathcal{R}_{\mathcal{A}}$ .  $\square$

The main difficulty with using (12) for analysis purpose is that it represents an infinite number of conditions. In fact, while the negativity of  $\dot{V}(x(t)) \forall x \in \tilde{\mathcal{P}}$  can be determined by assessing (12) at the vertices of the V-representation in (11), thus obtaining  $p$  conditions (one for each  $x_{(1)}, \dots, x_{(p)}$ ), which are equivalent to assessing it  $\forall x \in \tilde{\mathcal{P}}$ , doing so is obstructed by the dependency of these conditions on the varying parameter  $\theta$ , which varies within the set  $\Theta$  with infinite possible values, for each of which the condition should be assessed. Hence, similarly to the case of LPV systems [40], *polytopic* QPV systems should be considered in order to obtain a finite number of conditions.

In particular, the QPV system (5)-(6) will be referred to as *polytopic* if the matrix functions  $A(\theta(t))$ ,  $N_1(\theta(t))$ ,  $\dots$ ,  $N_{n_x}(\theta(t))$  satisfy:

$$\begin{pmatrix} A(\theta(t)) \\ N_1(\theta(t)) \\ \vdots \\ N_{n_x}(\theta(t)) \end{pmatrix} = \sum_{j=1}^N \mu_j(\theta(t)) \begin{pmatrix} A_j \\ N_{1,j} \\ \vdots \\ N_{n_x,j} \end{pmatrix} \quad (20)$$

with some finite  $N$  and:

$$\sum_{j=1}^N \mu_j(\theta) = 1, \quad \mu_j(\theta) \geq 0 \quad \begin{matrix} \forall j = 1, \dots, N \\ \forall \theta \in \Theta \end{matrix} \quad (21)$$

For polytopic QPV systems, the following corollary can be obtained:



**Corollary 1.** Let  $P \succ 0$  and  $0 < \gamma < 1$  be such that  $\forall i \in \{1, \dots, p\}, \forall j \in \{1, \dots, N\}$  and  $\forall k \in \{1, \dots, q\}$ :

$$He \left\{ \gamma P A_j + P \begin{pmatrix} x_{(i)}^T N_{1,j} \\ x_{(i)}^T N_{2,j} \\ \vdots \\ x_{(i)}^T N_{n_x,j} \end{pmatrix} \right\} \prec 0 \quad (22)$$

and (13)-(14) hold. Then,  $\mathcal{P} \subseteq \mathcal{R}_A$  for the QPV system (5)-(6) with matrices satisfying (20)-(21).

*Proof:* Taking into account the property of matrices [43] that any linear combination of (22) with non-negative coefficients, of which at least one different from zero, is negative definite, using the coefficients  $\mu_j(\theta(t))$ , and taking into account (20), then (22) is obtained.  $\square$

**Remark 1.** Notice that (22) represents a set of bilinear matrix inequalities (BMIs), due to the product  $\gamma P$ . However, it is possible to grid the interval of admissible values for  $\gamma$ , and apply Corollary 1 with each fixed  $\gamma$ . In this way, (22) becomes a set of linear matrix inequalities (LMIs), which can be solved efficiently using available toolboxes/solvers, e.g. YALMIP [44]/SeDuMi [45].

### 3. Controller design for a QPV system

Let us consider a QPV system with the following structure:

$$\dot{x}(t) = A(\theta(t))x(t) + N(\theta(t), x(t)) + B(\theta(t))u(t) + M(\theta(t), x(t), u(t)) \quad (23)$$

where  $x \in \mathbb{R}^{n_x}$  is the system state,  $u \in \mathbb{R}^{n_u}$  is the control input and  $\theta \in \Theta \subset \mathbb{R}^{n_\theta}$  is the varying parameter vector that schedules both the matrix functions  $A(\theta(t))$ ,  $B(\theta(t))$  and the nonlinear functions  $N(\theta(t), x(t))$  and  $M(\theta(t), x(t), u(t))$ , which are defined as (6) and:

$$M(\theta(t), x(t), u(t)) = \begin{pmatrix} x(t)^T M_1(\theta(t)) u(t) \\ x(t)^T M_2(\theta(t)) u(t) \\ \vdots \\ x(t)^T M_{n_x}(\theta(t)) u(t) \end{pmatrix} \quad (24)$$

respectively, where  $N_1(\theta(t)), \dots, N_{n_x}(\theta(t)), M_1(\theta(t)), \dots, M_{n_x}(\theta(t))$  are matrix functions of appropriate dimensions.

Hereafter, we will solve the problem of the state-feedback stabilization of the QPV system (23)-(24) over a given polytope  $\mathcal{P}$ , defined as in (11), i.e. finding a state-feedback controller in the form:

$$u(t) = K(\theta(t))x(t) \quad (25)$$

where  $K(\theta(t))$  is a matrix of appropriate dimensions to be designed, such that 0 is a stable equilibrium point for the closed-loop system obtained as the interconnection of (23)-(24) and (25), with  $\mathcal{P} \subseteq \mathcal{R}_A$ .

The following theorem provides sufficient conditions for the existence of a matrix function  $K(\theta(t))$  that solves the aforementioned problem.

**Theorem 2.** *Let  $Q \succ 0$ ,  $0 < \gamma < 1$  and the matrix function  $\Gamma(\theta(t))$  be such that  $\forall i \in \{1, \dots, p\}$ ,  $\forall k \in \{1, \dots, q\}$  and  $\forall \theta \in \Theta$ :*

$$He\{\gamma(A(\theta)Q + B(\theta)\Gamma(\theta))\} + He\left\{\begin{pmatrix} x_{(i)}^T(N_1(\theta)Q + M_1(\theta)\Gamma(\theta)) \\ x_{(i)}^T(N_2(\theta)Q + M_2(\theta)\Gamma(\theta)) \\ \vdots \\ x_{(i)}^T(N_{n_x}(\theta)Q + M_{n_x}(\theta)\Gamma(\theta)) \end{pmatrix}\right\} \prec 0 \quad (26)$$

$$\begin{pmatrix} 1 & x_{(i)}^T \\ x_{(i)} & Q \end{pmatrix} \succeq 0 \quad (27)$$

$$\begin{pmatrix} 1 & \gamma a_k^T Q \\ \gamma Q a_k & Q \end{pmatrix} \succeq 0 \quad (28)$$

Then,  $\mathcal{P} \subseteq \mathcal{R}_A$  for the closed-loop QPV system obtained as the interconnection of (23)-(24) and (25), with  $K(\theta(t)) = \Gamma(\theta(t))Q^{-1}$ .

*Proof:* The closed-loop system obtained as the interconnection of (23)-(24) and (25) is an autonomous QPV system, described by:

$$\dot{x}(t) = A_{cl}(\theta(t))x(t) + N_{cl}(\theta(t), x(t)) \quad (29)$$

with:

$$A_{cl}(\theta(t)) = A(\theta(t)) + B(\theta(t))K(\theta(t)) \quad (30)$$

$$N_{cl}(\theta(t), x(t)) = \begin{pmatrix} x(t)^T N_{cl,1}(\theta(t))x(t) \\ x(t)^T N_{cl,2}(\theta(t))x(t) \\ \vdots \\ x(t)^T N_{cl,n_x}(\theta(t))x(t) \end{pmatrix} \quad (31)$$

$$N_{cl,i}(\boldsymbol{\theta}(t)) = N_i(\boldsymbol{\theta}(t)) + M_i(\boldsymbol{\theta}(t))K(\boldsymbol{\theta}(t)) \quad i = 1, \dots, n_x \quad (32)$$

Hence, according to Theorem 1,  $\mathcal{P} \subseteq \mathcal{R}_A$  for (29) if there exist  $P \succ 0$ , and  $0 < \gamma < 1$  such that  $\forall i \in \{1, \dots, p\}$ ,  $\forall k \in \{1, \dots, q\}$  and  $\forall \boldsymbol{\theta} \in \Theta$ :

$$He \left\{ \gamma P A_{cl}(\boldsymbol{\theta}) + P \begin{pmatrix} x_{(i)}^T N_{cl,1}(\boldsymbol{\theta}) \\ x_{(i)}^T N_{cl,2}(\boldsymbol{\theta}) \\ \vdots \\ x_{(i)}^T N_{cl,n_x}(\boldsymbol{\theta}) \end{pmatrix} \right\} \prec 0 \quad (33)$$

and (13)-(14) hold.

It is straightforward to show that (26) is obtained by pre- and post-multiplying (33) by  $Q \triangleq P^{-1}$  (see [46]), and introducing the change of variables  $\Gamma(\boldsymbol{\theta}(t)) = K(\boldsymbol{\theta}(t))Q$  (thus,  $K(\boldsymbol{\theta}(t)) = \Gamma(\boldsymbol{\theta}(t))Q^{-1}$ ). Finally, (27)-(28) are obtained by applying Schur complements to (13) and pre-/post-multiplying both the resulting matrix inequality and (14) by  $\text{diag}(I, Q)$ .  $\square$

Also in this case, (26) is not useful from a practical point of view, since it represents an infinite number of constraints. However, if  $B(\boldsymbol{\theta}(t)), M_1(\boldsymbol{\theta}(t)), \dots, M_{n_x}(\boldsymbol{\theta}(t))$  are parameter-independent (i.e. constant), and the matrix functions  $A(\boldsymbol{\theta}(t)), N_1(\boldsymbol{\theta}(t)), \dots, N_{n_x}(\boldsymbol{\theta}(t))$  are polytopic, i.e. they are such that (20)-(21) hold, it is possible to choose the controller gain  $K(\boldsymbol{\theta}(t))$  in (25) as:

$$K(\boldsymbol{\theta}(t)) = \sum_{j=1}^N \mu_j(\boldsymbol{\theta}(t)) K_j \quad (34)$$

and reduce (26) to a finite number of conditions, as stated by the following corollary.

**Corollary 2.** Let  $Q \succ 0$ ,  $0 < \gamma < 1$  and the matrices  $\Gamma_j \in \mathbb{R}^{n_u \times n_x}$  be such that  $\forall i \in \{1, \dots, p\}$ ,  $\forall j \in \{1, \dots, N\}$  and  $\forall k \in \{1, \dots, q\}$ :

$$He \{ \gamma (A_j Q + B \Gamma_j) \} + He \left\{ \begin{pmatrix} x_{(i)}^T (N_{1,j} Q + M_1 \Gamma_j) \\ x_{(i)}^T (N_{2,j} Q + M_2 \Gamma_j) \\ \vdots \\ x_{(i)}^T (N_{n_x,j} Q + M_{n_x} \Gamma_j) \end{pmatrix} \right\} \prec 0 \quad (35)$$

and (27)-(28) hold. Then,  $\mathcal{P} \subseteq \mathcal{R}_A$  for the closed-loop QPV system obtained as the interconnection of (23)-(24), with parameter-independent input matrices  $B$ ,

$G_1, \dots, G_{n_x}$  and state matrices satisfying (20)-(21), and (25) with controller gain as in (34) and  $K_j = \Gamma_j Q^{-1}$ .

Proof: Similar to Corollary 1, thus omitted.  $\square$

**Remark 2.** In the case of a QPV system with parameter varying input matrices  $B(\theta(t)), M_1(\theta(t)), \dots, M_{n_x}(\theta(t))$ , it is possible to obtain a system with parameter-independent input matrices by pre-filtering the inputs  $u(t)$ , as proposed by [40] for LPV systems. More specifically, let us define a new input vector  $\tilde{u}(t)$  such that:

$$\dot{x}_u(t) = A_u x_u(t) + B_u \tilde{u}(t) \quad (36)$$

$$u(t) = C_u x_u(t) \quad (37)$$

with  $A_u$  stable. Then, the resulting QPV system is described by:

$$\begin{aligned} \begin{pmatrix} \dot{x}(t) \\ \dot{x}_u(t) \end{pmatrix} &= \begin{pmatrix} A(\theta(t)) & B(\theta(t))C_u \\ 0 & A_u \end{pmatrix} \begin{pmatrix} x(t) \\ x_u(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \tilde{B}_u \end{pmatrix} \tilde{u}(t) \\ + \begin{pmatrix} (x(t)^T \quad x_u(t)^T) \begin{pmatrix} N_1(\theta(t)) & M_1(\theta(t))C_u \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ x_u(t) \end{pmatrix} \\ (x(t)^T \quad x_u(t)^T) \begin{pmatrix} N_2(\theta(t)) & M_2(\theta(t))C_u \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ x_u(t) \end{pmatrix} \\ \vdots \\ (x(t)^T \quad x_u(t)^T) \begin{pmatrix} N_{n_x}(\theta(t)) & M_{n_x}(\theta(t))C_u \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ x_u(t) \end{pmatrix} \end{pmatrix} \end{aligned} \quad (38)$$

which has a suitable structure for performing the design. As an alternative, it is possible to relax the assumption that  $B(\theta(t)), M_1(\theta(t)), \dots, M_{n_x}(\theta(t))$  are parameter-independent by gridding  $\Theta$  using  $L$  points  $\theta_i, i = 1, \dots, L$ . By relying on the gridding approach, the theoretical properties would be guaranteed only at the gridding points. However, from a practical point of view, it is reasonable to assume that if the gridding of  $\Theta$  is dense enough, then they would still hold for values of  $\theta$  different from the gridding ones. A deep theoretical study of this fact is possible using the results in [47].

## 4. Examples

### 4.1. Academic example

Let us consider a QPV system as in (23), with:

$$A(\theta(t)) = \begin{pmatrix} -4 - \theta_1(t) & 10 & 2 + 2\theta_2(t) \\ -1 & -1 - \theta_2(t) & 1.5 + 2\theta_1(t) \\ 1 & 1 & -4 - 3\theta_1(t) \end{pmatrix}$$

$$\begin{aligned}
B &= \begin{pmatrix} -1.2 & 0 & 0.7 \\ 1 & 0.8 & 0 \\ 0 & 0 & 2 \end{pmatrix} \\
N_1(\theta(t)) &= \begin{pmatrix} 0.5 & 1+\theta_1(t) & 0 \\ 0 & 0 & -\theta_2(t) \\ 0 & 1+\theta_2(t) & 0 \end{pmatrix} \\
N_2(\theta(t)) &= \begin{pmatrix} -0.4 & 0 & 1-\theta_1(t) \\ 1.5 & 0 & 1+\theta_2(t) \\ 2+\theta_2(t) & 0 & 0 \end{pmatrix} \\
N_3(\theta(t)) &= \begin{pmatrix} 1.5 & -0.5-\theta_1(t) & 0 \\ 3+\theta_1(t) & 0 & 0 \\ 0 & 2-\theta_2(t) & 0 \end{pmatrix} \\
M_1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \quad M_2 = M_3 = 0_{3 \times 3}
\end{aligned}$$

with  $\theta_1, \theta_2 \in [0, 1]$ . By considering all the possible combinations of minimum and maximum values for the scheduling variables  $\theta_1$  and  $\theta_2$ , the matrix functions  $A(\theta(t)), N_1(\theta(t)), N_2(\theta(t)), N_3(\theta(t))$  can be expressed in the polytopic form (20) with  $N = 4$  and:

$$\begin{aligned}
\mu_1(\theta(t)) &= (1 - \theta_1(t))(1 - \theta_2(t)) \\
\mu_2(\theta(t)) &= (1 - \theta_1(t))\theta_2(t) \\
\mu_3(\theta(t)) &= \theta_1(t)(1 - \theta_2(t)) \\
\mu_4(\theta(t)) &= \theta_1(t)\theta_2(t)
\end{aligned}$$

$$\begin{aligned}
A_1 &= \begin{pmatrix} -4 & 10 & 2 \\ -1 & -1 & 1.5 \\ 1 & 1 & -4 \end{pmatrix} & A_2 &= \begin{pmatrix} -4 & 10 & 4 \\ -1 & -2 & 1.5 \\ 1 & 1 & -4 \end{pmatrix} \\
A_3 &= \begin{pmatrix} -5 & 10 & 2 \\ -1 & -1 & 3.5 \\ 1 & 1 & -7 \end{pmatrix} & A_4 &= \begin{pmatrix} -5 & 10 & 4 \\ -1 & -2 & 3.5 \\ 1 & 1 & -7 \end{pmatrix} \\
N_{1,1} &= \begin{pmatrix} 0.5 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & N_{1,2} &= \begin{pmatrix} 0.5 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 2 & 0 \end{pmatrix} \\
N_{1,3} &= \begin{pmatrix} 0.5 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & N_{1,4} &= \begin{pmatrix} 0.5 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 2 & 0 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
N_{2,1} &= \begin{pmatrix} -0.4 & 0 & 1 \\ 1.5 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix} & N_{2,2} &= \begin{pmatrix} -0.4 & 0 & 1 \\ 1.5 & 0 & 2 \\ 3 & 0 & 0 \end{pmatrix} \\
N_{2,3} &= \begin{pmatrix} -0.4 & 0 & 0 \\ 1.5 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix} & N_{2,4} &= \begin{pmatrix} -0.4 & 0 & 0 \\ 1.5 & 0 & 2 \\ 3 & 0 & 0 \end{pmatrix} \\
N_{3,1} &= \begin{pmatrix} 1.5 & -0.5 & 0 \\ 3 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} & N_{3,2} &= \begin{pmatrix} 1.5 & -0.5 & 0 \\ 3 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
N_{3,3} &= \begin{pmatrix} 1.5 & -1.5 & 0 \\ 4 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix} & N_{3,4} &= \begin{pmatrix} 1.5 & -1.5 & 0 \\ 4 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\end{aligned}$$

Let us first consider the case in which the system is in open-loop, i.e.  $u(t) = 0$ , and assume that we want to assess what is the minimum value of  $\delta > 0$  for which the cube:

$$P(\delta) = \left[-\frac{1}{\delta}, \frac{1}{\delta}\right] \times \left[-\frac{1}{\delta}, \frac{1}{\delta}\right] \times \left[-\frac{1}{\delta}, \frac{1}{\delta}\right] \quad (39)$$

belongs to  $\mathcal{R}_A$ . By applying Corollary 1, a minimum value  $\delta = 45$  is obtained with  $\gamma = 0.21$ . In order to check the system behavior, simulation runs with initial conditions corresponding to the  $p = 8$  vertices of  $\mathcal{P}(45)$  have been performed, with different trajectories of the scheduling variables  $\theta_1(t)$  and  $\theta_2(t)$ . The results are plotted in Fig. 1, where it can be seen clearly that all the trajectories converge to the origin, as expected from the theoretical analysis.

Now, let us consider the case in which the QPV system is controlled by the state-feedback controller (25). We want to assess the minimum value of  $\delta > 0$  for which it is possible to design the controller such that the cube  $P(\delta)$ , defined as in (39), belongs to  $\mathcal{R}_A$  for the closed-loop system. By applying Corollary 2, a minimum value  $\delta = 0.7$  is obtained with  $\gamma = 0.39$ . The simulation runs have demonstrated that indeed  $\mathcal{P}(0.7)$  does not belong to the region of attraction of the origin for the open-loop system. As a matter of fact, some of the obtained trajectories diverge, as shown in Fig. 2. On the other hand, the designed controller stabilizes successfully the QPV system, as shown in Fig. 3, where every trajectory starting in  $\mathcal{P}(0.7)$  reaches the origin of the state space.

Finally, it is worth highlighting, using the proposed example, the advantage of using a gain-scheduled controller instead of a robust one when the varying parameters are known. To this end, let us consider a state-feedback control given by  $u(t) = Kx(t)$ , and let us assess again the minimum value of  $\delta > 0$  for which

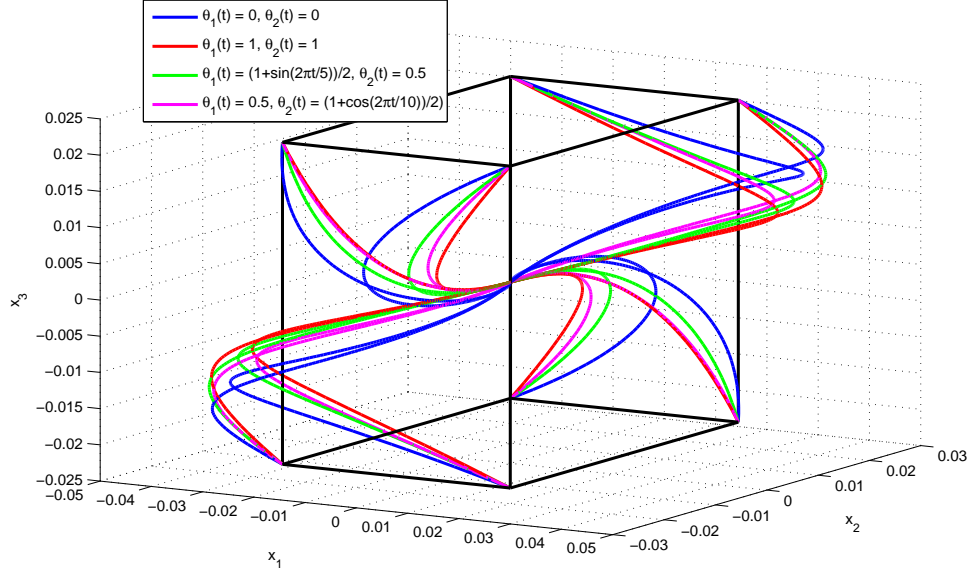


Figure 1: Academic example: open-loop trajectories starting from the vertices of  $\mathcal{P}(45)$ .

it is possible to solve the aforementioned controller design problem. By applying Corollary 2 with  $\Gamma_j = \Gamma = KQ$ , a minimum value  $\delta = 0.81$  is obtained with  $\gamma = 0.39$ , which means that a smaller polytopic attractive region is obtained.

#### 4.2. Two-joint planar robotic manipulator

Let us consider a two-joint planar robotic manipulator, whose dynamic equations can be expressed as<sup>1</sup> [48]:

$$a\ddot{q}_1 + b\cos(q_2 - q_1)\ddot{q}_2 - b\dot{q}_2^2\sin(q_2 - q_1) = \tau_1 \quad (40)$$

$$b\cos(q_2 - q_1)\ddot{q}_1 + c\ddot{q}_2 + b\dot{q}_1^2\sin(q_2 - q_1) = \tau_2 \quad (41)$$

where  $a$ ,  $b$ ,  $c$  are physical parameters that depend on the specific manipulator (in the following,  $a = 6\text{kgm}^2$ ,  $b = 1.5\text{kgm}^2$  and  $c = 2\text{kgm}^2$  will be considered),  $q_1$  and  $q_2$  are the joint positions and  $\tau_1$ ,  $\tau_2$  are the control torques. The equations (40)-(41) can be reshaped in state space form by considering the state vector  $x =$

<sup>1</sup>The dependence of the variables on time is omitted in order to ease the notation.

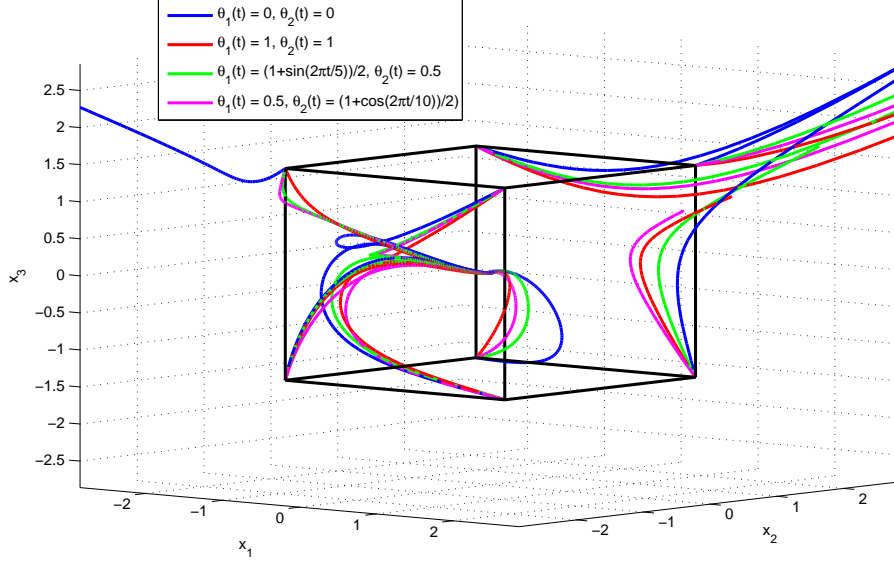


Figure 2: Academic example: open-loop trajectories starting from the vertices of  $\mathcal{P}(0.7)$ .

$[q_1, q_2, \dot{q}_1, \dot{q}_2]^T$  and the input vector  $u = [\tau_1, \tau_2]^T$ , thus obtaining:

$$\dot{x}_1 = x_3 \quad (42)$$

$$\dot{x}_2 = x_4 \quad (43)$$

$$\dot{x}_3 = \frac{b^2 \sin(x_2 - x_1) \cos(x_2 - x_1) x_3^2}{ac - b^2 \cos^2(x_2 - x_1)} + \frac{bc \sin(x_2 - x_1) x_4^2}{ac - b^2 \cos^2(x_2 - x_1)} \quad (44)$$

$$\begin{aligned} \dot{x}_4 = & \frac{cu_1}{ac - b^2 \cos^2(x_2 - x_1)} - \frac{b \cos(x_2 - x_1) u_2}{ac - b^2 \cos^2(x_2 - x_1)} \\ & - \frac{ab \sin(x_2 - x_1) x_3^2}{ac - b^2 \cos^2(x_2 - x_1)} - \frac{b^2 \sin(x_2 - x_1) \cos(x_2 - x_1) x_4^2}{ac - b^2 \cos^2(x_2 - x_1)} \\ & - \frac{b \cos(x_2 - x_1) u_1}{ac - b^2 \cos^2(x_2 - x_1)} + \frac{au_2}{ac - b^2 \cos^2(x_2 - x_1)} \end{aligned} \quad (45)$$

which can be put in the QPV form (23) by introducing the varying parameters:

$$\theta_1(t) = (ac - b^2 \cos^2(x_2 - x_1))^{-1} \quad (46)$$

$$\theta_2(t) = \sin(x_2 - x_1) \quad (47)$$

$$\theta_3(t) = \cos(x_2 - x_1) \quad (48)$$



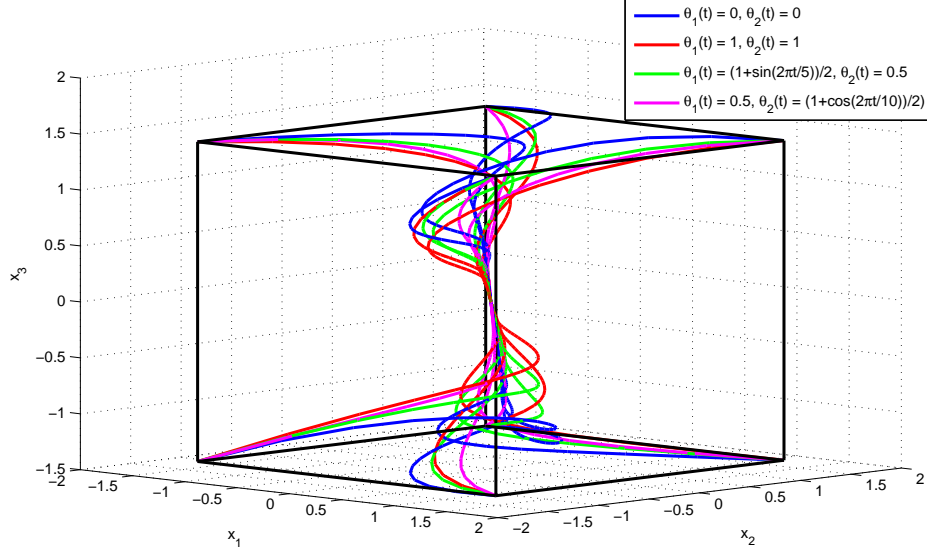


Figure 3: Academic example: closed-loop trajectories starting from the vertices of  $\mathcal{P}(0.7)$ .

such that:

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B(\theta(t)) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ c\theta_1(t) & -b\theta_1(t)\theta_3(t) \\ -b\theta_1(t)\theta_3(t) & a\theta_1(t) \end{pmatrix}$$

$$N_1 = N_2 = O_{4 \times 4} \quad N_3(\theta(t)) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & b^2\theta_1(t)\theta_2(t)\theta_3(t) & 0 \\ 0 & 0 & 0 & bc\theta_1(t)\theta_2(t) \end{pmatrix}$$

$$N_4(\theta(t)) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -ab\theta_1(t)\theta_2(t) & 0 \\ 0 & 0 & 0 & -b^2\theta_1(t)\theta_2(t)\theta_3(t) \end{pmatrix} \quad \begin{matrix} M_1 = O_{4 \times 2} \\ M_2 = O_{4 \times 2} \\ M_3 = O_{4 \times 2} \\ M_4 = O_{4 \times 2} \end{matrix}$$

By considering  $x_1, x_2 \in [-\pi/2, \pi/2]$  and  $x_3, x_4 \in [-2, 2]$ , the matrix functions  $N_3(\theta(t))$ ,  $N_4(\theta(t))$  can be expressed in the polytopic form (20) with  $N = 4$  and:

$$\mu_1(\theta(t)) = \frac{0.0833 - \theta_1(t)\theta_2(t)}{0.1666} \frac{1 - \theta_3(t)}{2}$$

$$\begin{aligned}\mu_2(\theta(t)) &= \frac{0.0833 - \theta_1(t)\theta_2(t)}{0.1666} \frac{1 + \theta_3(t)}{2} \\ \mu_3(\theta(t)) &= \frac{0.0833 + \theta_1(t)\theta_2(t)}{0.1666} \frac{1 - \theta_3(t)}{2} \\ \mu_4(\theta(t)) &= \frac{0.0833 + \theta_1(t)\theta_2(t)}{0.1666} \frac{1 + \theta_3(t)}{2}\end{aligned}$$

$$\begin{aligned}N_{3,1} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1874 & 0 \\ 0 & 0 & 0 & -0.2499 \end{pmatrix} & N_{3,2} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1874 & 0 \\ 0 & 0 & 0 & -0.2499 \end{pmatrix} \\ N_{3,3} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1874 & 0 \\ 0 & 0 & 0 & -0.2499 \end{pmatrix} & N_{3,4} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1874 & 0 \\ 0 & 0 & 0 & 0.2499 \end{pmatrix} \\ N_{4,1} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7498 & 0 \\ 0 & 0 & 0 & -0.1874 \end{pmatrix} & N_{4,2} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7498 & 0 \\ 0 & 0 & 0 & 0.1874 \end{pmatrix} \\ N_{4,3} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.7498 & 0 \\ 0 & 0 & 0 & 0.1874 \end{pmatrix} & N_{4,4} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -0.7498 & 0 \\ 0 & 0 & 0 & -0.1874 \end{pmatrix}\end{aligned}$$

By applying Corollary 2 with  $\gamma = 0.30$ , the Lyapunov matrix and the vertex controller gains are calculated by considering a grid of 121 values for the matrix function  $B(\theta(t))$ , obtaining:

$$Q = \begin{pmatrix} 27.3857 & -1.9578 & -4.0678 & -0.5027 \\ -1.9578 & 27.3954 & -0.4982 & -9.0549 \\ -4.0678 & -0.4982 & 11.4185 & 1.1866 \\ -0.5027 & -9.0549 & 1.1866 & 21.1250 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} -9.4805 & -2.3424 & -58.4594 & -2.5381 \\ -1.8158 & -5.1160 & -8.7053 & -11.6866 \end{pmatrix}$$

$$K_2 = \begin{pmatrix} -10.9459 & -6.5764 & -64.6494 & -10.8433 \\ 1.1693 & -5.6379 & 13.8149 & -14.9285 \end{pmatrix}$$

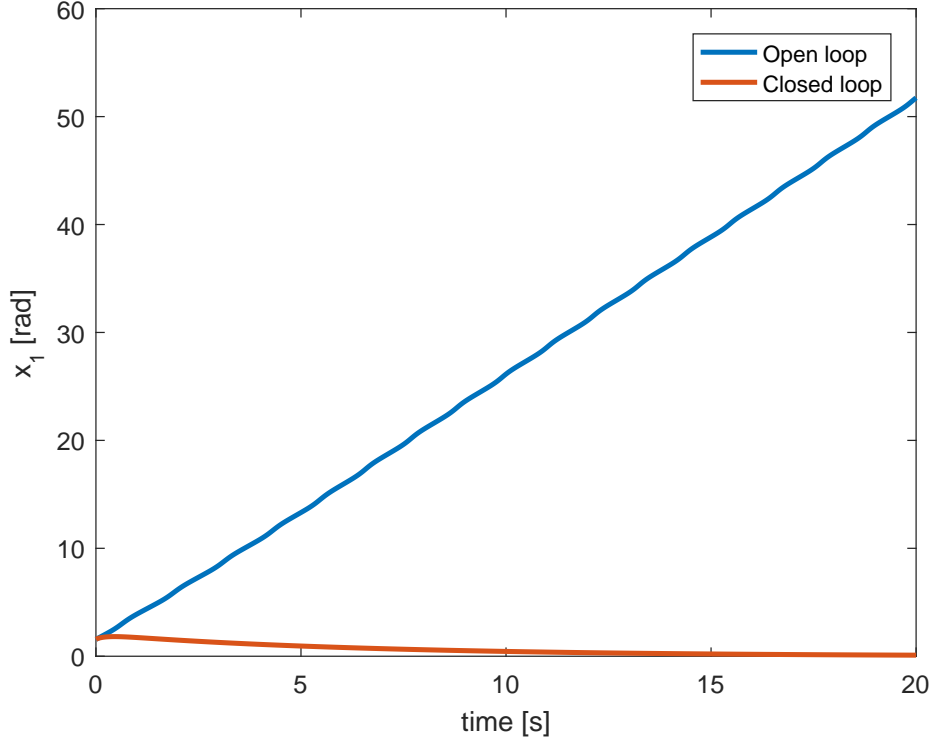


Figure 4: Robotic manipulator: trajectory of the state variable  $x_1$ .

$$K_3 = \begin{pmatrix} -7.6292 & -2.4167 & -45.9100 & -4.5673 \\ -1.5631 & -5.0779 & -7.2713 & -12.0615 \end{pmatrix}$$

$$K_4 = \begin{pmatrix} -11.3865 & -6.0586 & -67.7836 & -9.1867 \\ 1.1157 & -5.5208 & 13.3777 & -14.5624 \end{pmatrix}$$

In order to assess the effectiveness of the designed controller, a simulation lasting 20s where the manipulator's initial condition is set as  $x_0 = [\pi/2, -\pi/2, 2, -2]$  has been considered. The comparison of the open loop trajectories (blue lines) with the closed loop trajectories (red lines) in Figs. 4-7 shows that the designed controller stabilizes successfully the robotic manipulator.

## 5. Conclusions

This paper has proposed an analysis/design approach for QPV systems, i.e. a class of systems that are quadratic in the structure, but differ from the classical QTI

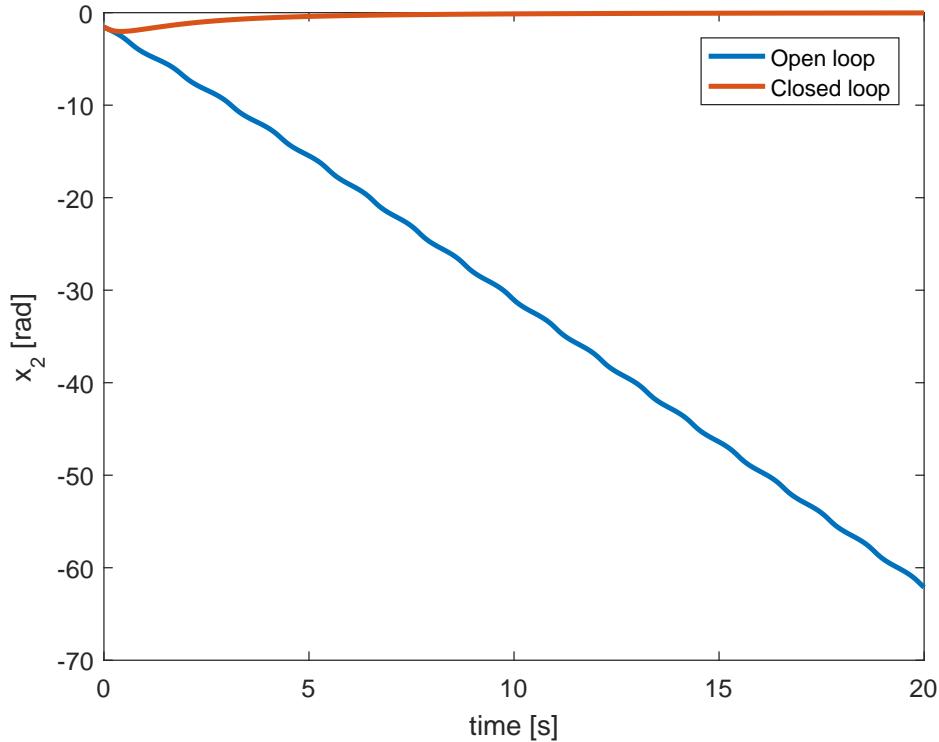


Figure 5: Robotic manipulator: trajectory of the state variable  $x_2$ .

formulation since the involved matrices are not constant, but parameter-varying. An LMI-based methodology for assessing whether a given polytope belongs to the region of attraction of the origin has been proposed. This methodology has also allowed providing results for the stabilization of a QPV system by means of a parameter-varying state-feedback controller. The main characteristics of the proposed approach have been illustrated by means of simulation results obtained using both an academic example and a two-joint planar robotic manipulator. In both cases, the validity of the theoretical results have been confirmed. Also, the advantage of a gain-scheduled controller with respect to a robust one when the varying parameters are known has been shown.

Future research concerning the analysis and control of QPV systems will follow several directions. Since global stability can be of interest, results that are able to assess this property for QPV systems with particular structures will be investigated. Moreover, the application of parameter-dependent Lyapunov func-

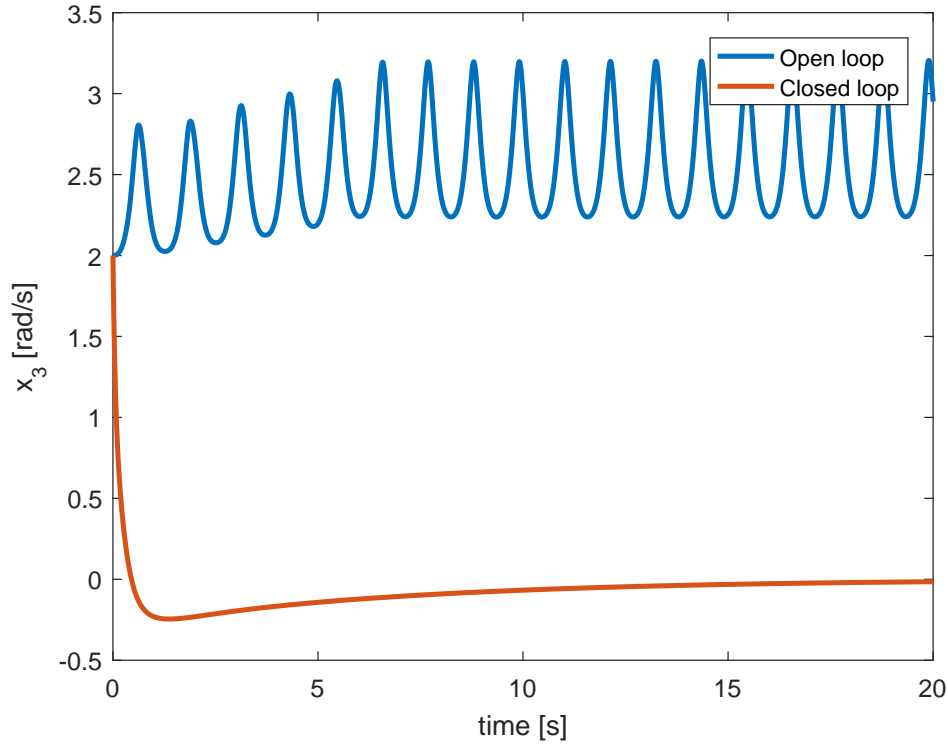


Figure 6: Robotic manipulator: trajectory of the state variable  $x_3$ .

tions will be investigated, with the aim of decreasing the conservativeness brought by the use of a constant Lyapunov matrix. To this regard, it is expected that parameter-dependent Lyapunov functions will lead to a larger number of conditions that have to be assessed, and they will need additional assumptions about the derivative of the varying parameters. However, they will provide feasibility of the matrix inequalities in cases where the approach proposed in this paper fails in assessing stability or designing a stabilizing controller for a given candidate polytopic region of attraction. Finally, the assumption of knowing perfectly the varying parameters or the matrices of the QPV model will be relaxed by applying results from the robust control theory, e.g. [49] and [50].

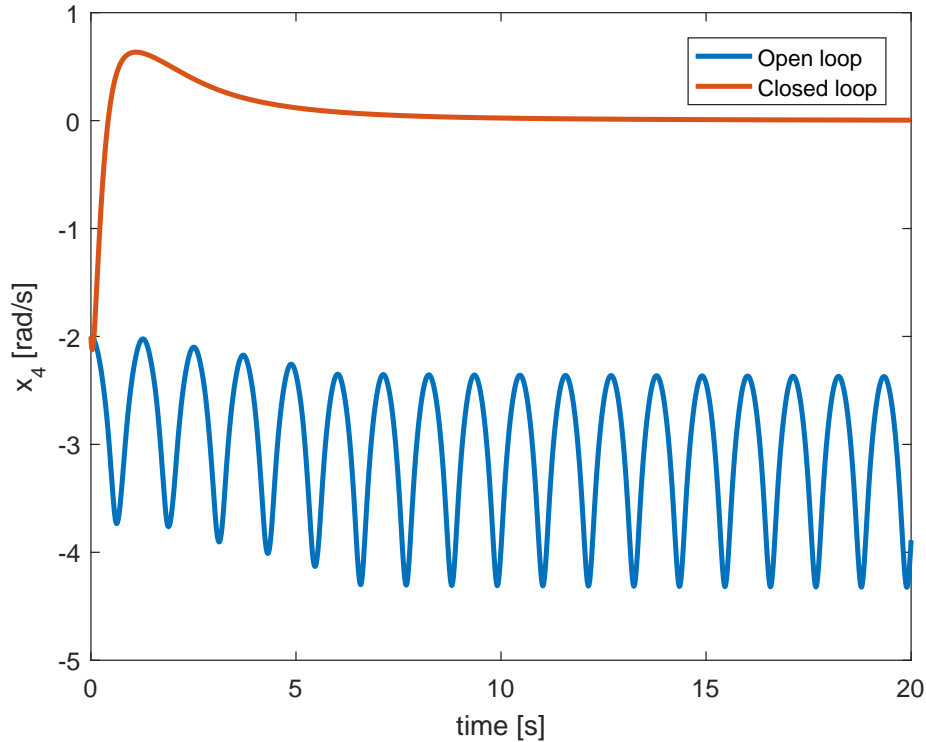


Figure 7: Robotic manipulator: trajectory of the state variable  $x_4$ .

## 6. Acknowledgements

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