The Effect of Seniority on the Pricing and Risk of Corporate Debt

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Preface

I would like to thank my supervisor Snorre Lindset for invaluable guidance, especially concerning the steps in deriving a theoretical model. I would also like to thank my friends and family for much appreciated help and support.

Abstract

This thesis looks at what effect seniority has on corporate debt. I have used a theoretical approach and expanded the model developed by Merton (1974). My analysis is based on the corporate debt being pure discount bonds. By introducing a set of assumptions I have a basis for my valuation of the bonds. I present two propositions, and their proofs, for the pricing of two bonds with different seniority. Through the replicating portfolio method, I am able to derive a pricing equation of a third bond which is subordinated the two other bonds. With the insights from the pricing equations developed for these three bonds, I am able to create a general pricing formula of a bond based on its seniority. This model needs the same input in order to price the bonds, as that needed to price European options in the Black and Scholes (1973) option pricing model.

I find that the appropriate measure of the risk of the bonds is the standard deviation of the daily continuous returns of the bonds. This is due to the bonds being issued by the same company, and not regarded as part of a portfolio. Given my assumptions, subordinated bonds have a higher risk, measured in the standard deviation of daily returns, than any equivalent senior bond. This added risk calls for a higher yield to maturity for the subordinated bond. This result is shown through the presentation of two simulations.

Sammendrag

Denne masteroppgaven ser på effekten av senioritet på prising og risiko på gjelden til bedrifter. Jeg bruker en teoretisk fremgangsmte, og utvider modellen til Merton (1974). Analysen min er basert på at bedrifts gjelden er en ren rabatt obligasjon. Ved å introdusere ett sett med antakelser har jeg ett utgangspunkt for å prise obligasjonene. Jeg presenterer to påstander, og deres tilhrende bevis, for prisingen av to obligasjoner med ulik senioritet. Gjennom en replikerende portefølje metode er jeg i stand til å kunne prise en tredje obligasjon som er underlagt de to andre obligasjonene. Ved å deretter bruke prisings funksjonene fra disse tre obligasjonene finner jeg fram til en generell formel for å prise obligasjoner basert på deres senioritet. Denne modellen bruker de samme variablene som er brukt for å prise en europeisk opsjon i Black og Scholes (1973) sin modell for opsjonsprising.

Jeg finner at det passende målet for risikoen til obligasjonene er standardavviket til den daglige kontinuerlige avkastningen til obligasjonene. Dette er grunnet at obligasjonene er utstedt av samme firma, og inngår ikke som en del av en portefølje. Gitt mine antakelser vil underlagte obligasjoner ha en høyere risiko enn en tilsvarende obligasjon med høyere senioritet. Denne økte risikoen kompenseres for med en høyere effektiv rente. Dette resultatet kommer frem ved bruk av to simuleringer.

Contents

1 Introduction

In this thesis I try to answer how seniority affects the pricing and risk of debt. In order to do this, I use a theoretical approach and expand the structural model proposed by Merton (1974) to allow for the pricing of several bonds with different seniority. Given my assumptions, I find that a higher seniority reduces the risk, and therefore lowers the yield to maturity of a bond.

In the U.S. alone the average daily trading volume of corporate bonds was equal to 30 billion dollars in 2016 (SIFMA Research Department, 2017, p. 37). This is a substantial amount, and shows the need for being able to effectively price corporate bonds. My paper looks at a small, but important part of a bond, which is the seniority. The seniority of a bond is its priority on the firm's assets if the firm that issued the bond declares bankruptcy.

Existing literature on the pricing and risk of corporate debt is substantial. However, the literature which looks exclusively at seniority is somewhat lacking, but there are a few key articles on the subject. Here it is natural to refer to the article on bond indentures written by Black and Cox (1976), which includes a section devoted to subordinated bonds. Subordinated bonds are bonds with a lower seniority. Black and Cox use a safety covenant boundary for the subordinated bonds in accordance with the rest of their article. I have created a model without a safety covenant to better be able to see the isolated effect of seniority.

In this thesis I assume the ownership of the firm is transferred to the debt-holders if the value of the firm's assets is less than the outstanding liabilities due to be paid at that time. Debt with high seniority will have a high priority of ownership on the firm's assets, in the event the firm defaulting on its debt. Some of the assumptions I use, such as the absolute priority rule, may be unrealistic. However, given the limited timeframe to conduct my research I found that using these assumptions allowed me to get a functional model. To be able to show how my model prices bonds, I simulate the price path of the firm and the price of two bonds with different seniority.

Traditional measures of risk for debt is the probability of default and the recovery rate. These are implicitly included in the pricing formulas of my model. I also use bonds instead of loans to represent corporate debt. Bonds are publicly traded, in contrast to loans which is private debt. Since bonds are standardized debt contracts, they are liquid, and can be traded before maturity. This means an investor can sell the bond at any time, and it makes sense to use a mark-to-market valuation of the bond. The relevant risk is thus obtained by estimating the standard deviation of the daily return of the bond (Merton, 1974). However, the conclusion that debt with higher seniority has less risk should, given my assumptions, be the same for loans.

This thesis is structured in such a way that section 2 provides a literary review, section 3 introduces my model, section 4 gives an analysis of the pricing and risk of bonds with a foundation in my model, and lastly section 5 concludes my findings.

2 Literary review

Black and Cox (1976) have previously addressed the effect of subordinated debt as a form on indenture on corporate debt. They found among other things that by adding subordinated debt the risk of the senior bond decreases. This decrease in risk in the senior bonds is due to their model including net-worth covenants. The net-worth covenants gives bondholders the option of declaring the firm bankrupt, if the value of firm's assets drops below a predetermined level before the bonds reach maturity. By declaring bankruptcy when the value of the firm reaches a certain predetermined level the senior bonds are protected from the value of the firm decreasing further. The increased value of the bond will be equal to the value of a down and in barrier call option on the firm's asset's, with the barrier being the predetermined level at which the bondholders can declare the firm bankrupt (Brockman & Turtle, 2003).

Ho and Singer (1982) argue that the time to maturity of the two bonds and which bond matures first largely effects the risk structure of the bonds. The bond that matures first will most likely get paid in full, and therefore have less risk than an equivalent bond that matures later. A senior bond that matures before any subordinated debt will therefore always be worth more than an equivalent junior bond issued by the same firm. If the junior bond matures first, the situation changes. As long as the payment to the junior bond is sufficiently small it will most likely be paid in full, and it is no longer possible to say with certainty that the junior debt is more risky than senior debt (Ho & Singer, 1982). The maturity of the bonds can have a large effect on the risk structure and thereby the value of bonds. Black and Cox (1976) also found that the value of the junior bond can be an increasing function of the time to maturity if the bonds mature at the same time. If maturity is approaching, and it seems that the junior bond will be worthless at maturity, it is in the best interest of the junior bondholders that all the bonds receive an increased time to maturity.

Warner (1977) defines bankruptcy as a firm being bankrupt at time T if its market value is below the amount due to bondholders at time T. So if the value of the firm's assets at time T is less than the face value of the bonds due, the ownership and value of the firm is transferred to the bondholders. In Warner's article he also discusses the implications of deviations from the absolute priority rule. The absolute priority rule states that senior debt gets paid in full before any subordinated debt or equity is paid. Later Eberhart, Moore and Roenfeldt (1990) conclude that there are in fact substantial deviations from the absolute priority rule. With this in mind it may seem that the assumption of absolute priority is somewhat unrealistic. However seniority is found to have a significant effect on the recovery rate in the event of the firm defaulting (Varma & Cantor, 2004). The recovery rate is the percentage of outstanding debt that is received by debt holders if the the firm defaults. So even though absolute priority may not always be the case, it allows for a way of incorporating seniority into a risk neutral pricing model.

According to Altman, Resti and Sironi (2004), credit pricing models can be divided into three categories: i) first generation structural form models, ii) second generation structural form models, and iii) reduced form models. The first generation structural form models are based on the Merton approach, and use the option pricing framework. Within these first generation structural form models it is the value of the firm's assets that is the foundation of the default process. The firm defaults at maturity if the value of the firm's assets is less than the face value of outstanding debt.

3 The model

In this section I first present the variables and assumptions in my model. I then present four different scenarios and their corresponding payoff for the shareholders, a senior bondholder, and a junior bondholder. My model is a first generation structural form model, since it uses the original Merton (1974) framework, and the default process depends on the value of the firm's assets.

3.1 Variables

By using the framework developed by Merton (1974), I am able to price bonds of different seniority using the face values of the bond and all senior bonds, time until maturity, the priorities of the bond in the event of the firm defaulting, and some characteristics of the firm. The face value of the bonds, time until maturity and the characteristics of the firm are all assumed to be known at the time of issuance.

Knowing the face value and the priority of the bonds I also need to know some characteristics of the firm. These characteristics are the same needed by Black and Scholes (1973) to price European options.

The volatility of the firm is its standard deviation, given by σ, and can be estimated through time series data of the firm's asset value. This is the sum of the idiosyncratic and market risk of the firm. The standard deviation of the firm's assets is assumed to be constant over time.

The risk free interest rate, given by r, is continuous and is assumed constant over the time period. This rate can be found by looking at the interest rate on close to default-free bonds such as U.S Treasury bonds (Berk & DeMarzo, 2014, p.156).

The bonds have the same maturity date which is time T. The time in which the pricing occurs is time 0.

The value of the firm's assets has the notation V_t, where the subscript denotes the time of valuation. I assume that the process for the value of the firm's assets follows a geometric Brownian motion[1] under the principle of risk neutral valuation,

$$V_T = V_0 e^{(r-0.5\sigma^2)T+\sigma\sqrt{T}z}, \tag{3.1}$$

where z is a standard normally distributed random variable. A variable with a standard normal distribution has a mean of 0 and a standard deviation of 1.

I assume a firm with a capital structure consisting of equity owned by shareholder, and debt in the form of pure discount bonds of different seniority. I start with two bonds with face values D_1 and D_2. The subscripts denote the seniority structure, with 1 being the most senior. Bond 1 is senior to bond 2, bond 2 is senior to bond 3 and so forth. P_1 and P_2 are the payoffs, at time T, of respectively bond 1 and 2. B_1 and B_2 are the prices of bond 1 and 2.

3.2 Assumptions

In my model I start with two bonds, one senior and one junior, which are referred to as respectively bond 1 and 2. These bonds are issued at the same date with the same time to maturity, at which point they pay out their face value if the firm does not default. Default happens at maturity if the value of the firm's assets is less than the aggregated value of all liabilities. I use the same definition of bankruptcy as Warner (1977), in that if the firm defaults on its debt the ownership of the firm is transferred to the bondholders.

In my model I have not included net-worth covenants in the form of barrier options. Introducing this complicates the model to such an extent that it is hard to derive anything intuitive about seniority from it. Therefore, I have chosen to not include this.

[1]Further explanation of the geometric Brownian motion can be found in appendix A.

The bonds do not pay any coupons, which means they are pure discount bonds. A pure discount bonds pays out its face value at maturity and is traded at a discount relative to its face value. The lower the price of the bond at issue relative to the promised face value, the higher the yield of the bond. With this framework as a foundation I use the following assumptions for my model:

Assumption 1:
I follow the assumption of absolute priority which states that senior bondholders get paid in full before junior bondholders can state their claim. Shareholders will own the residual value of the firm after all bondholders are paid in full at maturity. With the absolute priority assumption the shareholders will always be subordinated any debt.

Assumption 2:
In my model there are no bankruptcy costs. Once default occurs, the firm can continue to operate with the same management and be owned by the bondholders, or sold for the current market value of the firm. Adding bankruptcy costs means I would need to include taxes to justify the existence of debt in the firm's capital structure. However, if contracts are carefully specified as to minimize any legal costs associated with bankruptcy, the affects of bankruptcy costs need not be that significant (Black & Cox, 1976). With this in mind, I choose to exclude these costs in order to keep my model simple.

Assumption 3:
There is no renegotiation of debt. If the value of the firm's assets is less than the outstanding liabilities due at that time, the shareholders exercise their option of walking away, and bondholders receive ownership on the firm's assets. This means that the firm's ownership is transferred in the event of the firm defaulting on its debt.

Assumption 4:
The firm pays no dividend to bondholders. Dividend can be easily added to the model in accordance with how it has been done previously (Merton, 1974). However,

this will not add to the economic understanding of how the seniority affects the bonds, and I have therefore chosen to leave it out to simplify the model.

Assumption 5:
I assume there is no dilution, which means that all new bonds are subordinated existing bonds.

Assumption 6:
There exists a continuous risk free rate that is know and constant over time.

Assumption 7:
The capital structure of the firm consists only of equity and the pure discount bonds being priced.

Assumption 8:
In order to replicate the risky bond's payoff using options and a risk free asset, I also need to assume no short sale restriction and no transaction costs.

3.3 Payoff structures and scenarios

The firm is financed both through equity, held by shareholders, and debt in the form of pure discount bonds. For the senior bond the payoff structure is

$$P_1 = \min(D_1, V_T) \tag{3.2}$$

where P_1 is the payoff of bond 1, D_1 is the face value of bond 1, and V_T is the value of the firm in time T, while min stands for a minimizing function of the two values. The senior bond pays out its face value at time T if the value of the firm's assets exceeds the senior bond's face value. If the value of the firm's assets in time T is less than the face value of the senior bond, the senior bondholder receives the full value of the firm's assets.

The junior bond's payoff is given by

$$P_2 = \min(D_2, \max(V_T - D_1, 0)) \tag{3.3}$$

where P_2 is the payoff of bond 2, D_2 is the face value of bond 2. The max is a maximizing function. If the firm does not default on its debt, the junior bond gets paid its face value, D_2, at time T. If the firm defaults, but the value of the firm in time T exceeds D_1, then the junior bond gets the residual value of the firm after the senior bondholder is paid. If the firm defaults, and the value of the firm is less than D_1, then the senior bondholder gets the full value of the firm and the junior bondholder receives nothing.

The payoff for shareholders is given by

$$S_T = \max\left(V_T - \sum_{j=1}^{n} D_j, 0\right) \tag{3.4}$$

where S_T is the payoff at time T for the shareholders, the summation of D_j is the face value of all outstanding bonds. The shareholders payoff at time T is either zero or the residual value of the firm after the face value of all bonds are paid out. The shareholders are subordinated all debt. However, if the value of the firm exceeds the face value of bonds, the shareholders receive all surplus value.

From these payoff functions I can derive four scenarios and their corresponding payoff for both senior and junior bondholders, and the shareholders
Scenario A:

$$(D_1 + D_2) \leq V_T \tag{3.5}$$

In this scenario the value of the firm exceeds the amount due to the bondholders. Both the senior and junior bonds pay out their face value. Shareholders receive any residual value.
Scenario B:

$$D_1 < V_T \leq (D_1 + D_2) \tag{3.6}$$

The value of the firm exceeds the amount due to the senior bondholder, but is less than the total amount due to the bondholders. Senior bond gets paid in full and receives D_1. The junior bondholder gets the remaining value of the firm after the senior bondholder is paid, $V_T - D_1$. The shareholders receive nothing.

Scenario C:

$$V_T \leq D_1 \tag{3.7}$$

The value of the firm is less than the face value of the senior bond. In this scenario the senior bondholder gets the full value of the firm as compensation. The junior bondholder and the shareholders receives nothing.

Scenario AB:

$$D_1 \leq V_T \tag{3.8}$$

The value of the firm exceeds the face value of the senior bond. For the senior bond the payoff will be the same in scenario A and B, adding these together under the subscript AB, meaning that either scenario A or B occurs. This scenario is only relevant for the pricing of the senior bond. Adding this fourth scenario allows for the pricing of the senior bond to be equal to that proposed by Merton (1974).

I assign each scenario with an indicator function given by 1_A, 1_B, 1_C and 1_{AB}. These indicator functions have a value of one if the corresponding scenario, given by the subscript, happens and zero otherwise. The expectations of these functions will be between zero and one, and can be regarded as the probability of the scenario happening. Since I use the risk-neutral pricing principle, these will be the risk neutral probabilities and not the true probabilities.

With the assumptions and payoff functions in place I can now proceed to deriving the pricing equations for the bonds. The price of the bond will equals its discounted expected payoff. Using the risk neutral pricing principle, the correct rate for discounting is the risk free rate.

4 Analysis

4.1 Valuation

In this subsection I present two propositions and their corresponding proof. I then present a method for pricing further subordinated bonds.

4.1.1 The senior bond

Proposition 1 *The pricing of the most senior bond is given by the equation* $B_1 = D_1 e^{-rT} N(d_{1,2}) + V_0 N(-d_{1,1})$

where

$d_{1,1} = \frac{ln\left(\frac{V_0}{D_1}\right) + (r + 0{,}5\sigma^2)T}{\sigma\sqrt{T}}$

$d_{1,2} = \frac{ln\left(\frac{V_0}{D_1}\right) + (r - 0{,}5\sigma^2)T}{\sigma\sqrt{T}}$

and N *is the cumulative distribution function of the standard normal distribution.*

Proof. The proof of proposition 1 can be given through the mathematical steps for deriving it, and by a replicating portfolio of the bond's payoff. I will start by deriving the mathematical steps as proof of proposition 1.

The price of bond 1 will be its discounted expected payoff. The payoff from each scenario is discounted and multiplied with the indicator function for that scenario. The price of the most senior bond, bond 1, is therefore given by

$$B_1 = D_1 e^{-rT} E^Q(1_{AB}) + e^{-rT} E^Q(V_T 1_C) \tag{4.1}$$

where B_1 is the price of bond 1, D_1 is the face value of bond 1, E^Q is the expectation given the probability measure Q, 1_{AB} and 1_C are the indicator functions for respectively scenario AB and C. To get the expectation of the indicator function, 1_C, separated from the expectation of V_T, I need to perform a change of measure. To be able to perform a change of measure, I multiply and divide the last term by the expected value of the firm's assets at time T under the Q probability measure.

$$B_1 = D_1 e^{-rT} E^Q(1_{AB}) + e^{-rT} E^Q \left(\frac{V_T}{E(V_T)} 1_C \right) E^Q(V_T) \tag{4.2}$$

I cannot use the probability measure Q for the risk neutral probability of scenario C, while at the same time wanting the discounted process for the value of the firm's assets to be a martingale [2]. However, I can use the Radon-Nikodym derivative and Girsanov's theorem [3] to obtain a new probability measure. This new probability measure is necessary in able to derive the risk neutral probability of scenario C happening. Taking the process for the value of the firm's assets, given by equation (3.1), and dividing it by the expected value of the firm's assets I get

$$\zeta(T) = \frac{V_T}{E(V_T)} = \frac{V_0 e^{(r-0.5\sigma^2)T+\sigma\sqrt{T}z}}{V_0 e^{rT}} = e^{\sigma\sqrt{T}z-0.5\sigma^2 T} \tag{4.3}$$

where $\zeta(T)$ is the Radon-Nikodym derivative. Multiplying $\zeta(T)$ with the probability measure Q I get the new probability measure F,

$$F = Q\zeta(T). \tag{4.4}$$

By using this change of measure I can now write the equation for the pricing of bond 1 as

$$B_1 = D_1 e^{-rT} Q_{AB} + e^{-rT} F_C E(V_T) \tag{4.5}$$

[2]Martingales are explained in detail in appendix B.

[3]Girsanov's theorem and the Radon-Nikodym derivative are explained in detail in appendix C.

where F_C is the risk neutral probability of scenario C happening under the F probability measure. I can now derive the risk neutral probabilities for scenario AB and C happening. Assuming the lognormal return on the firm's assets, with a Brownian motion given by Z_t.

$$V_T = V_0 e^{(r-0.5\sigma^2)T+\sigma Z_T} \tag{4.6}$$

where

$$Z_T = \sqrt{T}z, \tag{4.7}$$

and z is a standard normally distributed random variable.

I take the natural log and get:

$$\ln(V_T) = \ln(V_0) + (r - 0.5\sigma^2)T + \sigma\sqrt{T}z \tag{4.8}$$

For scenario AB to happen the value of the firm must be higher than the face value of bond 1,

$$Q_{AB} = Q(D_1 \leq V_T) \tag{4.9}$$

where Q_{AB} is the risk neutral probability of scenario AB happening under the Q probability measure. Equation (4.9) is equal to

$$Q_{AB} = Q[\ln(D_1) \leq \ln(V_T)]. \tag{4.10}$$

Filling in for the natural logarithm of $\ln V_T$ from equation (4.8)

$$Q_{AB} = Q[\ln(D_1) \leq \ln(V_0) + (r - 0.5\sigma^2)T + \sigma\sqrt{T}z]. \tag{4.11}$$

By rearranging so as to get the standard normally distributed random variable, z, on one side

$$Q_{AB} = Q\left(-z \leq \frac{ln(\frac{V_0}{D_1}) + (r - 0.5\sigma^2)T}{\sigma\sqrt{T}}\right) \tag{4.12}$$

which, using the cumulative distribution function for a standard normal random variable, is

$$Q_{AB} = N\left(d_{1,2}\right), \tag{4.13}$$

where

$$d_{1,2} = \frac{ln(\frac{V_0}{D_1}) + (r - 0.5\sigma^2)T}{\sigma\sqrt{T}}. \tag{4.14}$$

Q_{AB} is the risk neutral probability of scenario AB happening, in which bond 1 pays out its face value at maturity. The risk neutral probability of the senior bondholder receiving the value of the firm at time T is given by F_C. To obtain F_C I use Girsanov's Theorem and add a drift to the Brownian process, Z_t to obtain the process

$$\tilde{Z}_t = Z_t + \sigma t, \tag{4.15}$$

which is used instead of Z_T in the process of V_T whenever I change the probability measure. Before I look at the risk neutral probability of scenario C, it is necessary to look at how the new process of $\tilde{Z}_t$ changes the process for the value of the firm's assets. The process for the value of the firm's assets is given by

$$V_T = V_0 e^{(r-0.5\sigma)T + \sigma Z_T}. \tag{4.16}$$

Substituting Z_T with $\tilde{Z}_T$ from equation (4.15) yields

$$V_T = V_0 e^{(r-0.5\sigma)T+\sigma(Z_T+\sigma T)} \tag{4.17}$$

$$= V_0 e^{(r+0.5\sigma)T+\sigma Z_T}, \tag{4.18}$$

from which I take the natural logarithm to obtain

$$\ln(V_T) = \ln(V_0) + (r + 0.5\sigma^2)T + \sigma z\sqrt{T}. \tag{4.19}$$

Going from equation (4.18) to (4.19) I fill in for Z_T from equation (4.7). Whenever I am to obtain the risk neutral probabilities using the F probability measure, I substitute Z_T with $\tilde{Z_T}$ in the process for V_T.

Now that the process for the value of the firm's assets under the F probability measure is established, I have a basis for calculating risk neutral probabilities under the new probability measure. The risk neutral probability of scenario C happening under the F probability measure is

$$F_C = F(V_T \leq D_1). \tag{4.20}$$

Take the natural logarithm of both sides

$$= F\left[ln(V_0) + (r + 0.5\sigma^2)T + z\sigma\sqrt{T} \leq ln(D_1)\right], \tag{4.21}$$

and isolating z on one side

$$= F\left[\frac{ln(\frac{V_0}{D_1}) + (r + 0.5\sigma^2)T}{\sigma\sqrt{T}} \leq -z\right] \tag{4.22}$$

yields

$$F_C = N(-d_{1,1}), \tag{4.23}$$

where

$$d_{1,1} = \frac{ln(\frac{V_0}{D_1}) + (r + 0.5\sigma^2)T}{\sigma\sqrt{T}}. \tag{4.24}$$

From the probabilities I have just derived, I can now write the discounted expected payoff for the senior bond. This will equal the price of the bond.

$$B_1 = e^{-rT} D_1 Q_{AB} + e^{-rT} E(V_T) F_C \tag{4.25}$$

Since the discounted process for the value of the firm is a martingale, the expected value of the firm at time T is

$$E(V_T) = V_0 e^{rT}. \tag{4.26}$$

Inserting for $E(V_T)$, F_C and Q_{AB} into equation (4.25) yields

$$B_1 = e^{-rT} D_1 N\left[\frac{ln(\frac{V_0}{D_1}) + (r - 0.5\sigma^2)T}{\sigma\sqrt{T}}\right] + V_0 N\left[-\frac{ln\left(\frac{V_0}{D_1}\right) + (r + 0.5\sigma^2)T}{\sigma\sqrt{T}}\right] \tag{4.27}$$

$$= D_1 e^{-rT} N(d_{1,2}) + V_0 N(-d_{1,1}). \tag{4.28}$$

Equation (4.28) is equal to Merton's (1974) equation for pricing of risky debt in the special case with only one seniority tranche, and is equal to the pricing equation given by proposition 1.

I will now show how proposition 1 can be obtained by copying the payoff of bond 1 using a replicating portfolio.

The payoff of bond 1 at time T is given by

$$P_1 = \min(D_1, V_T). \tag{4.29}$$

Bond 1 payoff

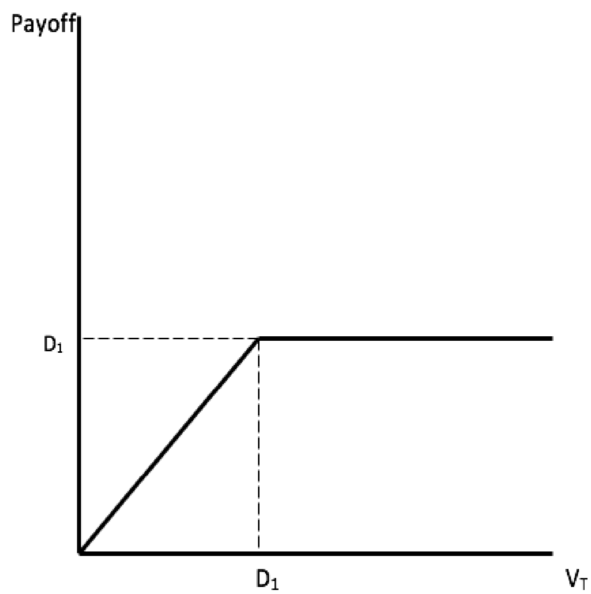

Figure 1: This shows the payoff structure of the most senior bond, bond 1. V_T is the value of the firm at time T. D_1 is the face value of bond 1.

Figure 1 depicts the payoff of the senior bond, bond 1, at time T. It shows that as the value of the firm drops below the face value of bond 1, the bond's payoff at time T equals the value of the firm's assets. To replicate the payoff of the most senior bond, bond 1, I need a long position of $e^{-rT}D_1$ in the risk free asset at time 0, which will pay D_1 at time T, and a short position in a European put on V_T with a strike price equal to the face value of the bond, D_1, to be exercised at time T.

The payoff for the senior bond is already done as replicating portfolio by Merton (1974).

The payoff at time T of the risk free asset and the short put can be graphically represented.

Payoff structure 1: Risk free asset and short put

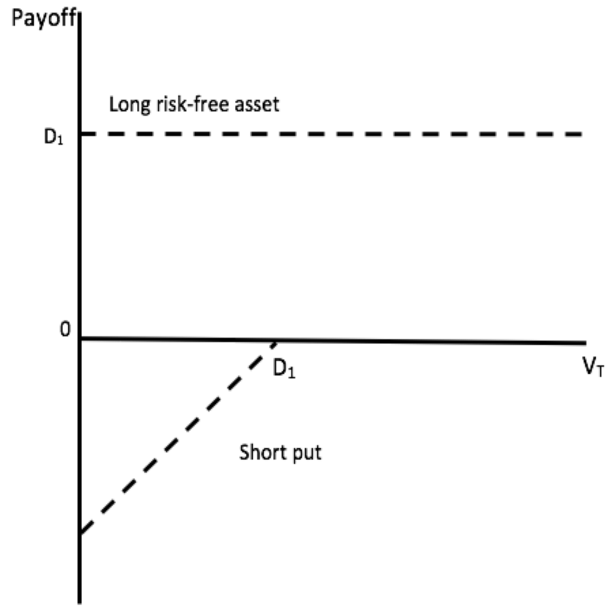

Figure 2: This figure shows the isolated payoffs at time T of a long position in the risk free asset of $e^{-rT}D_1$, and a short European put option on V_T with a strike price equal to the face value of bond 1, D_1.

Adding these payoffs together yields the payoff represented in figure 1. By adding together the value of the risk-free asset and the short put, I get the value of bond 1. The valuation of the short put is done in accordance with the framework set in place by Black and Scholes (1973) and Merton (1973)[4].

A replicating portfolio for bond 1 as described above yields a discounted expected payoff of

[4]Option pricing is explained in appendix D.

$$B_1 = D_1 e^{-rT} - D_1 e^{-rT} N \left[-\frac{ln\left(\frac{V_0}{D_s}\right) + (r - 0,5\sigma^2)T}{\sigma\sqrt{T}} \right] + V_0 N \left[-\frac{ln\left(\frac{V_0}{D_s}\right) + (r + 0,5\sigma^2)T}{\sigma\sqrt{T}} \right] \tag{4.30}$$

$$= D_1 e^{-rT} - D_1 e^{-rT} N\left(-d_{1,2}\right) + V_0 N\left(-d_{1,1}\right) \tag{4.31}$$

$$= D_1 e^{-rT} N\left(d_{1,2}\right) + V_0 N\left(-d_{1,1}\right), \tag{4.32}$$

where

$$d_{1,1} = \frac{ln\left(\frac{V_0}{D_1}\right) + (r + 0,5\sigma^2)T}{\sigma\sqrt{T}} \tag{4.33}$$

$$d_{1,2} = \frac{ln\left(\frac{V_0}{D_1}\right) + (r - 0,5\sigma^2)T}{\sigma\sqrt{T}}, \tag{4.34}$$

which is the same as proposition 1.

4.1.2 The junior bond

Proposition 2 *The pricing of bond 2 subordinated bond 1, is priced by:*
$B_2 = D_2 e^{-rT} N(d_{2,2}) + V_0 \left[N(d_{1,1}) - N(d_{2,1})\right] - D_1 e^{-rT} \left[N(d_{1,2}) - N(d_{2,2})\right]$

where

$d_{1,1} = \frac{ln\left(\frac{V_0}{D_1}\right) + (r+0,5\sigma^2)T}{\sigma\sqrt{T}}$

$d_{1,2} = \frac{ln\left(\frac{V_0}{D_1}\right) + (r-0,5\sigma^2)T}{\sigma\sqrt{T}}$

$d_{2,1} = \frac{ln(\frac{V_0}{D_1+D_2}) + (r+0.5\sigma^2)T}{\sigma\sqrt{T}}$

$d_{2,2} = \frac{ln(\frac{V_0}{D_1+D_2})+(r-0.5\sigma^2)T}{\sigma\sqrt{T}}$.

Proof. As with the senior bond, I will start with the mathematical proof, and then use a replicating portfolio for the payoff.

The price of the junior bond, bond 2, is given by

$$B_2 = e^{-rT}E^Q(1_A D_2) + e^{-rT}E^Q(1_B V_T) - e^{-rT}E^Q(1_B D_1). \tag{4.35}$$

By performing a change of measure, as done for proposition 1, where it is needed, and assigning the notations for the probabilities of each scenario under its respective probability measure, I get

$$B_2 = e^{-rT}Q_A D_2 + e^{-rT}F_B E(V_T) - e^{-rT}Q_B D_1. \tag{4.36}$$

Which in essence is that the junior bond gets paid its face value in scenario A. In scenario B the payoff is $(V_T - D_1)$. In scenario C the junior bondholder gets nothing.

For scenario A to happen V_T must be more than, or equal to, the sum of the face values, $(D_1 + D_2)$. The risk neutral probability of scenario A happening is given by the Q probability measure. The risk neutral probability of scenario A happening is therefore

$$Q_A = Q[\ln(D_1 + D_2) \leq \ln(V_T)], \tag{4.37}$$

where Q_A is the risk neutral probability of scenario A happening under the Q probability measure. Filling in for $\ln V_T$ from equation (4.8)

$$Q_A = Q[\ln(D_1 + D_2)] \leq [\ln(V_0) + (r - 0.5\sigma^2)T + z\sigma\sqrt{T}] \tag{4.38}$$

$$= Q\left[-z \leq \frac{\ln(\frac{V_0}{D_1+D_2}) + (r-0.5\sigma^2)T}{\sigma\sqrt{T}}\right] \quad (4.39)$$

which yields

$$Q_A = N\left[d_{2,2}\right], \quad (4.40)$$

where

$$d_{2,2} = \frac{\ln(\frac{V_0}{D_1+D_2}) + (r-0.5\sigma^2)T}{\sigma\sqrt{T}}. \quad (4.41)$$

Q_A is the risk neutral probability for scenario A happening.

The risk neutral probability of scenario B happening is a bit more complex. The term for the probability of scenario B which includes D_1, has a probability of happening which follows the Q probability measure. However, the term for the probability of scenario B term including $E(V_T)$, has a probability of happening which follows the F probability measure. The probabilities for scenario B will be different since I perform a change of measure.

For scenario B to happen, two requirements need to be fulfilled. The value of the firm's assets must be lower than the aggregated face value of the bonds, and must also exceed the face value of the senior bond, bond 1.

$$Q_B = Q(D_1 \leq V_T \leq D_1 + D_2) \quad (4.42)$$

$$= Q(V_T \leq D_1 + D_2) - Q(V_T \leq D_1) \quad (4.43)$$

$$= N\left(-\frac{ln(\frac{V_0}{D_1+D_2}) + (r-0.5\sigma^2)T}{\sigma\sqrt{T}}\right) - N\left(-\frac{ln\left(\frac{V_0}{D_1}\right) + (r-0.5\sigma^2)T}{\sigma\sqrt{T}}\right) \quad (4.44)$$

Using the property of the standard normal distribution

$$N(-d) = 1 - N(d), \tag{4.45}$$

I get

$$= \left[1 - N\left(\frac{ln(\frac{V_0}{D_1+D_2}) + (r - 0.5\sigma^2)T}{\sigma\sqrt{T}}\right)\right] - \left[1 - N\left(\frac{ln\left(\frac{V_0}{D_1}\right) + (r - 0.5\sigma^2)T}{\sigma\sqrt{T}}\right)\right] \tag{4.46}$$

$$Q(B) = N\left[\frac{ln\left(\frac{V_0}{D_1}\right) + (r - 0.5\sigma^2)T}{\sigma\sqrt{T}}\right] - N\left(\frac{ln\left(\frac{V_0}{D_1+D_2}\right) + (r - 0.5\sigma^2)T}{\sigma\sqrt{T}}\right) \tag{4.47}$$

$$= N(d_{1,2}) - N(d_{2,2}) \tag{4.48}$$

where

$$d_{1,2} = \frac{ln\left(\frac{V_0}{D_1}\right) + (r - 0.5\sigma^2)T}{\sigma\sqrt{T}} \tag{4.49}$$

and

$$d_{2,2} = \frac{ln(\frac{V_0}{D_1+D_2}) + (r - 0.5\sigma^2)T}{\sigma\sqrt{T}}. \tag{4.50}$$

Q_B is the risk neutral probability of scenario B happening under the Q probability measure.

Using the new process of $\tilde{Z_T}$ as a substitute for Z_t in the process of V_T, I can obtain the risk neutral probability of scenario B happening under the F probability measure, given by F_B.

$$F_B = N(d_{1,1}) - N(d_{2,1}) \tag{4.51}$$

where

$$d_{2,1} = \frac{ln(\frac{V_0}{D_1+D_2}) + (r + 0.5\sigma^2)T}{\sigma\sqrt{T}} \tag{4.52}$$

and

$$d_{1,1} = \frac{ln(\frac{V_0}{D_1}) + (r + 0.5\sigma^2)T}{\sigma\sqrt{T}}. \tag{4.53}$$

The steps in deriving F_B are the same as for Q_B, but substituting Z_t with the process of $\tilde{Z}_t$, as done above for F_C in equation (4.18).

The price of bond 2 is given by equation (4.36), from which I can now fill in for $E(V_T)$ and the risk neutral probabilities given by Q_A, Q_B and F_B.

$$B_2 = D_2 e^{-rT} N(d_{2,2}) + V_0 \left[N(d_{1,1}) - N(d_{2,1})\right] - D_1 e^{-rT} \left[N(d_{1,2}) - N(d_{2,2})\right] \tag{4.54}$$

where

$$d_{1,1} = \frac{ln(\frac{V_0}{D_1}) + (r + 0.5\sigma^2)T}{\sigma\sqrt{T}}, \tag{4.55}$$

$$d_{1,2} = \frac{ln\left(\frac{V_0}{D_1}\right) + (r - 0.5\sigma^2)T}{\sigma\sqrt{T}}, \tag{4.56}$$

$$d_{2,1} = \frac{ln(\frac{V_0}{D_1+D_2}) + (r + 0.5\sigma^2)T}{\sigma\sqrt{T}}, \tag{4.57}$$

$$d_{2,2} = \frac{ln(\frac{V_0}{D_1+D_2}) + (r - 0.5\sigma^2)T}{\sigma\sqrt{T}}. \tag{4.58}$$

Equation (4.54) is the same as the pricing given by proposition 2.

Proposition 2 can also be proved using a replicating portfolio. The payoff at time T of bond 2 is given by

$$P_2 = \min(D_2, \max(V_T - D_1, 0)), \tag{4.59}$$

which can be graphically presented:

Bond 2 payoff

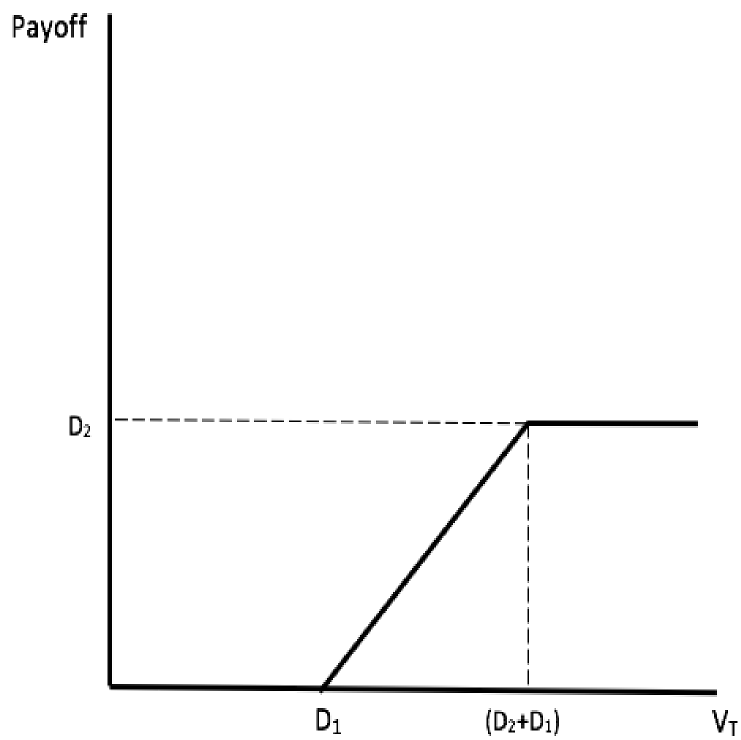

Figure 3: Shows the payoff structure at time T of bond 2, subordinated the senior bond 1. D_2 is the face value of bond 2.

Figure 3 shows the payoff of bond 2 at time T, which is a shift to the right from the payoff shown in figure 1. The payoff of bond 2 drops in accordance with V_T, when V_T drops below the sum of the face values of the two bonds. As the value of the firm drops to the face value of bond 1, the payoff for bond 2 equals zero.

It is possible, given my assumptions of no transaction costs and no restrictions on short sales, to replicate this payoff structure using the risk free asset and put options. To replicate this payoff I need a long position in the risk free asset equal to $D_2 e^{-rT}$ at time 0 which pays out D_2 at time T. A short position in a European put option with a strike price equal to the sum of the face values, $(D_1 + D_2)$. And lastly in order to make sure the payoff does not become negative, I need to add a long position in a put option with a strike price of D_1. Figure 1 and 3 are both previously illustrated by Hart and Moore (1990, p15). The isolated payoffs from the instruments in the replicated portfolio at time 0 can be graphically represented:

Payoff structure 2: Risk free, short put and long put

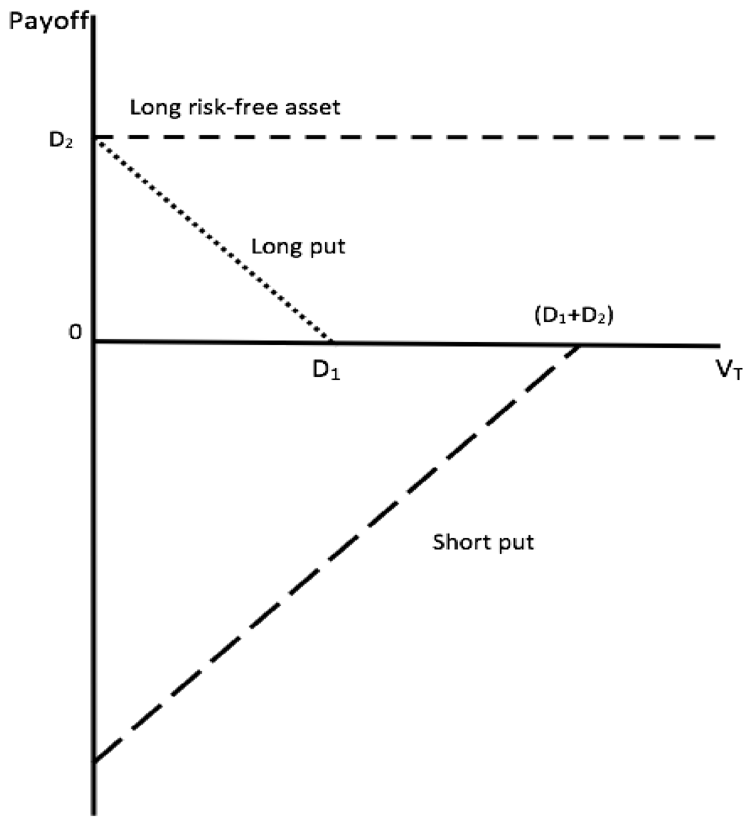

Figure 4: This figure shows each component needed to replicate the payoff of bond 2 at time T.

From figure 4 it is possible to see that the options will not be exercised if; $V_T > (D_1 + D_2)$. If none of the options are exercised, the payoff of bond 2 will only be given by the risk free asset, and will be equal to D_2. If V_T drops below the sum of the face values, the short put will be exercised, and the payoff of bond 2 will be the residual value of the firm, $V_T - D_1$. If V_T drops below D_1 the long put kicks in and cancels any extra effect from the short put, and the payoff is then zero. This portfolio has a payoff which corresponds with the payoffs given by scenario A,B and C for bond 2.

Using the option pricing theory by Black and Scholes (1973) to price the portfolio at time 0 yields

$$B_2 = D_2e^{-rT} + D_1e^{-rT}N\left[-d_{1,2}\right] - V_0N\left[-d_{1,1}\right] - (D_1+D_2)e^{-rT}N(-d_{2,2}) + V_0N(-d_{2,1}). \tag{4.60}$$

With some rearranging equation (4.60) yields the same pricing for the junior bond as that given by proposition 2

$$B_2 = D_2e^{-rT}N(d_{2,2}) + V_0\left[N(d_{1,1}) - N(d_{2,1})\right] - D_1e^{-rT}\left[N(d_{1,2}) - N(d_{2,2})\right]. \tag{4.61}$$

4.1.3 Additional subordinated bonds

By using the replicating portfolio method, I can now expand my model to include another bond that is subordinated the other two. This bond will be called bond 3, as it is subordinated bond 1 and 2. The payoff at time T of bond 3 is

$$P_3 = \min[D_3, \max(V_T - D_1 + D_2, 0)] \tag{4.62}$$

where P_3 is the payoff at time T for bond 3, and D_3 is the face value. This payoff structure can be graphically represented.

Bond 3 payoff

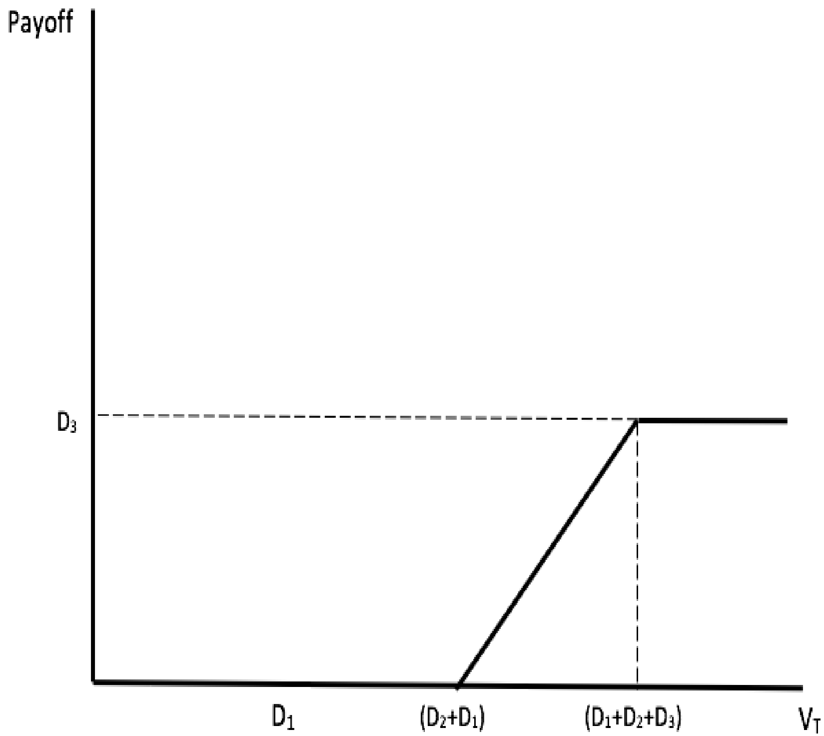

Figure 5: This shows the payoff structure of the third bond at time T, subordinated bond 1 and 2. D_3 is the face value of the third bond.

Figure 5 shows how the payoff of bond 3 is another shift to the right from that of bond 2, shown in figure 3. By comparing figure 1, 3 and 5, it is apparent that for each bond added the point in which the payoff decreases shifts to the right with an amount equal to the face value of the last bond added. The point in which the payoff moves in accordance with V_T, is when V_T drops below the aggregated face values of bond 3 and the senior bonds. When the value of the firm is depleted to the extent that is is equal to the aggregated face value of all senior bonds, bond 3 has a payoff of zero. To replicate the payoff of bond 3, I generate a portfolio consisting of: A long position in the risk free asset equal to the discounted face value of bond 3 at time 0, D_3e^{-rT}, a long put option on the firm with a strike price equal to the sum of the two senior bonds, $(D_1 + D_2)$, and a short put option with a strike price equal to the sum of the face values of all three bonds, $(D_1 + D_2 + D_3)$. Adding a third bond will have no negative effect on any senior bonds, since the new bond is subordinated the previous bonds, and I assume absolute priority and no dilution. It will however dissipate the equity of the firm, since this will be subordinated any new debt under the absolute priority assumption.

Payoff structure 3: Risk free, short put and long put

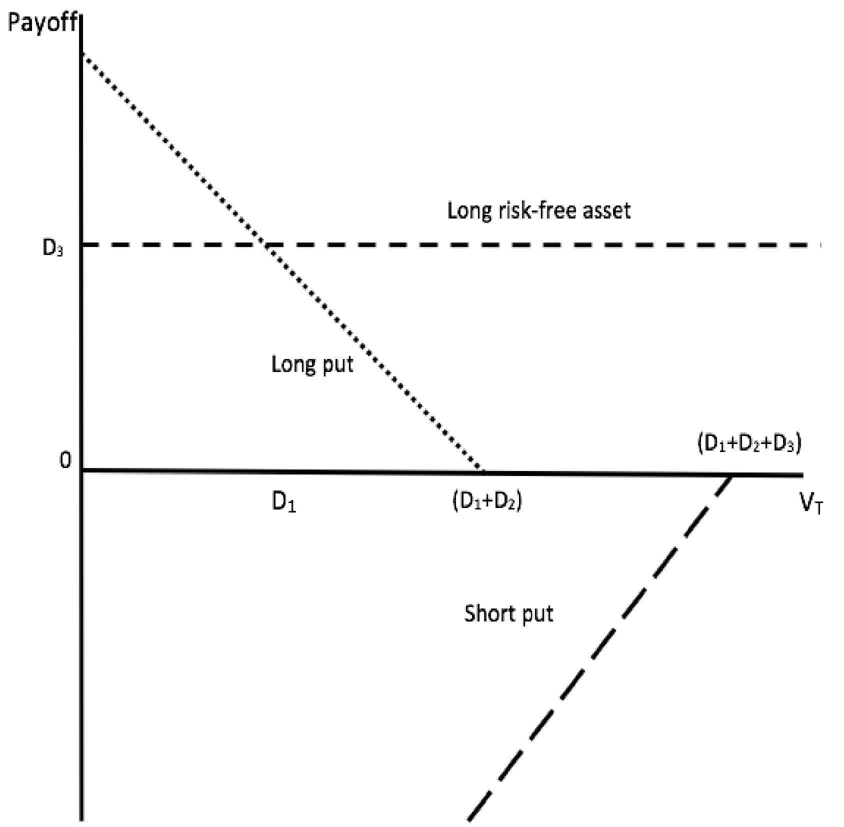

Figure 6: This figure shows each component needed to replicate the payoff of bond 3 at time T

By pricing each component of the replicating portfolio for bond 3, as done previously for bond 1 and 2, I get the following pricing function

$$B_3 = D_3 e^{-rT} N(d_{3,2}) + V_0[N(d_{2,1}) - N(d_{3,1})] - (D_1 + D_2)[N(d_{2,2}) - N(d_{3,2})], \quad (4.63)$$

where

$$d_{3,1} = \frac{ln\left(\frac{V_0}{(D_1+D_2+D_3)}\right) + (r + 0,5\sigma^2)T}{\sigma\sqrt{T}}, \quad (4.64)$$

$$d_{3,2} = \frac{ln\left(\frac{V_0}{(D_1+D_2+D_3)}\right) + (r - 0,5\sigma^2)T}{\sigma\sqrt{T}}, \quad (4.65)$$

and B_3 is the price of bond 3. Comparing proposition 2 to equation (4.63), the reason for assigning d_1 and d_2 from Black and Scholes (1973) to each bond becomes apparent. This assignment allows for an easy pricing of further subordinated bonds. By adding a third bond using the replicating portfolio a pattern is revealed which allows for a general pricing formula based on the seniority of the bond. For each bond added the payoff simply shifts to the right with an amount equal to the face value of the bond added. I can add further bonds and obtain a general payoff function for bond i. i is the bond number based on the level of seniority, where 1 is the most senior. I can write the payoff function for bond i as

$$P_i = \min[D_i, \max(V_T - \sum_{j=1}^{i-1} D_j, 0)] \quad (4.66)$$

where P_i is the payoff at time T of bond i, D_i is the face value of bond i, and $\sum_{j=1}^{i-1} D_j$ is the sum of the face values of all bonds senior to bond i. The payoff for bond i at time T can be illustrated.

Bond i payoff

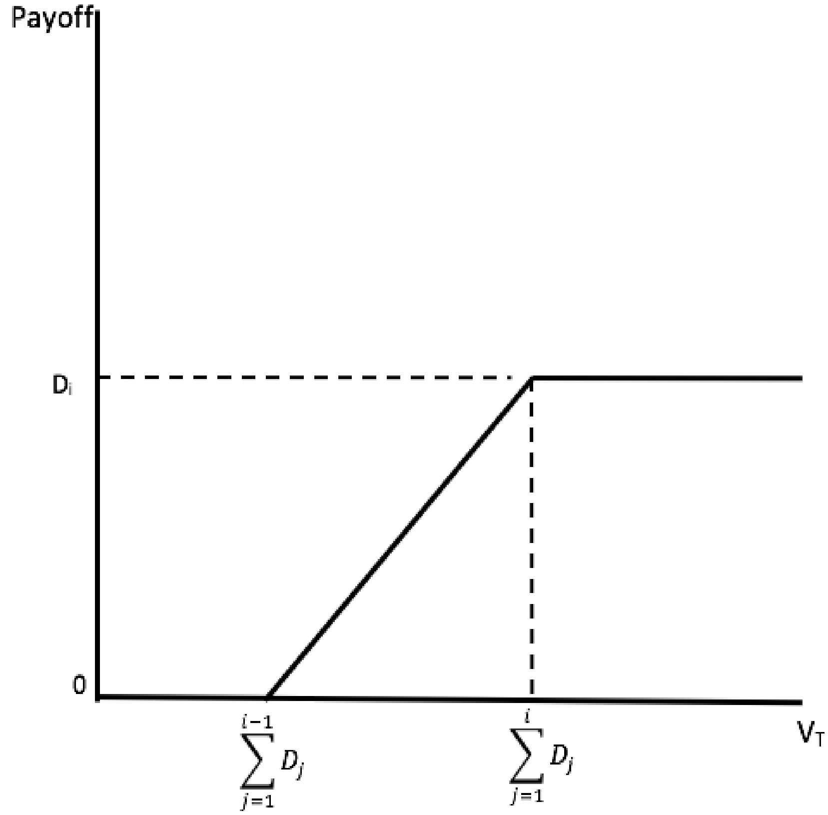

Figure 7: This shows the payoff structure of the bond i, subordinated all bonds that have a lower number than i. D_i is the face value of bond i.

The payoff structure of bond i, given by figure 7, can be replicated as a portfolio at time 0. The portfolio consists of a long position in the risk-free asset equal to the discounted face value of bond i at time 0, $D_i e^{-rT}$, a short put option on V_T with a strike price equal to the sum of the face values of all bonds i and senior, $\sum_{j=1}^{i} D_j$, and a long put option on V_T with a strike price equal to the sum of the face value of all bonds senior to bond i, $\sum_{j=1}^{i-1} D_j$.

Payoff structure i: Risk-free asset, short put and long put

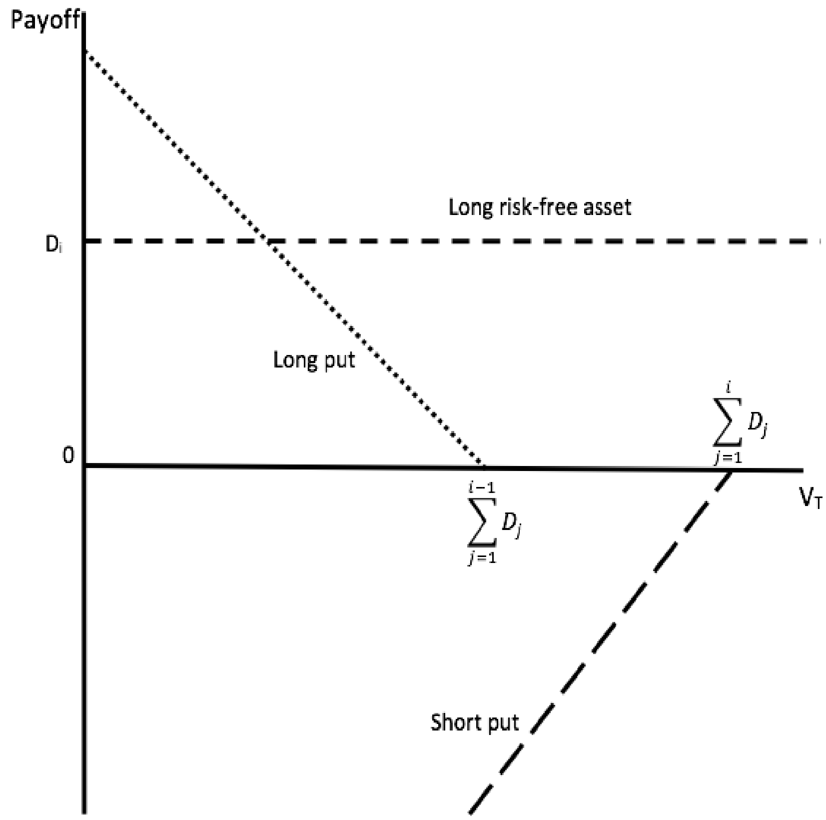

Figure 8: This figure shows the instruments needed to replicate the payoff of bond i at time T.

Generalizing d_1 and d_2 from Black and Scholes (1973) for each bond

$$d_{i,1} = \frac{ln\left(\frac{V_0}{K_i}\right) + (r + 0.5\sigma^2)T}{\sigma\sqrt{T}} \tag{4.67}$$

$$d_{i,2} = \frac{ln\left(\frac{V_0}{K_i}\right) + (r - 0.5\sigma^2)T}{\sigma\sqrt{T}} \tag{4.68}$$

where K_i is the strike price given by

$$K_i = \sum_{j=1}^{i} D_j. \tag{4.69}$$

The $d_{i,1}$ and $d_{i,2}$ stand for bond i's respective d_1 and d_2, where i is the order of the bond. Using equation (4.63) for the third bond and the different d_1 and d_2 for each bond, I can now derive a generalized pricing formula based on the bond's seniority.

$$B_i = D_i e^{-rT} N(d_{i,2}) + V_0[N(d_{i-1,1}) - N(d_{i,1})] - [N(d_{i-1,2}) - N(d_{i,2})] \sum_{j=1}^{i} D_j \tag{4.70}$$

With this formula it is possible to price a bond with regards to the seniority of the bond.

4.2 Risk

4.2.1 Measuring risk

The valuation of the bonds will be affected by their risk structure. In this subsection I will look at different measures of risk and how they are connected to the pricing.

Traditional risk terms for debt are probability of default (PD), and the recovery rate (RR). The recovery rate is the percentage of the face value the bondholders receive in the event of a default. The recovery rate has traditionally been seen to be dependent on individual features of the debt, such as seniority, and be independent of the probability of default. However, with the pricing framework developed by Merton (1974), the RR and PD have a tendency to move inversely. As the value of the firm's assets goes down, the probability of default increases and the expected recovery rate at default decreases (Altman, Resti & Sironi, 2003). With the case of different seniority tranches, the RR of each tranche of seniority will be different.

The PD will be solely dependent on the aggregated sum of debt due at any time and V_t.

The yield to maturity is the discount rate that gives a present value of the bond's face value equal to the price of the bond (Berk & DeMarzo, 2014, p. 171). The continuous yield to maturity of a bond is given by

$$YTM_i = \frac{1}{T} \ln \left(\frac{D_i}{B_i} \right) \tag{4.71}$$

where YTM_i is the continuous yield to maturity of bond i per year, D_i and B_i are respectively the face value and the price of bond i, T is time until maturity measured in years. Since the two initial bonds in my model are pure discount bond an increase in risk should lead to a lower price on the bond. This will give the bond a higher yield to maturity.

From equation (4.71) I can derive the yield spread, which is the difference between the yield to maturity and the risk free rate. This can also be regarded as a risk premium. Merton (1974) suggests using this yield spread as another measure of risk, given that it moves in accordance the volatility of the value of the bond when the pricing variables are changed. In other words it means that an investor would need a higher yield to accept a riskier investment. Since I assume a constant risk free rate over the period until maturity, the risk premium will only depend on the yield to maturity. I am able to show how seniority affects the yield to maturity using simulations later on this thesis.

Both the probability of default and the recovery rate are implicitly included in the pricing functions given by proposition 1 and 2. Therefore, it makes sense to use a measure of risk based on the price of the bonds. The relevant risk for investors will be the risk on their return using the mark-to-market approach. Mark-to-market means valuing assets based on their current market value rather than their book value (McDonald, 2013, p. 8).

The relevant measure of risk will also depend on whether the bonds are part of a portfolio. If the bonds are part of a portfolio of debt from different companies their values might be correlated, which will not be captured by my pricing model. If the bond is part of a portfolio, the correct measure for risk is the elasticity of the value of the bond with regards to V_t (Merton, 1974) (Ho & Singer, 1982). The relevant risk measure for a portfolio will only contain the systematic risk of the bond. Since I am looking at bonds issued by the same firm, the bonds should not be considered part of a portfolio. The correct measure for the risk is therefore the standard deviation of the value of the bonds (Merton, 1974). The standard deviation of the value of the bond will include both the systematic risk and the idiosyncratic risk of the bond.

4.3 Comparative statics and simulations

4.3.1 Comparative statics

Black and Cox (1976) state that, in a static analysis, the senior bond value will decrease in value as the volatility of the firm increases. The value of the junior bond can be both an increasing and decreasing function of the volatility, depending on the value of the firm. The infliction point they find, where B_2 becomes an increasing function of V_T, is given by

$$V* = [D_1(D_1 + D_2))]^{1/2} e^{-(r-\delta+0,5\sigma^2)T} \tag{4.72}$$

where $V*$ is the infliction point, δ is the continuous dividend rate. I have changed the other notations to suit those already specified in this thesis. When V_t is less than $V*$, the price of the junior bond is an increasing function of the volatility of V_t. When V_t is more than $V*$, B_2 is a decreasing function of σ. So the seniority of a bond could, according to Black & Cox(1976), effect the way the value of the bond is effected by a change in the volatility of the firm. Subordinated bonds could have an incentive for influencing the firm into taking riskier decisions.

The ratio of the discounted face value of the bond to the value of the firm, given by Merton (1974), will partly determine the risk structure of debt.

$$c = \frac{e^{-rT} D}{V_t} \tag{4.73}$$

where c is a ratio of the discounted face value divided by the value of the firm's assets. Merton (1974) uses d as a notation for the ratio given by equation (4.73). I have changed this to c since d is already used in the valuation. Since debt is discounted at the risk free rate the debt-to-value ratio given by c, is an upward bias estimate of the actual debt-to-firm value ratio.

If V_t goes to approximately zero, c will go towards infinity. As this happens the volatility of the bond will go toward the volatility of V_t. As the firm value increases, c will go toward zero, and the value of the bond will approach its face value discounted by the risk-free rate. The risk of the bond will therefore be approximately equivalent to a default-free bond, and the yearly yield to maturity should be equal to the risk free rate. When $(0 < c < \infty)$ the bond will behave as a combination of risk-less debt and equity and will continuously change. However, in general the riskiness of debt is less than that of the firm (Merton, 1974). Intuitively this seems correct. The shareholders payoff will be influenced by any change in V_T, whiles the payoff of the bondholders will only be influenced if V_T is less than the aggregated sum of the face values. This can be seen from the payoff equations of the bondholders and the shareholders in section 3.3. Therefore, there should be less variations in the value of the bonds than the value of the firm's assets, unless the value of the firm's assets goes toward zero and c approaches infinity.

In the case with only one seniority tranche, the value of the bonds are highly dependent on the ratio c. If $c \geq 1$ then the risk premium will be large with short time until maturity and will decrease exponentially as time to maturity increases. With $c \leq 1$ the risk premium will be zero as time to maturity is zero, and will increase as time to maturity increases, until it reaches a top when added time to maturity decreases the risk premium (Merton, 1974).

Applying this to the case with different tranches of seniority, I can see from proposition 1 that subordinated bonds are not included in the pricing of the senior bond. This exclusion of subordinated bonds means I can conclude, that although the overall c ratio increases, only the bonds of the same seniority or higher are relevant for the risk of the bond. This conclusion can only be drawn if the assumptions of my model are true. In other words, the risk of the senior bond is not affected by the inclusion of further subordinated bonds. This is in part due to the absence of net-worth covenants, but also the fact that the bonds mature at the same time.

By increasing the face value of more senior bonds, holding all else equal, the value of subordinated bonds will decrease. This is due to the expected recovery rate decreasing. The junior bonds will be the first affected by the initial decrease in value of the firm below the default threshold. The relevant c for the subordinated bonds will include the face value of all senior bonds.

4.3.2 Simulations

I will now show two simulations of the model done in Excel. I use the same parameters for both simulations. The simulations contains two bonds, where bond 1 is senior to bond 2. Using a continuous risk free interest rate of 1.5%, a standard deviation in the value of the firm's assets of 30%. The starting value of the firm, V_0, is 100, while the face values of bond 1 and 2 are both 45. I have chosen these values so that the firm will have a larger probability of defaulting, which makes it possible to see what happens with the value of the bonds when default is imminent. The currency of the values is arbitrary so I have chosen not to define this. The simulation lasts over three years where the bonds mature at the end. I have divided the simulation into 1095 days, assuming that the bonds can be traded 365 days a year.

V_0	100
D_1	45
D_2	45
r	1.5%
σ	30%
T	3

Table1: Shows the parameters used for the simulations.

The bonds are priced each day according to the time left until maturity and the firm value at the time. For the pricing of the bonds I have used proposition 1 and 2. The starting price for bond 1, with these parameters, is $42,29$, and the starting price for bond 2 is $30,89$. In accordance with equation (4.71) the yearly yield to maturity of bond 1 will be 0.0207. For bond 2 the yearly yield to maturity is 0.1254. With the face value and time to maturity of the two bonds being the same, the difference in the yield to maturity will be solely due to their difference in seniority.

By taking the continuous change in price from day to day, in the form of $\ln(B_{1,t}/B_{1,t+1})$ and $\ln(B_{2,t}/B_{2,t+1})$, I get a measure for the continuous daily return of each bond. I then obtain the standard deviation of the daily continuous return. I do this by calculating the standard deviation for the first twenty days, then moving one day ahead for each measure of the standard deviation. By doing this I get 1075 observations of the standard deviation of the continuous returns on B_1 and B_2.

Simulation 1

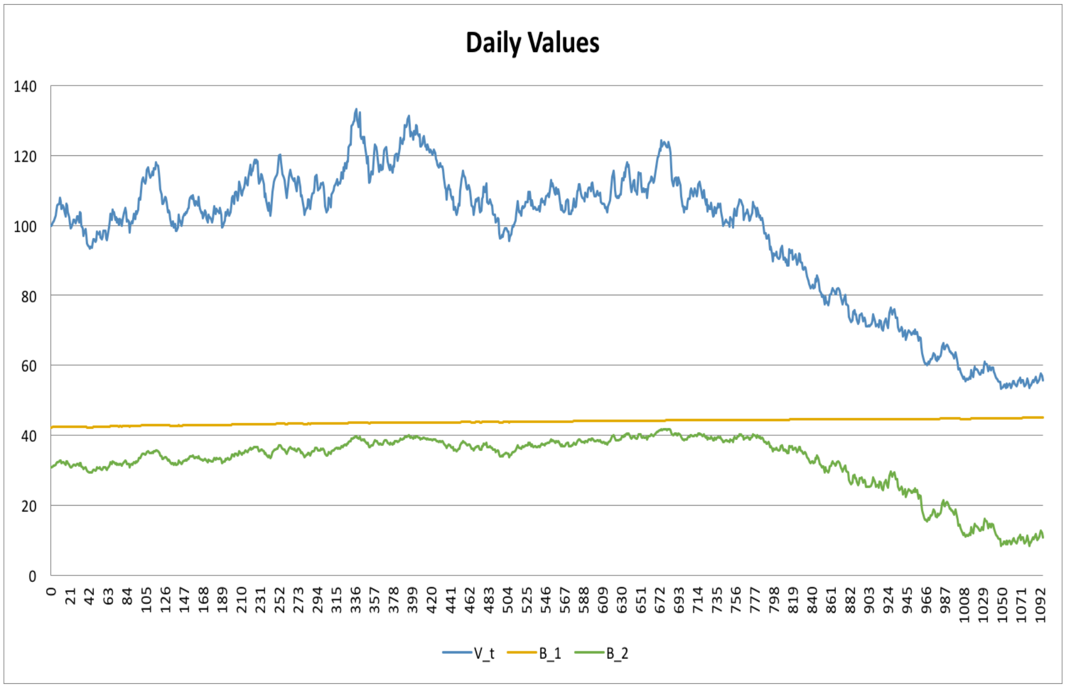

Figure 9: This shows a daily simulation over 3 years of the price of the bonds as a result of the value of the firm at that time. The blue line is the value of the firm. The yellow is the value of bond 1(senior). Green is the value of bond 2 (junior).

In the simulation given by figure 9, the value of the firm's assets stays consistently above 45, so the value of bond 1 remains fairly stable throughout the simulation. Bond 2 can be seen to shift in accordance with the value of the firm. However, since $V_t > 0$, c is finite for both bonds, the volatility of both bonds will be less than that of the firm's assets. c is the ratio given by equation (4.73). As the value V_t drops close to the sum of the face values, the payoff of bond 2 at maturity is likely to be be the residual value of the firm after the face value of bond 1 has been paid. This results in the value of bond 2 following V_t closely, which can be seen in figure 9. The payoffs for the bonds at time T from this simulation correspond with that of scenario B and AB.

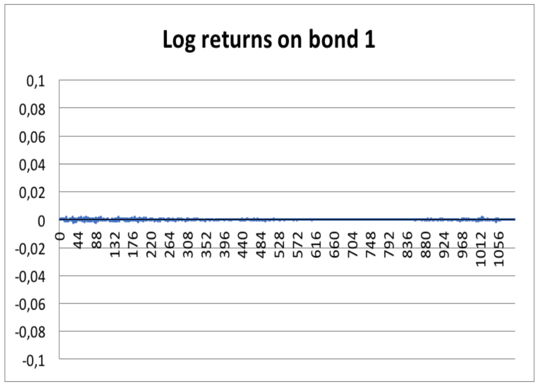

Figure 10: This figure shows the daily log returns of bond 1 (senior) over the 3 year simulation.

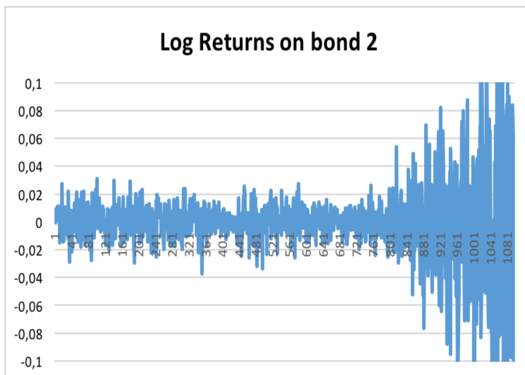

Figure 11: This figure shows the daily log returns of bond 2 (junior) over the 3 year simulation.

Figure 10 and 11 show the daily log returns of respectively bond 1 and bond 2. Since the variable V_t does not drop below 45, bond 1 will most likely get paid its face value at maturity. Therefore, the price of the bond remains stable and the day to day continuous return is close to zero. For bond 2 the daily continuous returns vary a lot more. This variation is due to V_t being close to or below the sum of the face values.

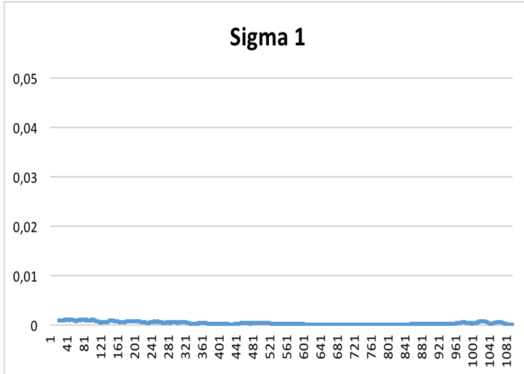

Figure 12: This figure shows the standard deviation of the log returns on bond 1(senior).

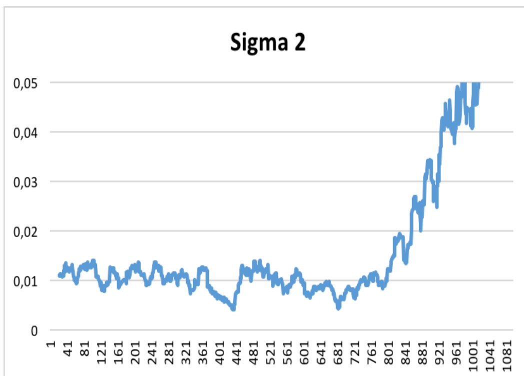

Figure 13: This figure shows the standard deviation of the log returns on bond 2(junior)

From figure 12 and 13, I see that the standard deviation of the continuous return is a lot larger for bond 2 than bond 1. This difference in standard deviation is due to B_1 not being as sensitive to changes in V_t as B_2 is.

Simulation 2

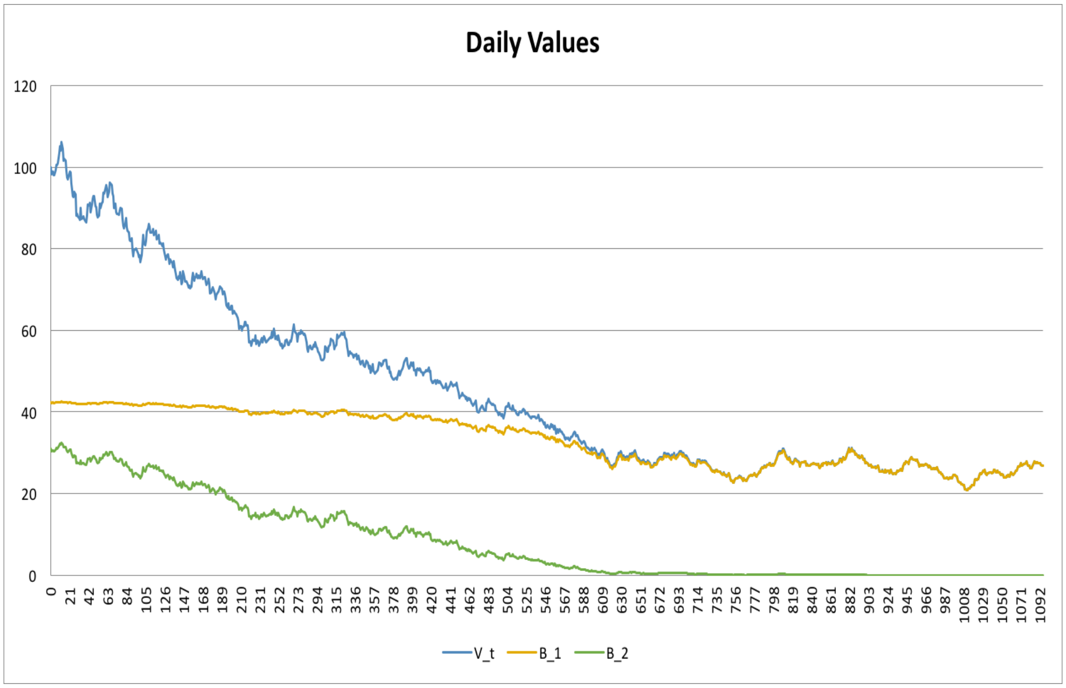

Figure 14: The second simulation. The blue line is the value of the firm. The yellow is the value of bond 1(senior). Green is the value of bond 2 (junior)

Figure 14 shows simulation 2. It shows a path for V_t which steadily declines over the period. The main difference from simulation 1 is that the value of the firm drops below the face value of the senior bond, bond 1. With V_t being less than 45 the value of bond 1 is affected in a much larger degree by changes in V_t than in simulation 1. The value of bond 2 drops close to zero. The end of the period yields a payoff for the bonds equal to the payoffs in scenario C.

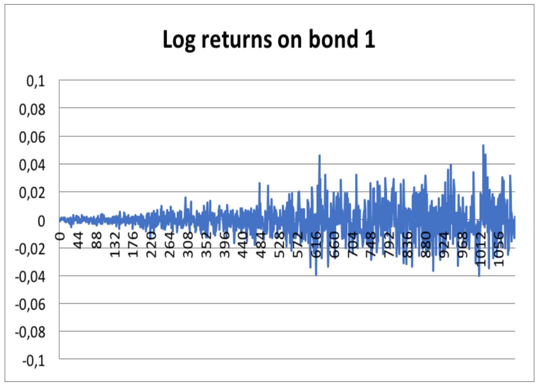

Figure 15: This figure shows the daily log returns of bond 1 (senior) over the 3 year simulation.

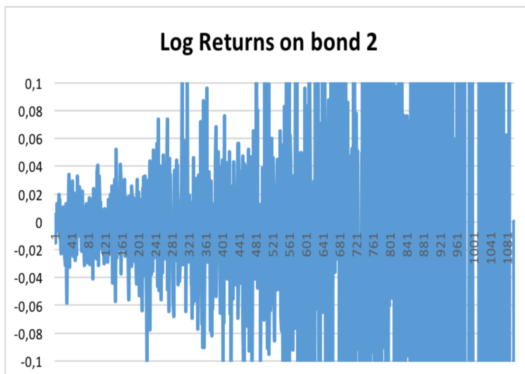

Figure 16: This figure shows the daily log returns of bond 2 (junior) over the 3 year simulation.

From figure 14 it can be seen that at the end the absolute variation in B_1 is larger than that of B_2. However, since the measure of return I use looks at the relative change, the variation in continuous daily return is larger for B_2. Since B_2 approaches zero any small change in value will have a large effect on the relative return. This large variation in the return can be seen in figure 16.

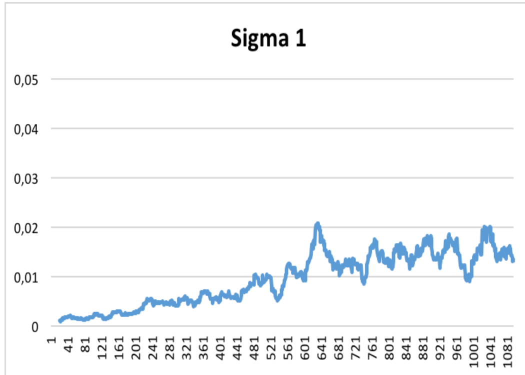

Figure 17: This figure shows the standard deviation of the log returns on bond 1 (senior).

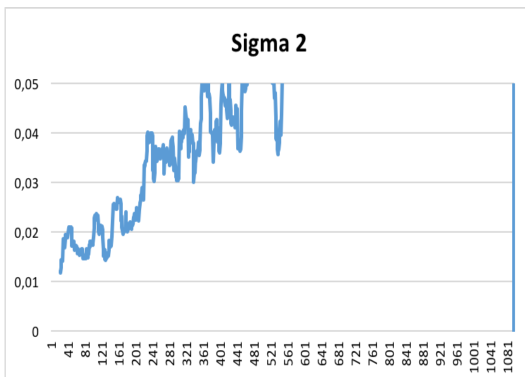

Figure 18: This figure shows the standard deviation of the log returns on bond 2 (junior).

From figure 17 and 18, I can see that the standard deviation of $\ln(B_{1,t}/B_{1,t+1})$ is higher than that of $\ln(B_{2,t}/B_{2,t+1})$. It can be seen that this is the case also in a simulation where the the value of the firm's assets drops below the face value of the most senior bond. These simulations help me conclude that, given my assumptions, a bond with higher seniority will be less risky than any subordinated bonds.

5 Conclusion

In this thesis I have looked at what effect seniority has on corporate debt. I have used a theoretical approach and expanded the model developed by Merton (1974). By introducing the world through a set of assumptions, I have been able to derive a simple model for the pricing of two bonds with regards to their seniority. This model needs the same input in order to price the bonds, as that needed to price European options in the Black and Scholes (1973) option pricing model. By using this option pricing theory, I replicate the payoffs of the two bonds using a portfolio consisting of the risk free asset and European put options on the firm's assets. Through the replicating portfolio method, I am able to derive a pricing equation for a third bond. With the insights from the pricing equations developed for the three bonds, I am able to create a general pricing formula of a bond based on its seniority.

With a foundation in my pricing model I look at the risk of the bonds. Looking at previously used measures of risk, I find that the appropriate measure of risk is the standard deviation of the daily logarithmic returns of the bonds. This is due to the bonds being issued by the same company, and not regarded as part of a portfolio. It is therefore the total risk of the bond that is relevant and not only the systematic risk.

I use two simulations to illustrate how the seniority will affect the yield and risk. The first simulation shows the value of the firm's assets dropping below the aggregated sum of the face values, but staying above the face value of the senior bond. In the second simulation the value of the firm's assets drops below the face value of the senior bond. These simulations show how the price of bonds with different seniorities can move, given that my assumptions hold. However, my assumptions are not applicable to the real world in that the risk free rate is not constant over time and the absolute priority rule is seldom followed.

Given my assumptions, subordinated bonds have a higher risk, measured in the standard deviation of daily returns, than any senior bond. This added risk calls for a higher yield to maturity for any subordinated bonds.

6 References

[1] Altman, E., Resti, A., & Sironi, A. (2004) Default Recovery Rates in Credit Risk Modeling: A Review of the Literature and Empirical Evidence. *Economic Notes*, *33*, 183 - 208

[2] Berk, J., & DeMarzo, P. (2014) *Corporate Finance.* (3rd ed). Harlow: Pearson Education Limited.

[3] Black, F., & Cox, J. C. (1976) Valuing corporate securities: Some effects of bond indenture provisions. *The Journal of Finance*, *31*, 351 - 361.

[4] Black, F., & Scholes, M. (1973) The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, *81*, 637-654. 10.1086/260062

[5] Brockman, P., & Turtle, H. J. (2003) A barrier option framework for corporate security valuation. *Journal of Financial Economics*, *67*, 511-529.

[6] Eberhart, A.C., Moore, W.T., & Roenfeldt, R.L. (1990) Security Pricing and Deviations from the Absolute Priority Rule in Bankruptcy Proceedings. *The Journal of Finance*, *45*, 1457-1469

[7] Hart, O., & Moore, J. (1995) A Theory of Corporate Financial Structure Based on the Seniority of Claims. *American Economic Review*, *85* (3431), 567 - 585 10.3386/w3431

[8] Ho, T.S., & Singer R. F. (1982) Bond indenture provisions and the risk of corporate debt. *Journal of Financial Economics*, *10*, 375 - 406.

[9] McDonald, R. L. (2013) *Derivatives Markets* (3rd ed). London: Pearson Education Limited.

[10] Merton, R.C. (1973) Theory of Rational Option Pricing *The Bell Journal of Economics and Management Science*, *4*, 141-183

[11] Merton, R.C. (1974) On the Pricing of Corporate Debt: The Risk Structure of Interest Rates. *The Journal of Finance*, *29*, 449 - 470

[12] SIFMA Research Department, 2017. (2017). 2017 Fact Book. Retrieved from https://www.sifma.org/wp-content/uploads/2016/10/US-Fact-Book-2017-SIFMA.pdf

[13] Varma, P., & Cantor R. (2004) Determinants of Recovery Rates on Defaulted Bonds and Loans for North American Corporate Issuers: 1983-2003. Retrieved from https://www.moodys.com/researchdocumentcontentpage.aspx?docid=PBC_90593

[14] Warner, J.B. (1977) Bankruptcy, Absolute Priority, and the Pricing of Risky Debt Claims. *The Journal of Financial Economics*, *4*, 239-276

Appendix

The parts in this section concerning the geometric Brownian motion, martingales and Girsanov's theorem are based on McDonald (2013, pages 617 - 673), the section on option pricing using the Black and Scholes (1973) framework is based on Berk and DeMarzo (2014, p. 747-753) and McDonald (2013, p. 366).

A Geometric Brownian Motion

I assume that the value of the firm's assets follows a geometric Brownian motion. The value of the firm is referred to as V_t, with the subscript denoting the time of valuation.

In the case where the asset drift, α, and the volatility, σ, depend on the asset value,

$$\frac{dV_t}{V_t} = \alpha dt + \sigma dZ_t. \tag{.1}$$

The percentage change in V_t has an instantaneous mean of α, and an instantaneous variance of σ^2. Z_T is given by

$$Z_T = \sqrt{T}z \tag{.2}$$

where z is a standard normally distributed random variable. The σ in front of dZ_T in equation (.1) is added as a scale factor. Since

$$\int_0^T dZ_t \tag{.3}$$

has a variance of T,

$$\int_0^T \sigma dZ_t \tag{.4}$$

will have a variance of $T\sigma^2$.
The process given by equation (.1) is the Geometric Brownian Motion. This process is not normally distributed as the mean and volatility depend on realized values of V_t, However $\ln[V_t]$ is normally distributed.

$$\ln[V_T] \sim \mathcal{N}(\ln[V_0] + (\alpha - 0,5\sigma^2)T, \sigma^2 T) \tag{.5}$$

Here $\ln[V_0] + (\alpha - 0,5\sigma^2)T$, is the mean of $\ln[V_T]$, and $\sigma^2 T$ is the variance.

The solution to the stochastic difference equation in (.1) is given by

$$V_T = V_0 e^{(\alpha - 0.5\sigma^2)T + \sigma\sqrt{T}z}, \tag{.6}$$

where V_0 is the value of the firms asset at time 0. I use the principle of risk neutral pricing, which means the asset drift, α, is replaced by the risk free rate, r, and I can now effectively price the bonds using the following equation for the value of the firm's assets.

$$V_T = V_0 e^{(r - 0.5\sigma^2)T + \sigma\sqrt{T}z} \tag{.7}$$

This equation for the value of the firm's assets at time T will be central in the pricing of debt, since the value of the bonds is linked to the value of the firm's assets.

B Martingales

A variable is said to follow a martingale if the conditional expectation of the variable given all previous realized values of the variable equals the most recent realized value

$$E(X_{n+1}|X_1, X_2, ..X_n) = X_n \tag{.8}$$

where X_n is the last realized value of X.

Assuming two assets with the values S_1 and S_2. The second asset is normally referred to as the numeraire.

$$E_t\left(\frac{S_{1,T}}{S_{2,T}}\right) = \frac{S_{1,t}}{S_{2,t}} \tag{.9}$$

which is equivalent to

$$S_{1,t} = S_{2,t}E_t\left(\frac{S_{1,T}}{S_{2,T}}\right) \tag{.10}$$

Now assuming that S_2 is a pure discount default-free bond, it should give a return equal to the risk free rate. Assuming that the value of $S_{2,T}$ is 1, then $S_{2,t}$ will equal $e^{-r(T-t)}$, assuming that the risk free rate is constant.

$$S_{1,t} = e^{-r(T-t)}E_t(S_{1,T}) \tag{.11}$$

Using a zero-coupon default-free bond as numeraire is consistent with the risk neutral valuation framework provided by Black and Scholes (1973) and Merton (1973). By adjusting equation (.11), I can get an expression for the expected future value of the firm.

$$E_t(S_{1,T}) = S_{1,t}e^{r(T-t)} \tag{.12}$$

The discounted process of S_1 is a martingale. I use the value of the firm's assets, given by V_t, as S_1 in this thesis.

C Girsanov's Theorem and the Radon-Nikodym derivative

The process for the discounted value of V_t is a martingale only for a specific distribution function. When I conduct a change of measure the original discounted process for the value of the firm's assets is no longer a martingale under the new probability measure. This can be corrected using Girsanov's theorem.

The Girsanov theorem can be broken down into three parts and goes as follows:
1: Given the process

$$Z_t = \sqrt{t}z, \tag{.13}$$

add a drift to it to obtain the process

$$\tilde{Z}_t = Z_t + \sigma t \tag{.14}$$

2: Then obtain the Radon-Nikodym derivative. The value of the firm is given by the process

$$V_T = V_0 e^{(r-0.5\sigma^2)T+\sigma Z_T} \tag{.15}$$

where the expected value value of the firm's assets is

$$E(V_T) = V_0 e^{rT}. \tag{.16}$$

By dividing the process for the value of the firm's assets with its expected value I get

$$\zeta(T) = \frac{V_T}{E(V_T)} = e^{\sigma Z_T - 0.5\sigma^2 T} \tag{.17}$$

where ζ(T) is the Radon-Nikodym derivative.

3: Then multiply the original probability distribution, Q, with the Radon-Nikodym derivative.

$$F = \zeta(T)Q \tag{.18}$$

By doing this we obtain a new probability measure given by F, in which the discounted process for the value of the firm's assets, containing the process $\tilde{Z}_t$, is a martingale.

D Option pricing

I use option pricing as a way of copying a payoff of a bond with a replicating portfolio. In my thesis I use a long and a short position in a European put option on the firm's assets. I will therefore show how to price a European put option of a non-dividend paying stock.

A European put option gives an option to sell an asset at a given time. The time the put option can be exercised is time T. If an investor is the one issuing the put option, that investor has a short position in the put option. A short position in a put option simply adds the negative value of the option.

The value of a put option with a strike price of K is

$$PO(S, K, \sigma, r, T, \delta) = CO(S, K, \sigma, r, T, \delta) + Ke^{-rT} - Se^{-\delta T} \tag{.19}$$

where PO is the value of the put option, S is the value of the asset at the time of pricing, σ is the volatility of the asset, CO is the value of a European call option on the same asset, δ is the continuous dividend rate paid to the owners of the asset.

Since the value of a call option is given by

$$CO(S, K, \sigma, r, T, \delta) = Se^{-\delta T} N(d_1) - Ke^{-rT} N(d_2) \tag{.20}$$

where

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r - \delta + 0,5\sigma^2)T}{\sigma\sqrt{T}} \tag{.21}$$

$$d_2 = \frac{\ln\left(\frac{S}{K}\right) + (r - \delta - 0,5\sigma^2)T}{\sigma\sqrt{T}}. \tag{.22}$$

Inserting this into equation (.19):

$$PO(S, K, \sigma, r, T, \delta) = Ke^{-rT}N(-d_2) - Se^{-\delta T}N(-d_1). \tag{.23}$$

Since I do not use dividends in my model, I can remove the δ and get the value of the put option used in the replicating portfolio

$$PO(S, K, \sigma, r, T) = Ke^{-rT}N(-d_2) - SN(-d_1). \tag{.24}$$

where δ is also removed from d_1 and d_2 in equation (.21) and (.22).

E Listing of notations

B = Price of the bond, subscript denotes which bond.
D = Face Value of the bond, subscript denotes which bond.
P= Payoff for the bond, subscript denotes which bond.
V= Value of the firm, subscript denotes the time of valuation.
Q= The original probability distribution.
F= The probability distribution given by Q multiplied by the Radon-Nikodym derivative.
$\mathrm{Z_t}$= The original Brownian process.
$\tilde{\mathrm{Z}}_\mathrm{t}$ = A new process consisting of the original Brownian process and a drift consisting of σT.

σ= The standard deviation of the value of the firm, also referred to as the volatility of the firm.

r=The continuous risk free interest rate.

δ=The continuous dividend rate.

0= The time in which the bonds are priced.

t= Any time between time 0 and T.

T= The time at which the bonds mature.

YTM =Yield to Maturity.

z=A standard normal distributed random variable.

N=The cumulative distribution function for a standard normally distributed random variable.

c=A ratio of the discounted face value of bonds to the value of the firm's assets.

$$d_{1,1} = \frac{ln\left(\frac{V_0}{D_1}\right)+(r+0,5\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_{1,2} = \frac{ln\left(\frac{V_0}{D_1}\right)+(r-0,5\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_{2,1} = \frac{ln(\frac{V_0}{D_1+D_2})+(r+0.5\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_{2,2} = \frac{ln(\frac{V_0}{D_1+D_2})+(r-0.5\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_{3,1} = \frac{ln\left(\frac{V_0}{(D_1+D_2+D_3)}\right)+(r+0,5\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_{3,2} = \frac{ln\left(\frac{V_0}{(D_1+D_2+D_3)}\right)+(r-0,5\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_{i,1} = \frac{ln\ \left(\frac{V_0}{K_i}\right)+(r+0.5\sigma^2)T}{\sigma\sqrt{T}}$$

$$d_{i,2} = \frac{ln\ \left(\frac{V_0}{K_i}\right)+(r-0.5\sigma^2)T}{\sigma\sqrt{T}}$$

$$K_i = \sum_{j=1}^{i} D_j$$