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Backtesting counterparty credit exposure based on the Heath, Jarrow and Morton framework for simulation of interest rates

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Master of Science in Physics and Mathematics
Submission date: June 2018
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Preface

This thesis is carried out at the Department of Mathematical Sciences at the Norwegian University of Science and Technology (NTNU), Trondheim, during the period of February 2018 to June 2018. The thesis concludes a 5 year study program in physics and mathematics, with spesialization in industrial mathematics and statistics, leading to the degree of Master of Science.

I would like to thank my supervisor Jacob Laading for productive discussions and constructive feedback through the process.

Abstract

In this thesis a framework for backtesting counterparty credit exposure is developed and implemented. Using the Heath, Jarrow and Morton model for simulation of interest rates, separate models are implemented for risk-neutral pricing of interest rate derivatives, and for simulation of future real-world interest rates. The models are combined to simulate distributions of credit exposure for a simple swap contract between a financial institution and a typical counterparty. The implemented framework is discussed with respect to practical use, model assumptions, and potential improvements.

The results show that the model performs well in most periods, but fails to capture the impact of the unprecedented low interest rates prevailing after the financial crisis. A proposed improvement of the model is to increase the volatility of the real-world interest rate model to better capture unexpected future events.

Sammendrag

I denne oppgaven er det utviklet og implementert et rammeverk for backtesting av motpartseksponering. Ved hjelp av Heath, Jarrow og Mortons modell for simulering av renter er det implementert separate modeller for risikonøytral prising av rentederivater, og for simulering av fremtidige renter. Modellene er deretter kombinert for å simulere fordelinger av kreditteksponering for en enkel swapkontrakt mellom en finansiell institusjon og en typisk motpart. Det implementerte rammeverket diskuteres deretter med hensyn til praktisk bruk, modellenes forutsetninger, og eventuelle forbedringer.

Resultatene viser at modellen fungerer bra i de fleste perioder, men den sliter med å fange effekten av de lave rentene som har vært rådende etter finanskrisen. Foreslåtte forbedringer av modellen er øke volatiliteten slik at modellen har bedre forutsetninger for fange fremtidige uforutsette hendelser.

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Chapter 1

Introduction

At the end of 2017, the world gross notional value of all outstanding derivative contracts in the over-the-counter (OTC) market totalled almost 532 trillion USD. For comparison, the total capitalisation of listed equities only totalled 79 trillion USD. The OTC derivatives market has experienced rapid growth in the later years from 70 trillion USD in 1998 to a peak of more than 700 trillion USD in 2013. Of today's total notional value of derivatives, roughly 80% are interest rate derivatives, meaning their value is dependent on interest rates of various types. Of all OTC interest rate derivatives, almost 75% are swap contracts [4].

Financial institutions like banks, insurers and pensions are major dealers in the OTC derivatives market. With the use of interest rate derivatives, financial institutions can help other corporations and individuals to hedge out uncertain cash-flows on loans by transferring the risk associated with changes in the interest rate to itself. Although derivatives provide this beneficial risk-sharing effect, their use often also leads to speculative bubbles and increased risks from the leverage these instruments provide. The leverage in derivatives can get large because contract values or prices are often small compared with the gross notional amount covered by the contract. Movements in the underlying risk-factors can then lead to large fluctuations in the contract value. A prime example of this was found in the market of credit defaults swaps (CDS) in the run-up to the financial crisis. The CDS were originally created as a hedge against mortgage backed securities, but ended up instead being used as speculative vehicles. The gross notional value of the CDS market, totalling over 62 trillion USD at its peak in 2007, made the whole market more intertwined and dependent. These exposures became visible when the housing market crashed, and lead to an escalation of the crisis and to the bankruptcy of some of the most prominent investment banks on Wall Street [21].

The CDS market under the financial crisis underlined a key characteristic of OTC derivatives: There is both an uncertainty in the future value of a derivatives contract, but also in the counterparty's ability to fulfil its obligations. Having a hedged position with two different counterparties is therefore not a safe position, because if one of the counterparties

defaults the now unhedged position is exposed to losses. From the standpoint of a financial institution, wishing to maximise profits while keeping the risk at at acceptable levels, the need for mathematical modelling to accurately estimate potential exposures and losses in the future are therefore of great importance.

The challenge when developing a framework to measure and forecast such *counterparty credit exposure* associated with future movement in derivative prices is twofold. Firstly, the derivatives traded needs to be priced correctly and consistent with the underlying risk-factors and observed market prices. One therefore needs a model for pricing derivatives, given possible future risk-factors. The second challenge is to model and generate these underlying future risk-factors. These two tasks, pricing and scenario generation, are highly related and are usually both done by stochastic modelling and simulations. Implementing and combining these two tasks will be the main focus of this thesis.

Although the mathematical approach to model interest rates as stochastic variables is relatively new, many different models and approaches exists. One of the first popular models was developed by Vasicek (1977), who derived a time homogeneous short-rate model. This model and several others, including the model by Cox, Ingersoll and Ross (1985), concerned only the modelling of the shortest interest rate, called one-factor models. This type of models gained popularity mainly due to their possibility of pricing bonds and bond options analytically, not necessarily because they reflected reality in an accurate way. The first important alternative to one-factor models was proposed by Ho and Lee (1986). They modelled the evolution of not only the short rate, but the entire yield curve in a binomial-tree setting. This multi-factor approach to interest rate modelling was further developed and the next big breakthrough came from Heath, Jarrow and Morton (HJM) (1992) and their celebrated framework for continuous time modelling of interest rate dynamics. By choosing the instantaneous forward rates as fundamental quantities to model, they derived an arbitrage-free framework for the stochastic evolution of the entire yield curve, where the forward-rates dynamics are fully specified through their instantaneous volatility structures. This framework is automatically fitted to market data, meaning that the model produces bond prices consistent with prices observed in the market. In this thesis, the HJM framework is implemented and used for simulation of risk-factors and counterparty credit exposure [6].

The mathematical rigour and elegance in the the derivation of the HJM model is impressive, but the the true test of any financial model is always its applicability for practical use. As for most financial models, the HJM framework relies on some highly idealised assumptions like deterministic volatility and normally distributed interest rates which is known to some degree to be incorrect [9]. To test the models applicability, it is common procedure to preform historical *backtests*. Using historical market data, it is possible to test how precise the model has been in the past to give an indication of what level of accuracy to expect in the future. Since this thesis concerns risk management with respect to counterparty credit exposure, the main part is dedicated to simulating the underlying risk-factors and the resulting credit exposures from previous historical periods before comparing with actual, realised risk-factors and exposures. The models predictive powers and usefulness

can then be assessed.

The thesis starts of in chapter 2 by introducing the powerful concept of Monte Carlo, a useful tool when modelling stochastic processes. Chapter 3 outlines the principles behind risk-neutral pricing, which is the fundamental concept used to price derivatives. Chapter 4 introduces formal definitions and some intuition behind interest rates, the market price of risk, and interest rate derivatives. Chapter 5 concerns interest rate modelling and the complete derivation and practical implementation of the HJM model, while volatility is the focus of chapter 6. Chapter 7 describes the specifics of counterparty credit risk, why it is important, and how the interest rate modelling framework can be applied in this setting. Chapter 8 details the Methodology and portfolios used in the backtesting simulations to come, while chapter 9 contains preliminary data analysis and estimation of parameters used in the model. The main results are presented in chapter 10. The chapter starts off by presenting results from the conducted backtests, before a discussion with respect to model assumptions and practical use. The thesis is then concluded in chapter 11 with some additional remarks on possible future work.

Chapter 2

Monte Carlo

2.1 The Principles of Monte Carlo

Monte Carlo simulations are used to model the probability of different outcomes in a process that cannot easily be predicted due to the effect from random variables. Monte Carlo is therefore a useful tool when studying phenomenons in finance with a stochastic nature like asset prices and interest rates where a simulation based approach is needed.

One simple example of Monte Carlo is the problem of solving

$$\alpha = \int_0^1 f(x)dx,$$

as an expected value $E[f(U)]$ when U is uniformly distributed between 0 and 1. By drawing U_i's independently from $[0, 1]$, the integral can be represented by

$$\hat{\alpha}_n = \frac{1}{n}\sum_{n=1}^{n} f(U_i),$$

also called the Monte Carlo estimate [14]. Assuming f is integrable over $[0, 1]$, the strong law of large numbers says that

$$\hat{\alpha}_n \to \alpha \quad \text{with probability} \quad 1 \quad \text{as} \quad n \to \infty.$$

The estimation error $\hat{\alpha}_n - \alpha$ will approximate a normal distribution with mean 0, and with error variance $\sigma_e^2 = \frac{\sigma_f^2}{n}$, where σ_f^2 can be estimated by the standard sample variance,

$$S_f^2 = \frac{1}{n-1}\sum_{n=1}^{n}(f(U_i) - \hat{\alpha}_n)^2.$$

It is also important to note that the convergence rate of a Monte Carlo algorithm is of order $\mathcal{O}(n^{-\frac{1}{2}})$. This means that by doubling the number of draws or simulated scenarios, the

estimation error is reduced with a factor of $\sqrt{2}$. Although its relative slow convergence, the Monte Carlo approach has significant advantages compared to other methods because its convergence is independent on the dimensions of the problem [14].

2.2 Order Statistics

A useful tool when evaluating a a random sample resulting from a Monte Carlo simulation is the concept of order statistics. The following formal definition is given in [13]:

Definition 2.2.1. *The order statistics of a random sample $X_1, ...X_n$ are the sample values placed in ascending order. They are denoted $X_{(1)}, ...X_{(n)}$. The order statistics are then random variables satisfying $X_{(1)} \leq ... \leq X_{(n)}$. In particular,*

$$X_{(1)} = \min_{1 \leq i \leq n} X_i$$
$$X_{(n)} = \max_{1 \leq i \leq n} X_i$$

The ordered sample can then be used to describe properties of the simulated distribution like the mode, the maximum and the minimum, in addition to other desired percentiles for creating confidence intervals.

Chapter 3

Derivative Pricing Theory

A central part of this thesis concerns the pricing of interest derivatives by simulations, using the Monte Carlo approach. This chapter starts out with an outline of the basic idea of derivatives pricing. The foundation of a model in discrete mathematical finance is then given. The chapter ends with a description of the principles of arbitrage and risk-neutrality, and how they can be applied to price derivatives. A recommended introduction into the general probability theory and stochastic processes used in the derivations can be found in appendix A. For a more in depth description and proofs, see Glasserman [14] and Bingham & Kiesel [3].

3.1 Principles of Derivatives Pricing

The mathematical theory of derivatives pricing is both elegant and practical. A financial derivative, also called a contingent claim is defined as a starting point.

Definition 3.1.1. *A derivative security, or a contingent claim, is a financial contract whose value at expiration date T i completely determined by the price of the underlying asset at time T.*

Glasserman [14] starts by outlining 3 core principles behind the theory of pricing contingent claims, and importantly how they can be applied to the Monte Carlo framework to evaluate prices.

1. If a derivative security can be perfectly *replicated* (equivalently, *hedged*) through trading in other assets, then the price of the derivative security is the cost of the replicating trading strategy.

2. Discounted (or *deflated*) asset prices are martingales under a probability measure associated with the choice of discount factor (or *numeraire*). Prices are expectations of discounted payoffs under this martingale measure.

3. In a *complete* market, any payoff (satisfying modest regularity conditions) can be synthesized through a trading strategy, and the martingale measure associated with a numeraire is unique. In an *incomplete* market there are derivative securities that cannot be perfectly hedged; the price of such a derivative is not completely determined by the prices of other assets.

Glasserman's first principle describes how to think of what the price of a derivative should be, but not how to calculate it. The second principle bridges the gap between theory and practise, and describes how to represent prices as expectations. Expectations can be evaluated by the Monte Carlo approach or other numerical methods. This is done by modelling the dynamics of the underlying asset price not as it is observed in the real world, but under a *risk-adjusted* probability measure. The third principle describes under which conditions the first and second principle can and cannot be applied.

3.2 Mathematical Finance in Discrete Time

To develop the framework for risk-neutral pricing of derivatives, discrete time is assumed. The notation and terminology used in the following model will be consistent with the ones introduced in appendix A, outlining probability theory and stochastic processes as in [3].

The time horizon for the model is specified as T, and the filtration $\mathcal{F} = \{\mathbb{F}_t\}_{t=0}^{T}$ consisting of σ algebras $\mathcal{F}_0 \in \mathcal{F}_1 \in ... \in \mathcal{F}_T$, and the finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is used [3].

As a starting point, the market is assumed to contain $d+1$ financial assets, often d risky assets and one risk-free asset indexed at 0. The asset prices are assumed to be random variables and at time t denoted $S_0(t,), S_1(t,), S_d(t)$, or just $S(t)$ as notation for the whole vector of asset prices. At least one of the asset prices is assumed to follow a strictly positive process.

Definition 3.2.1. *A numeraire is a price process $(X(t))_{t=0}^{T}$, (a sequence of random variables), which is strictly positive for all $t \in \{0, 1, ..., T\}$*

By dicounting or deflating all other prices, a numeraire expresses all the other prices on a relative basis. If for example $S_0(t)$ is chosen as numeraire, prices can now be represented as $S_i^{'}(t) = S_i(t)/S_0(t)$.

A *trading strategy* is defined by a $d+1$ dimensional vector $\phi = (\phi(t))_{t=1}^{T}$. $\phi_i(t)$ denotes the number of shares of asseet i which is held in the portfolio at time t. The investor determines the portfolio after observing the prices $S(t-1)$, but before the announcement of the prices $S(t)$. The components $\phi_i(t)$ can take both positive and negative values, meaning that short sale is permitted.

Definition 3.2.2. *The value of the portfolio at time t, denoted $V_\phi(t)$, is then defined as*

$$V_\phi(t) = \phi(t) \cdot S(t) := \sum_{i=0}^{d} \phi_i(t) S_i(t), \quad (t = 0, 1, ..., T)) \quad \textit{and} \quad V_\phi(0) = \phi(1)S(0).$$

The process $V_\phi(t)$, is then called the value process of the trading strategy ϕ, with $V_\phi(0)$ beeing the initial wealth.

Definition 3.2.3. *The gains process G_ϕ of a trading strategy is given as the change in the portfolio value from the portfolio weights ϕ is established at time $t-1$, and until the prices is realised at time t, mathematically*

$$G_\phi(t) := \sum_{\tau=1}^{t} \phi(\tau) \cdot (S(\tau) - S(\tau - 1)) \quad \textit{for} \quad (t = 1, 2, ..., T).$$

Definition 3.2.4. *A trading strategy ϕ is said to be self-financing if*

$$\phi(t) \cdot S(t) = \phi(t+1) \cdot S(t) \quad \textit{for} \quad (t = 1, 2, ..., T-1).$$

The intuition behind a self-financing strategy is that the investor adjust his portfolio between periods from $\phi(t)$ to $\phi(t+1)$, without bringing in or consuming any wealth.

3.3 Arbitrage

The absence of arbitrage means that there exist no investment strategies which makes an instant profit without taking on risk. Using the model outlined in the previous section, the following definitions and theorems relates arbitrage strategies and equivalent probability measures [3].

Definition 3.3.1. *Let $\tilde{\Phi} \in \Phi$ be a set of self-financing investment strategies. A strategy $\phi \in \tilde{\Phi}$ is called an arbitrage opportunity or arbitrage strategy with respect to $\tilde{\Phi}$ if $\mathbb{P}\{V_\phi(0) = 0\} = 1$, and the terminal wealth of ϕ satisfies*

$$\mathbb{P}\{V_\phi(T) \geq 0\} = 1 \quad \textit{and} \quad \mathbb{P}\{V_\phi(T) > 0\} > 0$$

An arbitrage opportunity means that there exists a self-financing strategies with zero initial wealth with non-negative final value with probability one, and a positive probability of a positive final value of the portfolio. The next definition is obtained by generalising this concept to the whole security market.

Definition 3.3.2. *A security market $\mathcal{M}$ is arbitrage-free if there are no arbitrage opportunities in the class Φ of trading strategies.*

The final definition required relates equivalent martingales and measures.

Definition 3.3.3. *A probability measure $\mathbb{P}^*$ on $(\Omega, \mathcal{F}_T)$ equivalent to $\mathbb{P}$ is called a martingale measure for $\tilde{S}$ if the process $\tilde{S}$ follows a $\mathbb{P}^*$ -martingale with respect to the filtration $\mathbb{F}$. We denote by $\mathcal{P}(\tilde{S})$ the class of equivalent martingale measures.*

Using the terminology introduced in definitions 3.3.2 and 3.3.3, the highly important no-arbitrage theorem can now be stated.

Theorem 3.3.1. *(No-arbitrage Theorem) The market $\mathcal{M}$ is arbitrage-free if and only if there exists a probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$ under which the discounted d-dimensional asset price process $\tilde{S}$ is a $\mathbb{P}^*$-martingale.*

3.4 Risk-Neutral Pricing

A contingent claim is according to Bimhham and Kiesel [3] said to be *attainable* if there exists a *replicating strategy* $\phi \in \Phi$ such that

$$V_\phi(T) = X.$$

This means that the replicating strategy generates the same cash-flow at time T as X. It is then possible to equate the discounted value of this contingent claim with the corresponding gain from a trading strategy. Using $\beta(T)$ as discount factor, and notation introduced in definitions 3.2.2 and 3.2.3 for the value and gains process, this equates to

$$\beta(T)X = \tilde{V}_\phi(T) = V(0) + \tilde{G}_\phi(T).$$

The equation states that the discounted value of the contingent claim is simply the cost of setting up the replicating strategy in addition to the gains from trading. In an arbitrage-free market $\mathcal{M}$, any attainable contingent claim X can be uniquely replicated [3]. This is the basic idea behind the arbitrage pricing theory and leads to the definition of an arbitrage price process.

Definition 3.4.1. *Suppose the market is arbitrage-free. Let X be any attainable contingent claim with time T maturity. Then the arbitrage price process π_X, $0 \leq t \leq T$ or simply arbitrage price of X is given by the value process of any replicating strategy ϕ for X.*

Observe that the pricing process do not rely on the individual preferences of the agents. As long as the no-arbitrage condition holds, meaning agents or investors prefer more to less, their tolerance of risk does not matter. An economy of risk-neutral investors would therefore price the contingent claims in the same way as in an economy were all investors were extremely risk-averse [8]. This insight simplifies the general pricing formula for a contingent claim to the discounted payoff with respect to an equivalent martingale measure.

Definition 3.4.2. *The arbitrage price process of any attainable contingent claim X is given by the risk-neutral valuation formula*

$$\pi_X(t) = \beta(t)^{-1}\mathbb{E}^*(X\beta(T)|\mathcal{F}_t) \quad \forall\, t = 0, 1, ..., T,$$

where $\mathbb{E}^*$ is the expectation operator with respect to an equivalent martingale measure $\mathbb{P}^*$.

This explicit formula shows how to price an attainable contingent claim using an equivalent martingale measure. It is therefore important to know under which conditions a contingent claim is attainable. This can be done by considering the definition of *completeness*, before stating the completeness theorem.

Definition 3.4.3. *A market $\mathcal{M}$ is complete if every contingent claim is attainable, i.e. for every $\mathcal{F}_T$-measurable random variable X, there exist a self-financing strategy $\phi \in \Phi$ such that $V_\phi(T) = X$*

Theorem 3.4.1. *(Completeness Theorem) An arbitrage-free market $\mathcal{M}$ is complete if and only if there exists a unique probability measure $\mathbb{P}^*$ equivalent to $\mathbb{P}$, under which discounted asset prices are martingales.*

Combining the no-arbitrage theorem 3.3.1 and the completeness theorem 3.4.1, the fundamental theorem of asset pricing is then stated.

Theorem 3.4.2. *(Fundamental Theorem of Asset Pricing) In an arbitrage-free complete market $\mathcal{M}$, there exists a unique equivalent martingale measure $\mathbb{P}^*$.*

Since it is the pricing of contingent claims which is the main concern, the equivalent martingale measure $\mathbb{P}^*$ is of great importance. Actually, the original measure $\mathbb{P}$ is irrelevant and one need only to know its null sets, so that the measures are equivalent. $\mathbb{P}^*$ is often called the *risk-neutral* measure, and all asset prices are martingales under this under this measure. This is summarised in the risk-neutral pricing theorem.

Theorem 3.4.3. *(Risk-neutral Pricing Theorem) In an arbitrage-free complete market $\mathcal{M}$, the arbitrage prices of contingent claims are their discounted expected values under the risk-neutral (equivalent martingale) measure $\mathbb{P}^*$*

Concluding this chapter, a short summary of how to apply the risk-neutral framework to price a financial derivative using the Monte Carlo approach is provided: The goal is to price a derivative security giving a payoff at time T as a function f of an underlying asset S. To price the derivative, the dynamics of the underlying asset is modelled, but under the risk-neutral approach. This means that a suitable numeraire is chosen to make the asset price into a martingale when discounted with the risk-free interest rate r. In practise, this often means that the real-world drift or growth-rate of the asset is substituted with the risk-free interest rate. The initial price $V(0)$ of the derivative is then given by the discounted payoff

$$V(0) = \mathbb{E}\big[\exp\{-rT\}f(S(T))\big]. \tag{3.1}$$

This expectation can be evaluated by simulating the underlying risk-neutral process repeatedly before taking the average by Monte Carlo [14].

Chapter 4

Interest Rates and Derivatives

The basic concept of interest rates is known to most people. If one deposits money in the bank, the money is expected to grow at a certain rate. If one borrows money to buy a house, one expects to pay an interest rate on the mortgage. Interest rates can therefore intuitively be thought of as the price of money, and because they evolve unpredictably over time they are often thought of as stochastic variables.

There exists many different types of interest rates. The magnitude of a particular rate varies with several different factors like risk and maturity. Risky loans like credit-card loans have higher interest rates than for example government loans because the chance of repayment is much lower for the former. Longer dated loans like the interest rate on a 30 year government bond is also generally higher than the interest rate on a 3 month government bill. This relationship between interest rates of differing maturities is very important and will be discussed more in depth both in later theory parts as well as in the analysis.

The interest rates chosen to model and study in this thesis are Norwegian swap rates of different maturities. A swap interest rate of a particular maturity is the fixed interest rate payed on a swap contract of that same maturity. The swap contract is an integral part of this thesis and is described later in this chapter. Swap rates are closely linked and derived from NIBOR rates, which are the Norwegian Interbank Offered Rate. The NIBOR, although slightly more risky than government bonds, is often assumed to be risk-free. A risk-free interest rate means that there is no default risk and a 100% chance of repayment of the loan or investment. The swap rates modelled in this thesis are also assumed to be completely free of risk. More details on government bonds and the NIBOR can be found at the home pages of Oslo Boers [5].

Although interest rates in this thesis are thought of and modelled as pure stochastic variables, some fundamental understanding of the market is still required. Some of the driving forces behind changes in interest rates includes varying macroeconomic conditions, central bank interventions via monetary policy, fiscal policy, market liquidity and the daily

emotions of market participants. Some of these factors will be discussed with respect to the results of this thesis, but for a more complete understanding of the subject a god textbook like Giavazzi and Blanchard [22] is recommended.

4.1 Interest Rates and Discount Factors

Before defining and formalising key relations between discount factors and different kinds of interest rates, some notes on the use of language and notation are required. In most textbooks and papers where risk-free interest rates are modelled, the focus is on government interest rates. These rates are often derived from zero-coupon government bonds, and it therefore makes sense to formulate interest rates in relation to prices and return of such bonds. In this thesis, the focus is on swap rates, and it therefore does not make as much sense to use bond prices and returns to derive relationships between for example spot rates and forward rates. The term *discount factor* will therefore consistently be used instead of the usual *bond prices*, and instead of *rate of return on a bond with a given maturity*, simply *the rate of return on investment until maturity* will be used. The latter in practise often meaning the risk-free interest rate on a loan between two banks with a given maturity.

Following definitions and the general notation from [6], the bank account is first defined.

Definition 4.1.1. *(Bank account) Let $B(t)$ be the value of a bank account at time t, and assume $B(0) = 1$ is the normalised value at time $t = 0$. The bank account then evolves according to*

$$dB = r_t B(t) dt,$$

with r_t being a positive function of time. Solving this equation with the normalised initial value $B(0) = 1$, the bank account value at time t is then

$$B(t) = exp\{ \int_0^t r_s ds \}. \tag{4.1}$$

The bank account grows exponentially with the instantaneous growth rate r_t. The instantaneous rate is often referred to as the instantaneous spot rate, or briefly as the short rate soon to be defined.

Discounting, or discount factors is an important concept. One dollar today is generally not worth one dollar in a year, because one can earn a risk free interest rate. This relative difference between the value of a dollar at different times is formalised using the bank account from definition (4.1) into the discount factor.

Definition 4.1.2. *(Discounting) A discount factor $D(t, T)$ between times t and T is the amount at time t which is equivalent to one unit of currency at time T, given by*

$$D(t, T) = \frac{B(t)}{B(T)} = exp\{ - \int_t^T r_s ds \} \tag{4.2}$$

It is important to note that both the bank account and the discount factors can be considered deterministic or stochastic, depending on how the interest rate is modelled. In this

thesis, one of the central goals is to model the interest rate as a stochastic variable, and all discount factors are therefore assumed to be stochastic.

Using the discount factor $D(t,T)$ between times t and T, this section continues with central definitions of some important interest rates.

Definition 4.1.3. *(Continuously-compounded spot interest rate) The Continuously-compounded spot interest rate, or just the spot rate at time t for maturity T denoted as $R(t,T)$, is the constant annual rate which a safe investment, for example a safe loan, grows with until maturity. Mathematically defined as*

$$R(t,T) = -\frac{ln(D(t,T))}{T-t}. \tag{4.3}$$

A curve showing $R(0,T)$ for a set of different maturities T is called the term structure of interest rates or the yield curve, and is often used as an illustration of the market's expectations of future interest rates.

Definition 4.1.4. *(The instantaneous spot rate) The instantaneous spot rate, often called the short rate is denoted $r(t)$ and is the continuously compounded spot interest rate $R(t,T)$, when $T \to t$.*

It is usual market practice to set the short rate as the 3 month spot rate $R(t, t+3\text{months})$, which is then used as r_s in the discount factor defined in equation 4.2 [1].

Definition 4.1.5. *(Forward interest rates) A forward interest rate $F(t,T_1,T_2)$, with $t \leq T_1 \leq T_2$ is the interest rate between T_1 and T_2, contracted at time t. The continuously forward interest rate is given by*

$$F(t,T_1,T_2) = -\frac{logD(t,T_2) - logD(t,T_1)}{T_2 - T_1}. \tag{4.4}$$

The following proof is provided to show how this interest rate between two future dates T_1 and T_2 with certainty can be determined at an earlier date t: Assuming absence of arbitrage, which is explained in chapter 3, two risk-free investment strategies is set up at time t. Since both strategies are risk-free, they should yield the same rate of return. Strategy 1 is buying 1 unit of a safe investment in the form of a loan with maturity at T_2, yielding a rate of return equivalent to the spot rate $R(t,T_2)$. By holding the investment to maturity and compounding continuously, this should yield a total return of

$$e^{R(t,T_2)[T_2-t]} = e^{-logD(t,T_2)} = \frac{1}{D(t,T_2)}, \tag{4.5}$$

by using the relationship between discount factors and and spot rate from equation (4.3). Strategy 2 is to buy 1 unit of a safe investment maturing at T_1 yielding a safe rate of return equal to $R(t,T_1)$, while simultaneously agreeing to invest the proceeds at time T_1 at the forward rate $F(t,T_1,T_2)$. This gives the total return on investment as

$$e^{R(t,T_1)[T_1-t]+F(t,T_1,T_2)[T_2-T_1]} = \frac{1}{D(t,T_1)} e^{F(t,T_1,T_2)[T_2-T_1]}. \tag{4.6}$$

Equating (4.5) to (4.6), and solving for the forward rate $F(t,T_1,T_2)$ gives (4.4).

Definition 4.1.6. *(Instantaneous forward interest rate) The Instantaneous forward interest rate, denoted $f(t,T)$ for $T > t$ is the continuously compounded interest rate contracted at time t for borrowing at time T. It is derived from the general forward interest rate as*

$$f(t,T) := \lim_{S \to T} F(t,T,S) = -\frac{\partial}{\partial T}\big(logD(t,T)\big),$$

were $f(t,t) = r(t)$ is simply the short rate.

This section is concluded by stating the discount factor equivalent to 4.2, but as a function of the instantaneous forward rate $f(t,T)$ as

$$D(t,T) = e^{-\int_t^T f(t,\tau)d\tau}, \tag{4.7}$$

and as the inverse of equation 4.3,

$$D(t,T) = e^{-R(t,T)[T-t]}. \tag{4.8}$$

4.2 Market Price of Risk and the Expectation Hypothesis

An important question when modelling the real path of interest rates is the following: What is the relation between a forward rate $f(t,T_1,T_2)$, and the expected equivalent future spot rate $E[R(T_1,T_2)]$ at time t? Starting by using the relationship between the instantaneous forward rate and current spot rates, 4.7 and 4.8 can be combined into

$$\int_{t_0}^{t_n} f(\tau,t_n)d\tau = R(t_0,t_n)[t_n - t_0]. \tag{4.9}$$

One theory described as *the pure expectation hypothesis* suggests forward and expected spot rates are equal, and by equation (4.9) is often stated as

$$R(t_0,t_n)[t_n - t_0] = \sum_{i=0}^{n-1} E[R(t_i,t_{i+1)}][t_{i+1} - t_i]. \tag{4.10}$$

Here, the total return on a safe investment between times t_0 and t_n, $\exp\{R(t_0,t_n)[t_n - t_0]\}$, is assumed to equal the expected total rolling return on investments with shorter maturities [20]. Unfortunately, the pure expectation hypothesis has been rejected by empirical studies of past market data [18]. A more accepted and modified version, often called the *traditional expectation hypothesis*, is mathematically described as

$$R(t_0,t_n)[t_n - t_0] = \sum_{i=0}^{n-1} E[R(t_i,t_{i+1)}][t_{i+1} - t_i] - \lambda_n. \tag{4.11}$$

This version has an extra term, the constant λ_n, known as the *market price of risk* [25]. On this form of the equation, the market price of risk is usually estimated to be negative, meaning realised future spot rates are on average lower than the equivalent forward rates. The market price of risk is also assumed to be dependent on maturity, with largest

magnitude for short maturities. Observing a typical upward sloping forward instantaneous interest rate curve $f(0,t)$ for $0 \leq t \leq 10$, this means that the difference between the 1 year forward rate $f(0,1)$ and the short rate $f(0,0)$ is usually larger than the difference between the 10-year $f(0,10)$ and the 9-year $f(0,9)$ forward rates [25].

There are several explanations for the existence of the market prize of risk, with the perhaps most recognised called the liquidity preference theory. The theory states that investors on average demands a premium for tying up capital for longer periods compared with short term investments. The theory was first stated by the late great economist John Maynard Keynes in his his famous *The General Theory of Employment, Interest and Money* [17]. Figure 4.1 illustrates this important point that on average, the forward rates $f(t, T_1, T_2)$ are higher than the expected equivalent spot rates $R(T_1, T_2)$. The market price of risk can be visualised as the space between expected spot and forward rates. The realised spot rates are itself stochastic and highly variable.

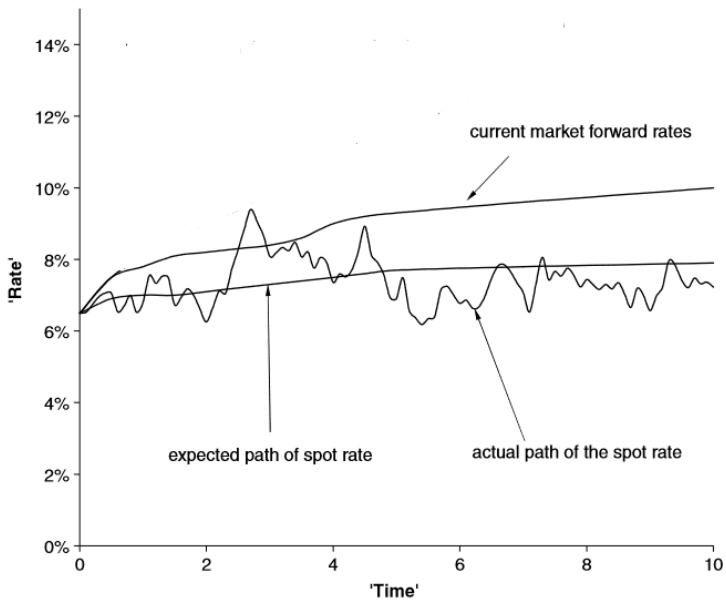


Figure 4.1: An example of the expected spot rate, compared with the equivalent forward rate and realised path of the spot rate. The figure is a modified illustration from [25].

One of the main challenges when modelling the real path of interest rates is therefore to accurately estimate the market price of risk. The problem, as for example shown by Willmott in [25], is that the market price of risk is itself highly variable and can sometimes even be positive. This apparent unpredictable, non-constant relationship between forward rates and expected spot rates means that also the traditional form of the expectation hypothesis in equation (4.11) is rejected. This often means that some idealised assumptions needs to be made when modelling the real path of interest rates [18] [15].

4.3 Interest Rate Derivatives

As in definition 3.1.1, a financial derivative is a security with a value that is dependent upon or derived from one or more underlying assets. This thesis focuses on interest rate derivatives, a class of derivatives dependent on present or future levels of some type of interest rate. The derivatives modelled in this thesis is assumed to be traded over-the-counter (OTC), as opposed to through an exchange. OTC derivatives are often agreements directly between counterparties and can be very flexible regarding the contract details [25]. As mentioned in the introduction, interest rate instruments make up the majority of all OTC derivatives and are of particular importance to banks and other large corporations who needs to hedge out risk associated with future interest payments on loans and cash-flows.

As defined in the previous section, there are many different types of interest rates like spot rates and forward rates of differing maturities. Since an interest rate derivative can depend on one or a combination of several different rates, the possible number of interest rate derivatives are almost infinite. This thesis focuses on the most simple and widely used interest rate derivatives, all with value derived from the short interest rate. Notation and definitions are as in Wilmott [25].

Cap

An interest rate cap, often just called a cap, is a contract which consist of several possible payoffs called caplets. The different caplets, each maturing at time t_i, gives a payoff V_c of

$$V_c(t_i, r_{l_i}, r_c) = N\alpha \max(r_{l_i} - r_c, 0). \tag{4.12}$$

r_{l_i} is a floating interest rate, often derived from the short rate, r_c is a fixed or capped rate, N is the notional value of the contract, and α is the day count fraction corresponding to the period which the rate r_l is set. From the payoff it is clear that an interest rate cap is a bullish bet on the interest rate, with increasing payoffs as the interest rate rises. Corporations and individuals with floating rate loans can therefore use an interest rate cap to hedge out the risk associated with rising interest rates.

Floor

An interest rate floor is similar to the cap, consisting of different floorlets with payoffs equal to

$$V_f(t_i, r_{l_i}, r_c) = N\alpha \max(r_c - r_{l_i}, 0). \tag{4.13}$$

The floor is a bearish bet on the interest rate, profiting when the interest rate falls below the capped rate. Interest rate floors can therefore be used by for example pension funds who invests in short dated interest-bearing securities to hedge out the risk associated with falling interest rates.

Swap

The last interest rate derivative considered in this thesis is the most important and widely used, namely the interest rate swap. A swap consists of swaplets and is a contract where

two counterparties agrees to exchange payments based on two different interest rates, often a fixed for a floating rate. By studying the payoffs from the caplets and floorlets, it can be seen that this is equivalent to buying a cap and selling a floor which gives payoffs in each period equivalent to

$$\begin{aligned} V_s(t_i, r_{l_i}, r_c) &= V_c - V_f \\ &= N\alpha\big(\max(r_{l_i} - r_c, 0) - \max(r_c - r_{l_i}, 0)\big) \\ &= N\alpha(r_{l_i} - r_c). \end{aligned} \tag{4.14}$$

The swap contract exchanging fixed for floating interest rate is a very common contract between a financial institution and another non-financial corporation. If the corporation has a floating interest rate loan, it could enter into a swap agreement with a bank, agreeing to pay the bank a fixed rate, while receiving a floating one. The party paying the fixed rate is said to have the *payer* position on the contract, receiving the cash-flow in equation (4.14) every payment date. The corporation then uses the floating rate received to pay off the interest rate on the loan. This way the the corporation has hedged its loan expenses by transferring the interest rate risk to the bank. The swap contract can therefore bee seen as an insurance policy against movement in interest rates issued by banks and financial institutions. Corporations wants to buy this insurance, often at a premium, to be able to focus on their core business without having to worry about interest rates movements affecting their results.

Pricing

To put the described derivatives and payoffs into perspective, one can return to chapter 3 and in particular the risk-neutral pricing equation 3.1. The earlier generalised function $f(S)$ is now specified through the desired payoff functions in 4.12, 4.13 and 4.14. The remaining challenge in pricing these derivative is to simulated the underlying interest rate in the risk-neutral measure. This will be the main topic of the next chapter.

Chapter 5

Interest Rate Modelling

5.1 Stochastic Interest Rates Modelling

There exists many models which attempts to describe possible movements in future interest rates. One class of models is called 1-factor models, and describes the development of one particular interest rate, often the risk-free short rate $r(t)$. A general 1-factor interest rate model is often described by a stochastic difference equation on the form.

$$dr = u(r,t)dt + w(r,t)dX. \tag{5.1}$$

dr represents the next increment in the interest rate over a period dt. dX is a normally distributed stochastic variable described by the Brownian motion in appendix B.1, with variance dt and represents the stochastic nature of the interest rate [14]. The function $w(r,t)$ represents the volatility and the term $u(r,t)$ is the underlying drift of the interest rate. The functions $w(r,t)$ and $u(r,t)$ depend on the particular model and may be dependent on both time and the current level of the interest rate as indicated by the notation. The form of the drift is also dependent on the measure which the model is implemented under. For a real-world measure, the market price of risk is often estimated and a term $\lambda(t)$ is therefore included.

Another class of interest rate models is called multi-factor interest rate models. A multi-factor model uses more than one source of randomness to describe the development of several different variables at the same time. This is often interest rates of differing maturities in an attempt to describe more than one point on the yield curve to better capture the dynamics of future interest rates. Such models can also be described by equation (5.1), with dr, $w(r,t)$, dX and $u(r,t)$ now being vectors.

5.2 Heath, Jarrow and Morton Model

In this thesis the multi-factor Heath, Jarrow and Morton model is explained and implemented as described by Glasserman in [14], with the only difference in notation being the continued use of the more general *discount factors* instead of *bond prices*. The more complete and original derivation by Heath, Jarrow and Morton can be found in [10]. The model is implemented in both the risk-neutral measure for pricing and in a simplified real-world measure for generation of real future interest rates. Important theorems and other mathematical tools used in the derivations can be found in appendix B, and will be referenced when needed.

5.2.1 Outline

The HJM interest rate model is used to describe the dynamics of the instantaneous forward interest rate curve, denoted $\{f(t,T), 0 \leq t \leq T \leq T^*\}$, where T^* is some ultimate maturity. $f(t,T)$ represent the instantaneous forward interest rate at time t for maturity T as defined in 4.1.6, and can be thought of as the continuously compounded interest rate at time t for risk-free borrowing at time T. This important relation between instantaneous forward rates and discount factors is restated as a starting point for the derivation of the model

$$f(t,T) = -\frac{\partial}{\partial T}\big(logD(t,T)\big). \tag{5.2}$$

It should again be noted that the forward rate $f(t,t)$ is simply the realised short rate $r(t)$ at time t. The evolution of the forward interest rate curve is described by a stochastic difference equation of the similar form to (5.1). Using the same notation as in [14], the development of the forward curve is governed by

$$df(t,T) = \mu(t,T)dt + \sigma(t,T)^\top dW(t). \tag{5.3}$$

It is important to remember that the change in forward interest rates, denoted df, is with respect to time t and not maturity T. $dW(t)$ is a standard Wiener process defined in appendix B.1, but in M dimensions, which is the number of factors in the model. The drift and volatility coefficients μ and σ are M dimensional scalars, and can be both stochastic or deterministic.

5.2.2 Risk-Neutral Measure

Realising from (5.2) that $df(t,T) = \frac{dD(t,T)}{D(T)}$, the evolution of discount factors in the risk neutral world is given by

$$\frac{dD(t,T)}{D(t,T)} = r(t)dt + \nu(t,T)^\top dW(t) \qquad 0 \leq t \leq T \leq T^*,$$

were ν is the discount factor volatility. Applying Itô's formula from equation (B.1) as formulated in [23] to (5.3), resulting in

$$d(logD(t,T)) = \big[r(t) - \frac{1}{2}\nu(t,T)^\top\nu(t,T)\big]dt + \nu(t,T)^\top dW(t).$$

Following [14], differentiation with respect to T before interchanging the order of differentiation between t and T from equation (5.2), the risk neutral formula for $df(t,T)$ is obtained as

$$\begin{aligned} df(t,T) &= -\frac{\partial}{\partial T} logD(t,T) \\ &= -\frac{\partial}{\partial T}\big[r(t) - \frac{1}{2}\nu(t,T)^\top \nu(t,T)\big]dt - \frac{\partial}{\partial T}\nu(t,T)^\top dW(t). \end{aligned} \tag{5.4}$$

Comparing (5.3) and (5.4), and realising that $r(t)$ is independent of T, leads to the following expressions for the risk neutral drift and volatility:

$$\begin{aligned} \sigma(t,T) &= -\frac{\partial}{\partial T}\nu(t,T) \\ \mu(t,T) &= \sigma(t,T)^\top \int_t^T \sigma(t,u)du. \end{aligned} \tag{5.5}$$

Substituting (5.5) into (5.3) the development of the forward rate is then given by

$$df(t,T) = \left(\sigma(t,T)^\top \int_t^T \sigma(t,u)du\right)dt + \sigma(t,T)^\top dW(t), \tag{5.6}$$

which is the centrepiece of the HJM framework, showing that the drift under the risk-neutral measure is fully determined by the volatility structure [14]. To investigate the risk-neutral volatility structure further, the change of measure from the real-world to the risk-neutral world via the Radon-Nikodym derivative described in appendix B.3 is useful. By applying this change of measure to Girasanov's theorem described in B.4, it is shown that the volatility in the risk-neutral HJM model described by (5.6) actually is equivalent to the volatility in the real-world measure. The volatility in (5.6) can therefore be estimated from historical data [14].

5.2.3 Real-World Measure

Having the risk-neutral measure to price the derivatives, a model to simulate real-world scenarios of interest rates is also needed. Generating realistic real paths of interest rates is actually more difficult than to price the derivatives. The volatility is as showed by Grishanov's therorem in appendix B.4.1 equal in the risk-neutral and the real world, but the drift of the interest rate is not. From the HJM model outlined in the previous section, the model uses observed forward rates as input to generate future spot rates. As discussed in section 4.2, the relation between forward rates and spot rates are complicated and often involves the market price of risk to adjust the drift. This market price of risk is itself highly variable and very difficult to accurately estimate. An example of an attempt at a real-world HJM framework estimating the market price of risk can be found in [26].

So how can real-world interest rates be generated while avoiding to estimate the market price of risk? One option is to use the risk-neutral implementation, and assume that the interest rate generated is an accurate enough representation of real-world interest rates.

Studying equation (5.6), the drift term in the risk-neutral implementation is actually positive, leading to realised spot rates being higher than indicated by forward rates. In practise this means that using the risk-neutral approach to simulate real-world interest rates is equivalent to assuming a positive market price of risk. This contradicts the generally accepted notion that the market price of risk is negative.

The approach chosen in this thesis is simple and pragmatic. By assuming the purest form of the expectation hypothesis, given by equation (4.10) holds true, the drift in the model is set to zero. This means the model will produce expected short rates $E(r(t)) = f(0,t)$ equal to the equivalent forward rates at initialisation for all times. The consequence of this simplification will be a central part of the discussion of the results

5.2.4 Discretization and Simulation

Simulation of the continuous model described in equation (5.6) in the previous section is impossible except for very special choices of σ [14]. To simulate from the general framework in (5.6) without restricting the form of the volatility, a discrete approximation is therefore needed. Let $\hat{f}(t_i, t_j)$ represent the discrete forward rate for $t = t_j$ at time t_i. Both t_i and t_j are discretized, and for convenience the same grid $0 = t_0 < t_1 < ... < t_M = T^*$ is used for both variables. By this approximation, the disocunt factor equation from (4.7) is written into discrete form

$$\hat{D}(t_i, t_j) = \exp\Big\{ \sum_{l=i}^{j-i} \hat{f}(t_i, t_l)\big[t_{l+1} - t_l\big] \Big\}. \tag{5.7}$$

To avoid a larger than necessary discretization error, the continuous discount factors from (4.7) is set equal to the discrete discount factors from (5.7) at time $t_i = 0$. This gives the condition

$$\int_0^{t_j} f(0,u)du = \sum_{l=0}^{j-i} \hat{f}(0, t_l)\big[t_{l+1} - t_l\big],$$

or equivalently

$$\begin{aligned} \hat{f}(0, t_l) &= \frac{1}{t_{l+1} - t_l} \int_{t_l}^{t_{l+1}} f(0,u) \\ &= \frac{1}{t_{l+1} - t_l} log \frac{D(0,t_l)}{D(0, t_{l+1})} \qquad \text{for all} \quad l = 1, 2, ..., M-1, \end{aligned} \tag{5.8}$$

were the discount factors $D(0,t)$ is calculated the usual way from observed market interest rates as in equation (4.8). The discrete version of (5.6), with M factors can now be formulated as

$$\hat{f}(t_i, t_j) = \hat{f}(t_{i-1}, t_j) + \hat{\mu}(t_{i-1}, t_j)[t_i - t_{i-1}] + \sum_{k=1}^{M} \hat{\sigma}_k(t_{i-1}, t_j)\sqrt{t_i - t_{i-1}} W_{ik}, \tag{5.9}$$

for all $i = 1, .., M$ and $j = i, .., M$ [14]. The W_i's are independent vectors of length M of random standard normal distributed variables. The drift terms in the risk-neutral implementation, $\hat{\mu}(t_{i-1}, t_j)$ are approximated by discretization of the expression for $\mu(t, T)$ in (5.5) and given by

$$\hat{\mu}(t_{i-1}, t_j)[t_{j+1} - t_j] = \sum_{k=1}^{M} \hat{\mu}_k(t_{i-1}, t_j),$$

were $\hat{\mu}_k(t_{i-1}, t_j)$ is given by

$$\hat{\mu}_k(t_{i-1}, t_j)[t_{j+1} - t_j] = \frac{1}{2}\left(\sum_{l=i}^{j} \hat{\sigma}_k(t_{i-1}, t_l)[t_{l+1} - t_l]\right)^2 - \frac{1}{2}\left(\sum_{l=i}^{j-1} \hat{\sigma}_k(t_{i-1}, t_l)[t_{l+1} - t_l]\right)^2.$$

In the simplified real-world model, all drift terms μ are set to zero.

To simulate from the discrete algorithm given by (5.9), only an initial forward curve $\hat{f}(0, t)$ for $0 < t < T^*$ and the volatility parameters $\hat{\sigma}_k$ are needed. The initial forward curve is calculated from observed market prices combining equations (5.8) and (4.8), with time steps Δt to transform the spot rates into forward rates as

$$\hat{f}(0, t) = \frac{1}{\Delta t}\Big(R(0, t + \Delta t) \cdot (1 + \Delta t) - R(0, t) \cdot t\Big). \tag{5.10}$$

The volatility structure used in the implementation of the model will be discussed in the next chapter.

5.2.5 Pricing Derivatives with HJM

Returning yet again to chapter 3 and equation 3.1, all components needed for evaluating prices are now in place. Using the payoff functions for the derivatives presented in chapter 4, and the risk-neutral implementation of the underlying interest rates presented in the current chapter, fair prices of derivatives is obtained. The final part of this chapter outlines explicitly how the output of the risk-neutral HJM implementation is used to evaluate prices.

Discount factors

The risk-neutral implementation of the HJM algorithm is automatically fitted to marked data by its initialisation from equation (5.8) [14]. This means that expected simulated discount factors will be equal to the observed discount factors in the market at initialisation, calculated from (4.8). As will be explained in chapter 7, counterparty credit risk concerns both expected prices and the distribution of future prices. Simulation and study of the distribution of discount factors is therefore still important.

The calculation of discount factors is done by using the simulated short rates $r(t) = \hat{f}(t, t)$. By continuously discounting the present values with the short rate at all intervals, relevant discount factors are found. The simulated discount factors, now denoted

$D_s(t_0, t_j)$ to avoid confusion with the observed discount factors $D(0, t)$ in the market, are for each period t_j, starting at $t = t_0$ given as

$$D_s(t_0, t_j) = \exp\Big(\sum_{l=0}^{j-1} \hat{f}(t_l, t_l)[t_{l+1} - t_l]\Big). \tag{5.11}$$

Floor, Cap and Swap Pricing

An interest rate floor is priced using the HJM algorithm by summing the discounted value of all floorlets over the floor's maturity. The discount factor for each floorlet is given as the discount factor for the period from initiation t_0 to cash-flow t_i denoted $D_s(t_0, t_i)$ as in (5.11). r_c is the strike rate on the contract. For each floorlet, with value at maturity t_i as in (4.13), the floating rate r_l needs to be determined. r_l is often called the reference rate, and is usually set constant over each time interval as the discreetly compounded rate over this interval and is estimated by

$$\hat{r}_d(t_i) = \frac{1}{t_{i+1} - t_i}\Big(\exp\{\hat{f}(t_i, t_i)[t_{i+1} - t_i]\} - 1\Big). \tag{5.12}$$

Replacing the interest rate r_l with the estimated discrete reference rate $\hat{r}_d$ from (5.12) into (4.13) and discounting, the following formula gives the present value of a floor P_f with n floorlets with discounted value V_i with maturity at time t_i:

$$P_f = \sum_{i=1}^{n} V_i = \sum_{i=1}^{n}\left(N\alpha_i \max(0, r_c - \hat{r}_d(t_i)) \cdot \exp\Big\{\sum_{l=0}^{i-1} \hat{f}(t_l, t_l)[t_{l+1} - t_l]\Big\}\right) \tag{5.13}$$

Having a formula to price interest rate floors using the output from the HJM algorithm, pricing an interest rate cap and a swap is trivial. Substituting the payoff from equation (4.12) instead of (4.13) the following cap price P_c as a sum of n caplets is obtained as

$$P_c = \sum_{i=1}^{n}\left(N\alpha_i \max(0, \hat{r}_d(t_i) - r_c) \cdot D_s(t_0, t_i)\right). \tag{5.14}$$

Similarly using equation (4.14), a swap is priced using the output of the algorithm as

$$P_s = \sum_{i=1}^{n}\left(N\alpha_i(\hat{r}_d(t_i) - r_c) \cdot D_s(t_0, t_i)\right). \tag{5.15}$$

Chapter 6

Volatility

Volatility is one of the most important concepts in financial modelling, and therefore also very important in this thesis. The HJM algorithm described in the previous chapter does not explicitly specify the form of the volatility, giving a wide range of choices to the implementer. This chapter will start off by describing volatility in general terms, followed by a discussion of some of the options to chose from. The choice of volatility-structure in the HJM model is then justified. Independent of the final choice, it is important to bear in mind that volatility is a highly unpredictable quantity and no method or approach of measuring it will be perfect.

6.1 Definitions and Different Formulations

Definitions and Metrics

Volatility is loosely defined as the standard deviation σ of the increments, measured with some frequency on some interval, in the price of a financial instrument. Fixing both a frequency and an interval, and defining a time series of an asset prices as $A = \{a_1, a_2, \dots, a_n\}$, the increments can be measured in two different ways before taking the standard deviation. Assuming the increments are independent on the current level of asset price, the natural choice for the differenced time series is $d_1 = \{a_2 - a_1, a_3 - a_2, \dots, a_n - a_{n-1}\}$. Assuming dependence on current level, which is often done for stocks and other assets with an expected underlying growth rate, the natural choice is $d_2 = \left\{\frac{a_2-a_1}{a_1}, \frac{a_3-a_2}{a_2}, \dots, \frac{a_n-a_{n-1}}{a_n}\right\}$. As for interest rates which are the concern in this thesis, the choice of metric is far from obvious, and the different formulations can have large consequences on the resulting behaviour of the model. In [25], Willmott summarises both how volatility is formulated in some of the popular interest rate models, in addition to some empirical research on the subject. Unfortunately, no conclusion can be reached with certainty. Although most of the research indicates that d_2 is a more accurate description of the behaviour of interest rates, a lot of this research was done when interest rates were much higher than today. By estimating volatility as a fraction of the underlying asset, a decline in the asset lowers the

absolute volatility. If the assets tends towards zero, as interest rates have done in recent years, volatility also tends to zero. In reality, interest rates have a larger chance of going from 0.5% to 0.55% than from 5% to 5.5% over the same time interval, implying inconsistency with the d_2 metric. Measuring volatility using d_1 captures the behaviour of low interest rates better, but can naturally overestimate volatility when interest rates rises. It is still probably a more robust estimator in today's low interest rate environment.

Regarding the choice of frequency with which the asset is measured, it is common practise to use daily measurements of the closing price. In most models, including HJM, the underlying interest rate is assumed to follow a random walk. From the definition of Brownian motion in the appendix B.1, this means volatility in addition to being constant can be scaled by $\sqrt{t}$ when t is the number of days into the preferred time horizon [14]. By scaling with $t = 252$, approximating the number of trading days in a year, the volatility is said to be annualised.

Historical, Implied and other Methods

In addition to *how* to measure volatility, a central question is over what *time horizon* to measure it, whether it is forward or backward looking. Since the instant volatility at a given time is impossible to measure, some amount of data is needed. *Historical volatility* is therefore a popular way of estimating the volatility in a model. By using the standard deviation of past observed prices over a specified time horizon, an estimate of the volatility is easy to obtain. The challenge with this approach is to choose an appropriate time horizon for measurement. This can often be a problem because one implicitly assumes historical data reflects the future. As will be shown in chapter 9 where the data is analysed, market conditions often changes and historical data is seldom capable of predicting the future in an accurate way.

Another way of estimating volatility is to use the *implied volatility*. Using the market prices of financial derivatives, for example a stock option or an interest rate derivative, it is possible to use analytic formulas and calculate backwards to obtain an estimate of what the market thinks the volatility will be over a future period. An advantage with using implied volatility is that all pricing done by the model is consistent with the market. A downside with this method is that the analytic formulas relating volatility and prices of derivatives do not exist for most multi-factor interest rate models. Another downside of using this approach is that it assumes the market knows the future volatility. In practise this is highly unlikely, because as with historical volatility the market's expectations changes all the time.

Given the challenge of modelling the volatility to a satisfactory extent, other approaches are also sometimes used. A common denominator for such methods is that they model the volatility as a stochastic or time dependent variable. This includes both Autoregressive Conditional Heteroskedasticity models (ARCH) and Generalized Auto Regressive Conditional Heteroskedasticity (GARCH) models. These models has had some success, but are often to complicated to be feasible alternatives to the more traditional methods described above. More information on volatility and the different ways of modelling it can be found

in [25].

Volatility in HJM

Based on the previous discussion, the choices made when implementing the HJM model are the following: Because of its assumed robustness when interest rates are low, the volatility is measured on an absolute basis, using metric d_1. Historical volatility is chosen over implied, because there exist no simple analytic formulas to calculate the implied volatility from in the multi-factor HJM model. The time horizon used for estimation will be discussed after analysing the data in section 9, and the method used for estimation is called *principal component analyis* and is presented in the next section.

6.2 Principal Component Analysis

Principal component analysis (PCA) is described as a statistical procedure that uses an orthogonal transformation to convert a set of observed correlated variables into a set of values of linearly uncorrelated variables called the principal components. The procedure is often used in cases where strong correlations between the observed variables exists. Since forward interest rates for different maturities are very correlated, PCA is an important tool when implementing and using historical volatility in multi-factor interest rate models [25].

The procedure starts by calculating the covariance matrix Σ, a $M \times M$ matrix for the M-factor model based on the observed data. After this is done, Σ is decomposed into

$$\Sigma = V \Lambda V^{-1},$$

were Λ is a diagonal matrix with the eigenvalues of Σ, λ_i on its diagonal. V contains the eigenvectors of Σ, namely V_i. The volatility factors are then defined as

$$\hat{\sigma}_j(t_{i-1}, t_j) = \sqrt{\lambda_j * 252} V_{ij} \tag{6.1}$$

were $j = 1...M$ represents the different maturities [25]. The factor $\sqrt{252}$ is included to scale the volatility from days into years which are the metric used for t in the implemented HJM model. The principal components can then be visualised by plotting the components $\hat{\sigma}_j(t_{i-1}, t_j)$ as a function of the maturity j. Applying PCA on yield curve data or spot rates, the economic interpretation of the first 3 components are clear. The first, and most significant component represents a parallel shift, the second one represents a twist, and the third component represents the bend. By transforming spot rates into forward rates, the interpretation when applying PCA is less intuitive.

To determine how many of the principal components to include in the model, the different components' total contribution to the variance in percent is measured by TC_i and given by

$$TC_i = \frac{\lambda_i}{\sum_{i=1}^{M} \lambda_i} 100\%. \tag{6.2}$$

The components to be included is chosen in descending size, usually with a criterion for how much of the variability in the data the chosen components should explain. There will always be a trade-off between accuracy and noise, and there are many different ways to determine the number of factors to include in the model as underlined in [16]. Measuring in percent as in equation (6.2), anything from 70% to 90% is generally considered as a rule of thumb, but sometimes a higher limit is needed. To be on the safe side, the criterion used in this thesis is set to include components until at least 95% of the variability in the data is explained.

A practical simplification of the expression for the volatility, which is used in this thesis, is obtained using the Musiela parametrization. The volatility in this parametrization is assumed to only be a function of time to maturity, and can therefore be simplified into $\hat{\sigma}_j(t_i, t_j) = \hat{\sigma}_j(t_j - t_i)$ [25].

Chapter 7

Counterparty Credit Risk

Counterparty credit risk (CCR) is the term used for the risk associated with a counterparty not living up to its contracted obligations. Counterparty credit risk is often divided into the following 3 parts.

Definition 7.0.1 (Probability of default)**.** *Probability of default is the probability that the counterparty defaults, and will not be able no meet its contractual obligations.*

Definition 7.0.2 (Loss given default)**.** *Loss given default is defined as the percentage amount of its obligations the counterparty is expected to not be able to pay back in case of a default.*

Definition 7.0.3 (Counterparty credit exposure)**.** *Counterparty credit exposure is the amount a company could potentially lose in the event of one of its counterparties defaulting.*

A complete framework for credit risk modelling naturally incorporates all three components, which is outlined in [7]. The concern of this thesis is exclusively the modelling and simulation of counterparty credit exposure, but it will still be useful to keep the other components in mind when discussing the results. The main source used in this chapter is Cesari et. al [11].

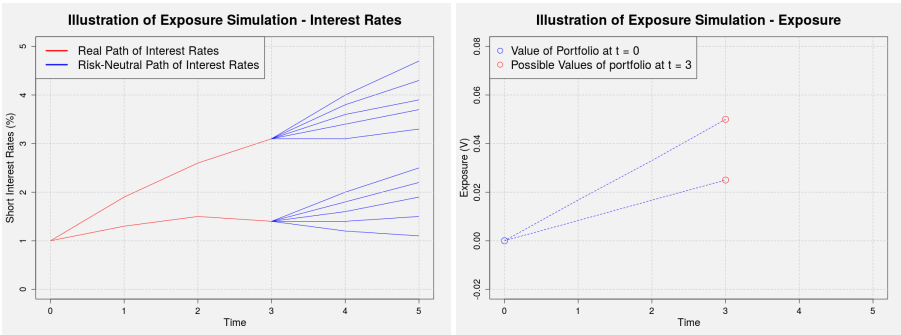
7.1 Counterparty Credit Exposure

7.1.1 Main idea

The starting point of the modelling is a situation where company A, say a financial institution has a portfolio of derivatives with a counterparty, say company B. The portfolio can consist of anything from interest rate swaps, to currency swaps and other more complicated derivatives. The value of the portfolio is denoted V_t for the time horizon $0 \leq t \leq T$, where all the derivatives in the portfolio is assumed to mature no later than $t = T$. Depending on the portfolio, V_t can assume both positive and negative values. An interest rate swap can for example yield cash-flows in both directions, depending on the movement in

the underlying interest rate. The goal is to simulate and gain understanding of the distribution of V_t for future times $0 \leq t \leq T$.

The value or the exposure V_t of the portfolio at each time step will depend on the price of the derivatives, which again are dependent on the underlying risk-factors. In this thesis this means the Norwegian short swap interest rates. The first step in simulating the distribution of V_t is therefore to simulate the underlying risk-factors, before pricing the portfolio in the different scenarios. The idea is illustrated in figure 7.1.



(a) Illustration of 2 possible paths of the real short interest rate, each used to price the portfolio at $t = 3$, using the risk-neutral simulated short interest rates.

(b) Illustration of 2 possible portfolio values 3 years after contract initialisation.

Figure 7.1: 2 possible simulated paths of counterparty credit exposure.

The figures shows how the distribution of V_3 can be obtained. Looking at figure 7.1a, the real-world interest rate using the zero-drift implementation is simulated until $t = 3$. Using the whole simulated real-world forward interest rate curve at $t = 3$ as input, the risk-neutral implementation is then used to price the portfolio by simulating paths to maturity and discounting payoffs as explained in chapter 5. Figure 7.1b shows the resulting portfolio values given the simulated paths in figure 7.1a. By repeating this procedure many times, the distribution of the exposure at various times is generated.

The precise relation between the figures is emphasised by the following example: At $t = 0$, the short interest rate in figure 7.1a is around 1%. Assuming the forward interest rate curve is flat, a portfolio of a simple payer swap with fixed rate of 1% will be valued around 0 at initialisation. Following the upper path of interest rates in figure 7.1a, the simulated short interest rate has risen to around 3% after 3 years. Still assuming a flat forward interest rate curve, the holder of the payer swap now has a profit and the contract is positive valued $V_3 > 0$ represented by the upper red circle in figure 7.1b. The holder of the payer swap still has to pay the 1% fixed rate, but expects a higher floating interest rate in return for the remainder of the contract. If the counterparty, the holder of the receiver swap now defaults at $t = 3$, the contract is suddenly worthless. The contract has to be replaced in the market

at the price V_3, incurring a potential loss for the holder of the payer swap.

7.1.2 Risk Measures

After simulating the portfolio values V_t, several statistical quantities is often used to describe the resulting counterparty credit exposure distributions. The Expected Positive Exposure (EPE) is defined as

$$\text{EPE}_t = \mathbb{E}[\max(V_t, 0)] = \mathbb{E}[V_t^+],$$

were the expectation only is taken over positive portfolio values [11]. This is because when the portfolio value is negative, there is no counterparty credit risk because a default by the counterparty wont result in a loss.

The Potential Future Exposure (PFE) is defined as

$$\text{PFE}_{\alpha,t} = \inf\{x : \mathbb{P}(V_t \leq x) \leq \alpha\},$$

where $\mathbb{P}$ is the simulated probability distribution of V_t and α is the significance level [11]. In this thesis the significance level is always 95%, which is also industry practise [1]. The PFE can intuitively be thought of as a one-sided confidence bound on the distribution of exposure V_t, and says how large the exposure is at minimum, in the worst $(1-\alpha)$ percentage of future outcomes. The 95% PFE is in practice calculated using the order statistics defined in 2.2.1.

As an alternative measure to PFE, the Expected Shortfall (ES) is sometimes used, which is defined as

$$\text{ES}_{\alpha,t} = \mathbb{E}[V_t | V_t > \text{PFE}_{\alpha,t}].$$

The ES provides more information regarding the tail of the distribution, in form of the average exposure in the $(1-\alpha)$ worst percentage of the outcomes. The ES is therefore always equal or bigger than the equivalent PFE. The focus in this thesis is on EPE and PFE.

7.2 Risk Mitigation

Several methods can be used to mitigate some of the counterparty credit risks and exposures which occurs when dealing in the OTC derivatives market. One such example is *netting agreements*. An example of netting is if the portfolio between counterparty A and B consist of n different products, each valued v_{t_i}. Since at every time t

$$\sum_{i=1}^{n} v_{t_i}^+ \geq \sum_{i=1}^{n} v_{t_i},$$

netting cash-flows in opposite directions with the same counterparty will reduce or keep the exposure equal. The use of *collateral* is also an important tool in reducing the counterparty exposure. Collateral can be posted as cash or other assets, and is used as protection

against default of the counterparty. In case of a default, the loss is partially or completely covered, and both potential exposures and losses becomes smaller than without collateral.

Another way of reducing exposure is by the use of regular, often daily, *settlements*. Even though the interval between cash-flows in a contract can be a year, the contract itself is typically valued on a day-to-day basis. By regularly netting out the contract value to zero, the exposures at default will be much lower.

The final risk-reducing tool discussed is the use of a *central counterparty*, also known as a clearinghouse. If company A and B are in a derivatives contract, the clearinghouse's role is to function as the counterparty for both A and B. This means that company A and B still has the same contract and exposure as before, but they both have the clearinghouse as counterparty. Companies A and B have thus no exposure to each other. The clearinghouse has a net zero position in the contract, and is responsible for collecting and maintaining collateral and settlements from the counterparties [1] [11].

7.3 Regulatory Requirements

Financial institutions must control risk and exposures for two reasons. The first reason is to minimise losses and maximise profits for its shareholders, and the second reason is for regulatory purposes. Regulatory requirements are imposed on financial institutions because the industry is so large and intertwined with the rest of the economy. Sector specific events such as a banking crisis can therefore have large spillover-effects into the real economy as seen in the great financial crisis in 2008-2009. Below is a short summary of some of the Norwegian Finanstilsynets regulatory rules which financial institutions has to comply with regarding the transactions of derivatives and counterparty credit risk. Many of the regulatory requirements are based on the Basel committee's standards.

The European Market infrastructure regulation (EMIR), is a set of rules implemented in the EU after the financial crisis to better control the risk in the OTC derivatives market. The rules determines that clearinghouses are to be used if the derivative contracts are between two financial institutions, or when the notional value of a contract with a particular counterparty rises above a certain threshold, which is 3 billion NOK for interest rate derivatives. Daily settlements of contract values are also mandatory between financial institutions.

The capital requirement regulations concerns business' reserve capital, risk management, public information and the maximal exposure against a single counterparty. The relevant part of the capital requirement regulation regarding this thesis is naturally the part concerning counterparty credit risk. The regulations says how potential credit exposure is used as an important component in determining how much reserve capital a financial institution needs to hold. Financial institutions are allowed to use their own models for estimating this exposure if they are approved by Finanstilsynet. Proving the validity of the internal models for exposure calculations is therefore an important task for any financial institution. More details on rules and regulations can be found on Finanstilsynet website [12], or

the website of the Basel committee [2].

Chapter 8

Methodology

Chapter 7 outlined and described how to simulate counterparty credit exposure using the models implemented in chapter 5. The next objective in this thesis is to develop a framework to test this model's accuracy. A common way of testing a models predictive properties is by running different kinds of backtests. A backtests often consists of running the implemented model over multiple historical periods. Using actual historical data, the results from the simulations and the real world can be compared, and the validity of the model can be assessed. Backtests are mandatory for a bank's internal risk model to show regulators that the model have performed well under previous, often stressed market conditions.

This chapter continues with a description of the portfolios used in the simulations, before the technical details of the backtests are described.

8.1 Portfolios

The portfolios selected for simulation needs to be relatively simple for efficiency purposes, but also realistic to produce applicable results. According to the Norwegian bank DNB, the interest rate swap derivative outlined in section 4, consisting of annual swaplets with cash-flows according to equation (4.14) is very common amoung their customers [1]. A bank often enters in to a short position in this contract, or in the position as the receiver of the fixed rate, to help corporations hedge out their interest rate risk associated with floating interest rate loans. In the contract, the bank is exposed to interest rate fluctuations. The contract value and the counterparty credit exposure increases if interest rates are falling unexpectedly, and decreases if rates are rising unexpectedly. It is assumed that the notional contract value is below 3 billion NOK, and that the counterparty is not a financial insitution. The contract is therefore not concerned with any of the regulatory requirements described in chapter 7, and it is also assumed to be no collateral or netting between counterparties in the further analysis.

Maturities on the swaps usually ranges from 3 to 10 years, sometimes even longer [1]. The contract maturities in the backtests are set to the lower end of this range, at 5 years. The strike rate, or the fixed rate on the contract is set so that the contract value at initiation is roughly equal to zero. This is done by a simple optimisation procedure, were the strike price is set so that the price of the swap is zero at roughly 3 significant digits at $t = 0$. The notional values of the contracts are normalised to 1 for simplicity.

8.2 Backtesting Procedure

The implemented framework for simulating counterparty credit exposure will be backtested two different ways. The first test, called a *risk-factor backtest*, is designed to test the underlying interest rates generated by the real-world implemented HJM model. The test compares the underlying risk factors generated by the model, in particular the short interest rate, with the realised interest rate over the same historical periods. The simulated distribution of interest rates can then be compared with the realised interest rate by looking at the difference between volatility and expected paths. The time horizon for the individual risk-factor tests are set equivalent to the maturity of the swaps, at 5 years.

The second and most important backtest is the *portfolio backtest*. This test simulates the exposure of a typical derivative portfolio, in this thesis the swap described in the previous section, over historical periods. The same portfolio is then evaluated, using the realised risk factors over the same periods to calculate *actual exposures*. The procedure of calculating actual exposure is equivalent to the method used for generating the theoretical exposures outlined in chapter 7, except that the historical realised interest rates are used instead of the interest rates generated by the real-world algorithm. The risk-neutral model is in both cases used to price the portfolio at every time step. A comparison between the theoretical simulated potential future exposure (PFE) and the expected positive exposure (EPE) with the actual exposure (AE) over the same periods will then be carried out.

All tests will be done out-of-sample, meaning that the historical volatility parameters are estimated over a period of time leading up to the start of every simulation. As will be justified in the data analysis in the next section, the length of each estimation period is set to 2 years. This length is chosen to avoid estimating short term noise while simultaneously trying to capture long term trends in the behaviour of the volatility. The tests will overlap by running a new simulation from every 2 years. The reason for this is that the typical potential exposure is largest, and therefore most interesting to study after approximately 2 years in the 5-years swap contracts. Even if there are some overlap between periods, the 2-year exposures will be independent, and running tests every 2 years will therefore maximise the number of independent samples of the 2-year exposures.

Chapter 9

Data Analysis and Parameter Estimation

9.1 Preliminary Data Analysis

The data-set used in this thesis is provided by DNB and consists of swap rates of differing maturities. The relevant rates are the 3-month rate used as the risk-free short rate, and other swap rates with maturities of up to 6 years which is the time horizon for the backtests in this thesis. The 6-year swap rate is needed in addition to the 5-year swap rate to calculate the 5-year forward swap rate according to (4.9). The data contains interest rates from 1.1.1995 until 1.1.2018, or 23 years worth of data. Table 9.1 shows descriptive statistics for the whole data-set. The standard deviation or volatility is calculated as the standard deviation of the absolute increments in daily interest rates, and measured in percentage points.

Table 9.1: Table of initial spot interest rate data.

Maturity	Mean yield (%)	Volatility (%-points)
3M/short rate	3.55	0.0590
1Y	3.78	0.0559
2Y	4.05	0.0535
3Y	4.19	0.0517
4Y	4.32	0.0515
5Y	4.44	0.0511
6Y	4.56	0.0469

Looking at the average yield for different maturities it can be seen that the interest rate

increases slightly with maturity, from 3.55% to 4.56% yield. This is consistent with the theory, predicting a negative market price of risk as discussed in chapter 4. The standard deviation of the daily changes is slightly decreasing as a function of maturity ranging from around 0.59 to 0.47 percentage points. This indicates that longer interest rates are on average slightly less volatile than shorter rates.

Continuing the initial data analysis, the short rate is investigated further. It should be noted that the short rate and the forward rates are highly correlated, so similar conclusions can be drawn for all of the interest rates. Figure 9.1 shows the historical weekly short rate between $1.1.1995$ and $1.1.2018$. Figure 9.2 shows the historical 30-day volatility of the short rate, meaning the standard deviation is measured over the previous 30 days, which is an often used metric for the current volatility. Figure 9.3 shows the historical 2-year volatility of the short rate, which is naturally a smoother curve and more useful for the time horizon used in this thesis. Both measures of volatility are annualized by scaling the daily volatility by a factor of $\sqrt{252}$, were 252 is the average number of trading days per year.

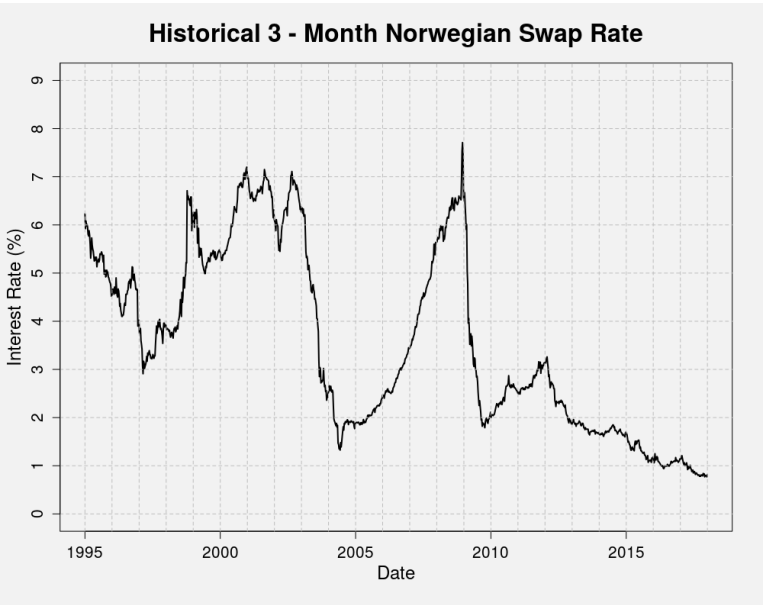


Figure 9.1: Historical Norwegian 3m swap rate.

As can be seen from figure 9.1, the interest rate varies greatly with no apparent mean-reverting level. Rates were generally higher before 2003, with interest rates of magnitudes

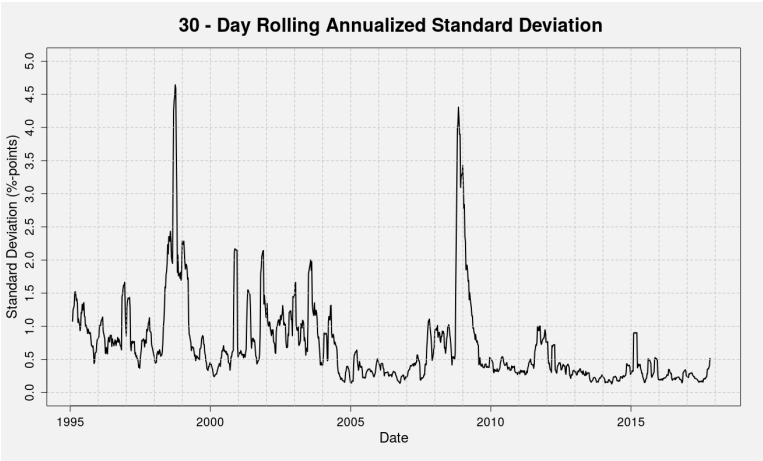


Figure 9.2: Historical 30-day volatility of Norwegian 3m swap rate.

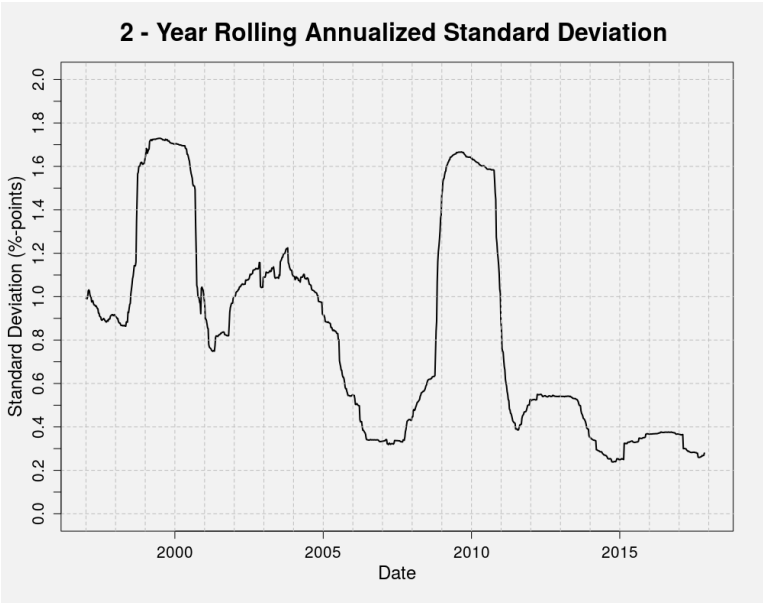


Figure 9.3: Historical 2-year volatility of Norwegian 3m swap rate.

around 5% to 7% being the norm. After a sharp decline in 2003 were rates dropped below 2%, interest rates climbed during the next few years before another sharp drop in the aftermath of the great financial crisis in 2008 - 2009. In the later years there have been a steady trend of lower interest rates, with rates even below 1%.

By looking at figure 9.2, two large spikes in volatility are observed. In both cases did

the volatility increase almost 10-fold over a short period of time. The first spike corresponds to the emerging markets and Russian debt crisis at the end of the 1990's, while the second spike corresponds to the financial crisis and its aftermath in $2008 - 2010$. The period $2001 - 2003$, following the bursting of the dot.com bubble in the U.S is also a notable period with a generally elevated level of volatility. These periods with high volatility is often called *stressed market conditions* and are of special importance. This is because a financial model is most likely to break down during such stressed conditions, and testing a model against historically volatile periods is therefore an important step in any model testing procedure.

Figure 9.3 shows the same patterns as figure 9.2. It should be noted that even when the volatility is smoothed over a 2 -year period, it still varies greatly. The highly variable nature of the volatility will have a large impact on the results when modelling using historical volatility as input. As an example of the variability of the volatility, the annualised 2-year volatility has since 2014 consistently been under 0.4%-points. This is less than a fourth of the volatility during both the emerging markets and Russian crisis, and the financial crisis on a 2-year rolling basis. The volatility seems to exhibit both short term mean reversion, and some longer term persistent trends. The choice of a 2-year horizon when estimating volatility as described in section 8.2 therefore seems justified.

9.2 Transforming Data and Discretization

As outlined in chapter 5, the HJM framework simulates instantaneous forward rates, not spot rates. The data therefore have to be transformed. This is done by applying the whole data set to equation (5.10). Table C.1 in appendix C lists the transformed instantaneous forward rate curves at the start of each year in the data set. The data in the table are both used as initial conditions in the different simulations, and in the calculations of the actual exposures in the different periods. Different types of forward curves representing different market conditions are observed through the period. A typical upward sloping curve in $1.1.1995$ indicated a fast growing economy and rising future short rates. A flat forward curve in $1.1.2003$ curve indicated that the market expected rates to stay at the current level because of a relative stable economy. The downward sloping or inverted curve in $1.1.2009$ meant market participants expected rates to fall further as the central bank was expected to cut interest rates, before rates would increase when the growth would pick up after the recession [19].

The time-steps chosen in the implementation of the algorithm is $\Delta t = 1$, implying annual steps. The relatively large time-steps is taken both for efficiency reasons, and because the derivatives priced and time intervals relevant for this thesis is of the same magnitude.

9.3 Volatility Estimation

The volatility is estimated following the principal component analysis procedure from the last section in chapter 6 on the transformed data. After using the whole data set to calculate

the initial principal components, the eigenvalues are used to calculate explained variability as in equation (6.2). The results are shown in table 9.2.

Table 9.2: Table of eigenvalues and explained variability.

Eigenvalue	Value	Explained cumulative variability
λ_1	2.25e-06	0.383
λ_2	1.39e-06	0.620
λ_3	8.65e-07	0.768
λ_4	5.94e-07	0.868
λ_5	4.74e-07	0.949
λ_6	3.03e-07	1.000

Although the forward rates are correlated, there are only 6 different variables to consider. To include over 95% of the variability in the data set all the component will have to be used. The HJM model implemented will therefore be implemented as a 6-factor model.

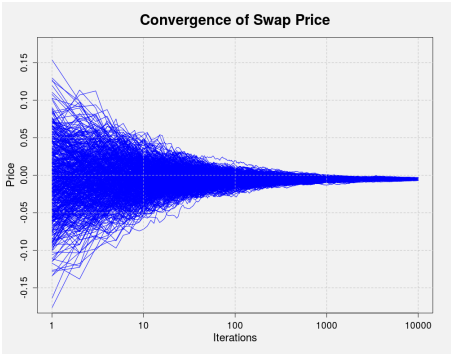
Having decided to use all the components, the principal components or parameters to be used in the individual backtesting periods are calculated. To estimate the historical 2-year volatility leading up to every testing period, the data set is divided into 2 year periods from $1.1.1995$ - $1.1.1997$, to $1.1.2011$ and $1.1.2013$. The resulting 9 sets of volatility estimates is then used as input in the algorithm for the different testing periods.

9.4 Convergence of Algorithm

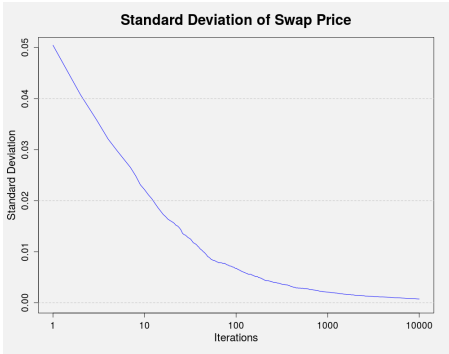
To get satisfying results for the exposure distributions, one wishes to simulate as many scenarios of the underlying risk factors as possible, while pricing each scenario as precisely as possible. The challenge is that both scenario generation and pricing is based on simulation, and the run-time of the algorithm therefore quickly grows as a function of both scenarios and pricing accuracy. Before conducting the backtests it is therefore essential to find out how many iterations the pricing algorithm needs to run to price the interest rate derivatives with satisfying accuracy. This is done by simply running the pricing algorithm on a simple swap, while plotting the price as a function of iterations to see were the algorithm converges. The swap is initiated at $1.1.2018$, has a maturity of 5 years, the swap rate is 0.017, and the volatility is estimated using the whole data set.

The results for 500 independent runs are shown in figure 9.4a, while figure 9.4b shows the standard deviation for the same 500 samples, both as function of the number of iterations. The convergence of the Monte Carlo method is as described in the theory of order $\mathcal{O}(n^{-\frac{1}{2}})$. The standard deviation in figure 9.4b is therefore decreasing by a factor of $\sqrt{2}$ for every doubling of the number of iterations. Seeing from figure 9.4a that the algorithm in fact converges towards a price, a tolerance for the error needs to be set to determine how

many iterations to run. From some trial and error, the normalised swap for different strike prices is priced roughly in 10^{-1} order of magnitude. The tolerance is then set so that the simulated standard deviation S must be smaller than 10^{-3}. This criteria is satisfied after around 5000 iterations, with $S = 0.994 * 10^{-3}$ for the simulated sample. 5000 is therefore chosen as the number of iterations in the risk-neutral algorithm when pricing the swaps.



(a) Simulated price of 500 5-year swap prices as a function of number of iterations in log scale.



(b) Simulated standard deviation of 500 5-years swap prices as a function of number of iterations in log scale.

Figure 9.4: Convergence and standard deviation of swap pricing algorithm.

Chapter 10

Analysis

10.1 Results

After estimating parameters and implementing the methods and models outlined in the previous chapters, the risk-factor backtest and the portfolio backtest are completed for all 9 overlapping periods. The risk-factor backtests are generated by simulating 100000 scenarios of the 3-month swap rate, while the portfolio backtests are generated by 5000 scenarios, each prised 5000 times.

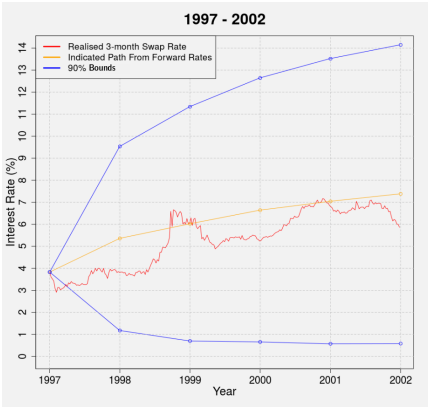
Descriptive tables and figures for all periods and for both backtests can be found in appendix C. Appendix C.2 describes complete results for the risk-factor backtest with table C.2 showing descriptive statistics and figure C.1 and C.2 showing plots of the test for all periods. The simulated 5 and 95-percentiles are used to show the volatility of the simulated short interest rates in the risk-factor test. These percentiles will be called 90% bounds in the further analysis, and are calculated using the order statistics from definition 2.2.1.

Appendix C.3 outlines the portfolio backtests, with table C.3 showing descriptive statistics of the simulated expected positive exposure (EPE) and potential future exposure (PFE), and the calulated actual exposure (AE). Figures C.3 and C.4 shows the exposures over the whole periods, while figures C.5 and C.6 shows the distributions of exposures at year 2. A normal density line is also plotted over the portfolio backtests to highlight the shape of the distributed exposures more clearly.

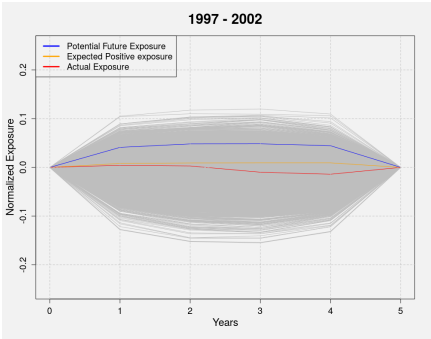
The main focus in the analysis will be on 4 key periods in the data set. All the chosen periods includes either stressed market conditions or other interesting features. The chosen periods are presented in chronological order, starting with the first period in the data set, which is from 1.1.1997 to 1.1.2002.

1997-2002

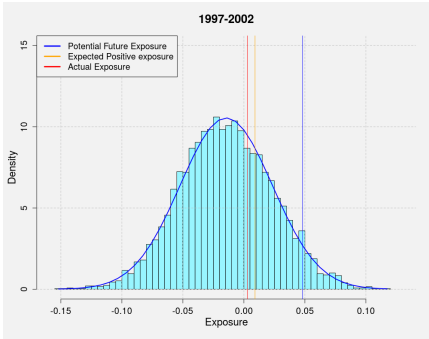
The first backtesting period chosen to present is the 1997 to 2002 period, with figure 10.1 showing the relevant results for both the risk-factor backtest and the portfolio backtest. The period is characterised by increasing interest rates and high volatility, especially in late 1998 to 1999 due to the emerging markets and Russiand debt crisis.



(a) Risk-factor backtest with expected path, simulated 90% bounds, and realised short rate.



(b) Portfolio backtest, exposure for all years with 95% PFE, EPE and AE.

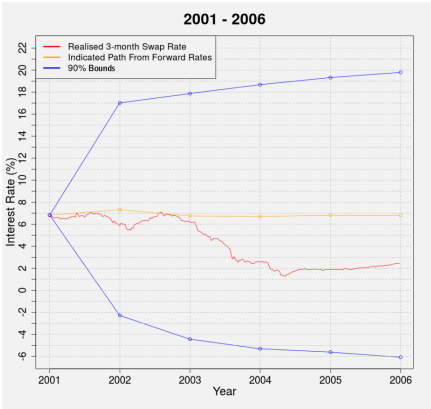


(c) Portfolio backtest, exposure at $t = 2$ with 95% PFE, EPE and AE.

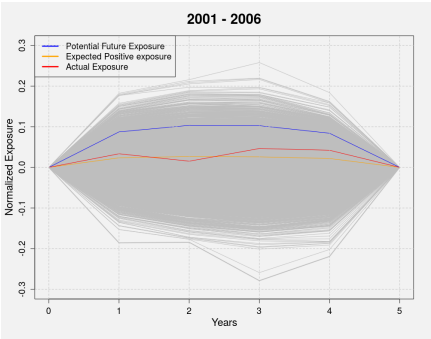
Figure 10.1: Backtests of the 1997 - 2002 period.

Starting with the risk-factor backtest in figure 10.1a, the realised short interest rate looks to be following its expected path indicated by the rising forward interest rate curve at initialisation. Except for the first month or so, the realised interest rates are well within the 90% bounds of the simulated distribution. The bounds predicts a fairly volatile period. with a 90% probability of short interest rates between 1% and 14% at the end of the simulation in 2002.

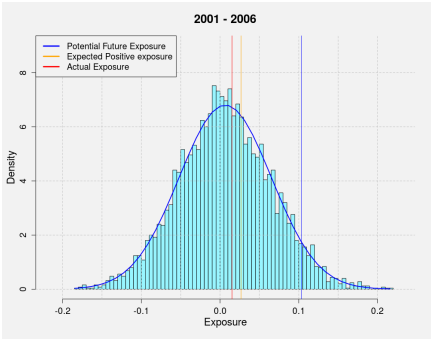
From the portfolio backtest in figure 10.1b, it is observed that the AE is close to zero during the whole period. The AE is therefore lower than the EPE, and also much lower than the 95% PFE for all years. Looking more closely at year 2, shown in figure 10.1c, it is seen that the EPE is around 0.01. This means an exposure of 1% of notional contract value 2 years after contract initiation. The 95% PFE is slightly below 0.05, predicting that in the worst 5% of the cases, the exposure is at least 5% of the notional contract value. The AE at year 2 is only 0.3%, indicating an insignificant exposure at that time.



(a) Risk-factor backtest with expected path, simulated 90% bounds, and realised short rate.



(b) Portfolio backtest, exposure for all years with 95% PFE, EPE and AE.



(c) Portfolio backtest, exposure at $t = 2$ with 95% PFE, EPE and AE.

Figure 10.2: Backtests of the 2001 - 2006 period.

2001-2006

Figure 10.2 shows the results from the risk-factor and portfolio backtest for the period 2001 - 2006. The period contain the lead up to, and the bursting of the dot.com bubble.

This resulted in falling interest rates and a generally elevated level of volatility during the period.

Again starting by considering the risk-factor test in figure 10.2a, the model predicts a highly volatile period for interest rates. The 90% bounds predict short interest rates either above 19% or below -6% with a 10% probability at the end of 2005. The forward rate curve at the start of the period indicates a more or less flat future path of the short rate, diverging from the realised path two years into the simulation when the realised short rates fell roughly 5%-points from 7% to just 2%. The realised interest rate are still well within the 90% bounds of the model for all times.

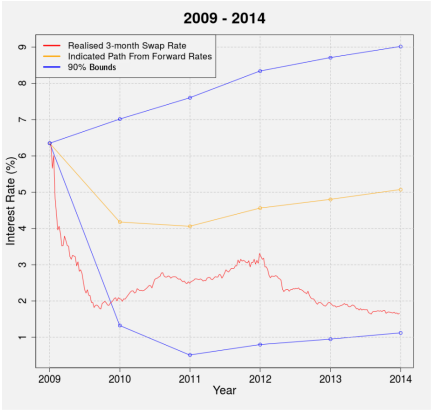
The full exposure profile for the portfolio backtest in figure 10.2b shows that the AE is positive and of the same magnitude as the EPE for the first 2 years, while rising in years 3 and 4 as a result of falling interest rates. The AE is still well below the 95% PFE for the whole period. Taking a closer look at year 2 in figure 10.2c, the model predicts an EPE of 0.027 and a PFE of 0.10. This indicates an EPE of under 3% with a 5% chance for an exposure of 10% or more of the notional value of the contract at year 2. The AE at the same time was only 0.015, or around 1.5% of notional contract value.

2009-2014

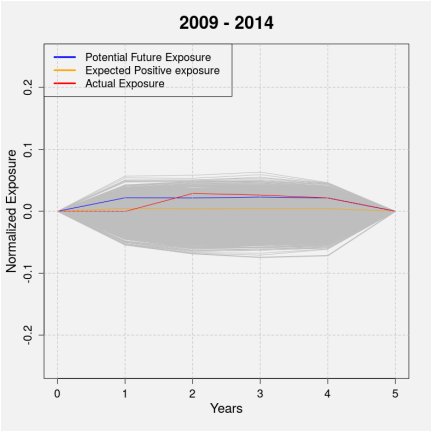
The next period of interest is the period between 2009 and 2014, with figure 10.3 showing the results. This period contains the end and aftermath of the financial crisis, where interest rates fell significantly at the start of the period. Volatility was also initially high in 2009, but decreased as markets stabilised after the crisis.

Figure 10.3a shows that the model assumes this period to be much less volatile than previous ones. Starting in 2009, the model predicts short interest rates in 2014 to be between 1% and 9% with 90% probability. The forward curve indicates at the start of the period that interest rates will fall a couple of percentage points within a year, before starting to drift slowly upwards again. The realised short interest rates over the same period dropped much more than predicted, dropping below the 90% bounds for a short period, and going from 6% to under 2% in only 9 months. In the last 4 years of the period, realised rates have been drifting lower and ending the period well below 2% which is very close to the lower 90% bound.

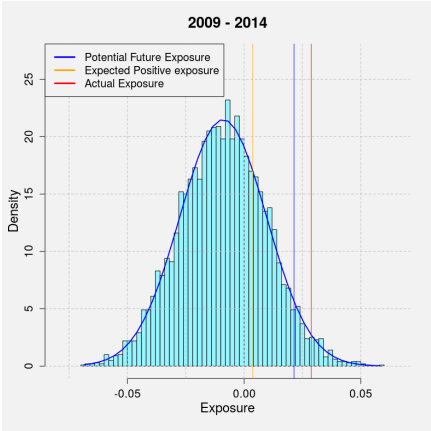
The portfolio backtest in figure 10.3b shows that with the exception of year 1, the AE is equivalent to or higher than the 95% PFE and of course also much higher than the EPE. The high AE is a result from several unexpected negative shifts in the whole forward curve during this period. Figure 10.3c shows that at year 2, the EPE is predicted to be 0.005 and the 95% PFE is 0.0273. The AE for year 2 of this period is 0.0285. This means that the AE is 2.85% of notional contract value, which is slightly larger than the predicted 95% PFE at 2.73%.



(a) Risk-factor backtest with expected path, simulated 90% bounds, and realised short rate.



(b) Portfolio backtest, exposure for all years with 95% PFE, EPE and AE.

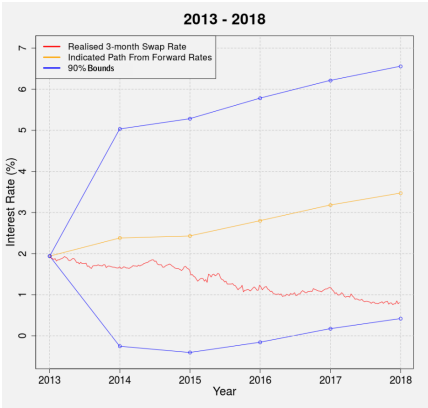


(c) Portfolio backtest, exposure at $t = 2$ with 95% PFE, EPE and AE.

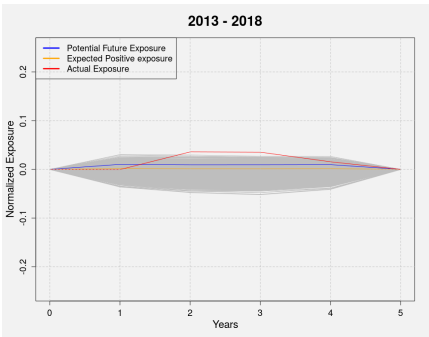
Figure 10.3: Backtests of the 2009 - 2014 period.

2013-2018

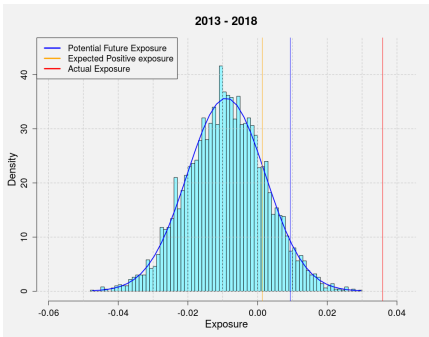
The final period studied in detail is the most recent period, starting in 2013 and ending at the start of 2018. Figure 10.4 outlines the results. The period is characterised with both extremely low levels of volatility and interest rates.



(a) Risk-factor backtest with expected path, simulated 90% bounds, and realised short rate.



(b) Portfolio backtest, exposure for all years with 95% PFE, EPE and AE.



(c) Portfolio backtest, exposure at $t = 2$ with 95% PFE, EPE and AE.

Figure 10.4: Backtests of the 2013 - 2018 period.

Based on the simulated risk-factors in figure 10.4a, the model predicts even lower volatility in interest rates than in the 2009 - 2014 period. At initialisation, the 90% bounds predicts short interest rates between 0.5% and 6.5% 5 years later. The interest rate forward curve indicates that interest rates are expected to rise through the whole period from their initial low level of below 2%. In reality, interest rates continued to drift steadily lower, ending the period with rates below 1%, and very close to the lower 90% bound.

Figure 10.4b shows that the AE is within the 95% PFE in year 1, but much bigger in the other years. A continually lower drifting forward curve during the whole period makes the AE large compared to the model predictions. Again, looking especially at year 2 in the distribution in figure 10.4c, the EPE is 0.001 and the 95% PFE is 0.009. The AE is much larger at 0.0359, indicating an exposure around 3.6% of notional contract value. Calculating the mean of the distribution at year 2 in figure 10.4c to -0.01, and the standard deviation to 0.01, the AE is according to the model a 4.6 standard deviation event. Such an event has roughly a 1-in-500000 chance of happening given that the AE is from the simulated distribution of exposures at year 2.

Summary

Before moving on to a more general discussion regarding the results, a short summary of the main results for all periods is provided. Table C.2 and figures C.1 and C.2 in appendix C.2 shows the results from the risk-factor backtest. The realised short interest rates are withing the simulated 90% bounds at every discrete, annual time step of the model in all periods. Another common denominator between all periods is that the realised short rate is below the expected path assumed by forward rates at the end of each period. By taking the 5-year instantaneous forward rate at initiation and subtracting the realised short rate 5 years later, the average difference is found to be 0.0263. The realised short interest rates 5 years after initialisation is therefore approximately 2.6% lower on average than predicted by the model.

The portfolio backtest, with results in table C.3 and figures C.3, C.4, C.5 and C.6 in appendix C.3 shows that 4 out of 9 periods has AE which is higher than the 95% PFE estimated by the model at least one time during the 5 years in each testing period. These periods are the 2003 - 2008 and 2011 - 2016 periods, in addition to the 2009 - 2014, and 2013 - 2018 periods already discussed. The same 4 out of 9 periods has an AE higher than the 95% PFE in year 2, and the 2003 - 2008 period has the highest measured AE with over 8% of notional contract value for year 2 and 3.

10.2 Discussion

After presenting the main results in the previous section, it is now time to discuss the model with respect to its central assumptions and practical use. Possible adjustments and improvements of the model will also be considered.

As seen from the results, in particular for the portfolio backtests, there are in some cases a big difference between model output and actual historical data. The model tends to underestimate future exposure as highlighted by the 4.6 standard deviation event of the portfolio backtest in the second year of the 2013 - 2018 period, and the fact that 4 out of 9 periods have larger AE than the 95% PFE.

It is still important to emphasize that underestimated exposure to a counterparty in itself do not automatically lead to losses. Losses are caused by the combination of *hedged* exposure and default by the counterparty. Remember that positive exposure on a contract is actually a gain seen from the banks viewpoint. This gain only turns into a loss if the counterparty defaults, and the bank has hedged this exposure with a now losing position to another counterparty. Underestimating exposures in periods of elevated default risks therefore increases the probability of large losses. It is therefore interesting to return and take a deeper look at the result for the 2013 to 2018 period seen in figure 10.4. In addition to the model underestimating the exposure, this period also contain some highly interesting macroeconomic events worth taking a closer look at.

At the start of 2013, the model predicted low volatility of interest rates. The swaps contracted in 2013 were therefore assumed to have little risk in the form of low future exposures compared with earlier periods. A typical Norwegian bank is and was heavily exposed to the oil and oil-service sectors. It is therefore reasonable to assume that around 2013, Norwegian banks had swap contracts with many corporations in this sector. The following collapse in oil prices from 115 USD per barrel in the summer of 2014 to below 30 USD in early 2016 led to defaults on both loans and derivative contracts in the mentioned industry sectors [1]. These defaults came at the worst possible time for the swaps contracted in 2013. In 2015 and 2016, corresponding to year 2 and 3 in the contracts, the realised actual exposure had risen to $3 - 4$ times the simulated 95% PFE on these swaps. This combination of much bigger than anticipated exposures and increased default rates would have led to large losses in the Norwegian banking sector if the banks had used the model implemented in this thesis.

In an attempt to compare the results from this thesis with relevant sources, the Norwegian bank DNB was consulted [1]. DNB provided some useful qualitative information of their internal market model, and how it performed in estimating exposure during the interesting 2013 to 2018 period. Their main takeaway was that the exposures in the 2013 - 2018 period discussed above were much higher than they had expected, resulting from rising forward rates and falling realised rates. The model used by the bank still seemed to performed better than the model implemented in this thesis, and were more successful in capturing the relative high exposure in this period. The reason for this, and one of the key differences between the two models is that the bank's model contains an estimated market

price of risk. This means that the bank's model estimates lower future expected interest rates than the model in this thesis. The true losses due to defaults on derivative contracts in this time period is therefore likely to have been lower for the bank than the model in this thesis would indicate.

After discussing the consequences of using the implemented model in practise, it is clear that some improvements are needed. The focus in the further analysis will be on the drift, the volatility and the resulting distributions produced by the model. Understanding how they affect the results is important to adjust and improve the model.

Distribution

Both the simulated risk-factors and the exposure distributions generated by the HJM model are normally distributed [14]. Most empirical studies contradicts this and shows that most asset and financial instruments have fat-tailed distributions, a concept discussed at length by Taleb [24]. One therefore has to both expect and accept that unlikely events with large consequences happen more often then predicted from a thin-tailed distribution like the normal distribution. This certainly seems to be the case for the actual exposures calculated in this thesis.

Normally distributed risk-factors also allows for the possibility of interest rates to go negative. This property of the HJM model used to be considered a drawback because general macroeconomic theory suggest that interest rates cannot fall below zero because of the liquidity trap [22]. Negative observed interest rates on for example short dated European and Japanese government debt in later years have contradicted this theory and given the normal distributed models more validity. Accepting that negative rates can occur, it is still likely that the model is exaggerating both the probability and the magnitude of the negative rates. A good example of this is the 2001-2006 period discussed in the analysis in figure 10.2a. The simulated distribution indicates that in 2001, there were a roughly 5% chance of the short interest rate being below -6% in 2006. The simulated periods starting in 1999, 2003 and 2005 also predicts negative interest rates of magnitudes around -1% to -3% percent with a probability of around 5%. Comparing with realised rates, the lowest observed short rate occurred in December of 2017, and was just below 1%. This supports the belief that the left tails of the model distribution of risk-factors do not accurately reflect reality.

A possible solution to the problem of negative interest rates of large magnitudes is to implement a floor. By simply removing all simulated paths lower than a given threshold, more realistic results can be obtained. By implementing a floor, the resulting simulated exposures will be lower because negative interest rates contributes positively to the exposure calculations. The effect of the floor is assumed to be small, particularly for short maturities, but will naturally depend on the level of the implemented floor. Setting the exact level is difficult, and must be done by a combination of common sense and empirical investigation. A floor set around -1% to -3% seems like a reasonable level, but the exact floor for the risk-free short interest rate is of course impossible to know [1].

Drift

Discussing the model drift, one should be careful to distinguish between the risk-neutral and real-world counterparts. The risk-neutral drift used to price the derivatives is of great interest from a theoretical standpoint, but is not as important when discussing the practical results. Since no comparison between theoretical and market prices are done, one just has to assume the pricing is done consistent by the model.

Considering the real-world implementation, an important simplification is setting the drift to zero. The implication of zero drift is that the average of future simulated short rates are equal to the equivalent forward rates at initialisation. This is done because of the complexity and uncertainty in estimating the market price of risk, especially with a multi-factor model. The market price of risk is generally considered to be negative, which coincides well with the results. The model overestimates future short interest rates by an average of 2.6%-points during the 5-year simulations of the risk-factors. This supports the well established theory of a liquidity premium in the interest rates market, and is definitely a contributing factor to the realised exposures generally being more positively skewed than predicted by the model.

Discussing the apparent large difference of 2.61%-points between average simulated and observed interest rates over the 5 year periods, the size and specific content of the data set needs to be considered. The data set contains two economic shocks or recessions, both the dot.com bust in the early 2000's and the financial crisis in 2008 - 2009. Interest rates fell in both cases over 5%-points within a year or two, and in none of the instances did the forward rate curve predict the fall in advance. The amount and size of these unexpected drops in rates have big impact on the size of the difference between simulated and observed interest rates. Another large contributor to the divergence between average observed and simulated rates are from the periods from 2009 until present day seen in in figure C.2 in appendix C.3. Even though forward rates during the last 9 years have continued to slope upwards, realised rates have continued to drift lower to historical unprecedented levels.

The data set used in this thesis is relatively small, containing only 24 years of market data. Considering the specific, and not necessary representative events in the data set, it can therefore be argued that by implementing a real-world drift adjustment one is equally likely to overstate the future market price of risk as understating it. Although the models used in the industry have included the market price of risk estimated from historical data, one should be careful in adopting this approach because the exact model and method of estimation is unknown [1]. With today's unusual low interest rates, implementing a model with a large negative market price of risk as indicated from past data will make the average simulated future short rates drift well into negative territory. The market price of risk estimates which produced the most accurate results in the past will therefore not necessary produce accurate results in the future.

Volatility

As for the real-world drift, drawing definite conclusions regarding the volatility in the model is difficult because of the limited sample size. As one would expect when using historical volatility, the model is better at capturing the variability in risk-factors and exposures in periods were the realised volatility is either stable or falling. This is typical for the 5 earliest periods in the data set, starting from 1997 and until the period starting in 2005. The only exception amongst these periods where the actual exposures are bigger than predicted, are in the 2003 - 2008 period where interest rates collapsed and volatility exploded. The actual exposure after 2 years rose to over 8% in this period, which is highest in all the portfolio backtests. Based on the discussion of empirical heavy tails in the distributions and the non-predictability of the volatility, one should not expect the model to capture such extreme movements.

More alarming is therefore the tendency of the model to underestimate the exposure when volatility is expected to be low, as is the case for the 3 last periods in the exposure tests. One could initially think that this is caused by low volatility in the 2 year tuning periods before initialisation. If the volatility then suddenly rises through the periods, the model could potentially underestimate the exposure. By the plots of the volatility in figure 9.2 and 9.3, and the persistent underestimation of exposure in subsequent periods in figure C.4, this seems an unlikely explanation. There should not be any apparent reason for the model to continually underestimate the exposure when volatility is both constant and low. The reason for poor model performance in these periods therefore have to be put on elevated market price of risk and persistent trends, not on the volatility.

It is difficult to say with certainty what causes the wrongly simulated exposures, be it volatility, market price of risk, trends, or fat tailed distributions. The easiest approach to model improvement is still often to adjust the real-world implemented volatility. By simply scaling up the volatility, more of the extreme measurements of exposures are covered by the model, no matter the initial causes of it. Although this is an artificial solution, it will prevent the bank from being over-exposed in stressed market conditions. The downside to scaling up the volatility is that the exposure on average will be overestimated, and that the bank ends up taking to little risk and do not make as much money in times of normal market conditions. This is the eternal trade-off between minimising risk and the probability of going bankrupt, and the objective of maximising profit.

Chapter 11

Conclusion

11.1 Concluding Remarks

In this thesis a framework for backtesting counterparty credit exposure is developed and implemented. Using the Heath, Jarrow and Morton model for simulation of interest rates, separate models are implemented for risk-neutral pricing of interest rate derivatives, and for simulation of future real-world interest rates. The models are combined to simulate distributions of future credit exposures for a simple swap contract between a financial institution and a typical counterparty.

The results obtained in this thesis shows a substantial difference between model output and actual historical data. This is partly as expected, because the model relies on some highly simplified assumptions including a zero market price of risk and constant volatility. The actual historical exposures, in particular from recent time periods, are much bigger than what the model predicts. The combination of underestimated exposures in periods of elevated default risk is shown to have potentially large consequences. The model in its current form is therefore not advised to be used uncritically for future counterparty exposure calculations before improvements are done.

Regarding potential improvements of the model, several are discussed. Implementing an interest rate floor, adjusting the drift to account for the market price of risk and scaling up the volatility are all common industry practises [1]. Based on the the analysis of the results, drift adjustments and scaling volatility will certainly improve the accuracy of the backtests.

It is worth noting that by making changes to the model parameters based on the results, the testing goes from out-of-sample to in-sample. Also taking into account the small amount of data tested, only 24 years and 9 correlated periods, the risk of data-mining leading to an over-fitted model is therefore substantial. All adjustments of the model will therefore have to be done with caution, realising that the past may not be an accurate reflection of the future.

With this in mind, one will seek to improve the model in a way that minimise its sensitivity to future changes and surprises. The most robust improvement, and the improvement recommended in this thesis is therefore to just scale up the volatility. This is because the potential harm caused by wrongfully scaling up the volatility, or not scaling it up enough, is much smaller than the potential consequences of a large wrongly estimated market price of risk.

11.2 Further work

As improvements to the model are discussed, a naturally theme for further work is to actually implement the discussed improvements. Another interesting topic of further work would be to investigate more in depth the relationship between credit exposure, probability of default and loss given default, to be able to see the results in a more realistic context.

A comparison between other models would also be interesting. Running two different models with the same input and parameters would help gain insight into the difference between specific model problems for HJM, and more general modelling problems. The LIBOR market model described in [14] would be a good candidate for such a comparison.

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Appendix A

Probability and Stochastic Processes

The following concepts and definitions from Bingham & Kiesel in [3] are used in the derivation of the derivative pricing models presented in chapter 3.

A.1 Measure

Definition A.1.1 (Algebra)**.** *A collection A_0 of subsets of a set Ω is called an algebra on Ω if:*

i. $\Omega \in A_0$,

ii. $A \in A_0 \Rightarrow A_c = \Omega \setminus A \in A_0$,

iii. $A, B \in A_0 \Rightarrow A \cup B \in A_0$

Definition A.1.2 (σ -Algebra)**.** *An algebra A of subsets of Ω is called a σ - algebra on Ω if for any sequence $A_n \in A$, ($n \in \mathbb{N}$), we have*

$$\bigcup_{n=1}^{\infty} A_n \in A.$$

Such a pair (Ω, A) is called a measurable space.

Definition A.1.3. *Let ($\Omega, \mathcal{A}$) be a measurable space. A countable additive map*

$$\mu : \mathcal{A} \Rightarrow [0, \infty]$$

is called a measure on ($\Omega, \mathcal{A}$). The triple ($\Omega, \mathcal{A}, \mu$) is called a measure space.

Definition A.1.4. *A measure $\mathbb{P}$ on a measurable space (Ω, A) is called a probability measure if*

$$\mathbb{P}(\Omega) = 1$$

The triple$(\Omega, A, \mathbb{P})$ is called a probability space.

Definition A.1.5. *Given two different measures, $\mathbb{P}$ and $\mathbb{Q}$, defined on the same σ-algebra $\mathcal{F}$, we say that $\mathbb{P}$ is absolutely continuous with respect to $\mathbb{Q}$, written $\mathbb{P} << \mathbb{Q}$ if $\mathbb{P}(A) = 0$, whenever $\mathbb{Q}(A) = 0$, $A \in \mathcal{F}$. If $\mathbb{P} << \mathbb{Q}$ and $\mathbb{P} >> \mathbb{Q}$, we call $\mathbb{P}$ and $\mathbb{Q}$ equivalent measures.*

A.2 Probability

To describe a random experiment mathematically, a sample space, which is the set of all possible outcomes in Ω is defined. Each point, say $\omega \in \Omega$ is then a sample point, and represents a possible and random outcome of the experiment. For a subset $A \subseteq \Omega$, of some points ω, the probabilities $\mathbb{P}(A)$ is desired, and the following properties are defined.

1. $\mathbb{P}(\emptyset) = 0, \mathbb{P}(\Omega) = 1$,
2. $\mathbb{P}(A \geq 0)$ for all A,
3. If A_1, A_2,...,A_∞ are disjoint, $\mathbb{P}(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$ (Countable additivity).
4. If $B \subseteq A$ and $\mathbb{P}(A) = 0$, then $\mathbb{P}(B) = 0$ (completeness)

Definition A.2.1. *A probability space. also called a Kolmogorov triple is a triple $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathcal{F}$ being a σ-algebra and $\mathbb{P}$ defined on $\mathcal{F}$ according to definition A.1.4, satisfying the Kolmogorov axioms $1, 2, 3, 4$ above.*

Having defined a probability space for the random experiment, it is possible to quantify outcomes ω. Defining the real-valued function X on Ω as $X : \Omega \to \mathbb{R}$. If such a function is measurable it is called a random variable. The following properties of the random variable X are then defined.

Definition A.2.2. *The Expectation $\mathbb{E}$ of a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ is defined by*

$$\mathbb{E}X := \int_{\Omega} X d\mathbb{P},$$

while the variance is defined as

$$\mathbb{V}ar(X) := \mathbb{E}\big[(X - \mathbb{E}(X))^2\big] = \mathbb{E}(X)^2 - (\mathbb{E}X)^2.$$

A.3 Information and Filtrations

The flow of information is an important concept in finance. As time passes, new information becomes available and financial agents updates their beliefs and portfolios according to this new information. A framework for modelling such dynamic situations is therefore

needed, and is given by the idea of filtration.

Modelling a situation involving randomness over time, it is assumed that information is never lost. The information arrives in integer steps $t = 0, 1, 2, ...$, either to a final maturity T, or with an infinite time horizon. As time progresses, more information is accumulated. From definition A.1.2, σ-algebras can be represented as knowledge or information, and a sequence of σ-algebras is given by $\mathbb{F} = \{\mathcal{F}_n : n = 0, 1, 2, ...\}$. With increasing amount of information, the sequence $\mathcal{F}_n \subset \mathcal{F}_{n+1} \quad (n = 0, 1, 2, ...)$ models the flow of information with $\mathcal{F}_n$ representing the information available at time n. The family $\mathbb{F} = \{\mathcal{F}_n : n = 0, 1, 2, ...\}$ is called a *filtration*, and a probability space endowed with a filtration $\{\Sigma, \mathbb{F}, \mathcal{F}, P\}$ is known as a stochastic basis or a *filtered* probability space.

A stochastic process is a family of stochastic variables on a common probability space, defined as $X = \{X_n : n \in I\}$, were I is representing time. The process $X = (X_n)_{n=0}^{\infty}$ is defined to be *adopted* to the filtration $\mathbb{F} = (\mathcal{F}_n)_{n=0}^{\infty}$ if

$$X_n \text{ is } \mathcal{F}_n \text{ - measurable for all } n.$$

This means that if X is adapted, the knowledge of the value X_n is available at time n. In addition, if

$$\mathcal{F}_n = \sigma\big(X_0, X_1, ..., X_n\big),$$

$\mathcal{F}_n$ is called the natural filtration of X. A process is therefore always adapted to its own natural filtration.

A.4 Martingales

Definition A.4.1. *A process $X = (X_n)$ is called a martingale relative to $(\{\mathcal{F}_n\}, \mathbb{P})$ if*

i. *X is adapted to $\{\mathcal{F}_n\}$,*

ii. *$\mathbb{E}\,|X_n| < \infty$ for all n,*

iii. *$\mathbb{E}[X_n|\mathcal{F}_{n-1}] = X_{n-1}$.*

Most intuitive and important is perhaps property iii, which states that the expected value of X_n, given what is already known of the process represented by the filtration $\mathcal{F}_{n-1}$, is simply X_{n-1}. A martingale can therefore be said to be *constant on average*, and is often used as a term for a *fair game*.

Appendix B

Important Theorems and Tools

B.1 Brownian Motion

A Brownian motion, also known as a Wiener process plays an important part in the financial models presented in this thesis. Defined in for example [14], a stochastic process $X_t = \{X(t)\}_{t=0}^{\infty}$ is said to be a Wiener process if it satisfies

- $X(0) = 0$ almost surely,
- The increments of X_t is stationary and independent,
- $X_t \sim N(0, t\sigma^2)$.

B.2 Itô's lemma

Itô's lemma is used to relate a small change in a function of a random variable, to the random variable itself. This is particularly useful in financial models, where prices are quoted in discrete time intervals dt, but the mathematical models work in continuous time as $dt \to 0$. Following the definition from [23], f is given as a twice differentiable function of the random variable G, which again is described by the stochastic difference equation

$$dG = A(X,t)dX + B(G,t)dt.$$

If X is a random variable, then Itô's lemma says that a small change df in the function $f(G)$ can be expressed as

$$df = A(X,t)\frac{\partial f}{\partial G}dX + \Big(B(G,t)\frac{\partial f}{\partial G} + \frac{1}{2}(A(X,t))^2\frac{\partial^2 f}{\partial G^2}\Big)dt \tag{B.1}$$

B.3 Change of numariere and the The Radon-Nikodym derivative

Using definition 3.2.1, a numeraire is a strictly positive process, often an asset like a stock or an interest rate, which is used to discount other assets with. Considering two different numeraires, $p(t)$ and $q(t)$, with different equivalent martingale measures $\mathbb{P}$ and $\mathbb{Q}$. The prices of a contigent claim X can then be equivalent stated as

$$p(t)\mathbb{E}^P\Big[\frac{X(T)}{p(T)}|\mathcal{F}_t\Big] = q(t)\mathbb{E}^Q\Big[\frac{X(T)}{q(T)}|\mathcal{F}_t\Big]. \tag{B.2}$$

$\mathbb{F}$ is the usual filtration and the expectations is is under the respective measures. Stating $G(T) = X(T)/p(T)$, (B.2) is rewritten into

$$\mathbb{E}^P\Big[G(T)|\mathcal{F}_t\Big] = \mathbb{E}^Q\Big[G(T)\frac{p(T)/p(t)}{q(T)/q(t)}|\mathcal{F}_t\Big].$$

The expectation of G under the measure $\mathbb{P}$ is equal to the expectation of $G \cdot \frac{p(T)/p(t)}{q(T)/q(t)}$ under the measure $\mathbb{Q}$. The random variable $\frac{p(T)/p(t)}{q(T)/q(t)}$ denoted $d\mathbb{P}/d\mathbb{Q}$ is known as the Radon-Nikodym derivative which changes the measure $\mathbb{P}$ into $\mathbb{Q}$ [14].

B.4 Girasanov's theorem

One of the most important results regarding stochastic calculus is Girasanov's Theorem. The theorem states the effect a change of measure has on the underlying stochastic process [14]. There are several versions of the theorem, with the following being the most relevant in this thesis:

Theorem B.4.1. *(Girsanov'ss Theorem) For any stochastic process $f(t)$ such that*

$$\mathbb{P}\left(\int_0^t f^2(\tau)d\tau < \infty\right) = 1,$$

consider the Radon-Nikodym derivative defined $\frac{dP^*}{dP} = g(t)$

$$g(t) = exp\left\{\int_0^t f(\tau)dW - \frac{1}{2}\int_0^t f(\tau)d\tau\right\},$$

where W is a Brownian motion as previously defined in appendix B.1 under the measure P. Under the measure P^* the process

$$W^*(t) = W(t) - \int_0^t f(\tau)d\tau$$

is then a Brownian motion.

Girsanov's theorem describes how the drift is adjusted when moving from the real-world dynamics of an asset price to the risk-neutral dynamics which is used for pricing. Another important consequence is that the diffusion, or volatility of the process remains unchanged through this change of measure. This means volatility can be estimated in the real-world, and still be used as input in risk-neutral models.

Appendix C

Complete Results

C.1 Results from Preliminary Analysis

Table C.1: Table of initial transformed instantaneous forward rates measured in %. The initial forward rates are used as input for the backtests and for calculations of actual exposures.

Year	Short rate	1Y	2Y	3Y	4Y	5Y
1.1.1995	5.98	8.20	8.64	9.34	8.62	9.56
1.1.1996	4.63	5.60	6.48	6.86	7.12	7.55
1.1.1997	3.83	5.36	6.02	6.64	7.04	7.38
1.1.1998	3.83	5.39	5.56	5.76	6.09	7.06
1.1.1999	6.19	5.87	5.45	5.65	5.82	5.90
1.1.2000	5.29	6.28	6.24	6.33	6.42	7.41
1.1.2001	6.83	7.32	6.76	6.70	6.83	6.81
1.1.2002	5.88	5.95	6.10	6.11	6.14	6.26
1.1.2003	6.22	6.48	6.46	6.57	6.51	6.55
1.1.2004	2.58	4.43	5.23	5.45	5.60	5.66
1.1.2005	1.90	3.13	3.91	4.48	4.82	5.01
1.1.2006	2.40	3.86	4.14	4.27	4.36	4.41
1.1.2007	3.50	4.66	4.69	4.69	4.65	4.65
1.1.2008	5.68	5.65	5.36	5.30	5.37	5.55
1.1.2009	6.35	4.18	4.06	4.56	4.80	5.07
1.1.2010	2.08	4.21	4.50	4.64	4.75	4.85
1.1.2011	2.49	3.25	3.51	3.90	4.23	4.45
1.1.2012	3.25	3.18	3.22	3.66	3.97	4.10
1.1.2013	1.94	2.38	2.43	2.80	3.18	3.47
1.1.2014	1.68	2.09	2.40	2.87	3.31	3.68
1.1.2015	1.59	1.30	1.40	1.62	1.89	2.11
1.1.2016	1.22	0.73	0.99	1.36	1.75	2.02
1.1.2017	1.19	1.16	1.38	1.64	1.89	2.08
1.1.2018	0.83	1.22	1.48	1.75	1.97	2.06

C.2 Risk-factor Backtest

Table C.2: Table of descriptive statistics for the simulated 3-month swap interest rate (in $\%$) with simulated 90% bounds for all periods. $t = 0$ is the observed short rate at initialisation. For $t > 0$ the 90% bounds are written as intervals centered around the expected rate.

Period	t = 0	t = 1	t = 2	t = 3	t = 4	t = 5
1997-2002	3.83	5.36 ± 4.17	6.02 ± 5.32	6.64 ± 5.98	7.04 ± 6.49	7.38 ± 6.81
1999-2004	6.19	5.87 ± 4.06	5.45 ± 5.44	5.65 ± 6.15	5.82 ± 6.77	5.90 ± 7.29
2001-2006	6.83	7.32 ± 9.69	67.6 ± 11.22	6.70 ± 11.97	6.83 ± 12.42	6.81 ± 12.85
2003-2008	6.22	6.48 ± 7.51	6.46 ± 8.34	6.57 ± 8.87	6.51 ± 9.25	6.55 ± 9.51
2005-2010	1.90	3.13 ± 4.04	3.91 ± 4.72	4.48 ± 5.15	4.81 ± 5.42	5.00 ± 5.62
2007-2012	3.50	4.66 ± 2.34	4.69 ± 2.94	4.69 ± 3.04	4.65 ± 3.11	4.65 ± 3.12
2009-2014	6.35	4.18 ± 2.84	4.06 ± 3.53	4.56 ± 3.73	4.80 ± 3.86	5.07 ± 3.91
2011-2016	2.49	3.25 ± 2.79	3.51 ± 3.33	3.90 ± 3.58	4.23 ± 3.70	4.45 ± 3.80
2013-2018	1.94	2.38 ± 2.64	2.43 ± 2.85	2.80 ± 2.96	3.18 ± 3.05	3.47 ± 3.06

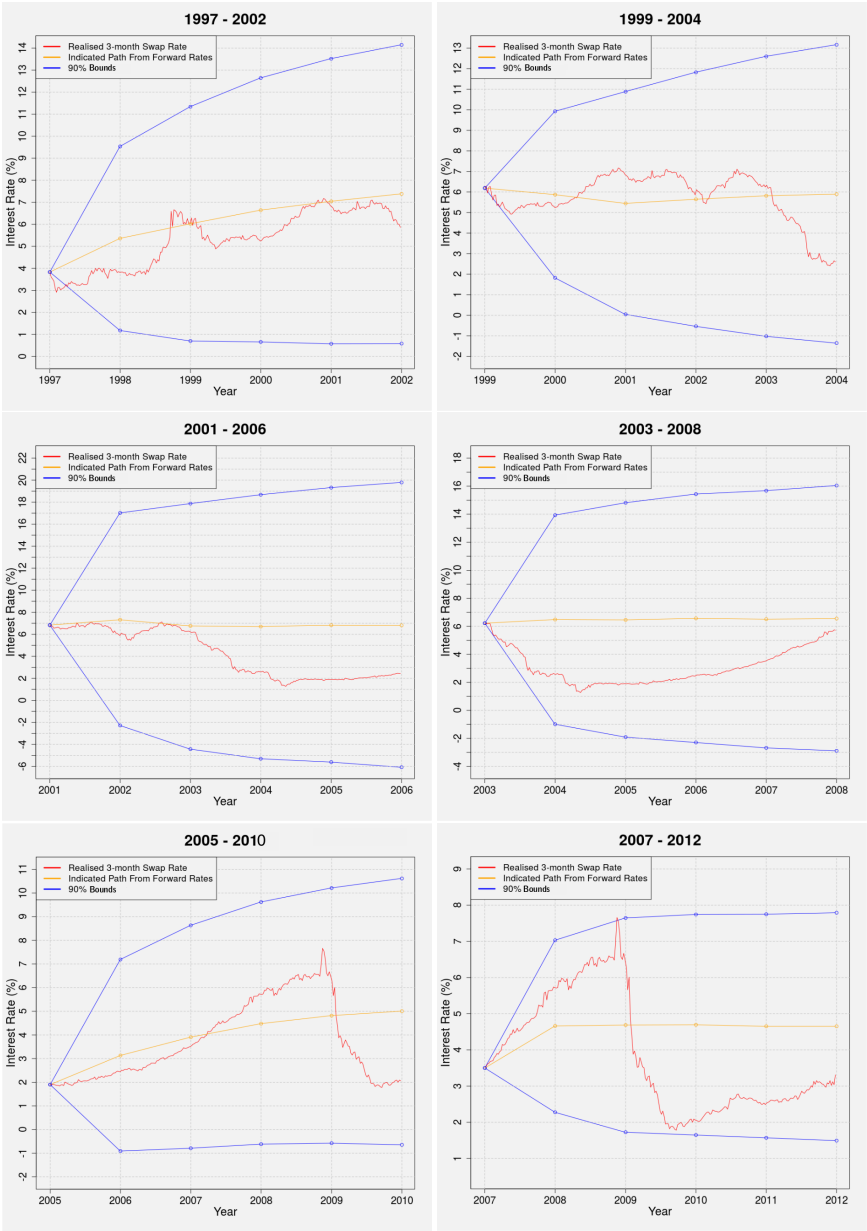

Figure C.1: Risk-factor backtest with expected path, simulated 90% bounds, and realised short rate.

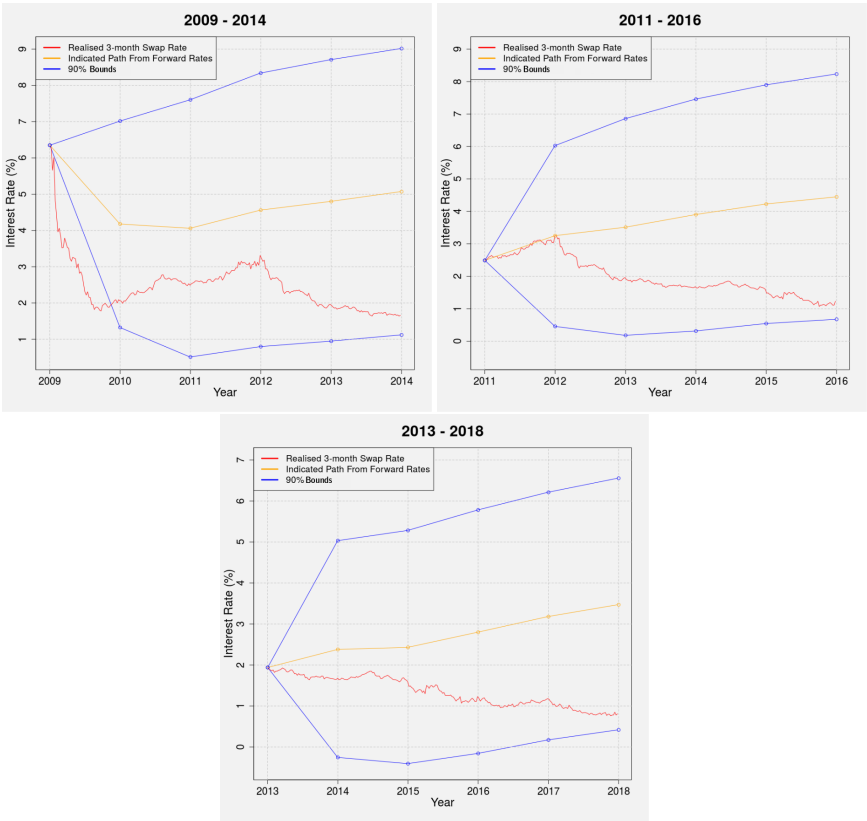

Figure C.2: Risk-factor backtest with expected path, simulated 90% bounds, and realised short rate.

C.3 Portfolio Backtest

Table C.3: Table of simulated EPE and 95% PFE in addition to realised actual exposures (AE) for all periods including strike price for the swap contracts. Gross notional values of all contracts are normalised to 1.

Period	Exposure	t = 1	t = 2	t = 3	t = 4	Strike (%)
	EPE	0.0081	0.0091	0.0093	0.0093	
1997-2002	PFE	0.0411	0.0482	0.0483	0.0445	6.79
	AE	0.0042	0.0029	-0.0099	-0.0140	
	EPE	0.0149	0.0170	0.0165	0.0145	
1999-2004	PFE	0.0579	0.0711	0.0689	0.0589	6.05
	AE	-0.0190	-0.0330	-0.0045	-0.0062	
	EPE	0.0236	0.0267	0.0259	0.0219	
2001- 2006	PFE	0.0879	0.1035	0.1027	0.0841	7.70
	AE	0.0335	0.0152	0.0464	0.0422	
	EPE	0.0126	0.0152	0.0164	0.0140	
2003 - 2008	PFE	0.0526	0.0633	0.0676	0.0571	7.00
	AE	0.0539	0.0801	0.0528	0.0210	
	EPE	0.0041	0.0051	0.0058	0.0062	
2005 - 2010	PFE	0.0248	0.0295	0.0326	0.0307	4.40
	AE	0.0039	-0.0126	-0.0246	0.0012	
	EPE	0.0046	0.0054	0.0055	0.0051	
2007 - 2012	PFE	0.0199	0.0225	0.0220	0.0204	4.80
	AE	-0.0283	0.0115	0.0065	0.0148	
	EPE	0.0062	0.0051	0.0046	0.0044	
2009 - 2014	PFE	0.0293	0.0273	0.0251	0.0227	4.65
	AE	0.0009	0.0285	0.0260	0.0216	
	EPE	0.0050	0.0044	0.0041	0.0040	
2011 - 2016	PFE	0.0254	0.0249	0.0237	0.0206	3.95
	AE	0.0122	0.0389	0.0328	0.0259	
	EPE	0.0018	0.0013	0.0013	0.0015	
2013 - 2018	PFE	0.0103	0.0094	0.0094	0.0097	2.29
	AE	-0.0003	0.0359	0.0353	0.0155	

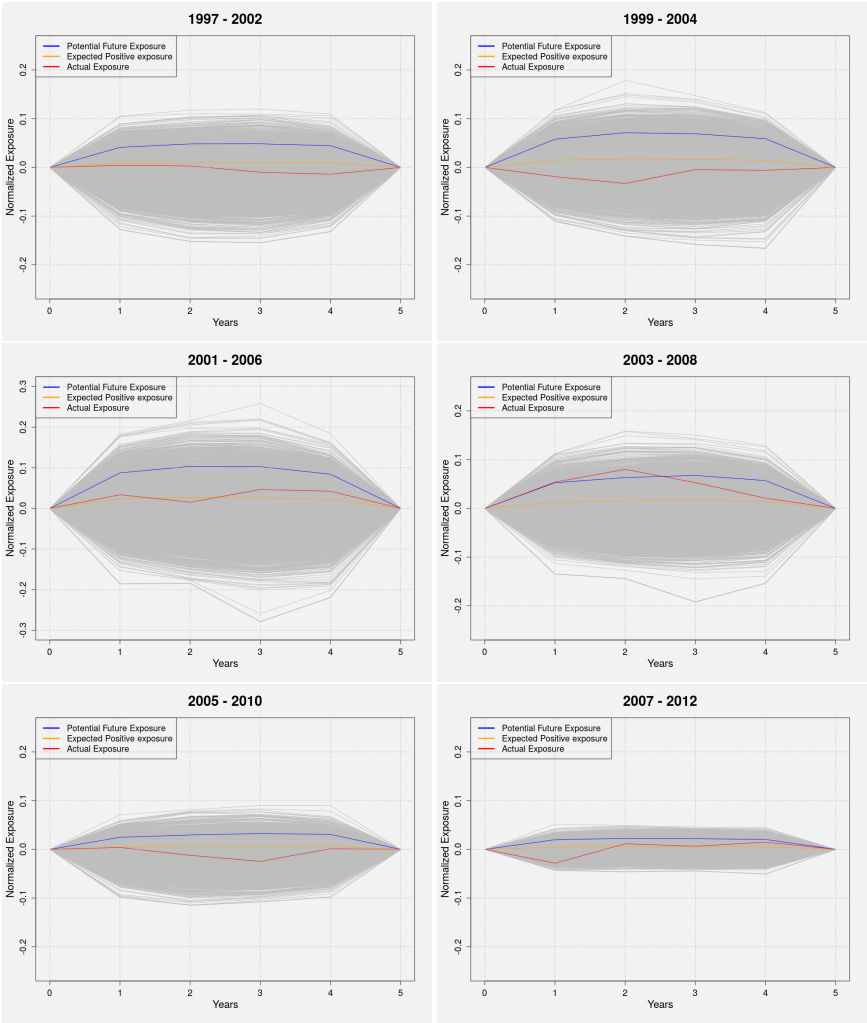

Figure C.3: Portfolio backtest, exposures for all 5 years with 95% PFE, EPE and AE.

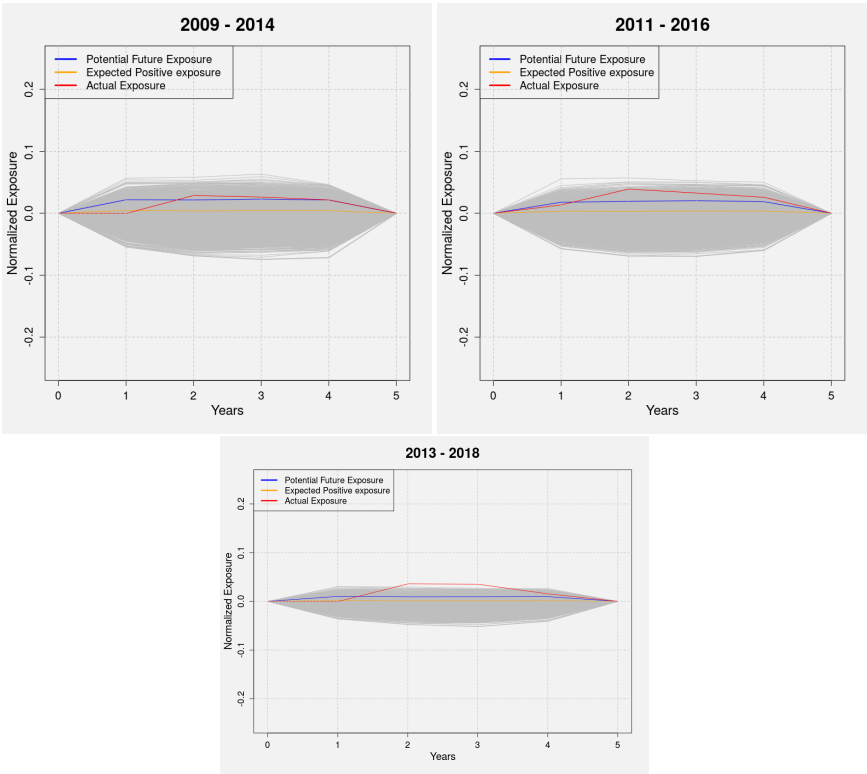

Figure C.4: Portfolio backtest, exposures for all 5 years with 95% PFE, EPE and AE.

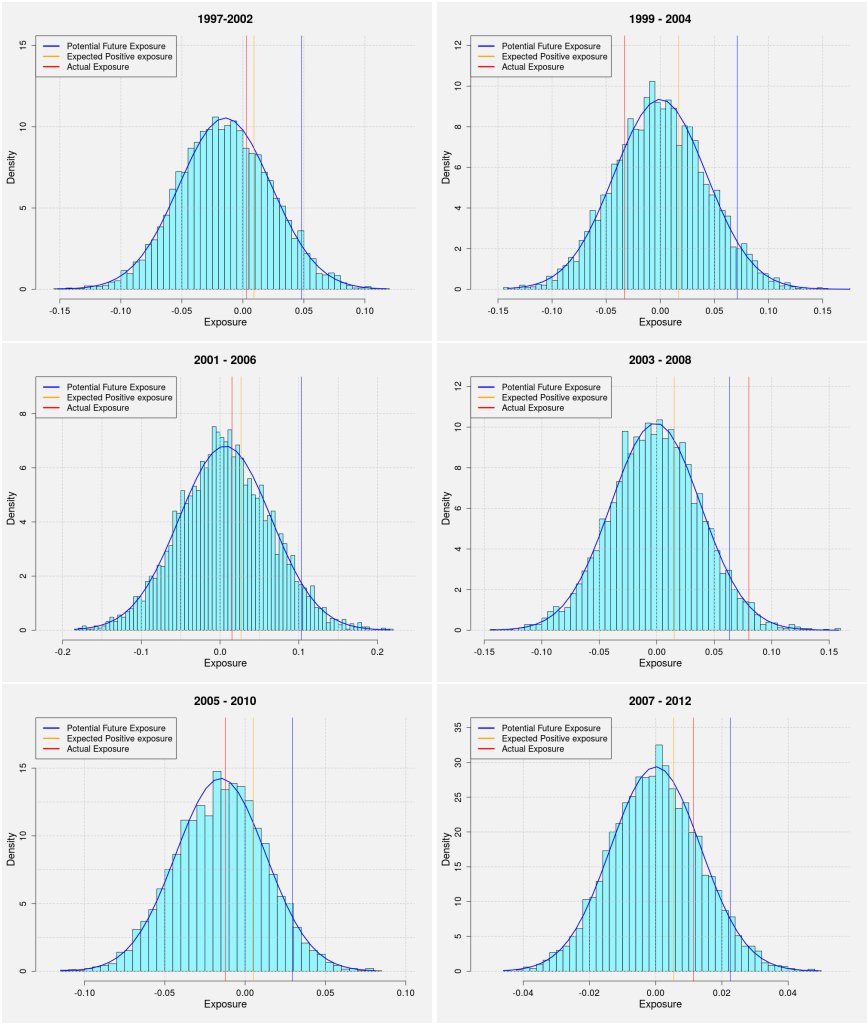

Figure C.5: Portfolio backtest, exposure distribution at t = 2 with 95% PFE, EPE and AE.

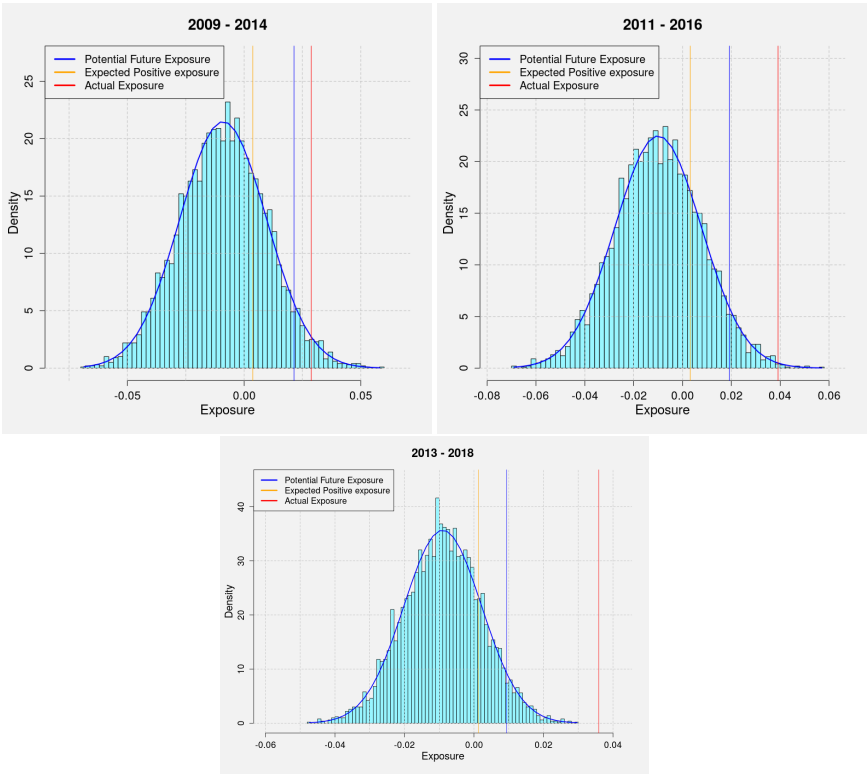

Figure C.6: Portfolio backtest, exposure distribution at t = 2 with 95% PFE, EPE and AE.