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Noncommutative Geometry in Wireless Communication Applications

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Abstract

The theory of time-frequency analysis is introduced, and basic results on modulation spaces are proved. We describe Gabor frames and prove results which are relevant to application in wireless communication. We outline a technique for transmitting data through a time-domain signal using time-frequency shifts, and show how pseudodifferential operators can be used to model a communication channel. The abstract structures of Hilbert C^*-modules are described and used to explain central aspects of Gabor theory, and we use a link to noncommutative geometry to prove the Balian-Low theorem.

Sammendrag

Teorien om tidsfrekvensanalyse blir introdusert, og grunnleggende resultater om modulasjonsrommene bevises. Vi beskriver Gabor-rammer og gir bevis for resultater som er nyttige innen utvikling av trådløs kommunikasjon. Vi skisserer en prosedyre for overføring av data via et signal i tidsdomenet, og viser hvordan pseudodifferensielle operatorer kan brukes til å modellere en kommunikasjonskanal. Videre beskrives den abstrakte strukturen Hilbert C^*-modul, og den brukes til å forklare sentrale aspekt ved Gabor-teori. Vi bruker i tillegg en kobling til ikke-kommutativ geometri til å gi ytterligere innsikt i tidsfrekvensanalysen og bevise Balian-Low-teoremet.

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During the autumn of 2017 I did a smaller project on Gabor frames together with Associate Professor Franz Luef. This project, which is common practice among technology students, set me up very well for writing this thesis, as it can be seen as a continuation of what I did in the previous semester. Because of this, it was natural to include parts of the autumn project in this thesis. More precisely, large parts of chapters 1 and 2 originate from this project. The thesis constitutes 30 Norwegian credits ("studiepoeng"), or half a year of studies, while the autumn project constituted 15.

I would like to thank my supervisor Franz Luef for over a year of supervision and support. Not only has he provided excellent professional guidance in our regular meetings, but he has also given moral support in times of little progress. I would also like to thank my friends and my family for the continued support throughout this thesis as well as my studies in general.

Sindre Nås,
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Contents

Introduction

In mobile wireless communication, the goal is representing data as analogue signals, and retrieving these data upon reception. A common method is called orthogonal frequency-division multiplexing (OFDM), and is used in a range of different systems, including Digital Audio Broadcasting (DAB radio), Wireless Local Area Networks (WLAN), and underwater acoustic communications.[18] In OFDM, discrete data is represented as a superposition of shifts in time and frequency, of a given function, or *window*, g. Now, this could be done by finding a basis among the time-frequency shifts of g, and hence a unique representation for every signal f. However, both theory and practice has shown this is not the most effective approach. In the practical context, a unique representation is vulnerable to noise and incomplete transmissions, since information contained in the signal can only be transmitted by one unique shift of the window. In theory, we shall see that by the Balian-Low theorem, for well-behaving windows g, it is in fact impossible to make such a system an orthonormal basis. By allowing for some redundancy of information in the representation, however, practitioners within the field have arrived at the concept of a *frame*. Instead of requiring unique representations, this generalisation of a basis allows for linearly dependent elements.

In his 1946 paper [6], Dennis Gabor introduced the idea to use time-frequency shifts of the Gaussian to represent a signal. In the 80's, his name was given to the *Gabor frames* of the form $\mathcal{G}(g, \Lambda) = \{M_\omega T_x g\}_{(x,\omega)\in\Lambda}$, where Λ is some discrete subset, usually a lattice, in $\mathbb{R}^2$. This has a natural connection to the OFDM of wireless signals. The signal construction and data retrieval - the synthesis and analysis - are neatly described by the so-called *Gabor frame operator*. Gabor theory consists of describing these systems, and their many properties and unsolved questions, as well as manipulating the frame operator to make such a transmission system more effective.

The Gabor frame operator leads us into the field of operator algebras, where, among the C^*-algebras [14, 11], we find the noncommutative tori, developed broadly by Alain Connes [1] and Marc Rieffel [20]. These can be seen as generalisations of the commutative algebra of continuous functions on the 2-torus, where the two generating unitary elements are subject to a different commutation relation. The noncommutative tori can be realised in $\mathcal{B}(L^2(\mathbb{R})$ both as a *twisted group C^*-algebra* and as a *crossed product*. Together with the abstract structure of a Hilbert C^*-module, these two different realisations give insight into the mathematical mechanics behind some mysterious properties of the Gabor frames, and in particular, representations of the Gabor frame operator.

The Serre-Swan Theorem, due to Jean-Pierre Serre and Richard Swan [22], states that there is an equivalence of categories between finitely generated, projective modules over an algebra, and vector bundles, via the concept of *sections*. With the geometrical structure of vector bundles, the Hilbert C^*-modules allow us to describe elements from Gabor analysis further. In particular, the Balian-Low theorem can be proved as a consequence of the differential structure added to smooth vector bundles over smooth manifolds.

In this thesis, chapter 1 introduces important concepts of time-frequency analysis, such as the time-frequency shift, the short-time Fourier transform, and the modulation spaces. Chapter 2 gives an introduction to general frame theory, before using time-frequency shifts to define the very central concept of Gabor frames. These are thoroughly discussed, and we consider representations of the operator, duality with the adjoint lattice, and multi-window Gabor frames. Chapter 3 introduces some insight to the appliance of representation of data through time-frequency shifts, and the pseudodifferential operators used to model communication channels are being discussed. In chapter 4, we introduce some abstract mathematics in the noncommutative tori, and we look at two ways to realise them as sets of operators on $L^2(\mathbb{R})$. Chapter 5 uses the concept of Morita equivalence between the C^*-modules of the previous chapter to show links to Gabor analysis. In particular, associativity of the inner products on two realisations of an equivalence bimodule between noncommutative tori with indices θ and $1/\theta$ is shown to be equivalent to the Walnut and Janssen representations, respectively. Finally, chapter 6 gives a geometrical aspect to the modules by introducing vector bundles, and through the concept of connections, we are able to give a proof of the Balian-Low theorem.

Chapter 1

Modulation Spaces

In this chapter, we introduce central concepts to the field of time-frequency analysis. In particular, we define and establish the properties of the modulation spaces, which will be important classes of functions throughout the thesis.

1.1 The Short-Time Fourier Transform

Central to time-frequency analysis are the two operators T_x and M_ω, where $x, \omega \in \mathbb{R}$, acting on $L^2(\mathbb{R})$ as

$$T_x f(t) = f(t-x),$$
$$M_\omega f(t) = e^{2\pi i \omega t} f(t).$$

These are called the *translation* and *modulation operators*, respectively. Some basic properties are given in the following lemma.

Lemma 1.1.1.

i) $M_\omega T_x = e^{2\pi i x\omega} T_x M_\omega$.

ii) $T_x^* = T_{-x}$ *and* $M_\omega^* = M_{-\omega}$.

iii) $(M_\omega T_x)(M_{\omega'} T_{x'}) = e^{2\pi i(\omega x' - \omega' x)}(M_{\omega'} T_{x'})(M_\omega T_x)$

Proof.

i) $M_\omega T_x f(t) = e^{2\pi i t\omega} f(t-x) = e^{2\pi i x\omega} e^{2\pi i (t-x)\omega} f(t-x) = e^{2\pi i x\omega} T_x M_\omega f(t)$.

ii)

$$\langle T_x f, g\rangle = \int_{\mathbb{R}} f(t-x)\overline{g(t)}dt = \int_{\mathbb{R}} f(u)\overline{g(u+x)}du = \langle f, T_{-x} g\rangle.$$

Similarly,

$$\langle M_\omega f, g\rangle = \int_{\mathbb{R}} e^{2\pi i\omega t} f(t)\overline{g(t)}dt = \int_{\mathbb{R}} f(t)\overline{e^{-2\pi i\omega t} g(t)}dt = \langle f, M_{-\omega} g\rangle.$$

iii)

$$\begin{aligned}(M_\omega T_x)(M_{\omega'} T_{x'})f(t) &= e^{2\pi i\omega t} e^{2\pi i\omega'(t-x)} f(t-x'-x)\\ &= e^{2\pi i(\omega' t+\omega t-\omega x')} f(t-x'-x)\\ &= e^{2\pi i(\omega x'-\omega' x)} e^{2\pi i(\omega' t+\omega t-\omega x')} f(t-x'-x)\\ &= e^{2\pi i(\omega x'-\omega' x)}(M_{\omega'} T_{x'})(M_\omega T_x)\end{aligned}$$

□

Note that *ii* implies that $M_\omega T_x$ is a unitary operator, since clearly $M_\omega^{-1} = M_{-\omega}$ and $T_x^{-1} = T_{-x}$. We will often use both operators together, making it convenient to write $\pi(\lambda) = M_\omega T_x$ for $\lambda = (x, \omega) \in \mathbb{R}^2$. We shall call the operator $\pi(\lambda)$ a *time-frequency shift*. The time-frequency shift is used on a so-called *window function* $g \in L^2(\mathbb{R})$, usually a well-concentrated pulse, which can be manipulated by $\pi(\lambda)$ to carry information as a signal. Now, we define the *Fourier transform* of a function $f \in L^2(\mathbb{R})$ by

$$\mathcal{F}(f)(\omega) = \hat{f}(\omega) = \int_{\mathbb{R}} f(t)e^{-2\pi i\omega t}.$$

Definition 1.1.1. Fix a window function $g \in L^2(\mathbb{R})$. The *short-time Fourier transform* (STFT) of a function $f \in L^2(\mathbb{R})$ with respect to g is the function $V_g f : \mathbb{R}^2 \to \mathbb{C}$ given by

$$V_g f(x,\omega) = \langle f, \pi(x,\omega)g\rangle = \int_{\mathbb{R}} f(t)\overline{g(t-x)}e^{-2\pi i\omega t}dt = \mathcal{F}(fT_x g(\omega)).$$

It can be shown that for $f \in L^2(\mathbb{R})$, we have $V_g f \in L^2(\mathbb{R}^2)$. The STFT can be thought of as a way to describe a signal f with regards to both time and frequency. As a function of time, $f(t)$ only tells us the amplitude at time t, and nothing about the frequencies. Similarly, as a function of frequency, the Fourier transform $\hat{f}(\omega)$ only tells us the amplitude of a component with frequency ω, and nothing about the time at which this frequency occurs. The STFT yields information about both time and frequency in one function, and is therefore often useful for describing e.g. music, for which both time and frequency information is vital.

Proposition 1.1.2 (Moyal's Identity)**.** *Let g, γ, f and ϕ be functions in $L^2(\mathbb{R})$. Then*

$$\langle V_g f, V_\gamma \phi\rangle = \langle f, \phi\rangle \overline{\langle g, \gamma\rangle}.$$

The proof of this can be found in [7].
We shall investigate how the STFT behaves under a switch of the roles of f and g. For $\lambda = (x, \omega) \in \mathbb{R}^2$, using lemma (1.1.1), we get

$$\begin{aligned} V_g f(\lambda) &= \langle f, \pi(\lambda)g\rangle \\ &= \overline{\langle \pi(x,\omega)g, f\rangle} = \overline{\langle M_\omega T_x g, f\rangle} \\ &= \overline{\langle g, T_{-x}M_{-\omega} f\rangle} = \overline{\langle g, \pi(-\lambda) f\rangle e^{-2\pi i\omega x}} \\ &= \overline{V_f g(-\lambda)} e^{2\pi i \omega x}. \end{aligned}$$

Lemma 1.1.3. *The adjoint $V_g^* : L^2(\mathbb{R}^2) \to L^2(\mathbb{R})$ of the STFT is given by*

$$V_g^* F(t) = \int_{\mathbb{R}^2} F(\lambda)\pi(\lambda)g(t)d\lambda.$$

Proof.

$$\begin{aligned} \langle V_g f, F\rangle_{L^2(\mathbb{R}^2)} &= \int_{\mathbb{R}}\int_{\mathbb{R}} V_g f(x,\omega)F(x,\omega)dxd\omega \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} f(t)\overline{g(t-x)}e^{-2\pi i t\omega}dt\overline{F(x,\omega)}dxd\omega \\ &= \int_{\mathbb{R}} f(t)\int_{\mathbb{R}}\int_{\mathbb{R}} \overline{F(x,\omega)g(t-x)}e^{-2\pi i t\omega}dxd\omega dt \\ &= \langle f, V_g^* F\rangle_{L^2(\mathbb{R})}. \end{aligned}$$

□

Using the adjoint we can describe inversion of the STFT:

Lemma 1.1.4. *Given functions $f, g, \gamma \in L^2(\mathbb{R})$,*

$$V_\gamma^* V_g f = \langle \gamma, g\rangle f.$$

Proof. Take any function $\phi \in L^2(\mathbb{R})$. Then

$$\langle V_\gamma^* V_g f, \phi\rangle = \langle V_g f, V_\gamma \phi\rangle = \langle f, \phi\rangle\overline{\langle g, \gamma\rangle} = \langle\langle \gamma, g\rangle f, \phi\rangle,$$

using Moyal's Identity in the second equality. Since this holds for any $\phi \in L^2(\mathbb{R})$, we must have $V_\gamma^* V_g f = \langle \gamma, g\rangle f$. □

1.2 Definition of the Modulation Spaces

A *weight function* – a non-negative, continuous function $v : \mathbb{R}^2 \to \mathbb{R}$ – is said to be *submultiplicative* if

$$v(z+w) \leq v(z)v(w)$$

for all $z, w \in \mathbb{R}^2$. A weight function $m : \mathbb{R}^2 \to \mathbb{R}$ is *v-moderate* if

$$m(z+w) \leq Cv(z)m(w)$$

for all $z, w \in \mathbb{R}^2$.

Definition 1.2.1. For $p, q \in [1, \infty]$, $g \in L^2(\mathbb{R})$ and a v-moderate weight m, the *modulation spaces* are defined as follows:

$$M_m^{p,q}(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : ||f||_{M_m^{p,q}} := ||(V_g f)||_{L_m^{p,q}(\mathbb{R}^2)} < \infty\},$$

where $L_m^{p,q}(\mathbb{R}^2)$ is the mixed-norm space of all functions $F : \mathbb{R}^2 \to \mathbb{C}$ such that

$$||F||_{L_m^{p,q}(\mathbb{R}^2)} := \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |F(x,\omega)|^p m(x,\omega)^p dx)^{q/p} d\omega\Big)^{1/q} < \infty.$$

Perhaps surprisingly, the definition of the modulation spaces does in fact not depend on the choice of the atom g, as long as we set some restrictions on the choice. Recall that the *Schwartz space* is the subspace of $C^\infty(\mathbb{R})$ of infinitely differentiable functions whose derivatives are decaying fast:

$$\mathcal{S}(\mathbb{R}) = \left\{ f \in C^\infty(\mathbb{R}) \mid \sup_{t \in \mathbb{R}} \left|(2\pi i t)^m f^{(n)}(t)\right| < \infty \quad \forall m, n \in \mathbb{N} \right\}.$$

Proposition 1.2.1. *For different choices of $g \in \mathcal{S}(\mathbb{R})$, the norms on $M_m^{p,q}(\mathbb{R})$ are equivalent. Consequently, the definition of $M_m^{p,q}(\mathbb{R})$ is independent on the choice of window function $g \in \mathcal{S}(\mathbb{R})$.*

In order to prove this result, we need a somewhat technical lemma concerning the norm of convolutions in $L_m^{p,q}(\mathbb{R}^2)$:

Lemma 1.2.2. *Let m be a v-moderate weight function, and take $F \in L_v^1(\mathbb{R}^2)$ and $G \in L_m^{p,q}(\mathbb{R}^2)$. Then*

$$||F * G||_{L_m^{p,q}} \leq C||F||_{L_v^1}||G||_{L_m^{p,q}}.$$

Proof. Let H be a function in $L^{p',q'}_{1/m}(\mathbb{R}^2)$, where $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$. Then

$$\begin{aligned}
|\langle F*G,H\rangle| &= \left|\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} F(w)G(z-w)\overline{H(z)}dwdz\right| \\
&\leq \int_{\mathbb{R}^2}|F(w)|\left(\int_{\mathbb{R}^2}|T_wG(z)|\ |H(z)|dz\right)dw \\
&\leq \int_{\mathbb{R}^2}|F(w)|\,||T_wG||_{L^{p,q}_m}\,||H||_{L^{p',q'}_{1/m}}dw \quad \text{(by Hölder's Inequality)} \\
&\leq C\int_{\mathbb{R}^2}|F(w)|v(w)dw||G||_{L^{p,q}_m}||H||_{L^{p',q'}_{1/m}}.
\end{aligned}$$

Thus,

$$||F*G||_{L^{p,q}_m}=\sup\left\{|\langle F*G,H\rangle| : ||H||_{L^{p',q'}_{1/m}}\leq 1\right\}\leq C||F||_{L^\infty_v}||G||_{L^{p,q}_m}.$$

□

We are now ready to prove the equivalence of norms on $M^{p,q}_m(\mathbb{R})$, by using the results above for the STFT:

Proof of proposition 1.2.1. Take $F\in L^{p,q}_m(\mathbb{R}^2)$. With $\lambda=(x,\omega)$, we have

$$\begin{aligned}
V_gV_g^*F(\lambda) &= \langle V_g^*F,\pi(\lambda)g\rangle \\
&= \iint_{\mathbb{R}^2} F(u,\eta)\overline{V_g[\pi(\lambda)g](u,\eta)}\,du\,d\eta \\
&= \iint_{\mathbb{R}^2} F(u,\eta)V_gg(x-u,\omega-\eta)e^{-2\pi ix(\omega-\eta)}\,du\,d\eta.
\end{aligned}$$

Thus we have the pointwise norm estimate

$$|V_gV_g^*F(\lambda)|\leq(|F|*|V_gg|)(\lambda).$$

Thus, by lemma 1.2.2, we have

$$||V_g(V_g^*F)||_{L^{p,q}_m}\leq C||F||_{L^{p,q}_m}\,||V_gg||_{L^1_v}. \tag{1.1}$$

With $||g||_2=1$, and using the Gaussian g_0 for comparison, (1.1) finally gives

$$||V_{g_0}f||_{L^{p,q}_m}=||V_{g_0}(V_g^*V_gf)||_{L^{p,q}_m}\leq C||V_{g_0}g||_{L^1_v}\,||V_gf||_{L^{p,q}_m}=C'||V_gf||_{L^{p,q}_m}$$

and

$$||V_g f||_{L^{p,q}_m} = ||V_g(V^*_{g_0} V_{g_0} f)||_{L^{p,q}_m} \leq c||V_g g_0||_{L^1_v} \, ||V_{g_0} f||_{L^{p,q}_m} = c'||V_{g_0} f||_{L^{p,q}_m}.$$

This shows that for any $g \in \mathcal{S}(\mathbb{R})$,the norms $||V_g f||_{L^{p,q}_m}$ and $||V_{g_0} f||_{L^{p,q}_m}$ are equivalent on $M^{p,q}_m(\mathbb{R})$. Consequently, for any $g \in \mathcal{S}(\mathbb{R})$, $||V_g f||_{L^{p,q}_m} < \infty$ if and only if $||V_{g_0} f||_{L^{p,q}_m} < \infty$, so the definition of the modulation spaces is independent of the window function $g \in \mathcal{S}(\mathbb{R})$. □

1.3 Particular Examples

We shall concentrate on the modulation space $M^{1,1}_1(\mathbb{R})$, i.e. the case where $p = q = 1$, and $m = 1$, which we shall denote $S_0(\mathbb{R}) = M^{1,1}_1(\mathbb{R})$. The space is also called Feichtinger's algebra, first introduced by Hans G. Feichtinger in [3].
We also write $M^1_s(\mathbb{R}) := M^{1,1}_{v_s}(\mathbb{R})$, where v_s is the weight function defined by $v_s(x, \omega) = (1 + |x|^2 + |\omega|^2)^{s/2}$.

Proposition 1.3.1. *$M^1_s(\mathbb{R})$ is invariant under Fourier transform.*

Proof. Let $f \in M^1_s(\mathbb{R})$. Since the modulation spaces are independent on the choice of $g \in \mathcal{S}$, it is sufficient to consider the Gaussian $g = g_0$. Then

$$\begin{aligned} ||\hat{f}||_{M^1_s(\mathbb{R})} &= ||V_{g_0}\hat{f}||_{L^1_s(\mathbb{R}^2)} \\ &= \iint_{\mathbb{R}^2} |\langle \hat{f}, \pi(\lambda)g_0\rangle| v_s(\lambda)\, d\lambda \\ &= \iint_{\mathbb{R}^2} |\langle f, \widehat{\pi(\lambda)g_0}\rangle| v_s(\lambda)\, d\lambda \quad \text{(by Parseval's formula)} \end{aligned} \tag{1.2}$$

The Fourier transform of a time-frequency shift by $\lambda = (x, \omega) \in \mathbb{R}^2$ is given by

$$\begin{aligned} \widehat{\pi(x,\omega)}g(\eta) &= \int_{\mathbb{R}} e^{2\pi i \omega t} g(t-x) e^{-2\pi i \eta t}\, dt \\ &= \int_{\mathbb{R}} g(t) e^{2\pi i (t+x)(\omega-\eta)}\, dt \\ &= \int_{\mathbb{R}} g(t) e^{2\pi i t(\omega-\eta)}\, dt\, e^{2\pi i x(\omega-\eta)} \\ &= \hat{g}(\eta-\omega) e^{-2\pi i x(\eta-\omega)} \\ &= T_\omega M_{-x}\hat{g}(\eta). \end{aligned}$$

Inserted into (1.2), and using that $\hat{g}_0 = g_0$, we finally obtain

$$\begin{aligned} ||\hat{f}||_{M^1_s(\mathbb{R})} &= \iint_{\mathbb{R}^2} |\langle f, T_\omega M_{-x} g_0\rangle| v_s(\lambda)\, d\lambda \\ &= \iint_{\mathbb{R}^2} |V_{g_0} f(\omega, -x)| v_s(\lambda)\, d\lambda \\ &\le C ||V_{g_0} f||_{L^1_s(\mathbb{R}^2)} \\ &= C||f||_{M^1_s(\mathbb{R})}. \end{aligned}$$

□

Now, when looking at the intersection of all the spaces $M^1_s(\mathbb{R})$, it turns out that this is in fact the Schwartz space:

$$\mathcal{S}(\mathbb{R}) = \bigcap_{s\ge 0} M^1_s(\mathbb{R}). \tag{1.3}$$

A proof of this can be found in [23].

Chapter 2

Gabor Frames

When transmitting discrete data through an analogue signal f, we need a well-defined procedure for constructing such a signal - the *synthesis* - as well as a one for breaking it down to study its components - the *analysis*. One way of transmitting data through a signal is to express a signal f as a discrete sum of translated and modulated versions of some simple, well-located window function g. By indexing our data by the time-frequency shifts, this method allows us to transmit the data through the analogue signal, and retrieve the data upon reception.
In this chapter we shall introduce the abstract concept of frames for a Hilbert space as a generalisation of, and alternative to, orthonormal bases. Then we will look at the concrete case of Gabor frames for $L^2(\mathbb{R})$, for which we establish important properties and well-known results. In particular, we will look closely at the Gabor frame operator, which describes the synthesis and analysis procedures of a data transmission.

2.1 Frames in Hilbert Spaces

Let H be a Hilbert space. A *frame* for H is a sequence $\{e_j\}_{j\in J}$ such that there are $A, B > 0$ satisfying the so-called *frame inequality*,

$$A||x||^2 \leq \sum_{j\in J} |\langle x, e_j\rangle|^2 \leq B||x||^2, \tag{2.1}$$

for all $x \in H$. Note that a frame is always a *Bessel sequence*, that is, it satisfies

$$\sum_{j\in J} |\langle x, e_j\rangle|^2 < \infty$$

for all $x \in H$. Therefore, the upper bound B is often called the *Bessel bound* of a frame. If (2.1) holds with $A = B$, $\{e_j\}_{j\in J}$ is said to be *tight*, and if additionally $A = B = 1$ satisfies the equation, the frame is called a *Parseval frame*. Parseval frames are particularly useful, as they admit an expansion

$$x = \sum_{j\in J} \langle x, e_j\rangle e_j$$

for all $x \in H$. Note that the coefficients $\langle x, e_j\rangle$ need not be unique. Although a frame is in many ways like a basis, this property distinguishes the two concepts: a frame allows linearly dependent elements, and thus supporting a notion of redundancy - overlapping of information.

Definition 2.1.1. For a frame $\{e_j\}_{j\in J}$ for a Hilbert space H, a *dual frame* is any frame $\{e'_j\}_{j\in J}$ satisfying

$$x = \sum_{j\in J} \langle x, e'_j\rangle e_j \qquad \forall x \in H.$$

Note that the concept of duality is symmetric, i.e. any frame is a dual frame for its own dual: suppose $\{e'_j\}_{j\in J}$ is a dual frame for $\{e_j\}_{j\in J}$, i.e. that (2.1.1) holds. Then consider the inner product of any two elements $x, y \in H$:

$$\langle x, y\rangle = \left\langle \sum_{j\in J} \langle x, e'_j\rangle e_j, y \right\rangle = \sum_{j\in J} \langle x, e'_j\rangle \langle e_j, y\rangle = \langle x, \sum_{j\in J} \langle y, e_j\rangle e'_j\rangle$$

must hold for all $x, y \in H$. This implies $y = \sum_{j\in J}\langle y, e_j\rangle e'_j$ for all $y \in H$, meaning $\{e_j\}_{j\in J}$ is a dual frame for $\{e'_j\}_{j\in J}$. Thus we may talk about pairs of dual frames without concern.

Note also that we may have more than one dual frame associated to a particular frame - again, we see a certain freedom in working with frames.

Definition 2.1.2. For a frame $\{e_j\}_{j\in J}$ for a Hilbert space H, the associated *analysis operator* is defined by

$$\begin{aligned} C : H &\to l^2(J) \\ x &\mapsto \{\langle x, e_j\rangle\}_{j\in J}. \end{aligned}$$

The associated *synthesis operator* is defined by

$$\begin{aligned} D : l^2(J) &\to H \\ \{c_j\}_{j\in J} &\mapsto \sum_{j\in J} c_j e_j. \end{aligned}$$

The analysis and synthesis operators can be shown to be adjoints of each other, and their composition make up the so-called *frame operator* associated to the frame $\{e_j\}$:

$$\begin{aligned} S := DC = C^*C = DD^* : & H \to H \\ & x \mapsto \sum_{j\in J} \langle x, e_j\rangle e_j. \end{aligned}$$

The following lemma lists a few basic properties of the frame operator.

Lemma 2.1.1. *Let $\{e_j\}$ be a frame with corresponding frame operator S. Then S is a bounded, invertible, positive, self-adjoint operator.*

Proof.

$$\langle Sx, x\rangle = \left\langle \sum_{j\in J} \langle x, e_j\rangle e_j, x \right\rangle = \sum_{j\in J} \langle x, e_j\rangle\langle e_j, x\rangle = \sum_{j\in J} |\langle x, e_j\rangle|^2 \le B||x||^2,$$

proving the boundedness of S. In fact, this shows that being a Bessel sequence is sufficient. The frame inequality (2.1) implies $A||x||^2 \le \langle Sx, x\rangle \le B||x||^2$, proving positivity, and hence invertibility. Self-adjointness follows from C and D being adjoints. □

The frame operator guarantees a dual frame whenever we have a frame $\{e_j\}_{j\in J}$: by considering the sequence $\{S^{-1}e_j\}_{j\in J}$, we see that

$$\sum_{j\in J} \langle x, S^{-1}e_j\rangle e_j = S^{-1} \sum_{j\in J} \langle x, e_j\rangle e_j = S^{-1}Sx = x \qquad \forall x \in H,$$

showing that $\{S^{-1}e_j\}_{j\in J}$ is always a dual frame for $\{e_j\}_{j\in J}$. We call this the *canonical dual frame* for $\{e_j\}_{j\in J}$. S also guarantees a *canonical tight frame*: for a given frame $\{e_j\}_{j\in J}$, since S is a positive operator, we can consider the sequence $\{S^{-1/2}e_j\}_{j\in J}$:

$$\sum_{j\in J} \langle x, S^{-1/2}e_j\rangle S^{-1/2}e_j = S^{-1} \sum_{j\in J} \langle x, e_j\rangle e_j = S^{-1}Sx = x \qquad \forall x \in H,$$

so $\{S^{-1/2}e_j\}_{j\in J}$ is a tight frame whenever $\{e_j\}_{j\in J}$ is a frame.

2.2 Gabor Frames in $L^2(\mathbb{R})$

We shall look at a particular type of frame for the Hilbert space $H = L^2(\mathbb{R})$. Recall the time-frequency shift $\pi(x, \omega) = M_\omega T_x$ of functions in $L^2(\mathbb{R})$, defined in section 1.1. Fix a function $g \in L^2(\mathbb{R})$ and let $\lambda = (x, \omega)$ run through a lattice $\Lambda = Q\mathbb{Z}^2$, where Q is an invertible 2×2 matrix. The resulting set $\mathcal{G}(g, \Lambda) := \{\pi(\lambda)g \mid \lambda \in \Lambda\}$ of time-frequency shifts of g is called a *Gabor system* with *atom* g. If the set also satisfies the requirement for being a frame for $L^2(\mathbb{R})$, i.e. if there exist $A, B > 0$ such that

$$A||f||^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g\rangle|^2 \leq B||f||^2 \qquad \forall f \in L^2(\mathbb{R}),$$

the system is called a *Gabor Frame*. We define the *volume* of the lattice Λ to be the determinant of the matrix Q, and denote it $\text{vol}(\Lambda)$.
Given a Gabor frame $\mathcal{G}(g, \Lambda)$, a function $\gamma \in L^2(\mathbb{R})$ is called a *dual atom* or *dual window* of g if $\mathcal{G}(\gamma, \Lambda)$ is a dual frame for $\mathcal{G}(g, \Lambda)$, i.e. if

$$f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)\gamma = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\gamma\rangle \pi(\lambda)g \quad \forall f \in L^2(\mathbb{R}).$$

The Gabor frame operator is given by

$$\begin{aligned} S_{g,g,\Lambda} : & L^2(\mathbb{R}) \to L^2(\mathbb{R}) \\ & f \mapsto \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)g. \end{aligned}$$

We use this notation to allow the slightly more general version of the Gabor frame operator which we shall use more frequently. This is defined by

$$\begin{aligned} S_{g,\gamma,\Lambda} : & L^2(\mathbb{R}) \to L^2(\mathbb{R}) \\ & f \mapsto \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)\gamma. \end{aligned}$$

From the definition of dual frames, we see that $S_{g,\gamma,\Lambda}$ is the identity operator exactly when g and γ are dual atoms. In this case a function f can be decomposed into a linear combination of elements of the frame, i.e. time-frequency shifts of any of the window functions g and γ.

Lemma 2.2.1. *$S_{g,\gamma,\Lambda}$ commutes with all time-frequency shifts $\pi(\lambda)$ for $\lambda \in \Lambda$.*

Proof. Let $\mu = (u, \eta) \in \Lambda$ and $f \in L^2(\mathbb{R})$. Then, using the adjoint and commutation properties of time-frequency shifts from lemma 1.1.1,

$$\begin{aligned}
\pi(\mu)^* S_{g,\gamma,\Lambda}\pi(\mu) f &= \pi(\mu)^* \sum_{\lambda\in\Lambda} \langle \pi(\mu) f, \pi(\lambda) g\rangle \pi(\lambda)\gamma \\
&= \sum_{\lambda\in\Lambda} \langle f, \pi(\mu)^*\pi(\lambda) g\rangle \pi(\mu)^*\pi(\lambda)\gamma \\
&= \sum_{\lambda\in\Lambda} \langle f, e^{2\pi i\eta(x-u)}\pi(\lambda-\mu) g\rangle e^{2\pi i\eta(x-u)}\pi(\lambda-\mu)\gamma \\
&= \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda-\mu) g\rangle \pi(\lambda-\mu)\gamma \\
&= \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda) g\rangle \pi(\lambda)\gamma \\
&= S_{g,\gamma,\Lambda} f,
\end{aligned}$$

using the re-indexing $\lambda - \mu \mapsto \lambda$ in the last equality. Due to the unitarity of the time-frequency shift, this implies that $S_{g,\gamma,\Lambda}\pi(\mu)f = \pi(\mu)S_{g,\gamma,\Lambda}f$ for all $f \in L^2(\mathbb{R})$. $\square$

For Gabor frames, the canonical dual frame $\{S^{-1}\pi(\lambda)g\}_{\lambda\in\Lambda}$ is a much more manageable set than in the case of general frames: since the frame operator commutes with time-frequency shifts, this is in fact the frame $\{\pi(\lambda)S^{-1}g\}_{\lambda\in\Lambda}$. This means only the lattice point λ of the time-frequency shift varies, so only the computation of the *canonical dual atom* $S^{-1}g$ is needed. This constitutes a significant advantage in computational efficiency.

2.3 The Fundamental Identity of Gabor Analysis

Let Λ be any lattice. Then we define the *adjoint lattice* Λ° by the following commutation criterion:

$$\Lambda^\circ = \left\{\lambda^\circ \in \mathbb{R}^2 \;\middle|\; \pi(\lambda)\pi(\lambda^\circ) = \pi(\lambda^\circ)\pi(\lambda) \;\; \forall\lambda \in \Lambda\right\}.$$

It is easily shown that for a lattice $\Lambda = A\mathbb{Z}^2$, the adjoint lattice is given by

$$\Lambda^\circ = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (A^T)^{-1}\mathbb{Z}^2,$$

and that, in particular, the adjoint of a separable lattice $\alpha\mathbb{Z} \times \beta\mathbb{Z}$ is $\beta^{-1}\mathbb{Z} \times \alpha^{-1}\mathbb{Z}$.

We shall see that the concept of an adjoint lattice yields several important results regarding the Gabor frames we have discussed. To lay the foundations for the connection between a frame-generating lattice Λ and its adjoint Λ°, we will use a version of the Poisson summation formula in which the sum is being taken over a lattice in $\mathbb{R}^2$. In this we shall see a connection between the duality of a lattice to its adjoint lattice, and the duality of a function and its symplectic Fourier transform, as observed in [5].

Definition 2.3.1. The *symplectic Fourier transform* of a function $F \in L^2(\mathbb{R}^2)$ at the point $\lambda = (x, \omega)$ is defined by

$$\hat{F}^s(\lambda) = \iint_{\mathbb{R}^2} e^{2\pi i(u\omega - x\eta)} F(u, \eta)\, du\, d\eta.$$

An important result in the following is a version of the Poisson summation formula where sum is being taken over a lattice in $\mathbb{R}^2$:

Proposition 2.3.1 (Poisson Summation Formula over a lattice in $\mathbb{R}^2$)**.** *Let F be a function in $M_s^1(\mathbb{R}^2)$ and Λ be a lattice in $\mathbb{R}^2$. Then*

$$\sum_{\lambda\in\Lambda} F(\lambda) = \mathrm{vol}(\Lambda)^{-1} \sum_{\lambda^\circ\in\Lambda^\circ} \hat{F}^s(\lambda^\circ), \tag{2.2}$$

where both sums converge absolutely.

Proof. Let $\lambda = (x, \omega), \mu = (u, \eta) \in \Lambda$, and let $\Phi(\mu) = \sum_{\lambda\in\Lambda} F(\mu + \lambda)$. Since the left-hand side of (2.2), $\Phi(0)$, is Λ-periodic, we get the following euclidean Fourier expansion over the dual lattice $\Lambda^\perp$:

$$\Phi(\mu) = \sum_{\lambda^\perp\in\Lambda^\perp} \hat{\Phi}_{\lambda^\perp} e^{2\pi i(\lambda^\perp\cdot\mu)},$$

where the Fourier coefficients are

$$\begin{aligned}
\hat{\Phi}_{k,n} &= \mathrm{vol}(\Lambda)^{-1} \iint_V \sum_{\lambda\in\Lambda} F(\mu + \lambda) e^{-2\pi i(\lambda^\perp\cdot\mu))}\, du\, d\eta \\
&= \mathrm{vol}(\Lambda)^{-1} \iint_{\mathbb{R}^2} F(\mu) e^{-2\pi i(\lambda^\perp\cdot\mu))}\, du\, d\eta \\
&= \mathrm{vol}(\Lambda)^{-1} \hat{F}(\lambda^\perp).
\end{aligned}$$

with $\hat{F}$ meaning the euclidean Fourier transform on $\mathbb{R}^2$. Thus,

$$\sum_{\lambda\in\Lambda} F(\mu + \lambda) = \mathrm{vol}(\Lambda)^{-1} \sum_{\lambda^\perp\in\Lambda^\perp} \hat{F}(\lambda^\perp) e^{2\pi i(\lambda^\perp\cdot\mu)}.$$

Now, from definition 2.3.1 of the symplectic Fourier transform, we see that $\hat{F}(\mu) = \hat{F}^s(J(\mu))$, where $J(u,\eta) = (-\eta, u)$ is a rotation in the time-frequency plane. By rotating the lattice similarly, we get the lattice $J\Lambda^{\perp} = \Lambda^{\circ}$, so

$$\sum_{\lambda\in\Lambda} F(\mu+\lambda) = \mathrm{vol}(\Lambda)^{-1} \sum_{\lambda^{\circ}\in\Lambda^{\circ}} \hat{F}^s(\lambda^{\circ}) e^{2\pi i(x^{\circ}\eta - u\omega^{\circ}))}$$

for $\lambda^{\circ} = (x^{\circ}, \omega^{\circ}) \in \Lambda^{\circ}$. Evaluating at $\mu = (u,\eta) = 0$, we obtain the desired result. □

We include a statement of the Poisson summation formula for functions in $L^2(\mathbb{R})$. This is a simpler version of the result 2.3.1 proved above, and follows a simpler, yet similar proof as the one just given.

Lemma 2.3.2 (Poisson Summation Formula)**.** *Let ϕ be a function in $M_s^1(\mathbb{R})$. Then*

$$\sum_{n\in\mathbb{Z}} \phi(t+nT) = \frac{1}{T} \sum_{n\in\mathbb{Z}} \hat{\phi}\left(\frac{n}{T}\right) e^{2\pi i n t/T}.$$

The powerful connection between a lattice and its adjoint through the use of the symplectic Fourier transform leads us directly to the following central theorem: [21, 13, 2, 4]

Theorem 2.3.3 (Fundamental Identity of Gabor Analysis)**.** *Let f and γ be functions in $L^2(\mathbb{R})$, ϕ and g functions in $M^1(\mathbb{R})$ and Λ a lattice in $\mathbb{R}^2$. Then*

$$\sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \overline{\langle \phi, \pi(\lambda)\gamma\rangle} = \mathrm{vol}(\Lambda)^{-1} \sum_{\lambda^{\circ}\in\Lambda^{\circ}} \langle \gamma, \pi(\lambda^{\circ})g\rangle \overline{\langle \phi, \pi(\lambda^{\circ})f\rangle}, \quad (2.3)$$

where both sums converge absolutely.

Proof. Let $F(\lambda) = \langle f, \pi(\lambda)g\rangle\langle \pi(\lambda)\gamma, \phi\rangle$. Then by referring to [5], we have that $F \in M^1(\mathbb{R}^2)$. Now, with $\lambda = (x,\omega)$ and $\lambda^{\circ} = (u,\eta)$ in $\mathbb{R}^2$, the symplectic Fourier transform of F is

$$\begin{aligned}
\hat{F}^s(\lambda^{\circ}) &= \iint_{\mathbb{R}^2} e^{2\pi i(x\eta - u\omega)} \langle f, \pi(\lambda)g\rangle\langle \pi(\lambda)\gamma, \phi\rangle \, d\lambda \\
&= \iint_{\mathbb{R}^2} \langle \pi(\lambda^{\circ})f, e^{-2\pi i(x\eta - u\omega)} \pi(\lambda^{\circ})\pi(\lambda)g\rangle\langle \pi(\lambda)\gamma, \phi\rangle \, d\lambda \\
&= \iint_{\mathbb{R}^2} \langle \pi(\lambda^{\circ})f, \pi(\lambda)\pi(\lambda^{\circ})g\rangle\langle \pi(\lambda)\gamma, \phi\rangle \, d\lambda \\
&= \iint_{\mathbb{R}^2} V_{\pi(\lambda^{\circ})g}[\pi(\lambda^{\circ})f](\lambda) \overline{V_{\gamma}\phi} \, d\lambda \\
&= \langle V_{\pi(\lambda^{\circ})g}[\pi(\lambda^{\circ})f], V_{\gamma}\phi\rangle \\
&= \langle \pi(\lambda^{\circ})f, \phi\rangle \overline{\langle \pi(\lambda^{\circ}), \gamma\rangle}.
\end{aligned}$$

By using the Poisson summation formula (proposition 2.3.1), we arrive at (2.3). □

2.4 Representations of the Gabor Frame Operator

Two very useful representations of a Gabor frame operator $S_{g,h,\Lambda}$ are the Walnut and the Janssen representations. We shall first derive the Walnut representation for separable lattices $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$, where $\alpha, \beta \in \mathbb{R}$, and then show how it leads to the Janssen representation. Then we generalise the Janssen representation to general lattices in $\mathbb{R}^2$, and show how the Walnut representation is obtained from this.
We start looking at the Gabor frame operator. For $\lambda \in \alpha\mathbb{Z} \times \beta\mathbb{Z}$, it can be written

$$\begin{aligned}
S_{g,\gamma,\Lambda} f(t) &= \sum_{k,n\in\mathbb{Z}} \langle f, \pi(\alpha k, \beta n) g\rangle \pi(\alpha k, \beta n)\gamma(t) \\
&= \sum_{k,n\in\mathbb{Z}} \left(\int_{\mathbb{R}} f(\tau) e^{-2\pi i \beta n \tau} g(\tau - \alpha k) d\tau \right) e^{-2\pi i \beta n t} \gamma(t - \alpha k) \\
&= \sum_{k,n\in\mathbb{Z}} \mathcal{F}\big[f(\tau) g(\tau - \alpha k)\big](\beta n) e^{-2\pi i \beta n t} \gamma(t - \alpha k) \\
&= \sum_{k\in\mathbb{Z}} \gamma(t - \alpha k) \left[\sum_{n\in\mathbb{Z}} \mathcal{F}\big[f(\tau) g(\tau - \alpha k)\big](\beta n) e^{-2\pi i \beta n t} \right]
\end{aligned}$$

Now, using the Poisson summation formula, lemma 2.3.2, on the bracket with $\phi(\tau) = f(\tau) g(\tau - \alpha k)$ and $T = 1/\beta$, we get

$$\begin{aligned}
S_{g,\gamma,\Lambda} f(t) &= \beta^{-1} \sum_{k\in\mathbb{Z}} \gamma(t - \alpha k) \sum_{n\in\mathbb{Z}} f(t + \beta^{-1} n) g(t + \beta^{-1} n - \alpha k) \\
&= \beta^{-1} \sum_{n\in\mathbb{Z}} \left[\sum_{k\in\mathbb{Z}} \gamma(t - \alpha k) \overline{g(t + \beta^{-1} n - \alpha k)} \right] f(t + \beta^{-1} n)
\end{aligned}$$

Letting

$$G_n(t) = \sum_{k\in\mathbb{Z}} \gamma(t - \alpha k) \overline{g(t + \beta^{-1} n - \alpha k)},$$

we see that $S_{g,\gamma,\Lambda}f$ can be expressed in terms of time-shifts of f:

$$S_{g,\gamma,\Lambda}f(t) = \beta^{-1} \sum_{n\in\mathbb{Z}} G_n(t) f(t - \beta^{-1}n)$$

This is the *Walnut representation*, which was first described by David Walnut [24]. Note that the function G_n is not dependent on f, only on the dual atoms g and γ. Thus, the Walnut representation allows us to write Sf as a linear combination of time shifts of f only. Note also that G_n is periodic with period α. Consequently, the Fourier series representation is

$$G_n(t) = \sum_{l\in\mathbb{Z}} \hat{G_n}(l) e^{2\pi i l t/\alpha},$$

with Fourier coefficients

$$\begin{aligned} \hat{G_n}(l) &= \alpha^{-1} \int_0^{\alpha} G_n(t) e^{-2\pi i l t/\alpha} dt \\ &= \alpha^{-1} \int_{-\infty}^{\infty} \overline{g(t - \beta^{-1}n)} \gamma(t) e^{-2\pi i l t/\alpha} dt \\ &= \alpha^{-1} \langle \gamma, \pi(\beta^{-1}n, \alpha^{-1}l) g \rangle. \end{aligned}$$

This leads us to the *Janssen representation* for separable lattices:

$$S_{g,\gamma,\Lambda}f = (\alpha\beta)^{-1} \sum_{n,l\in\mathbb{Z}} \langle \gamma, \pi(\beta^{-1}n, \alpha^{-1}l) g \rangle \pi(\beta^{-1}n, \alpha^{-1}l) f, \tag{2.4}$$

The Janssen representation was discovered by both Janssen and Daubechies, Landau and Landau [13, 2]. Here a frame operator associated to a frame with lattice $\alpha\mathbb{Z} \times \beta\mathbb{Z}$ is represented by a similar operator associated to a frame with lattice $\beta^{-1}\mathbb{Z} \times \alpha^{-1}\mathbb{Z}$. The Janssen representation is a very useful tool in Gabor analysis. Without it, one would have to compute the inner product of every signal function f, but this representation allows us to write the frame operator as a linear combination of time-frequency shifts of the signal function.
Having introduced the adjoint lattice, we see from the Janssen representation (2.4) for separable lattices that this is simply the adjoint of the separable lattice. In fact, we can generalise (2.4) to hold for all lattices and their adjoints:

Proposition 2.4.1 (Janssen Representation)**.** *Let g and γ be functions in M^1_v. Then for any $f \in L^2(\mathbb{R})$,*

$$S_{g,\gamma,\Lambda}f = \mathrm{vol}(\Lambda)^{-1} \sum_{\lambda^\circ\in\Lambda^\circ} \langle \gamma, \pi(\lambda^\circ) g \rangle \pi(\lambda^\circ) f. \tag{2.5}$$

Proof. Take $\phi \in L^2(\mathbb{R})$. By the Fundamental Identity of Gabor Analysis, we get

$$\begin{aligned} S_{g,\gamma,\Lambda} f &= \left\langle \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda) g\rangle \pi(\lambda)\gamma, \phi \right\rangle \\ &= \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda) g\rangle \overline{\langle \phi, \pi(\lambda)\gamma\rangle} \\ &= \mathrm{vol}(\Lambda)^{-1} \sum_{\lambda^\circ\in\Lambda^\circ} \langle \gamma, \pi(\lambda^\circ) g\rangle \langle \pi(\lambda^\circ) f, \phi\rangle \\ &= \left\langle \mathrm{vol}(\Lambda)^{-1} \sum_{\lambda^\circ\in\Lambda^\circ} \langle \gamma, \pi(\lambda^\circ) g\rangle \pi(\lambda^\circ) f, \phi \right\rangle. \end{aligned}$$

Since this holds for all $\phi \in L^2(\mathbb{R})$, the first components of the respective inner product must be equal, and so we obtain the desired conclusion. □

We will now show the transition back into the Walnut representation. Let $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$ be a separable lattice, so that $\Lambda^\circ = \beta^{-1}\mathbb{Z} \times \alpha^{-1}\mathbb{Z}$ is its adjoint lattice. For $f \in L^2(\mathbb{R})$, the Janssen representation gives:

$$\begin{aligned} S_{g,\gamma,\Lambda} f(t) &= (\alpha\beta)^{-1} \sum_{n,l\in\mathbb{Z}} \langle \gamma, \pi(\beta^{-1}n, \alpha^{-1}l) g\rangle e^{2\pi i l t/\alpha} f(t-\beta^{-1}n) \\ &= \beta^{-1} \sum_{n\in\mathbb{Z}} \left[\alpha^{-1} \sum_{l\in\mathbb{Z}} \langle \gamma, \pi(\beta^{-1}n, \alpha^{-1}l) g\rangle e^{2\pi i l t/\alpha} \right] f(t-\beta^{-1}n), \end{aligned}$$

where we recognise the bracket as $G_n(t) = \sum_{l\in\mathbb{Z}} \hat{G}_n(l) e^{2\pi i l t/\alpha}$, as defined earlier. This shows that the Walnut representation is in essence a way of writing the more general Janssen representation in the separable case.

2.5 The Wexler-Raz Biorthogonality Condition

The connection of a lattice with its adjoint lattice can also be used to characterise dual atoms for a given lattice Λ. The following result describes this relationship.

Proposition 2.5.1 (Wexler-Raz). *Let Λ be a lattice in $\mathbb{R}^2$ with adjoint lattice Λ°, and g and γ be functions in $L^2(\mathbb{R})$ generating Gabor frames $\mathcal{G}(g, \Lambda)$ and*

$\mathcal{G}(\gamma, \Lambda)$. *Then these are dual frames frames, i.e. g and γ are dual atoms, if and only if*

$$\langle \gamma, \pi(\lambda^\circ) g\rangle = \begin{cases} \mathrm{vol}(\Lambda) & \text{if } \lambda^\circ = 0 \\ 0 & \text{otherwise} \end{cases}. \tag{2.6}$$

Proof. We first prove that (2.6) implies duality of g and γ. Take any functions f and ϕ in $L^2(\mathbb{R})$. Then

$$\begin{aligned} \langle S_{\gamma,g} f, \phi\rangle &= \left\langle \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda) g\rangle \pi(\lambda)\gamma, \phi \right\rangle \\ &= \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda) g\rangle \overline{\langle \phi, \pi(\lambda)\gamma\rangle} \\ &= \mathrm{vol}(\Lambda)^{-1} \sum_{\lambda^\circ\in\Lambda^\circ} \langle \gamma, \pi(\lambda^\circ) g\rangle \overline{\langle \phi, \pi(\lambda^\circ) f\rangle} \qquad \text{(by the FIGA)} \\ &= \overline{\langle \phi, f\rangle} = \langle f, \phi\rangle, \end{aligned}$$

where the last line comes from the assumption of equation (2.6). Thus, $S_{\gamma,g} f = f$ for all $f \in L^2(\mathbb{R})$, so g and γ are dual atoms.

Conversely, suppose g and γ are dual atoms, so that $S_{\gamma,g} f = f$ for all $f \in L^2(\mathbb{R})$. Take functions f and ϕ in $L^2(\mathbb{R})$. Then, similar to the above, we get

$$\begin{aligned} \langle f, \phi\rangle &= \langle S_{\gamma,g} f, \phi\rangle \\ &= \mathrm{vol}(\Lambda)^{-1} \sum_{\lambda^\circ\in\Lambda^\circ} \langle \gamma, \pi(\lambda^\circ) g\rangle \overline{\langle \phi, \pi(\lambda^\circ) f\rangle} \\ &= \mathrm{vol}(\Lambda)^{-1} \left\langle \sum_{\lambda^\circ\in\Lambda^\circ} \langle \gamma, \pi(\lambda^\circ) g\rangle \pi(\lambda^\circ) f, \phi \right\rangle. \end{aligned}$$

Since this holds for all $\phi \in L^2(\mathbb{R})$, we get

$$\sum_{\lambda^\circ\in\Lambda^\circ} \langle \gamma, \pi(\lambda^\circ) g\rangle \pi(\lambda^\circ) f = \mathrm{vol}(\Lambda) f \quad \forall f \in L^2(\mathbb{R}).$$

Writing in operator form, and using that $\pi(0) = I$, we get

$$\sum_{\lambda^\circ\in\Lambda^\circ,\, \lambda^\circ\neq 0} \langle \gamma, \pi(\lambda^\circ) g\rangle \pi(\lambda^\circ) + \langle \gamma, g\rangle - \mathrm{vol}(\Lambda) = 0.$$

Now, π being a faithful representation, all the terms in the left sum must equal zero, so

$$\langle \gamma, \pi(\lambda^\circ) g \rangle = \begin{cases} \mathrm{vol}(\Lambda) & \text{if } \lambda^\circ = 0 \\ 0 & \text{otherwise} \end{cases}.$$

□

2.6 Frame-Generating Atoms and Lattices

We have introduced and described properties of Gabor frames, but we do not yet know anything about which Gabor systems $\mathcal{G}(g, \Lambda) = \{\pi(\lambda) g\}_{\lambda \in \Lambda}$ are indeed frames, if any at all. More precisely, a description of which atoms g and lattices Λ can produce Gabor frames, is needed. This has proved, however, a difficult task, and contemporary research is being done on such characterisations.[10] Here, we will give a few relevant concepts and results related to the matter.

Theorem 2.6.1 (Balian-Low)**.** *Let g be a function in $L^2(\mathbb{R})$ such that both $tg(t)$ and $\omega \hat{g}(\omega)$ are also in $L^2(\mathbb{R})$. Then $\mathcal{G}(g, \Lambda)$ cannot be an orthonormal basis for $L^2(\mathbb{R})$.*

This theorem describes a limit as to how "well-behaved" functions we can use as a Gabor frame atom if we wish to have the often-wanted property of being an orthonormal basis. It tells us that we cannot use functions which are well-concentrated in both time and in frequency.

One way of systematising a characterisation of Gabor frames is, given an atom g, to ask which lattices $\Lambda \subset \mathbb{R}^2$ would make $\mathcal{G}(g, \Lambda)$ a frame. We therefore define the *frame set* for a function g as follows:

$$\mathcal{F}(g) = \left\{ \Lambda \subset \mathbb{R}^2 \text{ lattice } \mid \mathcal{G}(g, \Lambda) \text{ is a Gabor frame} \right\}.$$

The following result gives a necessary condition for a lattice to be in $\mathcal{F}(g)$ for a function $g \in L^2(\mathbb{R})$.

Proposition 2.6.2. *Let g be a function in $L^2(\mathbb{R})$. If $\mathcal{G}(g, \Lambda)$ is a frame, then* $\mathrm{vol}(\Lambda) \leq 1$.

A natural example to consider is the Gaussian $g_0(t) = e^{-t^2}$, for which we have $\mathcal{F}(g_0) = \{\Lambda \subset \mathbb{R}^2 \text{ lattice } \mid \mathrm{vol}(\Lambda) < 1\}$, for which the proof g_0 can be found in [17].

2.7 Multi-Window Gabor Frames

Given a window $g \in L^2(\mathbb{R})$ and a lattice $\Lambda \subset \mathbb{R}^2$, we have seen that the corresponding Gabor system $\{\pi(\lambda)g, \Lambda\}$ is sometimes a Gabor frame, and sometimes not. In some of the latter cases, however, all we need to do to obtain a frame is to add time-frequency shifts of some additional windows to the system.[25] We therefore introduce the following generalisation of Gabor frames.

Definition 2.7.1. A *multi-window Gabor frame* is a frame

$$\{\pi(\lambda)g_j \mid \lambda \in \Lambda, 1 \leq j \leq n\}$$

for $L^2(\mathbb{R})$ consisting of time-frequency shifts of a finite number of windows $g_1, \cdots g_n \in L^2(\mathbb{R})$ over a lattice Λ in $\mathbb{R}^2$.

The explicit frame condition for such a set is

$$A||f||^2 \leq \sum_{j=1}^{n} \sum_{\lambda \in \Lambda} |\langle f, \pi(\lambda)g_j \rangle|^2 \leq B||f||^2 \quad \forall f \in L^2(\mathbb{R})$$

for some $A, B > 0$, and the (general) Gabor frame operator for such a frame is given by

$$S_{\{g_j\},\{\gamma_j\},\Lambda} f = \sum_{j=1}^{n} \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g_j \rangle \pi(\lambda)\gamma_j,$$

where $\gamma_1, \cdots, \gamma_n \in L^2(\mathbb{R})$ and $f \in L^2(\mathbb{R})$. The concept of allowing more than one window to make a Gabor frame will show its importance later.

Chapter 3

Orthogonal Frequency-Division Multiplexing

Orthogonal frequency-division multiplexing (OFDM) is a method for the process of transmitting data through a time-domain signal. Examples of appliance of this technique are wi-fi and 4G signals. The key principle in general frequency-division multiplexing is dividing the frequency spectrum into usually non-overlapping divisions, and letting each division carry a part of the signal. Among the advantages is a reduced risk of losing an entire signal, since the data is spread into several *subcarriers*, or *subchannels*. When using OFDM, we let the subcarriers be orthogonal to each other. This allows them to transmit on overlapping parts of the frequency spectrum without interference.

In this chapter we describe how a transmitted signal is perturbed by the channel through which it propagates, and introduce some terms used by engineers when constructing a signal, transmitting it, and retrieving data.

3.1 Multipath Propagation and Doppler Spread

In mobile wireless communication applications, suppose a signal $x(t)$ is transmitted via a channel, modelled as an operator H, and is received as the perturbed signal $y(t) = (Hx)(t)$. Typically, there are two major ways in which such a signal is perturbed by this channel.

First, nearby reflectors causes there to be several paths for the signal. This so-called *multipath propagation* leads to time dispersion; the signal pulses arrive several times, spread out in time. Since the data symbols are separated by their placement in time, this may cause *intersymbol interference*, or *ISI*.

Second, any relative motion between transmitter, receiver and reflectors leads to *Doppler spread*, a spread of modulated versions of the transmitted signal. Because different subchannels are separated by frequency and confined to different frequency intervals, the Doppler spread may result in *interchannel interference*, or *ICI*. In general, the input-output relation of a wireless signal can be represented as

$$y(t) = (Hx)(t) = \int_{-\infty}^{\infty} h_t(s)x(t-s)\,ds,$$

where, in contrast to a time-invariant channel, the impulse response $h_t(s) = h(t,s)$ is dependent on time and can be considered a function of two variables. By rearranging the variables, we can write

$$(Hx)(t) = \int_{-\infty}^{\infty} h(t,t-s)x(s)\,ds. \tag{3.1}$$

By letting

$$\sigma(t,\omega) = \mathcal{F}_2 h(t,\omega) := \int_{-\infty}^{\infty} h(t,s)e^{-2\pi i\omega s}\,ds$$

be the regular Fourier transform of h in the second variable, and using the convolution theorem for Fourier transforms, we can rewrite (3.1) as

$$(Hx)(t) = \int_{-\infty}^{\infty} \sigma(t,\omega)\hat{x}(\omega)e^{2\pi i\omega t}\,d\omega. \tag{3.2}$$

Since H depends on the so-called *Kohn-Nirenberg symbol* σ, we write $H = H_\sigma$, and we name the mapping $\sigma \mapsto H_\sigma$ the *Kohn-Nirenberg correspondence*. To emphasise the time-frequency aspect of the operator, we use (3.2) and rewrite the operator in the following way:

$$\begin{aligned} H_\sigma x(t) &= \iint_{\mathbb{R}^2} \sigma(t,\omega)e^{2\pi i(t-y)\omega}x(y)\,d\omega\,dy \\ &= \iint_{\mathbb{R}^2} \hat{\sigma}(\eta,y-t)e^{2\pi i\eta t}x(y)\,dy\,d\eta \\ &= \iint_{\mathbb{R}^2} \hat{\sigma}(\eta,u)e^{2\pi i\eta t}x(t+u)\,dy\,d\eta \\ &= \iint_{\mathbb{R}^2} \hat{\sigma}(\eta,u)M_\eta T_{-u}x(t)\,du\,d\eta. \end{aligned} \tag{3.3}$$

In this representation, it becomes clear how the channel acts on the signal as a continuous superposition of time-frequency shifts. The function $\hat{\sigma}$ is called the *spreading function* of the channel, and describes how the shift of the signal is weighted.

Definition 3.1.1. A channel operator H_σ is said to be *underspread* if its spreading function $\hat{\sigma}$ is compactly supported in $[-\tau_{\max}, \tau_{\max}] \times [-\omega_{\max}, \omega_{\max}]$ with $\tau_{\max} \cdot \omega_{\max} < 1$.

Often, though $\hat{\sigma}$ is not truly compactly supported in u, one may assume approximate compact support due to its rapid decay.

To study the channel operator, we need to establish some properties of the spreading function $\hat{\sigma}$. In particular, it would be useful to identify boundaries with respect to both variables. We therefore look at the properties of a realistic modelling of a mobile wireless signal.
The energy loss of a signal is more severe in a realistic environment than the theoretically achievable $|h(t,s)| \propto 1/s^2$. In fact, a common way to model it is by an exponential decay, i.e.

$$|h(t,s)| \leq ce^{-a|s|}$$

for some positive constants a and c. Since $\hat{\sigma} = \mathcal{F}_1 \mathcal{I} h$, where $\mathcal{F}_1$ is the regular Fourier transform in the first variable and $\mathcal{I}f(x,y) = f(-x,y)$, the above implies $|\hat{\sigma}(\eta, u)| \leq ce^{-a|u|}$.
The Doppler shift ω_d is given by

$$\omega_d = \frac{v}{\lambda} \cos\phi,$$

where v is the relative velocity of the object, ϕ is the angle between the direction of movement of the object and the direction of the signal wave, and λ is the wavelength of the signal. This implies that $\hat{\sigma}$ is compactly supported in $[-\omega_{\max}, \omega_{\max}]$ with respect to its first variable, where $\omega_{\max} = v/\lambda$.
As a conclusion, the symbol σ of a mobile wireless channel operator H_σ must satisfy

$$\hat{\sigma}(\eta, u) = 0 \quad \text{for} \quad |\eta| > \omega_{\max} \quad \text{and} \quad |\hat{\sigma}(\eta, u)| \leq ce^{-a|u|} \tag{3.4}$$

for some $c, a > 0$.

3.2 Multicarrier Communication Systems

When constructing the transmit signal $x(t)$ in multicarrier communication systems, one divides the information to be transmitted into several signals $g_l, l = 1 \cdots N$, or *subchannels*, of which the transmitted signal is a superposition. Commonly, as in frequency-division multiplexing, the different subchannels are separated by frequency: each has its frequency interval, where

the other channels will hopefully not interfere. A typical set of subchannels is $\{g_l(t) = g(t)e^{2\pi ilbt}, l = 1...N\}$, where $g \in L^2(\mathbb{R})$ is a fixed pulse and $0 < b \in \mathbb{R}$ is called the *carrier separation* of the signal construction.
The data $\{c_n\}_{n\in\mathbb{Z}}$ which is to be transmitted is rearranged and divided into data blocks (indexed by $k \in \mathbb{Z}$), which are being transmitted subsequently with a time delay, or *symbol period* of $0 < a \in \mathbb{R}$. Each data block is transmitted as a superposition of the already mentioned frequency-divided subchannel signals $g_l(t)$. We shall therefore write the data as $\{c_{k,l}\}$, and the resulting transmit signal is built in the following way:

$$x(t) = \sum_{k\in\mathbb{Z}} \sum_{l=1}^{N} c_{k,l} g(t - ka)e^{2\pi ilbt} = \sum_{k\in\mathbb{Z}} \sum_{l=1}^{N} c_{k,l} M_{lb} T_{ka} g(t).$$

We recognise the discrete superposition of time-frequency shifts from Gabor theory. When using OFDM, the time-frequency shifted versions of the basic pulse g, denoted by $g_{k,l}$, are mutually orthogonal, and we will assume them to be mutually orthonormal. The received signal $y(t) = H_\sigma x(t)$ needs to be analysed to recover the original data. The following discrete data set is received:

$$d_{k,l} = \langle H_\sigma x, g_{k,l}\rangle = \sum_{k',l'} c_{k,l} \langle H_\sigma g_{k',l'}, g_{k,l}\rangle.$$

Write $c = \{c_{k,l}\}_{k,l\in\mathbb{Z}}$, $d = \{d_{k,l}\}_{k,l\in\mathbb{Z}}$, and let $R = R(\sigma, g)$ be the matrix with entries $\langle H_\sigma g_{k',l'}, g_{k,l}\rangle$ for $k, l, k', l' \in \mathbb{Z}$. This is called the *channel matrix* of a communication channel, and depends on the symbol σ of the channel operator and the fixed pulse g. The task of recovering the original information c is now reduced to solving the linear system

$$Rc = d.$$

Consequently, the structure of R is highly relevant for the recovery of the data; a "nice" matrix would allow us to solve the linear system more efficiently. In particular, we would like the matrix to be diagonal (or "almost diagonal", which will soon be specified). Since the channel is modelled by time-frequency shifts of the original transmit signal, it is natural to use time-frequency shifts of a simple pulse to make a basis for our purpose. Indeed, the following important theorem shows that Gabor frames plays a role in regards to the channel matrix. It is due to Gröchenig, and a proof can be found in [8].

Theorem 3.2.1 (Almost diagonalisation)**.** *Let* Λ *be a lattice in* $\mathbb{R}^2$ *and let* $g \in M_v^1(\mathbb{R})$ *such that* $\mathcal{G}(g, \Lambda)$ *is a Gabor frame. Let* j *denote the rotation*

$j(u,\eta) = (\eta, u)$ in the time-frequency plane. Then $\sigma \in M^{\infty,1}_{v\circ j^{-1}}$ if and only if there exists $h \in \ell^1_v(\Lambda)$ such that

$$|R(\sigma, g)_{\lambda,\mu}| \leq h(\lambda - \mu) \tag{3.5}$$

for all $\lambda, \mu \in \Lambda$. Furthermore, $\inf||h||_{\ell^1_v}$ taken over all $h \in \ell^1_v(\Lambda)$ satisfying (3.5) is an equivalent norm on $M^{\infty,1}_{v\circ j^{-1}}$.

This theorem states that although $R(\sigma, g)$ is not truly a diagonal matrix, the decay away from the diagonal is significant, allowing us to increase computational efficiency by approximating the channel matrix by only considering the super- and subdiagonals.

3.3 Pseudodifferential Operators

The operator H_σ is an example of a *pseudodifferential operator*. In general, given a *symbol* σ, a pseudodifferential operator is an operator acting on $L^2(\mathbb{R})$ as

$$H_\sigma f(t) = \int_{\mathbb{R}} \sigma(t,\omega)\hat{f}(\omega)e^{2\pi it\omega}d\omega.$$

As seen in (3.3), the operator can be rewritten using the translation operator T_x and the modulation operator M_ω:

$$H_\sigma f(t) = \iint \hat{\sigma}(\eta, u)M_\eta T_{-u}x(t)\, du\, d\eta.$$

Due to the time-frequency shifts arising in the channel operator, we shall use Gabor analysis, and introduce the notion of a *Gabor multiplier*, as done in [9], to consider the operator further. Assume that we have $g \in L^2(\mathbb{R})$ and a lattice $\Lambda \subset \mathbb{R}^2$ such that $\mathcal{G}(g, \Lambda)$ is a Parseval frame. Recall that Parseval frames admit expansions to elements of the Hilbert space. We shall expand both f and $H_\sigma f$ with respect to this Gabor frame. This gives

$$f = \sum_{\mu\in\Lambda} \langle f, \pi(\mu)g\rangle \pi(\mu)g$$

and

$$\begin{aligned}
H_\sigma f &= \sum_{\lambda\in\Lambda} \langle H_\sigma f, \pi(\lambda)g\rangle \pi(\lambda)g \\
&= \sum_{\lambda\in\Lambda} \left\langle H_\sigma \sum_{\mu\in\Lambda} \langle f, \pi(\mu)g\rangle \pi(\mu)g, \pi(\lambda)g \right\rangle \pi(\lambda)g \\
&= \sum_{\lambda\in\Lambda}\sum_{\mu\in\Lambda} \langle H_\sigma \pi(\mu)g, \pi(\lambda)g\rangle \langle f, \pi(\mu)g\rangle \pi(\lambda)g \\
&= \sum_{\nu\in\Lambda}\sum_{\mu\in\Lambda} \langle H_\sigma \pi(\mu)g, \pi(\mu+\nu)g\rangle \langle f, \pi(\mu)g\rangle \pi(\mu+\nu)g \qquad (\text{via } \lambda = \mu+\nu) \\
&= \sum_{\nu\in\Lambda} \pi(\nu) \left(\sum_{\mu\in\Lambda} \langle H_\sigma \pi(\mu)g, \pi(\mu+\nu)g\rangle e^{2\pi i \nu_1\mu_2} \langle f, \pi(\mu)g\rangle \pi(\mu)g \right),
\end{aligned}$$

where $\mu = (\mu_1, \mu_2)$ and $\nu = (\nu_1, \nu_2)$. By defining the *sequence of symbols* $a(\nu,\mu) = \langle H_\sigma \pi(\mu)g, \pi(\mu+\nu)g\rangle e^{2\pi i \nu_1\mu_2}$ for $\nu, \mu \in \Lambda$, the parenthesis can be written

$$M_a = \sum_{\mu\in\Lambda} a(\nu,\mu) \langle f, \pi(\mu)g\rangle \pi(\mu)g.$$

This is what is called a *Gabor multiplier with symbol a.* We see that the pseudodifferential operator can be written as a sum of shifted Gabor multipliers:

$$H_\sigma = \sum_{\nu\in\Lambda} \pi(\nu) M_a. \tag{3.6}$$

Lemma 3.3.1. *Assume $\mathcal{G}(g,\Lambda)$ is a Parseval frame and $g \in M_v^1(\mathbb{R})$. If the sequence of symbols a satisfies*

$$\sum_{\nu\in\Lambda} ||a(\nu,\cdot)||_\infty v(\nu) = \sum_{\nu\in\Lambda} \sup_{\mu\in\Lambda} |a(\nu,\mu)| v(\nu) < \infty,$$

then the sum (3.6) *of shifted Gabor multipliers converges in the operator norm on $M_m^{p,q}$ for every $1 \le p, q \le \infty$ and every v-moderate weight m.*

Having established that H_σ can be represented through infinitely many Gabor multipliers, it is natural to ask whether, and how well, it can be approximated by truncating the sum (3.6). To do this, we first need a result on what H_σ looks like, assuming the well-behavedness argued for in the previous section.

Theorem 3.3.2. *Let H_σ be a channel operator satisfying* (3.4)*. Then $\sigma \in M_w^{\infty,1}(\mathbb{R}^2)$, where $w(t,\omega) = e^{|(t,\omega)|^\alpha}$ with $\alpha < 1$.*

Now, for such an operator, we have the following approximation result, due to Gröchenig [9]:

Theorem 3.3.3. *If $g \in M_v^1$ and $\sigma \in M_v^{\infty,1}$, then*

$$E_N \leq C||\sigma||_{M_v^{\infty,1}} \sup_{|\nu|>N} v(\nu)^{-1},$$

where E_N is the truncation error when truncating at $N \in \mathbb{N}$, given by

$$E_N := ||H_\sigma - \sum_{|\nu|\leq N} \pi(\nu) M_a||_{M^{p,q} \to M^{p,q}}.$$

Chapter 4

Noncommutative Tori

We have seen that a channel operator can be modelled by a continuous superposition of time-frequency shifts. In this chapter, we shall focus on a countable superpositions of such shifts. It links more abstract mathematics to some of the applied mathematics we have seen in the preceding ones, and will provide a basis for the important chapter 5 by considering the structure of some sets of operators recognisable from Gabor analysis.

4.1 Twisted Group C^*-algebras

Recall the *translation* and *modulation* operators on $L^2(\mathbb{R})$, given by

$$T_x f(t) = f(t-x),$$
$$M_\omega f(t) = e^{2\pi i\omega t} f(t),$$

respectively, and the way we write them together as $\pi(\lambda) = M_\omega T_x$ for $\lambda = (x,\omega) \in \Lambda$ for a lattice $\Lambda \subset \mathbb{R}^2$. Here we shall take Λ to be a separable lattice $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$, where $\alpha, \beta \in \mathbb{R}$.

Definition 4.1.1. For $\theta = \alpha\beta = \text{vol}(\Lambda)$, define the set

$$A_\theta^1 := \left\{ a = \sum_{\lambda\in\Lambda} a_\lambda \pi(\lambda),\ (a_\lambda) \in \ell^1(\Lambda) \right\} \subset \mathcal{B}(L^2(\mathbb{R})).$$

Define the norm

$$||a||_{A_\theta^1} := ||a||_{\ell^1(\Lambda)} = \sum_{\lambda\in\Lambda} |a_\lambda|.$$

Let multiplication be given by

$$\left(\sum_{\lambda\in\Lambda} a_\lambda \pi(\lambda)\right)\left(\sum_{\lambda\in\Lambda} b_\lambda \pi(\lambda)\right) = \sum_{\lambda\in\Lambda} (a\natural b)_\lambda \pi(\lambda),$$

where the sequence $(a\natural b)_\lambda$ is the *twisted convolution* of the sequences (a_λ) and (b_λ), defined by

$$(a\natural b)_\lambda := \sum_{\mu\in\Lambda} a_\mu b_{\lambda-\mu} e^{-2\pi i\theta(x-u)\eta}, \tag{4.1}$$

for $\lambda = (x,\omega)$ and $\mu = (u,\eta)$ are in Λ, and involution as in $\mathcal{B}(L^2(\mathbb{R}))$.

Lemma 4.1.1. *If* $\sum_{\lambda\in\Lambda} a_\lambda \pi(\lambda) = 0$*, then* $a_\lambda = 0$ *for all* $\lambda \in \Lambda$*.*

Proof. Let $g, \gamma \in S_0(\mathbb{R})$ and $\mu = (u,\eta) \in \mathbb{R}^2$. Then $\pi(\mu)g, \pi(\mu)\gamma \in S_0(\mathbb{R})$, so by assumption, we have

$$\sum_{\lambda\in\Lambda} a_\lambda \langle \pi(\lambda)\pi(\mu)g, \pi(\mu)\gamma\rangle = 0$$

Now, $\pi(\mu)^*\pi(\lambda)\pi(\mu) = e^{2\pi i(u\omega - x\eta)}\pi(\lambda)$, so

$$\sum_{\lambda\in\Lambda} a_\lambda \langle \pi(\lambda)g, \gamma\rangle e^{2\pi i(u\omega - x\eta)} = 0$$

for all $g, \gamma \in S_0(\mathbb{R}), \mu \in \mathbb{R}^2$. We recognise this series as an absolutely convergent Fourier series on $\mathbb{R}^2/\Lambda$. By uniqueness of Fourier series, we must have

$$a_\lambda \langle \pi(\lambda)g, \gamma\rangle = 0$$

for all $\lambda \in \Lambda$, so $a_\lambda = 0$ for all λ. □

A^1_θ is an involutive Banach algebra, and is isomorphic to the *twisted group algebra* $\ell^1(\alpha\mathbb{Z} \times \beta\mathbb{Z}, c)$ of ℓ^1-sequences over the lattice Λ, equipped with the twisted convolution (4.1) with associated 2-cocycle $c((x,\omega),(u,\eta)) = e^{2\pi i x\eta}$.

Definition 4.1.2. The *enveloping* C^**-algebra* of an involutive Banach algebra B is the completion of B in the norm

$$||b|| = \sup_\rho ||\rho(b)||,$$

where the supremum is taken over all involutive representations ρ of B - i.e. all $*$-homomorphisms from B to $\mathcal{B}(H_\rho)$ for some Hilbert space H_ρ.

The enveloping C^*-algebra of A_θ^1 is A_θ, a C^*-algebra generated by the two unitaries M_β and T_α in $\mathcal{B}(L^2(\mathbb{R}))$. We shall call this the *twisted group C^*-algebra*. We shall see that the structure of A_θ is known by a different name in the abstract setting of C^*-algebras:

Definition 4.1.3. For $\theta \in \mathbb{R}$, the *noncommutative torus* $\mathcal{A}_\theta$ with parameter θ is the universal algebra generated by two unitaries U and V such that

$$UV = e^{2\pi i\theta} VU.$$

Proposition 4.1.2. *The twisted group C^*-algebra A_θ is isomorphic to the noncommutative torus $\mathcal{A}_\theta$ for $\theta = \alpha\beta$.*

Proof. Consider the homomorphism $\kappa : \mathcal{A}_\theta \to A_\theta$ defined by $U \mapsto M_\beta, V \mapsto T_\alpha$. From lemma 1.1.1 we have that

$$\kappa(U)\kappa(V) = M_\beta T_\alpha = e^{2\pi i\alpha\beta} T_\alpha M_\beta = e^{2\pi i\theta}\kappa(V)\kappa(U),$$

so the commutation relation is preserved. Since A_θ is generated by M_β and T_α, it follows that κ is surjective. Now, suppose $a = \sum_{m,n\in\mathbb{N}} a_{m,n}U^nV^m$ and $b = \sum_{k,l\in\mathbb{N}} b_{k,l}U^lV^k$ are elements in $\mathcal{A}_\theta$ such that

$$\kappa\left(a\right) = \kappa\left(b\right).$$

Then

$$\sum_{m,n\in\mathbb{N}} (a_{m,n} - b_{m,n})\pi(\alpha m, \beta n) = \sum_{m,n\in\mathbb{N}} (a_{m,n} - b_{m,n})\kappa(U)^n\kappa(V)^m = 0.$$

By lemma 4.1.1, $a_{m,n} = b_{m,n}$ for all $m, n \in \mathbb{Z}$, so $a = b$. Thus, κ is injective, and hence an isomorphism between $\mathcal{A}_\theta$ and A_θ. □

Note that this isomorphism shows that A_θ is fully determined by the volume θ of the underlying lattice Λ, as all twisted group C^*-algebras of lattices with volume θ are isomorphic to the same universal C^*-algebra.
The name "noncommutative torus" is often used about the representation A_θ in $\mathcal{B}(L^2(\mathbb{R}))$ as well as the abstract, universal algebra.

Proposition 4.1.3. *Any noncommutative torus is isomorphic to a noncommutative torus with parameter $\theta \in [0, 1/2]$.*

Proof. Take $n \in \mathbb{Z}$. Then $\mathcal{A}_{\theta+n}$ is the universal C^*-algebra generated by two unitaries U and V such that $UV = e^{2\pi i(\theta n)}VU = e^{2\pi i\theta}VU$. This coincides with the definition of $\theta \in \mathbb{R}$, so we have that $\mathcal{A}_\theta \cong \mathcal{A}_{\theta+n}$ for all integers n.

Furthermore, $\mathcal{A}_{-\theta}$ is the universal C^*-algebra generated by two unitaries U and V such that $UV = e^{-2\pi i\theta}VU$. By considering the map $U \mapsto V, V \mapsto U$, we see that this is an isomorphism into the universal C^*-algebra generated by two unitaries such that $VU = e^{-2\pi i\theta}UV$, and we recognise the definition of $\mathcal{A}_\theta$.
Thus, all noncommutative tori are isomorphic to some noncommutative torus with parameter $\theta \in [0, 1/2]$. □

It is also interesting to consider other versions of A^1_θ, by modifying the criteria for the sequence $(a_\lambda)_{\lambda\in\Lambda}$ of coefficients. For $s \geq 0$, define

$$A^{1,s}_\theta := \left\{ a = \sum_{\lambda\in\Lambda} a_\lambda \pi(\lambda),\ (a_\lambda) \in \ell^1_s(\Lambda) \right\},$$

where $\ell^1_s(\Lambda)$ is a *weighted ℓ^1 space*, in this case the space of sequences $a = (a_\lambda)_{\lambda\in\Lambda}$ satisfying

$$||a||_s := \sum_{\lambda\in\Lambda} a_\lambda (1 + |\lambda|^2)^{s/2} < \infty.$$

By taking the intersection of the sequence spaces, we get only the collection of sequences which are bounded by *every* polynomial, so

$$\bigcap_{s\geq 0} \ell^1_s(\Lambda) = \mathcal{S}(\Lambda).$$

where $\mathcal{S}(\Lambda)$ denotes the Schwartz space on Λ. Thus, the similar intersection of algebras becomes

$$\bigcap_{s\geq 0} A^{1,s}_\theta = A^\infty_\theta := \left\{ a = \sum_{\lambda\in\Lambda} a_\lambda \pi(\lambda),\ (a_\lambda) \in \mathcal{S}(\Lambda) \right\}. \tag{4.2}$$

This is also an involutive algebra, albeit not a Banach algebra in general.

4.2 Crossed Products

Another useful representation of the noncommutative torus is a so-called crossed product. In this section, we shall introduce the term, and show isomorphism between one particular crossed product and $\mathcal{A}_\theta$.

Definition 4.2.1. An *action of a discrete group G on a C^*-algebra A* is a group homomorphism $\alpha : G \to \mathrm{Aut}(A)$. The triple (A, G, α) is called a *C^*-dynamical system.*

We write the automorphism $\alpha(g)$ on A as α_g. For a given system (A, G, α), there are several associated involutive algebras. One is the algebra AG of formal sums $\sum_{g\in G} a_g\delta_g$, where the a_g are elements of A, and δ_g can be thought of as the Dirac delta function at the group element g. This algebra comes equipped with algebra operations convolution, given by

$$\left(\sum_{g\in G} a_g\delta_g\right) * \left(\sum_{h\in G} b_h\delta_h\right) := \sum_{g\in G}\sum_{h\in G} a_g\alpha_g(b_{g^{-1}h})\delta_h,$$

and involution, given by

$$(a\delta_g)^* := \alpha_{g^{-1}}(a^*)\delta_{g^{-1}}.$$

We also realise that

$$\left|\left|\sum_G a_g\delta_g\right|\right|_1 := \sum_G ||a_g||_A$$

defines a norm on AG. This is an involutive Banach algebra, but in general not a C^*-algebra. Given a C^*-dynamical system (A, G, α), the enveloping C^*-algebra of AG is called the *crossed product of A by the action α of G*, denoted by $A \rtimes_\alpha G$. Having established the meaning of a crossed product, we shall now look at one particular choice of A and G, and thus arrive at the following central example.

Definition 4.2.2. Let θ be a real number, and define the automorphism $(\alpha_\theta f)(t) = f(t+\theta)$. The *rotation algebra* is the crossed product

$$C(\mathbb{T}) \rtimes_\theta \mathbb{Z}.$$

We refer to [20] for a proof of the following proposition.

Proposition 4.2.1. *The rotation algebra $C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ is isomorphic to the noncommutative torus $\mathcal{A}_\theta^2$.*

We shall consider a representation $\delta_k \mapsto T_{\alpha k}$ of $C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ into $\mathcal{B}(L^2(\mathbb{R}))$, where the elements are sums of the form

$$\sum_{k\in\mathbb{Z}} a_k(t)T_{\alpha k},$$

where the functions $a_k \in C(\mathbb{T})$ are of the form

$$a_k(t) = \sum_{l\in\mathbb{Z}} a_{k,l}e^{2\pi i\beta lt}.$$

Consequently, convolution is given by

$$(a_k(t)T_{\alpha k})^* = \alpha_\theta(a_k^*(t))T_{-\alpha k} = \overline{a_k(t-\theta)}T_{-\alpha k},$$

and multiplication is given by

$$\begin{aligned}\left(\sum_{k\in\mathbb{Z}} a_k(t)T_{\alpha k}\right) * \left(\sum_{m\in\mathbb{Z}} b_m(t)T_{\alpha m}\right) &= \sum_{k,m\in\mathbb{Z}} a_k(t)\alpha_\theta(b_{m-k}(t))T_{\alpha m} \\ &= \sum_{k,m\in\mathbb{Z}} a_k(t)b_{m-k}(t-\theta)T_{\alpha m}.\end{aligned}$$

We shall see that the two realisations of the noncommutative torus will be very useful to describe different properties of the Gabor frame operator.

Chapter 5

Hilbert C^*-modules

In chapter 4 we looked at C^*-algebras of operators acting on functions in $L^2(\mathbb{R})$. In this chapter we let these algebras act on our previously defined function spaces by introducing the abstract structure of a Hilbert C^*-module. We define so-called *Morita equivalence* for C^*-algebras, and we shall see that the added structure allows us to draw links to important aspects of Gabor Analysis.

5.1 Hilbert C^*-modules and Morita Equivalence

Definition 5.1.1. For a unital C^*-algebra A, a *left inner product A-module* is a left A-module $\mathcal{E}$ with an A-valued inner product $\langle \cdot, \cdot \rangle$ such that the following hold for all $x, y, z \in \mathcal{E}$ and $a, b, \in A$:

1. $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$
2. $\langle x, y \rangle^* = \langle y, x \rangle$
3. $\langle x, x \rangle \geq 0$, with equality if and only if $x = 0$.

A *right inner product A-module* is defined likewise, with the exception of being linear in the second argument instead of the first.
A *Hilbert C^*-module over A* or a *Hilbert A-module* is an inner product A-module which is complete with respect to the norm

$$||x||_{\mathcal{E}} := ||\langle x, x \rangle||_A^{1/2}.$$

Definition 5.1.2. A Hilbert C^*-module $\mathcal{E}$ is said to be *full* if $\overline{span}\{\langle x, y \rangle \mid x, y \in \mathcal{E}\} = A$.

Definition 5.1.3. For C^*-algebras A and B, an *equivalence A-B-bimodule* is an A-B-module $\mathcal{E}$ satisfying the following:

1. $\mathcal{E}$ is a full left Hilbert A-module with respect to an inner product ${}_\bullet\langle\cdot,\cdot\rangle$ and a full right Hilbert B-module with respect to an inner product $\langle\cdot,\cdot\rangle_\bullet$.

2. For $x, y \in \mathcal{E}, a \in A$ and $b \in B$,
$$\langle ax, y\rangle_\bullet = \langle x, a^*y\rangle_\bullet$$
and
$${}_\bullet\langle xb, y\rangle = {}_\bullet\langle x, yb^*\rangle.$$

3. For $x, y, z \in \mathcal{E}$,
$${}_\bullet\langle x, y\rangle z = x\langle y, z\rangle_\bullet. \tag{5.1}$$

If such an equivalence A-B-bimodule exists for C^*-algebras A and B, then A and B are said to be *Morita equivalent.*

We shall denote by ${}_\bullet\mathcal{E}$ and $\mathcal{E}_\bullet$ the left and right modules, respectively, when there is risk of confusion. The associativity condition (5.1) linking the two inner products will play a major role in the following, as it enables us to draw parallels to Gabor frames. Before we get into a concrete example, however, we give this simple lemma arising from the associativity.

Lemma 5.1.1. *Let x and y be elements of an equivalence A-B-module $\mathcal{E}$, and $\langle\cdot,\cdot\rangle$ be any inner product on $\mathcal{E}$. Then the following identities hold:*

i) $\langle x, x\langle y, y\rangle_\bullet\rangle = \langle y\langle x, x\rangle_\bullet, y\rangle$

ii) $\langle {}_\bullet\langle y, y\rangle x, x\rangle = \langle y, {}_\bullet\langle x, x\rangle y\rangle$

iii) $\langle x\langle y, y\rangle_\bullet, x\rangle = \langle y, y\langle x, x\rangle_\bullet\rangle$

iv) $\langle x, {}_\bullet\langle y, y\rangle x\rangle = \langle {}_\bullet\langle x, x\rangle y, y\rangle$

Proof. We only show the first case, as the other ones are very similar. By using the associativity and involution rules of the inner products, we have
$$\begin{aligned}\langle x, x\langle y, y\rangle_\bullet\rangle &= \langle x, {}_\bullet\langle x, y\rangle y\rangle \\ &= \langle {}_\bullet\langle y, x\rangle x, y\rangle \\ &= \langle y\langle x, x\rangle_\bullet, y\rangle.\end{aligned}$$

□

5.2 Feichtinger's Algebra as an Equivalence Bimodule

In light of these definitions, we wish to consider the Banach algebras A^1_θ and $A^1_{1/\theta}$ acting on Feichtinger's algebra S_0, following Luef [14]. First, take $g \in S_0(\mathbb{R})$, $a \in A^1_\theta$ and $b \in A^1_{1/\theta}$. Let Λ be a lattice in $\mathbb{R}^2$, and let Λ° be its adjoint lattice. Define a left action by A^1_θ on $S_0(\mathbb{R})$ by

$$ag := \sum_{\lambda \in \Lambda} a_\lambda \pi(\lambda) g, \tag{5.2}$$

and a right action by $A^1_{-1/\theta}$ on $S_0(\mathbb{R})$ by

$$gb := \sum_{\lambda^\circ \in \Lambda^\circ} b_{\lambda^\circ} \pi(\lambda^\circ)^* g. \tag{5.3}$$

Now, define the following algebra-valued inner products on $S_0(\mathbb{R})$:

$${}_\bullet\langle f, g\rangle := \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g\rangle \pi(\lambda)$$

and

$$\langle f, g\rangle_\bullet := \mathrm{vol}(\Lambda)^{-1} \sum_{\lambda^\circ \in \Lambda^\circ} \langle g, \pi(\lambda^\circ)^* f\rangle \pi(\lambda^\circ)^*.$$

Lemma 5.2.1.

i) $S_0(\mathbb{R})$ *is a left inner product* A^1_θ*-module with respect to the left action of* A^1_θ *and the* A^1_θ*-valued inner product defined above.*

ii) $S_0(\mathbb{R})$ *is a right inner product* $A^1_{1/\theta}$*-module with respect to the right action of* $A^1_{1/\theta}$ *and the* $A^1_{1/\theta}$*-valued inner product defined above.*

Proof. We will only prove that $S_0(\mathbb{R})$ is a left inner product A^1_θ-module, as the second proof is analogous. Since $\pi(\lambda)g \in S_0(\mathbb{R})$ for all $\lambda \in \mathbb{R}^2$, we have that $ag \in S_0(\mathbb{R})$. Linearity of the inner product goes as follows, where the indexes of the lattice are $\lambda = (x, \omega)$ and $\mu = (u, \eta)$:

$$\begin{aligned}
{}_\bullet\langle af, g\rangle &= \sum_{\lambda\in\Lambda} \langle af, \pi(\lambda)g\rangle \pi(\lambda) \\
&= \sum_{\lambda\in\Lambda} \left\langle \sum_{\mu\in\Lambda} a_\mu \pi(\mu) f, \pi(\lambda) g \right\rangle \pi(\lambda) \\
&= \sum_{\lambda\in\Lambda}\sum_{\mu\in\Lambda} a_\mu \left\langle \pi(\mu) f, \pi(\lambda) g\right\rangle \pi(\lambda) \\
&= \sum_{\lambda\in\Lambda}\sum_{\mu\in\Lambda} a_\mu \left\langle f, \pi(\mu)^*\pi(\lambda) g\right\rangle \pi(\lambda) \\
&= \sum_{\lambda\in\Lambda}\sum_{\mu\in\Lambda} a_\mu \left\langle f, e^{2\pi i\theta(x-u)\eta}\pi(\lambda-\mu) g\right\rangle \pi(\lambda) \\
&= \sum_{\lambda\in\Lambda}\sum_{\mu\in\Lambda} a_\mu \langle f, \pi(\lambda-\mu) g\rangle e^{-2\pi i\theta(x-u)\eta}\pi(\lambda) \\
&= \sum_{\lambda\in\Lambda} (a \natural {}_\bullet\langle f, g\rangle)_\lambda \pi(\lambda) \\
&= a\, {}_\bullet\langle f, g\rangle,
\end{aligned}$$

where we recognise the twisted convolution of the operators a and ${}_\bullet\langle f, g\rangle$. For conjugate-symmetry, we use that $\pi(\lambda)^* = e^{-2\pi i x\omega}\pi(-\lambda)$, and get that

$$\begin{aligned}
{}_\bullet\langle f, g\rangle^* &= \sum_{\lambda\in\Lambda} \overline{\langle f, \pi(\lambda) g\rangle} \pi(\lambda)^* \\
&= \sum_{\lambda\in\Lambda} \langle \pi(\lambda) g, f\rangle \pi(\lambda)^* \\
&= \sum_{\lambda\in\Lambda} \langle g, \pi(\lambda)^* f\rangle \pi(\lambda)^* \\
&= \sum_{\lambda\in\Lambda} \langle g, e^{-2\pi i x\omega}\pi(-\lambda) f\rangle e^{-2\pi i x\omega}\pi(-\lambda) \\
&= \sum_{\lambda\in\Lambda} \langle g, \pi(\lambda) f\rangle \pi(\lambda) \qquad (\text{via } \ \lambda \mapsto -\lambda) \\
&= {}_\bullet\langle g, f\rangle.
\end{aligned}$$

For positive-definiteness, ${}_\bullet\langle f, f\rangle$ is a positive operator if and only if the $L^2(\mathbb{R})$ inner product $\langle {}_\bullet\langle f, f\rangle g, g\rangle_{L^2(\mathbb{R})}$ is positive for all $g \in S_0(\mathbb{R})$.

For any $g \in S_0(\mathbb{R})$, we have

$$\begin{aligned}
\langle {}_\bullet\langle f, f\rangle g, g\rangle_{L^2(\mathbb{R})} &= \langle \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda) f\rangle \pi(\lambda) g, g\rangle_{L^2(\mathbb{R})} \\
&= \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda) f\rangle \langle \pi(\lambda) g, g\rangle \\
&= \mathrm{vol}(\Lambda)^{-1} \sum_{\lambda^\circ\in\Lambda^\circ} \langle g, \pi(\lambda^\circ) f\rangle \langle \pi(\lambda^\circ) f, g\rangle \qquad \text{(by the FIGA)} \\
&= \sum_{\lambda\in\Lambda} |\langle g, \pi(\lambda^\circ) f\rangle|^2 \geq 0,
\end{aligned}$$

where we use the Fundamental Identity of Gabor Analysis (theorem 2.3.3) in line three. This completes the proof. □

Considering the inner products on $S_0(\mathbb{R})$, Gabor analysis provides useful insight into the abstract algebras. Let $g \in S_0(\mathbb{R})$ be a function such that $\mathcal{G}(g, \Lambda)$ is a Parseval frame for some lattice $\Lambda \subset \mathbb{R}^2$. Take $f \in S_0(\mathbb{R})$. By the frame expansion and associativity of the inner products on $S_0(\mathbb{R})$, we get that

$$f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda) g\rangle \pi(\lambda) g = {}_\bullet\langle f, g\rangle g = f \langle g, g\rangle_\bullet .$$

This implies that $\langle g, g\rangle_\bullet \in A^1_{1/\theta}$ is the identity. Under the same assumptions, ${}_\bullet\langle g, g\rangle$ turns out to be a projection in A^1_θ:

$${}_\bullet\langle g, g\rangle^2 = {}_\bullet\langle {}_\bullet\langle g, g\rangle g, g\rangle = {}_\bullet\langle g \langle g, g\rangle_\bullet, g\rangle = {}_\bullet\langle g, g\rangle .$$

This is the so-called *Rieffel projection* in A_θ, and thus abstractly in the C^*-algebra $\mathcal{A}_\theta$. The awareness that the two problems finding a projection in the noncommutative torus and constructing a Parseval frame for $L^2(\mathbb{R})$ are in fact connected, was first discussed by Franz Luef [15]. We summarise this as a proposition:

Proposition 5.2.2. *Suppose $g \in S_0(\mathbb{R})$. Then the following are equivalent:*

i) $\mathcal{G}(g, \Lambda)$ is a Parseval frame.

ii) $\langle g, g\rangle_\bullet$ is the identity operator on $S_0(\mathbb{R})$.

iii) ${}_\bullet\langle g, g\rangle$ is a projection on $S_0(\mathbb{R})$.

Definition 5.2.1. The *left module norm* on $S_0(\mathbb{R})$ is given by

$$||g||_\Lambda := ||\,{}_\bullet\langle g, g\rangle||^{1/2}_{\mathcal{A}_\theta}$$

and the *right module norm* on $S_0(\mathbb{R})$ is given by

$$||g||_{\Lambda^\circ} := ||\langle g, g\rangle_\bullet||^{1/2}_{\mathcal{A}_{1/\theta}}.$$

The norm properties of these are inherited from the inner products from which they are defined. The following lemma from Rieffel's 1988 paper [21] is slightly technical, and necessary for a later, important result.

Lemma 5.2.3. *For $a \in A^1_\theta$, $b \in A^1_{1/\theta}$ and $g \in S_0(\mathbb{R})$, we have*

$$\langle ag, ag\rangle_\bullet \leq ||a||^2 \langle g, g\rangle_\bullet$$

and

$${}_\bullet\langle gb, gb\rangle \leq ||b||^2\, {}_\bullet\langle g, g\rangle.$$

Proof. We only show the first case, as the other one is shown in a very similar fashion. For $f \in S_0(\mathbb{R})$ and using the standard inner product $\langle \cdot, \cdot\rangle$ of $L^2(\mathbb{R})$, we have

$$\begin{aligned}
\langle f\langle ag, ag\rangle_\bullet, f\rangle &= \langle ag, ag\langle f, f\rangle_\bullet\rangle \\
&= \langle a(g\langle f, f\rangle^{1/2}_\bullet), a(g\langle f, f\rangle^{1/2}_\bullet)\rangle \\
&\leq ||a||^2\langle g\langle f, f\rangle^{1/2}_\bullet, g\langle f, f\rangle^{1/2}_\bullet\rangle \\
&= ||a||^2\langle g, g\langle f, f\rangle_\bullet\rangle \\
&= ||a||^2\langle f\langle g, g\rangle_\bullet, f\rangle,
\end{aligned}$$

using the identities from lemma 5.1.1 in the first and the last equality. Since this holds for all $f \in S_0(\mathbb{R})$, we arrive at the desired conclusion. $\square$

We are now able to show the following significant result:

Proposition 5.2.4. *The two module norms coincide on $S_0(\mathbb{R})$.*

Proof. We follow the proof given in [19]. Take $g \in S_0(\mathbb{R})$. We use the C^* property of the operator algebras, the associativity and linearity of the inner products and the Cauchy-Schwartz inequality to get

$$\begin{aligned}
||\langle g, g\rangle_\bullet||^2 &= ||\langle g, g\rangle_\bullet\langle g, g\rangle_\bullet|| \\
&= ||\langle g, g\langle g, g\rangle_\bullet\rangle_\bullet|| \\
&= ||\langle g, {}_\bullet\langle g, g\rangle g\rangle_\bullet|| \\
&\leq ||\langle g, g\rangle_\bullet||^{1/2}||\langle {}_\bullet\langle g, g\rangle g, {}_\bullet\langle g, g\rangle g\rangle_\bullet||^{1/2} \\
&\leq ||\langle g, g\rangle_\bullet||^{1/2}||\,{}_\bullet\langle g, g\rangle||\; ||\langle g, g\rangle_\bullet||^{1/2},
\end{aligned}$$

where we used lemma 5.2.3 in the last inequality. By dividing by $||\langle g, g\rangle_\bullet||$, we get $||\langle g, g\rangle_\bullet|| \leq ||{}_\bullet\langle g, g\rangle||$. An analogous computation gives $||{}_\bullet\langle g, g\rangle|| \leq ||\langle g, g\rangle_\bullet||$. Thus, we conclude that $||g||_\Lambda = ||g||_{\Lambda^\circ}$. □

The equality of the module norms leads to a very interesting result on the two underlying Gabor systems:[12]

Proposition 5.2.5. *The following are equivalent:*

i) $||g||_\Lambda$ *is finite.*

ii) $||g||_{\Lambda^\circ}$ *is finite.*

iii) $\{\pi(\lambda)g\}_{\lambda\in\Lambda}$ *is a Bessel system with Bessel bound* $||g||_{\Lambda^\circ}$.

iv) $\{\pi(\lambda^\circ)g\}_{\lambda^\circ\in\Lambda^\circ}$ *is a Bessel system with Bessel bound* $\mathrm{vol}(\Lambda)||g||_\Lambda$.

Proof. The equivalence of *i)* and *ii)* follows directly from proposition 5.2.4. To show the equivalence of *i)* and *iv)*, note that the module norm of g in $S_0(\mathbb{R})$ is the operator norm

$$||g||_\Lambda = \sup_{\gamma\neq 0}\left\{\frac{\langle {}_\bullet\langle g, g\rangle\gamma, \gamma\rangle}{||\gamma||^2_{L^2(\mathbb{R})}}\right\}.$$

By the Fundamental Identity of Gabor Analysis,

$$\begin{aligned}\langle {}_\bullet\langle g, g\rangle\gamma, \gamma\rangle &= \left\langle \sum_{\lambda\in\Lambda}\langle g, \pi(\lambda)g\rangle\pi(\lambda)\gamma, \gamma\right\rangle \\ &= \sum_{\lambda\in\Lambda}\langle g, \pi(\lambda)g\rangle\langle\pi(\lambda)\gamma, \gamma\rangle \\ &= \mathrm{vol}(\Lambda)^{-1}\sum_{\lambda^\circ\in\Lambda^\circ}\langle\gamma, \pi(\lambda^\circ)g\rangle\langle\pi(\lambda^\circ)g, \gamma\rangle \\ &= \mathrm{vol}(\Lambda)^{-1}\sum_{\lambda^\circ\in\Lambda^\circ}|\langle\gamma, \pi(\lambda^\circ)g\rangle|^2,\end{aligned}$$

so $\sum_{\lambda^\circ\in\Lambda^\circ}|\langle\gamma, \pi(\lambda^\circ)g\rangle|^2 \leq \mathrm{vol}(\Lambda)||g||_\Lambda$. A similar computation shows that $\sum_{\lambda\in\Lambda}|\langle\gamma, \pi(\lambda)g\rangle|^2 \leq ||g||_{\Lambda^\circ}$, which shows the equivalence of *ii)* and *iii)*. □

The following result is central to this section, and constitutes the reason for our introduction of Hilbert C^*-modules and its links to Gabor analysis.

Proposition 5.2.6. *The completion $\mathcal{E}$ of $S_0(\mathbb{R})$ with respect to the module norm is an equivalence $\mathcal{A}_\theta$-$\mathcal{A}_{1/\theta}$-bimodule.*

Proof. The proof consists of showing that $S_0(\mathbb{R})$ fulfils the requirements of definition 5.1.3 both as a left A^1_θ-module and as a right $A^1_{1/\theta}$-module, except completeness. It then follows that the completion $\mathcal{E}$ is an equivalence $\mathcal{A}_\theta$-$\mathcal{A}_{1/\theta}$-bimodule. We will not show fullness om the module here, but refer to theorem 3.4 of [14] for a proof. Now, take $f, g \in S_0(\mathbb{R})$ and $a \in A_\theta$. Then we have

$$\begin{aligned}
\langle af, g\rangle_\bullet &= \mathrm{vol}(\Lambda)^{-1} \sum_{\lambda^\circ \in \Lambda^\circ} \langle af, \pi(\lambda^\circ) g\rangle \pi(\lambda^\circ) \\
&= \mathrm{vol}(\Lambda)^{-1} \sum_{\lambda^\circ \in \Lambda^\circ} \left\langle \sum_{\lambda \in \Lambda} a_\lambda \pi(\lambda) f, \pi(\lambda^\circ) g \right\rangle \pi(\lambda^\circ) \\
&= \mathrm{vol}(\Lambda)^{-1} \sum_{\lambda^\circ \in \Lambda^\circ} \left\langle f, \sum_{\lambda \in \Lambda} \overline{a_\lambda} \pi(\lambda)^* \pi(\lambda^\circ) g \right\rangle \pi(\lambda^\circ) \\
&= \mathrm{vol}(\Lambda)^{-1} \sum_{\lambda^\circ \in \Lambda^\circ} \left\langle f, \pi(\lambda^\circ) \sum_{\lambda \in \Lambda} \overline{a_\lambda} \pi(\lambda)^* g \right\rangle \pi(\lambda^\circ) \\
&= \mathrm{vol}(\Lambda)^{-1} \sum_{\lambda^\circ \in \Lambda^\circ} \langle f, \pi(\lambda^\circ) a^* g\rangle \, \pi(\lambda^\circ) \\
&= \langle f, a^* g\rangle_\bullet .
\end{aligned}$$

Here we take advantage of the fact that Λ and Λ° are adjoint lattices. A similar computation shows that ${}_\bullet\langle fb, g\rangle = {}_\bullet\langle f, gb^*\rangle$ for $b \in A_{1/\theta}$.
For the last requirement, we need only point out the connection between the inner products in $S_0(\mathbb{R})$ and the Gabor frame operator introduced in section 2:

$$S_{g,\gamma,\Lambda} f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g\rangle \pi(\lambda)\gamma = {}_\bullet\langle f, g\rangle \gamma$$

and

$$S_{g,f,\Lambda^\circ} \gamma = \sum_{\lambda^\circ \in \Lambda^\circ} \langle \gamma, \pi(\lambda^\circ) g\rangle \pi(\lambda^\circ) f = \mathrm{vol}(\Lambda)\, f \langle g, \gamma\rangle_\bullet .$$

The requirement now follows by the Janssen representation of the Gabor frame operator (proposition 2.5), namely $S_{g,\gamma,\Lambda} f = \mathrm{vol}(\Lambda)^{-1} S_{g,f,\Lambda^\circ}\gamma$. □

The proof of the last requirement shows that the associativity condition of the equivalence $\mathcal{A}_\theta$-$\mathcal{A}_{1/\theta}$-bimodule is indeed equivalent to the Janssen representation (2.5). This shows how the abstract structure of the Hilbert C^*-module has parallels to the applied ones in Gabor analysis.

5.3 An Equivalence Bimodule Over the Crossed Product

We have seen how the completion $\mathcal{E}$ of Feichtinger's algebra can be seen as an equivalence bimodule between the noncommutative tori $\mathcal{A}_\theta$ and $\mathcal{A}_{1/\theta}$ realised as the operator algebras A^1_θ and $A^1_{1/\theta}$, respectively. However, as discussed in section 4, the noncommutative torus can also be considered as a crossed product. We shall consider a module over $C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ and $C(\mathbb{T}) \rtimes_{1/\theta} \mathbb{Z}$, and show how these and the modules over the twisted group C^*-algebras link to the two previously discussed representations of the Gabor frame operator from Gabor theory. Let $a = \sum_{k\in\mathbb{Z}} a_k(t)T_{\alpha k}$ be an element of the crossed product $C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$, and $g \in S_0(\mathbb{R})$. Then we define the left action

$$ag(t) := \sum_{k\in\mathbb{Z}} a_k(t)T_{\alpha k}g(t) = \sum_{k\in\mathbb{Z}} a_k(t)g(t - \alpha k), \tag{5.4}$$

where the function a_k is periodic and has the form

$$a_k(t) = \sum_{l\in\mathbb{Z}} a_{k,l}e^{2\pi i\beta lt},$$

with $(a_{k,l}) \in \ell^1(\mathbb{Z}^2)$. Similarly, the right action from the crossed product $C(\mathbb{T}) \rtimes_{1/\theta} \mathbb{Z}$ is defined by

$$gb(t) := \sum_{k\in\mathbb{Z}} b_k(t)T_{\beta^{-1}k}g(t) = \sum_{k\in\mathbb{Z}} b_k(t)g(t - \beta^{-1}k), \tag{5.5}$$

where b_k has the form

$$b_k(t) = \sum_{l\in\mathbb{Z}} b_{k,l}e^{2\pi i\alpha^{-1}lt},$$

with $(b_{k,l}) \in \ell^1(\mathbb{Z})$. The algebra-valued inner products ${}^\bullet\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle^\bullet$ on $S_0(\mathbb{R})$ are defined by the following sequences of coefficients in the Fourier expansions of the continuous, periodic functions:

$$({}^\bullet\langle f, g\rangle)_{k,l} = \alpha\langle f, \pi(\alpha k, \beta l)g\rangle$$

and

$$(\langle f, g\rangle^\bullet)_{k,l} = \beta^{-1}\langle g, \pi(\beta^{-1}k, \alpha^{-1}l)f\rangle.$$

When considering $\mathcal{E}$ as a module over the twisted group C^*-algebras A^1_θ and A^∞_θ, we saw that it was an equivalence bimodule for the two algebras, making them Morita equivalent. We wish to do the same when $\mathcal{E}$ is considered a module over the crossed products, and see that it is indeed an equivalence bimodule for $C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ and $C(\mathbb{T}) \rtimes_{1/\theta} \mathbb{Z}$. Instead of showing the equivalence bimodule properties explicitly, however, we show the strong link between the twisted group C^*-algebra and the crossed product.

We first make the observation that the crossed product algebra actions on $S_0(\mathbb{R})$ only differ from the twisted group C^*-algebra actions by a partial Fourier transform; the coefficients $a_{k,l}$ and $b_{k,l}$ from (5.2) and (5.3), respectively, are correspond to the periodic, continuous functions $a_k(t)$ and $b_k(t)$ from (5.4) and (5.5), respectively. By writing out the Fourier expansion of $a_k(t)$, the left case is easily shown:

$$ag(t) = \sum_{k\in\mathbb{Z}} a_k(t) T_{\alpha k} g(t) = \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} a_{k,l} e^{2\pi i\beta l} g(t-\alpha k) = \sum_{k,l\in\mathbb{Z}} a_{k,l} \pi(\alpha k, \beta l) g(t).$$

The right case is analogous. Also, the two sets of algebra-valued inner products on $S_0(\mathbb{R})$ are strongly linked:

$$\begin{aligned}
{}^{\bullet}\langle f, g\rangle &= \sum_{k\in\mathbb{Z}} ({}^{\bullet}\langle f, g\rangle)_{k,l} T_{\alpha k} \\
&= \alpha \sum_{k\in\mathbb{Z}} \sum_{l\in\mathbb{Z}} \langle f, \pi(\alpha k, \beta l) g\rangle e^{2\pi i\beta l t} T_{\alpha k} \\
&= \alpha \sum_{k,l\in\mathbb{Z}} \langle f, \pi(\alpha k, \beta l)\rangle \pi(\alpha k, \beta l),
\end{aligned}$$

again with the right case being analogous. Similarly as for the inner product A_θ^1-module, we can define norms on the crossed product module from the inner products and show them to coincide, hence allowing us to simply write $||\cdot||$. From the above, it then follows that the completion $\mathcal{E}$ of $S_0(\mathbb{R})$ with respect to the module norm $||\cdot||$ satisfies the requirements of definition 5.1.3, and is hence an equivalence $(C(\mathbb{T}) \rtimes_\theta \mathbb{Z})$-$(C(\mathbb{T}) \rtimes_{1/\theta} \mathbb{Z})$-bimodule.

Clearly, the two ways of making $\mathcal{E}$ an equivalence bimodule are very similar. However, the difference between them does shed light onto a certain part of Gabor analysis. We saw in section 5.2 that the associativity of the inner product was equivalent to the Janssen representation, proposition 2.5. Here we shall see that the associativity of the crossed-product-valued inner product is equivalent to the Walnut representation. Letting $f, g, \gamma \in S_0(\mathbb{R})$, we start with the assumption of associativity of the inner products on the crossed product module. Since

$$\begin{aligned}
{}^{\bullet}\langle f,g\rangle\gamma(t) &= \sum_{k\in\mathbb{Z}} ({}^{\bullet}\langle f,g\rangle)_k(t)T_{\alpha k}\gamma(t) \\
&= \sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}} ({}^{\bullet}\langle f,g\rangle)_{k,l}e^{2\pi i\beta lt}\gamma(t-\alpha k) \\
&= \alpha\sum_{k,l\in\mathbb{Z}}\langle f,\pi(\alpha k,\beta l)g\rangle\pi(\alpha k,\beta l)\gamma(t) \\
&= \alpha S_{g,\gamma,\alpha\mathbb{Z}\times\beta\mathbb{Z}}f(t),
\end{aligned}$$

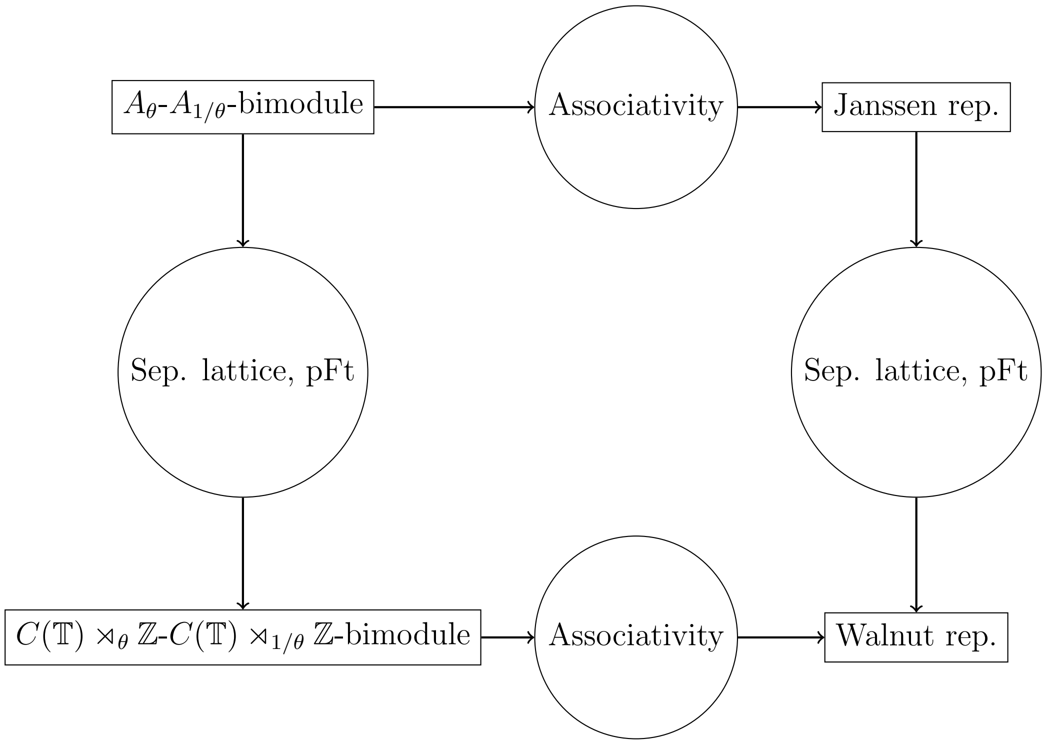
associativity of the inner products is equivalent to

$$\begin{aligned}
S_{g,\gamma,\alpha\mathbb{Z}\times\beta\mathbb{Z}}f(t) &= \alpha^{-1}f\langle g,\gamma\rangle^{\bullet}(t) \\
&= (\alpha\beta)^{-1}\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}\langle\gamma,\pi(\beta^{-1}k,\alpha^{-1}l)g\rangle e^{2\pi i\alpha^{-1}lt}T_{\beta^{-1}k}f(t) \\
&= \beta^{-1}\sum_{k\in\mathbb{Z}}G_k(t)f(t-\beta^{-1}k),
\end{aligned}$$

where G_k are given by

$$G_k(t) = \alpha^{-1}\sum_{l\in\mathbb{Z}}\langle\gamma,\pi(\beta^{-1}k,\alpha^{-1}l)g\rangle e^{2\pi i\alpha^{-1}lt}.$$

We recognise this as the Walnut representation of the Gabor frame operator for a separable lattice, as presented in section 2.4. In the previous section we saw the equivalence between inner product associativity in the twisted group C^*-algebra module and the Janssen representation. This shows that we have an analogous equivalence between inner product associativity in the crossed product module. Thus, each of these two representations of the Gabor frame operator has its realisation of the abstract structure of the noncommutative torus. As we have seen, the way to convert the twisted group C^*-algebra module into the crossed product module, is restriction of our lattice to the separable case $\Lambda = \alpha\mathbb{Z}\times\beta\mathbb{Z}$, and the partial Fourier transform. Analogously, the Janssen representation was shown in section 2.4 to convert to the Walnut representation by restriction to separable lattices and use of the Fundamental identity of Gabor analysis (theorem 2.3.3), which in turn is based on a Fourier transform. We summarise these findings in the following diagram, where pFt refers to the partial Fourier transform:



5.4 Finitely Generated, Projective Modules

We shall look at a particular class of modules. The finitely generated, projective modules are of special interest to us, as they have a peculiar connection to geometry, through the concept of vector bundles. Also, we shall see that all the modules that we have seen until now fall within this category.

Definition 5.4.1. Given a ring R and an R-module $\mathcal{E}$, we say $\mathcal{E}$ is *finitely generated* if there exist finitely many elements $e_1, ..., e_n \in \mathcal{E}$ such that for every $e \in \mathcal{E}$,

$$e = \sum_{j=1}^{n} r_j e_j$$

for some $r_j \in R$. We say that $\mathcal{E}$ is *projective* if there exists an R-module $\mathcal{F}$ such that the direct product $\mathcal{E} \oplus \mathcal{F}$ is a free R-module, i.e. an R-module with a basis.

Definition 5.4.2. Given a C^*-algebra A, a left inner product A-module $\mathcal{E}$ and elements $x, y \in \mathcal{E}$, a *rank-one operator* on $\mathcal{E}$ is an operator $\Theta : \mathcal{E} \to \mathcal{E}$ of the form

$$\Theta_{x,y} z = \langle z, x \rangle y$$

for $z \in \mathcal{E}$.

Rank-one operators can be shown to be bounded and module-linear, and to be adjointable, i.e. have an adjoint operator Θ^* such that $\langle \Theta x, y \rangle = \langle x, \Theta^* y \rangle$. We call finite linear combinations of rank-one operators *finite-rank operators* on $\mathcal{E}$, and define the *compact operators* on $\mathcal{E}$ to be the set

$$\mathcal{K}(\mathcal{E}) := \overline{span}\{\Theta_{x,y} \mid x, y \in \mathcal{E}\}.$$

The compact operators make a closed ideal in the C^*-algebra of all bounded, linear operators on $\mathcal{E}$. Also, the set of finite-rank operators is an ideal in the set of compact operators. The following lemma, for which we refer to [21] for a proof, connects the equivalence bimodules to the compact operators:

Lemma 5.4.1. *If $\mathcal{E}$ is an equivalence A-B-bimodule for C^*-algebras A and B, then $A \cong \mathcal{K}(\mathcal{E}_\bullet)$ and $B \cong \mathcal{K}({}_\bullet\mathcal{E})$.*

With this result we are able to show an important result on the equivalence $\mathcal{A}_\theta$-$\mathcal{A}_{1/\theta}$-bimodule $\mathcal{E}$:

Proposition 5.4.2. *$\mathcal{E}$ is a finitely generated, projective left module over $\mathcal{A}_\theta$, and a finitely generated, projective right module over $\mathcal{A}_{1/\theta}$.*

Proof. We show only the left case of ${}_\bullet\mathcal{E}$ as an left $\mathcal{A}_\theta$-module. The right case is analogous. $\mathcal{A}_\theta$ is a unital C^*-algebra. Hence by lemma 5.4.1, the identity operator is in $\mathcal{K}(\mathcal{E}_\bullet)$, so there are rank-one operators $\Theta_1, \cdots, \Theta_n$ on $\mathcal{E}_\bullet$ such that $\Theta_1 + \cdots + \Theta_n = I$. By fullness of $\mathcal{E}$, there must then exist a finite set of window functions $g_1, \cdots, g_n, \gamma_1, \cdots, \gamma_n \in \mathcal{E}$ such that for any $f \in \mathcal{E}$,

$$f\langle \gamma_1, g_1 \rangle_\bullet + \cdots + f\langle \gamma_n, g_n \rangle_\bullet = f.$$

By the associativity of the inner products $\langle \cdot, \cdot \rangle_\bullet$ and ${}_\bullet\langle \cdot, \cdot \rangle$, we have that

$${}_\bullet\langle f, \gamma_1 \rangle g_1 + \cdots {}_\bullet\langle f, \gamma_n \rangle g_n = f.$$

Thus, ${}_\bullet\mathcal{E}$ is a finitely generated module over $\mathcal{A}_\theta$, with generators $g_1, \cdots, g_n$. Now, since $g_1, \cdots, g_n$ generate $\mathcal{E}$, the matrix $P = [\langle g_j, g_j \rangle]_{1 \leq j \leq n}$ is a projection in $M_n(\mathcal{A}_\theta)$, and $\mathcal{E} \cong P\mathcal{A}_\theta^n$, where $\mathcal{A}_\theta^n$ is a free $\mathcal{A}_\theta$-module, so we get that $\mathcal{E}$ is projective. □

We want to prove the same result for the module $S_0(\mathbb{R})$, but for this we need a particular property of the Banach subalgebras A^1_θ and $A^1_{1/\theta}$. We say that a subalgebra $A_0 \subset A$ is *inverse-closed* in A if $a \in A_0$ implies $a^{-1} \in A_0$. The following theorem is a major result, e.g. for proving that $S_0(\mathbb{R})$ is finitely generated, and is due to Gröchenig and Leinert. We refer to their paper [11] for a proof.

Theorem 5.4.3. *A^1_θ and $A^1_{1/\theta}$ are inverse-closed in $\mathcal{A}_\theta$.*

Proposition 5.4.4 (Luef)**.** *$S_0(\mathbb{R})$ is a finitely generated, projective left module over A^1_θ, and a finitely generated, projective right module over $A^1_{1/\theta}$.*

Proof. We follow the proof of proposition 3.7 from Rieffel's 1988 paper [21], and show only the case for $S_0(\mathbb{R})$ as a left A^1_θ-module. The case as a right $A^1_{1/\theta}$-module is similar. $\mathcal{E}_\bullet$ is finitely generated and projective, so $1_{\mathcal{A}_{1/\theta}} = \sum_{j=1}^n \langle f_j, g_j\rangle_\bullet$ for some $f_1, \cdots, f_n, g_n, \cdots, g_n \in \mathcal{E}$. $S_0(\mathbb{R})$ is dense in $\mathcal{E}$, so we can approximate an invertible element in $\mathcal{E}$ by a finite sum of right inner products of elements in $S_0(\mathbb{R})$. This is an element of $A^1_{1/\theta}$, so by lemma 5.4.3, its inverse is in $A^1_{1/\theta}$. By multiplying the two, we get that $1_{\mathcal{A}_{1/\theta}}$ can be expressed by a finite sum of inner products of elements in $S_0(\mathbb{R})$. So there exist some $\gamma_1, \cdots, \gamma_n, g_1, \cdots, g_n \in S_0(\mathbb{R})$ such that

$$\langle \gamma_1, g_1\rangle_\bullet + \cdots + \langle \gamma_n, g_n\rangle_\bullet = 1_{\mathcal{A}_\theta}.$$

Thus, for any $f \in S_0(\mathbb{R})$, by associativity of the inner products,

$$f = \sum_{j=1}^n f\langle \gamma_j, g_j\rangle_\bullet = \sum_{j=1}^n {}_\bullet\langle f, \gamma_j\rangle g_j,$$

so $S_0(\mathbb{R})$ is finitely generated as a left A^1_θ-module. By the same argument as in proposition 5.4.2, we get that $S_0(\mathbb{R})$ is also projective. □

This result is significant to the field of Gabor analysis. Since $S_0(\mathbb{R})$ is finitely generated, then for any lattice Λ, there is always a pair of dual multi-window frames $\mathcal{G}(g_1, \cdots, g_n, \Lambda)$ and $\mathcal{G}(\gamma_1, \cdots, \gamma_n, \Lambda)$ of finitely many atoms, so any f can be written as

$$f = \sum_{k=1}^n {}_\bullet\langle f, g_j\rangle \gamma_j = \sum_{k=1}^n \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda) g_j\rangle \pi(\lambda)\gamma_j.$$

Indeed, by doing the "$S^{-1/2}$ trick" mentioned in section 2.1, we get that for any lattice Λ, there is a corresponding tight frame, so that any f can be written as

$$f = \sum_{j=1}^{n} \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\phi_j \rangle \pi(\lambda)\phi_j.$$

where $\phi_j = S^{-1/2}_{\{g_j\},\{\gamma_j\},\Lambda} g_j$.

As we have seen, $M^1_1(\mathbb{R}) = S_0(\mathbb{R})$ is a finitely generated, projective module over the Banach algebras A^1_θ and $A^1_{1/\theta}$. As we saw in (1.3), the Schwartz space $\mathcal{S}(\mathbb{R})$ is the intersection of the weighted modulation spaces $M^1_s(\mathbb{R})$ for $s \geq 0$. Also, in (4.2), the involutive algebra A^∞_θ was defined as the intersection of all the weighted algebras A^s_θ for $s \geq 0$. It can be proved that for all $s \geq 0$, $M^1_s(\mathbb{R})$ is finitely generated and projective as a left module over A^s_θ and as a right module over $A^s_{1/\theta}$, with a proof analogous to the one given for proposition 5.4.4. It follows that the intersection $\mathcal{S}(\mathbb{R})$ is finitely generated and projective as a left module over S^∞_θ and as a right module over $A^\infty_{1/\theta}$.

Chapter 6

Vector Bundles

We have seen how adding structure to the discussion of Gabor frames through the concept of Hilbert C^*-modules has provided new insights. In this chapter, we shall introduce the concept of vector bundles, which, roughly speaking, can be considered as the assignment of a vector space for every point in a topological space. By the well-known result of the Serre-Swan theorem, linking vector bundles to finitely generated, projective modules, we shall see that the modules of chapter 5 get a geometrical aspect to them, and see how this useful, additional structure can benefit our understanding of Gabor analysis.

6.1 Definition of Vector Bundles

Definition 6.1.1. For a topological space X (called the *base space*), a *complex vector bundle over* X is a topological space E (called the *total space*) equipped with a surjective, continuous map $\pi : E \to X$ (called the *bundle projection*) such that the following hold:

1. There exists an open covering $\mathcal{U}$ of X, and for every $U \in \mathcal{U}$ there exists a finite-dimensional complex vector space V_U and a homeomorphism $h_U : \pi^{-1}(U) \to U \times V_U$ such that $p_1 \circ h_U = \pi$ on $\pi^{-1}(U)$ for some map p_1.

2. For every $x \in X$, the set $E_X := \pi^{-1}(\{x\})$ (called a *fibre*) is a finite-dimensional complex vector space, and h_U restricts to an isomorphism of vector spaces $\pi^{-1}(\{x\}) \to \{x\} \times V_U \cong V_U$.

We often refer to E as the vector bundle, when the base space and bundle projection is clear from the context. If all the vector spaces V_U have the

same dimension r, we say the vector bundle has rank r. A vector bundle of rank 1 is often called a *line bundle*. Given a base space X, a *morphism* from one vector bundle E with bundle projection π to another vector bundle F with bundle projection ρ, both over X, is a continuous map $f : E \to F$ such that $\rho \circ f = \pi$, and the restrictions $f : \pi^{-1}(x) \to \rho^{-1}(x)$ are linear maps of vector spaces for all $x \in X$. If $X = M$ is a smooth manifold, E is a smooth manifold, π is a smooth map of manifolds and the homeomorphisms h_U are all diffeomorphisms, then we say that $E \to M$ is a *smooth vector bundle*.
One example of a vector bundle over a smooth manifold M is the *tangent bundle* $\pi : TM \to M$, with fibres T_pM given by the tangent spaces at the points $p \in M$. From this, we can construct another example: the *cotangent bundle* $\pi : T^*M \to M$ has fibres T_p^*M given by the cotangent spaces, i.e. the dual spaces of the tangent spaces, at the points $p \in M$.

6.2 Sections and the Serre-Swan Theorem

Having introduced the abstract concept of a vector bundle, we shall give it some context by providing an important link to the useful modules discussed in chapter 5. In particular, we will give the classical result that finitely generated, projective modules can be considered as vector bundles over certain topological spaces.

Definition 6.2.1. A *section* of a vector bundle E is a map $\sigma : X \to E$ such that $\pi \circ \sigma = \mathrm{id}_X$. Furthermore, we define sections to be *continuous* if they are continuous maps in the topological sense, and to be *smooth* if they are smooth maps of smooth manifolds.

We denote by $\Gamma_X(E)$ the set of continuous sections of a vector bundle E over X, and by $\Gamma_M^\infty(E)$ the set of smooth sections of a smooth vector bundle E over a smooth manifold M.
We shall consider how the ring $C(X)$ of continuous functions $f : X \to \mathbb{C}$ act on the continuous sections. By invoking the rule

$$(f\xi)(x) = f(x)\xi(x) \tag{6.1}$$

for $\xi \in \Gamma_X(E)$, we find that $\Gamma_X(E)$ has the structure of a left $C(X)$-module. By the same rule for $f \in C^\infty(M)$ and $\xi \in \Gamma_M^\infty(E)$, we find that $\Gamma_M^\infty(E)$ is a left $C^\infty(M)$-module. Thus, Γ_X and Γ_M^∞ can be viewed as functors between the category of vector bundles and the category of modules, and it can be shown to preserve operations on vector bundles into the corresponding modules. In fact, we have the following important result about Γ in the following proposition, due to Jean-Pierre Serre and Richard Swan [22]:

Proposition 6.2.1 (Serre-Swan)**.** *For a compact, Hausdorff topological space X, the functor Γ_X between the category of vector bundles over X and the category of finitely generated, projective left modules over $C(X)$ is an equivalence of categories. Furthermore, for a smooth manifold M, the functor Γ_M^∞ between the category of smooth vector bundles over M and the category of finitely generated, projective left modules over $C^\infty(M)$ is an equivalence of categories.*

Due to this result, we may use the terms of vector bundles and finitely generated, projective modules interchangeably. This equivalence between the categories allows us to make use of some geometrical concepts on our modules, which will be important in the following section.

6.3 Connections

In this section we will explore the vector bundles further, while keeping in mind how they can be counted as finitely generated, projective modules. In particular, we consider a way of, roughly speaking, moving between fibres within a smooth vector bundle, by introducing the concept of a connection. First, however, we need some definitions.

Definition 6.3.1. Let M be a smooth manifold. A *derivation* on M is a $\mathbb{C}$-linear map

$$\partial : C^\infty(M) \to \mathbb{R}$$

satisfying the Leibniz rule

$$\partial(fg) = (\partial f)g + f(\partial g)$$

for all $f, g \in C^\infty(M)$.

Definition 6.3.2. Let $E \to M$ be a smooth vector bundle over a smooth manifold M. Let $E \otimes T^*M$ denote the *tensor product* of E and T^*M, with fibres $(E \otimes T^*M)_x = E_x \otimes (T^*M)_x$ as vector spaces. Suppose ∂ is a derivation on M. A *connection* on E is a $\mathbb{C}$-linear map

$$\nabla : \Gamma_M^\infty(E) \to \Gamma_M^\infty(E \otimes T^*M),$$

satisfying the Leibniz rule

$$\nabla(f\sigma) = f(\nabla\sigma) + (\partial f)\sigma.$$

for all $f \in C^\infty(M), \sigma \in \Gamma_M^\infty(E)$.

Connections can be seen as parallel transport between the fibres over the smooth manifold. We shall look at one particular connection on a vector bundle over a smooth manifold. Keeping the link to finitely generated, projective modules in mind, we wish to define an inner product on the module $\Gamma_M^\infty(E)$. For a given Borel measure μ on M, we define the inner product by

$$\langle \sigma, \tau \rangle = \int_M \langle \sigma(x), \tau(x) \rangle d\mu(x).$$

Definition 6.3.3. Let $E \to M$ be a smooth vector bundle over a smooth manifold M, equipped with a derivation ∂. A *covariant derivative* on E is a connection ∇ on E satisfying so-called *compatibility* with the inner product of $\Gamma_M^\infty(E)$, i.e.

$$\partial(\,\langle \sigma, \tau \rangle) = \,\langle \nabla\sigma, \tau \rangle + \,\langle \sigma, \nabla\tau \rangle.$$

for all $\sigma, \tau \in \Gamma_M^\infty(E)$. We say that ∇ *lifts* ∂ to $\Gamma_M^\infty(E)$.

Definition 6.3.4. For two covariant derivatives ∇_1 and ∇_2 with commuting derivations ∂_1 and ∂_2, the *curvature* is defined to be $F_{1,2} = \nabla_1\nabla_2 - \nabla_2\nabla_1$. We say that two covariant derivatives have a *constant curvature*, and that $F_{1,2}$ is a *constant curvature connection*, whenever the curvature is a multiple of the identity.

Recall from the discussion preceding proposition 6.2.1 that $\Gamma_M^\infty(E)$ could be considered a module over the $*$-algebra $C^\infty(M)$ by defining scalar multiplication as in (6.1). By proposition 6.2.1, any finitely generated, projective module $\mathcal{E}_0$ over a ring A_0 is isomorphic to the module $\Gamma_M^\infty(E)$ of smooth sections on a smooth vector bundle $E \to M$, over the ring $C^\infty(M)$. Thus definition 6.3.2 can translate to the following for a left finitely generated, projective module $\mathcal{E}_0$ over a ring A_0: a *connection* on $\mathcal{E}_0$ is a $\mathbb{C}$-linear map $\nabla : \mathcal{E}_0 \to \mathcal{E}_0$ such that

$$\nabla(ax) = (\partial a)x + a(\nabla x)$$

for all $x \in \mathcal{E}_0, a \in A_0$. Similarly, we can define the covariant derivatives as connections satisfying the compatibility condition with the inner product on the module.

We shall look at concrete examples of connections defined on a finitely generated, projective module. Let $\mathcal{E}_0 = \mathcal{S}(\mathbb{R})$, let $A_0 = A_\theta^\infty$, and let $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$ with $\theta = \alpha\beta$. Let ∂_1 and ∂_2 be derivations on A_θ^∞, given by

$$\partial_1 \left(\sum_{k,l\in\mathbb{Z}} a_{k,l}\pi(\alpha k, \beta l) \right) = 2\pi i \sum_{k,l\in\mathbb{Z}} k a_{k,l}\pi(\alpha k, \beta l)$$

and

$$\partial_2\left(\sum_{k,l\in\mathbb{Z}} a_{k,l}\pi(\alpha k,\beta l)\right) = 2\pi i \sum_{k,l\in\mathbb{Z}} l a_{k,l}\pi(\alpha k,\beta l).$$

Then let ∇_1 and ∇_2 be two operators on $C^\infty(\mathbb{R})$ defined by

$$(\nabla_1 g)(t) = \frac{2\pi i t}{\alpha} g(t) \quad \text{and} \quad (\nabla_2 g)(t) = \frac{1}{\beta} g'(t).$$

Lemma 6.3.1. *∇_1 and ∇_2 are covariant derivatives lifting ∂_1 and ∂_2, respectively.*

Proof. They satisfy the Leibniz rule:

$$\begin{aligned}
(\nabla_1(ag))(t) &= \frac{2\pi i t}{\alpha}(ag)(t) \\
&= \frac{2\pi i t}{\alpha} \sum_{k,l\in\mathbb{Z}} a_{k,l}\pi(\alpha k,\beta l) g(t) \\
&= \frac{2\pi i t}{\alpha} \sum_{k,l\in\mathbb{Z}} a_{k,l} e^{2\pi i \beta l t} g(t-\alpha k) \\
&= 2\pi i \left(\sum_{k,l\in\mathbb{Z}} a_{k,l}(\frac{t}{\alpha} - k + k) e^{2\pi i \beta l t} g(t-\alpha k)\right) \\
&= 2\pi i \sum_{k,l\in\mathbb{Z}} \alpha k a_{k,l} e^{2\pi i \beta l t} g(t-\alpha k) \\
&\qquad + \sum_{k,l\in\mathbb{Z}} k a_{k,l} e^{2\pi i \beta l t} (2\pi i (t-\alpha k) g(t-\alpha k)) \\
&= 2\pi i \sum_{k,l\in\mathbb{Z}} \alpha k a_{k,l}\pi(\alpha k,\beta l) g(t) + \sum_{k,l\in\mathbb{Z}} k a_{k,l}\pi(\alpha k,\beta l)(2\pi i t g(t)) \\
&= (\partial_1 a) g(t) + a(\nabla_1 g)(t)
\end{aligned}$$

and

$$\begin{aligned}
(\nabla_2(ag))(t) &= \frac{1}{\beta}(ag)'(t) \\
&= \frac{1}{\beta}\sum_{k,l\in\mathbb{Z}} \left(a_{k,l}\pi(\alpha k,\beta l)g\right)'(t) \\
&= \frac{1}{\beta}\sum_{k,l\in\mathbb{Z}} \left(2\pi i\beta l a_{k,l}e^{2\pi i\beta lt}g(t-\alpha k) + a_{k,l}e^{2\pi i\beta lt}g'(t-\alpha k)\right) \\
&= 2\pi i\sum_{k,l\in\mathbb{Z}} l a_{k,l}\pi(\alpha k,\beta l)g(t) + \sum_{k,l\in\mathbb{Z}} a_{k,l}\pi(\alpha k,\beta l)(\frac{1}{\beta}g'(t)) \\
&= (\partial_2 a)g(t) + a(\nabla_2 g)(t).
\end{aligned}$$

They are also compatible with the inner product:

$$\begin{aligned}
\partial_1({}_\bullet\langle f,g\rangle) &= \partial_1\left(\sum_{k,l\in\mathbb{Z}}\langle f,\pi(\alpha k,\beta l)g\rangle\pi(\alpha k,\beta l)\right) \\
&= 2\pi i\sum_{k,l\in\mathbb{Z}} k\langle f,\pi(\alpha k,\beta l)g\rangle\,\pi(\alpha k,\beta l) \\
&= \frac{2\pi i}{\alpha}\sum_{k,l\in\mathbb{Z}} \alpha k\int_{\mathbb{R}} f(t)e^{-2\pi i\beta lt}g(t-\alpha k)\,dt\,\pi(\alpha k,\beta l) \\
&= \frac{2\pi i}{\alpha}\sum_{k,l\in\mathbb{Z}} \int_{\mathbb{R}}(t-\overline{(t-\alpha k)})f(t)e^{-2\pi i\beta lt}\overline{g(t-\alpha k)}\,dt\,\pi(\alpha k,\beta l) \\
&= \frac{2\pi i}{\alpha}\sum_{k,l\in\mathbb{Z}}\Bigg(\int_{\mathbb{R}} tf(t)e^{-2\pi i\beta lt}\overline{g(t-\alpha k)}\,dt \\
&\qquad - \int_{\mathbb{R}} f(t)e^{-2\pi i\beta lt}\overline{(t-\alpha k)g(t-\alpha k)}\,dt\Bigg)\pi(\alpha k,\beta l) \\
&= \frac{2\pi i}{\alpha}\sum_{k,l\in\mathbb{Z}}\left(\langle tf,\pi(\alpha k,\beta l)g\rangle - \langle f,\pi(\alpha k,\beta l)(tg)\rangle\right)\pi(\alpha k,\beta l) \\
&= {}_\bullet\langle\frac{2\pi it}{\alpha}f(t),g(t)\rangle + {}_\bullet\langle f(t),\frac{2\pi it}{\alpha}g(t)\rangle \\
&= {}_\bullet\langle\nabla_1 f,g\rangle + {}_\bullet\langle f,\nabla_1 g\rangle,
\end{aligned}$$

and, by writing out the inner products and integrating by parts,

$$
\begin{aligned}
&{}_\bullet\langle \nabla_2 f, g\rangle + {}_\bullet\langle f, \nabla_2 g\rangle \\
&\quad = \frac{1}{\beta}\sum_{k,l\in\mathbb{Z}} \langle f', \pi(\alpha k, \beta l) g\rangle \pi(\alpha k, \beta l) + \sum_{k,l\in\mathbb{Z}} \langle f, \pi(\alpha k, \beta l) g'\rangle \pi(\alpha k, \beta l) \\
&\quad = \frac{1}{\beta}\sum_{k,l\in\mathbb{Z}} \left(\int_{\mathbb{R}} f'(t) e^{-2\pi i \beta l t} \overline{g(t-\alpha k)}\, dt \right. \\
&\qquad\quad \left. + \int_{\mathbb{R}} f(t) e^{-2\pi i \beta l t} \overline{g'(t-\alpha k)}\, dt \right) \pi(\alpha k, \beta l) \\
&\quad = \frac{1}{\beta}\sum_{k,l\in\mathbb{Z}} \left(\left[f(t) e^{-2\pi i \beta l t} \overline{g(t-\alpha k)} \right]_{-\infty}^{\infty} \right. \\
&\qquad\quad - \int_{\mathbb{R}} f(t) \left(-2\pi i \beta l e^{-2\pi i \beta l t} \overline{g(t-\alpha k)} + e^{-2\pi i \beta l t} \overline{g'(t-\alpha k)} \right) dt \\
&\qquad\quad \left. + \int_{\mathbb{R}} f(t) e^{-2\pi i \beta l t} \overline{g'(t-\alpha k)}\, dt \right) \pi(\alpha k, \beta l) \\
&\quad = \sum_{k,l\in\mathbb{Z}} \left(\int_{\mathbb{R}} f(t) 2\pi i l e^{-2\pi i \beta l t} \overline{g(t-\alpha k)}\, dt \right) \pi(\alpha k, \beta l) \\
&\quad = 2\pi i \sum_{k,l\in\mathbb{Z}} l \langle f, \pi(\alpha k, \beta l) g\rangle \pi(\alpha k, \beta l) \\
&\quad = \partial_2({}_\bullet\langle f, g\rangle).
\end{aligned}
$$

□

Furthermore, ∇_1 and ∇_2 have a constant curvature:

$$F_{1,2} = \nabla_1\nabla_2 - \nabla_2\nabla_1 g(t) = 2\pi i(tg'(t) - g(t) - tg'(t)) = 2\pi i g(t),$$

so $F_{1,2} = -2\pi i\, I$. We shall see how the fact that $S_0(\mathbb{R})$ has a constant curvature connection play a major role, as we will revisit the Balian-Low theorem from section 2. We will give a proof of the theorem, showing a link between Gabor theory and noncommutative geometry. First we give a lemma linking the Fourier transformation with the covariant derivatives.

Lemma 6.3.2. *For $g \in L^2(\mathbb{R})$, the following identities hold:*

i) $\nabla_1 \hat{g} = \frac{\beta}{\alpha}\widehat{\nabla_2 g}$.

ii) $\nabla_2 \hat{g} = -\frac{\alpha}{\beta}\widehat{\nabla_1 g}$.

Proof.

i)

$$\begin{aligned}
\nabla_1 \hat{g}(\omega) &= \frac{2\pi i}{\alpha}\omega \hat{g}(\omega) \\
&= \frac{2\pi i}{\alpha}\omega \int_{\mathbb{R}} g(t)e^{-2\pi i\omega t}\,dt \\
&= \frac{2\pi i}{\alpha}\omega \left(\left[\frac{e^{-2\pi i\omega t}}{-2\pi i\omega} g(t) \right]_{-\infty}^{\infty} - \int_{\mathbb{R}} g'(t)\frac{e^{-2\pi i\omega t}}{-2\pi i\omega}\,dt \right) \\
&= \frac{1}{\alpha}\int_{\mathbb{R}} g'(t)e^{-2\pi i\omega t}\,dt \\
&= \frac{\beta}{\alpha}\widehat{\nabla_2 g}(\omega).
\end{aligned}$$

ii)

$$\begin{aligned}
\nabla_2 \hat{g}(\omega) &= \frac{1}{\beta}(\hat{g})'(\omega) \\
&= \frac{1}{\beta}\frac{d}{d\omega}\int_{\mathbb{R}} g(t)e^{-2\pi i\omega t}\,dt \\
&= \frac{1}{\beta}\int_{\mathbb{R}} \frac{\partial}{\partial\omega} g(t)e^{-2\pi i\omega t}\,dt \\
&= -\frac{2\pi i}{\beta}\int_{\mathbb{R}} t g(t)e^{-2\pi i\omega t}\,dt \\
&= -\frac{2\pi i}{\beta}\widehat{[tg(t)]}(\omega) \\
&= -\frac{\alpha}{\beta}\widehat{\nabla_1 g}(\omega).
\end{aligned}$$

□

We will now prove the Balian-Low theorem on the foundations of the geometric concepts provided by the vector bundle connections. Recall the Balian-Low theorem (theorem 2.6.1), here stated with the updated notation:

Theorem 6.3.3 (Balian-Low)**.** *Let g be a function in $L^2(\mathbb{R})$ such that both $\nabla_1 g$ and $\nabla_1 \hat{g}$ are in $L^2(\mathbb{R})$. Then $\mathcal{G}(g, \Lambda)$ cannot be an orthonormal basis for $L^2(\mathbb{R})$.*

The idea that the restrictions on well-behavedness of Gabor-frame generating functions originate from the geometric structure provided by vector bundles was one of Franz Luef's. The following proof follows the lines of the one in his paper [16].

Proof of the Balian-Low theorem. Assume that $\mathcal{G}(g,\Lambda)$ is an orthonormal basis for $S_0(\mathbb{R})$, and at the same time that both $\nabla_1 g$ and $\nabla_1 \hat{g}$ are functions in $L^2(\mathbb{R})$. By lemma 6.3.2, we have $\nabla_1 \hat{g} = \frac{\beta}{\alpha}\widehat{\nabla_2 g}$, so this implies $\nabla_2 g \in L^2(\mathbb{R})$. We also have $\nabla_1^* = -\nabla_1$ and $\nabla_2^* = -\nabla_2$. Using the orthogonal expansions of $\nabla_1 g$ and $\nabla_2 g$ in the frame, we get

$$\begin{aligned}
\langle \nabla_1 g, \nabla_2 g\rangle &= \left\langle \sum_{k,l\in\mathbb{Z}} \langle \nabla_1 g, \pi(\alpha k, \beta l) g\rangle \pi(\lambda) g, \nabla_2 g \right\rangle \\
&= \sum_{k,l\in\mathbb{Z}} \langle \nabla_1 g, \pi(\alpha k, \beta l) g\rangle \langle \pi(\lambda) g, \nabla_2 g\rangle \\
&= \sum_{k,l\in\mathbb{Z}} \langle \pi(-\alpha k, -\beta l) g, \nabla_1 g\rangle \langle \nabla_2 g, \pi(-\alpha k, -\beta l) g\rangle \\
&= \langle \nabla_2 g, \sum_{k,l\in\mathbb{Z}} \langle \nabla_1 g, \pi(\alpha k, \beta l) g\rangle \pi(\alpha k, \beta l) g\rangle \\
&= \langle \nabla_2 g, \nabla_1 g\rangle.
\end{aligned}$$

This shows that

$$\langle F_{1,2} g, g\rangle = \langle (\nabla_1\nabla_2 - \nabla_2\nabla_1) g, g\rangle = 0.$$

However, since $F_{1,2}$ is a constant curvature connection, we would expect $\langle F_{1,2} g, g\rangle = -2\pi i \langle g, g\rangle \neq 0$ for a necessarily non-zero window g. This contradiction shows our assumptions to be false. Hence, $\mathcal{G}(g,\Lambda)$ cannot be an orthonormal basis if both $t\hat{g}(t)$ and $\omega\hat{g}(\omega)$ are in $L^2(\mathbb{R})$. $\square$

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