3 MATHEMATICAL MODELLING OF DIGITAL CONTROL SYSTEMS FOR ELECTRONIC ENERGY CONVERTERS (30.10.01)

A digital computer based controller is concerned with processing of number sequences. These sequences are discrete time signals generated by sampling a continuous signal at constant time intervals. In order to analyse and design such a control system it is necessary to have a mathematical model of these signals, the digital control system components, and the controlled power electronics plant.

This chapter contains a review of the theoretical aspects concerning discrete and sampled system and signals. No attempt is made to present the complete theory in this report. Rather, attention is focused on a limited number of topics which can serve as an adequate background necessary for the understanding and guiding of practical design of a digital control system in power electronics.

The introduction of the discrete time concept is done as a logical and gradual extension of the classical philosophy for continuous systems. The presentation is based both on the input-output model approach for mono-variable systems, where the transform technique is used, and the state space representation.

3.1 System overview, terminology, and basic assumptions

This section serves two purposes. The first is to give an overview of a power electronic system controlled by a digital computer. The second is to identify system characteristics and to introduce terminology related to signals and systems. The discussion will be done with reference to Figure 3.1, which is a revision of figure 1.1 where the basic element and signals in the closed loop control system are highlighted.

The part of the system that contains the manipulated and measured variables is called the process or plant. In this text we are dealing with processes in connection with power electronics.

The process/plant in a power electronic system is most often some kind of energy conversion equipment. Depending on the actual application, the manipulated variables may be rotating speed, current, voltage, temperature, etc. In some applications it is convenient to include the electrical power converter as part of the process while in other applications it can be regarded as a separate unit.

The measured output from the process y(t) is a continuous time signal. A computer is able to work with digital coded information only. The output from the process must therefore be converted into digital form by some kind of analog to digital (A/D) converter. The digital output signal from the A/D converter is discrete in both time and amplitude level. The discretizing in time is called sampling. The discretizing in amplitude level is called quantizing.

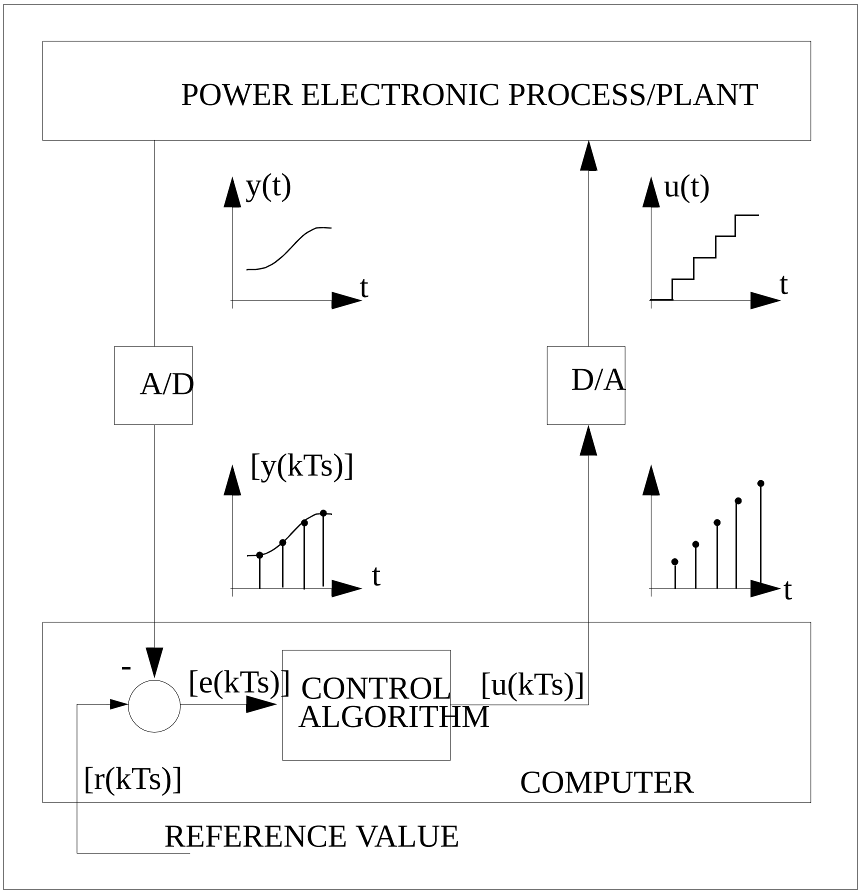


Figure 3.1 Schematic diagram illustrating the interface signal between the system elements.

The computer interprets the converted signal $y(kT_s)$ as a sequence of numbers. These discrete signals are manipulated by the computer algorithm which solves a difference equation implementing the desired control law. This algorithm generates a new sequence of numbers $u(kT_s)$. This sequence is passed on to the process through some kind of a reconstruction device represented here by a digital to analog (D/A) converter. The D/A converter must produce a continuous-time signal. This is normally done by keeping the control signal constant between the conversions (zero order hold). Note that the system runs open loop between the sampling instants.

Ideally the A/D and D/A conversions are done at the sampling instants $t = kT_S$ where $k = 0,1,2....$ The time interval between these instants are usually constant and are denoted by the sampling period T_s. The inverse of T_s is the sampling frequency

$$f_s = 1/T_s \tag{3.1}$$

The execution of the control program may be started by a clock which gives an interrupt signal to the computer at each sampling instant.

The events that takes place in the system are illustrated in figure (3.2) and and figure (3.3).

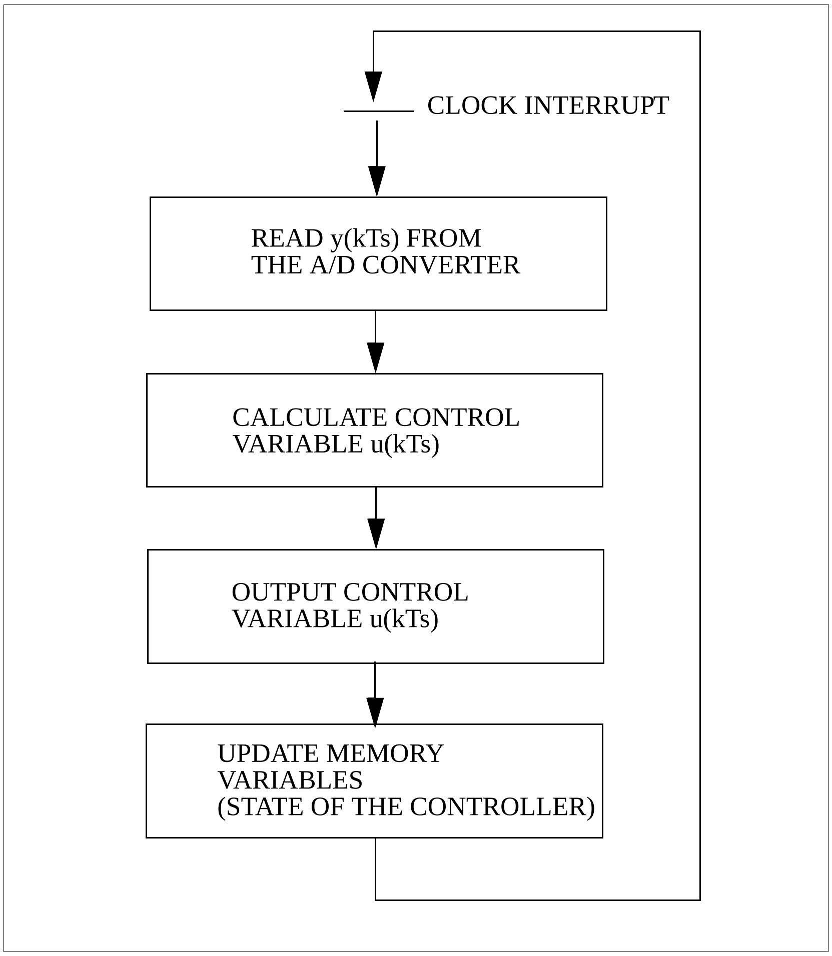


Figure 3.2 Graphical illustration of the events that take place in a program that represents a digital controller.

Since the computer is performing the tasks in sequence, there will be always a time delay due to the computing time. The A/D and D/A conversion also takes time. These time delays must often be taken into account when designing the control system. We will show later how this can be done.

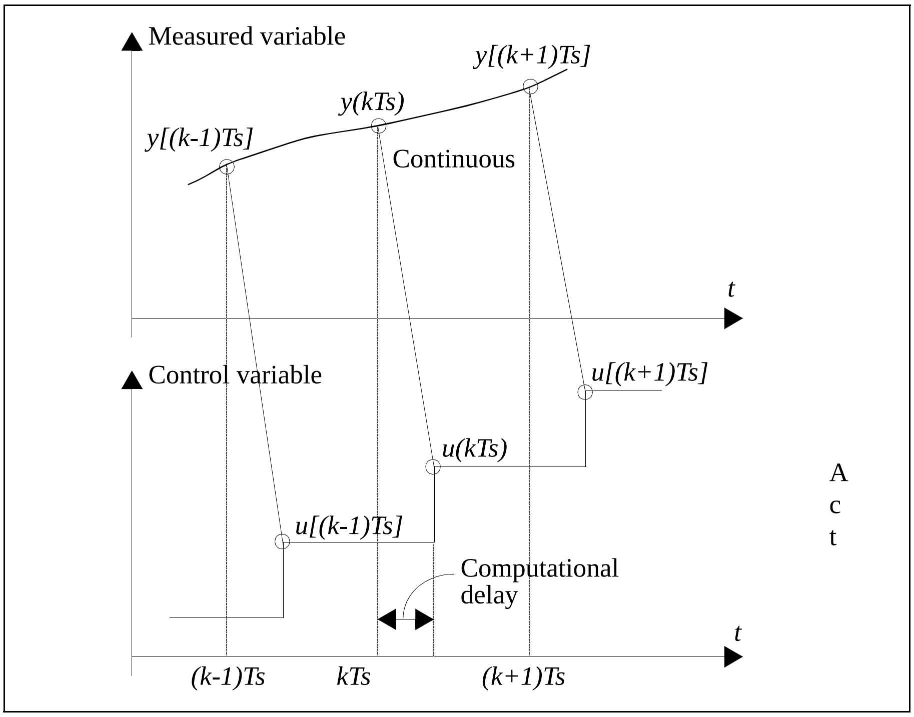


Figure 3.3 Synchronizing input and output. $y(kT_S)$ and $u(kT_S)$ are numbers stored in the computer.

In order to be concise, let us give the applied definition of the various type of signals and systems that will be used in the following sections:

Continuous-time signals (CT-signals) are defined over a continuous span of time. The amplitudes of these signals range either over a continuous range of values or a finite number of possible values. Sometimes one will use the term analog signals to denote CT-signals.

Discrete-time signals or sequences (DT-signals) are defined over only a particular set of discrete values of time, which means that such signals can be represented as sequences of numbers. A sequence of numbers, x, is denoted as

$$x = \{x(kT_s)\}, k = 0, 1, 2\ldots \qquad (3.2)$$

Although this implies that $x(kT_S)$ is actually the k'th member, it is convenient to denote the sequence itself by $x(kT_S)$. The short form of notation

$$x(kT_s) = x(k) = x_k \tag{3.3}$$

will also be used interchangeably throughout the text.

Sampled signals represent discretized version of CT-signals. They are a special case of DT-signals which are pulse amplitude modulated and denoted as

$$x^*(t) = \sum_{k=0}^{n} x(kT_s)\delta(t - kT_s)$$

Digital signals are signals where the information is in some kind of coded form. They are quantized in amplitude and discrete in time. Thus, they are a special case of DT-signals.

Systems are classified by the same criteria as signals.

Continuous (or analog) systems are systems where both input and output signals are CT-signals.

Discrete-time systems are systems whose input and output are DT-signals.

Sampled-data systems contain both discrete- and continuous time signals.

Digital Control system is a system that contains both digital and continuous time signals.

From the previous discussion we have seen that a power electronic system that contains a computer for control, operates on signals that are of digital and continuous nature. It may therefore be characterized as a digital control system.

If we assume that the computer and the signal converters have a sufficiently large word length, we may neglect the effect of amplitude quantizing. With this approximation our system can be said to be a sampled system consisting of a purely discrete component, (the computer algorithm), and a continuous system component, (the process or plant). Between these two sub-systems there must be some kind of signal conversion component, logically represented by the A/D and the D/A converters.

In order to analyse this system, we must have a mathematical representation of the various system elements. The aim of this chapter is to appropriately model each element in order to connect these in an overall representation of the complete system.

We will assume that the system components can be modelled as linear and time invariant components. Components in power electronic systems are, as will be shown, not all linear. A first approximation of the system's behaviour may be considered essentially linear, or at least having a linear working domain. Because of this approximation it is necessary to do computer simulations of the designed system to check the behaviour outside the linear domain.

3.2 Modelling of discrete time systems - The computer control algorithm represented as a difference equation

Assume that the input to the digital processor up to the time $t=kT_s$ has been $e(0)$, $e(T_s)$, $e(2T_s)$, ..., $e(kT_s)$ and the output sequence prior to that time was $u(0)$, $u(T_s)$, $u(2T_s)$, ..., $u((k-1)T_s)$. The next output at $t=kT_s$ is written as:

$$u_k = f(e_k, e_{k-1}, e_{k-2}, \ldots e_{k-n}; u_{k-1}, u_{k-2} \ldots u_{k-n})$$

We will assume that the computer algorithm performs a linear combination of the input and the past control output. Thus we write:

$$u_k = g_0 e_k + g_1 e_{k-1} + \ldots + g_n e_{k-n} - f_1 u_{k-1} - \ldots - f_n u_{k-n} \tag{3.4}$$

This is a linear difference equation. If the coefficients are constant it is said to be time invariant. The order of the equation is n if the signals from only the last n sampling instants enter the equation. If not all f_i are zero the equation is said to be recursive because it specifies a recursive procedure for determining the output in terms of the inputs and previous outputs. If all f_i are zero the equation is said to be non-recursive.

The algorithm of the general form (3.4) can adequately perform most control tasks in power electronic systems. The aim of the control system design procedure is to select the order of the equation, the sampling rate, and give values to g_i and f_i. This must be done in such a way that the overall system attains the desired dynamic properties.

3.2.1 A discrete PI-control algorithm

One of the most common control algorithms in power electronic systems is the proportional-integral (PI) control action. As an example of the origins of a difference equation we will consider a discrete approximation to the continuous PI-control law. Suppose we have a continuous input signal, *e(t)*, of which a segment is sketched in figure

(3.4), and we wish to compute an approximation to the continuous PI control law given by

$$u(t) = K_p\left(e(t) + \frac{1}{T_i}\int_{-\infty}^{t} e(\tau)d\tau\right) \tag{3.5}$$

with the transfer function

$$h_r(s) = K_p\frac{1 + sT_i}{sT_i} \tag{3.6}$$

Using only the discrete values *e(0),......,e(k-1), e(k)* the output of the discrete PI controller can be written as

$$u(k) = K_p\left(e(k) + \frac{1}{T_i}I(k)\right) \tag{3.7}$$

where $I(k)$ is the approximation to the integral up to the time $t = kT_s$. We will assume that we have an approximation of the integral up to the time $t=(k-1)T_s$ and we call it $I(k-1)$ the integral then can be written

$$I(k) = I(k-1) + \Delta I(k) \tag{3.8}$$

The problem is to obtain *I(k)* from this information. By interpreting the integral as the area under the curve $e(t)$ the problem reduces to finding an approximation to the area under the curve between $(k-1)T_s$ and kT_s. See figure (3.4). By the trapezoid approximation we can write:

$$\Delta I(k) = \frac{e(kT_s) + e((k-1)T_s)}{2}\cdot T_s \tag{3.9}$$

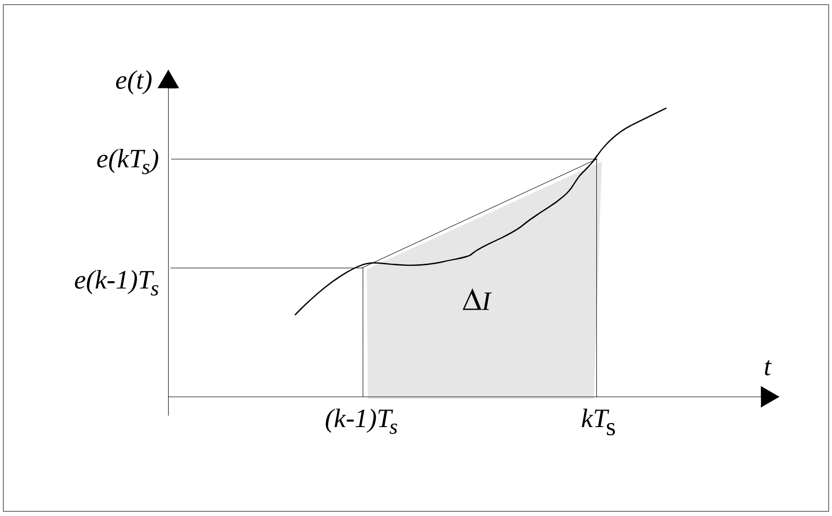


Figure 3.4 Trapezoid approximation of the integral.

By observing that

$$\frac{K_P}{T_i}(I(k-1)) = u(k-1)-K_p e(k-1) \tag{3.10}$$

and combining equations (3.7) through (3.10) we find

$$u(k) = u(k-1) - K_p e(k-1) + \frac{K_p}{T_i} \cdot \frac{T_s}{2}(e(k) + e(k+1)) + K_p e(k) \tag{3.11}$$

By collecting terms, this recursive differential equation describing a discrete PI control algorithm can be expressed as

$$u(k) = u(k-1) + g_0 e(k) + g_1 e(k-1) \tag{3.12}$$

where the coefficients are

$$g_0 = K_p\left(1 + \frac{T_S}{2T_i}\right) \tag{3.13}$$

$$g_1 = -K_p\left(1 - \frac{T_s}{2T_i}\right) \tag{3.14}$$

If we approximate the area under the curve *e(t)* in the time interval from $t=(k-1)T_S$ to $t=kT_S$ by the rectangle of height *e(k-1)* the resulting formula for the integral is called the Forward Rectangular Rule of integration and is given by

$$I(k) = I(k-1) + T_s e(k-1) \tag{3.15}$$

A third possible integration method is the Backward Rectangular Rule, given by

$$I(k) = I(k-1) + T_s e(k) \tag{3.16}$$

A block diagram representation of equation (3.12) is shown in Figure 3.5. Note that the data storage is represented by a time delay which delays the input data for one sampling period.

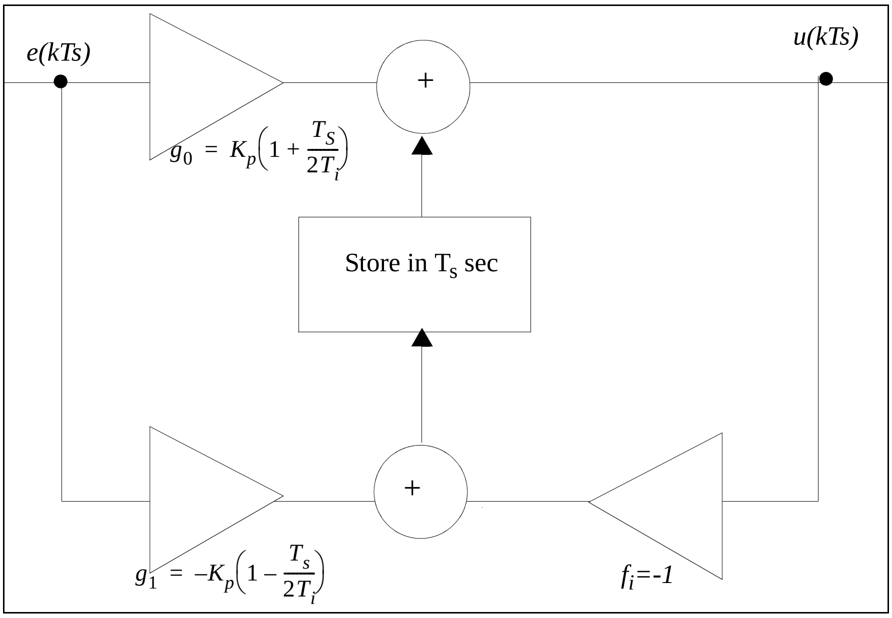


Figure 3.5 Block diagram representation of equation (3.12)

Recall that in a block diagram representation of CT-systems the basic element is the integrator. In DT-systems the basic element is the time delay (or storage) of T_s seconds.

3.2.2 Discretizing of a continuous system

We have seen that the computer control algorithm can be represented as a difference equation. As another example of the origins of a difference equation we will consider the situation where we have a time-discrete input control of a continuous plant. Suppose we have a continuous input signal, $u(t)$, and the output signal from the CT-plant, $y(t)$ as shown in Figure (3.6).

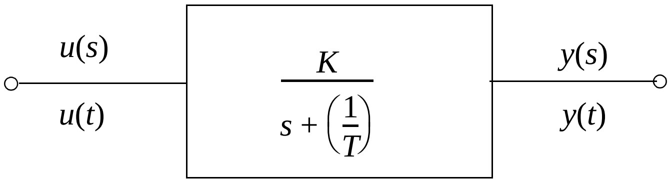


Figure 3.6 Block diagram representation of a first order plant.

The system differential equation is given by

$$\frac{d}{dt}y(t) + \left(\frac{1}{T}\right)y(t) = Ku(t) \tag{3.17}$$

The solution to this equation is known to be

$$y(t) = e^{-\left(\frac{1}{T}\right)(t-t_0)}y(t_0) + \int_{t_0}^{t} e^{-\left(\frac{1}{T}\right)(t-\tau)}Ku(\tau)d\tau \tag{3.18}$$

We will now assume that the input signal to the plant is generated by a computer which work with a sampling period of T_s seconds. We will further assume that the computer has a A/D converter which holds the value $u(t)$ constant between the sampling instants. The output signal $y(t)$ is sampled at the same instants as placement of a new input takes place. The situation is illustrated in Figure (3.7)

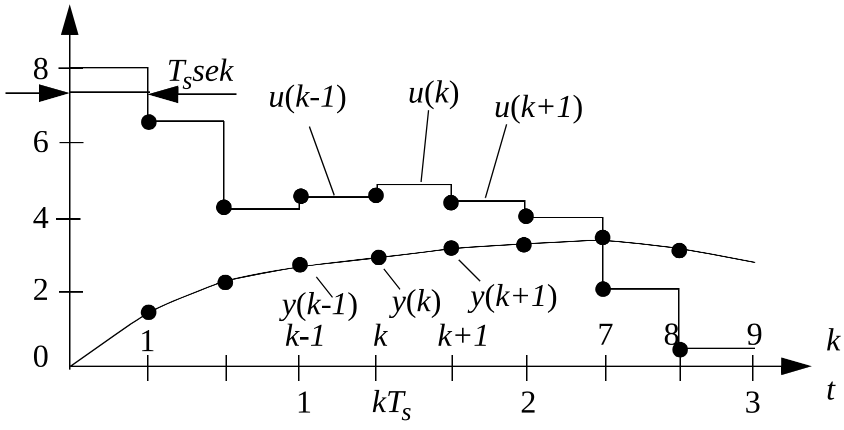


Figure 3.7 Input output signals in a continuous plant.

If we are not interested in the variables between the sampling instants but we want to focus on the values at the sampling instants we may define the following time discrete variables which are marked in the figure.

$$y(k) = y(kT_s) \quad ; k = 0, 1, 2, 3, ...$$

$$u(k) = u(t) \quad ; kT_s \le t < (k+1)T_s$$

By inserting $t_0 = kT_s$ and $t = (k+1)T_s$ in equation (3.18) we find:

$$y(k+1) = ay(k) + bu(k) \tag{3.19}$$

Where

$$a = e^{\left(-\left(\frac{1}{T}\right)\right)T_s} \tag{3.20}$$

$$b = K\int_0^{T_s} e^{-\left(\frac{1}{T}\right)\tau} d\tau = \frac{K\left(1 - e^{-\left(\frac{1}{T}\right)T_s}\right)}{\frac{1}{T}} \tag{3.21}$$

The difference equation (3.19) represents the result from discretizing of the continuous plant given by (3.17). The discretizing was carried out based on constant sampling period T_s and constant input $u(t)$ during the sampling interval.

One particular simple approximate method for discretizing differential equations which work fine for short sample intervals, is the Eulers method (also called the forward rectangular rule).

$$\frac{dy}{dt} \cong \frac{y(k+1) - y(k)}{T_s} \tag{3.22}$$

The approximation given in equation (3.22) will be used in place of all derivatives that appear in the differential equation.

Applying Eulers method to equation (3.17) we arrive to a difference equation given by (3.19) where the constant coefficients are given by:

$$a = 1 - \left(\frac{1}{T}\right)T_s \tag{3.23}$$

$$b = KT_s \tag{3.24}$$

By comparing (3.23) with (3.20) and (3.24) with (3.21), we see that the coefficients calculated by the approximate method are similar to the result we get if we take only the first two terms in the series expansion of the exponential function of the exact solution.

Example 3.1

Given a first order low-pass analog passive filter with low frequency gain 1 and time constant 1s. The filter topology and the transfer function is given in figure (3.8)

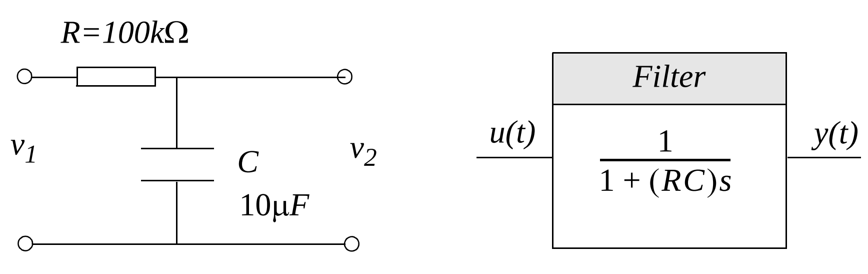


Figure 3.8 Analog low-pass filter

The output signal is sampled 3 times in a second. $\left(T_s = \frac{1}{3}\right)$. *The input signal is constant during the sampling interval* T_S. *Find the difference equation describing the filter.*

3.2.3 Difference equations

We have seen that discrete systems can be modelled in the time domain by difference equations. We have seen that the difference equation can be evaluated directly by a computer and that they can also represent discrete models of physical processes defined at the sampling instants. If the system is linear and time invariant the equation will be a *constant coefficients difference equation* (CCDE).

A general linear difference equation of order *n* with constant coefficients can be written as:

$$y(k) + a_1 y(k-1) + \ldots + a_n y(k-n) = b_1 u(k-1) + \ldots + b_n u(k-n) \quad (3.25)$$

For solving linear time-invariant difference equation there are different techniques that can be used. One approach consists of finding the complementary and the particular parts of the solution, in a manner similar to that used in the classical solution of linear differential equations. We will not take that approach here, but use a direct method which is a sequential procedure simular to the method used in the digital computer solution of difference equations. The method will be illustrated by examples.

To solve a specific CCDE we need a starting time (value of k) and a number of initial values depending on the order of the equation. The initial conditions represents the state of the system characterized by the computer memory at that time. For a physical process (for example a power electronics plant) the initial state may represent the energy stored in the system at starting time.

Example 3.2 First order difference equation

It is desired to find the unit step response y(k) for the differenc equation

$$y(k) = \frac{1}{2} y(k-1) + u(k-1) \quad \text{for} \quad k \geq 0$$

Compared with the general equation (equation 3.25) we see that $a_1=-0.5$ *and* $b_1=1$.

The y(k) can be determined by solving the difference equation first for k=0, then for k=1, k=2 and so on. Thus

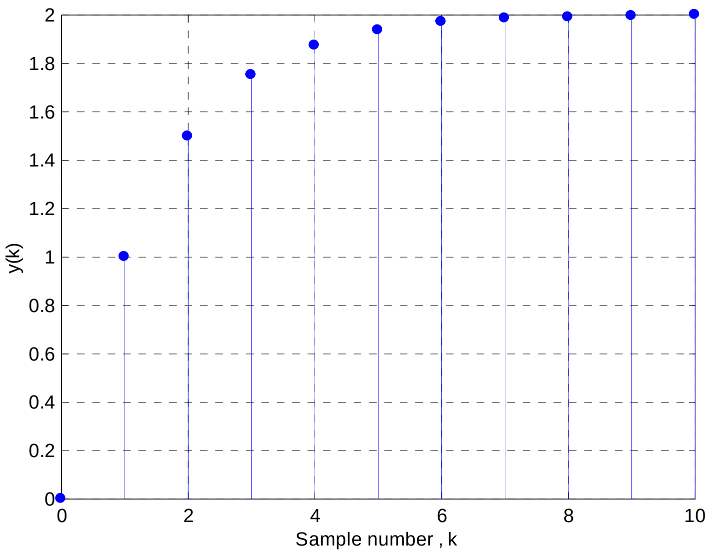
Table 3.1

k	*u(k-1)*	*0.5y(k-1)*	*y(k)*
0	0	0	0
1	1	0	1
2	1	0.5	1.5
3	1	0.75	1.75
4	1	0.875	1.875
5	1	0.975	1.9375

Note the sequential nature of the solution process. Continuing this procedure we can find y(k) for any value of k. This technique is not practical except when implemented on a digital computer. For this example a matlab script which solve the equation is :

```
k=0:10;  % 11 samples
u=ones(size(k)); % step input
bcoeff=[0 1];
acoeff=[1 -0.5];
y=filter(bcoeff,acoeff,u);
[k' y'] % display values of k and y
stem(k,y,); % plot responce
xlabel('Sample number, k');
ylabel('y(k)');
grid;
```

Plot from the above code is:



Example 3.3 Second order difference equation

Consider the difference equation

$$y(k) - y(k-1) + y(k-2) = u(k-2) \tag{3.26}$$

Calculate the impulse response (u(0)=1; u(k)=0 for $k \neq 0$ *)*

Table 3.2

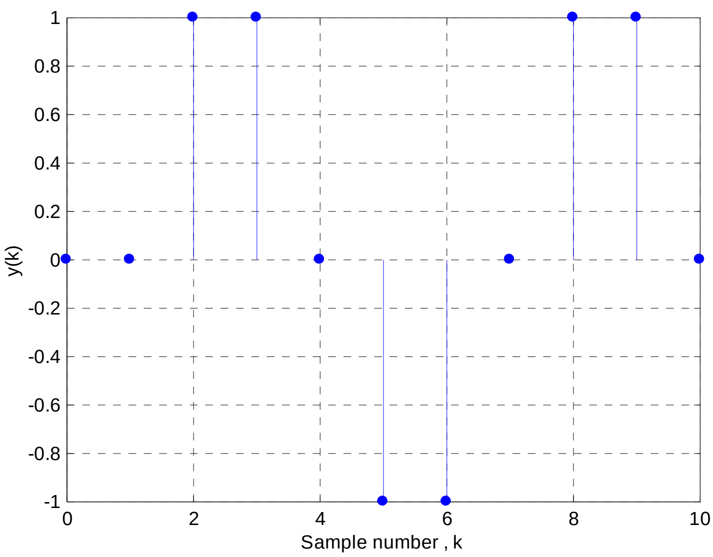
k	*u(k-2)*	*y(k-1)-y(k-2)*	*y(k)*
0	0	0	0
1	0	0	0
2	1	0	1
3	0	1	1
4	0	0	0
5	0	-1	-1
6	0	-1	-1

A matlab script for the solution is given below

```
k=0:10;
u=zeros(size(k));
```

```
u(1)=1;
bcoeff=[0 0 1];
acoeff=[1 -1 1];
y=filter(bcoeff,acoeff,u);
[k' y'] % display values of k and y
stem(k,y,); % plot responce
xlabel('Sample number, k');
ylabel('y(k)');
grid;
```

Plot from the above code is:



We see from the last examples that the solution to difference equations, simular to differential equations, can have stable response and approach a specific limit (the limit is 2 in example (3.2)) or be oscillating with a constant amplitude. In some case the response can be growing without bound. If the response of a dynamic system to any finite initial conditions can grow without bound, we call the system *unstable*. We would like to be able to examine equations like (3.25) and, without having to solve them explicitly, see if the response will be stable and even understand the general shape of the solution.

SOLUTION OF CCDE

The classical methods for solving difference equation is similar to the methods of differential equations. These methods require the prior determination of the homogeneous solution. The homogeneous difference equation give us the characteristic equation of the corresponding difference equation.

The solution to the characteristic equations describes the natural behaviour of the solution and predicts the stability of the solution to the difference equation.

Given the CCDE

$$\begin{aligned} y(k) = & -a_1 y(k-1) - a_2 y(k-2) - \ldots - a_n y(k-n) \\ & + b_0 u(k) + \ldots + b_m u(k-m) \end{aligned} \tag{3.27}$$

The homogeneous equation is

$$y(k) + a_1 y(k-1) + a_2 y(k-2) + \ldots + a_n y(k-n) = 0 \tag{3.28}$$

Assuming $y(k) = A\lambda^k$ we get the equation

$$A\lambda^k + a_1 A\lambda^{k-1} + \ldots + a_n A\lambda^{k-n} = 0 \tag{3.29}$$

$$A\lambda^k(\lambda^n + a_1\lambda^{n-1} + \ldots a_n\lambda^0) = 0 \tag{3.30}$$

The characteristic equation is:

$$\lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0 \tag{3.31}$$

Note that λ^k plays the same role in DE as $e^{\lambda t}$ in LTI differential equations.

The geneal solution to the CCDE is a linear combination of solutions, based on the roots of the characteristic equation.

For example

$$y(k) = 0{,}9y(k-1) - 0{,}2y(k-2) + b_0 u(k) \tag{3.32}$$

The homogeneous equation

$$\lambda^2 - 0{,}9\lambda + 0{,}2 = 0 \tag{3.33}$$

The roots are

$\lambda_1 = 0{,}5$ and $\lambda_2 = 0{,}4$

and the solution is

$$y(k) = A_1(0{,}5)^k + A_2(0{,}4)^k \tag{3.34}$$

Since both roots are inside the unit circle, equation (3.32) is stable. The value of the roots describe the natural behaviour of the solution $y(k)$.

Assuming that one of the roots is real.

$$y(k) = A\lambda^k \tag{3.35}$$

The numerical value of $y(k)$ is summarized in table 3.3.

Table 3.3

λ	value of $y(k)$ for $k = 0, 1, 2\ldots$
$\lambda > 1$	increasing
$\lambda = 1$	constant
$0 < \lambda < 1$	decreasing
$-1 < \lambda < 0$	decreasing, alternating sign
$\lambda = -1$	oscillating between +A and -A
$\lambda < -1$	increasing, alternating sign

Complex or imaginary roots always occur in conjugate pairs and give solution of the form:

$$y(k) = A_1(\lambda)^k + A_2(\lambda')^k \tag{3.36}$$

where

$\lambda = a + jb$

$\lambda' = a - jb$

Multiple real roots generate behaviour which consists of the term $k\lambda^k$. In general, if the repeated roots are indicated by the factor $(\lambda - r_k)^q$, terms of the form:

$$A_1 k^{q-1}(r_k)^k + A_2 k^{q-2}(r_k)^k + \dots + A_q (r_k)^k$$

will appear in the solution.

A practical method of finding the particular solution to the difference equation is to use the z-transform approach which will be presented later.

3.3 Z-transform representation of discrete systems.

So far we have treated the DT-system in the time domain. We will now introduce the Z-transform which is a convenient tool when dealing with problems of a discrete nature. The role of the Z-transform (ZT) in discrete systems is similar to that of the Laplace transform in continuous systems.

3.3.1 Definition of the Z-transform.

The Z-transform maps a time sequence into the complex *z*-plane.

Given a time sequence $\{x(kT_s)\}$. The basic one sided Z-transform is defined as [15]:

$$x(z) = Z\{x(kT_s)\} = \sum_{k=0}^{\infty} x(kT_s) z^{-k} \tag{3.37}$$

$x(kT_s) = 0$ for $k < 0$.

That is a causal sequence. Since in practical situations the sequence is causal, this one-sided ZT will be satisfactory for most engineering applications. To assure that the ZT exists we assume that $\lim_{k \to \infty} \sum x(kT_s) z^{-k}$ exists in some region in the complex z-plane.

A fundamental property of the ZT is that there is a one-to-one correspondence between the sequence $\{x(kT_s)\}$, and its ZT. That is, given $x(z)$ we can recover $\{x(kT_s)\}$ via the inverse Z-transform (IZT) which we denote as:

$$\{x(kT_s)\} = Z^{-1}\{x(z)\} = \frac{1}{2\pi i}\int_C z^{k-1}x(z)dz \tag{3.38}$$

where the integral is taken along a closed path, C, in the complex z-plane which must contain the origin. FT and IFT form a transform pair.

If the sequence $x(kT_s)$ is the result of sampling a CT function $x(t)$ we must observe that there is no a unique correspondence between $x(x)$ and $x(t)$. This occurs because many CT-signals can give the same $\{x(kT_s)\}$ and therefore $x(x)$. The ZT represents a sampled CT-signals $x(t)$, with samples $x(kT_s)$, at the sampling instants only.

From the definition of the ZT, equation (3.37), many useful properties of the ZT can be derived. The most importants ones are summarized in appendix A.

3.3.2 The Z-transform of some elementary time functions/sequences.

For larger reference we will in this section calculate the ZT for some elementary time sequences. The calculations will be based on the ZT-definition.

(a) The unit impulse.

$$\delta(k) = \begin{cases} 1 & \text{for k=0} \\ 0 & \text{for k=1,2,...,}\infty \end{cases}$$

$$Z[\delta(k)] = \sum_{k=0}^{\infty} 1z^{-k} = 1$$

(b) The unit step.

$$u(k) = 1 \qquad for \qquad 0 \le k \le \infty$$

$$Z[u(k)] = 1 + z^{-1} + z^{-2} + \dots$$

$$= \frac{1}{1-z^{-1}} = \frac{z}{z-1} \qquad |z| > 1$$

(c) The discrete ramp function = $f(kT_s) = kT_s$

$$F(z) = \sum_{k=0}^{\infty} (kT_s)z^{-k} = T_s z^{-1} + 2T_s z^{-2} + 3T_s z^{-3} + \qquad (3.39)$$

To express $F(z)$ in closed form, we first multiply both sides by z^{-1}, resulting in

$$z^{-1}F(z) = T_s z^{-2} + 2T_s z^{-3} + \dots \qquad (3.40)$$

Subtracting the last equation from the first we get

$$\begin{aligned}(1 - z^{-1})F(z) &= T_s z^{-1} + T_s z^{-2} + T_s z^{-3} + \dots \qquad (3.41)\\ &= T_s z^{-1}(1 + z^{-1} + z^{-2} + \dots)\\ &= T_s z^{-1}\left(\frac{1}{1 - z^{-1}}\right)\end{aligned}$$

$$F(z) = \frac{T_s z^{-1}}{(1 - z^{-1})^2} = \frac{T_s z}{(z-1)^2} \qquad (3.42)$$

(d) The discrete exponential function

$x(k) = a^k$

$$\begin{aligned}Z[x(k)] = Z[a^k] &= \sum_{k=0}^{\infty} a^k \cdot z^k \qquad (3.43)\\ &= 1 + \left(\frac{z}{a}\right)^{-1} + \left(\frac{z}{a}\right)^{-2} + \dots\\ &= \frac{1}{1 - az^{-1}} = \frac{z}{z-a} \qquad |z| > a\end{aligned}$$

Another form of the discrete exponential function is:

$x(k) = e^{-akT_s}$ where a is a real constant.

$$x(z) = Z[e^{-akT_s}] = \sum_{k=0}^{\infty} e^{-akT_s} z^{-k} \qquad (3.44)$$

$$= 1 + e^{-aT_s} z^{-1} + (e^{-2aT_s} z^{-2}) + \dots$$

$$= 1 + e^{-aT_s} z^{-1} + (e^{-aT_s} z^{-1})^2 + \dots$$

This infinite series converges for all values of *z* that satisfy

$\left|e^{-aT_s} z^{-1}\right| < 1$

$$= \frac{1}{1 - e^{-aT_s} z^{-1}} = \frac{z}{z - e^{-aT_s}}$$

for $\left|e^{-aT_s} z^{-1}\right| < 1 \qquad or \qquad \left|z^{-1}\right| < e^{aT_s}$

(e) The discrete sinusoidal function

$f(kT_s) = \sin(\omega kT_s)$

$$F(z) = \sum_{k=0}^{\infty} (\sin \omega kT_s) z^{-k} \qquad (3.45)$$

It is convenient to express $\sin \omega kT_s$ in the exponential form

$$F(z) = \sum_{k=0}^{\infty} \left(\frac{e^{j\omega kT_s} - e^{-j\omega kT_s}}{2j}\right) z^{-k} \qquad (3.46)$$

Since the z transform of the exponential function is:

$$Z\left(e^{-akT_s}\right) = \frac{1}{1 - e^{-aT_s} z^{-1}}$$

then we get:

$$F(z) = \frac{1}{2j}\left[\frac{1}{1-e^{j\omega T_s}z^{-1}} - \frac{1}{1-e^{-j\omega T_s}z^{-1}}\right]$$

$$= \frac{1}{2j}\frac{\left(e^{j\omega T_s} - e^{-j\omega T_s}\right)z^{-1}}{1-\left(e^{j\omega T_s} + e^{-j\omega T_s}\right)z^{-1} + z^{-2}}$$

$$= \frac{z^{-1}\sin\omega T_s}{1-2z^{-1}\cos\omega T_s + z^{-2}}$$

Therefore:

$$F(z) = \frac{z\sin\omega T_s}{z^2 - 2z\cos\omega T_s + 1} \qquad \text{for } |z| > 1 \tag{3.47}$$

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3.3.3 The transfer function for discrete time systems.

We know that for CT-systems the Laplace transform (LT) enables us to represent these systems in terms of transfer functions. We will now show that difference equations and the ZT enable us to do so for DT-systems.

We have seen that the computer algorithm in Figure 3.1 represents a DT-system expressed by the following input-output relationship:

$$u_k = g_0 e_k + g_1 e_{k-1} + \ldots + g_n e_{k-n} + -f_1 u_{k-1}(-f_n)u_{k-n} \tag{3.48}$$

Assume that the system is initially at rest, that is, all initial conditions are zero.

If we multiply (3.48) by z^k and sum over k we get:

$$\sum_{k=0}^{\infty} u_k z^{-k} = g_0 \sum_{k=0}^{\infty} e_k z^{-k} + g_1 \sum_{k=0}^{\infty} e_{k-1} z^{-k} + \ldots + g_n \sum_{k=0}^{\infty} e_{k-n} z^{-k}$$

$$-f_1\sum_{k=0}^{\infty}u_{k-1}z^{-k}-f_2\sum_{k=0}^{\infty}u_{k-2}z^{-k}-\ldots-f_n\sum_{k=0}^{\infty}u_{k-n}z^{-k}$$

For a general term which is of the form:

$$\alpha_j\sum_{k=0}^{\infty}k_{k-j}z^{-k} \tag{3.49}$$

Using the definition of the ZT which defines $x_i = 0$ for $i < 0$

$\sum_{i=0}^{\infty}x(iT_s)z^{-i} = x(z)$ and letting k-j = i we obtain

$$\alpha_j z^{-j}\sum_{i=-j}^{\infty}x_i x^{-i} = \alpha_i z^{-j}\sum_{i=0}^{\infty}x(iT_s)z^{-i} = \alpha_j z^{-j}x(z) \tag{3.50}$$

Thus (3.48) is now transformed to an algebraic equation in the z-domain

$$u(z) = g_0e(z)+\ldots+g_nz^{-n}e(z)-f_1z^{-1}u(z)-\ldots-f_nz^{-n}u(z) \tag{3.51}$$

Rearranging the terms gives:

$$u(z) = \frac{g_0+g_1z^{-1}+\ldots+g_2z^{-n}}{1+f_1z^{-1}+\ldots f_nz^{-n}}e(z) \tag{3.52}$$

This result could also been obtained directly by the backward shift theorem.

The ratio

$$\frac{u(z)}{e(z)} = h_R(z) = \frac{g_0+g_1z^{-1}+\ldots+g_nz^{-n}}{1+f_1z^{-1}+\ldots+f_nz^{-n}} \tag{3.53}$$

is defined as the transfer function $h_R(z)$ which also can be written as

$$h_R(z) = \frac{g_0 z^n + g_1 z^{n-1} + \dots g_n}{z^n + f_1 z^{n-1} + \dots f_n} \tag{3.54}$$

The causality condition, or practical realization of equation (3.54) requires that no future values can be used for calculation of *uk*. This is reflected in the ZT function (3.54) by the fact that the order of the numerator must be less than or equal to the order of denominator.

Factoring the polynomials in the numerator and denominator of $h_R(z)$, we obtain

$$h_R(z) = \frac{g_0(z-z_1)(z-z_2)\dots(z-z_n)}{(z-p_1)(z-p_2)\dots(z-p_n)} \tag{3.55}$$

where z_i and p_i denote the ith zero and pole, respectively, of $h_R(z)$.

The impulse response of a DT-system is the output that results when the input is the unit impulse sequence {δ(k)} =1 for k = 0 and zero elsewhere.

The ZT of the unit impulse is:

$$Z[\delta(k)] = \sum_{k=0}^{\infty} 1z^{-k} = 1 \tag{3.56}$$

The corresponding output is:

$y(z) = h(z) * 1 = h(z)$

Upon inverting the z-transform the result is

$h(k) = Z^{-1} h(z)$

Thus the inverse transform of the transfer function is the impulse response sequence.

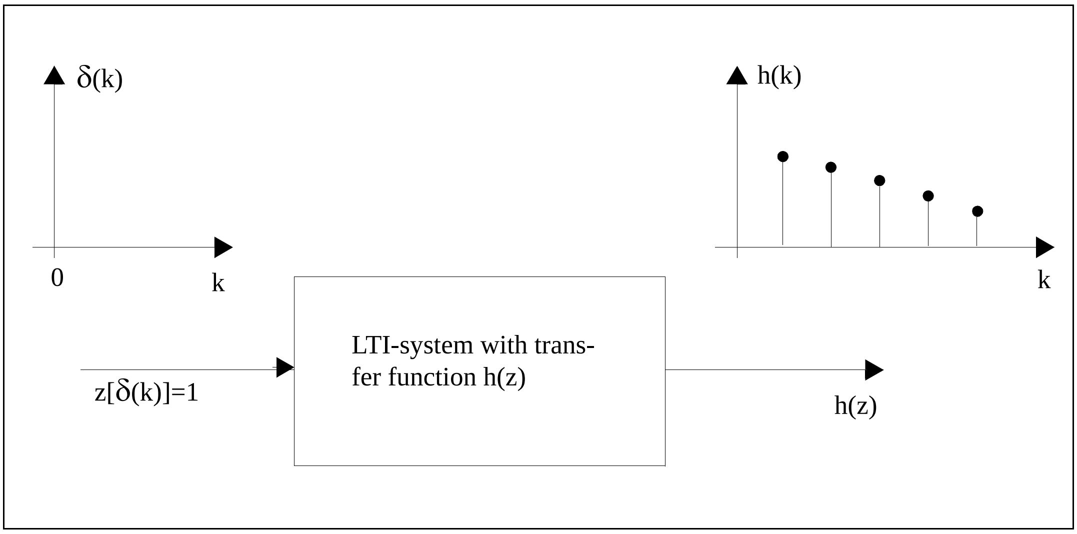

Figure 3.9 The impulse response of a discrete time invariant system

3.4 Representation of the conversion between discrete and continuous signals

Special electronic devices are necessary to make the digital controller able to exchange information with the CT-system which is to be controlled. These devices are logically represented in Figure 3.1 as D/A and A/D converters. The aim of this section is to present a mathematical representation of these conversion processes.

The D/A converter acts on the DT-sequence from the computer.

This signal, u_k, represented as a numerical content of some register in the processor, is converted to a CT-signal u(t) by keeping the amplitude constant between the sampling instants.

That is to say:

$$u(t) = u(kT_s) \qquad kT_s \le t \le (k+1)T_s$$

From a functional viewpoint it may be regarded as a device which consists of a decoder and a zero order hold unit as shown in Figure 3.10. The decoder acts simply as a constant gain.

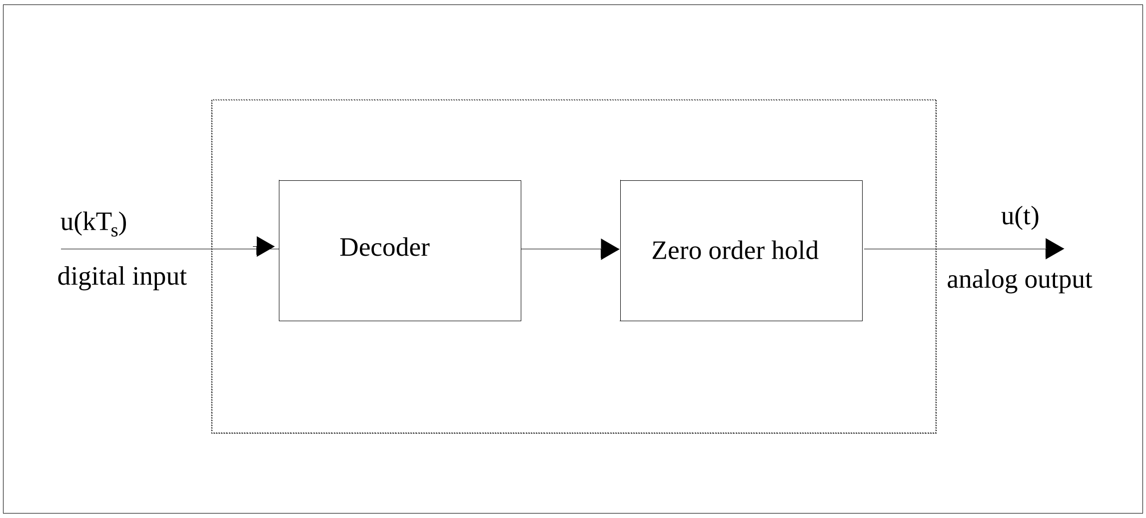

Figure 3.10 Functional block diagram representation of D/A conversion

The A/D conversion process is the transformation of information contained in a CT-signal into a digital-coded word. This involves the following sequential operations: sample, quantizing, and encoding.

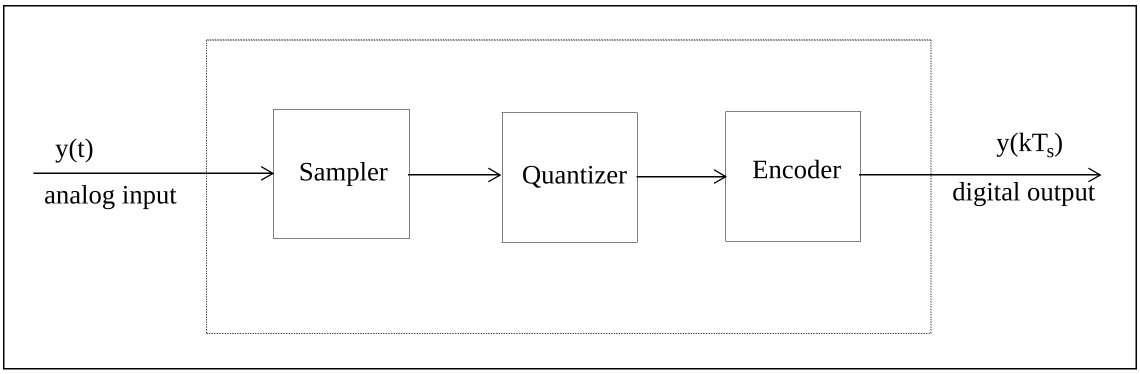

Figure 3.11 Functional block diagram representation of A/D conversion

If we assume that the resolution of the A/D conversion is very high, quantizing can be neglected. The encoder also acts simply as a constant gain, normally a unity gain. Thus for analytical purposes the block diagram can be reduced to a sampler.

We are now going to establish a mathematical model for the basic elements sampler and hold element.

The sampler can be considered as a switch, which is closed in a very short time interval, of equal length, at every sampling instant. The pulses thus generated will have a strength or area proportional to the magnitude of the input signal at the sampling instants.

We can model this process by a mathematical idealization where the CT signal, x(t), is amplitude modulated by the impulse train

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s) \tag{3.57}$$

as shown in Figure 3.12.

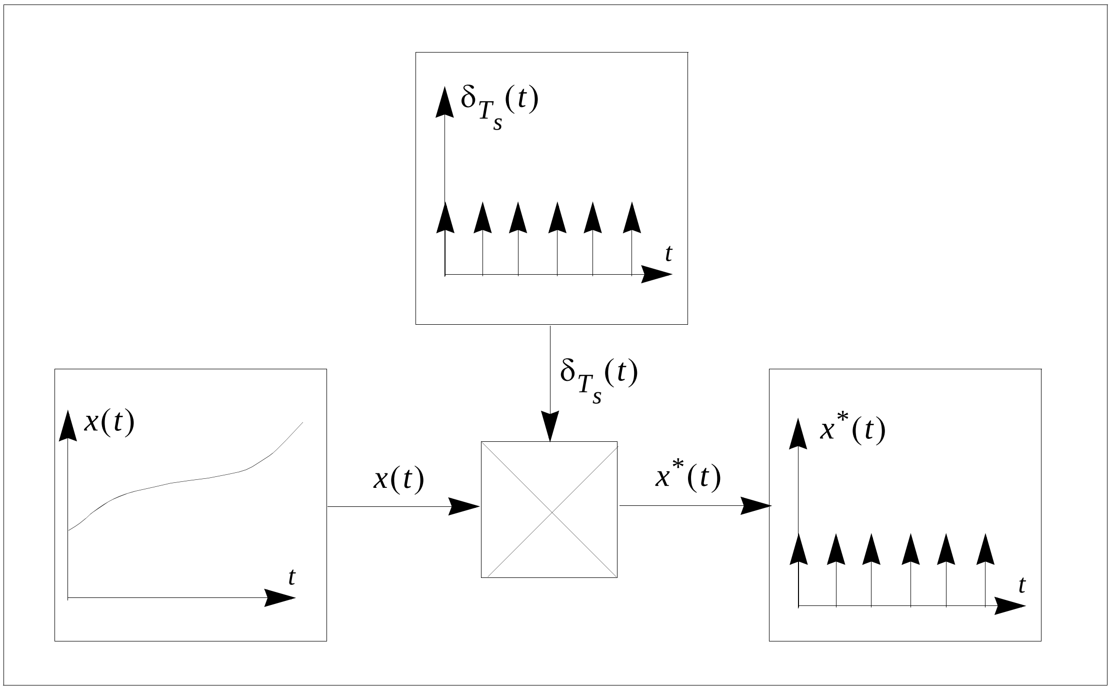

Figure 3.12 Representation of ideal sampling

The output from this ideal sampler is a train of impulses with strengths equal to $x(kT_s)$:

$$x^*(t) = x(t) \sum_{k=-\infty}^{\infty} \delta(t - kT_s) = \sum_{k=-\infty}^{\infty} x(kT_s)\delta(t - kT_s) \tag{3.58}$$

The symbol (*) is a common way of denoting a signal sampled by ideal impulse sampling. Later in the text, a switch with the sampling period indicated below, will be used to represent an ideal sampler.

Since the train of impulses is a periodic function with period T_s and angular frequency $\omega_s = \frac{2\pi}{T_s}$ the function can be represented by the complex Fourier series

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_s t} \tag{3.59}$$

where the complex Fourier coefficient is given by

$$c_n = \frac{1}{T_s} \int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta_T(t) e^{-jn\omega_s t} dt = \frac{1}{T_s} \tag{3.60}$$

Thus the infinite train of impulses may be written as:

$$\delta_T(t) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} e^{jn\omega_s t} \tag{3.61}$$

If we substitute (3.61) for the impulse train in (3.58), for the sampled signal we get:

$$x^*(t) = \frac{x(t)}{T_s} \sum_{n=-\infty}^{\infty} e^{jn\omega_s t} \tag{3.62}$$

From the definition of the Laplace transform we can now calculate the LT of the signal x*(t)

$$x^*(s) = \int_0^{\infty} x^*(t) e^{-st} dt = \frac{1}{T_s} \int_0^{\infty} x(t) \sum_{n=-\infty}^{\infty} e^{jn\omega_s t} e^{-st} dt \tag{3.63}$$

which is equal to

$$x^*(s) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} x(s - jn\omega_s) \tag{3.64}$$

where x(s) is the LT of the CT signal x(t).

Another expression for the LT of the sampled signal x*(t) can be derived directly from (3.58)

$$x^*(s) = \int_0^{\infty} \sum_{k=-\infty}^{\infty} x(kT_s)\delta(t - kT_s)e^{-st}dt$$

$$x^*(s) = \sum_{k=0}^{\infty} x(kT_s)e^{-skT_s} \tag{3.65}$$

The expressions (3.64) and (3.65) have many interesting implications. We will return to eq. (3.65) in a later section. One of the most interesting implications of (3.64) is the important sampling theorem which we are now going to discuss.

The operation of sampling a CT signal gives a sampled frequency domain function which can be evaluated by simply evaluating (3.64) on the s=jω axis. This gives:

$$x^*(j\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} x[j(\omega - n\omega_s)] \tag{3.66}$$

where x(jω) is the Fourier transform of the unsampled function x(t).

The equation (3.66) makes the connection between the FT of x(t) and the FT of x*(t) very clear. Consider a band limited signal x(t) which has a amplitude spectrum |x(jω)| and a cut off frequency ω_0 as shown in Figure 3.13.

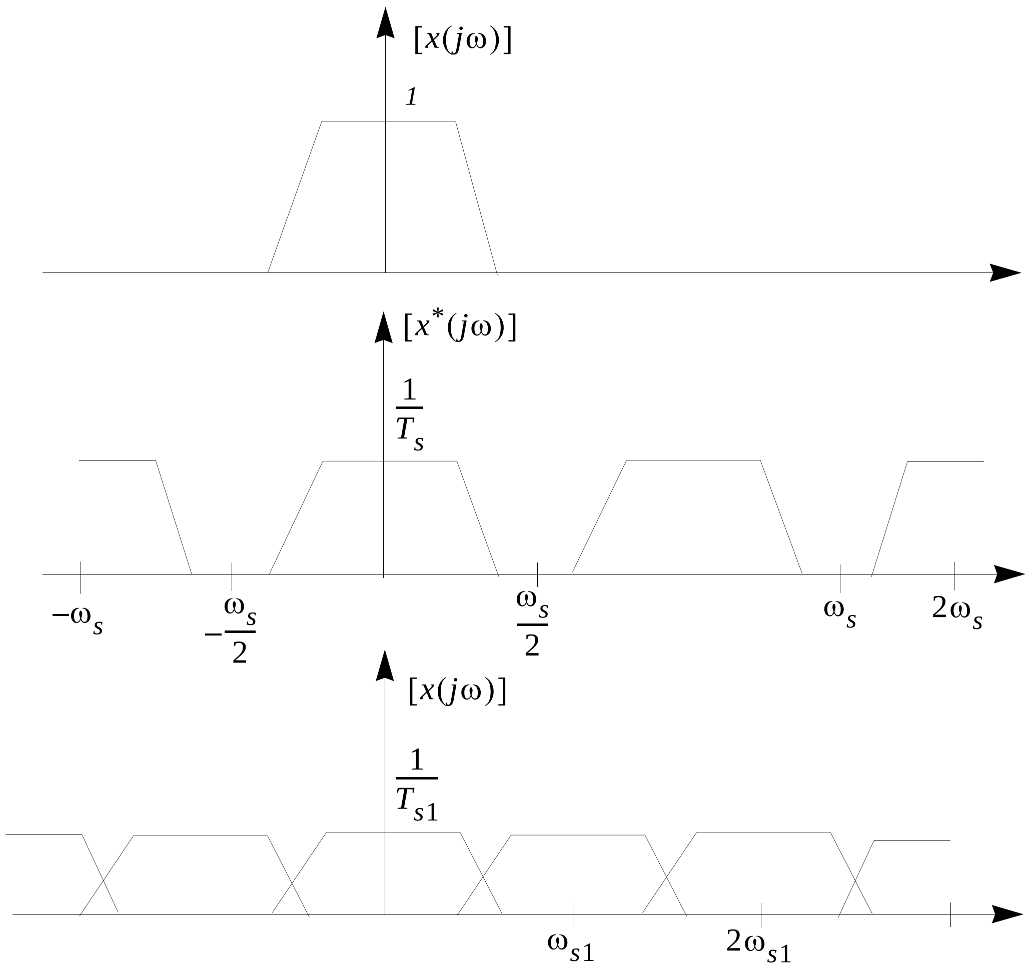

Figure 3.13 The effect of sampling on the amplitude spectrum of a band limited signal.

As we see from the figure and equation (3.66) the amplitude spectrum of x(t) is repeated with period ω_s along the ω-axis. When the sampling frequency $\omega_s > 2\omega_0$ the spectrum $|x^*(j\omega)|$ will be periodic with no overlap. The sampler acts as a harmonic generator.

If we lower the sampling frequency so that $\omega_s < 2\omega_0$ we get overlap between the base band and the frequency shifted bands. This is called aliasing. The higher frequencies are folded back into the low frequencies. Information in the CT signal spectrum is now lost. It is not possible to reconstruct the original frequency spectrum. Figure 3.14 clearly shows what happens when a single frequency signal is sampled at a rate which is too low (i.e. aliasing occurs).

We will recall here that the derivation of the frequency spectrum $|x^*(j\omega)|$ was based on ideal sampling. Non-ideal sampling (i.e. finite-width pulse amplitude sampling) will give the same frequency distribution of $|X^*(j\omega)|$, but the amplitudes will have a factor $\frac{p}{T_s}\left|\frac{\sin(n\omega_s p/2)}{(n\omega_s p/2)}\right|$ instead of $\frac{1}{T_s}$ where p is the pulse width. [16].

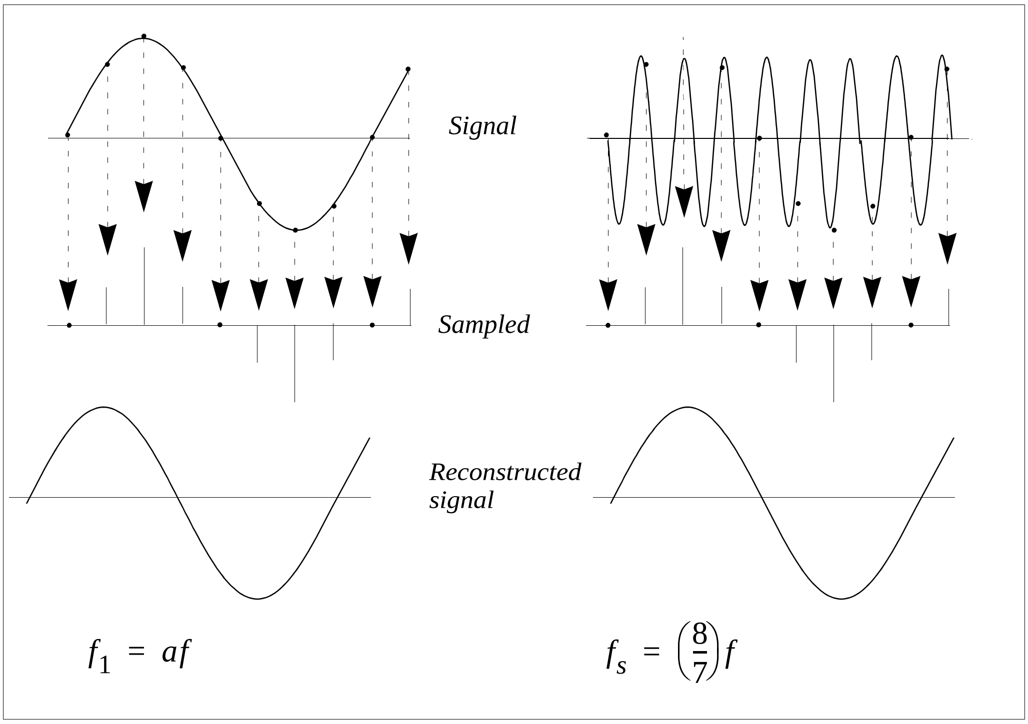

Figure 3.14 The higher frequency signal is folded down to a low frequency signal due to sampling

We can now conclude with the Shannon's Sampling Theorem which states: If a signal contains no frequency higher than ω_0 radians per second, it is completely characterized by the values of the signal sampled with a frequency $\omega_s > 2\omega_0$. The frequency $\frac{\omega_s}{2} = \frac{\pi}{T_s}$ is often called the Nyquist frequency.

In practice however, stability of the closed-loop system and other practical considerations may make it necessary to sample at a rate higher than this theoretical minimum.

Strictly speaking, a band-limited signal rarely exists in a physically control system. Only approximate band limited signals are found. Therefore, in practice when sampling a CT signal one must:

1. Choose a sampling frequency $\omega_s > 2\omega_b$ where ω_b is the highest frequency of interest in the CT signal.
2. Implement a low pass analog pre-sampling filter with cross-over frequency, ω_f where $\omega_b < \omega_f < \frac{\omega_s}{2}$

The highest frequency of interest is related to the bandwidth of the close-loop system. The selection of sampling rate can then be based on bandwidth, or equivalently, on the rise time of the closed loop system. A rule of thumb often used is to choose the sampling frequency about 10 times the bandwidth, or 3-4 samples during the rise time [17].

Because of the high frequency components inherently present, it is not desirable to apply the pulse sampled or discrete signal directly to an analog system. An equivalent time domain explanation is that these short pulses are not able to control the continuous plant without some sort of a reconstruction device. It is the strength of these pulses that are of interest. In a practical situation the sampled signal must therefore be followed by some sort of data hold, most often a zero-order hold, which is integrating each pulse.

The purpose of the zero order hold reconstruction filter is to hold the $u(kT_s)$ value until the succeeding sampling instant, as illustrated in Figure 3.15.

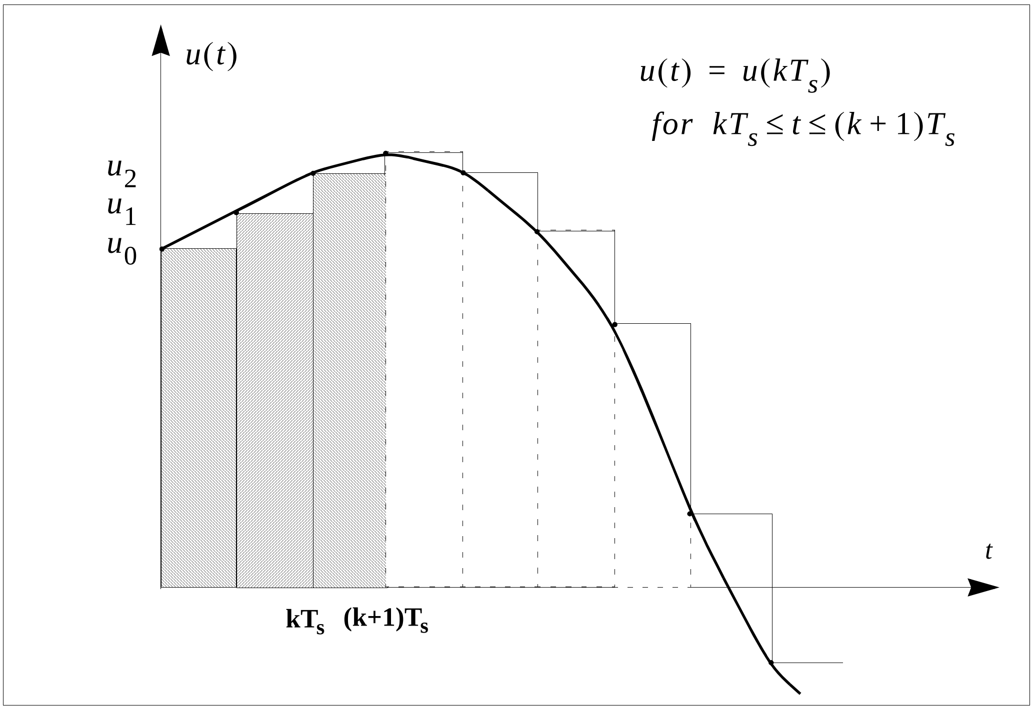

Figure 3.15 Reconstruction with a zero order hold

The impulse response of the zero order hold filter can be written as:

$$h(t) = u_1(t) - u_1(t - T_s) \tag{3.67}$$

where u(t) denotes a unit step function.

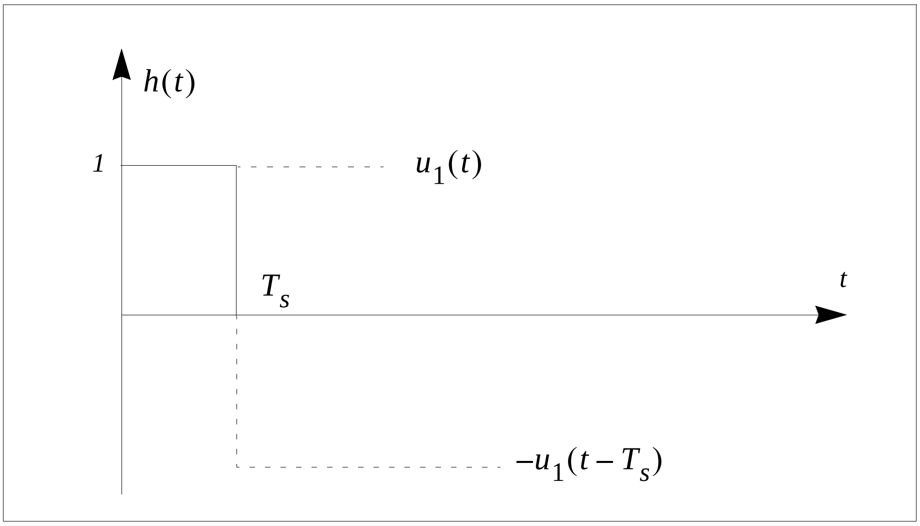

Figure 3.16 The impulse response of zero order hold.

For the LT we have:

$$h_h(s) = \frac{1}{s} + \frac{-e^{-sT_s}}{s} = \frac{1-e^{-sT_s}}{s} \tag{3.68}$$

And the frequency response is:

$$\begin{aligned} h_h(j\omega) &= \frac{1-e^{-j\omega T_s}}{j\omega} = e^{-j\omega\frac{T_s}{2}}\left\{\frac{e^{j\omega\frac{T_s}{2}} - e^{-j\omega\frac{T_s}{2}}}{2j}\right\}\frac{2j}{j\omega} \\ &= T_s e^{-j\omega\frac{T_s}{2}}\frac{\sin\frac{\omega T_s}{2}}{\frac{\omega T_s}{2}} \end{aligned} \tag{3.69}$$

The amplitudes and phase plots are shown in Figure 3.17.

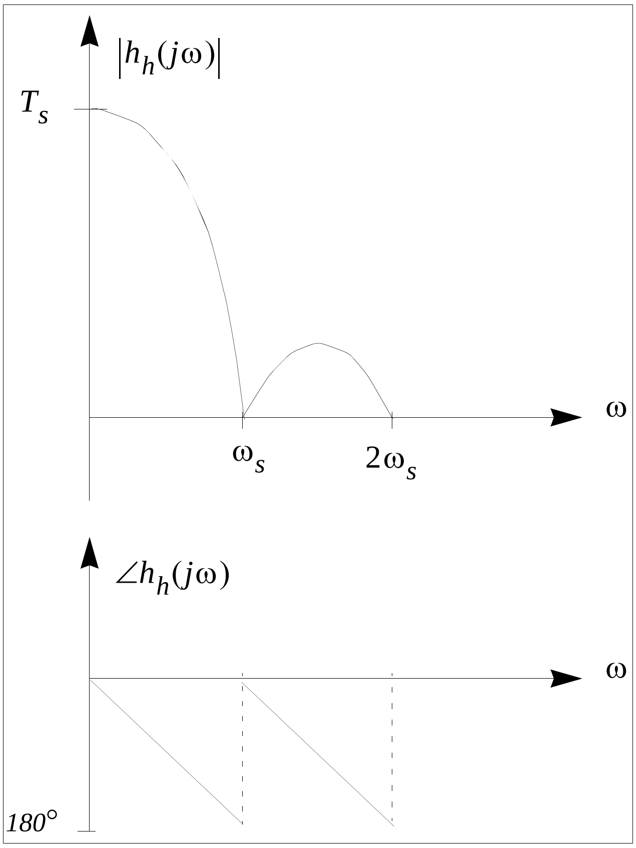

Figure 3.17 Frequency response of the zero order hold

We observe that the magnitude becomes zero at the frequency equal to the sampling frequency and at integral multiples of the sampling frequency. Notice that the steady-state gain of the zero order hold is $h_h(s=0)=T_s$. We have seen, eq. (3.64), that the ideal sampling process has a gain $\frac{1}{T_s}$. Thus the combination of a sampler and a zero order hold has unit steady state gain. We also observe that for very fast sampling, ω_s » $\frac{\pi}{T_s}$, a series connection of a sampler and zero order hold element act as a CT system with unit transfer function.

The ideal sampler is not a physical device but a mathematical approximation. In a physical situation the sampler is always followed by a hold element. Thus the combination of the ideal sampler and the zero order hold element accurately models a physical transfer operation. See Figure 3.18.

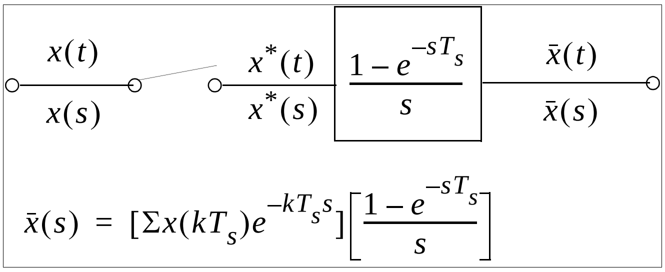

Figure 3.18 Representation of sampler and data hold

As can be seen from equation (3.65) sampling is a linear but time-variant operation. Many different input signals can result in the same output signal.

Thus it is not possible to find a transfer function for the ideal sampler. These properties complicate the analysis of a discrete control system, especially if we are interesting in the exact behaviour between sampling instants.

We have now found the LT, x*(s), of a sampled signal x*(t). In section 3.3 we introduced the ZT for a discrete time sequence $\{x(kT_s)\}$.

We may now ask tourselves: Is there a connection between these two transforms?

If we can establish such a relationship then the DT and the CT system models can be linked together. The CT-system part may then be modelled as a discretized or sampled system. In this way we may consider the complete Control System as discrete, and use the ZT representation.

The similarities between the transforms X(z) and x*(s) is obvious. In fact, if we assume the number sequence $\{x(kT_s)\}$ is obtained from sampling a time function x(t) and $e^{sT_s} = z$ in (3.65), then it becomes the z-transform. In this case we have

$$x(z) = x^*(s)\Big|_{e^{sT_s} = z} \tag{3.70}$$

We will use the following change in variable:

$$e^{sT_s} = z \qquad , s = \frac{1}{T_s} lnz \tag{3.71}$$

In general, the ZT instead of the LT* in our analysis of digital control system will be used.

One advantage of this approach is that according to equation (3.64) x*(s) has an infinity number of poles and zeroes in the s-plane. However, x(z) has a limited number of poles and zeroes. In this way analysis or design procedures that utilize a pole-zero approach are greatly simplified through the use of the ZT.

3.5 MATHEMATICALREPRESENTATION OF THE CONTINUOUS PLANT BY DISCRETIZATION

Now that we have described the conversion of signals between the computer and the CT-plant we turn to the fundamental problem of finding a discrete time equivalent representation of the CT-plant.

This representation will give the relation between the output sequence of the plant {y(kT} and the input {u(kT)}. Using this model representation the system variables are considered only at the sampling instants because the system is the time invariant.

We will consider the situation when a sample and hold device is connected to a LTI-system with the transfer function $h_p(s)$.

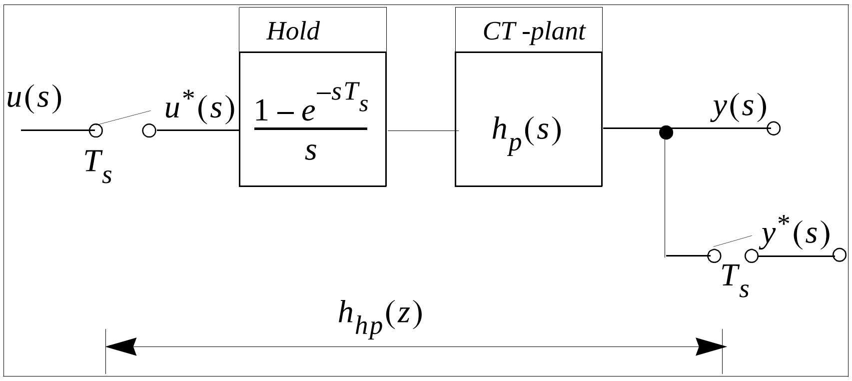

Figure 3.19 Forward loop for CT-process driven through zero order hold.

If we let $h_{hp}(s)$ be the transfer function of the zero-order hold and the process, i.e.

$$h_{hp}(s) = \frac{1}{s}\left(1 - e^{-sT_s}\right)h_p(s) \tag{3.72}$$

The LT of the sampled input signal u*(t) is given by

$$u^*(s) = \sum_{k=0}^{\infty} u(kT_s)e^{-skT_s}$$

The LT of the CT-output signal from the process is:

$$y(s) = h_{hp}(s)\sum_{k=0}^{\infty} u(kT_s)e^{-skT_s}$$

$$y(s) = h_{hp}(s)u^*(s) \tag{3.73}$$

Thus we have calculated the LT of the CT-output signal. As (3.73) shows, it is not possible to factor out the LT of the CT-signal u(t). So, we cannot find a transfer function between the input u(t) and the output y(t). This is because the system is not time invariant.

If we are interested in the output signal only at the sampling instants then we are able to define a transfer function. By adding a fictive synchronous sampler to the system as shown in the figure, the input output relation is then given by:

$y^*(s) = [h_{hp}(s)\ u^*(s)]^*$

From (3.64) we know

$$y^*(s) = \frac{1}{T_s}\sum_n y(s+jn\omega_s) = \frac{1}{T_s}\sum_n h_{hp}(s+jn\omega_s)u^*(s+jn\omega_s)$$

Because $u^*(s+jn\omega_s) = u^*(s)$ we can write

$$y^*(s) = u^*(s)\frac{1}{T_s}\sum_n h_{hp}(s+jn\omega_s)$$

$$y^*(s) = u^*(s)h_{hp}^*(s) \tag{3.74}$$

By using the relationship between x*(s) and x(z) equation we may write

$y(z) = u(z)\, h_{hp}(z)$

and

$$\frac{y(z)}{u(z)} = h_{hp}(z) \tag{3.75}$$

We have now seen that when sampling a CT-plant which is driven by a zero order hold reconstruction device, the discretized plant can be represented by the pulse transfer function $h_{hp}(z)$. This is a very important relation for the digital control of a CT plant.

3.6 COMPUTING THE PULSE TRANSFER FUNCTION FROM THE CT-TRANSFER FUNCTION $h_p(s)$.

The pulse transfer function, $h_{hp}(z)$, can be derived directly from the CT transfer function $h_p(s)$ by the following argumentation:

The signal at the output of the zero order hold could be decomposed into a series of gate functions as indicated in Figure 3.15. Each of these gate function could be further decomposed into step function as shown in Figure 3.16. The continuous time response to the first step of magnitude u_0 is

$$y(t) = u_0 L^{-1}\left[\frac{h_p(s)}{s}\right]$$

where L^{-1} is inverse Laplace transform. The ZT of the output sampled sequence y(k) in Figure 3.16 is given by:

$$y(z) = u_0 Z\left[L^{-1}\left[\frac{hp(s)}{s}\right]\right]$$

the response due to the negative step increment is similar except for the delay of one sampling period. The total ZT due to the first gate function is:

$$y(z) = u_0\left(Z\left[L^{-1}\left[\frac{h_p(s)}{s}\right]\right] - z^{-1} Z\left[L^{-1}\left[\frac{h_p(s)}{s}\right]\right]\right)$$

where z^{-1} represents the delay of one sample period. Extending this procedure to the entire series of gate functions representing u(t), the ZT of the sampled output y(kT) is:

$$y(z) = \left[\sum_{k=0}^{\infty} u(kT)z^{-k}\right](1 - z^{-1})Z\left[L^{-1}\left[\frac{h_p(s)}{s}\right]\right]$$

We recognize the first term as the ZT of the {u(kT)} sequence, so the DT transfer function between input and output, the pulse transfer function, is:

$$h_{hp}(z) = \frac{y(z)}{u(z)} = (1 - z^{-1})Z\left[L^{-1}\left[\frac{h_p(s)}{s}\right]\right] \tag{3.76}$$

We can summarize the method for obtaining the Pulse transfer function by the following steps:

1. Determine the time function corresponding to $\frac{h_p(s)}{s}$
2. Determine the corresponding ZT (Usually from a table which can be found in many text-books [16].)
3. Multiply by $(1 - z^{-1})$ to get the pulse transfer function including the zero order hold.

Another method for calculating an approximate value of the pulse transfer function $h_{hp}(z)$ is to use the so-called Tustin transformation

$$s = \frac{2}{T_s}\frac{z-1}{z+1}$$

An evaluation of this approximation is given in [18].

3.7 MATHEMATICAL REPRESENTATION OF A DC-MACHINE

As we have seen, the starting point for determining the discrete pulse transfer function for a continuous plant is a model representation of the plant given by the state space equations or the CT transfer function of the plant.

As the DC-machine will be used in the case studies that follow, the dynamic equations of this machine will be reviewed in this section. The power converter feeding the machine which is assumed to be a thyristor or transistor converter is also a part of the CT plant. The mathematical representation of the converter is discussed in the next section.

The dynamic state equations and the CT transfer function block diagram representation of the DC machine is shown in Figure 3.20. The model is linearized and no connection between the armature and field windings is assumed.

In the block diagram and equations, all voltages, currents, and torque are referenced to their rated values. The speed *n* is referenced to the value that appears when an unloaded machine is fed with rated voltage. Thus, the electro-mechanical system is characterized by the gain and the time constant of the armature circuit and by the integration time of the mechanical inertia:

$$K_a = \frac{U_{an}}{I_{an}\sum R_a} = \frac{I_{ak}}{I_{an}} = \frac{U_{an}I_{an}}{I_{an}^2\sum R_a}$$

is the ratio of nominal armature voltage and voltage drop across the armature resistance carrying rated current I_{an}. This is equal to the ratio of the current I_{ak} and I_{an}. I_{ak} is the current we get when the armature voltage is equal to U_{an} at zero speed. K_a can also be expressed as the ratio of nominal input armature terminal power and armature power loss. Typical values of K_a are in the range 8 to 15.

$$T_r = \frac{2\pi n_0 J}{M_n}$$

is the mechanical run up time of the machine. It is the time the machine will use to reach the unloaded or rated speed n_0 when it starts at zero speed and the torque is equal to the rated value M_n. J is the moment of inertia and includes all masses involved in the turning motion.

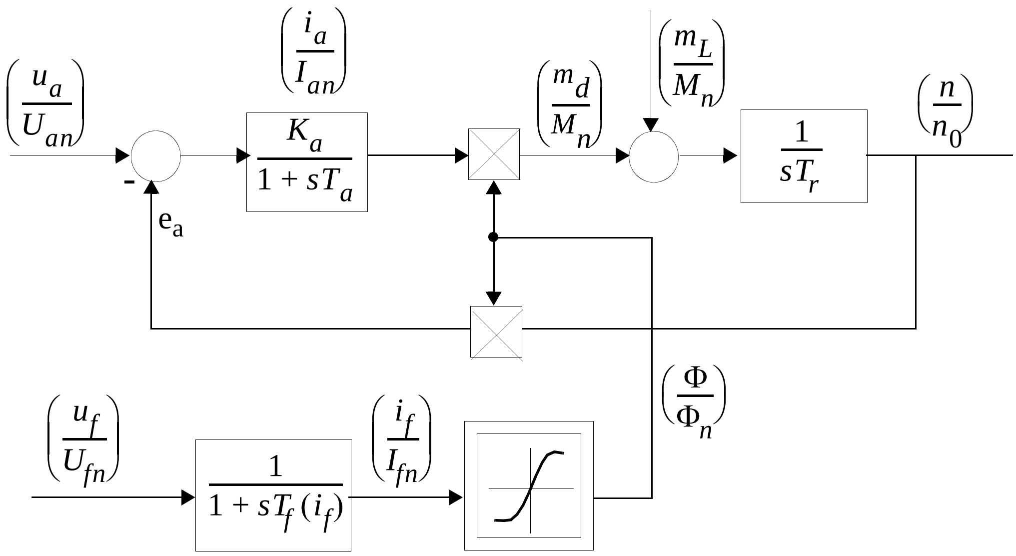

Figure 3.20 Mathematical representation of a DC-machine

$$T_m = \frac{T_r}{K_a\left(\frac{\phi}{\phi_n}\right)^2}$$

Sometimes T_m will be used instead of T_r. It is called the mechanical time constant because of its physical background: when the electrical time constant T_a is neglected and the machine thereby becomes a first order system, the time constant T_m characterizes all transient behaviour of the machine.

The given block diagram representation of the armature circuit of the DC machine is in accordance with the representation given in most text books on electrical machines [19], [20] and [21].

The discussion which follows will be concerned with the field circuits.

The effect of a rapid changing excitation voltage, which produces eddy currents in the iron laminations is not reflected in the diagram. The effects of eddy currents are often accounted for by means of a resistance R_ω in parallel with the inductance L_e of the field circuits [20]. In this case the corresponding transfer function also includes a differentiation term giving a phase advance:

$$\frac{\left(\frac{i_f}{I_{fn}}\right)}{\left(\frac{u_f}{U_{fn}}\right)} = \frac{1 + s\frac{L_e}{R_\omega}}{1 + sL_e\frac{R_e + R_\omega}{R_e R_\omega}} \tag{3.77}$$

Due to the saturation of the iron, the magnetic flux in the machine is a non-linear function of the excitation current. The slope at each point of the magnetization curve is the effective amplification in the field circuit. This slope is also a measure of the permeability of the iron and also the winding inductance. The following relationship is obtained for any point on the magnetization characteristic:

$$\frac{\Delta\phi}{\Delta i_e} = KL_e \tag{3.78}$$

Since the winding resistance is almost constant, the ratio of the amplification factor and the time constant is approximately constant over the operating range:

$$\frac{T_e}{\frac{\Delta\phi}{\Delta i_e}} = \frac{k}{R_e} = \text{constant} \tag{3.79}$$

For the purpose of an excitation current controller design the mean value of the excitation time constant is usually adequate. As will be shown later, the correct controller coefficients can be chosen on the basis of the relation between loop amplification and time constant (symmetrical optimum). Thus, in these cases the variation of the time constant has no influence.

3.8 MATHEMATICAL REPRESENTATION OF THE POWER CONVERTER

We will now face the problem of developing a mathematical model for a power converter. For the purpose of closed loop control this model should be linear and represent the converter as a CT transfer function. The input will be a coded digital word representing the wanted mean value of the voltage at the output terminal of the converter.

Let us first consider a DC chopper converter. The input reference to the chopper is the sequence $\{u(kT_s)\}$. The mean value of the output voltage of the chopper is a CT voltage with the mean value

$$u_d(t) = U_b \frac{t_{on}(nT_{ch})}{T_{ch}} \qquad nT_{ch} \le t < (n+1)T_{ch} \tag{3.80}$$

where U_b is the DC-supply voltage, t_{on} is the on-time of the switching element and T_{ch} is the chopper period.

The system is normally designed so that t_{on} is proportional to the last input reference value, at the start of the chopper period. Assume that the start of chopper period is synchronized to the sampling interval. Thus:

$$\frac{t_{on}(t)}{T_{ch}} = K_{ch} \frac{u(kT_s)}{u_{cmax}} \qquad for \;\; kT_s \le t < (k+1)T_s$$

For the mean value of output voltage we have:

$$\frac{u_d(t)}{U_b} = K_{ch} \frac{u(kT_s)}{u_{cmax}} \qquad for \;\; kT_s \le t < (k+1)T_s \tag{3.81}$$

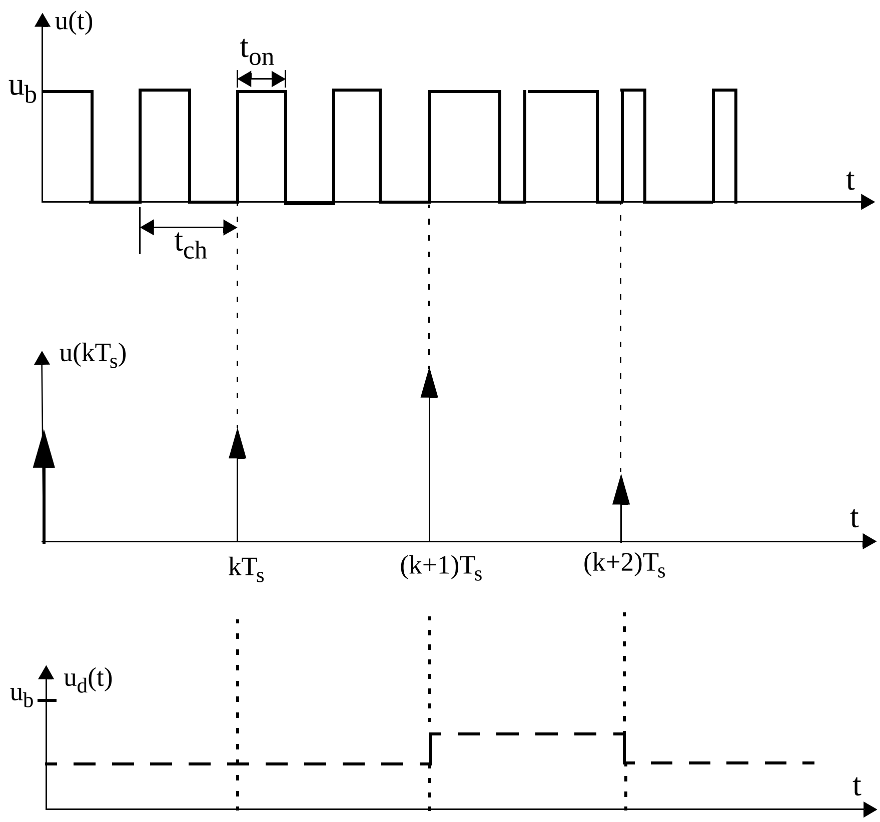

Figure 3.21 DC-chopper operation

The chopper will be latched to this value until the next sampling interval, as illustrated in Figure 3.21. We see that the PWM chopper acts as a zero order hold. As described in section 3.4 a dynamic model can then be developed as for the ZOH element.

If the sampling instant is not synchronized to the moment for switching of the chopping element, we will get a dead time with the average value equal to $\frac{T_{ch}}{2}$. The complete transfer function of the chopper is then:

$$h_{ch}(s) = K_{ch}\left[\frac{1-e^{-sT_s}}{s}\right]e^{-s\frac{T_{ch}}{2}} \tag{3.82}$$

Similar development can be carried out for a six pulse thyristor bridge. The gain of the bridge $\frac{du_d}{d\alpha}$ is not linear, but depends on the actual firing delay α. It will be shown in detail in chapter 7 how a gate firing algorithm can be designed to compensate for this non-linearity. The stationary gain, K_{br}, of the bridge can then be considered constant.

To establish the dynamic part of the transfer function the following argument will be used:

Assume that the sampling period of the input sequence is T_s=T/6 where T is the period of the power line voltages. The bridge will receive a new firing reference, α, every 1/6 of the power line period. Between the sampling instants the mean value of the output voltage, $u_d(t)$, will be kept constant according to the firing delay given at the last sampling instant. This is illustrated in Figure 3.22.

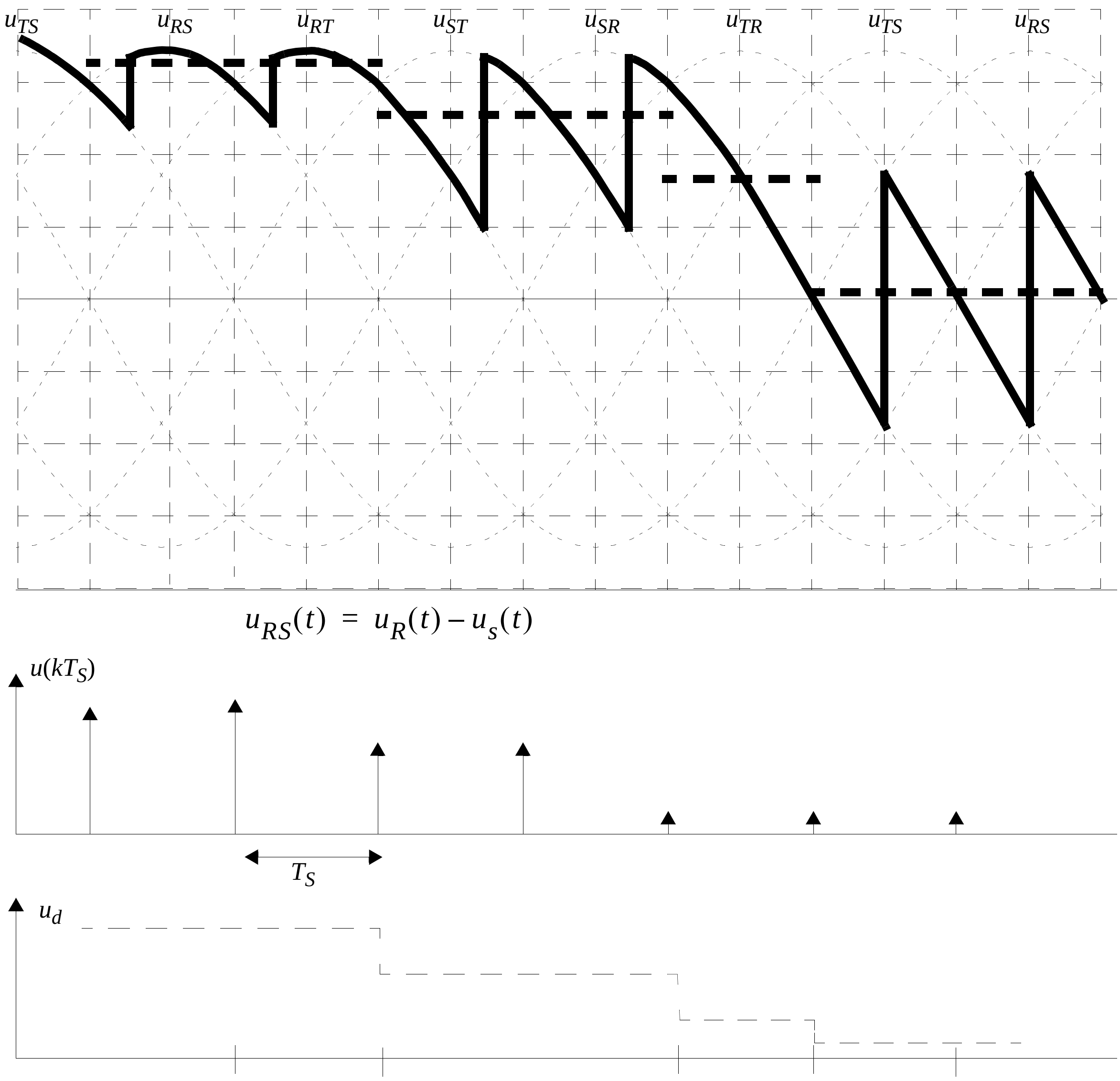

Figure 3.22 Thyristor bridge waveforms.

The thyristor bridge rectifier is inherently a sample and hold element in the control loop. It may, therefore be modelled as a ZOH element.

The thyristors are not necessarily fired at the sampling instant, but the firing is delayed according to the actual α. This can be modelled by a dead time with mean value $T_t = T/12 = \frac{T_s}{2}$ Thus the complete transfer function for the rectifier bridge is:

$$h_{br}(s) = K_{br}\left[\frac{1-e^{-sT_s}}{s}\right]e^{-s\frac{T_s}{2}} \tag{3.83}$$

3.9 PULSE TRANSFER FUNCTION FOR SYSTEMS WITH DEAD TIME

The method to determine the pulse transfer function of discrete or discretized systems so far does not apply to systems containing dead time.

As we have seen in previous sections dead times are encountered in digital power electronic control systems due to:

- computation time of the digital control algorithm
- dead time in the power electronic converter

In order to analyse system with dead time, a modification of the ZT is necessary. A continuous system containing a dead time element can be written as:

$$h_p(s) = h(s)*e^{-sT_d} \tag{3.84}$$

where h(s) is a rational function. In general, the dead time T_d can be expressed by an integer multiple of the sampling period T_s and a difference term εT_s:

$$T_d = mT_s - \varepsilon T_s = (m-\varepsilon)T_s \tag{3.85}$$

where m = 1, 2, 3.... and $0 < \varepsilon \le 1$ as illustrated in Figure 3.23.

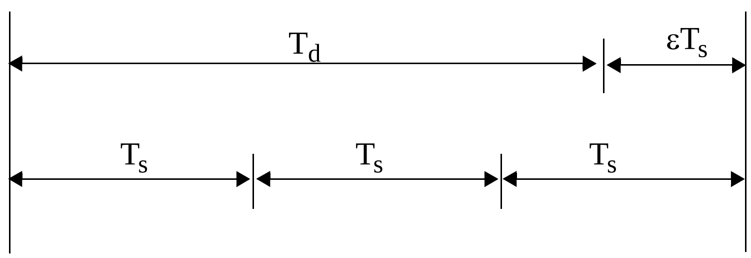

Figure 3.23 Expression of dead time

If we include the hold element, the CT transfer function is:

$$h_{hp}(s) = \frac{1 - e^{-sT_s}}{s} e^{-sT_d} h(s) \tag{3.86}$$

Let the step response for the rational part be

$$v(t) = L^{-1}\left[\frac{h(s)}{s}\right]$$

Then we may write:

$$\begin{aligned} h_{hp}(t) &= L^{-1}\left[e^{-sT_d}\left(\frac{h(s)}{s}\right) - e^{-s(T_s + T_d)}\left(\frac{h(s)}{s}\right)\right] \\ &= v(t - T_d) - v(t - T_d - T_s) \end{aligned}$$

Using equation (3.85) and the shift theorem for ZT we get

$$\begin{aligned} h_{hp}(z) &= (1 - z^{-1}) Z[v(kT_s - mT_s + \varepsilon T_s)] \\ &= (1 - z^{-1}) z^{-m} Z[v(kT_s + \varepsilon T_s)] \\ &= (1 - z^{-1}) z^{-m} Z\left[L^{-1}\left[\frac{h(s)}{s}\right]_{t = kT_s + \varepsilon T_s}\right] \end{aligned} \tag{3.87}$$

This z-transform will be denoted by

$$h(z, \varepsilon) = Z_{\varepsilon}\left[\frac{h(s)}{s}\right] \tag{3.88}$$

This new transform now introduced for the sequence $\{v(kT_s+\varepsilon T_s)\}$ will be called the advanced ZT. In general we write

$$x(z, \varepsilon) = Z_{\varepsilon}\{x(t)\} = \sum_{k=0}^{\infty} x(kT_s + \varepsilon T_s) z^{-k} \tag{3.89}$$

Tables of this transform can be found in [22]. It is not necessary to consider this transform as a new transform since all the rules of the ordinary ZT also apply to this transform.

In most literature on sampled data systems the modified ZT is introduced and denoted as:

$$X(z, m) = Z_m\{x(t)\} = z^{-1}\sum_{k=0}^{\infty} x(kT_s + mT_s)z^{-k} \tag{3.90}$$

for $0 \le m \le 1$

Note that m has another meaning than used earlier. Here m is a nymber between 0 and 1. Tables of the modified ZT can be found in [23] and [24].

The advanced ZT can be calculated from the modified ZT by the substitution m=1+ε or equivalently, multiplication by the factor z^{-1} and use of the substitution m=ε.

For ε=0 or m=1 both transforms become the ordinary ZT.

For later reference we will calculate the pulse transfer function for the plant:

$$h_{hp}(s) = \frac{1 - e^{-sT_d}}{s} K \frac{e^{-sT_d}}{1 + sT} \tag{3.91}$$

where $T_d = (m-\varepsilon)T_s$ $m = 1, 2, ...,$ and $0 < \varepsilon \le 1$.

This CT transfer function can represent a power converter feeding a inductive load with time constant T (example, field circuit of a DC motor). For the pulse transfer function we have

$$h_{hp}(z) = (1 - z^{-1}) z^{-m} Z_{\varepsilon}\left[L^{-1}\left[\frac{K}{s(1 + sT)}\right]\right]$$

From a table giving the advanced-ZT or from a table giving the modified-ZT we find:

$$h_{hp}(z) = \frac{z-1}{z} z^{-m} K \frac{z\left(\left(1 - e^{-\varepsilon\frac{T_s}{T}}\right)z + \left(e^{-\varepsilon\frac{T_s}{T}} - e^{-\varepsilon\frac{T_s}{T}}\right)\right)}{(z-1)\left(z - e^{-\frac{T_s}{T}}\right)}$$

Thus we have:

$$h_{hp}(z) = z^{-m} K \frac{\left(1 - e^{-\varepsilon\frac{T_s}{T}}\right)z + \left(e^{-\varepsilon\frac{T_s}{T}} - e^{-\varepsilon\frac{T_s}{T}}\right)}{\left(z - e^{-\frac{T_s}{T}}\right)} \tag{3.92}$$

3.10 MATHEMATICAL MODEL OF THE COMPLETE CLOSED

LOOP SYSTEM INCLUDING THE DISCRETIZED CT-PLANT

We have now established the necessary theoretical tool to represent a complete model for the closed loop digital system. This model will be used for analysis and design of the digital control system.

To assemble the different system elements into the complete model the following systematic approach will be given. Again Figure 3.1 is used as a starting point.

From the physical representation, we identify the elements that represent the signal conversion between the analog and digital parts of the system. These elements have been logically described as AD- and DA-converters.

- The AD converters are represented by ideal samplers.
- The DA converters are represented by ideal samplers followed by zero-order-hold elements.
- It is assumed that the samplers are perfectly synchronized.

We then get the block diagram representation given in Figure 3.24.

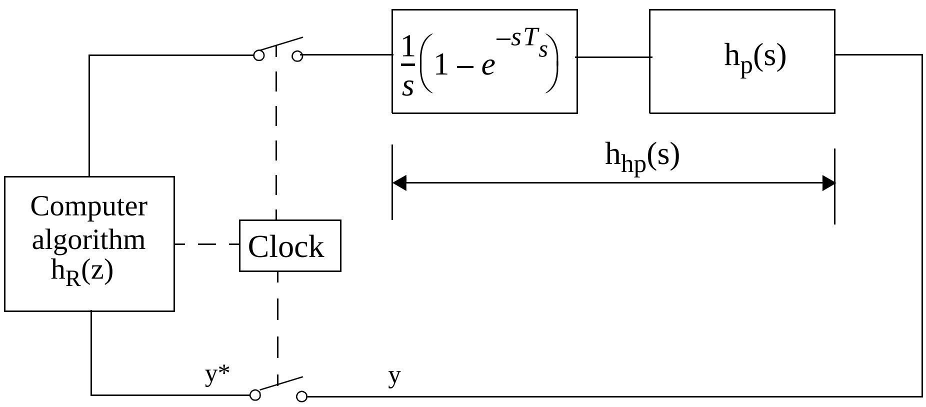

Figure 3.24 Model of the system. The computer algorithm must be described by impulse modulated signal.

- In this model the computer must be represented as a system component that transforms an impulse modulated signal to another impulse modulated signal.
- The analog parts are the zero order hold and the process.

The transfer function of the analog part is

$$h_{hp}(s) = \left(\frac{1 - e^{-sT_s}}{s}\right) h_p(s)$$

The LT y(s) of the output y(t) is

$y(s) = h_{hp}(s)*u(s)*$

The sampled output has the LT

$y*(s) = [h_{hp}(s)u*(s)]* = h*_{hp}(s)\ u*(s)$

If we represent the sampled signals as sequences we get a block diagram of the system as shown in Figure 3.25.

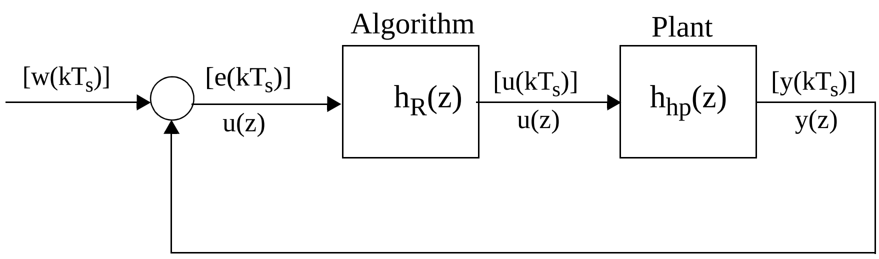

Figure 3.25 Model of the system, represented by pulse transfer functions and sequences.

- In this model the calculations in the computer are modelled by the Pulse transfer function $h_R(z)$.
- The CT part is represented by the Pulse transfer function

$$h_{hp}(z) = h_{hp}*(s)\big|_{e^{sT_s} = z}$$

This block diagram gives the properties of the system as seen by the computer.

It is an input-output representation that gives the relationship between the system variables at the sampling instants only.

Based on this representation we are able to find the transfer function from the input *w(k)*, and the output y(k), and the characteristic equation of the system, valid at the sampling instants.

At the summation point we have

$$e(z) = w(z) - y(z)$$

Substituting the equation $u(z) = h_R(z)e(z)$ for the controller, and $y(z) = h_{hp}(z)u(z)$ for the plant we get:

$$m(z) = \frac{y(z)}{w(z)} = \frac{h_R(z)h_{hp}(z)}{1 + h_R(z)h_{hp}(z)} \tag{3.93}$$

The characteristic equation of the system is defined as:

$$1 + h_0(z) = 0 \tag{3.94}$$

where $h_0(z)$ is the open loop transfer function. In this case we have

$$h_0(z) = h_R(z)h_{hp}(z) \tag{3.95}$$

When the system contains cascaded elements, care must be taken when deriving DT transfer function for the complete system. It is not always possible to write a transfer function between variables [25].

The following rules may be helpful:

- The ZT of two CT elements separated by a sampler is the product of the two Z-transforms.

$$Z[h_1(s)h_2(s)] = Z[h_1h_2(s)] \tag{3.96}$$

- In general no transfer function can be found for a system if the input is applied to a CT element before being sampled.

$$Z(h_1h_2) \neq (h_1(z)*h_2(z)) \tag{3.97}$$

In complex situations with feedback and many samplers the algebraic manipulations can be tedious. In such situations the signal flow graph method [23] may be used.

3.11 TRANSIENT BEHAVIOUR OF THE CLOSED LOOP SYSTEM

The roots of the characteristic equation (3.94) are the poles of the closed loop transfer function.The location of the roots of the characteristic equation in the z-plane determine the dynamic characteristic of the closed loop system. Thus the controller $h_R(z)$ must be designed so that the roots receive preferable locations.

For the output of the system we may write

$$y(z) = \frac{h_0(z)w(z)}{1 + h_0(z)} = \frac{\prod^{m}(z - z_i)}{\prod^{n}(z - p_i)}\omega(z) \tag{3.98}$$

If the input sequence w_k is the unit impulse at k=0 the output y(z) can be expressed by using a partial fraction expansion:

$$y(z) = \frac{k_1 z}{z - p_1} + \dots \frac{k_n z}{z - p_n}$$

The system is stable if the output sequence {y(k)} approaches zero as time increases. By taking the IZT of y(z) we find:

$$\{y(k)\} = Z^{-1}\left(\sum_{i=1}^{n} \frac{k_i z}{z - p_i}\right) = \sum_{i=1}^{n} k_i (p_i)^k \tag{3.99}$$

for $|p_i| < 1$ we see that y(k) converges to zero.

Thus, the system is stable if all the roots of the characteristic equation

$1 + h_0(z) = 0$

lie inside the unit circle in the z-plane. The output sequence {y(k)} will then be bounded if the input is a bounded signal. We see that the unit circle in the complex z-plane is the stability boundary, similar to the imaginary axis of the s-plane for CT systems.

Valuable insight into the dynamic behaviour of a discrete or discretized system can be gained by studying the locations of the system poles in the z-plane. As can be seen from equation (3.99) each pole in the transfer function contribute separate dynamic modes to the resulting response sequence at the sampling instants.

If we have a real pole $p_i < -1$ the associated time sequence will oscillate and increase in an oscillatory fashion. If $0 < p_i < 1$ it will decay in an exponential manner as k become large. If $p_i > 1$, the associated sequence will grow exponential without bound.

If the characteristic equation has complex roots, p and p* similar results can be derived. In both cases the decay for $|p| < 1$ will be faster if the poles lies close to zero.

A summary of various pole locations in the z-plane and the type of response sequences they represent is given in Figure 3.26.

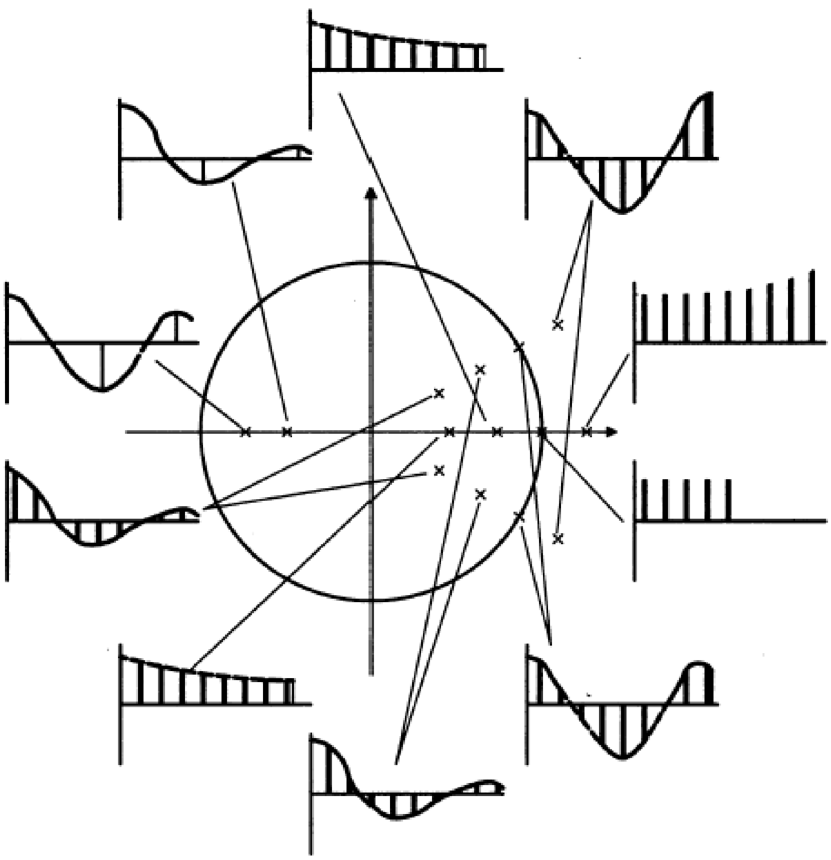

Figure 3.26 Pole locations and associated response.

For subsequent reference and to illustrate the methods presented we will investigate the digital control system shown in Figure 3.27. The block diagram shows a microprocessor controlled power converter driving a passive inductive load with time constant T. Assume a current controller with proportional control law and negligible dead time of the converter.

The pulse transfer function of the power converter, including the load, is given according to equation (3.76) and (3.82).

$$h_{hp}(z) = (1 - z^{-1}) Z\left\{ L^{-1}\left[\frac{K}{s(1 + sT)}\right]\right\} \tag{3.100}$$

From a ZT table we get:

$$h_{hp}(z) = K\frac{z^{-1}\left(1 - e^{-\frac{T_s}{T}}\right)}{1 - e^{-\frac{T_s}{T}} z^{-1}} \tag{3.101}$$

This pulse transfer function could also been obtained by letting m=1 and ε=1 in equation (4.49).

Choosing $T_s/T=1/3$ we get:

$$h_{hp}(z) = K\frac{0{,}2835z^{-1}}{1 - 0{,}7165z^{-1}}$$

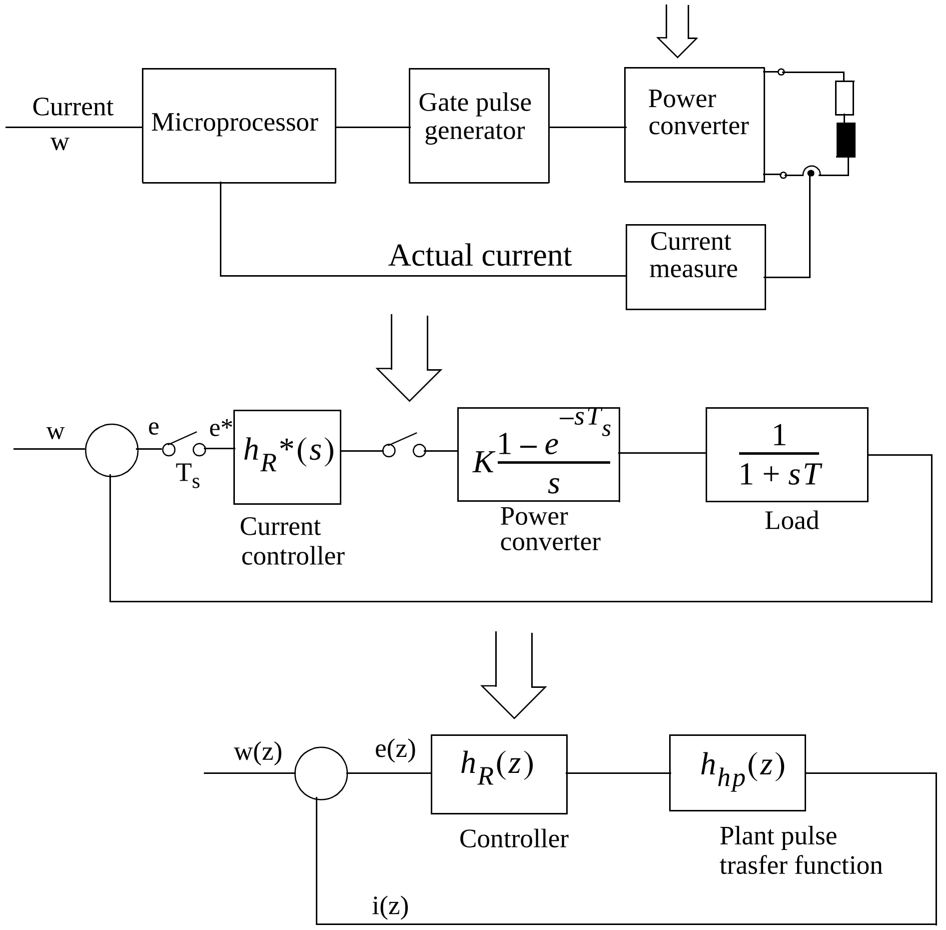

Figure 3.27 Block diagram of a microprocessor based current control system

The characteristic equation of the system is then:

$$1 + h_0(z) = 1 + h_R(z)h_{hp}(z) = 0$$

$$1 + \frac{K_pK(0{,}2835)z^{-1}}{1 - 0{,}7165*z^{-1}} = 0$$

The root of this equation is

$$p = 0{,}7165 - 0{,}2835K_0$$

where K_0 = KpK is the total loop gain.

As K_0 increases from zero, the system pole p moves to left from the location z=0.7165 for K_0=0 and eventually leaves the unit circle as illustrated in Figure 3.28. The value of the gain K_{crit} for which the system become unstable is of interest. The system is marginal stable for p=-1. We have K_{ocrit} = 6.05.

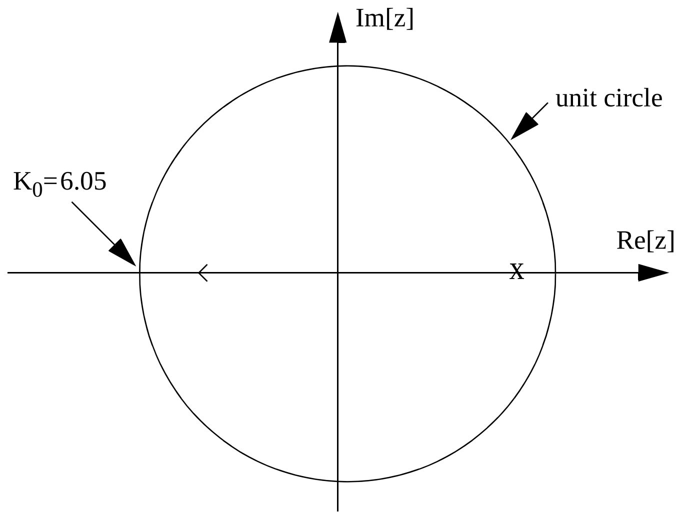

Figure 3.28 Root locus of the system

Important properties of a digital control system can be seen from this simple example:

- Note that the pole at s= - 1/T of the CT plant is transformed to a pole $z = e^{-\frac{T_s}{T}}$ in the discretized plant transfer function. The relation $z_p = e^{s_p T_s}$ between s-domain poles and poles of the pulse transfer function in the z-plane is generally valid. This sort of relationship does not necessarily hold for the zeros.
- The z domain poles location are in addition to the original s-domain location also dependent of the sampling interval T_s poles close to z=1 correspond to fast sampling or a short time constant for the plant.
- Another interesting observation is that though this system, can be unstable, the same plant under action of a CT control is always stable. The reason for this is the basic nature of a digital control sys-

tem. The digital controller receives only samples of the error signal at discrete time instances. Thus the control action must be performed based on a limited amount of information while for a CT control system an infinite amount of information about the error signal is available. As the DT system runs open loop between sampling instants, when the controller does take action it overcompensates by generating too high gain.

- Higher gain in the loop can be tolerated by increasing the sampling frequency before the system becomes unstable.

The relation given in equation (3.71) is a transformation between the s-plane pole locations and between the z-plane locations. To gain further insight into the characteristics of pole locations in the z-plane, several mappings will be considered.

The complex variable s may be written as s=α+jω. Hence, according to (3.71) we have:

$$z = e^{(\alpha + j\omega)T_s} = e^{\alpha T_s} e^{j\omega T_s} = e^{\alpha T_s} e^{(j\omega T_s + n2\pi)} \tag{3.102}$$

From equation (3.64) we see that the poles on the s-plane, whose frequencies differ in integral multiple of the sampling frequency, are transformed to the same position on the z-plane. So studying the so-called primary strip in the s-plane (n=0) is sufficient.

The mapping of the left half plane portion of the primary strip is mapped into the interior of the unit circle (since α<0, magnitude is less than 1). The imaginary axis in the s-plane (α=0) corresponds to |z|=1.

The right half portion maps into the exterior of the unit circle. (See Figure 3.29). This result is in agreement with the stability boundaries of the two planes. Figure 3.29 also shows how poles with constant damping, constant frequency, and constant relative damping factor are mapped into the z-plane.

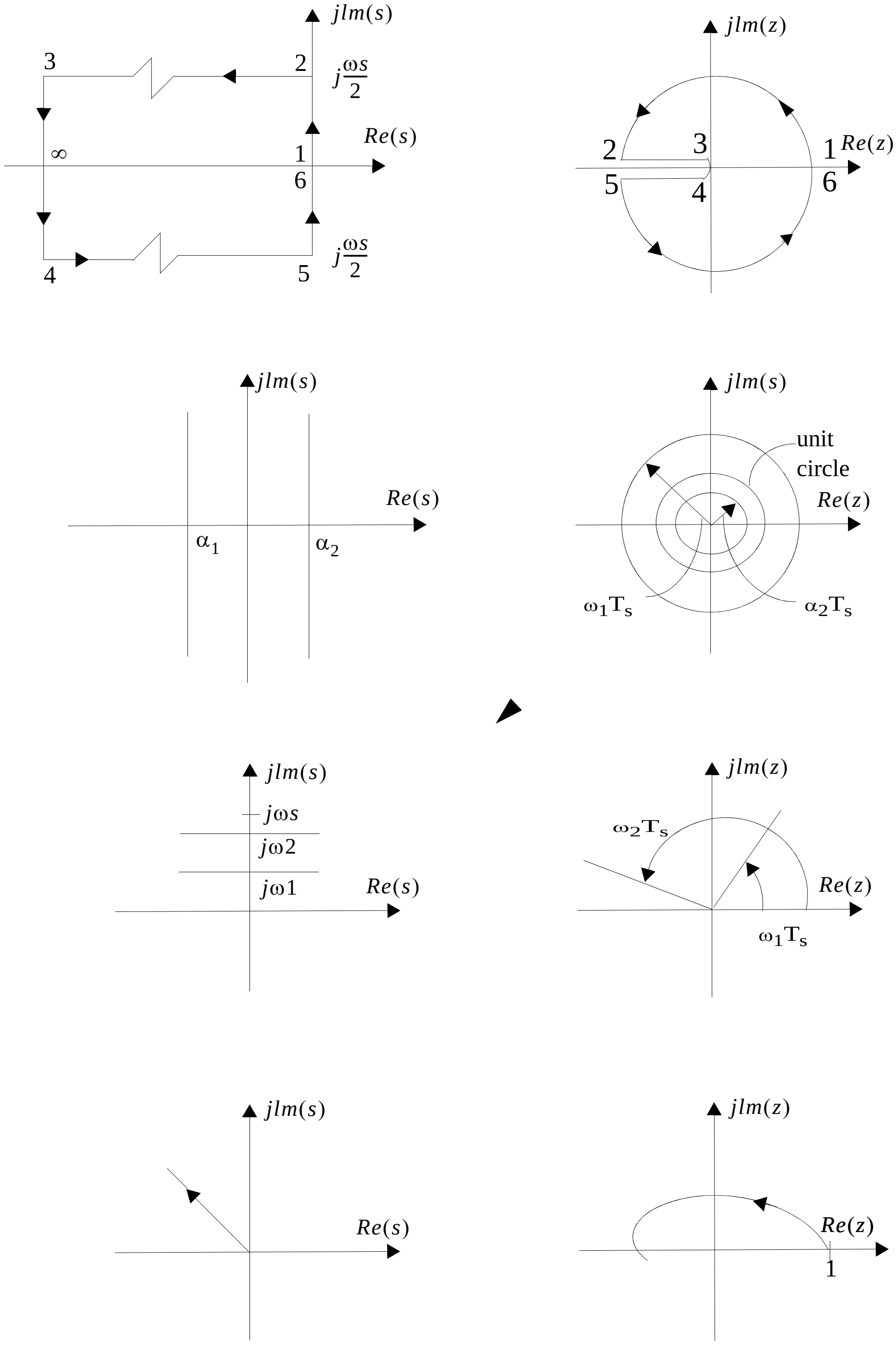

Figure 3.29 Mapping of s-plane into the z-plane

Complex poles on the s-plane can be written in the standard form:

$$s_{1,2} = \alpha_{1,2} + j\omega_{1,2} = -\xi\omega_0 \pm j\omega_0\sqrt{1-\xi^2} \tag{3.103}$$

The constant relative damping factor (ξ = constant) path on the z-plane is a family of logarithmic spirals, except for $\xi = 0$ and $\xi = 1$. This can be seen as follows: From equation (3.103), if we let ω_d be defined as $\omega_d = \omega_0\sqrt{1-\xi^2}$, then in the z-plane, the line with constant ξ becomes:

$$\begin{aligned} z &= e^{sT_s} = \exp(-\xi\omega_0 T_s + j\omega_d T_s) \\ &= \exp\left(-\frac{2\pi\xi}{\sqrt{1-\xi^2}}\,\frac{\omega_d}{\omega_s} + j2\pi\frac{\omega_d}{\omega_s}\right) \end{aligned}$$

Hence,

$$|z| = \exp\left(-\frac{2\pi\xi}{\sqrt{1-\xi^2}}\,\frac{\omega_d}{\omega_s}\right)$$

$$\angle z = 2\pi\left(\frac{\omega_d}{\omega_s}\right)$$

This implies that magnitude of z decreases and the angle of z increases as ω_d increases. The locus becomes a logarithmic spiral in the z-plane.The path for $\xi = 0.5$ and $\omega_d < \frac{\omega_s}{2}$ is shown in Figure 3.26 where ξ is the relative damping factor equal to $\xi = -\frac{\alpha}{\omega_0} = \sin\varphi$ and ω_0 is the undamped resonance frequency given by

$$\omega_0 = \sqrt{\omega_d{}^2 + \alpha^2} = \frac{\omega_d}{\cos\varphi}$$

3.12 STEADY STATE BEHAVIOUR OF THE CLOSED LOOP SYSTEM

The steady state performance of the digital control system can be determined through use of the final value theorem of the ZT.

The steady state error at the sampling instants is defined as:

$$e_{ss} = \lim_{k \to \infty} e(kT) = \lim_{z \to 1} (1 - z^{-1}) e(z) \tag{3.104}$$

where we assumed that $(1 - z^{-1})$ e(z) does not have any pole on/or outside the unit circle. For the system given in Figure 3.25 the ZT of the error signal is:

$$e(z) = \frac{w(z)}{1 + h_R(z) h_{hp}(z)} = \frac{w(z)}{1 + h_0(z)} \tag{3.105}$$

Thus we have

$$e_{ss} = \lim_{k \to \infty} e(kT) = \lim_{z \to 1} (1 - z^{-1}) \frac{w(z)}{1 + h_0(z)} \tag{3.106}$$

This equation shows that the steady state error depends on the reference input w(z), as well as the open loop transfer function $h_0(z)$. In the following we will calculate the steady state error for three basic type of input signals: step sequence, ramp sequence and parabolic sequence. The results are presented in table 4.1.

The limit as $z \to 1$ of the open loop transfer function $h_0(z)$ can always be expressed as:

$$\lim_{z \to 1} h_0(z) = \lim_{z \to 1} \frac{a_0 \prod^{m} (z - z_i)}{(z - 1) \prod^{p} (z - p_i)} = \lim_{z \to 1} \frac{K_{dc}}{(z - 1)^N} \qquad p_i \neq 1$$

where $K_{dc} = \left. \dfrac{a_0 \prod^{m} (z - z_i)}{\prod^{p} (z - p_i)} \right|_{z = 1}$ is the open loop dc gain when all poles of z = 1 are removed

Table 3.4 Steady state error of a plant with N poles of $h_0(z)$ at z = 1

Input sequence {w(kT)}	w(z)	Steady state error $e_{ss} = \lim\limits_{k \to \infty} e(kT)$	Number of poles at z = 1 in order to get $e_{ss} = 0$
unit step	$\dfrac{z}{z-1}$	$\dfrac{1}{1 + \lim\limits_{z \to 1} h_0(z)} = \left. \left(\dfrac{1}{1 + K_{dc}}\right)(z-1)^N \right\|_{z = 1}$	$N \geq 1$
ramp $\{kT_s\}$	$\dfrac{T_s z}{(z-1)^2}$	$\dfrac{1}{1 + \lim\limits_{z \to 1} (z-1) h_0(z)} = \left. \left(\dfrac{T_s}{K_{dc}}\right)(z-1)^{N-1} \right\|_{z = 1}$	$N \geq 2$
parabolic $\left\{\dfrac{(kT_s)^2}{2}\right\}$	$\dfrac{T_s^2 z(z+1)}{2z(z-1)^3}$	$\dfrac{1}{\dfrac{1}{T_s^2} \lim\limits_{z \to 1} (z-1)^2 h_0(z)} = \left. \left(\dfrac{T_s^2}{K_{dc}}\right)(z-1)^{N-2} \right\|_{z = 1}$	$N \geq 3$

The above calculations illustrate that, in general, increased system gain and/or addition of poles at z = 1 in the open loop transfer function $h_0(z)$ decreases the steady state error. It will also be noted that the designer must find a compromise between small steady state error and system stability because in general, both large system gain and poles of $h_0(z)$ at z = 1 have destabilizing effects on the system.

3.13 QUANTIZATION ERROR NOISE MODEL

When a band limited CT-signal is sampled and then converted to an N-bit digital signal by an ADC, an uncertainty in the signal level exists. This statistical uncertainty in the digital signal amplitude can be considered to be equivalent to a broadband quantization noise added to the CT-signal input before sampling. The CT input signal is assumed to digitized by an infinite-bit ADC.

To illustrate the properties of quantization noise, refer to the quantization error generating model of Figure 3.30. An analog signal, $x(t)$, is sampled and digitized by an N-bit DAC of the round-off type.

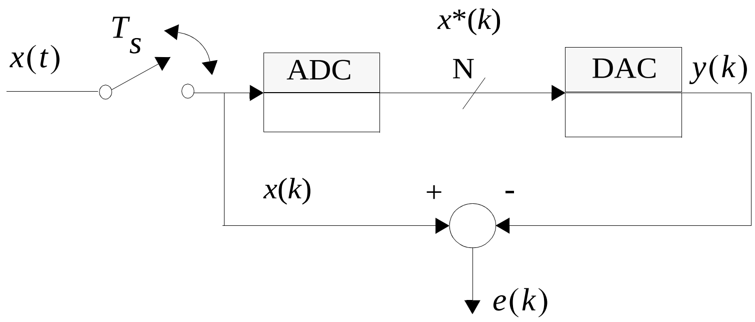

Figure 3.30 Block diagram of the system used to model quantization error for an ideal N-bit ADC driving an N-bit ideal DAC

The ADC's numerical output, x*(k), is the input to an N-bit DAC. The quantization error, e(k), is defined at sampling instants as the difference between the sampled analog input signal, x(k), and the analog DAC output, y(k). The ADC/DAC channel has unity gain. Thus the quantization error is:

$$e(k) = x(k) - y(k) \tag{3.107}$$

Figure 3.31 shows a bipolar rounding quantization process relating analog sampler output, $x(k)$, to the binary DAC output, $x^*(k)$. In this example, $N = 4$.

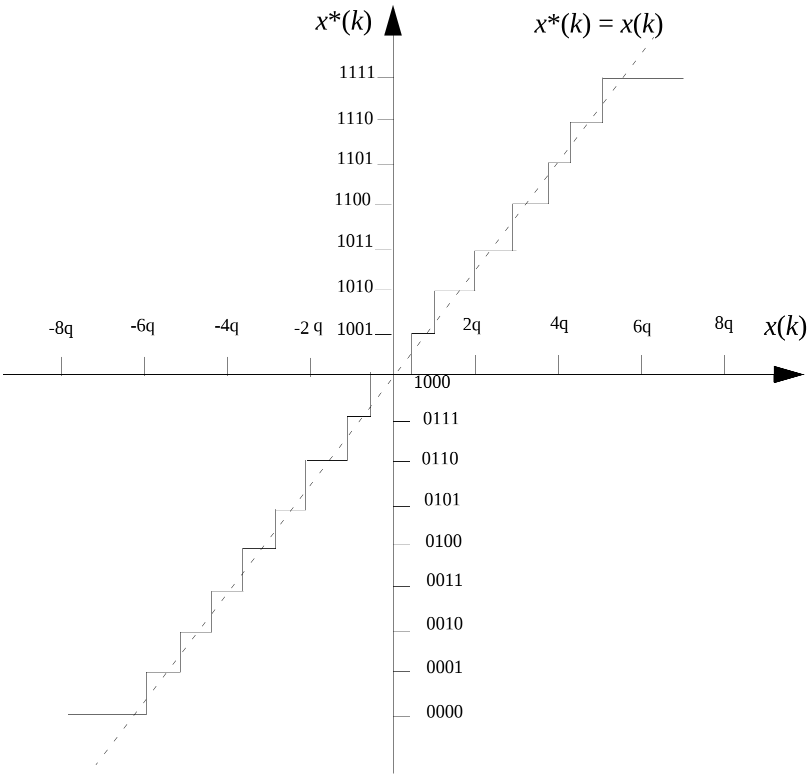

Figure 3.31 A 4-bit rounding quantizer I/O function

When $y(k)$ is compared to the direct path, the error, $e(k)$, can range over $\pm q/2$ in the center of the range, where q is the voltage step size of the ADC/DAC. It is easy to see that for full dynamic range, q should be:

$$q = \frac{V_m}{(2^N - 1)} \tag{3.108}$$

where V_m is the maximum (peak-to-peak) valtage value of the input, $x(t)$, to the ADC/DAC system. For example, if a 10-bit ADC is used to convert a signal ranging from -5 to +5 V, then by (3.108), q = 9.775 mV.

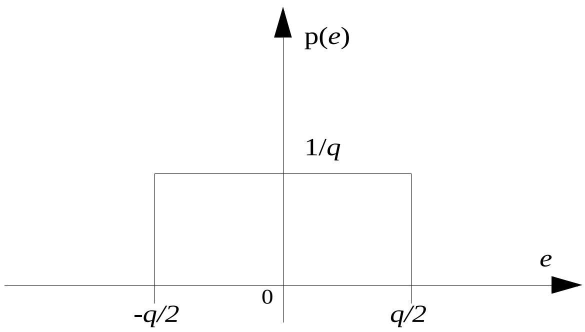

Figure 3.32 The rectangular probability density function generally assumed for quantization noise.

If $x(t)$ has zero mean and its probability density function has a standard deviation $\sigma_x > q$, then the probability density function of $e(n)$ can be modeled by a rectangular density, $p(e)$, for $e = \pm q/2$. This rectangular density function is shown in Figure 3.32; it has a peak of $1/q$. The mean-squared error voltage is found from the expectation value:

$$E\{e^2\} = \int_{-\infty}^{\infty} e^2 p(e) de = \frac{1}{q}\int_{-\frac{q}{2}}^{\frac{q}{2}} e^2 de = \frac{q^2}{12} = \sigma_q^{\ 2} \tag{3.109}$$

Equationit (3.109) shows that it is possible to treat the quantization error as a zero-mean broad-band noise with a standard deviation of $\sigma_q = q/\sqrt{12}$ added to the input signal $x(k)$. The quantization noise spectral bandwidth is assumed to be flat over $\pm fs/2$, where fs is the sampling frequency.

In order to minimize the effects of quantization noise for an N-bit ADC, it is important that the analog input signal, $x(t)$, span the full dynamic range of the ADC. In the case of a zero-mean, time-varying signal that is Nyquist band limited, gains and sensitivities should be chosen so that the peak expected $x(t)$ does not exceed the input range of the ADC. If $x(t)$ has a Gaussian probability density function with zero mean, the dynamic range of the ADC should be about ±3 standard deviations (rms value) of the signal. Under this particular condition, it is possible to derive an expression for the mean-squared signal-to-noise ratio of the ADC and its quantization noise.

Let the signal have a rms value or standard deviation equal to $x_{rms} = \sigma_x$. From (3.108) the quantization step size can be written:

$$q = \frac{6x_{rms}}{(2^N - 1)} \tag{3.110}$$

The mean-squared or the variance of output noise is:

$$\sigma_q^2 = \frac{q^2}{12} = \frac{36x_{rms}^2}{12(2^N - 1)^2} = \frac{3x_{rms}^2}{(2^N - 1)^2} \tag{3.111}$$

The signal-noise ratio (SNR) is the ratio of the signal power and noise power. Thus, the signal-to-noise ratio of the N-bit rounding quantizer is:

$$SNR_q = \frac{x_{rms}^2}{\sigma_q^2} = \frac{(2^N - 1)^2}{3} = \frac{(2^N - 1)}{\sqrt{3}} \tag{3.112}$$

Expressed i n dB we have:

$$SNR_{qdB} = 20\log(2^N - 1) - 10\log 3 \tag{3.113}$$

Table 3.5 on page 68 summarizes the SNR the quantizer for different bit values.

Table 3.5 *SNR Values for an N-Bit ADC Treated as a Quantizer. Input span is assumed to be* $6x_{rms}$

N	dB SNRq
6	31.2
8	43.4
10	55.4
12	67.5
14	79.5
16	91.6

Note: Input span is assumed to be $6x_{rms}$
About 6 dB of SNR improvement occurs for every bit added to the ADC word length.

The equivalent quantization noise is added to the ideal sampled signal at the input to some digital filter, $H(z)$. Note that the quantization error sequence, $e(k)$, is assumed to be from a stationary white-noise process, where each sample, $e(k)$, is uniformly distributed over the quantization error. The error sequence is also assumed to be uncorrelated with the corresponding input sequence, $x(k)$. Furthermore, the input sequence is assumed to be a sample sequence of a stationary random process, $\{\mathbf{x}\}$.

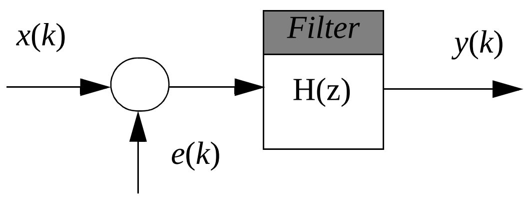

Figure 3.33 Block diagram of a model in which quantization noise is added to a noise-free sampled signal at the input to a digital filter.

Note that $e(k)$ is treated as white sampled noise (as opposed to sampled white noise). The auto-power density spectrum of $e(k)$ is assumed to be flat (constant) over the Nyquist range: $-\frac{\pi}{T_s} \leq \omega \leq \frac{\pi}{T_s}$

As $e(k)$ propagates through the digital filter; in the time domain this can be written as a real discrete convolution:

$$y(k) = \sum_{m=-\infty}^{\infty} e(m)h(k-m) \tag{3.114}$$

$e_{rms} = \sigma_q$ is the standard deviation (rms value) of the white quantization noise and the variance of the filter's output noise can be to be expressed as:

$${\sigma_y}^2 = {\sigma_q}^2 \sum_{n=0}^{\infty} h^2(n) \tag{3.115}$$