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# Shaping Stable Oscillation of a Pendulum on a Cart around the Horizontal ⋆

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**Abstract:** Planning and stabilizing induced oscillations for underactuated mechanical systems are challenging tasks. Available analytical solutions are primary linked to formats of representation of feasible trajectories and can give a rather limited perception of a variety of possibilities for particular systems. The paper provides new insights to the tasks exploring the classical and popular robotic benchmark set-up. In particular, the case study illustrates the procedure for generating a periodic behaviour of a pendulum on a cart, when the pendulum oscillates around the horizontal. Planning such a behaviour requires novel arguments for establishing a presence of a forced cycle. Furthermore, if found, the orbital stabilization of the cycle requires an alternative set of transverse coordinates. Both assignments are successfully solved. The analytical contributions are discussed and supported by numerical simulations.



*Keywords:* underactuated mechanical system, trajectory planning, orbital stabilization, transverse dynamics, transverse linearization, cart-pendulum system

## 1. INTRODUCTION

Most of methods developed for searching feasible behaviours of underactuated systems rely on structural assumptions, which can potentially restrict sets of trajectories that such methods allow to recover. Comprehensive understanding of limitations of different methods due to artificially introduced properties is, therefore, important and motivated. To this end, we can question and explore one of widely used representations of a trajectory of $n$-DoF mechanical system $q(t) = [q_1(t); \ldots; q_n(t)]$, $t \in [0, T]$, where the time evolution of one of degrees of freedom, let say $q_n(\cdot)$, can reproduce the behaviour through geometrical parametrization (virtual holonomic constraints - VHC)

$$q_1(t) = \phi_1(q_n(t)), \; \ldots, \; q_{n-1}(t) = \phi_{n-1}(q_n(t)), \quad t \in [0, T].$$

The format is obviously not universal and assumes structural properties. For instance, if the behaviour is periodic $q(t) = q(t + T)$ and the associated VHC-representation is found, then the motion generator – the scalar variable $q_n(\cdot)$ – cannot be monotonic. Furthermore, if the functions $\phi_i(\cdot)$ are smooth, then for those time moments, where velocities $\dot{q}(\cdot)$ along the behaviour are well defined, and the following relations

$$\dot{q}_i(t) = \tfrac{d}{dq_n}\phi_i(q_n(t))\dot{q}_n(t), \quad i = 1, \ldots, n-1.$$

hold. Meanwhile, since the behaviour of $q_n(\cdot)$ for periodic trajectory is not monotonic, therefore its velocity should become zero at some time moments, $\dot{q}_n(t_j) = 0$. In turn, the previous formula and smoothness of $\phi_i(\cdot)$ in the geometric representation imply that for this behaviour the velocities of all degrees of freedom at these time instants $t_j$ should be zeros as well. Clearly such synchronization property is rather demanding and might not necessary hold for some behaviours. This limitation can be partly mitigated by the following argument: this and other abnormal features of trajectories and the possibility for their smooth geometric (VHC) representations come from the fact that searching trajectories for different tasks is done in generic set of coordinates chosen in advance. Instead, the analysis of an individual trajectory might require an alternative set of variables and case by case study. Furthermore, one can question the necessity in searching trajectories that require one motion generator and one VHC-parametrization of the behaviour for the whole time interval $[0, T]$ without its decomposition and sequential parametrization on subintervals *etc.*

The contribution of the paper complements the discussion with an attempt to illustrate another limitation of the VHC-parametrization. As argued below, it appears in solving the planning and stabilization tasks for the pendulum on a cart system if an engineer is searching for an induced periodic behaviour of the system with oscillation of the pendulum around the horizontal. The VHC-approach successfully re-used for planning induced cycles for a variety of the case studies, see Shiriaev (2005); Freidovich (2008); Mettin (2008); Shiriaev (2010); Surov (2015); Grizzle (2015), relied on a qualitative analysis of a specific second order system (the reduced dynamics) and the oscillations were in part found as a consequence of Poincare-Bendixon statement. They encircled one or several equilibriums of the reduced dynamics. If a smooth VHC parametrization for any of induced oscillations was found, then these stationary points of reduced dynamics become equilibriums of the full (open loop) dynamics as well.

However, such conclusion for the induced oscillation of the pendulum around the horizontal for the underactuated

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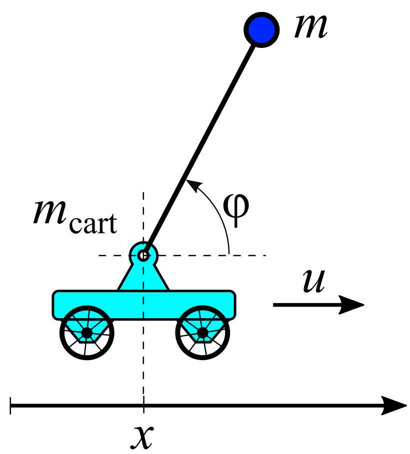


Fig. 1. Cart-pendulum system

cart-pendulum system contradicts the physics of the open loop dynamics. Indeed, if valid, it requires the cart to accelerate for keeping any constant inclination of the pendulum from the vertical. Therefore, the periodicity in the coordinate of the cart is impossible both for the constant inclination and for nearby oscillations of the pendulum. The lack of the well defined smooth VHC parametrization of a cycle for the case study has a number of consequences. Besides new steps in trajectory planning, it demands new arguments and analytical constructions for defining transverse coordinates necessary for orbital stabilization of the cycle. The main contributions of the paper suggest and exemplify new methods for planning the requested oscillation of the cart-pendulum system and for introducing the complete set of transverse coordinates for each of the found cycles.

The paper is organized as follows. Section 2 includes problem statement and the detailed analysis of the challenges which arise in solving the problem through the search of geometrical parametrization. New method for searching a requested periodic trajectory is discussed in Section 3. Section 4 describes steps for defining transverse coordinates and their use in synthesis of a stabilizing controller. The simulation results are presented in Section 5. Concluding remarks are given in Section 6.

## 2. PROBLEM FORMULATION

Dynamics of the cart-pendulum system is described by the Lagrange's equations (see, for example, Shiriaev (2005)):

$$\begin{aligned} a\ddot{x} - \sin\varphi\ddot{\varphi} - \cos\varphi\dot{\varphi}^2 &= u \\ -\sin\varphi\ddot{x} + l\ddot{\varphi} + g\cos\varphi &= 0, \end{aligned} \tag{1}$$

where $q = (x, \varphi)^T$, $x$ – cart horizontal position, $\varphi$ – angular coordinate of the pendulum (see Fig. 1), $m$ – mass of pendulum, $m_{\text{cart}}$ – mass of cart, $l$ – length of pendulum, $g$ – gravity acceleration, $a = (m + m_{\text{cart}})/ml$ – a constant, $f = uml$ – external force acting on the cart. Notice, that for real system $al > 1$, thus the dynamics is not singular. The topological properties of phase space are not important for us, because of we search trajectories in a small area. For that reason, we assume $\varphi \in \mathbb{R}$, and the configuration space is just $\mathbb{R}^2$. Using these notation, we can formulate the problem of trajectory planning as follows:

*Problem 1.* Does there exist a periodic solution $x_\star(t)$, $\varphi_\star(t)$, $u_\star(t)$ of system (1), such that $x_\star(t) \in C^2(\mathbb{R})$, $\varphi_\star(t) \in C^2(\mathbb{R})$, $u_\star(t) \in C^0(\mathbb{R})$, and $0 \in \varphi_\star(\mathbb{R}) \subseteq (-\frac{\pi}{2}, \frac{\pi}{2})$?

### 2.1 Virtual Holonomic Constraint Approach

*Search of the periodic trajectories.* Let us try to find a requested trajectory of the cart-pendulum system (1) using the VHC method. At first, we assume there are three function $x_\star(t)$, $\varphi_\star(t)$, $u_\star(t)$ satisfying the conditions of Problem 1, and there exists a smooth function $X \in C^2(\mathbb{R})$ (virtual constraint) satisfying

$$x_\star(t) = X(\varphi_\star(t)) \quad \forall\, t \in \mathbb{R}.$$

Substituting $x = X(\varphi)$ into (1) we obtain 2-nd order differential equation:

$$\alpha(\varphi)\ddot{\varphi} + \beta(\varphi)\dot{\varphi}^2 + \gamma(\varphi) = 0, \tag{2}$$

where $\alpha(\varphi) = l - \frac{dX}{d\phi}\sin\varphi$, $\beta(\varphi) = -\frac{d^2X}{d\phi^2}\sin\varphi$, $\gamma(\varphi) = g\cos\varphi$. Due to used notation, the equation (2) is often referred as $\alpha\beta\gamma$-equation. For any smooth $X(\varphi)$, such that

$$\alpha(\varphi) \neq 0 \tag{3}$$

holds on some interval $(\varphi_1, \varphi_2)$, equation (2) has only 2 different equilibrium points: $\varphi_0 = \pm\frac{\pi}{2}$. According to the Theorem 33 in Andronov (1966) a periodic trajectory must envelop at least one equilibrium point, thus the theorem forbids existence of periodic trajectories defined on interval $(-\pi/2, \pi/2)$ . So, if a periodic solution $\varphi_\star(t)$ of (2) exists, then

$$\alpha(\varphi_\star(t)) = 0 \tag{4}$$

for some $t$. Briefly, the VHC leads to a singular $\alpha\beta\gamma$ - equation (2).

*Transverse Linearization.* According to Shiriaev (2005) the linearised transverse dynamics (see equation (41) there) has the following form:

$$\dot{\zeta} = A(t)\zeta + b(t)v, \tag{5}$$

where $\zeta$ is the set of transverse coordinates, and $A : \mathbb{R} \to \mathbb{R}^{3\times3}$, $b : \mathbb{R} \to \mathbb{R}^3$ are matrix functions. Some components of matrix $A$ have $\alpha(\varphi_\star(t))$ in their denominators. As was shown, if system (2) has periodic trajectory, then the coefficient $\alpha(\varphi_\star(t))$ vanishes at some $t$. This leads to singularities in $A(t)$, and therefore the transverse coordinates $\zeta$ cannot be used to describe dynamics of the system (1) in a neighborhood of desired trajectory.

As show next, the problem appears due to a specific set of transverse coordinates used in the analysis, and there is another set of transverse coordinates, which are well-defined in a tubular neighborhood of a desired trajectory.

## 3. SEARCH OF THE PERIODIC TRAJECTORIES

Rewrite the system (1) in the explicit form:

$$\ddot{x} = \frac{l\cos\varphi\dot{\varphi}^2 - g\sin\varphi\cos\varphi + lu}{al - \sin^2\varphi} \tag{6}$$

$$\ddot{\varphi} = \frac{\sin\varphi\cos\varphi\dot{\varphi}^2 - ag\cos\varphi + \sin\varphi u}{al - \sin^2\varphi}. \tag{7}$$

If we choose the control as $u = u(\varphi)$, then the equation (7) is of well-studied type

$$\ddot{\varphi} = -b(\varphi)\dot{\varphi}^2 - c(\varphi). \tag{8}$$

The following lemma formulates the conditions under which the equation (8) has a set of periodic solutions.

*Lemma 2.* Assume that there is a pair of real numbers $\varphi_1, \varphi_2$ ($\varphi_1 < \varphi_2$), such that functions $b, c$ are twice continuously differentiable on interval $(\varphi_1, \varphi_2)$ and satisfy the following conditions:

- there exists a $\varphi_e \in (\varphi_1, \varphi_2)$: $c(\varphi_e) = 0$,
$$c(\varphi) < 0 \quad \forall \varphi \in [\varphi_1, \varphi_e)$$
$$c(\varphi) > 0 \quad \forall \varphi \in (\varphi_e, \varphi_2]$$
- the function
$$\mu(\varphi) = \int_{\varphi_e}^{\varphi} c(\eta)\, e^{2\int_{\varphi_e}^{\eta} b(\xi)d\xi} d\eta$$
has the same values at endpoints:
$$\mu(\varphi_1) = \mu(\varphi_2).$$

Then for any $\varphi_s \in (\varphi_1, \varphi_e)$ solution of the initial value problem (8) with $\varphi(0) = \varphi_s, \dot{\varphi}(0) = 0$ is a periodic function.

The number $\varphi_s$ "enumerates" periodic solutions. All of them encircle the only one equilibrium point $\varphi_e$ of type "centre". It can be shown that the control input of the form

$$u(\varphi) = 2k \sin\varphi, \tag{9}$$

with large enough $k \in \mathbb{R}$ ensures that the conditions of Lemma 2 hold on some interval $(\varphi_1, \varphi_2)$. The corresponding closed loop system (8, 9) has equilibrium $\varphi_e$ satisfying

$$\sin^2\varphi_e / \cos\varphi_e = \frac{ag}{2k}. \tag{10}$$

The functions

$$b(\varphi) = -\frac{\sin\varphi\cos\varphi}{al - \sin^2\varphi}$$
$$c(\varphi) = \frac{ag\cos\varphi - 2k\sin^2\varphi}{al - \sin^2\varphi}$$
$$\mu(\varphi) = ag\frac{\sin\varphi_e - \sin\varphi}{\sin^2\varphi_e - al} + k\frac{\varphi - \varphi_e - \sin\varphi\cos\varphi + \sin\varphi_e\cos\varphi_e}{\sin^2\varphi_e - al}$$

are continuous everywhere. Thus, the area between $\varphi_1, \varphi_e$ is filled by periodic trajectories. The values $\varphi_1, \varphi_2, \varphi_e$ can be estimated numerically (we will show this in Section 5).

Each value $\varphi_s$ from the interval $(\varphi_1, \varphi_e)$ generates a periodic solution $\varphi_\star = \varphi_\star(t, \varphi_s)$ with the corresponding control input $u_\star = u_\star(t, \varphi_s) = 2k\sin\varphi_\star$. Each trajectory $\varphi_\star$ generates a set of solutions $x_\star(t, \varphi_s, \dot{x}_0, x_0)$ of equation (6). These solutions depend on parameter $\varphi_s$, and initial values $x_0$, $\dot{x}_0$ as follows:

$$\dot{x}_\star(t, \varphi_s, \dot{x}_0) = \dot{x}_0 + \int_0^t \frac{l\cos\varphi_\star\dot{\varphi}_\star^2 - g\sin\varphi_\star\cos\varphi_\star + lu_\star}{al - \sin^2\varphi_\star} d\tau$$
$$x_\star(t, \varphi_s, \dot{x}_0, x_0) = x_0 + \int_0^t \dot{x}_\star(t, \varphi_s, \dot{x}_0)\, d\tau.$$

The last step is to show that for some $\varphi_s$ and $x_0, \dot{x}_0$ the function $x_\star(t, \varphi_s, \dot{x}_0, x_0)$ is periodic. Then, the functions $\varphi_\star(t, \varphi_s)$, $x_\star(t, \varphi_s, \dot{x}_0, x_0)$, $u_\star(t, \varphi_s)$ be a solution of the Problem 1.

*Theorem 3.* Let $T(\varphi_s)$ be the period of $\varphi_\star(t, \varphi_s)$. If there exists a $\varphi_s \in (\varphi_1, \varphi_e)$ such that

$$F(\varphi_s) \overset{\text{def}}{=} \int_0^{T(\varphi_s)} \frac{l\cos\varphi_\star\dot{\varphi}_\star^2 - g\sin\varphi_\star\cos\varphi_\star + lu_\star}{al - \sin^2\varphi_\star} d\tau = 0 \tag{11}$$

then $(x_\star(t, \varphi_s, \dot{x}_0, x_0), \varphi_\star(t, \varphi_s))$ with $x_0 \in \mathbb{R}$ and

$$\dot{x}_0 = \frac{-1}{T(\varphi_s)} \int_0^{T(\varphi_s)} \int_0^{\xi} \frac{l\cos\varphi_\star\dot{\varphi}_\star^2 - g\sin\varphi_\star\cos\varphi_\star + lu_\star}{al - \sin^2\varphi_\star} d\tau d\xi$$

is a periodic solution of (6,7,9).

Since function $T(\varphi_s)$ is continuous (see Lemma 13 in Andronov (1966)), the function $F$ is also continuous. So, to check the condition of Theorem 3 it is sufficient to find $\varphi_a, \varphi_b \in (\varphi_1, \varphi_e)$, such that $F$ has different signs at these points. After that the value $\varphi_s$ can be found by solving numerically equation (11), the equation (12) gives the initial values $x_0, \dot{x}_0$.

## 4. ORBITAL STABILIZATION

This section presents an approach for orbital stabilization of a found periodic trajectory. As shown, the control input (9) generates a periodic trajectory. Consider the control input of the form

$$u(\varphi) = 2k\sin\varphi + v.$$

with an additional (stabilizing) term $v$. In this way, the feedback system looks like

$$\dot{z} = f(z) + g(z)v \tag{12}$$

where

$$f(z) = \begin{pmatrix} \dot{x} \\ \dfrac{l\cos\varphi\dot{\varphi}^2 - g\sin\varphi\cos\varphi + 2lk\sin\varphi}{al - \sin^2\varphi} \\ \dot{\varphi} \\ \dfrac{\sin\varphi\cos\varphi\dot{\varphi}^2 - ag\cos\varphi + 2k\sin^2\varphi}{al - \sin^2\varphi} \end{pmatrix},$$
$$g(z) = \frac{1}{al - \sin^2\varphi}(0,\, l,\, 0,\, \sin\varphi)^T \tag{13}$$

$z = (x, \dot{x}, \varphi, \dot{\varphi})^T$ is the state vector, $f, g$ are smooth functions.

Let $z_\star$ be a closed trajectory of the system (6,7,9). As can be seen, the function $z_\star$ satisfies

$$\dot{z}_\star(t) = f(z_\star(t)) \quad \forall t \in \mathbb{R}. \tag{14}$$

The goal is to find $v$ as a function of $z$

$$v = v(z),$$

which makes trajectory $z_\star$ orbitally stable.

Our solution of the stabilization problem is a variation of transverse linearization approach Banaszuk (1995).

The function $z_\star$ can be considered as a parametric representation of a curve in phase space. All the points lying in a neighbourhood of the curve can be located using a moving along the curve affine frame. This frame at each point consists of tangent vector of the curve and its orthogonal complement. Basis vectors of the orthogonal complement define a hyperplane that acts as a Poincaré section.

The moving affine frame at point $z_\star(t)$ consists of 4 vectors: the normalized tangent vector $\bar{f}_\star(t) = \frac{f(z_\star(t))}{\|f(z_\star(t))\|}$ and 3 other vectors orthogonal to $\bar{f}_\star(t)$ and to each other. The standard approach of constructing of the frame is the Frenet-Serret formulas Manfredo P. do Carmo (1995). But it has some disadvantages which make it difficult to apply in practice. Firstly, this approach leads to singularities if curvature of the trajectory vanishes at some points. Secondly, the basis vectors are expressed through high-order derivatives of function $f$, which usually cannot be estimated well due to parameters uncertainty. Below we offer different approach for constructing of orthogonal complement.

Let us define vector fields

$$e_1 = P_1 \bar{f}_\star,\ e_2 = P_2 \bar{f}_\star,\ e_3 = P_3 \bar{f}_\star, \tag{15}$$

where $P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, P_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, $P_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$. Direct calculations show that vectors $\bar{f}_\star, e_1, e_2, e_3$ are orthonormal for any non-zero $\bar{f}_\star$. Due to the facts that the trajectory $z_\star$ does not contain stationary points (otherwise it cannot be periodic), and the differential equation (14) defines a smooth vector field, the affine frame $(\bar{f}_\star, e_1, e_2, e_3)$ is smooth and orthonormal for all $t$. The Poincaré-section hyperplane is spanned on the vectors $e_1, e_2, e_3$.

*Remark 4.* An analogous method for the constructing of orthonormal system of vectors exists also for $\mathbb{R}^2$ and $\mathbb{R}^8$. This allows to apply the same approach for the constructing of Poincaré sections for dynamical system that can be embedded into $\mathbb{R}^8$. According to Adams (1962), the exact upper bound of independent vector fields on sphere $S^{n-1}$ is less than $n$ for any $n > 8$. It means that the proposed method cannot be used when $n > 8$.

Any state vector $z$ lying in a small enough neighbourhood of curve $z_\star$ can be decomposed into:

$$z = z_\star(\tau) + e_1(\tau)\xi_1 + e_2(\tau)\xi_2 + e_3(\tau)\xi_3$$

where $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$ are components of transverse vector $\xi$, and $\tau$ defines the closest curve's point:

$$\tau = \arg\min_{t\in[0,T)} \|z, z_\star(t)\|$$

(in a small enough neighbourhood of $z_\star$, this equation has a unique solution). Use for short:

$$Q = (e_1, e_2, e_3) = \frac{1}{\|f_\star\|}(P_1 f_\star, P_2 f_\star, P_3 f_\star), \\ f_\star = f(z_\star(\tau)).$$

Then

$$z = z_\star(\tau) + Q\xi \tag{16}$$

and

$$\dot{z} = \frac{dz_\star}{d\tau}\dot{\tau} + \frac{dQ}{d\tau}\dot{\tau}\xi + Q_\tau\dot{\xi}. \tag{17}$$

Multiplying (17) by $Q^T$ and considering that $\frac{dz_\star}{d\tau} = f_\star$, and $Q^T Q = I_{3\times 3}$, we obtain dynamics of the system (12) in coordinates $\xi, \tau$:

$$\dot{\xi} = Q^T f(z) + Q^T g(z)v - Q^T f_\star\dot{\tau} - Q^T\frac{dQ}{d\tau}\dot{\tau}\xi \tag{18}$$

We linearise (18) using approximations

$$\dot{\tau} \approx 1 \\ f(z) \approx f_\star + \frac{\partial f}{\partial z}|_{z_\star} Q\xi$$

The linearised equation takes the form:

$$\frac{d\xi}{d\tau} = A(\tau)\xi + B(\tau)v, \tag{19}$$

where

$$A(\tau) = Q^T\frac{\partial f}{\partial z}|_{z_\star}Q - Q^T\frac{dQ}{d\tau} \\ B(\tau) = Q^T g_\star.$$

The following theorem formulates method of orbital stabilization. Its proof establishes the correctness of linearization (19).

*Theorem 5.* Assume there is a feedback $v$ of the form

$$v(\tau, \xi) = K(\tau)\xi, \tag{20}$$

$K : \mathbb{R} \to \mathbb{R}^{1\times 3}$, which provides exponential stability of trivial solution of system (19). Then, the feedback

$$u(z) = 2k\sin\varphi + v(\tau(z), \xi(z)), \text{ with} \\ \xi(z) = Q^T z - Q^T z_\star(\tau(z)) \\ \tau(z) = \arg\min_{t\in[0,T)} \|z - z_\star(t)\|$$

provides orbital stability of solution $z_\star$ of system (12).

*Remark 6.* The matrix $K(\tau)$ can be found by solving matrix Riccati differential equation with periodic coefficients as it was shown in Shiriaev (2005).

## 5. NUMERICAL EXPERIMENTS

Using the approach presented in Section 3, we found a periodic trajectory of system (6,7,9) with parameters $m = 0.4$, $m_{\text{cart}} = 0.1$, $l = 1.0$, $g = 9.8$, $k = 80.0$.

The functions $c(\varphi), \mu(\varphi)$ of equation (8) are depicted on Figure 2. The numerically evaluated parameters (see Lemma 2) are: $\varphi_1 \approx -0.558411$, $\varphi_2 \approx 0.273904$, $\varphi_e \approx -0.274904$. Calculating $F(\varphi_1) = 23.5913$ and $F(\varphi_1) = 23.5913$, we see that system (6,7,9) has a periodic solution according to Theorem 3. Using Brent's method Brent (1972) we found a root of function $F$: $\varphi_s \approx -0.557983$, and taking $x_0 = 0$ evaluated initial velocity of the cart according to (11): $\dot{x}_0 \approx 2.77290 \cdot 10^{-9}$.

To be sure, that the found initial values $\varphi_s, \dot{x}_0$ are correct, we integrated equations of motion (6,7,9) with

$$\varphi(0) = \varphi_s, x(0) = 0, \dot{x}(0) = \dot{x}_0, \dot{\varphi}(0) = 0.$$

The obtained trajectory depicted on Figure 3. The trajectory is periodic within error margin of numerical integration method.

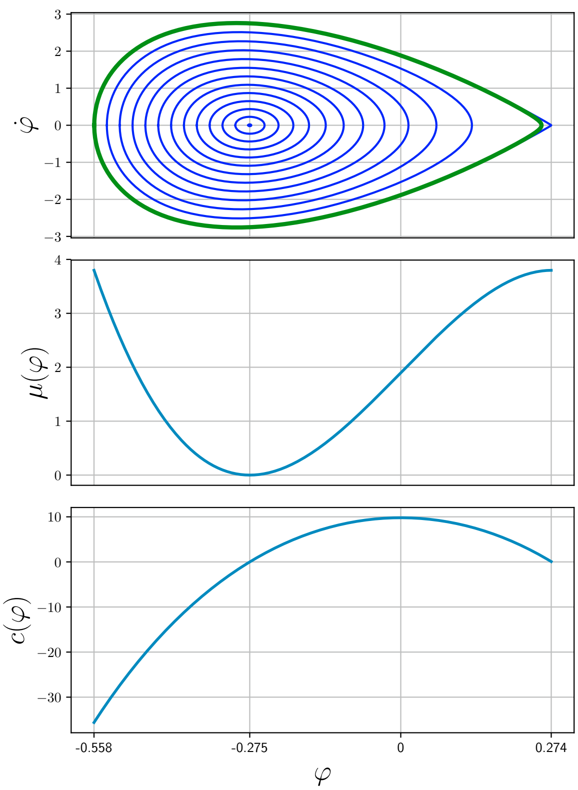


Fig. 2. Phase portrait of equation (8), the green trajectory crosses x-axis at $\varphi_s$; the functions $c(\varphi)$, and $\mu(\varphi)$

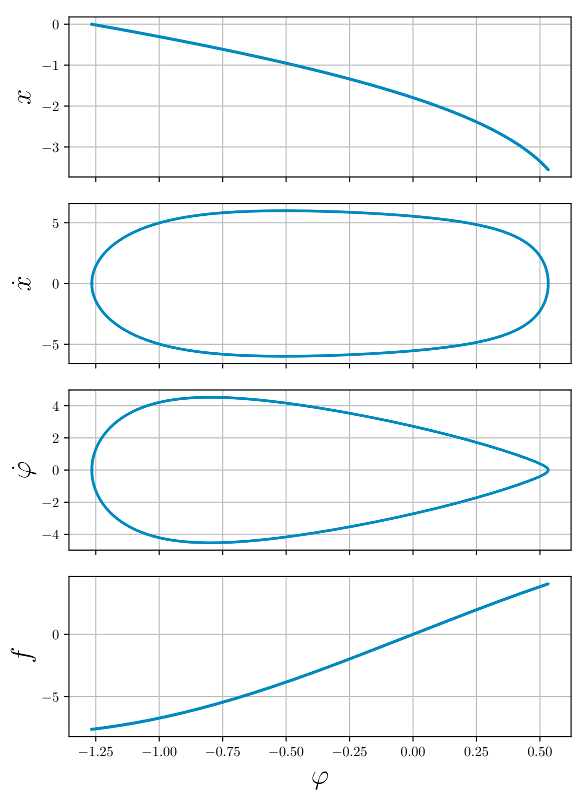


Fig. 3. Desired periodic trajectory (projections onto the different planes)

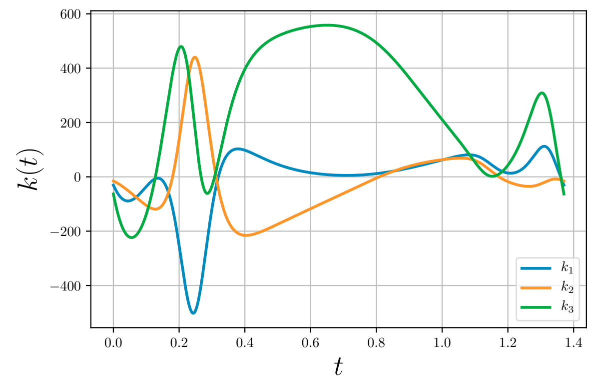


Fig. 4. Coefficients of LQR

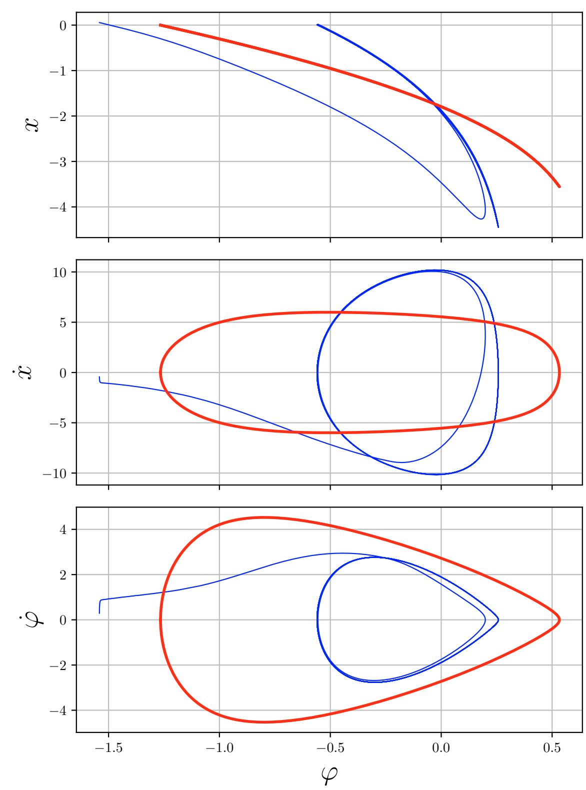


Fig. 5. Evolution of phase coordinates of feedback controlled cart-pendulum

The coefficients $K(\tau)$ of LQR (see equation (20)) where obtained by solving (numerically) a corresponding matrix Riccati differential equation with periodic coefficients. They are depicted on Figure 4.

The results of numerical simulations of feedback controlled system are depicted on Figure 5. As can be seen, the system begins it's motion from the bottom stable equilibrium point (i.e. $\varphi = -\frac{\pi}{2}$), and reaches the desired trajectory very fast. Besides, the control input depicted on Figure 6 is a smooth (and of course, bounded) function of time.

Finally, we compare the trajectory found by our approach, and the corresponding VHC phase portrait. As can be seen on Figure 3, the projection of the phase trajectory on plane $x, \varphi$ is a smooth curve which can be parametrized

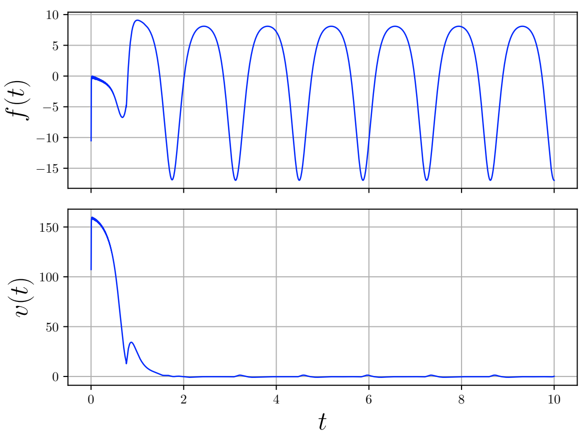


Fig. 6. Control input of the system

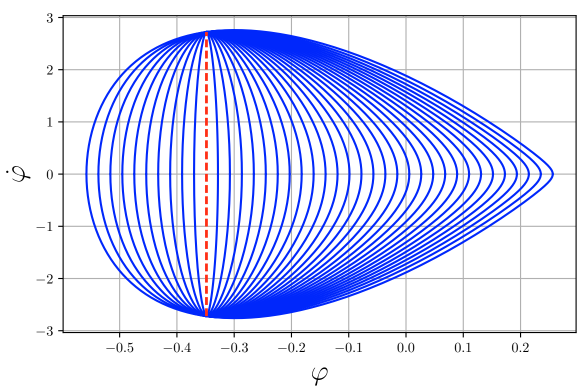


Fig. 7. Phase portrait of equation (2)

by any of generalized coordinates $x, \varphi$. This means, that this curve can be considered as a virtual constraint $X(\varphi)$. We interpolated the curve by a B-spline, and substituted into (2). The phase portrait of the obtained equation is depicted on Figure 7. The dashed red line corresponds to singular point $\varphi_0$ of this equation, i.e. $\alpha(\varphi_0) = 0$. As can be seen, the phase portrait does not contain equilibriums.

The animated results of numerical simulations can be found at `https://youtu.be/NIXzwuEEPwM` and `https://youtu.be/hkMllQsUrP8`.

## 6. DISCUSSION AND CONCLUDING REMARKS

The main results of the paper are the following:

- The method of cart-pendulum periodic motions planning is developed. It can find trajectories that are unreachable by the VHC. Changing the feedback $u(\varphi)$ one can find variety of desired trajectories. One periodic trajectory is investigated in details.
- The new simple and numerically efficient method of transverse coordinates construction is proposed. The method can be used for any smooth non-singular trajectory for systems with up to four generalized coordinates.
- The computer simulation verifies the obtained results and demonstrates their robustness.

The paper presents an example of mechanical system that has unexpected periodic motion and describes the controller stabilizing this motion. The aim of the example is to inspire investigation of problems, where the VHC leads to singular $\alpha\beta\gamma$-equation. The phase portrait of $\alpha\beta\gamma$-equation for considered example is shown on Figure 7. The dashed red line corresponds to the value of coordinate $\varphi$, where coefficient $\alpha$ of equation (2) vanishes. All the phase trajectories are crossing this line in two point. Such a behaviour of trajectories is impossible in case of non-singular equation. This figure illustrates how singularity of $\alpha\beta\gamma$-equation leads to genesis of periodic trajectories. Usually the restrictions on admissible trajectories are applied to avoid singularities in $\alpha\beta\gamma$-equation when the VHC is being used. Our example shows that the singularity of this equation should rather be considered as an opportunity for search of new periodic trajectories, but not the obstacle.

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