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Distributed Quasi-Nonlinear Model Predictive Control with Contractive Constraint

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Abstract: An approach to low complexity distributed MPC of nonlinear interconnected systems with coupled dynamics subject to both state and input constraints is proposed. It is based on the idea of introducing a contractive constraint in the centralized NMPC problem formulation, which would guarantee the closed-loop system stability when using a small prediction horizon. Particularly, the one step ahead NMPC problem is considered. Further, a quasi-NMPC method is developed, which is based on a sequential linearization of the nonlinear system dynamics and finding distributedly a suboptimal solution of the resulting convex Quadratically Constrained Quadratic Programming problem. The suggested approach would be appropriate for distributed convex NMPC of some cyber-physical systems, since it will reduce the complexity of the on-line NMPC computations, simplify the software implementation, and reduce the requirements for available memory. The proposed method is illustrated with simulations on the model of a quadruple-tank system.

Keywords: Predictive control, Distributed control, Interconnected systems, Nonlinear systems, Constraints.

1. INTRODUCTION

Model predictive control (MPC) involves the solution at each sampling instant of a finite horizon optimal control problem subject to the system dynamics, and state and input constraints. Several methods for distributed/decentralized MPC of interconnected systems have been developed (Christofides et al. (2013), Maestre and Negenborn (2014)) which allow the computation of the control inputs to be done by the systems without the need for centralized optimization. In Venkat et al. (2008), Alessio et al. (2011), Giselsson et al. (2013), Giselsson and Rantzer (2014), approaches for distributed/decentralized MPC for systems consisting of *linear* interconnected subsystems have been developed. In Giselsson and Rantzer (2014), the assumption for guaranteeing the stability of the overall system is that the prediction horizon is sufficiently large, such that the relaxed dynamic programming condition in Grüne and Rantzer (2008) is fulfilled. Methods for distributed MPC for systems composed of several *nonlinear* subsystems have also been proposed (e.g. Raimondo et al. (2007), Dunbar (2007), Grancharova et al. (2016), Heidarinejad et al. (2011)).

Recently, several methods for reducing the complexity of the MPC controllers have been developed. In Hovd et al. (2014), an approach to design a low complexity *centralized* MPC for linear systems by using contractive set constraint is proposed. The idea of adding a contractive constraint has been widely used in MPC to guarantee the stability of the closed-loop system (e.g. De Oliveira and Morari (2000)). On a broad scope, various Lyapunov-based MPC algorithms (Christofides et al. (2013)) have been developed.

In Grancharova et al. (2016), an approach to distributed quasi-NMPC for interconnected nonlinear systems has been proposed, where the stability of the closed-loop system can be achieved by choosing a sufficiently large prediction horizon. In this paper, a low complexity distributed quasi-NMPC approach is proposed, which applies the method in Murillo et al. (2016) of introducing a contractive constraint in the centralized NMPC problem formulation. This would help the closed-loop system stability when using a small prediction horizon and at the same time will reduce the complexity of the optimization problem and the requirements for available memory. Here, the one step ahead NMPC problem is considered. Then, a quasi-NMPC method is developed, which uses a sequential linearization of the nonlinear system dynamics and finds a suboptimal solution of the resulting convex Quadratically Constrained Quadratic Programming (QCQP) problem by using distributed iterations of the dual accelerated gradient method.

2. FORMULATION OF CONTRACTIVE NMPC PROBLEM WITH ONE STEP AHEAD PREDICTION

Consider a system composed by the interconnection of M subsystems with overall state and overall control input:

$$x(t) = [x_1(t), x_2(t), \dots, x_M(t)] \in \mathbb{R}^n, \; n = \sum_{i=1}^{M} n_i \tag{1}$$

$$u(t) = [u_1(t), u_2(t), \dots, u_M(t)] \in \mathbb{R}^m, \; m = \sum_{i=1}^{M} m_i \tag{2}$$

where $x_i(t) \in \mathbb{R}^{n_i}$ and $u_i(t) \in \mathbb{R}^{m_i}$ are the state and the control input, related to the i-th subsystem. The general case when the subsystems are coupled both through their states and inputs is considered and it is assumed that their dynamics are described by the nonlinear discrete-time models:

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$$x_i(t+1) = f_i(x(t), u(t)) \ , \ i = 1, 2, \dots, M \quad (3)$$

Here, $f_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n_i}$ is a nonlinear function of the overall system state $x(t)$ and control input $u(t)$. The constraints imposed on the subsystems are:

$$x_i(t) \in \mathcal{X}_i \ , \ u_i(t) \in \mathcal{U}_i \ , \ i = 1, 2, \dots, M \quad (4)$$

where $\mathcal{X}_i$ and $\mathcal{U}_i$ are the admissible sets, and the following assumptions are made:

A1. The functions f_i, $i = 1, \dots, M$ are continuously differentiable with $f_i(0,0) = 0$.

A2. The admissible sets $\mathcal{X}_i$ and $\mathcal{U}_i$ are bounded polyhedral sets, i.e. they are defined by:

$$\mathcal{X}_i = \{x_i \in \mathbb{R}^{n_i} \mid C_i^x x_i \le d_i^x\} \ , \ \mathcal{U}_i = \{u_i \in \mathbb{R}^{m_i} \mid C_i^u u_i \le d_i^u\} \quad (5)$$

and they include the origin in their interior. Here, $C_i^x \in \mathbb{R}^{n_{c,x_i} \times n_i}$, $C_i^u \in \mathbb{R}^{n_{c,u_i} \times m_i}$, $d_i^x \in \mathbb{R}^{n_{c,x_i}}$, $d_i^u \in \mathbb{R}^{n_{c,u_i}}$ and n_{c,x_i} and n_{c,u_i} are the number of constraints imposed on x_i and u_i.

Here, the optimal regulation problem is considered where the goal is to steer the overall state of the system (3) to the origin. It is supposed that a full measurement $\overline{x} = [\overline{x}_1, \overline{x}_2, \dots, \overline{x}_M]$ of the overall state is available at the current time t. In order to reduce the on-line computational complexity of the optimization, a NMPC problem based on one step ahead prediction is formulated. For the current overall state $\overline{x}$, the regulation NMPC solves the optimization problem:

Problem P1 (Centralized NMPC):

$$V^*(\overline{x}) = \min_{u_t} J(u_t, \overline{x}) \quad (6)$$

subject to:

$$x_{i,t+1|t} \in \mathcal{X}_i \ , \ i = 1, \dots, M \quad (7)$$

$$u_{i,t} \in \mathcal{U}_i \ , \ i = 1, \dots, M \quad (8)$$

$$x_{i,t+1|t} = f_i(\overline{x}, u_t), \ i = 1, \dots, M \quad (9)$$

$$x_{t+1|t} = [x_{1,t+1|t}, x_{2,t+1|t}, \dots, x_{M,t+1|t}] \quad (10)$$

$$u_t = [u_{1,t}, u_{2,t}, \dots, u_{M,t}] \quad (11)$$

with the cost function given by:

$$J(u_t, \overline{x}) = \frac{1}{2} \sum_{i=1}^{M} [x_{i,t+1|t}^{\mathrm{T}} Q_i x_{i,t+1|t} + u_{i,t}^{\mathrm{T}} R_i u_{i,t}] \quad (12)$$

Here, $Q_i, R_i \succ 0$ are symmetric weighting matrices for the i-th subsystem. For guaranteeing the stability of the closed-loop system, the approach in Murillo et al. (2016) is applied and the following contractive constraint is added to the centralized NMPC problem formulation:

$$J(u_t, \overline{x}) \le J(u_t^0, \overline{x}) \quad (13)$$

where $u_t^0 = 0$ and:

$$J(u_t^0, \overline{x}) = \frac{1}{2} \sum_{i=1}^{M} [x_{i,t+1|t}^{0,\mathrm{T}} Q_i x_{i,t+1|t}^0] \quad (14)$$

$$x_{i,t+1|t}^0 = f_i(\overline{x}, u_t^0), \ i = 1, \dots, M \quad (15)$$

The contractive constraint (13) can be represented in the form:

$$\frac{1}{2} \sum_{i=1}^{M} [f_i^{\mathrm{T}}(\overline{x}, u_t) Q_i f_i(\overline{x}, u_t) + u_{i,t}^{\mathrm{T}} R_i u_{i,t}] \le \frac{1}{2} \sum_{i=1}^{M} [f_i^{\mathrm{T}}(\overline{x}, 0) Q_i f_i(\overline{x}, 0)] \quad (16)$$

Note that in Murillo et al. (2016) the contractive constraint formulation considers the more general case with horizon $N > 1$.

3. AN APPROACH TO DISTRIBUTED QUASI-NMPC WITH CONTRACTIVE CONSTRAINT

3.1 Approximation of the NMPC problem with one step ahead prediction by a linear MPC problem

The first step of the quasi-NMPC approach is to locally approximate the dynamics of the subsystems (3) by linear models for one step ahead prediction. Let at time t, $\overline{x}_i$ and $\tilde{u}_{i,t}$ be the known state and the predicted update of the control input of the i-th subsystem. Taylor series expansion of the right-hand side of the model (3) about the point $(\overline{x}_i, \tilde{u}_{i,t})$ leads to the locally linear prediction models of the subsystems:

$$x_{i,t+1} = \sum_{j=1}^{M} B_{ij,t} u_{j,t} + g_{i,t} \ , \ i = 1, \dots, M \quad (17)$$

where the matrix $B_{ij,t}$ and the vector $g_{i,t}$ are computed as:

$$B_{ij,t} = \nabla_{u_j} f_i(\overline{x}, \tilde{u}_t), \ g_{i,t} = -\sum_{j=1}^{M} B_{ij,t} \tilde{u}_{j,t} + f_i(\overline{x}, \tilde{u}_t) \quad (18)$$
$$i, j = 1, \dots, M$$

In (18), $\tilde{u}_t = [\tilde{u}_{1,t}, \tilde{u}_{2,t}, \dots, \tilde{u}_{M,t}]$ and $\overline{x}$ is the known state of the whole system. The prediction model (17)-(18) is a linear time-varying approximation of the nonlinear model (3). The one step ahead linearized prediction model of the overall systems is:

$$x_{t+1} = B_t(\overline{x}, \tilde{u}_t) u_t + g_t(\overline{x}, \tilde{u}_t) \quad (19)$$

where:

$$B_t(\overline{x}, \tilde{u}_t) = \begin{bmatrix} B_{11,t} & B_{12,t} & \cdots & B_{1M,t} \\ B_{21,t} & B_{22,t} & \cdots & B_{2M,t} \\ \vdots & \vdots & \ddots & \vdots \\ B_{M1,t} & B_{M2,t} & \cdots & B_{MM,t} \end{bmatrix}$$
$$g_t(\overline{x}, \tilde{u}_t) = [g_{1,t}, g_{2,t}, \dots, g_{M,t}]^{\mathrm{T}} \quad (20)$$

The following assumption is made:

A3. The system (19) is uniformly controllable.

As in Giselsson and Rantzer (2014), the following tightened constraint sets are introduced:

$$(1-\delta)\mathcal{X}_i = \{x_i \in \mathbb{R}^{n_i} \mid C_i^x x_i \le (1-\delta) d_i^x\} \quad (21)$$

$$(1-\delta)\mathcal{U}_i = \{u_i \in \mathbb{R}^{m_i} \mid C_i^u u_i \le (1-\delta) d_i^u\} \quad (22)$$

where $\delta \in (0, 1)$ is the amount of relative constraint tightening. The reason for the tightening is related to the fact that a suboptimal solution of the NMPC problem will be found (see Section 3.3) and it should be ensured that it will keep the original constraints. Let the constraints sets for the overall system be denoted:

$$\mathcal{X} = (1-\delta)(\mathcal{X}_1 \times \dots \times \mathcal{X}_M) \ , \ \mathcal{U} = (1-\delta)(\mathcal{U}_1 \times \dots \times \mathcal{U}_M) \quad (23)$$

Then, for the locally linear dynamics (17)-(18) with initial state $\overline{x}$, the linear MPC problem is formulated:

Problem P2 (Centralized linearized MPC):

$$V^*(\overline{x}) = \min_{u_t} J(u_t, \overline{x}) \quad (24)$$

subject to the equality constraints (17)-(18), the admissible sets constraints:

$$x_{i,t+1|t} \in (1-\delta)\mathcal{X}_i \ , \ i = 1,...,M \quad (25)$$

$$u_{i,t} \in (1-\delta)\mathcal{U}_i \ , \ i = 1,...,M \quad (26)$$

and the contractive constraint:

$$\frac{1}{2}\left\{[B_t u_t + g_t]^{\mathrm{T}} Q [B_t u_t + g_t] + u_t^{\mathrm{T}} R u_t\right\} \le \frac{1}{2} g_t^{\mathrm{T}} Q g_t \quad (27)$$

In (27) the overall weighting matrices are:

$$Q = \text{blockdiag}\{Q_1, \dots, Q_M\},\ R = \text{blockdiag}\{R_1, \dots, R_M\} \quad (28)$$

In (24) the cost function $J(u_t, \bar{x})$ is defined by (12). The contractive constraint (27) is a coupling constraint and it can be represented in a more compact form:

$$u_t^{\mathrm{T}}(B_t^{\mathrm{T}} Q B_t + R) u_t + 2 g_t^{\mathrm{T}} Q B_t u_t \le 0 \quad (29)$$

Assume $Q, R \succ 0$, then $(B_t^{\mathrm{T}} Q B_t + R)$ is a positive definite matrix. Therefore, (29) is a convex quadratic constraint and the problem P2 is a convex Quadratically Constrained Quadratic Programming (QCQP) problem. The following assumption is also made:

A4. The matrices B_t, Q, R and the vector g_t are such that for any $\bar{x} \in \mathcal{X}$ there exists $u_t \in \mathcal{U}$ for which the contractive constraint (29) is satisfied.

3.2 Distributed Quadratically Constrained Quadratic Programming problem

Before representing the optimization problem P2 as a convex QCQP problem, the following notation is introduced for the *i*-th subsystem. Let the vector $Y_i \in \mathbb{R}^{n_{Y_i}}$ ($n_{Y_i} = n_i + m_i$) include the decision variables (the control input at current time and the predicted successor state) for the *i*-th subsystem, i.e.:

$$Y_i = [x_{i,t+1|t}, u_{i,t}] \quad (30)$$

The matrix $\bar{H}_i \in \mathbb{R}^{n_{Y_i} \times n_{Y_i}}$ ($n_{Y_i} = n_i + m_i$), including the cost matrices for the *i*-th subsystem, is defined as:

$$\bar{H}_i = \begin{bmatrix} Q_i & 0_{m_i} \\ 0_{n_i} & R_i \end{bmatrix} \quad (31)$$

where 0_{n_i}, 0_{m_i} are square zero matrices with dimensions n_i, m_i. The time-varying matrices $\bar{A}_i \in \mathbb{R}^{n_i \times n_Y}$ and $\bar{G}_i \in \mathbb{R}^{n_i}$, related to the equality constraints (17)-(18) for the *i*-th subsystem are introduced:

$$\bar{A}_i = \begin{bmatrix} \bar{A}_{i1} & \bar{A}_{i2} & \cdots & \bar{A}_{iM} \end{bmatrix},\ \bar{G}_i = g_{i,t} \quad (32)$$

where for given *i*, $\bar{A}_{ij}$, $j = 1, 2, \dots, M$ are:

$$\bar{A}_{ij} = [0_{n_i, n_j}, -B_{ij}],\ i \ne j;\ \bar{A}_{ii} = [I_{n_i}, -B_{ii}] \quad (33)$$

Here, $0_{n_i, n_j}$ is a zero matrix with dimensions $n_i \times n_j$, and I_{n_i} is the identity matrix with dimension n_i. The matrix $\bar{C}_{i,\text{dec}} \in \mathbb{R}^{n_{i,\text{dec}} \times n_{Y_i}}$ and the vector $\bar{d}_{i,\text{dec}} \in \mathbb{R}^{n_{i,\text{dec}}}$, associated to the decoupled inequality constraints (25)-(26) for the *i*-th subsystem, are introduced:

$$\bar{C}_{i,\text{dec}} = \begin{bmatrix} C_i^x & 0_{m_i} \\ 0_{n_i} & C_i^u \end{bmatrix},\ \bar{d}_{i,\text{dec}} = [(1-\delta) d_i^x, (1-\delta) d_i^u]^{\mathrm{T}} \quad (34)$$

Similarly, the matrix $\bar{C}_{i,\text{contr}} \in \mathbb{R}^{n_{Y_i} \times n_{Y_i}}$ and the vector $\bar{c}_{i,\text{contr}} \in \mathbb{R}^{1 \times n_{Y_i}}$ ($n_{i,\text{dec}} = n_{c,x_i} + n_{c,u_i}$) $i = 1, \dots, M$, related to the contractive constraint (29), are defined as:

$$\bar{C}_{i,\text{contr}} = \begin{bmatrix} 0_{n_i} & (B_{i,t}^{\mathrm{T}} Q_i B_{i,t} + R_i) \\ 0_{n_i} & 0_{m_i} \end{bmatrix},\ \bar{c}_{i,\text{contr}} = \begin{bmatrix} 0_{n_i} & 2 g_{i,t}^{\mathrm{T}} Q_i B_{i,t} \end{bmatrix} \quad (35)$$

where the matrix $B_{i,t}$ is:

$$B_{i,t} = \begin{bmatrix} B_{i1,t} & B_{i2,t} & \cdots & B_{iM,t} \end{bmatrix}^{\mathrm{T}} \quad (36)$$

By taking into account this notation, the problem P2 is represented as a convex QCQP problem in the following way. The decision variables for the overall system are stacked into one vector $Y \in \mathbb{R}^{n_Y}$ with dimension $n_Y = \sum_{i=1}^{M} n_{Y_i}$:

$$Y = [Y_1, Y_2, \dots, Y_M]^{\mathrm{T}} \quad (37)$$

The matrices and vectors, associated to the global cost function (12), the equality constraints, the inequality constraints, and the contractive constraint for the overall system, are denoted as $\bar{H} \in \mathbb{R}^{n_Y \times n_Y}$, $\bar{A} \in \mathbb{R}^{n \times n_Y}$, $\bar{G} \in \mathbb{R}^{n}$, $\bar{C}_{\text{dec}} \in \mathbb{R}^{n_{\text{dec}} \times n_Y}$, $\bar{d}_{\text{dec}} \in \mathbb{R}^{n_{\text{dec}}}$, $\bar{C}_{\text{contr}} \in \mathbb{R}^{n_Y \times n_Y}$, $\bar{c}_{\text{contr}} \in \mathbb{R}^{1 \times n_Y}$, ($n = \sum_{i=1}^{M} n_i$, $n_{\text{dec}} = \sum_{i=1}^{M} n_{i,\text{dec}}$), and are defined by:

$$\bar{H} = \text{blockdiag}\{\bar{H}_1, \bar{H}_2, \dots, \bar{H}_M\} \quad (38)$$

$$\bar{A} = [\bar{A}_1 | \bar{A}_2 | \dots | \bar{A}_M]^{\mathrm{T}},\ \bar{G} = [\bar{G}_1 | \bar{G}_2 | \dots | \bar{G}_M]^{\mathrm{T}} \quad (39)$$

$$\begin{aligned} &\bar{C}_{\text{dec}} = \text{blockdiag}\{\bar{C}_{1,\text{dec}}, \bar{C}_{2,\text{dec}}, \dots, \bar{C}_{M,\text{dec}}\} \\ &\bar{d}_{\text{dec}} = [\bar{d}_{1,\text{dec}} | \bar{d}_{2,\text{dec}} | \dots | \bar{d}_{M,\text{dec}}]^{\mathrm{T}} \end{aligned} \quad (40)$$

$$\begin{aligned} &\bar{C}_{\text{contr}} = \text{blockdiag}\{\bar{C}_{1,\text{contr}}, \bar{C}_{2,\text{contr}}, \dots, \bar{C}_{M,\text{contr}}\} \\ &\bar{c}_{\text{contr}} = [\bar{c}_{1,\text{contr}} | \bar{c}_{2,\text{contr}} | \dots | \bar{c}_{M,\text{contr}}]^{\mathrm{T}} \end{aligned} \quad (41)$$

Then, the optimization problem P2 can be written as the following convex QCQP problem:

Problem P3 (QCQP problem):

$$V^*(\bar{x}) = \min_{Y} \frac{1}{2} Y^{\mathrm{T}} \bar{H} Y \quad (42)$$

subject to:

$$\bar{A} Y = \bar{G} \quad (43)$$

$$\bar{C}_{\text{dec}} Y \le \bar{d}_{\text{dec}} \quad (44)$$

$$Y^{\mathrm{T}} \bar{C}_{\text{contr}} Y + \bar{c}_{\text{contr}} Y \le 0 \quad (45)$$

The problem P3 is convex since $\bar{H} \succ 0$, $\bar{C}_{\text{contr}} \succ 0$.

The convex QCQP problem P3 can be solved distributedly by applying the dual accelerated gradient algorithm in Giselsson et al. (2013). The decomposition is enabled by formulating the dual problem to problem P3, which is created by introducing dual variables $\lambda \in \mathbb{R}^{n}$ for the equality constraints (43), dual variables $\mu_{\text{dec}} \in \mathbb{R}^{n_{\text{dec}}}$ for the linear inequality constraints (44), and dual variables $\mu_{\text{contr}} \in \mathbb{R}$ for the quadratic inequality constraint (45). The dual problem can be written as:

$$\max_{\lambda, \mu_{\text{dec}} \ge 0, \mu_{\text{contr}} \ge 0} D(\bar{x}, \lambda, \mu_{\text{dec}}, \mu_{\text{contr}}) \quad (46)$$

where $D(\bar{x}, \lambda, \mu_{\text{dec}}, \mu_{\text{contr}})$ is the dual cost function (Boyd and Vandenberghe (2004)):

$$D(\bar{x},\lambda,\mu_{\text{dec}},\mu_{\text{contr}}) = -\frac{1}{2}(\bar{A}^{\text{T}}\lambda+\bar{C}_{\text{dec}}^{\text{T}}\mu_{\text{dec}}+\bar{c}_{\text{contr}}^{\text{T}}\mu_{\text{contr}})^{\text{T}}(\bar{H}+\mu_{\text{contr}}\bar{C}_{\text{contr}})^{-1} (\bar{A}^{\text{T}}\lambda+\bar{C}_{\text{dec}}^{\text{T}}\mu_{\text{dec}}+\bar{c}_{\text{contr}}^{\text{T}}\mu_{\text{contr}})-\lambda^{\text{T}}\bar{G}-\mu_{\text{dec}}^{\text{T}}\bar{d}_{\text{dec}} \quad (47)$$

The dual problem (46) is solved by applying a modified version of the dual accelerated gradient method (Giselsson et al. (2013) and the references therein) that is here adapted to maximize a dual cost function of the form (47).

In order to distribute the iterations of the dual gradient method, let $\lambda_i \in \mathbb{R}^{n_i}$, $\mu_{i,\text{dec}} \in \mathbb{R}^{n_{i,\text{dec}}}$ be the dual variables for the equality and the linear inequality constraints, related to the *i*-th subsystem. Then, the distributed iterations of the modified dual accelerated gradient method, applied to solve the convex QCQP problem P3, are:

$$Y_i^r = -(\bar{H}_i+\bar{C}_{i,\text{contr}}\mu_{\text{contr}}^r)^{-1}(\sum_{j=1}^{M}\bar{A}_j^{i\text{T}}\lambda_j^r+\bar{C}_{i,\text{dec}}^{\text{T}}\mu_{i,\text{dec}}^r+\bar{c}_{i,\text{contr}}^{\text{T}}\mu_{\text{contr}}^r) \quad (48)$$

$$\bar{Y}_i^r = Y_i^r + \frac{r-1}{r+2}(Y_i^r - Y_i^{r-1}) \quad (49)$$

$$\lambda_i^{r+1} = \lambda_i^r + \frac{r-1}{r+2}(\lambda_i^r - \lambda_i^{r-1}) + \frac{1}{L}(\bar{A}_i\bar{Y}^r - \bar{G}_i)) \quad (50)$$

$$\mu_{i,\text{dec}}^{r+1} = \max\{0, \mu_{i,\text{dec}}^r + \frac{r-1}{r+2}(\mu_{i,\text{dec}}^r - \mu_{i,\text{dec}}^{r-1}) + \frac{1}{L}(\bar{C}_{i,\text{dec}}\bar{Y}_i^r - \bar{d}_{i,\text{dec}})\} \quad (51)$$

$i = 1, 2, \ldots, M$

$$\mu_{\text{contr}}^{r+1} = \max\{0, \mu_{\text{contr}}^r + \frac{r-1}{r+2}(\mu_{\text{contr}}^r - \mu_{\text{contr}}^{r-1}) + \frac{1}{L}\bar{C}_{\text{contr}}\bar{Y}^r\} \quad (52)$$

where $\bar{A}_j^i$ are the columns of the matrix $\bar{A}_j$ corresponding to the decision vector Y_i. Here, *r* is the iteration index and $1/L$ determines the step size of improving the solution of the dual problem (usually L is the Lipschitz constant to the gradient of the dual function, which can be estimated by off-line computations). Because of the couplings in the dynamics models of the subsystems, the computation of the decision variables Y_i^r for the *i*-th subsystem requires to have information about the dual variables λ_j^r of the subsystems interacting with it. Also, the update of the dual variables λ_i for the *i*-th subsystem uses the information about the decision variables $\bar{Y}_j^r$ of those subsystems. In (48) and (50) this is reflected with a corresponding construction of the matrices $\bar{A}_j^i$ and $\bar{A}_i$. Since there are both decoupled inequality constraints (cf. (25)-(26)) and coupled inequality constraint (cf. (27)), the respective dual variables $\mu_{i,\text{dec}}$ and μ_{contr} are updated by separate formulas. The update of $\mu_{i,\text{dec}}$ requires information only about the decision variables $\bar{Y}_i^r$ for the *i*-th subsystem, while for updating μ_{contr} it is necessary to have information about the decision variables $\bar{Y}^r$ for the whole system. Therefore, the distributed solution of the dual problem (46) requires for the interacting subsystems to exchange information about the current updates of their dual variables and decision variables.

3.3 Algorithm for distributed quasi-NMPC with contractive constraint

Here, a suboptimal algorithm is proposed that differs from the one in Grancharova et al. (2016) in two aspects: 1) the quasi-NMPC approach is based on one step ahead prediction of the system behaviour and including contractive constraint, 2) the resulting optimization problem is a convex Quadratically Constrained Quadratic Programming problem, which is solved distributedly by applying the dual accelerated gradient method. The suggested algorithm for distributed NMPC includes two loops. In the outer loop, the dynamics of the nonlinear system (3) is locally approximated with a linear model (17)-(18) about the known state $\bar{x}$ and the current update $\tilde{u}_t$ of the control input at time t. Then, in the inner loop, a suboptimal solution to the resulting convex QCQP problem P3 is found by applying the distributed iterations (48)-(52) of the dual accelerated gradient method.

Let $\tilde{u}_t$ be the current update of the control input and denote with $x_{t+1|t}$ the corresponding predicted state of the nonlinear system (3) obtained for initial state $x_{t|t} = \bar{x}$, i.e.:

$$x_{t+1|t} = f(\bar{x}, \tilde{u}_t) \quad (53)$$

Then, the current update $Y(t)$ of the decision variables can be easily constructed according to (30). Respectively, if updates Y^r are obtained by performing the iterations (48)-(52), the corresponding update u_t^r of the control input can be extracted from it. Further, assume that a relative tolerance $\varepsilon > 0$ of achieving the extremum of the cost function is specified, i.e. the iterations in the outer loop will terminate if the following condition is satisfied:

$$|J(u_2,\bar{x}) - J(u_1,\bar{x})| / J(u_1,\bar{x}) \le \varepsilon \quad (54)$$

Here, u_1, u_2 and $J(u_1,\bar{x})$, $J(u_2,\bar{x})$ are the control inputs and cost function values obtained in two consecutive iterations in the outer loop of the algorithm.

Suppose that the relative constraint tightening δ, the relative tolerance ε and the number N_r of iterations (48)-(52) are specified. Then, the algorithm for distributed quasi-NMPC with one step ahead prediction is described as follows.

Algorithm 1:

1. Given δ, ε and N_r. Let $t = 0$, $\tilde{u}_t = 0$.

2. Let the state at time t be $x(t) = \bar{x} = [\bar{x}_1, \ldots, \bar{x}_M]$.

3. Compute the predicted state $x_{t+1|t}$ (53) of the nonlinear system corresponding to initial state $\bar{x}$ and control input $\tilde{u}_t$, and the associated cost function value $J_2 := J(\tilde{u}_t, \bar{x})$ by using (12). Form the vector $Y(t)$ of decision variables.

4. Do

5. $J_1 := J_2$

6. Obtain simultaneously the locally linear prediction models (17)-(18) of the subsystems about the point $(\bar{x}, \tilde{u}_t)$.

7. For $r = 0, 1, \ldots, N_r$ **do**

8. **If** $r = 0$ **then**

9. Initialize iterations (48)-(52) with $Y^{-1} = Y(t)$, $\lambda^0 = \lambda^{-1} = 0$, $\mu^0_{\text{dec}} = \mu^{-1}_{\text{dec}} = 0$, $\mu^0_{\text{contr}} = \mu^{-1}_{\text{contr}} = 0$.

10. **else**

11. Let $Y^{r-1} := Y^r$, $\lambda^{r-1} := \lambda^r$, $\lambda^r := \lambda^{r+1}$, $\mu^{r-1}_{\text{dec}} := \mu^r_{\text{dec}}$, $\mu^r_{\text{dec}} := \mu^{r+1}_{\text{dec}}$, $\mu^{r-1}_{\text{contr}} := \mu^r_{\text{contr}}$, $\mu^r_{\text{contr}} := \mu^{r+1}_{\text{contr}}$.

12. **end**

13. For i-th subsystem, $i = 1, 2, \dots, M$, communicate the dual variables λ^r_j and the decision variables $\bar{Y}^r_j$, $j = 1, 2, \dots, M$, $j \neq i$ of the interconnected subsystems.

14. Run iterations (48)-(52) *distributedly* and obtain Y^r, λ^{r+1}, μ^{r+1}_{dec}, μ^{r+1}_{contr}. Extract u^r_t from Y^r.

15. **end**

16. Let $\tilde{u}_t = u_t^{N_r}$.

17. Compute the predicted state $x_{t+1|t}$ (53) of the nonlinear system corresponding to initial state $\bar{x}$ and control input $\tilde{u}_t$ and the cost function value $J_2 := J(\tilde{u}_t, \bar{x})$ by using (12). Form the vector $Y(t)$ of decision variables.

18. **while** $|J_2 - J_1| / J_1 > \varepsilon$

19. Apply to the overall system the control input $u(t) = \tilde{u}_t$.

20. Let $t = t + 1$ and go to step 2.

It would be necessary to perform an offline study of the performance of the Algorithm 1 with different values of the parameters δ, ε and N_r in order to ensure that the computed suboptimal NMPC in closed-loop with the nonlinear system (3) will lead to feasibility, stability and desired performance. It also should be noted that the suggested approach can be easily modified so to relax the independent constraints on states and inputs (4) to linear constraints in the extended input-state space.

4. EXAMPLE

4.1 System description

As an example, the quadruple-tank system in Johansson (2000) is considered, which is schematically shown in Fig. 1.

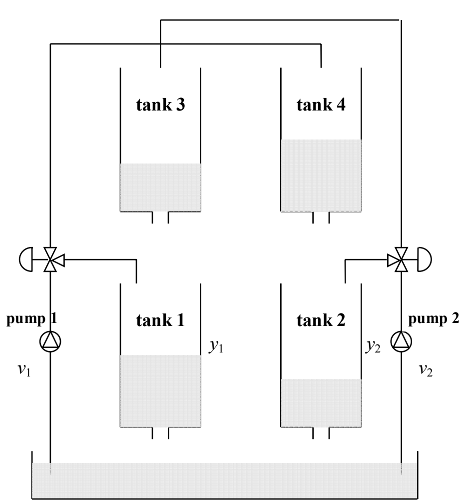

Fig. 1. Quadruple-tank system (Johansson (2000)).

The objective is to control the level in the lower two tanks with two pumps. The control inputs are v_1 and v_2 (input voltages to the pumps) and the outputs are y_1 and y_2 (voltages from level measurement devices). The first-principles model of the system is (Johansson (2000)):

$$\dot{h}_1 = -\frac{a_1}{A_1}\sqrt{2gh_1} + \frac{a_3}{A_1}\sqrt{2gh_3} + \frac{\gamma_1 k_1}{A_1} v_1 \quad (55)$$

$$\dot{h}_2 = -\frac{a_2}{A_2}\sqrt{2gh_2} + \frac{a_4}{A_2}\sqrt{2gh_4} + \frac{\gamma_2 k_2}{A_2} v_2 \quad (56)$$

$$\dot{h}_3 = -\frac{a_3}{A_3}\sqrt{2gh_3} + \frac{(1-\gamma_2)k_2}{A_3} v_2 \quad (57)$$

$$\dot{h}_4 = -\frac{a_4}{A_4}\sqrt{2gh_4} + \frac{(1-\gamma_1)k_1}{A_4} v_1 \quad (58)$$

In (55)-(58), A_i is the cross-sectional area of tank i, a_i is the cross-sectional area of the outlet hole of tank i, h_i is the water level in tank i (Johansson (2000)). The voltage applied to pump i is v_i and the corresponding flow is $k_i v_i$. The parameters γ_1, $\gamma_2 \in (0, 1)$ are determined from the positions of the two valves. In the simulation experiments, it is chosen that $\gamma_1 = 0.7$ $\gamma_2 = 0.6$, which lead to a minimum-phase behavior of the plant (Johansson (2000)). The flow to tank 1 is $\gamma_1 k_1 v_1$ and the flow to tank 4 is $(1-\gamma_1)k_1 v_1$. The flows to tanks 2 and 3 are $\gamma_2 k_2 v_2$ and $(1-\gamma_2)k_2 v_2$, respectively. The acceleration of gravity is denoted g. The measured level signals are $y_1 = k_c h_1$ and $y_2 = k_c h_2$, where k_c is a constant. The parameter values of the quadruple-tank system are given in (Johansson (2000)). The control objective is to keep the water levels h_1 and h_2 at the set-points:

$$h_1^* = 12.4 \text{ cm}, \; h_2^* = 12.7 \text{ cm} \quad (59)$$

The steady-state values of h_3, h_4, v_1, v_2, corresponding to these set-points are:

$$h_3^* = 1.6 \text{ cm}, \; h_4^* = 1.45 \text{ cm}, \; v_1^* = 3.04 \text{ V}, \; v_2^* = 2.97 \text{ V} \quad (60)$$

The following variables are introduced:

$$x_{1,1} = h_1 - h_1^*, \; x_{1,2} = h_4 - h_4^*, \; x_{2,1} = h_2 - h_2^*, \; x_{2,2} = h_3 - h_3^* \quad (61)$$

$$u_i = v_i - v_i^*, \; i = 1, 2 \quad (62)$$

Then, the quadruple-tank system can be considered as consisting of two interconnected sub-systems, which are described by:

<u>Subsystem S_1:</u>

$$\dot{x}_{1,1} = -(a_1/A_1)\sqrt{2g(x_{1,1} + h_1^*)} + (a_3/A_1)\sqrt{2g(x_{2,2} + h_3^*)} + (\gamma_1 k_1/A_1)(u_1 + v_1^*) \quad (63)$$

$$\dot{x}_{1,2} = -(a_4/A_4)\sqrt{2g(x_{1,2} + h_4^*)} + [(1-\gamma_1)k_1/A_4](u_1 + v_1^*) \quad (64)$$

<u>Subsystem S_2:</u>

$$\dot{x}_{2,1} = -(a_2/A_2)\sqrt{2g(x_{2,1} + h_2^*)} + (a_4/A_2)\sqrt{2g(x_{1,2} + h_4^*)} + (\gamma_2 k_2/A_2)(u_2 + v_2^*) \quad (65)$$

$$\dot{x}_{2,2} = -(a_3/A_3)\sqrt{2g(x_{2,2} + h_3^*)} + [(1-\gamma_2)k_2/A_3](u_2 + v_2^*) \quad (66)$$

The subsystem S_1 influences the dynamics of the subsystem S_2 with the expression $(a_4/A_2)\sqrt{2g(x_{1,2} + h_4^*)}$, while the subsystem S_2 influences the dynamics of the subsystem S_1 with the expression $(a_3/A_1)\sqrt{2g(x_{2,2}+h_3^*)}$.

4.2 Simulation results

The performance of the proposed distributed NMPC approach and algorithm is studied by simulations for the quadruple-tank system described above. The ordinary differential equations (63)-(66) are discretized with sampling time of 1 s by applying the Euler's method with step 0.1 s. The constraints imposed on the system (55)-(58) are:

$$0 \le v_i(t) \le 6 \text{ V},\ i=1,2 \qquad (67)$$

$$0 \le h_i(t) \le 20 \text{ cm},\ i=1,2\ ,\ 0 \le h_i(t) \le 3 \text{ cm},\ i=3,4 \qquad (68)$$

which by taking into account (59)-(62) become:

$$-3.04 \le u_1(t) \le 2.96 \text{ V}\ ,\ -2.97 \le u_2(t) \le 3.03 \text{ V} \qquad (69)$$

$$-12.4 \le x_{1,1}(t) \le 7.6 \text{ cm}\ ,\ -1.45 \le x_{1,2}(t) \le 1.55 \text{ cm} \qquad (70)$$

$$-12.7 \le x_{2,1}(t) \le 7.3 \text{ cm}\ ,\ -1.60 \le x_{2,2}(t) \le 1.40 \text{ cm} \qquad (71)$$

The following two cases of weighting matrices in the centralized NMPC problem P1 are considered:

a) $Q_1 = Q_2 = \text{diag}(50,\ 1)$, $R_1 = R_2 = 0.1$;

b) $Q_1 = Q_2 = \text{diag}(5,\ 1)$, $R_1 = R_2 = 0.1$.

In both cases, the Algorithm 1 is used to generate the control inputs for the following initial states of the subsystems:

$$[x_{1,1}(0)\ x_{1,2}(0)\ x_{2,1}(0)\ x_{2,2}(0)] = [-4.4\ -1.35\ -4.7\ -1.5] \qquad (72)$$

The trajectories obtained with the distributed contractive NMPC with horizon $N=1$ are compared to those corresponding to the distributed non-contractive NMPC approach (Grancharova et al. (2016)) with horizon $N=15$. The transients in Fig. 2 – Fig. 5 correspond to the weighting matrices in case a), while those in Fig. 6 – Fig. 9 are related to case b). Both distributed approaches use the same values of the parameters in the algorithms.

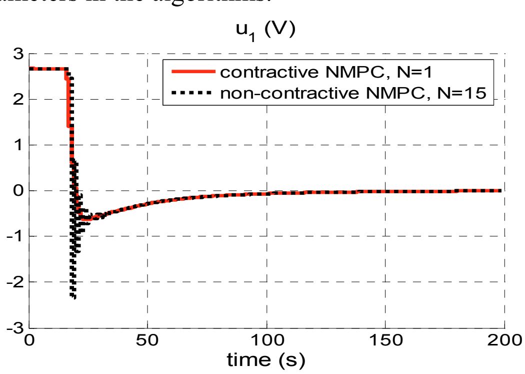

Fig. 2. The control input for subsystem S_1 - case a).

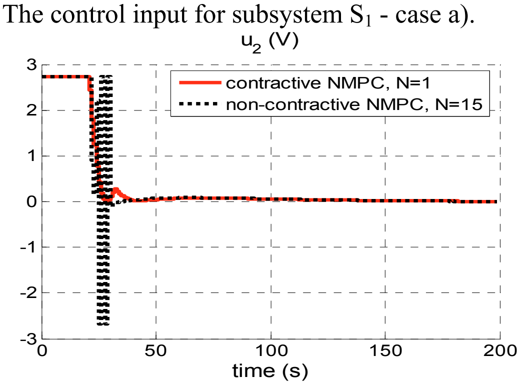

Fig. 3. The control input for subsystem S_2 - case a).

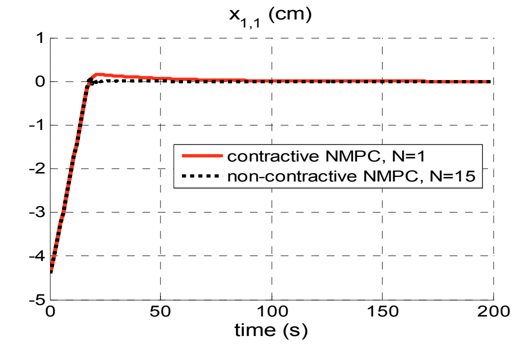

Fig. 4. The state $x_{1,1} = h_1 - h_1^*$ of subsystem S_1 - case a).

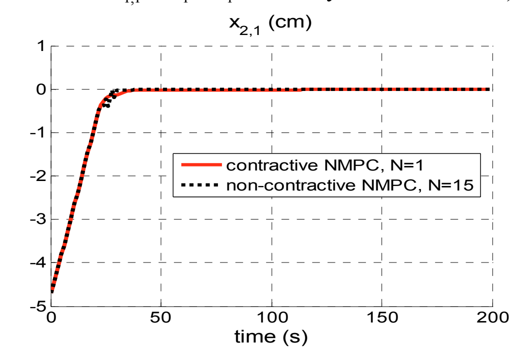

Fig. 5. The state $x_{2,1} = h_2 - h_2^*$ of subsystem S_2 - case a).

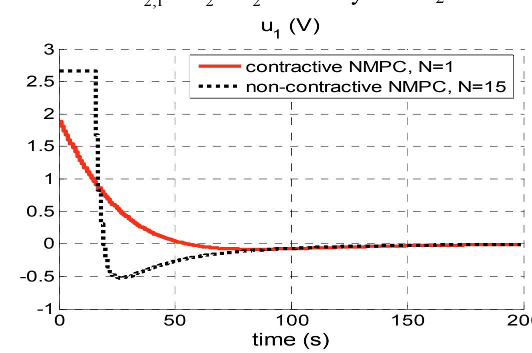

Fig. 6. The control input for subsystem S_1 - case b).

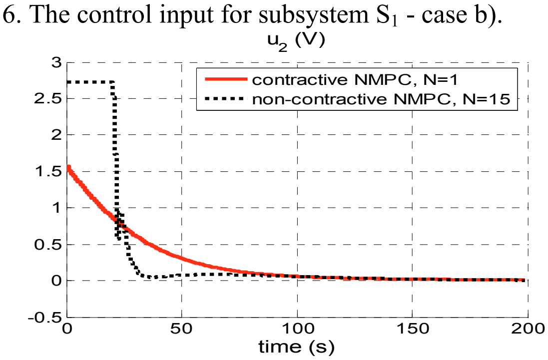

Fig. 7. The control input for subsystem S_2 - case b).

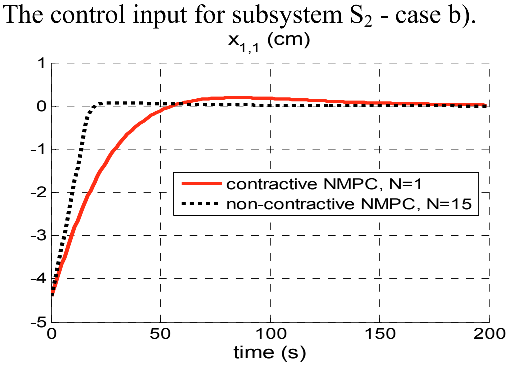

Fig. 8. The state $x_{1,1} = h_1 - h_1^*$ of subsystem S_1 - case b).

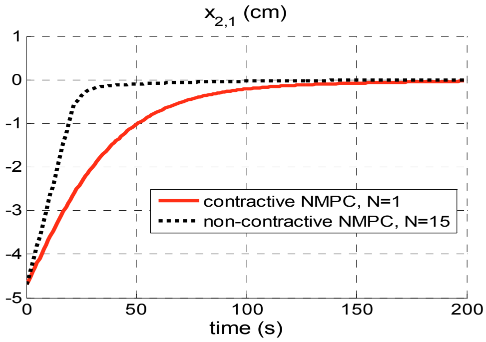

Fig. 9. The state $x_{2,1} = h_2 - h_2^*$ of subsystem S_2 - case b).

It can be seen from Fig. 2 – Fig. 9 that both distributed NMPC approaches lead to feasible trajectories, but different state weighting matrices should be chosen in order to obtain similar quality of performance in terms of regulation time and smoothness of the control inputs trajectories. A good control quality with the contractive NMPC with $N = 1$ is achieved for $Q_1 = Q_2 = \text{diag}(50, 1)$, while the non-contractive NMPC with $N = 15$ has good performance for $Q_1 = Q_2 = \text{diag}(10, 1)$. However, the significant advantage of the suggested contractive distributed NMPC approach is the small size of the optimization problems solved by the subsystems, as it is shown in Table 1. This would allow an efficient online implementation with less memory requirements.

Table 1. Comparison of QP sub-problems dimensions.

Distributed quasi-NMPC approaches	Dimension of the QP for each subsystem
with contractive constraint, $N = 1$	3 decision variables, 6 inequality constraints, 2 equality constraints 1 contractive constraint
without contractive constraint, $N = 15$	45 decision variables, 90 inequality constraints, 30 equality constraints

5. CONCLUSION

An approach to low complexity distributed MPC of nonlinear interconnected systems is proposed by including a contractive constraint in the MPC problem formulation and using one step ahead prediction. The advantages of the suggested approach in comparison to the distributed NMPC without contractive constraint and large horizon are that it is computationally less expensive and the dimension of the optimization sub-problem solved by each subsystem is much smaller. For these reasons, the suggested approach would be appropriate for *distributed* NMPC of cyber-physical systems.

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