

Estimation of UAV Position, Velocity and Attitude Using Tightly Coupled Integration of IMU and a Dual GNSS Receiver Setup

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PROJECT DESCRIPTION SHEET

Name:	Martin Lysvand Sollie
Department:	Engineering Cybernetics
Thesis title (Norwegian):	Estimering av UAV posisjon, hastighet og orientering ved bruk av tett integrasjon av IMU og et to-GNSS-mottaker oppsett
Thesis title (English):	Estimation of UAV Position, Velocity and Attitude Using Tightly Coupled Integration of IMU and a Dual GNSS Receiver Setup

Thesis Description: During unmanned aerial vehicle (UAV) operations, access to reliable position, velocity and attitude (PVA) information of vehicle is a vital. An aided inertial navigation system (INS), based on inertial measurement units (IMUs), can be applied to provide this. The purpose of the thesis is to estimate the position, velocity and attitude of a UAV implementing an aided INS. The INS should be based on measurements collected and time-stamped using the sensor and timing board (SenTiBoard) developed at the UAVlab.

The following tasks should be considered:

- Perform a literature review on PVA determination based on INS and Global Navigation Satellite System (GNSS) and provide relevant background information on such systems.
- 2. Derive and implement an extended Kalman filter (EKF) applicable for estimating position, velocity and attitude (PVA) of the UAV based on IMU and dual GNSS measurements.
- 3. Test the EKF using data collected from one of the UAVlab's fixed-wing UAVs. Use logged flight controller estimates and high-precision (real time kinematic (RTK)) calculated antenna positions to derive relevant reference signals for comparison.
- 4. Discuss results and error sources.
- 5. Conclude your results and suggest future work.

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Abstract

Increasing use of UAVs in high-precision applications, such as georeferencing and photogrammetry, increases the requirements on the accuracy of the estimated position, velocity and attitude of the vehicle. Commercial systems that utilize magnetometers in the heading estimates are cheap, but are affected by disturbances from both the vehicle itself and variations in Earth's magnetic field. On the other side, commercial dual-antenna satellite navigation systems can provide the required accuracy, but are expensive. This thesis explores the use of a low-cost setup using two independent GPS receivers, aiding an inertial navigation system. A multiplicative extended Kalman filter using unit quaternions as the nominal attitude parametrization and an error state based on the Gibbs vector is derived and implemented as the estimation algorithm. The system model is driven by measurements from a Sensonor STIM300 inertial measurement unit. Raw GPS pseudorange, Doppler frequency and carrier phase measurements from two longitudinally separated U-Blox NEO-M8T receivers are used for corrections, taking into account the lever arm to each antenna. The filter uses carrier phase interferometry with double differencing of the carrier phase measurements, and estimates the related ambiguities as float values which are fixed to integer values using the LAMBDA algorithm. The results show that carrier phase interferometry enables accurate heading estimation in static conditions, but in highly dynamic conditions the heading is already observable without it, and is estimated well using only a single antenna in the test performed. Compared to the estimates from the flight controller, the velocity has less noise and is accurate on a centimeter level. The position from the implemented algorithm and the flight controller is quite consistent except for some drift in altitude, despite showing an eastward bias compared to RTKLIB.

Keywords: Inertial Navigation System, Inertial Measurement Unit, Global positioning system, Global Navigation Satellite System, Integer ambiguity resolution, Multiplicative Extended Kalman Filter, State estimation, Unmanned aerial vehicle

Sammendrag

Økende bruk av ubemannede fly i høypresisjons-anvendelser, som georeferering og fotogrammetri, øker kravene til korrektheten til estimatene på fartøyets posisjon, hastighet og orientering. Kommersielle systemer som benytter magnetometere for girvinkelestimering er billige, men påvirkes av forstyrrelser både fra fartøyet selv og variasjoner i jordens magnetfelt. På den andre siden kan kommersielle to-antenne satellittnavigasjonssystemer gi den påkrevde estimeringstreffsikkerheten, men er kostbare. Denne masteroppgaven utforsker bruken av et lavkostnads oppsett med to uavhengige GPS-mottakere, for assistering av et treghetsnavigasjonssystem. Et multiplikativt utvidet Kalman-filter som bruker enhetskvaternioner som parametrisering av den nominelle orienteringen og en feiltilstand basert på Gibbs-vektoren utledes og implementeres som estimeringsalgoritme. Systemmodellen drives av målinger fra en Sensonor STIM300 treghetssensorenhet. Rå GPS-målinger av pseudoavstand, Dopplerfrekvens og bærebølgefase fra to U-Blox NEO-M8T mottakere separert i lengderetningen på flyet brukes for korreksjoner, og tar høyde for armlengden til hver antenne. Filteret bruker bærebølge-interferometri med dobbeldifferensiering av bærebølgemålingene, og estimerer de relaterte tvetydighetene som flyttall før heltallsverdier finnes ved bruk av LAMBDA-algoritmen. Resultatene viser at bærebølge-interferometri muliggjør treffsikker estimering av girvinkel i statiske tilfeller, men i tilfeller med høy dynamikk er girvinkelen allerede observerbar uten, og estimeres godt selv med bare én antenne i den gjennomførte testen. Sammenlignet med resultatene fra flyets kontrollsystem har hastigheten mindre støy og er treffsikkert på et centimeter-nivå. Posisjonen fra den implementerte algoritmen og flyets kontrollsystem er ganske konsistent bortsett fra litt drift i høyde, selv om den har et konstant avvik mot øst sammenlignet med RTKLIB.

Preface

The work on this thesis was carried out in the Spring of 2018 at the Department of Engineering Cybernetics at the Norwegian University of Science and Technology. The work amounts to a full semester (30 credits). This is a continuation of my project thesis work (Sollie, 2017) which focused on heading determination using only dual-antenna GNSS with the main focus on carrier phase measurements, using pseudoranges (without atmospheric corrections) only to obtain relative clock errors between receivers and line-of-sight vectors. In the project thesis measured Doppler frequency was used to extrapolate the carrier phase measurements from two receivers, measured at different times, to the same instant to allow differencing. This thesis replaces this with INS-smoothed estimates of the carrier phase rate, and integrates this with IMU measurements, pseudorange (with atmospheric corrections) and Doppler shift for accurate estimation of position, velocity and complete attitude.

The GPS dataset used here is the same as the one used in the project thesis. The software routines for extraction of GPS ephemerides and other parameters from the navigation message, which was written by me for the project thesis, is reused, but has been expanded to also extract ionospheric correction parameters. The routine for computation of satellite position, velocity and clock errors is based on work by Hansen (2017b), but some minor improvements were made for better handling of satellite clock errors. Some functions from the MSS GNC Toolbox of Fossen and Perez (2004) has been used for small calculations like conversion from ECEF coordinates to latitude-longitude representation. RTKLIB has been used to generate reference positions for both receivers, for comparison with the estimates obtained with the proposed algorithm. The Matlab implementation of the LAMBDA algorithm from Verhagen et al. (2012) was used for integer ambiguity resolution.

I would like to thank my co-supervisors Torleiv H. Bryne and Kristoffer Gryte for the advice they have given me during the semester, and especially the feedback they have given on draft versions of this thesis. I would also like to thank Postdoc Frederik Leira, Lars Semb and Pål Kvaløy of the NTNU UAV Lab for the logged data provided from a UAV flight, and Sigurd Albrektsen for guidance on the timing data provided by the SenTiBoard. Frederik Leira and Håkon Helgesen provided measurements of lever arms for the GNSS antennas on the UAV.

Martin Lysvand Sollie Trondheim, June 2018

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Abbreviations

6-DOF six-degree-of-freedom.

ADR accumulated doppler range.

AHRS attitude and heading reference systems.

AP autopilot.

BDS BeiDou Navigation Satellite System.

BPSK binary phase shift keying.

CAI cold atom interferometry.

CDMA code division multiple access.

DDCP double differenced carrier phase.

DDPR double differenced pseudorange.

DLL delay lock loop.

ECEF Earth Centered Earth Fixed.

ECI Earth Centered Inertial.

EGNOS European Geostationary Navigation Overlay System.

EKF extended Kalman filter.

FOG fiber optic gyroscope.

GEO geostationary Earth orbit.

GNC guidance, navigation and control.

GNSS global navigation satellite system.

GPS Global Positioning System.

GPST Global Positioning System Time.

HOW handover word.

IGSO inclined geosynchronous orbit.

IMU inertial measurement unit.

INS inertial navigation system.

ISA inertial sensor assembly.

KF Kalman filter.

LOS line of sight.

LS least squares.

MEKF multiplicative extended Kalman filter.

MEMS micro-electro-mechanical system.

MEO medium Earth orbit.

MIT Massachusetts Institute of Technology.

NED North East Down.

NLO nonlinear observer.

NMR nuclear magnetic resonance.

OCXO ovenized crystal oscillator.

PIGA pendulous integrating gyroscopic accelerometer.

PLL phase lock loop.

PPK post-processed kinematic.

PPS pulse per second.

PRN pseudorandom noise.

PVA position, velocity and attitude.

PVTA position, velocity, time and attitude.

RLG ring laser gyroscope.

RMS root mean square.

RTK real-time kinematic.

SBAS space based augmentation system.

- **SV** space vehicle.
- **TCXO** temperature compensated crystal oscillator.
- **TOV** time of validity.
- UAV unmanned aerial vehicle.
- **USNO** United States Naval Observatory.
- UTC universal coordinated time.
- XO crystal oscillator.

Nomenclature

- \boldsymbol{b}^{b} Baseline vector decomposed in $\{b\}$.
- \mathbf{B}^{b} Baseline matrix decomposed in $\{b\}$.
- \mathbf{L}^{e} Line of sight matrix decomposed in $\{e\}$.
- l^e Line of sight vector decomposed in $\{e\}$.
- A Between satellite differencing matrix.
- $\mathbf{S}(\boldsymbol{\omega}_{eb}^{b})$ Skew symmetrix matrix representation of the vector $\boldsymbol{\omega}_{eb}^{b}$.
- I Identity matrix.
- \vec{b} Coordinate free baseline vector representation.
- \vec{l} Coordinate free line of sight vector representation.
- $\| \cdot \|_2 \,$ The Euclidean norm giving the length of $\cdot.$
- q_a^b Unit quaternion from frame $\{a\}$ to frame $\{b\}$.
- ⊗ Quaternion product operator.
- \mathbf{R}_{a}^{b} Rotation matrix from frame {*a*} to frame {*b*}.
- $\mathbf{R}(\boldsymbol{q}_a^b) \ \boldsymbol{q}_a^b$ converted to a rotation matrix.
- *{i}* Inertial coordinate frame.
- *{e}* Earth Centered Earth Fixed coordinate frame.
- $\{n\}$ North East Down coordinate frame.
- $\{b\}$ BODY coordinate frame.
- ϕ Roll angle.
- θ Pitch angle.
- ψ Yaw angle.
- Θ_{nb} Euler angle vector representing the attitude of BODY relative NED.
- f_{L1} GPS L1 signal carrier frequency 1575.42 MHz.
- μ LAMBDA ratio test threshold.

- $\Delta \phi_{\alpha\beta,s}$ Carrier phase of satellite *s* differenced between receivers α and β . The notation ϕ_{sd} is also used for unspecified satellites and receivers.
- $\nabla \Delta \phi_{\alpha\beta,s_1s_2}$ Carrier phase differenced between receivers α and β and between satellites s_1 and s_2 . The notation ϕ_{dd} is also used for unspecified satellites and receivers.
- ϕ Carrier phase.
- λ Wavelength of the L1 carrier (approximately 19cm).
- Δf Doppler frequency.
- δt General clock error.
- $t_{\varepsilon f}$ Clock bias of front GNSS receiver.
- $t_{\varepsilon\beta}$ Clock bias of back GNSS receiver.
- t_{df} Clock drift rate of front GNSS receiver.
- $t_{d\beta}$ Clock drift rate of back GNSS receiver.
- P Pseudorange.
- ρ Geometric range.
- *c* Speed of light.
- $I_{\alpha,s}$ Ionospheric delay and phase advance between satellite *s* and receiver α .
- $T_{\alpha,s}$ Tropospheric delay between satellite *s* and receiver α .
- ϵ Noise and other unmodelled errors.
- N Carrier phase ambiguity vector.
- *x* True state.
- \mathbf{P}_k A posteriori state covariance matrix at timestep *k*.
- $f(x_k, u_k)$ Nonlinear state transition model.
- F Linearized continous state dynamics matrix.
- $\mathbf{\Phi}_k$ Linearized state transition matrix from timestep k 1 to k.
- $h(\cdot)$ Nonlinear measurement model.
- H_k Linearized measurement matrix at timestep k.
- \mathbf{K}_k Kalman gain at timestep k.

- \mathbf{Q}_k Discrete time process noise covariance matrix at timestep *k*.
- \mathbf{R}_k Measurement covariance matrix at timestep k.
- *k* Time step.
- \hat{x} Nominal state.
- ∂ Partial derivative.
- Δt Sampling time.
- $\sigma\,$ Standard deviation.
- δx Error state.
- \hat{x}_k^- A priori state estimate at timestep *k*.
- \hat{x}_k A posteriori state estimate at timestep k.
- u_k System input at timestep k.
- z_k Measurement at timestep k.
- w_k Process noise vector at timestep k.
- \mathbf{P}_k^- A priori state covariance matrix at timestep *k*.
- p_{bc}^{a} Position vector of frame $\{c\}$ relative frame $\{b\}$ decomposed in frame $\{a\}$.
- v_{bc}^{a} Linear velocity vector of frame $\{c\}$ relative frame $\{b\}$ decomposed in frame $\{a\}$.
- f_{bc}^{a} Specific force of frame $\{c\}$ relative frame $\{b\}$ decomposed in frame $\{a\}$.
- ω_{bc}^{a} Angular velocity of frame $\{c\}$ relative frame $\{b\}$ decomposed in frame $\{a\}$.
- g^a Gravity vector decomposed in frame $\{a\}$.
- *a* Attitude error parametrization.

Part I

Introduction and Background

Introduction

1.1 Problem background and motivation

The small UAVs used by hobbyists and researchers today are commonly equipped with an autopilot system estimating its position, velocity and attitude using one or more inertial measurement units (IMUs) containing gyros and accelerometers, a single global navigation satellite system (GNSS) receiver, a barometer and a magnetometer (magnetic compass). Examples of such systems are the Pixhawk series of flight controllers (PX4 Dev team, 2018). GNSSs provides position and velocity measurements which are free of long-term drift. However, there are many challenges with GNSS as a source of position and velocity measurements. Firstly, measurements are normally available at a rate too low for feedback control in highly dynamic systems, such as UAVs, and the measurements can be noisy. Secondly, because the signals received from the satellites are very weak, the receivers can be disturbed by jamming (deliberately increasing the background noise floor, making it difficult or impossible for the receiver to track the signal) or spoofing (transmitting a fake signal, deceiving the receiver). Obstructions between the receiver and satellites can block the signals, making it less suitable in valleys or dense urban environments and basically unsuitable for indoor navigation.

The use of an inertial navigation system (INS), consisting of an IMU and the processing required to estimate position, velocity and attitude, offers several advantages. As they are completely self contained and do not rely on any external signals, they cannot be disturbed by external effects. The measurements also typically have low noise and are available at a high rate, giving smooth position and attitude outputs. When an IMU is used the dynamic model for a vehicle can be replaced by simple kinematic equations driven by the IMU outputs, meaning that the kinetics of the vehicle, including torques and forces, can be omitted (Roumeliotis et al., 1999). The drawback of inertial navigation is that all IMUs experience slowly varying errors that cause position and attitude estimates, based on the integration of angular rate and specific force, to drift over time. The drift can be reduced by using better, more expensive sensors, but cannot be eliminated completely. Due to the complementary nature of INS and GNSS, combining the measurements in an estimation algorithm such as the Kalman filter (Kalman, 1960) can give the best of both worlds. The long-term drift is eliminated by estimating and compensating the INS errors using GNSS measurements, while the INS is used to smooth the output and provide position, velocity and attitude (PVA) estimates even when the GNSS receiver experiences signal problems.

UAVs using today's low cost autopilot systems are mostly used for waypoint flying and other tasks where small attitude errors pose no large problem as a deviation from the desired flight path, which is visible in the observed position and velocity errors, is handled using feedback control. For these uses having a correct course (direction of movement) is more important than a correct heading. For other uses however, having an accurate estimate of the full attitude is important: A telephoto camera or narrow-beam high-gain antenna mounted on a UAV which is to be pointed on a location on the ground can miss the target completely if the attitude is inaccurate.

The accelerometer in an IMU can be used to initialize the pitch and roll estimates and maintain these as reasonably accurate estimates, in addition to providing linear acceleration, used as input to the kinematic equations. While in flight, the observed errors between the GNSS measurements and the INS/barometer estimates are used to make corrections to the position, velocity, attitude, and IMU biases and any other states, provided the vehicle movement is sufficiently exciting for the error state to be observable. This correction improves the accuracy of the estimates.

For UAVs flying in a steady state, or hovering, with low acceleration and angular rate, the errors in attitude and IMU biases are not observable with only a single GNSS antenna, and the estimate will rely on the magnetic compass, which is used to initialize the heading and keep it reasonably accurate. The magnetometers typically used are susceptible to disturbances from irregularities in the Earth's magnetic field, or ferrous materials or electrical currents close to the sensor (Gade, 2016). A magnetic compass is also not very useful when navigating near the magnetic poles, and the local magnetic declination values for the areas of operation must be known. Improved estimates of heading can be obtainted by the use of dual-antenna GNSS, and with three or more antennas full attitude can also be found. Commercial systems using GNSS for heading or attitude, such as the Vectornav VN-300 (Vectornav, 2017), are however significantly more expensive than the autopilot systems discussed. The use of dual-antenna GNSS for UAV heading determination was explored in Sollie (2017), but the lack of other measurements resulted in long convergence times, or no convergence at all, if the initial heading estimate was incorrect. The integration of multi-antenna GNSS with INS, also utilizing the pseudorange and Doppler frequency measurements should improve this, and also enable estimation of full attitude.

1.2 A short historic review of inertial navigation

Professor Charles Stark Draper and his team at the Massachusetts Institute of Technology (MIT) Instrumentation Laboratory is commonly credited with the development of the first complete INS estimating position, velocity and attitude (Wrigley, 1977; Wildenberg, 2016). Draper had been thinking about this since the early 1930s, but at the end of WWII he saw the opportunity to gain support and funding for his idea (Draper, 1981; Wildenberg, 2016). Because of the gyro drift in the sensors of the time, Draper initially added a celestial tracking system to provide corrections, but he only saw this as a temporary solution to the limited gyro

accuracy, well aware that celestial tracking was subject to interference (MacKenzie, 1990). After submitting an initial study, the green light was given by the Air Force in 1947 to proceed with a program to test the feasibility of contructing an INS. An early conclusion was that new sensor technology was needed to give the required accuracy over the flight duration of 5-10 hours, with performance improvements of about 10,000 times that of common devices (Draper, 1981). Because of the low acceleration environment of a bomber aircraft and the long flight time, the main source of positioning accuracy of the system was the heading error cause by gyro drift (Wildenberg, 2016). The system used was a platform-based INS using rate gyros controlling servos in the suspension, keeping the plaform aligned with the horizontal. A sun tracker and magnetic compass was also used (Wrigley, 1977; Draper, 1981). Two accelerometers were placed on the plaform with the sensitive directions in the north-south and east-west directions (Wildenberg, 2016). Schuler tuning was used to maintain a level platform as the aircraft flew. The first system tested in flight was known as FEBE. This was done in 1949 and while it did not give sufficient accuracy for bombing missions, the results were encouraging enough that a follow on project was started to develop a purely inertial system (Wrigley, 1977), known as Space Inertial Reference Equipment (SPIRE). Three gyros with increased accuracy, developed by the lab, were used in the platform design. Testing in 1953 on a 12 hour flight were successful, showing that purely inertial navigation over long distances was feasible (Wildenberg, 2016).

Development of INSs for ballistic missiles and the Apollo program followed in the 1950s and 60s. The Apollo INS used for the moon landings was developed under supervision of the MIT Instrumentation Laboratory (Wrigley, 1977; Draper, 1981). In 1958 the submarine Nautilus navigated to the North Pole under the polar ice, also using systems from Draper's lab (Lawrence, 1993). The use of INS is important for submarine navigation to this day, as GNSS cannot be used. The first INS to be certified by the Federal Aviation Administration (FAA) for commercial aviation was the Litton LTN-51 in 1968 (Tazartes, 2014). It first entered service on a Boeing 707 (Potocki De Montalk, 1991).

The ring laser gyroscope (RLG) was first demonstrated experimentally by Macek and Davis (1963) and the fiber optic gyroscope (FOG) was proposed by Vali and Shorthill (1976). The rapidly increasing computing power available and the development of gyroscopes with high dynamic range made strapdown systems possible at the end of the 1970s (Tazartes, 2014). The introduction of semiconductor device fabrication, integrated circuits and micro-electro-mechanical system (MEMS) technology made it possible to create inertial sensors on silicon wafers (Southwest Center for Microsystems Education, 2001). The first such instrument was developed by The Charles Stark Draper Laboratory (renamed from the MIT Instrumentation Laboratory) in 1991 (Greiff et al., 1991). The spinning mass gyro is not well suitable for implementation using MEMS technology because small low friction bearings are difficult to manufacture (Trusov, 2011). Most MEMS gyros

use the vibrating mass principle. The first MEMS accelerometer was produced in high volume by Analog Devices in 1993 (Southwest Center for Microsystems Education, 2001), finding use in car airbag deplyment. The low cost of MEMS sensors has lead to their use in many consumer devices such as cellphones, but also development of new products such as self balancing vehicles and unmanned aerial vehicles used in the industry and by consumers.

In recent years work has been done on the development of new inertial sensor technologies such as nuclear magnetic resonance (NMR) gyros (Meyer and Larsen, 2014) and cold atom inertial sensors (Battelier et al., 2016).

1.3 Recent work in the field of UAV navigation

The reduction in size and cost of IMUs has lead to small unmanned aerial vehicles being readily available for consumers. Because of the many uses for this technology, research into indoor navigation and robust autonomous flight has been increasing (Zheng et al., 2017). While systems navigating outdoors can rely on GNSS most of the time, using other sensors for attitude estimation and tolerance of GNSS denied environments (Perez-Grau et al., 2018; Tang et al., 2018; Layh and Gebre-Egziabher, 2017), indoor navigation cannot usually use GNSS at all. Increasing available processing power and better camera technology has lead to significant research in the field of visual navigation (Lu et al., 2018; Fink et al., 2017; Brahmbhatt et al., 2017; Cerón et al., 2018). Other aiding sources can be used to increase the robustness of GNSS by detecting spoofing (Qiao et al., 2017). LIDARs are also being used for INS aiding (Tang et al., 2015; Zheng et al., 2017).

New integration filters have received attention in the last decades. Nonlinear observers, where stability properties can be verified theoretically before implementation, which is generally not the case for the variants of the nonlinear extention of the Kalman filter known as the extended Kalman filter, has been researched by i.e. Vik et al. (1999); Hansen (2017a). The proven stability properties of nonlinear observers has been combined with the near-optimality of the linearized Kalman filter (KF) in the eXogenous Kalman Filter (XKF) (Johansen and Fossen, 2017), and its variants such as the multiplicative XKF (MXKF) (Stovner et al., 2018), where the KF linearizes the dynamics around a state estimate from a nonlinear observer which can be sub-optimal but have proven stability properties. Neural networks have also been applied to INS/GNSS integration (Noureldin et al., 2011).

Cooperation of multiple vehicles, swarms, for improved navigation is currectly being researched (Vetrella et al., 2018; Yang et al., 2018).

1.4 Multi-antenna GNSS heading and attitude determination

A brief review of the history and previous work in the field of GNSS attitude determination can be found in Sollie (2017), and is repeated here:

The first prototype Global Positioning System (GPS) satellite was launched by the United States government in February of 1978 (Pace et al., 1995). The use of the system for attitude determination was suggested even before this. V.W. Spinney, an employee of Rockwell International, the company responsible for building the first eleven prototype GPS Block I satellites (Pace et al., 1995), proposed the use of interferometric principles (the use of differential signal phase shift) in a receiver configuration with multiple antennas, focusing on how attitude could be found from measurements of differential antenna-tosatellite range (Spinney, 1976). Spinney did however not go into detail on how to compute the differential range from the GPS signals.

Greenspan et al. (1982) presented experimental results showing that interferometric processing of carrier phase observables, using *single* differencing (differencing between receivers), is a feasible way to survey short baselines. A breadboard prototype GPS receiver able to track 4 satellites for each end of a single baseline (8 channels) was used. The focus here was however precise relative position, not heading determination. Brown et al. (1982) proposed using single differenced carrier phase for differential ranging for heading determination, aided by an IMU, and proposed a design of a GPS interferometer using three antennas with 1m baselines. Computer simulation was used to assess the performance.

The first real-time test of full attitude determination using three antennas was performed in 1988 on the guided-missile cruiser USS Yorktown (Kruczynski et al., 1989). Two baselines with lengths 60cm and 40cm mounted perpendicularly were used, with a protoype 18-channel GPS receiver. Testing of longer baselines was recommended, even though the increased integer search space resulting from this was seen as a challenge.

Purcell et al. (1989) performed the first test of GPS attitude determination from an aircraft, using a single 23m baseline with two receivers placed longitudinally on the fuselage of a DC-8. The first multi-baseline test on an aircraft was done by Van Graas and Braasch (1991) using double differenced carrier phase (DDCP) with four antennas and a 24-channel receiver (double differencing involves first differencing between receivers, then between satellites).

The PhD thesis of Cohen (1992) describes the design of a GPS receiver that can determine position, velocity, time and attitude (PVTA). Test data from flights on a Piper Dakota aircraft is presented. Cohen et al. (1994) presents results from testing of GPS attitude determination of a spacecraft in low Earth orbit. In 1993 Cohen filed a patent for a "System and Method for Generating Attitude Determinations Using GPS" (Cohen, 1996), where four antennas and carrier phase is used. The patent was granted in 1996.Posisjonen din

In 1993, the LAMBDA method for efficient integer ambiguity resolution was introduced (Joosten and Tiberius, 2002). Integer ambiguity resolution is an

inportant step when using carrier phase, since the carrier phase contains range information which is ambigous by whole wavelengths. The resolved integers are needed to transform the differenced *carrier* phase to *range* differences.

The PhD thesis of Lu (1995) describes the development of a multi-antenna GPS system for attitude determination using multiple off-the-shelf receivers. Chu and van Woerkom (1997) discusses the fusion of attitude from low-cost GPS receivers with other low-cost attitude sensors.

Bar-Itzhack et al. (1997) and Nadler et al. (2000) discusses techniques and algorithms for attitude determination using differential carrier phase measurements, assuming that integer ambiguities have been resolved. They focus on the operational attitude calculations and how this is done when a unit quaternion is used as attitude representation.

Garcia et al. (2005) explores the use of double differencing with independent receivers for a **stationary** baseline, correcting for the difference in measurement time resulting from the use of the local clock, normally only approximately aligned with Global Positioning System Time (GPST), to schedule the measurement outputs.

A performance evaluation of a completely different method for attitude determination is presented in Wang et al. (2007). A single precisely calibrated antenna is used by comparing the carrier-to-noise ratio c/n_0 of the received signals to the known gain pattern of the antenna. This is tested with several GNSS constellations, but the results shows that the accuracy is significantly worse than what the interferometric method provides.

Jurkowski et al. (2012) describes the development of integer ambiguity resolution methods that utilize a priori statistical and deterministic baseline information, showing faster integer resolution than unconstrained LAMBDA and more robust results than contrained LAMBDA.

Currently the International Space Station uses GPS for attitude determination (ISS Handbook ADCO, 2015) with an array of four antennas in a 3×1.5 m rectangle (Gomez and Lammers, 2004). The Russian Soyuz-MS spacecraft also uses GPS and GLONASS for attitude determination with its ASN-K satellite navigation system (Zak, 2017).

1.5 Main contribution

The main contributions of this thesis are

- Derivation of models for system dynamics and measurements for tightly coupled integration of dual-antenna GNSS with INS using a multiplicative extended Kalman filter (MEKF).
- Handling of relative time delay between IMU and GNSS measurements.

- Formulation of a measurement model for double differenced carrier phase measurements where measurements from each receiver are taken at different times. This results in extrapolation of the measurement from one receiver to the measurement time of the other receiver using estimated range rate rather than the more noisy Doppler frequency measurements. This enables the usage of multiple receivers not specifically designed for attitude detimination to estimate the UAV heading.
- Handling changes in tracked satellites while keeping the covariance matrix and state vector as small as possible by removing the ambiguity values for satellites where carrier phase lock is lost.
- Testing of the proposed estimation algorithm using data collected during a UAV flight.
- Comparison of the resulting estimates to logged estimates from the Pixhawk flight controller and post-processed kinematic (PPK) results from RTKLIB (Takasu, 2017).

1.6 Outline of thesis

This thesis is organized in 4 parts with a total of 8 chapters:

Part 1: Introduction and Background

Chapter 1: Introduction. The background and motivation for the work is presented. The main contributions are listed and an outline of the thesis is given.

Chapter 2: INS Preliminaries. This chapter presents the strapdown navigation equations using the unit quaternion as attitude parametrization and cartesian position coordinates using the Earth Centered Earth Fixed (ECEF) reference frame are derived. Different sensor technologies and IMU error sources are also discussed.

Chapter 3: GNSS Preliminaries. A short introduction to GNSS and the signal structure of GPS is given, and the pseudorange, Doppler frequency and carrier phase observables, and their measurement models, are introduced. The most important error sources are also presented.

Chapter 4: Integration of INS and GNSS. Different integration architectures are explained, including the tightly coupled integration implemented here. The requirements for all errors to be observable are briefly discussed, and the motivation of the MEKF is explained.

Part 2: Aided INS Design

Chapter 5: Algorithm design. The choice of attitude error parametrization for

the MEKF is discussed. The nominal system dynamics, linearized error state dynamics, discrete time versions of these and measurement models are derived. The handling of different measurement times, and of changes in the tracked satellites, is explained.

Chapter 6: Experimental Setup. The equipment used for the UAV flight is presented, with measurements of the GNSS antenna lever arms and baseline.

Part 3: Results

Chapter 7: Results and Discussion. Position, velocity and attitude estimates are compared with estimates from both the Pixhawk flight controller and PPK results using RTKLIB.

Chapter 8: Closing Remarks The work and results are concluded and possible further work is discussed.
INS Preliminaries

This chapter introduces the sensors used in an INS and the motivation for inertial navigation. The strapdown navigation equations used for predicting position, velocity and attitude over time are derived, and INS error sources are presented. The chapter ends with an explanation of how attitude can be determined by the use of accelerometer leveling and gyrocompassing.

2.1 Inertial sensors theory of operation

An INS is a system consisiting of an IMU, which is a sensor assembly with interface hardware and low-level downsampling, measurement calibration and error compensation software, and software that computes the attitude, position and velocity from the measurements. The inertial sensor assembly (ISA) of the IMU commonly consists of three orthogonally mounted gyros and accelerometers, allowing full six-degree-of-freedom (6-DOF) sensing.

2.1.1 Gyro

The gyroscope is an inertial sensor that measures either orientation directly, referred to as an angle gyros, or angular rates, referred to as rate gyros. For rate gyros the attitude can be found by integration of the rates if the initial attitude is known. The classical mechanical gyro uses a spinning flywheel suspended in a gimbal that maintains its orientation in inertial space by the conservation of angular momentum. The attitude of the base of the gimbal, which for example can be fixed to the fuselage of an aircraft, relative to the flywheel itself can be read directly from the gimbal angles. RLGs and FOGs are optical gyroscopes utilizing the Sagnac effect to measure angular velocity. Light is transmitted in both directions of a closed path, and light traveling with the rotation of the gyro travels a longer distance and thus takes longer time to make it around. A FOG send light in an optical fiber, while the RLG used a mirror assembly. Coriolis Vibratory Gyroscopes uses a mass vibrating linearly, such as a tuning fork, which will tend to continue vibrating is the same plane in inertial space (like Foucault's pendulum) by excerting a force on its support if this rotates. By measuring the force the angular rate can be determined. This method is commonly used in MEMS gyros.

Gyro technologies currently being researched includes the nuclear magnetic resonance (NMR) gyro (Meyer and Larsen, 2014) and cold atom sensors (both gyro and accelerometer) (Battelier et al., 2016). NMR gyros uses clouds of gas such as xenon (which has an atom that is a magnetic dipole) in a constant magnetic field \vec{B}_0 . The spin axis of the atoms will precess around \vec{B}_0 with a frequency known as the *Larmor* frequency. The gas is polarized using a light source with a technique known as *optical pumping*, which creates a net magnetic moment in the

gas. If a sinusoid magnetic field is applied in addition to \vec{B}_0 , orthogonal to it and with a frequency which is a function of the Larmor frequency, the net magnetic moment will also precess around \vec{B}_0 . The frequency of this precession, which can be measured, is nominally the Larmor frequency, but with a deviation dependent on the angular rate around \vec{B}_0 .

Cold atom inertial sensors use cold atom interferometry (CAI), where interferometry is done using the wave functions of atoms (in stead of waves of light as for optical gyros). Clouds of atoms are cooled close to absolute zero using lasers (The 1997 Nobel Price in Physics was awarded for this technique (Nobel Media AB, 2014)). Lasers are then used to split atomic waves which are later recombined (Cronin and Trubko, 2015).

2.1.2 Accelerometer

An accelerometer measures specific force, which is acceleration relative to free-fall, or acceleration resulting from real applied nongravitational forces (Jekeli, 2001).

The simplest directional accelerometer consists of a proof mass attached to two springs in a casing. As a force is applied on the casing, the inertia of the proof mass causes it to deflect from its neutral position. The deflection can then be used as a measure of the force applied. In a gravitational field both the casing and the proof mass experience the same acceleration, and the proof mass maintains its neutral position, thus measuring no specific force. An improvement to this simple model is to make the proof mass maintain its neutral position by applying a force from i.e. an electromagnet, making it a closed loop system. The input signal to the actuator is then a function of the measured specific force. This method can be implemented using MEMS technology.

Other types are vibratory accelerometers which use the change in frequency of a quartz beam under variyng tension as a way to measure force, and surface acoustic wave (SAW) accelerometers (Hartemann and Meunier, 1981). The pendulous integrating gyroscopic accelerometer (PIGA) uses a pendulous mass, a spinning gyro and a torque motor to measure acceleration while at the same time integrating it mechanically to give velocity. CAI can also be used to create accelerometers (Battelier et al., 2016).

2.1.3 MEMS sensors

MEMS IMUs, while not being a different operating principle, enables gyros to be manufactured in large numbers on silicon wafers along with interface and low level processing hardware (Trusov, 2011). This makes them chip-scale and cheap to manufacture, allowing them to be placed in basically all electronic devices today, even cheap toys. It also makes in possible to make very small self-stabilizing vehicles and handheld camera stabilizing systems. Most current MEMS IMUs are typically not suitable for unaided navigation, but because GNSS is available for most outdoor consumer applications, they have been sufficient for many aided navigation systems.

2.2 Dynamic model replacement - the motivation for INS

INS is not the only method capable of predicting the attitude, velocity and position of a vehicle without external inputs. By modeling the complete kinematics and kinetics of a vehicle, including actuator dynamics, aerodynamics, vehicle mass, moment of inertia, the location of the center of mass, and external environmental effects such as wind, the state can, at least in theory, be predicted. Such models can be highly complex, requiring a large number of states, and difficult to create accurate enough to give sufficient performance. They would also be tailored for the specific vehicle setup and would need modification when using the system on different vehicles. The external environmental effects are not easy to model, and not necessarily observable. While possible in theory, dynamic models do not necessarily give sufficiently good performance, as shown in Lefferts and Markley (1976).

The main advantage of using INS is that the model of vehicle dynamics can be simplified to a simple general kinematic model, independent on the physical properties of the vehicle. This means that IMU-driven systems can be adapted to any vehicle easily. External disturbances such as wind or ocean currents does not have to be modelled, as we measure the acceleration and angular velocity of the vehicle directly.

2.3 Continous time inertial sensor modeling

2.3.1 Rate gyro

A rate gyro is commonly modelled as

$$\boldsymbol{\omega}_{\rm IMU} = \boldsymbol{\omega}_{im}^m + \boldsymbol{b}_g + \boldsymbol{w}_g \tag{2.1}$$

where $\{m\}$ is the measurement frame which can differ from the body frame depending on how the body frame is defined and how the gyro is mounted, b_g is a slowly varying error commonly referred to as the gyro bias and w_g is Gaussian noise.

2.3.2 Accelerometer

The specific force in an arbitrary frame $\{a\}$ is the acceleration relative to the gravitational acceleration,

$$f_{ia}^{a} = a_{ia}^{a} - g_{ia}^{a}, (2.2)$$

where a_{ia}^a is the coordinate acceleration in frame $\{a\}$ and g_{ia}^a is called plumb bob gravity as it points in the direction of a plumb line. This is not directed towards the center of the earth, but better modelled as being normal to the reference model describing the Earth as an ellipsoid. This gravity is the combination of the gravitational pull towards the center of the Earth, γ^a , and the outward centripetal force normal to the Earth's axis of rotation an object experiences, which can be written as

$$g^{a} = \gamma^{a} - \mathbf{S}^{2}(\boldsymbol{\omega}_{ie}^{a})\boldsymbol{p}_{ia}^{a}.$$
 (2.3)

The negative sign for gravity in (2.2) is because it is not gravity itself we are measuring, but the force counteracting gravity when acceleration is zero. For example an item laying on ground senses a force from the ground upwards preventing it from free-falling, which for the accelerometer is equivalent to accelerating upwards in the absence of gravity. An assembly of three orthogonal accemelometers in a measurement frame $\{m\}$ measures the specific force f_{im}^m , but also includes low and high frequency errors. A commonly used model is

$$f_{\rm IMU} = f_{im}^m + \boldsymbol{b}_a^m + \boldsymbol{w}_a^m \tag{2.4}$$

$$= a_{im}^m - g^m + b_a^m + w_a^m \tag{2.5}$$

where \boldsymbol{b}_a^m is a slowly varying error and \boldsymbol{w}_a^m is presumed white Gaussian noise.

For both the accelerometer and the gyro the random noise w^m can be assumed to be isotropic is the three sensors used in the assembly are identical. This means that the noise vector has a covariance matrix invariant to rotation.

2.4 Strapdown and platform systems

INSs are normally split in two different types: *platform systems* and *strapdown systems*. In platform systems, the accelerometers are mounted on a plaform placed in a stabilizing gimbal. Mechanical gyros, or rate gyros and actuators, are used to keep the platform aligned with the reference frame. The attitude can then be read directly as the gimbal angles, and the position can be found by integrating the accelerometer measurement twice. Because the acceleration is measured in the reference frame directly, no rotation of the acceleration measurement in necessary, they can be integrated directly. Because the platform is stabilized it experiences very low angular velocities, meaning that the gyros do not have to have a high maximum rate. A disadvantage is that the gimbal wears over time (Lawrence, 1993). Platform systems have the advantage of better isolation of the accelerometer from the vehicle body, reducing measurement noise.

In strapdown systems the gyros and accelerometers are attached directly to the vehicle body. This is a lot simpler mechanically and gives a lower cost system, but the navigation equations become more complex. Because there are no large and heavy moving parts with bearings and slip rings the system is more rugged and

reliable. The lack of actuators means that less power is needed to run the system. Strapdown systems can be made very small, with MEMS technology the entire attitude and heading reference systems (AHRS) including the navigation computer can be made as a single chip. Strapdown systems also have disadvantages related to cross-axis errors, alignment and sensor calibration (Lawrence, 1993). In the remainder of this thesis the focus will be on strapdown INS.

2.5 Strapdown navigation equations

2.5.1 Attitude

The rate of change of the attitude quaternion q_b^a , relating the body frame {*b*} and an arbitrary reference frame {*a*}, can be written as the limit of the average rate of the change from time *t* to *t* + Δt as the time increment approaches 0,

$$\dot{\boldsymbol{q}}_{b}^{a} = \lim_{\Delta t \to 0} \frac{\boldsymbol{q}_{b}^{a}(t + \Delta t) - \boldsymbol{q}_{b}^{a}(t)}{\Delta t}.$$
(2.6)

The quaternion $q_b^a(t + \Delta t)$ is equal to the quaternion product of $q_b^a(t)$ and the small rotation Δq^b which occurs during the time Δt . Dropping the explicit use of time indices this can be written as

$$\dot{\boldsymbol{q}}_{b}^{a} = \lim_{\Delta t \to 0} \frac{\boldsymbol{q}_{b}^{a} \otimes \Delta \boldsymbol{q}^{b} - \boldsymbol{q}_{b}^{a}}{\Delta t}.$$
(2.7)

This assumes that the small rotation is given in the body frame. If it is given in the reference frame, Δq^a , is would instead be multiplied on the left ($\Delta q^a \otimes q_b^a$). Using the axis-angle quaternion expression (E.13), the small angle approximations $\sin \phi = \phi$ and $\cos \phi = 1$ with the rotation vector $\Delta \theta$ gives $\Delta q = \begin{bmatrix} 1 \\ \Delta \theta/2 \end{bmatrix}$ for infinitesimal rotations. Using this and factoring out q_b^a results in

$$\dot{\boldsymbol{q}}_{b}^{a} = \lim_{\Delta t \to 0} \frac{\boldsymbol{q}_{b}^{a} \otimes \left(\begin{bmatrix} 1 \\ \Delta \theta/2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)}{\Delta t}.$$
(2.8)

Moving q_b^a outside the limit and simplifying gives

$$\dot{\boldsymbol{q}}_{b}^{a} = \boldsymbol{q}_{b}^{a} \otimes \lim_{\Delta t \to 0} \frac{\begin{bmatrix} 0 \\ \Delta \theta/2 \end{bmatrix}}{\Delta t}$$
(2.9)

The last limit is simply equal to a vector containing half the angular velocity in body-fixed coordinates,

$$\dot{\boldsymbol{q}}_{b}^{a} = \frac{1}{2} \boldsymbol{q}_{b}^{a} \otimes \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{\omega}_{ab}^{b} \end{bmatrix}, \qquad (2.10)$$

which will also be written as $\dot{q}_b^a = \frac{1}{2}q_b^a \otimes \omega_{ab}^b$ for simplicity. This is valid for any frame *a*, but depending on how this frame rotates relative to inertial space,

terms other than the angular velocity relative to inertial space will appear in ω_{ab}^b . For instance when using ECEF, {*e*}, coordinates, Earth's rate of rotation will be included,

$$\boldsymbol{\omega}_{eb}^{b} = \boldsymbol{\omega}_{ib}^{b} - \boldsymbol{\omega}_{ie}^{b} = \boldsymbol{\omega}_{ib}^{b} - \mathbf{R}_{e}^{b} \boldsymbol{\omega}_{ie}^{e}.$$
(2.11)

If the local North East Down (NED) frame is used as reference, the rotation resulting from Earth surface velocities must also be included,

$$\boldsymbol{\omega}_{nb}^{b} = \boldsymbol{\omega}_{ib}^{b} - \boldsymbol{\omega}_{ie}^{b} - \boldsymbol{\omega}_{en}^{b} = \boldsymbol{\omega}_{ib}^{b} - \mathbf{R}_{e}^{b} \boldsymbol{\omega}_{ie}^{e} - \mathbf{R}_{e}^{b} \mathbf{R}_{n}^{e}(\boldsymbol{p}) \boldsymbol{\omega}_{en}^{n}.$$
(2.12)

As long as the gyro is fixed to a vehicle and the vehicle is rigid, lever arms will have no effect except for potential g-errors (which can be corrected internally in higher quality sensors), since all points on a rigid body experience the same rotation. Misalignments between the measurement frame used by the gyro and the body frame must however be taken into account. With the measurement frame *m* and body frame *b* related with the rotation matrix \mathbf{R}_b^m , the rotation between inertial and body is

$$\mathbf{R}_{b}^{i} = \mathbf{R}_{m}^{i} \mathbf{R}_{b}^{m} \tag{2.13}$$

Using the rotation matrix derivative from Fossen (2011, p. 25), we have

$$\dot{\mathbf{R}}_{m}^{i} = \mathbf{R}_{m}^{i} \mathbf{S}(\boldsymbol{\omega}_{im}^{m}) \qquad \dot{\mathbf{R}}_{b}^{i} = \mathbf{R}_{b}^{i} \mathbf{S}(\boldsymbol{\omega}_{ib}^{b}) \qquad (2.14)$$

Differentiating (2.13), the latter of these can also be written as

$$\dot{\mathbf{R}}_{b}^{i} = \dot{\mathbf{R}}_{m}^{i} \mathbf{R}_{b}^{m} + \mathbf{R}_{m}^{i} \dot{\mathbf{R}}_{b}^{m} = \mathbf{R}_{m}^{i} \mathbf{S}(\boldsymbol{\omega}_{im}^{m}) \mathbf{R}_{b}^{m}, \qquad (2.15)$$

since $\dot{\mathbf{R}}_{b}^{m} = \mathbf{0}$. Setting the two expressions for $\dot{\mathbf{R}}_{b}^{i}$ equal and postmultiplying with \mathbf{R}_{i}^{b} gives

$$\dot{\mathbf{R}}_{m}^{i}\mathbf{S}(\boldsymbol{\omega}_{im}^{m})\dot{\mathbf{R}}_{i}^{m} = \dot{\mathbf{R}}_{b}^{i}\mathbf{S}(\boldsymbol{\omega}_{ib}^{b})\dot{\mathbf{R}}_{i}^{b}, \qquad (2.16)$$

which can be simplified to

$$\boldsymbol{\omega}_{im}^{i} = \boldsymbol{\omega}_{ib}^{i}. \tag{2.17}$$

The body frame angular velocity can thus be found by simply using \mathbf{R}_{m}^{b} .

2.5.2 Translation

Just like for attitude, there are multiple parametrizations that can be used for position. Cartesian coordinates are commonly used when the reference and resolving frames are Earth Centered Inertial (ECI) or ECEF. When local frames such as NED or wander azimuth (local frame like NED, but the x-axis is not forced to align with north making it nonsingular at the poles) are used as resolving frames, geodetic latitude, longitude and height are more commonly used. The use of cartesian coordinates with a fixed local frame (a tangent frame) as reference can be suitable for navigation over small distances such as for indoor navigation. Other possibilities, when representing the horizontal position, are the n-vector parametrization (Gade, 2010) and the rotation matrix from NED to ECEF, \mathbf{R}_n^e . Because of its use by GPS, strapdown navigation equations using cartesian ECEF coordinates as the navigation frame are suitable for integration of INS and GPS. For this reason the translational kinematics will be derived using cartesian coordinates, decomposed in ECEF as the reference frame.

Because the resolving and reference frame of the position is the same, the time derivative is simply the velocity,

$$\dot{p}_{eb}^{e} = v_{eb}^{e}.$$
 (2.18)

The same can be done for the accelerometer, $\dot{v}_{eb}^e = a_{eb}^e$, but because the accelerometer measures relative to inertial space this is not really interesting. Instead we use that

$$\boldsymbol{p}_{eb}^{e} = \boldsymbol{p}_{ib}^{e} \tag{2.19}$$

because the ECEF and ECI frames have the same origin. This is equivalent to

$$\boldsymbol{p}_{ib}^{e} = \mathbf{R}_{i}^{e} \boldsymbol{p}_{ib}^{i}. \tag{2.20}$$

Taking the time derivative results in

$$\dot{\boldsymbol{p}}_{ib}^{e} = \dot{\mathbf{R}}_{i}^{e} \boldsymbol{p}_{ib}^{i} + \mathbf{R}_{i}^{e} \dot{\boldsymbol{p}}_{ib}^{i}$$

$$= \mathbf{R}_{i}^{e} \mathbf{S}(\boldsymbol{\omega}_{ei}^{i}) \boldsymbol{p}_{ib}^{i} + \mathbf{R}_{i}^{e} \boldsymbol{v}_{ib}^{i}$$

$$= -\mathbf{S}(\boldsymbol{\omega}_{ie}^{e}) \boldsymbol{p}_{ib}^{e} + \boldsymbol{v}_{ib}^{e}.$$
(2.21)

Differentiating this with respect to time a second time,

$$\ddot{\boldsymbol{p}}_{ib}^{e} = \dot{\mathbf{R}}_{i}^{e} \mathbf{S}(\boldsymbol{\omega}_{ei}^{i}) \boldsymbol{p}_{ib}^{i} + \mathbf{R}_{i}^{e} \mathbf{S}(\dot{\boldsymbol{\omega}}_{ei}^{i}) \boldsymbol{p}_{ib}^{i} + \mathbf{R}_{i}^{e} \mathbf{S}(\boldsymbol{\omega}_{ei}^{i}) \dot{\boldsymbol{p}}_{ib}^{i} + \dot{\mathbf{R}}_{i}^{e} \boldsymbol{\upsilon}_{ib}^{i} + \mathbf{R}_{i}^{e} \dot{\boldsymbol{\upsilon}}_{ib}^{i}.$$
(2.22)

Using the approximation of constant angular rate for Earth's rotation, $\dot{\omega}_{ei}^{i} = \mathbf{0}$ (International GNSS Service (2018) Earth rotation products can be used for applications where this approximation is not accurate enough), this can be rewritten as

$$\ddot{\boldsymbol{p}}_{ib}^{e} = \mathbf{R}_{i}^{e} \mathbf{S}(\boldsymbol{\omega}_{ei}^{i}) \mathbf{S}(\boldsymbol{\omega}_{ei}^{i}) \boldsymbol{p}_{ib}^{i} + \mathbf{R}_{i}^{e} \mathbf{S}(\boldsymbol{\omega}_{ei}^{i}) \boldsymbol{v}_{ib}^{i} + \mathbf{R}_{i}^{e} \mathbf{S}(\boldsymbol{\omega}_{ei}^{i}) \boldsymbol{v}_{ib}^{i} + \mathbf{R}_{i}^{e} \boldsymbol{a}_{ib}^{i}.$$
(2.23)

This can be simplified to

$$\ddot{p}_{ib}^{e} = \mathbf{S}^{2}(\boldsymbol{\omega}_{ie}^{e})p_{ib}^{e} - 2\mathbf{S}(\boldsymbol{\omega}_{ie}^{e})v_{ib}^{e} + a_{ib}^{e}, \qquad (2.24)$$

exploiting that $v_{ib}^i = \mathbf{R}_e^i v_{ib}^e$ and that $\omega_{ei}^e = -\omega_{ie}^e$. Solving (2.21) for v_{ib}^e and substituting this in gives

$$\ddot{\boldsymbol{p}}_{ib}^{e} = -\mathbf{S}^{2}(\boldsymbol{\omega}_{ie}^{e})\boldsymbol{p}_{ib}^{e} - 2\mathbf{S}(\boldsymbol{\omega}_{ie}^{e})\dot{\boldsymbol{p}}_{ib}^{e} + \boldsymbol{a}_{ib}^{e}$$
(2.25)

Changing reference frames for the positions back to e using (2.19), we can write this as

$$\dot{v}_{eb}^{e} = -\mathbf{S}^{2}(\omega_{ie}^{e})\boldsymbol{p}_{eb}^{e} - 2\mathbf{S}(\omega_{ie}^{e})\boldsymbol{v}_{eb}^{e} + \mathbf{R}_{b}^{e}\boldsymbol{a}_{ib}^{b}, \qquad (2.26)$$

where the a_{ib}^{b} signal is the IMU measurement, corrected for gravity, in the absence of IMU errors.

Lever arm

Unlike the gyro, the accelerometer measurement is dependent on the position of the sensor assembly on the vehicle. Assuming that the origin of the measurement frame $\{m\}$ is fixed at an arbitrary location in the body frame, we have

$$\boldsymbol{p}_{im}^{i} = \boldsymbol{p}_{ib}^{i} + \mathbf{R}_{b}^{i} \boldsymbol{p}_{bm}^{b}, \qquad (2.27)$$

where p_{bm}^{b} is a constant lever arm. Taking the time derivative of this and using $\dot{p}_{bm}^{b} = 0$ gives

$$\dot{\boldsymbol{p}}_{im}^{i} = \dot{\boldsymbol{p}}_{ib}^{i} + \dot{\mathbf{R}}_{b}^{i} \boldsymbol{p}_{bm}^{b}$$
$$= \dot{\boldsymbol{p}}_{ib}^{i} + \mathbf{R}_{b}^{i} \mathbf{S}(\boldsymbol{\omega}_{ib}^{b}) \boldsymbol{p}_{bm}^{b}.$$
(2.28)

Taking the time derivative a second time yields

$$\ddot{\boldsymbol{p}}_{im}^{i} = \ddot{\boldsymbol{p}}_{ib}^{i} + \dot{\mathbf{R}}_{b}^{i} \mathbf{S}(\boldsymbol{\omega}_{ib}^{b}) \boldsymbol{p}_{bm}^{b} + \mathbf{R}_{b}^{i} \mathbf{S}(\dot{\boldsymbol{\omega}}_{ib}^{b}) \boldsymbol{p}_{bm}^{b}$$
$$= \ddot{\boldsymbol{p}}_{ib}^{i} + \mathbf{R}_{b}^{i} \mathbf{S}^{2}(\boldsymbol{\omega}_{ib}^{b}) \boldsymbol{p}_{bm}^{b} + \mathbf{R}_{b}^{i} \mathbf{S}(\dot{\boldsymbol{\omega}}_{ib}^{b}) \boldsymbol{p}_{bm}^{b}.$$
(2.29)

Since the reference and resolving axes of the twice differentiated positions are the same, we can substitude $a_{ib}^i = \ddot{p}_{ib}^i$ and $a_{im}^i = \ddot{p}_{im}^i$. By also premultiplying both sides by \mathbf{R}_i^b , we get

$$a_{im}^{b} = a_{ib}^{b} + \mathbf{S}^{2}(\omega_{ib}^{b})p_{bm}^{b} + \mathbf{S}(\dot{\omega}_{ib}^{b})p_{bm}^{b}.$$
 (2.30)

Solving this for the acceleration of the body frame, a_{ib}^b , and substituting $a_{im}^b = \mathbf{R}_m^b a_{im}^m$ gives the acceleration used in (2.26),

$$\boldsymbol{a}_{ib}^{b} = \mathbf{R}_{m}^{b} \boldsymbol{a}_{im}^{m} - \mathbf{S}^{2}(\boldsymbol{\omega}_{ib}^{b}) \boldsymbol{p}_{bm}^{b} - \mathbf{S}(\dot{\boldsymbol{\omega}}_{ib}^{b}) \boldsymbol{p}_{bm}^{b}.$$
(2.31)

By using the expression for specific force (2.2) and assuming that the lever arm is short enough that gravity can be considered equal at the origins of $\{b\}$ and $\{m\}$, we get

$$\boldsymbol{f}_{ib}^{b} = \mathbf{R}_{m}^{b} \boldsymbol{f}_{im}^{m} - \mathbf{S}^{2}(\boldsymbol{\omega}_{ib}^{b}) \boldsymbol{p}_{bm}^{b} - \mathbf{S}(\dot{\boldsymbol{\omega}}_{ib}^{b}) \boldsymbol{p}_{bm}^{b}.$$
(2.32)

If there is no lever arm, i.e. by defining the body frame origin to coincide with the accelerometer measurement frame, this simply becomes

$$\boldsymbol{f}_{ib}^{b} = \mathbf{R}_{m}^{b} \boldsymbol{f}_{im}^{m}. \tag{2.33}$$

2.6 IMU errors

The accelerometer and gyro measurements are not perfect and contains several errors. Some of these are normally corrected by the processing software in the IMU, and some remain in the measurement output. A short section of gyro



Figure 2.1: Measurements from stationary STIM300 gyro. The gyro is configured integrate the internal 2kHz measurements and output angle increments in degrees at a 250Hz. With this unit the angular velocity of the earth $\omega_{ie}^{e} = 7.292 \times 10^{-5} \frac{\text{rad}}{\text{s}}$ is 1.671×10^{-5} degrees per sample. The offset of the z-axis measurement is around 5×10^{-4} , significantly larger.

measurements from a Sensonor STIM300 IMU in a UAV stationary (or very close to it) on the ground is plotted in Figure 2.1. The plotted signal contains both random noise and an offset which appear constant over a short time interval. Over longer periods of time this can change, having the appearance of a random-walk-like process.

2.6.1 Random noise

Basically all sensors exhibit some random component in the measurement resulting from electrical noise in the analog signals prior to the analog-to-digital sampling. Higher quality, more expensive sensors typically have less noise. This noise is in many cases approximated as Gaussian white noise where each sample is independent of each other with a Gaussian distribution, although no real sensor noise is actually white becase it would mean that the spectral density is constant for all frequencies, implying infinite noise energy.

For IMUs where the three accelerometers and gyros are identical, the random noise on the axes have the same amplitude. The noise is then reasonably assumed to be isotropic.

2.6.2 Bias

The slowly changing offset as shown in Figure 2.1 is present for both gyros and accelerometers and is often referred to as the sensor bias. These biases are in most cases estimated in the implementation of an aided INS, because disregarding them

will lead to drift in the INS position, velocity and attitude estimates. The longer an INS is operating without aiding the less the biases can change while maintaing satisfactory the estimation accuracy.. For operation intervals that are short compared to the drift rates of the biases, the bias can reasonable be approximated as a constant vector.

For use in a KF, the bias terms in (2.5) and (2.1) can be modelled either as a first-order Gauss-Markov process (Lefferts et al., 1982),

$$\dot{b}_{\star} = -T^{-1}b_{\star} + w_{\star}, \tag{2.34}$$

or the special case of it where the time constant is set to infinity, resulting in the so called Wiener process (Brown and Hwang, 2012),

$$\dot{b}_{\star} = w_{\star}.\tag{2.35}$$

The driving noise *w* in these models is Gaussian white noise. Datasheets for IMUs normally do not include values directly usable in such models, but rather use Allan variance.

2.6.3 Other errors

Depending on the accuracy requirements in a INS application, additional systematic errors might need to be included either by a static calibration or by modeling and estimation. Due to inaccuracies in the manufacturing of each sensor and the mounting of these in the IMU assembly, errors in the orthogonality of the measurement axes and misalignment of the actual measurement axes and the markings on the IMU casing can occur. Other error sources are scale factor errors, which cause errors that are proportional to the rate/acceleration, and nonlinearities in the sensor response.

Cross coupling is an error where a sensor, which is supposed to be sensitive only along one axis, also gives a measureable response along axes orthogonal to this. Because the attitude of a vehicle depends on the order of rotations, angular oscillations with different phases around different axes can lead to errors, known as coning errors. Similar combinations of angular rate and acceleration can lead to erroneous net changes in estimated velocity, known as sculling error. Sensors can also be affected by variations in temperature, including heating from the internal electronics. Other errors such as gyro g-sensitivity, hysteresis and deadband are explained in Lawrence (1993).

IMUs normally need some time from being powered on before providing completely stable measurements. The measurement biases can for instance change more in the first minutes after being powered on, than in the subsequent hours of operation. This type of bias stability in IMU datasheets are commonly noted as *in-run* biases. For best performance the sensors should thus be powered on some time before use.

2.7 Gravity

Because the force of gravity appears in the accelerometer measurement (2.5), gravity must be compensated for in measurements to get the coordinate acceleration a_{ia}^{a} . Different options with differing level of complexity exist, depending on the accuracy requirements and the aiding sensors available. INSs which operates with little or no external aiding requires more accurate gravity knowledge than systems relying on external aiding except for short periods of purely inertial navigation. Even an INS with perfect error free accelerometers will give position and velocity errors if the gravity compoensation used is inaccurate. As an example, gravitational anomalies and gravitational field mapping received significant attention during the development of the Trident D5 intercontinental ballistic missile (MacKenzie, 1990), also looking into the use sensors for gravity gradiometry. For aided systems gravity can also be estimated, basically performing gravimetry. It will however not always be observable, i.e. with a constant attitude the accelerometer biases and gravity cannot be separated.

A reasonable approximation for some systems is that gravity only works along the NED frame *z*-axis, which is normal to the reference ellipsoid,

$$\boldsymbol{g}^n = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{g} \end{bmatrix}.$$
(2.36)

where g is chosen as an appropriate constant value, or modelled depending on altitude. The direction of this is an approximation of the plumb line, but does not take local variations into consideration. Modeling gravity as altitude-dependent is more important for vehicles operating over larger altitude spans.

For the experiment performed in this thesis, the altitude variation will only be a few hundred meters, and a constant gravitational acceleration in the NED z-direction will be assumed.

2.8 Accelerometer leveling

If the gravity vector g^n can be measured in the body frame, the modelled gravity can be used to find the pitch and roll angles. This is commonly used as a part of the initialization of an INS, and used by AHRSs operating without aiding input. The relation can be written as

$$\boldsymbol{g}^{b} = \mathbf{R}_{n}^{b} \boldsymbol{g}^{n}. \tag{2.37}$$

Applying the assumption (2.36) and the rotation matrix from Euler angles (E.15), this becomes

$$\boldsymbol{g}^{b} = \begin{bmatrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{bmatrix} \boldsymbol{g}, \qquad (2.38)$$

where ϕ is the roll angle and θ is the pitch angle. For a pure INS with no external aiding, true acceleration, gravity and accelerometer bias cannot be separated. We thus have to tolerate errors caused by acceleration and biases, knowing the limitations of the system. With the measurement model (2.5), assuming either no lever arm (2.33), or that the angular rate and acceleration in (2.32) are negligible, and that the accelerometer bias is zero, an accelerometer measurement averaged over time should be

$$\bar{f}_{\rm IMU} = -g^b. \tag{2.39}$$

$$\bar{f}_{\rm IMU} = \begin{bmatrix} \sin \theta \\ -\cos \theta \sin \phi \\ -\cos \theta \cos \phi \end{bmatrix} g.$$
(2.40)

Estimates of the roll and pitch angles are then (Farrell, 2008, p. 416)

$$\hat{\phi} = \operatorname{atan2}(-\bar{f}_{\mathrm{IMU},y}, -\bar{f}_{\mathrm{IMU},z})$$
(2.41)

$$\hat{\theta} = \operatorname{atan2}(\bar{f}_{\mathrm{IMU},x'}\sqrt{\bar{f}_{\mathrm{IMU},y}^2 + \bar{f}_{\mathrm{IMU},z}^2}).$$
(2.42)

In unaided AHRSs the attitude errors caused by gyro drift are limited by using a low gain feedback from the accelerometers, approximating that the accelerometer measurement will be dominated by the force of gravity over time (Savage, 2014). This results in errors in the estimated attitude in long turning maneuvers or accelerations which users should be aware of. For instance a fixed wing aircraft in a coordinated turn will measure specific force only along the body z-direction, even though is has a nonzero roll angle. Such error can be reduced by using external acceleration measurements or infering the acceleration from changes in measured velocity or position.

2.9 Gyrocompassing

Gyrocompassing involves observation of the horizontal component of Earth's axis of rotation using gyroscopes. This makes it possible to find the direction of true north and thus the heading. These are commonly used on large ships where magnetic compasses are unsuitable due to the ferrous steel hull disturbing the compass. The true north provided by gyrocompasses is also more useful for navigation than magnetic north. Gyrocompassing is usually done using spinning mass gyros or optical gyros (Gade, 2016), but is also possible with MEMS gyros if techniques such as carouseling or maytagging are used (Prikhodko et al., 2013). Carouseling involves continous rotation of the gyro on a horizontal platform and maytagging involves taking measurements separated by a 180 degree horizontal angle.

Gyrocompassing can be divided in two different methods: direct and indirect. In direct gyrocompassing the rotation of the Earth is measured directly using gyroscopes. If the gyro is mounted to a vehicle which is not stationary, but subject to rotation relative to the Earth or vibrations, it is necessary to time average the measurement while requiring that the disturbances can actually be removed by averaging (Groves, 2013). The averaging times required can be long, making this unsuitable for highly dynamic aircraft operating over short periods of time. Indirect gyrocompassing involves observing the rotation of the gravity vector in inertial space, using the gyros to maintain an attitude estimate of the vehicle with inertial resolving axes. This is a faster process, but still requires very accurate gyros (Groves, 2013). In general gyrocompassing is most suitable for vehicles with slow dynamics and long operating times, and is generally not used in aircraft (Ray, 2017).

3

GNSS Preliminaries

This chapter presents background information on the use of GNSS, with a focus on the legacy civilian GPS signal. GNSS systems and the signal structure of the civilian GPS signal is introduced, before presenting the observables provided by GNSS receivers and the errors they contain. The calculation of satellite position, velocity and clock errors is then briefly discussed, before the concept of carrier phase interferometry and the related integer ambiguity resolution is explained. The chapter ends with a brief explanation of real-time kinematic (RTK) and PPK positioning.

3.1 Introduction to GNSS

A GNSS is a system using a constellation of satellites orbiting the Earth to provide global navigation and timing capabilities for receivers on or near the Earth (Groves, 2013). There currently exist several such systems: the American GPS, Russian GLONASS, Chinese BeiDou Navigation Satellite System (BDS), also known as COMPASS, and the European Galileo. All systems have satellites in medium Earth orbit (MEO), but BDS also utilizes satellites in geostationary Earth orbit (GEO) and inclined geosynchronous orbit (IGSO). The satellites, which will also be referred to as space vehicles (SVs), have atomic clocks on board and transmit L-band (1-2 GHz) signals to the user containing orbital data for the satellites in addition to the time of transmission of the signal, allowing users to solve for their position and receiver clock offset (Misra and Enge, 2012). Several systems that augment GNSS using networks of ground monitoring stations for estimation of ionospheric and tropospheric delays also exist, providing corrections to the user for increased accuracy, but these will not be dealt with further in this thesis. GPS does provide correction data itself in the Navigation Message Correction Table (NMCT), but under normal conditions this is encrypted even in the open L1 signal (GPS Directorate, 2015).

This thesis will focus on GPS as it will be used for the experimental testing, and as such the terms GNSS and GPS may sometimes be used interchangeably. The focus will also be on the use of the legacy civilian L1 C/A signal (see Section 3.2), while new civilian GPS signals are being introduced. Multi-GNSS navigation algorithms increase complexity since the different GNSSs use different data formats and orbital data, in addition to having different time references.

Because GNSS relies strongly on accurate timing and synchronization, both for measuring the travel time of signals from each satellite to receivers, but also for calculating the SV positions and velocities at the correct time of signal transmission, each GNSS has its own time reference. For GPS the United States Naval Observatory (USNO) maintains the GPST based on atomic clocks onboard the satellites and at the GPS monitoring stations. The offset, drift rate and aging of the SV clocks relative to GPST are estimated by the ground segment and broadcast by the satellites in the navigation message (USNO, 2018; Misra and Enge, 2012).

One of the weaknesses of using GNSS is that the signals received from the satellites are very weak. For example, the minimum received signal power for the GPS L1 C/A signal is -160 dBW (GPS Interface Specification (2015)), which is equivalent to 10^{-16} watts. The receivers are thus vulnerable to jamming, whether intentional or not, by noise from entities transmitting on the GPS frequencies. The weak signals also make GNSS difficult to use indoors as the signals cannot penetrate walls particularly well.

3.2 GPS signal structure and tracking

The legacy signal used for civilian navigation is the Coarse / Aquisition (C/A) signal modulated onto the L1 carrier which has a nominal frequency of 1575.42 MHz (Misra and Enge, 2012). The signal structure is documented in the GPS signal specification (GPS Interface Specification (2015)). Data is modulated onto the carrier at a rate of 50 bits per second (20ms per bit). To allow all satellites to transmit at the same carrier frequency without disturbing each other, a technique called code division multiple access (CDMA) is used. Each satellite is assigned a pseudorandom noise (PRN) sequence which is 1023 bits long. These are called the *gold codes* and are generated using linear-feedback shift registers. These are known to the receivers and necessary for signal tracking. Since the bits in these codes are not actual data, they are commonly referred to as *chips*, while the term bit is reserved for the data content. The codes have the nice properties that they are close to *orthogonal*, meaning that their cross correlation is low, and in addition their autocorrelation has a sharp peak when the signal is perfectly aligned. This is illustrated in Figure 3.1.

For a bit value of 0, 20 repetitions of the SVs PRN is modulated onto the carrier, while a bit value of 1 corresponds to the inverse of the chip sequence. While the bitrate is very low, the chipping rate is 1.024 MHz. The modulation of these codes onto the carrier is using binary phase shift keying (BPSK), meaning that the phase of the carrier is shifted by 180 degrees whenever a transition between 0 and 1 occurs. This spreads the signal power around the nominal frequency, resulting in a *spread spectrum* signal. The increased bandwidth this produces reduces the power spectral density below that of the background noise (Misra and Enge, 2012) and makes the signal more resistant to interference.

To receive and demodulate the signal, the receiver uses the known PRN sequences and attempts to correlate the internally generated code to the received signal. When the receiver searches for a specific satellite, the signal received from the others appear simply as noise, as the signal received from these do not correlate well with the PRN being used. The aquisition stage aligns the received and generated signals, by searching in two dimension for code offset and carrier frequency



Figure 3.1: Crosscorrelation between a section of the SV1 PRN and the same shifted by 30 chips. Note that in reality the received signal also includes noise, and the values do not change instantaneously. The blue plot is the first 60 chips of the PRN for SV1, the red plot is the same PRN shifted 30 chips to the right (and the first 30 chips are the last 30 chips wrapping around).

shift (mainly due to the Doppler effect) to within a half chip period, modulo one code period (1ms). The tracking is then taken over by a delay lock loop (DLL) which does closed loop tracking of the carrier. Because each PRN only lasts 1ms, and the signal propagation time from transmission to reception is normally in the range of about 70ms to 90 ms (Misra and Enge, 2012), there is an integer ambiguity connected with the tracking of the code due to the repeating nature of the signal (Rao and Falco, 2012). Once this lock without resolved ambiguity is aquired, the data bits of the navigation message can be demodulated. The structure of the messages, being divided into frames and subframes, and then further into bits and chips helps resolve the ambiguity. Counters are used to keep track of the location of each sample within a chip of the full message.

The navigation message broadcast by the satellites contains the *ephemeris* for that satellite, which are the Keplerian orbital parameters needed to calculate the position and velocity of the satellite at any time within the validity period of the data. It also containts the clock error parameters, the data relating GPST to universal coordinated time (UTC), parameters for an ionospheric correction model and the *almanac*. The almanac contains coarse orbital parameters for all satellites.

3.3 GNSS observables

3.3.1 Pseudorange

The signal transmitted from GNSS satellites have data modulated onto the carrier that contains the time of transmission of specific parts of the signal according to the satellite clocks. By taking the difference in time between the time of signal reception according to the receiver and the time of signal transmission according to the satellite, multiplied by the speed of light *c*, we get the observable called *pseudorange*. The *pseudo*- part of the name is used because this range measurement has the effect of several errors included. Clock errors for both the satellite and receiver, ionospheric and tropospheric delays, multipath and other sources all cause errors in this ranging. These errors are shown in Figure 3.2.



Figure 3.2: Pseudorange errors, not drawn to scale. The parts of the receiver hardware delay that is common to all satellites, such as the antenna cable, is normally just considered a part of the receiver clock error. Errors in the tracking of each satellite is not modelled and just considered noise. The satellite group delay and relativistic clock correction will be considered part of the clock error in the rest of the chapter.

The true geometric range between the antenna of a SV *s* and receiver α is, using ECEF coordinates,

$$\rho_{\alpha,s} = \|\boldsymbol{p}_{e\alpha}^{e} - \boldsymbol{p}_{es}^{e}\|_{2} = \sqrt{(x_{\alpha} - x_{s})^{2} + (y_{\alpha} - y_{s})^{2} + (z_{\alpha} - z_{s})^{2}},$$
(3.1)

with $p_{e\alpha}^e = [x_\alpha \quad y_\alpha \quad z_\alpha]^\top$ being the position of the receiver at the time of signal reception in and p_{es}^e is the position of the satellite antenna at the time of signal transmission. These times are important, as this is (in the absence of ionospheric and tropospheric delays) the path travelled by the signal, as shown in Figure 3.5. With the effects ionospheric and tropospheric delays, and satellite and receiver clock errors included, the psedorange for a single satellite can be modelled as (Misra and Enge, 2012, p. 386)

$$P_{\alpha,s} = \rho_{\alpha,s} + c(\delta t_{\alpha} - \delta t_s) + I_{\alpha,s} + T_{\alpha,s} + \epsilon_P, \qquad (3.2)$$

where $P_{\alpha,s}$ is the measured pseudorange, δt_{α} and δt_s are the receiver and satellite clock errors, respectively, $I_{\alpha,s}$ and $T_{\alpha,s}$ are the ionospheric and tropospheric delays, respectively, *c* is the speed of light in vacuum and ϵ_P in noise and unmodelled errors. Since the travel time of the signal from satellite to receiver depends on the distance, signals that are transmitted simultaneously are not received at the same time. Thus when measuring the pseudorange from multiple satellites, a choice must be made of what constitutes simultaneous pseudoranges. The two main options are used are: Common transmission time and common reception time. These both give the same statistical properties (Rao and Falco, 2012).

3.3.2 Doppler frequency

The receiver tracks the carrier frequency of the signal. The frequency received at the receiver antenna depends on the relative movement of the satellite and the receiver due to the Doppler shift. Figure 3.3 shows the compression of the carrier waves in the direction of movement of the satellite causing an increase in frequency in this direction, and a reduction in the opposite direction. The L1 carrier has a



Figure 3.3: Doppler shift. Not drawn to scale, the shift is exaggerated.

nominal frequency of 1575.42 MHz, but the relative movement of the satellite and receiver can lead to shifting of the received frequency by up to around ± 6 kHz (Groves, 2013). The drift rate, or frequency offset, of the receiver and satellite

clocks also affect the frequency measured by the receiver (Acharya, 2014, p. 140). The actual carrier frequency of the signal transmitted from a satellite is

$$f_{\text{transmitted}} = f_{L1}(1 + \delta t_s), \qquad (3.3)$$

where f_{L1} is the nominal frequency of 1575.42 MHz and δt_s is the drift rate of the satellite clock. The drift rate of the satellite clock is small (in the orders of $10^{-13} - 10^{-12}$), but it is known from the parameters in the navigation message, and can thus easily be accounted for. The Doppler shift of the signal can be modelled using

$$f_{\text{received}} = \left(1 - \frac{(\boldsymbol{v}_{is}^{e} - \boldsymbol{v}_{i\alpha}^{e}) \cdot \boldsymbol{l}_{\alpha,s}^{e}}{c}\right) f_{\text{transmitted}}, \qquad (3.4)$$

where $I_{\alpha,s}^{e}$ is the LOS vector from the receiver to the satellite. v_{is}^{e} and $v_{i\alpha}^{e}$ are the intertial frame velocities of the satellite and receiver, decomposed in the ECEF frame, respectively. When the range between the receiver and satellite increases, the value $(v_{is}^{e} - v_{i\alpha}^{e}) \cdot I_{\alpha,s}^{e}$ is positive. This is an approximation of the true relation found in Bahrami and Ziebart (2010), which is valid when relative velocities of the transmitter and receiver along the LOS is significantly lower than the wave propagation speed. In this case it is clear that this is the case. This approximation, with *c* replaced by the speed of sound, would for instance not be very suitable for a fast moving aircraft receiving/listening to audio as it approaches a stationary audio transmitter.

The frequency stability of clocks commonly used in receivers are normally significantly worse than the atomic GPS clocks. Since the receiver measures the frequency of the signal received relative to its internal reference oscillator, the frequency measured by the receiver is (Kaplan and Hegarty, 2006, p. 60)

$$f_{\text{measured}} = \frac{f_{\text{received}}}{1 + \delta t_{\alpha}}.$$
 (3.5)

Common drift rates for receiver clocks are on the order of 10^{-7} (note that the drift rate is unitless, intuitively it can be considered [s/s]), thus still small enough to a justify a linearization of this about $\delta t_{\alpha} = 0$. This yields the model used in Acharya (2014),

$$f_{\text{measured}} = f_{\text{received}}(1 - \dot{\delta t}_{\alpha}).$$
 (3.6)

A further simplification of this gives the more commonly used

$$f_{\text{measured}} = f_{\text{received}} - f_{L1}\delta t_{\alpha}$$
(3.7)

which only gives a range rate error around 0.6mm/s assuming a drift rate of 5×10^{-7} and a Doppler shift of 6kHz, which is used to set the boundaries of the Doppler search space during signal aquisition in e.g. Groves (2013). While the frequency of the received signal is measured, it is common to output the relative

doppler frequency, that is the shift relative to the nominal L1 frequency. This is done by i.e. U-Blox receivers capable of raw measurement output. The output is then supposed to be

$$\Delta f = f_{\text{received}} - f_{\text{transmitted}}.$$
 (3.8)

Because raw measurements do not include any of the corrections available in the navigation message, the actual output is

$$\Delta f_{\text{measurement}} = f_{\text{measured}} - f_{L1}. \tag{3.9}$$

With the approximation in (3.7) and the satellite frequency error, this becomes

$$\Delta f_{\text{measurement}} = \Delta f + f_{L1} (\delta t_s - \delta t_\alpha). \tag{3.10}$$

The frequency measured relates to the pseudorange rate and carrier phase by (Groves, 2013)

$$\dot{P} = \lambda \dot{\phi} = -\lambda \Delta f_{\text{measurement}}, \qquad (3.11)$$

where λ is the wavelength of the L1 carrier, which is approximately 19cm. Combining this and including the effects of ionospheric and tropospheric delay rates, we write the model

$$\lambda \dot{\phi}_{\alpha,s} = \dot{\rho}_{\alpha,s} - \dot{I}_{\alpha,s} + \dot{T}_{\alpha,s} + c(\delta t_{\alpha} - \delta t_{s}) + \epsilon_{\dot{\phi}}.$$
(3.12)

where $\epsilon_{\dot{\phi}}$ represents noise and unmodelled errors.

The satellite clock drift rate is given by the elements a_{f1} and a_{f2} in the GPS navigation message. Correction of residual relativistic time dilation effects should be applied for user with a high velocity with respect to the Earth (Groves, 2013). The satellite velocity is calculated from the ephemeris parameters at the time of transmission. It is, however, important to process the velocities in an inertial frame, such that linear velocities due to Earth's rotation is taken into account for both the satellite and receiver.

3.3.3 Carrier phase

The carrier phase of the signal is tracked by phase lock loops (PLLs) in the receiver. Because of the data modulated onto the carrier it is necessary to either wipe all the data off the carrier after demodulation of the data bits, to provide a data free carrier for the PLL, or use a so called Costas loop (Costas, 1956) which is insensitive to 180 degree phase shifts (using only the in-phase component of the signal after splitting it from the quadrature component), and thus able to handle the carrier with data. The carrier phase measurement is based on the accumulation of Doppler frequency shift (O'Driscoll, 2010) in addition to fractional phase measurements (Kaplan and Hegarty, 2006), and the observable is therefore also known as accumulated Doppler range (ADR). Because the receiver only starts counting cycles from the time at which it locks onto the signal from each satellite, the carrier phase does not provide absolute range measurements, but is ambiguous. Due to the insensitivity of the Costas loops to half cycle shifts, there is a half cycle ambiguity that will be discussed more in Section 3.4.7.

From Kaplan and Hegarty (2006, p. 400) the accumulated phase at epoch *n* is

$$\phi_n = \phi_{n-1} = \int_{t_{n-1}}^{t_n} \dot{\phi}(\tau) d\tau + \phi_r, \qquad (3.13)$$

where ϕ_r is the fractional phase measured by the PLL at the epoch and $\dot{\phi}$ is given by the measured Doppler frequency from (3.11). The fractional phase is measured by comparing the measured signal with the receiver generated signal (Misra and Enge, 2012)

$$\phi = \phi(t_0) + f_0(t - t_0), \tag{3.14}$$

where f_0 is the nominal frequency given by the imperfect receiver oscillator. The drift in the receiver clock affects both the Doppler frequency measurement (3.12) and the passage of time according to the receiver, and clock drift will thus cause the carrier phase measurement to drift over time.

Including ionospheric and tropospheric effects, we have the carrier phase model (Garcia et al., 2005)

$$\lambda \phi_{\alpha,s} = \rho_{\alpha,s} - \lambda N_{\alpha,s} + \lambda \phi_{\text{initial},s} - I_{\alpha,s} + T_{\alpha,s} + c(\delta t_{\alpha} - \delta t_{s}) + \epsilon_{\phi}, \qquad (3.15)$$

where $\lambda \phi_{\text{initial},s}$ is the phase offset between the receiver generated signal and the signal transmitted from the satellite. A simplified illustration of the carrier phase measurement showing the ambiguity is found in Figure 3.4.

Note that the effect of the ionosphere on the carrier phase is opposite to that on the code-based pseudorange calculation. While it causes a delay in the received signal, it also advances the phase of the carrier. This is explained further in Section 3.4.2.

The carrier phase provides ambiguous range information, but is a lot more accurate and precise than the pseudorange in measuring *changes* in range. Carrier phase is more susceptible to tracking issues than pseudorange, and loss of lock most often causes jumps in the measurements, explained in Section 3.4.6.



Figure 3.4: Illustration of the carrier phase (not drawn to scale). The integer ambiguity $N_{\alpha,s}$ is constant as long as the carrier is tracked.

3.4 GNSS error sources and compensation

3.4.1 Sagnac - Earth rotation

The rotation of the Earth while the signal propagates from the satellite to the receiver is important to take into account. The true range is the distance between the satellite at the time of signal transmission and the receiver at the time of signal reception as shown in Figure 3.5. In an inertial reference frame this simply

$$\rho = \|\boldsymbol{p}_{is}^{i}(t_{tx}) - \boldsymbol{p}_{i\alpha}^{i}(t_{rx})\|_{2}, \qquad (3.16)$$

where the positions are written as functions of time, with t_{tx} and t_{rx} being the time of signal transmission and reception, respectively. Since the earth rotates while the signal travels through space, care must be taken when working in ECEF coordinates, as the transformation between this and ECI is time dependent. The satellite position calculation algorithm in the GPS signal specification, also included in Appendix C, provides the satellite position at a chosen time in the ECEF frame *of the same instant*. However, the receiver calculates its position in an ECEF frame defined at the time of signal reception. Thus we are working with two different ECEF frames as shown in Figure 3.5. Simply calculating the range using the position, which has its maximum of around 41m at the equator (Groves, 2013). This can be handled by chosing one of the two frames to do the position calculations in. By doing this we are essentially using an inertial frame aligned with ECEF at a single time (Groves, 2013). The two ECEF frames have the same origin, so a simple rotation can be used to fransform between them,

$$\mathbf{R}_{e,tx}^{e,rx} = \mathbf{R}_z(\omega_{ie}(t_{rx} - t_{tx})) = \begin{bmatrix} \cos \omega_{ie}(t_{rx} - t_{tx}) & -\sin \omega_{ie}(t_{rx} - t_{tx}) & 0\\ \sin \omega_{ie}(t_{rx} - t_{tx}) & \cos \omega_{ie}(t_{rx} - t_{tx}) & 0\\ 0 & 0 & 1 \end{bmatrix}, \quad (3.17)$$

where the notation $\{e, tx\}$ and $\{e, rx\}$ is used for the ECEF frames of the times of transmission and reception. ω_{ie} is the rotation rate of the Earth around the polar



Figure 3.5: GNSS signal propagation (not drawn to scale). The signal travels approximately along the blue dashed line, but can deviate due do refraction caused by the ionosphere and troposphere.

axis, which when multiplied by the signal propagation time $t_{rx} - t_{tx}$ yields the angle rotated by the Earth during this time. **R**_z is given by (E.6). Since the rotation of the Earth during the signal propagation time is very small, around 5×10^{-6} radians, we can reasonably apply the small angle assumption $\cos \theta = 1$, $\sin \theta = \theta$:

$$\mathbf{R}_{e,tx}^{e,rx} \approx \begin{bmatrix} 1 & -\omega_{ie}(t_{rx} - t_{tx}) & 0\\ \omega_{ie}(t_{rx} - t_{tx}) & 1 & 0\\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\omega_{ie}\frac{\rho}{c} & 0\\ \omega_{ie}\frac{\rho}{c} & 1 & 0\\ 0 & 0 & 1 \end{bmatrix}$$
(3.18)

The true range can then be written with positions in the two different ECEF frames. In this case the satellite position is transformed into the ECEF frame of the time of signal reception.

$$\rho = \|\mathbf{R}_{e,tx}^{e,rx} \boldsymbol{p}_{es}^{e,tx}(t_{tx}) - \boldsymbol{p}_{e\alpha}^{e,rx}(t_{rx})\|_2$$
(3.19)

Earth rotation also needs to be taken into consideration when using range rate to estimate the antenna velocity. The velocity contribution of the rotating ECEF frame relative to inertial space, with the lever arms from the center of the Earth, should be included in the model (Groves, 2013),

$$\dot{\rho} = (\boldsymbol{l}^{e,rx})^{\mathsf{T}} \underbrace{(\underbrace{\mathbf{R}_{e,tx}^{e,rx}(\boldsymbol{v}_{es}^{e,tx} + \mathbf{S}(\boldsymbol{\omega}_{ie}^{e})\boldsymbol{p}_{es}^{e,tx}(t_{tx}))}_{\text{Satellite velocity}} - \underbrace{(\underbrace{\boldsymbol{v}_{e\alpha}^{e,rx} + \mathbf{S}(\boldsymbol{\omega}_{ie}^{e})\boldsymbol{p}_{e\alpha}^{e,rx}(t_{rx}))}_{\text{Antenna velocity}})), \quad (3.20)$$

where $l^{e,rx}$ is the LOS vector from $p_{e\alpha}^{e,rx}(t_{rx})$ to $p_{es}^{e,rx}(t_{tx})$.

3.4.2 Ionospheric delay and phase advance

The ionosphere is a part of the Earth's upper atmosphere streching from around 50km above the Earth's surface up to about 1000km (Groves, 2013, p. 287) where solar radiation ionize gas molecules, releasing free electrons (Kaplan and Hegarty, 2006). The presence of these particles cause the radio waves passing through it to be refracted because the wave propagation speed is reduced, also leading to the signal path being longer than the straight line between satellite and receiver. For the ionosphere this refraction is frequency dependent in the L-band and it is thus said to be frequency dispersive for the frequencies used for GNSS (Misra and Enge, 2012).

Because the ionospheric delay is frequency dispersive it can be measured by receivers capable of receiving signals on multiple frequencies. The introduction of new civilian GPS signals has this as one of its advantages, as single frequency users have to settle with models of the ionosphere which are less accurate than measurements.

The ionospheric delay causes the signal to reach the receiver later than the geometric range and speed of light in vacuum would imply, making the measured pseudorange too large. While some of the error (the smallest value, the zenith delay) would end up in the calculated clock error if not corrected, the dependence on the elevation causes the ionospheric delay to be larger for satellites visible at lower elevations. This means that the error in position will depend on the azimuth of the satellites included in the navigation calculations. Assuming for example 3 satellites evenly spread at a high elevation, with a single satellite at low elevation close to the horizon, the calculated position would end up being "pushed" away from the low elevation satellite, and the clock error would be calculated as to large.

Because the ionospheric signal refraction is caused by solar radiation, the effect varies with the solar activity. The magnetic activity of the sun is cyclic with a period of about eleven years. Thus some years will be more affected by the ionospheric errors than others. In addition, rapid fluctuations which are both local and unpredictable can occur, called ionospheric scintillation. This error can not be compensated for by single frequency users.

While the ionosphere causes the signal to arrive later at the receiver, another effect of the ionosphere is that the carriers in the L-band have a higher phase velocity in the ionosphere than in vacuum (Misra and Enge, 2012). This leads to an advance in the carrier phase, causing the sign of the ionospheric errors in (3.2) and (3.15) to have different signs, although the magnitude is the same. This difference in effect on the pseudorange and carrier phase is known as *code-carrier divergence*.

Klobuchar model

The Klobuchar ionospheric model (Klobuchar, 1987) found in Appendix F, also called the *half cosine model* is the ionospheric model which the GPS navigation message contains parameters for, intended to be used by the L1 C/A standard positioning service (SPS). It is computationally lightweight and requires few parameters, but is able to reduce the effect of the ionospheric errors on L1 position calculation by at least 50% RMS (GPS Interface Specification (2015)). The model works by assuming a constant magnitude of the ionospheric delay at night and following a half-cosine function increasing the magnitude of the delay at daytime. The eight parameters received in the navigation message are coefficients for two polynomials describing the amplitude and period of the half-cosine as a function of geomagnetic latitude.

3.4.3 Tropospheric delay

The troposphere is the lowest part of the Earth's atmosphere from the surface up to about 12 km (Groves, 2013, p. 287), containing gases and water vapour which refracts the signal similarly to the ionosphere. Unlike the ionosphere however, the troposphere is not frequency dispersive, so the signal delay can not be measured and corrected by multi-frequency receivers. The compensation for tropospheric delay is especially important for low elevation satellites, as the slant angle / obliquity factor is significantly larger close to the horizon than for ionospheric delay. This means that a user that does not correct for the troposphere should choose a higher elevation mask angle than a user using a tropospheric model. The addition or removal of signals from a low elevation satellite can give a change in the calculated user position of several meters if tropospheric compensation is not used, making the result highly dependent on the SVs used.

Unlike the ionosphere the troposphere also does not not cause carrier phase advance, as pseudorange and carrier phase are affected with the same sign. Since the error is not measureable for any receiver even if it can receive signals on multiple frequencies (unless it is at a known position), one way to do accurate compensation of it is by using space based augmentation system (SBAS) services such as European Geostationary Navigation Overlay System (EGNOS) which can estimate the error by using receiver placed at accurately surveyed locations. These corrections are then uploaded to GEO satellites and transmitted to the user. Most of today's commercial receivers are capable of using SBAS, increasing positioning accuracy.

Several models also exist for estimation of tropospheric error if SBAS is not used. The best models take the currect atmospheric conditions such as temperature and humidity into consideration, but standard atmospheric models can be used if measurements are not available or weather forecast data cannot be received. The orthometric height (height above the geoid, which is basically height above mean sea level) of the receivers is normally used in models, as the gas and water vapour content depends on altitude. Examples of tropospheric include the Saastamoinen model, the Hopfield model, the UNB3 model and the NATO STANAG model (Misra and Enge, 2012; Groves, 2013).

NATO STANAG troposphere model

A simple model which can be used without any atmospheric measurement is the NATO Standardization Agreement (STANAG) model (Groves, 2013, p. 394). While not as accurate as more advanced models, is still provides a decent improvement, in particular for satellites with low elevation. The model uses the orthometric height, h_{ortho} , and uses a few different models for the zenith delay for specific height ranges. The zenith delay for heights under 1000m is modelled as

$$P_{\text{zenith}} = 2.464 - 3.248 \times 10^{-4} h_{\text{ortho}} + 2.2395 \times 10^{-8} h_{\text{ortho}}^2$$
(3.21)

and the totalt tropospheric delay including satellite elevation is

$$P_{\text{tropo}} = \frac{P_{\text{zenith}}}{\sin(\text{elevation}) + \frac{0.00143}{\tan(\text{elevation}) \pm 0.0455}}.$$
(3.22)

The expected residual tropospheric delay when using this model are of the order of 0.6m (Groves, 2013).

3.4.4 Ephemeris and satellite clock errors

The GPS Control Segment estimates the position, velocity and clock errors of each satellite, and parameters of prediction models for these, based on measurements taken at monitoring stations spread around the world. The prediction model parameters are uploaded to the satellites and broadcast to the user. These values are used to calculate the predicted state of each satellite at times needed by a receiver for navigation calculations. The error between the true satellite states and the predictions from the broadcast prediction model grows as the age of data (AoD) increases. New parameters are uploaded by the ground segment regularly, but they also monitor the error growth by comparing the broadcast model with the best estimate available for each satellite. If the range error estimated exceeds a threhold, new parameters are uploaded to the satellite (Misra and Enge, 2012).

For users requiring lower ephemeris and clock errors than that provided in the broadcast navigation message, third parties such as the International GNSS Service (IGS) (International GNSS Service, 2018) provide precise ephemerides based on a large number of grond station, with improved values being available for real-time use, and even better values for post-processing use.

3.4.5 Group delay bias

The clock error parameters a_{f0} , a_{f1} and a_{f2} included in the navigation message (shown in Appendix A) are valid for the two frequency pseudorange

$$P = \frac{P_{L2P(Y)} - \gamma P_{L1P(Y)}}{1 - \gamma}$$
(3.23)

$$\gamma = \left(\frac{f_{L1}}{f_{L2}}\right)^2 = \left(\frac{1575.42}{1227.6}\right)^2 = \left(\frac{77}{60}\right)^2.$$
(3.24)

Because the L1 and L2 signals passes through different hardware in the satellites, the use on single frequency pseudoranges required additional correction. The T_{GD} parameter transmitted in the LNAV message, which is based on measurements done by the contractor during SV manufacture, as well as monitoring done by the Jet Propulsion Laboratory (Hegarty et al., 2004), are calibrated for the use of single frequency P(Y)-codes, with the following corrections:

$$(\Delta t_{SV})_{L1P(Y)} = \Delta t_{SV} - T_{GD} \tag{3.25}$$

$$(\Delta t_{SV})_{L2P(Y)} = \Delta t_{SV} - \gamma T_{GD}$$
(3.26)

However, if single frequency L1 C/A is to be used, the group delay correction requires the additional term $ISC_{L1C/A}$,

$$(\Delta t_{SV})_{L1C/A} = \Delta t_{SV} - T_{GD} + \text{ISC}_{L1C/A}, \qquad (3.27)$$

which is not transmitted in the LNAV message, only the newer CNAV message (GPS Interface Specification (2015)). This error is specified to be less than 10ns, but observed errors are normally less than 3ns (Hegarty et al., 2004). Using the T_{GD} parameters even for L1 C/A should give a reduction in the error at most times. The group delay differences and the correction values are shown illustrated in Figure 3.6.



Figure 3.6: Group delay (illustration). Note that the chipping rate of the P(Y) code is 10 times higher than for the C/A code, but the data rate is the same. In the illustration we have $T_{GD} > 0$, but the value can be either positive or negative depending on the satellite. It has also been exaggerated in the figure.

3.4.6 Carrier phase cycle slips

The integer ambiguity value N in (3.15) only remains constant as long as the receiver PLL maintains good signal lock. Losing carrier lock for any reason, no matter for how short amount of time, can lead to N having a different value when lock is reaquired. Due to the low signal power received by the receiver this can happen a lot, and thus methods for handling this well is basically a requirement for the use of the carrier phase observable. This is one of the drawbacks of the use of carrier phase. The loss of carrier lock can occur due to many reasons: signal blockage due to satellites temporarily being obscured by trees, buildings, mountains or anything else the signal cannot penetrate well. Severe ionospheric conditions such as scintillation (Sickle and Dutton, 2017), or issues in the receiver hardware or software can also cause this. High-noise environments such as in areas with high electro-magentic disturbances in the GNSS frequency bands, or in areas with GNSS jamming, can increase the rate of cycle slips.

Because *N* is independent for each satellite and receiver pair, the effect of cycle slips is not eliminated by double differencing of the measurements, and thus the $A\Delta N$ term (details are presented in Section 3.6.1) is also affected.

3.4.7 Carrier phase half cycle errors

Because the navigation data is modulated onto the L1 GPS carrier by the use of BPSK as described in Section 3.2, the bit values are initially ambiguous for the receiver. It does not know whether each bit has a value of 0 or 1, it only reads the points at which the bit values change. This is illustrated in Figure 3.8. The receiver can initially only make an assumption to which sequence is correct, and only when enough data has been received that the known structure of the navigation message can be recognized in the data, can this ambiguity be resolved.

For receivers using the popular Costas-type tracking loops which are insentivite to half cycle (180 degrees phase) offsets, a PLL does not know whether it is tracking



Figure 3.7: Illustration of a cycle slip: Because the change in carrier phase per sample can be several hundre cycles, a cycle slip is not necessarily visible in a plot like this using real data. The slip has been exaggerated to make it visible. The Figure is taken from Sollie (2017).

the carrier in phase or in antiphase. This creates a *half cycle ambiguity* in the carrier phase. If data cannot be demodulated from the received signal, this half cycle ambiguity cannot be resolved. When this is the case, the carrier phase model (3.15) will instead take the form (Kirkko-Jaakkola et al., 2009)

$$\lambda \phi_{\alpha,s} = \rho_{\alpha,s} - \frac{\lambda}{2} N_{\alpha,s} + \lambda \phi_{\text{initial},s} - I_{\alpha,s} + T_{\alpha,s} + c(\delta t_{\alpha} - \delta t_{s}) + \epsilon_{\phi}.$$
(3.28)

The bit value and half cycle ambiguity are essentially connected, such that finding out which of the two phases of the BPSK that corresponds to each bit value also makes it known whether the PLL is tracking the carrier in phase or antiphase. Thus after locking onto the carrier some time might be needed before the half cycle ambiguity is resolved, after which the carrier phase ambiguity can be assumed to have integer values.

3.4.8 Multipath

Multipath occurs when signals are reflected to the receiver antenna after first hitting i.e. buildings, the ground or a water surface. This can make it more difficult for the receiver to measure the correct pseudoranges and carrier phases, as the same signal is received more than once only slightly shifted in time. This is considered bad for positioning and makes it harder to navigate using only GNSS in urban environments with many tall buildings.



Figure 3.8: BPSK demodulation. Demodulating the BPSK can give either of the shown bit sequences. Also note that in reality there are 1540 cycles per chip on f_{L1} C/A.

3.4.9 Time correlated errors and choice of measurement sampling rate

Because of the way the observables are tracked over time in the receivers, the output measurements do have errors that are correlated over time. The tracking loops essentially act as filters. Even if no additional low pass filtering is applied to the receiver output, the assumption of uncorrelated Gaussian noise for the use of the measurements in a KF does not actually hold. Figure 3.9 shows the result of double differencing real pseudorange measurements taken at 5 Hz (the same measurements as used in Chapter 7) compared to the known double differenced range from PPK antenna positions. The result of this is that when the measurements are to be used in a Kalman filter, a higher measurement rate does not necessarily give better estimation results. Actually, even if a receiver is capable of a high output rate, it may be beneficial to deliberately skip measurements to reduce the rate at which measurements are used, as this will reduce the effect of time correlation.



Figure 3.9: Time correlation of double differenced pseudorange errors.

3.5 Calculating GPS satellite positions, velocities and clock errors

In order to use the pseudorange, Doppler frequency or carrier phase for navigation purposes, information about the states of the satellites at the time of each measurement must be known. The ephemeris parameters must be extracted from the navigation messages and used to calculate satellite positions, velocities and clock errors. The steps needed are:

- 1. Extract parameters from binary format navigation message (every time a new parameter set is received).
- 2. Find the time of signal transmission in satellite clock time, and use clock correction parameters to find this time relative GPST.
- 3. Calculate the position and velocity at this time.
- 4. Transform the position and velocity to the ECEF frame of the receiver.

3.5.1 Extracting ephemeris and clock error parameters from the GPS LNAV navigation message

The GPS LNAV navigation message is broadcast as 25 frames with a length of 1500 bits giving a total length of 37500 bits. This is called a master frame. These frames can be divided into five 300 bit subframes, each taking 6 seconds to transmit at the data rate of 50 bits per second. The ephemeris and clock error parameters of the broadcasting satellite are found in the first three subframes, which are repeated in each frame between data uploads from the ground segment. The parameters for the Klobuchar ionospheric model are found in subframe four, of which the content is not repeated in each frame, but is only transmitted once per master frame. The position of each parameter within a message can be found in Appendix A. The

method used to extract each parameters is described in Appendix B, where there is also a data example.

3.5.2 Time of transmission and clock errors

The receiver receives the time of transmission of specific parts of the navigation message as read from the clock onboard the transmitting satellite, and estimates the signal delay using DLLs. Finding the time of signal transmission of the specific part of the signal corresponding to a pseudorange measurement can be done by using the receiver measurement time and the pseudorange. A pseudorange measurement has the form

$$P = c(t_{rx,rec} - t_{tx,sat}), \tag{3.29}$$

where $t_{rx,rec}$ is the time of reception according to the receiver clock and $t_{tx,sat}$ is the time of transmission according to the satellite clock, from which the satellite time of transmission ca be calculated as

$$t_{tx,sat} = t_{rx,rec} - \frac{P}{c}.$$
(3.30)

The correction of this time to the GPST time of transmission can be done by using the broadcast clock correction parameters as well as a few of the ephemeris parameters used to calculate the relativistic correction. The equations to solve for this are

$$t = t_{SV} - \Delta t_{SV} \tag{3.31}$$

where *t* is time referenced to GPST, t_{SV} is satellite time and Δt_{SV} is the error, and

$$\Delta t_{SV} = a_{f0} + a_{f1}(t - t_{oc}) + a_{f2}(t - t_{oc})^2 + \Delta t_r - T_{GD}$$
(3.32)

from the GPS signal specification (GPS Interface Specification (2015)). a_{f0} , a_{f1} , a_{f2} are the offset, drift rate and aging at the reference time t_{oc} , $\Delta t_r = -2e \frac{\sqrt{A\mu}}{c^2} \sin E_k$ is a relativistic correction and T_{GD} is the group delay parameters described in Section 3.4.5. The values in the expressions for the relativistic correction are ephemeris parameters (see Appendix A) and values found as intermediate values in the position calculation algorithm (see Appendix C). The coupling between these equations means that we cannot simply solve for an explicit expression for *t*. The sensitivy of (3.32) to *t* is small, so it can be approximated by t_{SV} , but we can also quickly solve this iteratively by calculating Δt_{SV} starting at time t_{SV} , calculating *t* from this error, and then repeating the calculation using the new estimate of *t*. This only needs a few iterations to converge.

Neglecting the rate of change of the relativistic correction, the drift rate or frequency deviation of the satellite clock can be written as

$$\Delta t_{SV} = a_{f1} + a_{f2}(t - t_{oc}). \tag{3.33}$$

3.5.3 Position and velocity calculation

The algorithm used to calculate the satellite positions is given in the GPS Interface Specification (2015), and can also be found in Appendix C. After calculating the GPST time of transmission, the position algorithm can be used to compute the satellite position at this time. This position can then be transformed into the frame of the receiver as described in Section 3.4.1.

The equations needed for velocity calculation are not included in the GPS Interface Specification (2015). These are needed to use the measured Doppler frequency shift to calculate the receiver velocity. They are however not hard to derive, and can be found for example in Remondi (2004). As for the position, the velocities are calculated in the ECEF frame of the time of signal transmission.

3.6 Carrier phase interferometry

The estimation of attitude when baselines are short can be done using an interferometric method with the measured carrier phase. Due to the large distance from a user to the MEO satellites, antennas fixed at different locations on a vehicle can be reasonably approximated as having parallel LOS vectors. This is illustrated in Figure 3.10. This approximation means that the signal from a satellite can be seen as a plane wave.



Figure 3.10: Parallel LOS vectors. The LOS for receivers α and β , \vec{l}_{α} and \vec{l}_{β} , can be reasonably approximated as equal, $\vec{l}_{\alpha} \approx \vec{l}_{\beta}$. $\Delta \rho$ is the difference in range from each receiver to the satellite, which can be split into an integer wavelength part ΔN and a fractional part $\Delta \phi$.

Because each receiver can start tracking a satellite at different times, or because cycle slips can occur causing the integer ambiguity to change (and some receivers adds an integer to the measurement in an attempt to align the measurement with the range measured using code), the difference between the measurements from two receivers to the same satellite will not be the actual difference in phase shown in Figure 3.10.

Full attitude determination using this method requires at least three antennas, but a dual-antenna setup with the antennas placed longitudinally on an aircraft can make the heading and pitch observable, as shown in Sollie (2017).

3.6.1 Carrier phase differencing

An assumption here is that the measurements are taken at the exact same time. For a single dedicated receiver with multiple antenna channels, all using the same internal oscillator, this is easy to ensure. If non-dedicated independent receivers are used, this assumption does not necessarily hold. If the receivers schedule their measurement epochs using their clock corrected for the estimated clock error, this would approximately hold, but requires that a navigation message subframe has been received, so the clock error can be solved for. To allow raw measurement outputs as early as possible, and because handling of clock errors are normal part of using raw data, measurements from i.e. U-Blox receivers (U-Blox website, u-blox.com) are scheduled using the internal clock directly, which is not steered to a common reference. If access to receiver firmware is not available, but raw outputs of closed-source receivers are used, one must take it into account that the measurements are not necessarily obtained at the same time.

Single differencing of the carrier phase is differencing the measurements from a single SV between receivers. For a receiver pair constructing a single baseline, only two possible choices exist for the single differenced measurements, $\phi_{\alpha,s} - \phi_{\beta,s}$ or $\phi_{\beta,s} - \phi_{\alpha,s}$, differing only by a change in sign. A baseline is defined here as a vector between a receiver pair. If an array of more than two receivers are used, the baselines which should be used must be selected. There is no point in using all possible baselines, as a baseline which is a linear combination of others already included, provides no additional information. Differencing (3.15) for two receivers α and β results in

$$\lambda \Delta \phi_{\alpha\beta,s} = \lambda (\phi_{\alpha,s} - \phi_{\beta,s}) = \Delta \rho_{\alpha\beta,s} - \lambda \Delta N_{\alpha\beta,s} + c(\delta t_{\alpha} - \delta t_{\beta}) + \Delta \epsilon_{\phi}, \qquad (3.34)$$

where Δ is used as a symbol for single differences values between a pair of receivers. The single differencing cancels the error terms in the model that are common for both receivers, including the satellite clock errors and the initial phase offset. Because the atmospheric errors are higly spatially correlated, the errors for two receivers mounted on the same vehicle would be close to identical, such that these essentially cancels with single differencing.

Double differencing is differencing the measurement between receivers as above, and then between satellites. For this approach many options exist, because we normally have many satellites in view. The simplest is choosing one reference satellite, i.e. the one with lowest noise level (normally the one with highest eleva-

tion). Differencing (3.34) for two satellites s_1 and s_2 , we get

$$\lambda \nabla \Delta \phi_{\alpha\beta,s_1s_2} = \lambda (\Delta \phi_{\alpha\beta,s_1} - \Delta \phi_{\alpha\beta,s_2}) = \nabla \Delta \rho_{\alpha\beta,s_1s_2} - \lambda \nabla \Delta N_{\alpha\beta,s_1s_2} + \nabla \Delta \epsilon_{\phi}, \quad (3.35)$$

where ∇ is used to indicate differencing between a pair of satellites. The double differencing cancels the errors that are common for all satellites for the same receiver. This means that the receiver clock errors and timing biases caused by different antenna cable lengths are cancelled. With the measurements written in vector form, the operation of differencing can be done using a matrix. Because we are only using a dual-antenna setup this will not be done for the single differencing. In vector form (3.35) can be written as

$$\lambda \mathbf{A} \Delta \boldsymbol{\phi}_{\alpha\beta} = \mathbf{A} (\Delta \boldsymbol{\rho}_{\alpha\beta} - \lambda \Delta N_{\alpha\beta} + \Delta \boldsymbol{\epsilon}_{\phi}), \qquad (3.36)$$

with the differencing matrix $\mathbf{A} \in \mathbb{R}^{(k-1) \times k}$ (for *k* single differenced measurements) which can be for example

$$\mathbf{A} = \begin{bmatrix} 1 & \dots & 0 & -1 \\ 1 & & -1 \\ & \ddots & & \vdots \\ 0 & & 1 & -1 \end{bmatrix}.$$
 (3.37)

Differencing matrices must have rows that sum to 0. They do not need to have all integer values, but non-integer values in the matrix would remove the integer property of the double differences ambiguities.

3.6.2 Correlated noise in double differenced carrier phase

The error term ϵ_{ϕ} in the carrier phase model (3.15) contains both components which are common for measurements to a satellite for multiple receivers (satellite and atmospheric error residuals), components which are common for measurements from a single receivers to multiple satellites (unmodelled/uncorrected receiver errors), and noise which is not correlated between any measurements. The common error residuals will cancel with double differencing, and thus will not influence the double differenced error term $\nabla \Delta \epsilon_{\phi}$ in (3.35). The properties of the uncorrelated measurement noise however is worth some attention. The covariance matrix of ϵ_{ϕ} for measurements for multiple satellites from a single receiver can be reasonably assumed to have the form

$$\mathbf{R} = \begin{bmatrix} \sigma_{\phi,1}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{\phi,2}^2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \sigma_{\phi,n}^2 \end{bmatrix}$$
(3.38)
If a pair of receivers are used, the single differences values takes the form

$$\begin{bmatrix} \phi_{\alpha\beta}^{1} \\ \vdots \\ \phi_{\alpha\beta}^{n} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{I}_{n \times n} & -\mathbf{I}_{n \times n} \end{bmatrix}}_{\mathbf{A}_{sd} \in \mathbb{R}^{n \times 2n}} \begin{bmatrix} \phi_{\alpha}^{1} \\ \vdots \\ \phi_{\alpha}^{n} \\ \phi_{\beta}^{1} \\ \vdots \\ \phi_{\beta}^{n} \end{bmatrix}, \qquad (3.39)$$

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where it is also possible to move the negative sign in the differencing matrix to the other identity submatrix. When using only two receivers the single difference in normally not written in matrix form, as it can be written using only a vector difference. Because each column in the differencing matrix is only used once, the noise from a single measurement only ends up in one single differenced measurement, and the uncorrelated property of the noise remains:

$$\mathbf{R}_{sd} = \mathbf{A}_{sd} \begin{bmatrix} \mathbf{R}_{\alpha} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{\beta} \end{bmatrix} \mathbf{A}_{sd}^{\top} = \mathbf{R}_{\alpha} + \mathbf{R}_{\beta}.$$
(3.40)

This will however not be the case if more than two antennas are used, and the baselines chosen have overlapping receivers. There would then exist at least one measurement that is used in more than one single differenced value. This is the same as what normally occurs when double differencing. The differencing matrix (3.37) which can be used to difference all but one of the single differences against the last one, obviously leads to the noise from the reference satellite making its way into all the double differences. This causes the double difference noise vector $\nabla \Delta \epsilon_{\phi}$ to have correlated elements. This is the case if the matrix **AA**^T is not diagonal. For (3.37) this becomes

$$\mathbf{A}\mathbf{A}^{\top} = \begin{bmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & & \vdots \\ \vdots & & \ddots & 1 \\ 1 & \dots & 1 & 2 \end{bmatrix}.$$
 (3.41)

Some clever differencing matrices with dimension $(k - 1) \times k$ do exist that gives uncorrelated noise, but have other disadvantages such as having elements which are not integers, causing the ambiguities to loose their integer properties if special methods are not used.

3.6.3 Relation between baseline and carrier phase measurement

From Figure 3.10 it can be seen that the ideal single differenced carrier phase plus an integer number of is equal to the difference in range (in unit of wavelengths)

between the satellite and each receiver

$$\Delta \rho_{\alpha\beta,s} = \lambda (\Delta \phi_{\alpha\beta,s} + \Delta N_{\alpha\beta,s}). \tag{3.42}$$

The difference in range can also be written as the projection of the baseline onto the LOS vector,

$$\Delta \rho_{\alpha\beta,s} = \vec{l}_s \cdot \vec{b} = \|\vec{b}\|_2 \cos \theta_{\vec{l}_s,\vec{b}}, \qquad (3.43)$$

where \vec{l}_s and \vec{b} are the LOS and baseline vectors written as coordinate free vectors. This is valid regardless of the frame the vectors are decomposed in. Setting these expressions equal, assuming a single baseline \boldsymbol{b}^b known and constant in the body frame $\{b\}$, k satellites and ECEF used as reference frame, this can be written as

$$\Delta \boldsymbol{\rho} = \lambda (\Delta \boldsymbol{\Phi} + \Delta \boldsymbol{N}) = (\mathbf{L}^e)^\top \mathbf{R}_b^e \boldsymbol{b}^b$$
(3.44)

$$\Delta \boldsymbol{\Phi} = \begin{bmatrix} \Delta \phi^1 \\ \vdots \\ \Delta \phi^k \end{bmatrix} \qquad \Delta \boldsymbol{N} = \begin{bmatrix} \Delta N^1 \\ \vdots \\ \Delta N^k \end{bmatrix} \qquad (3.45)$$

$$\mathbf{L}^{e} = \left[\boldsymbol{l}_{1}^{e} \dots \boldsymbol{l}_{k}^{e} \right] \in \mathbb{R}^{3 \times k}.$$
(3.46)

The reason why actual single differenced measurements do not satisfy this equation if independent receivers, each with its own clock, are used is the receiver clock error difference present in (3.34), in addition to any difference in the reported measurement time for each receiver. If a dedicated multi-antenna receiver with only a single clock is used, single differencing the measurements would be sufficient. In the experimental testing in this thesis independent receivers will be used, and each side must then be multiplied with a valid differencing matrix,

$$\mathbf{A}\lambda(\Delta \mathbf{\Phi} + \Delta N) = \mathbf{A}(\mathbf{L}^e)^{\mathsf{T}} \mathbf{R}_b^e \mathbf{b}^b.$$
(3.47)

 L^e can be found by calculating the positions of each satellite and an estimating the position of the vehicle using pseudoranges. $\Delta \Phi$ is found from the carrier phase measurements and ΔN is the unknown integer ambiguity which must be resolved.

3.7 Integer ambiguity resolution for double differenced carrier phase

In (3.47) The vector $A\Delta N$ must be identified if the carrier phase measurement is to be used to its full potential.

3.7.1 Using pseudorange measurements

Since the pseudorange observable provides unambiguous range information, it can be used at least as an initial guess for the ambiguities. Single differencing the pseudorange model (3.2) results in

$$\Delta P_{\alpha\beta,s} = P_{\alpha,s} - P_{\beta,s} = \Delta \rho_{\alpha\beta,s} + c(\delta t_{\alpha} - \delta t_{\beta}) + \Delta \epsilon_P.$$
(3.48)

We then difference between satellites to cancel the difference in clock errors,

$$\nabla \Delta P_{\alpha\beta,s_1s_2} = \Delta P_{\alpha\beta,s_1} - \Delta P_{\alpha\beta,s_2} = \nabla \Delta \rho_{\alpha\beta,s_1s_2} + \nabla \Delta \epsilon_P. \tag{3.49}$$

With the carrier phase double difference from (3.35) the difference between the two results in the double differenced integer ambiguity and noise from both observables.

$$\nabla \Delta P_{\alpha\beta,s_1s_2} - \lambda \nabla \Delta \phi_{\alpha\beta,s_1s_2} = \lambda \nabla \Delta N_{\alpha\beta,s_1s_2} + \nabla \Delta \epsilon_P - \nabla \Delta \epsilon_\phi \tag{3.50}$$

Solving this for $\nabla \Delta N_{\alpha\beta,s_1s_2}$ results in the expression

$$\nabla \Delta N_{\alpha\beta,s_1s_2} = \frac{\nabla \Delta P_{\alpha\beta,s_1s_2}}{\lambda} - \nabla \Delta \phi_{\alpha\beta,s_1s_2} + \frac{\nabla \Delta \epsilon_P - \nabla \Delta \epsilon_\phi}{\lambda}, \quad (3.51)$$

where an estimate of this is

$$\widehat{\nabla\Delta N}_{\alpha\beta,s_1s_2} = \frac{\nabla\Delta P_{\alpha\beta,s_1s_2}}{\lambda} - \nabla\Delta\phi_{\alpha\beta,s_1s_2}.$$
(3.52)

This can thus be used as an initial estimate of the integer ambiguity. The usability of this depends on the length of the baseline due to the noise present, mostly from the pseudorange. Note that if the difference in measurement time between receivers is the same for the pseudorange and carrier phase, the error caused by this will cancel in (3.52). If this is estimate is averaged over time to reduce noise, it should be considered that due to the way the signal is tracked by the receivers, the noise in (3.51) is correlated over time. A plot showing the ambiguity estimates using this method is found in Figure 3.11.

3.7.2 Initializing ambiguities for added satellites using estimated attitude

If an estimate of the vehicle attitude is available, the known body-frame baseline can be projected onto the LOS vector of a satellite, using (3.43), giving an estimate of the range difference between the receivers along this direction. The double differenced range can be estimated the same way using

$$\widehat{\nabla \Delta \rho}_{\alpha\beta,s_1s_2} = (\boldsymbol{l}_{s_1}^e - \boldsymbol{l}_{s_2}^e)^{\mathsf{T}} \widehat{\mathbf{R}}_b^e \boldsymbol{b}^b.$$
(3.53)

The double differenced pseudorange can then be subtracted, just as for the pseudorange,

$$\widehat{\nabla\Delta N}_{\alpha\beta,s_1s_2} = \frac{\overline{\nabla\Delta\rho}_{\alpha\beta,s_1s_2}}{\lambda} - \nabla\Delta\phi_{\alpha\beta,s_1s_2}.$$
(3.54)

In this case the different measurement times for the carrier phases from each receiver must be considered, because the estimated range difference does not include it. The effect of using this method with a good knowledge of the baseline is illustrated in Figure 3.12. If a good estimate of the attitude is available and the baseline is short this is a better option than using the pseudorange. The pseudorange is however very suitable for long baselines, i.e. a longitudinal baseline on a long cargo ship or when using carrier phase for relative positioning as explained in Section 3.8.

3.7.3 Float ambiguity estimation using carrier phase

Because the DDCP integer ambiguity appears in the measurement (3.35), the state vector of an estimation algorithm can be augmented with all the ambiguities, which are then estimated without being constrained to be integers. Provided that the input is persistently exciting (PE), the ambiguities should be observable. A method for finding the best integer fit to these estimated should then be used for the best results.

3.7.4 LAMBDA

The LAMBDA algorithm, short for Least-squares AMBiguity Decorrelation Adjustment, is a integer least squares algorithm introduced by Teunissen (1993). It was developed for resolving integer ambiguities for GPS double differenced carrier phase from the beginning, but is a general integer least squares algorithm with many uses.

The need for such an algorithm appears because simply rounding the float values to the nearest integers independenly does not necessarily lead to the correct integer values. This is only valid if the values are independent, as is not correlated. The LAMBDA algorithm takes any vector of real values with a corresponding covariance matrix as input, taking the correlation into account to produce the integer vector which is optimal in the statistical sense (Joosten and Tiberius, 2002).

In estimation algorithms such as a KF, the ambiguities are commonly estimated without constraining them to be integers, and an attempt to find the best fitting integers is done separately. Assume that \hat{N} is such an estimate from a KF, with estimated covariance matrix $\Sigma_{\hat{N}}$. LAMBDA then searches for an integer vector N minimizing the cost function (Misra and Enge, 2012)

$$cost(N) = (N - \hat{N})^{\top} \Sigma_{\hat{N}}^{-1} (N - \hat{N}),$$
(3.55)

which is the sum of squared residuals weighted with the inverse of the covariance matrix. LAMBDA searches within the ellipsoid

$$\operatorname{cost}(N) \le d, \quad \text{for} \quad d > 0, \tag{3.56}$$

where *d* is a tuning parameter. Lower values of *d* gives faster results, but it must of course be selected so the optimal result lies within the search space. Instead of searching by brute force, LAMBDA attemps to transform the search to one with a diagonal covariance matrix (or close to it). A simple explanation of the concept can be done if diagonalization is possible using a suitable linear transformation **Z**. The float and integer vectors can then be transformed as $M = \mathbf{Z}N$ and $\hat{M} = \mathbf{Z}\hat{N}$. To maintain the integer property of M, **Z** must be an integer matrix. The covariance matrix of the transformed float value is then $\Sigma_{\hat{M}} = \mathbf{Z}\Sigma_{\hat{N}}\mathbf{Z}^{\mathsf{T}}$, which ideally is a diagonal matrix. The corresponding cost function is

$$\operatorname{cost}(M) = (M - \hat{M})^{\top} \underbrace{(\mathbf{Z}^{-\top} \Sigma_{\hat{N}}^{-1} \mathbf{Z}^{-1})(M - \hat{M})}_{\Sigma_{\hat{M}}^{-1}}.$$
(3.57)

Because the transformed variables are uncorrelated, finding **M** is trivial, **M** = round(\hat{M}). As long as **Z**⁻¹ exists and also is an integer matrix, the optimal integer for the original problem is found by the inverse transformation, $N = Z^{-1}M$.

The following simple example from Sollie (2017) illustrates the concept (note that the cost parameter has been changed here):

As an example, assume that we have a float solution $\hat{N} = [1.75 \quad 4.1]^{\top}$, with a covariance matrix $\Sigma_{\hat{N}} = \begin{bmatrix} 8.44 & 19.22 \\ 19.22 & 43.90 \end{bmatrix}$. The search space for this is shown in Figure 3.13a, with an ellipse drawn for d = 1. Color has been used to help show the value of the cost function (note that the same color does not represent the same cost in figures 3.13a and 3.13b). It can be seen that due to the very diagonally elongated search space, the integer vector N minimizing the cost function is not at all among the nearest neighbours of \hat{N} .

The transform $\mathbf{Z} = \begin{bmatrix} -7 & 3 \\ -2 & 1 \end{bmatrix}$ (with inverse $\mathbf{Z}^{-1} = \begin{bmatrix} -1 & 3 \\ -2 & 7 \end{bmatrix}$) gives $\hat{\mathbf{M}} = \begin{bmatrix} 0.05 & 0.6 \end{bmatrix}^{\top}$ and the diagonal covariance matrix $\Sigma_{\hat{\mathbf{M}}} = \begin{bmatrix} 1.42 & 0 \\ 0 & 0.78 \end{bmatrix}$. The search space for \mathbf{M} is shown in Figure 3.13b. \mathbf{M} can be found by simple rounding of $\hat{\mathbf{M}}, \mathbf{M} = \begin{bmatrix} 0 & 1 \end{bmatrix}^{\top}$. Transforming this back gives $\mathbf{N} = \mathbf{Z}^{-1}\mathbf{M} = \begin{bmatrix} 3 & 7 \end{bmatrix}^{\top}$.



The integer output of LAMBDA should not be blindly accepted. Because of the

Ine integer output of LAMBDA should not be blindly accepted. Because of the uncertainty of the float ambiguities, given by the covariance matrix, the integer result is not guaranteed to be correct. It should only be accepted if the integer value is sufficiently better than the other possible solutions. If integer values are not found which are certain enough, the float ambiguities are used directly for the for the interferometry, giving reduced accuracy. One acceptance test that can be used is the ratio test,

$$\frac{\operatorname{cost}(N_1)}{\operatorname{cost}(N_2)} \le \mu,\tag{3.58}$$

where $0 \le \mu \le 1$ is a parameter and $c(N_1)$ and $c(N_2)$ are the cost of the best and second best integer result, respectively. The second best integer solution in the above example is given by $M_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, which corresponds to $N_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$. $\cos(N_1) = 0.2069$ and $\cos(N_2) = 0.4633$. This gives the ratio 0.4466. This means that the best integer found has 44.66% of the cost of the second best. As this value approaches 1 the probability of the two best solutions become equal. A different and newer validation method also exist, called the Fixed Failure Rate Method which is explained in Verhagen and Teunissen (2013).



Figure 3.11: Ambiguity initialization using pseudorange: The integer ambiguity estimate using (3.52) for a section of real measurements is plotted, showing the large amount of noise from the pseudorange. Along with it are raw and measurement-time-corrected DDCP values, the predicted double differenced range from PPK positioning, and ambiguity values corresponsing to these. The true integer value is -14, approximately shown by the green line. The satellites used are SV 32 and SV 1.



Figure 3.12: Effect of differences in measurement time on DDCP: This is an enlargement of the same plots visible in Figure 3.11. The differences between the measured DDCP, with and without correction for different measurement times for the two receivers, and the predicted double differenced range from PPK are plotted. The blue uncorrected value is clearly affected by the clock drift of the two receivers. The jump at around 1450 seconds is caused by the front receiver changing its local measurement time by 1ms (it is not a jump in the clock error). The red corrected value is very close to the integer value -14.

3.8 Real-time and post-processed kinematic positioning

The carrier phase interferometry principle explained in Section 3.6 can be used to estimate the baseline between a stationary receiver and a second receiver which is free to move. This baseline, which is the same as the relative position, can be estimated with high accuracy and precision if the integer ambiguities can be resolved. If the position of the base station antenna is known accurately (i.e. by surveying) then the position of the moving receiver will also be known with high accuracy. This is illustrated in Figure 3.14.



Figure 3.14: Relative positioning using RTK. Not to scale. Both receivers track the carrier phase of the signal, but the base receiver also transmits this in real-time to the moving receiver (called the *rover*) using using i.e. radio communication. The rover can then estimate its position to the base with high accuracy and precision.

The term RTK is used when a real-time data link is used to transmit the raw carrier phase measurements from one receiver to the other, to do the baseline estimation process in real time. If the raw measurements are logged locally at the base and rover and the estimation process is performed at a later time, it is referred to as PPK. If estimates are not needed in real-time, the use of PPK can have some advantages. Since we have access to all the data when running the estimation, improved estimates can be obtained if we run through the data in both directions, in the form of i.e. the Rauch–Tung–Striebel (RTS) smoothing method (Rauch et al., 1965). Running in both directions also allow us to get more certain estimates for the integer ambiguities.



Integration of INS and GNSS

This chapter first introduces and discusses common architectures for integration of GNSS and INS, before ending with a brief look at observability of attitude and biases in aided INS.

4.1 Integration architectures

Different architectures exist for integration of GNSS and INS, differing in what measurements the integration filter uses, what it estimates and how the estimates are used. The sensors used can limit the integration options possible for a system. For example, many GNSS receivers made for the consumer market are unable to output raw observables, ruling out the use of architectures based on such measurements. Integration filters can be divided into error state and full state filters Groves (2013), depending on whether they estimate errors from a nominal estimate, commonly the IMU-driven estimate, or the complete state directly. The systems can also be divided into *centralized* and *decentralized* systems depending on whether a single or multiple estimation algorithms are used. Decentralized architectures where the output of one navigation filter is input to another are called cascaded. We also have the distiction between open loop and closed loop error state systems, depending on how the estimated errors are used to correct the INS errors (Groves, 2013). These distictions will be explained further in the following sections, where the most common architectures are explained. The different architecture are not strictly defined in the literature, and the names used differs slightly. In the following the architectures will be examplified by assuming a single GNSS receiver, but the concepts can be expanded to accommodate multiple receivers.

4.1.1 Loosely coupled integration

Loosely coupled integration architectures have in common that they use position and velocity solutions, and less commonly attitude, from a navigation processor in the GNSS receiver. When GNSS receivers without raw output capabilities are used, this is the only option. Using the definition from Groves (2013), the term loosely coupled is used both when the integration filter is input raw IMU measurements, or PVA estimates from an INS implementing the navigation equations externally. The combination of the GNSS navigation processor and an integration filter results in a cascaded, decentralized system.

An example of a loosely coupled system is illustrated in Figure 4.1, where the navigation equations are implemented in the INS, which receives error estimates from the integration filter. This feedback yields a closed loop architecture. Another option would be that the integration filter corrected the PVA output of the INS directly, which would be a open loop system (if no other feedback loops were

used for aiding the GNSS receiver). Feedback to the GNSS receiver navigation processor or aiding of the tracking loops can also be implemented if supported by the receiver, which can help make the signal tracking more robust. One of the advantages of the use of a separate INS implementing the strapdown equations, and GNSS receiver calculating its own state, is that estimates are still available if the integration filter, or the computer it runs on, should fail.

Another advantage of loosely coupled integration is that in can support basically any GNSS receiver and INS. Most receivers support outputting their estimates using the NMEA 0183 protocol (NMEA, 2018), making the replacement of a receiver with one from another manufacturer easy.



Figure 4.1: Closed loop, loosely coupled GNSS/INS integration framework

4.1.2 Tightly coupled integration

Tightly couples integration does not rely on the GNSS receiver to estimate PVA on its own before the integration filter, instead the integration filter uses raw GNSS observables. If raw IMU measurements are used, as illustrated in Figure 4.2, this results in centralized processing, using only a single estimation algorithm. It is however also possible to use raw GNSS observables with an INS implementaing the interial navigation equations outside the integration filter, but it would then still be a decentralized system. In centralized integration, the integration filter, commonly an extended Kalman filter (EKF) or nonlinear observer (NLO), uses the strapdown equations and models for the GNSS observables internally. A downside to the centralized integration is that any failure of the computer where the integration filter is implemented means that no estimates are available, as no independent estimates are available. A significant advantage of tightly coupled integration is that GNSS can be used to correct the INS errors even if less than four satellites, which is required for a standalone GNSS position estimate, are available (Groves, 2013).



Figure 4.2: Centralized, tightly coupled GNSS/INS integration framework.

4.1.3 Deeply coupled integration

Deeply coupled integration moves the integration filter into the GNSS receiver tracking loops. The GNSS measurements used by the integration filter are then the signal components from the receiver correlation channels directly. The integration filter aids the signal tracking by controlling the numerically controlled oscillators (NCOs) used to generate the receiver reference signals (Groves, 2013). Deeply coupled integration cannot be used unless it is supported by the GNSS receiver, which basically requires access to the receiver firmware, and the capability of connecting inertial sensors to the receiver hardware.

4.2 Observability of attitude and biases

The use of aiding sensors in an integrated navigation system enables observation of the IMU biases and attitude errors under certain conditions, depending on the aiding measurements used. For a pure INS without aiding, sensor biases cannot be observed without constraining movement, by e.g. performing a static calibration when the vehicle is known to be stationary with a known attitude (Farrell, 2008). If the vehicle is only known to be stationary with unknown attitude, which is normally the case when performing accelerometer leveling, the attitude and accelerometer biases cannot be separated (Batista et al., 2009; Groves, 2013). For systems without full instantaneous observability, it is interesting to see what vehicle maneuvering is required to make the errors observable. For an INS aided by single antenna GNSS, attitude errors and accelerometer biases are both observed through the errors they produce in the velocity and position. Both of these errors cause a linearly increasing velocity error when considering short time intervals and small attitude errors. As they are coupled, maneuvering is required for them to be separated. While the gyro bias also causes errors in velocity through the resulting attitude error, the effect on the velocity error from the gyro bias error is quadratic, which makes it obervable over time. In steady flight with constant velocity and direction, the pitch and roll errors are coupled with the body frame x- and y-axis accelerometer errors, respectively (Groves, 2013), and an error in heading does not cause any error in the velocity and position. Heading errors only cause errors in velocity when there is acceleration in the horizontal plane. A combination of angular velocity and linear acceleration is required to separate the attitude errors and accelerometer bias (Groves, 2013).

By using a longitudinal dual-antenna setup the heading and pitch angles can be made observable independently of maneuvering, as shown in Sollie (2017). By using three antennas forming two nonparallel baselines, the full attitude can be observed, then also leading to the accelerometer biases being observable.

4.3 Attitude parametrizations and MEKF motivation

When estimating the attitude of a vehicle that has the potential to experience any orientation, such as fighter aircraft, it is beneficial to use an attitude parametrization which is globally nonsingular. Stuelpnagel (1964) shows that it is topologically impossible to have this property for a three dimensional parametrization. The four dimensional unit quaternion, q (see Appendix E), is the smallest representation with this property, although it parametrizes the rotation group in a 2-1 way, with q and -q representing the same orientation. Since attitude only has three degrees of freedom, this means that when using only a single parametrization it must either be redundant or have at least one singularity. Using a redundant parametrization in the state vector of a Kalman filter (Kalman, 1960; Markley, 2003b) is not ideal since the attitude parameters. Thus no single ideal attitude parametrization exist.

Furthermore it is problematic to represent each of the four quaternion components by a Gaussian probability distribution because of the unit constraint, which would be the case if a quaternion is used in the state vector estimated by a Kalman filter. Restricting the probability distributions to only have nonzero probabilities for quaternions that satisfy the unit constraint would lead to the expected value violating the unit constraint, by lying inside the unit sphere in four-dimensional Cartesian space. The expected value operation would have to be done in non-Cartesian coordinates for this problem to be solved, but using the MEKF effectively produces the same result by working with a 3-dimensional attitude parametrization (Markley, 2003b).

The unit quaternion as a global nominal attitude parametrization has several advantages: It has a linear differential equation, unlike the 5-dimensional representation (Stuelpnagel, 1964). Creating the rotation matrix from a quaternion or rotating a vector can be done without any trancendental functions (sin/cos), and normalizing the quaternion to correct violations of the unity constraint caused by numerical errors when performing kinematic propagation in a computer can be done trivially by dividing the quaternion by its norm.

The way this is solved by the MEKF is to split the attitude into a nominal quaternion part which is propagated outside the filter, and a three dimensional error part which is estimated by the MEKF (Markley, 2003a). By injecting the estimated errors into the nominal attitude after the measurement corrections and resetting the error to zero, the error will be kept small after the initial transient, staying far away from any singularities.

The *multiplicative* part of the MEKF is the injection of the attitude error into the nominal quaternion. This cannot be done using an additive injection, as the resulting estimator would provide biased attitude estimates (Markley and Crassidis, 2014). This is illustrated by defining the quaternion error δq as the difference between the true quaternion q and the estimated quaternion \hat{q} ,

$$\delta q = q - \hat{q}. \tag{4.1}$$

The norm of the estimated quaternion becomes

$$\|\hat{q}\|^{2} = \|q - \delta q\|^{2} = \|q\|^{2} - 2\delta q^{\top} q + \|\delta q\|^{2}.$$
(4.2)

For an unbiased estimator the expectation of the error δq would be zero, giving the estimate norm expectation

$$E(\|\hat{\boldsymbol{q}}\|^2) = 1 + E(\|\delta \boldsymbol{q}\|^2), \tag{4.3}$$

which differs from 1 unless the variance of the error components is zero. Several methods have been proposed to solve the problems with additive quaternion filtering, but none are completely satisfactory (Markley and Crassidis, 2014). Replacing the additive error by a multiplicative error using the quaternion product (E.10),

$$q = \hat{q} \otimes \delta q(a), \tag{4.4}$$

however, does solve this issue.

Part II

Aided INS design

This chapter presents the derivation of an estimation algoritm for estimation of PVA using strapdown INS and raw pseudorange, Doppler frequency and carrier phase from two GPS receivers.

The estimation algorithm that is used to estimate PVA is the MEKF (Markley, 2003a), because of the advantages mention in Section 4.3. Many options for three dimentional attitude error parametrizations does however exist, and will be discussed here. Velocity and angular increments (also known as delta-velocity and delta-angle) (Titterton and Weston, 2004) from an IMU will be used as inputs to the system dynamics model. Increments are outputted in stead of specific force and angular rates, as this allows for internal sampling at maximum-IMU-sampling rate (up to several kilo Hertz), while outputting measurements on a reduced frequency. This allows usage for of all measured information, while maintaing the computational burden of estimation algorithms managable. GPS pseudorange, Doppler shift and double differenced carrier phase from two receivers will be used for corrections. This results in tightly coupled integration, as explained in Section 4.1.2. All GPS measurements were processed taking antenna lever arms into account. GLONASS was not utilized in the aided INS, however it was used in calculating the references to compare results. Since independent GNSS receivers are used, the measurement corrections also take into account the differences in measurement timing.

The nominal states of the aided INS are initialized by running single epoch least squares (LS) estimation using pseudoranges and Doppler frequency measurements for both receivers. This initializes position, velocity, clock biases and clock drift rates, and works for initialization independently of whether the UAV is on the ground or in-air. Attitude is initialized using a combination of accelerometer leveling and pseudorange position for each antenna. If the UAV is identified to be on the ground (has velocity close to zero), gyro bias will be initialized by assuming $\omega_{ch}^{b} = 0$.

5.0.1 MEKF steps

The steps performed by the MEKF is the following:

- 1. Prediction
 - a) Nonlinear nominal state prediction of the INS

$$\hat{\boldsymbol{x}}_{k}^{-} \leftarrow f(\hat{\boldsymbol{x}}_{k-1}, \boldsymbol{u}, t_{k}) \tag{5.1}$$

(if no correction was performed last time step \hat{x}_{k-1} is replaced by \hat{x}_{k-1}^{-}).

b) Calculate the linearized transition matrix $\mathbf{\Phi}_k$ and the discrete process noise covariance matrix \mathbf{Q}_k .

c) Covariance propagation

$$\mathbf{P}_{k}^{-} = \mathbf{\Phi}_{k} \mathbf{P}_{k-1} \mathbf{\Phi}_{k}^{\top} + \mathbf{Q}_{k}$$
(5.2)

2. Correction

a) Calculate Kalman gain

$$\mathbf{K}_{k} = \mathbf{P}_{k}^{-} \mathbf{H}_{k}^{\top} (\mathbf{H}_{k} \mathbf{P}_{k}^{-} \mathbf{H}_{k}^{\top} + \mathbf{R}_{k})^{-1}$$
(5.3)

b) Observe the error-state using measurement correction.

$$\delta \boldsymbol{x}_k = \mathbf{K}_k (\boldsymbol{z}_k - \boldsymbol{h}(\hat{\boldsymbol{x}}_k^-)) \tag{5.4}$$

c) Covariance correction

$$\mathbf{P}_{k} = (\mathbf{I} - \mathbf{K}_{k}\mathbf{H}_{k})\mathbf{P}_{k}^{-}(\mathbf{I} - \mathbf{K}_{k}\mathbf{H}_{k})^{\top} + \mathbf{K}_{k}\mathbf{R}_{k}\mathbf{K}_{k}^{\top}$$
(5.5)

This is the so called Joseph form of the covariance measurement update which keeps **P** positive definite and symmetric.

d) Injection of the error estimates into the nominal state,

$$\hat{\boldsymbol{x}}_k = \hat{\boldsymbol{x}}_k^- \oplus \delta \boldsymbol{x}_k, \tag{5.6}$$

where \oplus is a the composition applicable (additive or multiplicative) for each of the nominal and error states.

e) Reset the error to zero

$$\delta \boldsymbol{x}_{k+1}^{-} = \boldsymbol{0}. \tag{5.7}$$

If several different sets of measurements are available at the same time, the correction part can be run multiple times in succession. The nominal state is output every iteration, even if no measurement correction is performed. The injection of the observed error into the nominal state does not change the value of the estimate, it simply moves the error component. The covariance matrix does not change, and is not reset with the error state.

5.0.2 Choice of attitude error parametrization

Many possible choices for minimal (three component) attitude error parametrizations exist. Euler angles (see Appendix E) can be used, as well as vectorial parametrizations defined as

$$a = a(\phi)u, \tag{5.8}$$

with the unit vector u, the rotation ϕ around the axis defined by u and the odd generating function $a(\phi)$, (Bauchau and Trainelli, 2003) with

$$\lim_{\phi \to 0} \frac{a(\phi)}{\phi} = \kappa, \tag{5.9}$$

where κ is a real number. The simplest of these is the rotation vector ($a(\phi) = \phi$, $\kappa = 1$). Vectorial parametrizations with generating functions in the sine and tangent families,

$$a(\phi) = \xi \sin \frac{\phi}{m}$$
 and $a(\phi) = \xi \tan \frac{\phi}{m}$, (5.10)

with $\kappa = \frac{\xi}{m}$, are closely related to the unit quaterion

$$q(u,\phi) = \begin{bmatrix} \cos\frac{\phi}{2} \\ u\sin\frac{\phi}{2} \end{bmatrix},$$
(5.11)

with the advantageous property that the conversion q(a) requires no transcendental functions. Parametrizations in these families include the Gibbs vector $(a(\phi) = 2 \tan \frac{\phi}{2})$, the modified Rodrigues parameters $(a(\phi) = 4 \tan \frac{\phi}{4})$ and the reduced Euler-Rodrigues parameters $(a(\phi) = \sin \frac{\phi}{2}, \kappa = \frac{1}{2})$, which is the vector part of the quaternion (5.11), (Bauchau and Trainelli, 2003). The most important parameters in (5.10) are the denominators *m* in the sin/tan arguments, as we can use scaled versions of these representations, which can give some advantages. The mentioned options with some relevant properties are shown in Table 5.1. Many different names exist for the vectorial parametrizations as shown in the table.

Table 5.1: Three component attitude parametrizations

Parametrization	Symbol	Singularities	$\delta q(a)$	Invalid errors
Euler angles	$\boldsymbol{\Theta} = [\phi, \theta, \psi]^{\top}$	$\theta = \pm \frac{\pi}{2}$	$ \begin{bmatrix} c \frac{\phi}{2} c \frac{\phi}{2} c \frac{\psi}{2} + s \frac{\phi}{2} s \frac{\phi}{2} s \frac{\psi}{2} \\ s \frac{\phi}{2} c \frac{\phi}{2} c \frac{\psi}{2} - c \frac{\phi}{2} s \frac{\phi}{2} s \frac{\psi}{2} \\ c \frac{\phi}{2} s \frac{\phi}{2} c \frac{\psi}{2} + s \frac{\phi}{2} c \frac{\phi}{2} s \frac{\psi}{2} \\ c \frac{\phi}{2} s \frac{\phi}{2} c \frac{\psi}{2} + s \frac{\phi}{2} c \frac{\phi}{2} s \frac{\phi}{2} \\ c \frac{\phi}{2} c \frac{\phi}{2} s \frac{\phi}{2} - s \frac{\phi}{2} s \frac{\phi}{2} c \frac{\psi}{2} \end{bmatrix} $	$\ \phi\ , \ \theta\ , \ \psi\ > \frac{\pi}{2}$
Rotation vector (exponential map)	$\phi = e\phi$	$\phi=2\pi n$	$\begin{bmatrix} \cos \frac{\ \phi\ }{2} \\ (\frac{\phi}{\ \phi\ }) \sin \frac{\ \phi\ }{2} \end{bmatrix}$	$\ \boldsymbol{\phi} \ > \pi$
Rodrigues / Gibbs / Cayley parametrization	$a_g = \frac{e}{\eta}$	$\phi=\pm\pi$	$\frac{1}{\sqrt{1+\ \boldsymbol{a}_{\mathcal{S}}\ _{2}^{2}}} \begin{bmatrix} 1 \\ \boldsymbol{a}_{\mathcal{S}} \end{bmatrix}$	None
Wiener / Milenkovic / Modified Rodrigues parameters / conformal rotation vector (CRV)	$a_p = \frac{\epsilon}{1+\eta}$	$\phi=\pm 2\pi$	$\frac{1}{1+\ a_p\ _2^2} \begin{bmatrix} 1-\ a_p\ _2^2\\ 2a_p \end{bmatrix}$	$\ \boldsymbol{a}_p\ >1$
Vector/imaginary part of quaternion, re- duced Euler Rodrigues parameters	e	$\phi=\pm\pi$	$\begin{bmatrix} \sqrt{1 - \ \boldsymbol{\epsilon}\ _2^2} \\ \boldsymbol{\epsilon} \end{bmatrix}$	$\ \boldsymbol{\epsilon}\ > 1$

Different sources also use different scaling factors, i.e. Bauchau and Trainelli (2003) uses c = m such that $\kappa = 1$, while Markley (2003a) uses c = 1, giving the Gibbs and modified Rodrigues parameters the simple quaternion relations

$$a_g = \frac{\epsilon}{\eta} = u \tan \frac{\phi}{2} \tag{5.12}$$

$$a_p = \frac{\epsilon}{1+\eta} = \frac{u\sin\frac{\phi}{2}}{1+\cos\frac{\phi}{2}} = u\tan\frac{\phi}{4}$$
(5.13)

The requirement (5.9) met by the rotation vector and sine and tangent families means that these parametrization all provide the same first order quaternon approximation if ξ is chosen such that κ is the same. They are also equal to the Euler angles to first order if $\kappa = 1$, meaning that these parametrizations are all equal in a first-order filter. In fact the vectorial parametrizations even provide the same second order quaternion approximation (Markley, 2003a). Scaling the vector parametrizations to give $\kappa = 1$ as done Bauchau and Trainelli (2003) has the advantage of giving the variances in the Kalman filter the unit of radians squared.

The Gibbs vector in particular has some nice features which makes it very suitable for use in a MEKF: As the attitude error approaches π , the Gibbs vector approaches infinity. This means that attitude error estimates in the MEKF state vector over π , which would make no sense due to the maximum attitude error being 180 degrees, cannot occur and all possible values give sensible quaternion resets. For the other options an insensible error can occur. Also since all estimates in the MEKF are given by Gaussian distributions with an expected value in the error state δx and variances in **P**, and the Gaussian distribution has infinite tails, the other parametrizations than the Gibbs vector will always have non-zero probabilities of insensible errors even if the expected value is sensible. The Gibbs vector is clearly more suitable for an error with a Gaussian distribution.

Another advantage of the Gibbs vector is in the injection of the observed error into the nominal state: since $(bq_1) \otimes q_2 = b(q_1 \otimes q_2)$ for any scalar *b*, we can create an unnormalized quaternion from the Gibbs vector simply as $\begin{bmatrix} 1 & a_g^\top \end{bmatrix}^\top$, inject this into the nominal quaternion and then normalize afterwards.

In the MEKF implementation in this thesis the attitude error is defined as twice the Gibbs vector,

$$a \coloneqq 2\delta a_g = 2\frac{\delta\epsilon}{\delta\eta},\tag{5.14}$$

since this will give variances in radians squared. A quaternion can be created by this as

$$\delta q(a) = \underbrace{\frac{1}{\sqrt{4 + \|a\|_2^2}}}_{\text{Normalizing factor}} \begin{bmatrix} 2\\ a \end{bmatrix}, \qquad (5.15)$$

found by solving $||a_g||_2 = \frac{||\epsilon||_2}{\eta} \iff \frac{1}{2} ||a||_2 = \frac{\sqrt{1-\eta^2}}{\eta}$ for η and using $\delta q(a) = \begin{bmatrix} \eta \\ \frac{1}{2}a\eta \end{bmatrix}$ (here assuming we are using $\eta \ge 0$). This error will represent the attitude error given in body-frame, but it could also have been given in the global ECEF frame (Solà, 2017). This would change the order of the nominal quaternion and the estimated error in the error injection.

5.1 System dynamics

The true state vector is chosen as

$$\boldsymbol{x} = [(\boldsymbol{q}_b^e)^\top \quad \boldsymbol{b}_g^\top \quad (\boldsymbol{p}_{eb}^e)^\top \quad (\boldsymbol{v}_{eb}^e)^\top \quad \boldsymbol{b}_a^\top \quad t_{\varepsilon f} \quad t_{\varepsilon \beta} \quad t_{df} \quad t_{d\beta} \quad \boldsymbol{N}_{dd}^\top]^\top, \quad (5.16)$$

where q_b^e is the quaternion parametrization of the rotation from the body-frame to the ECEF frame, b_g is the gyro bias, p_{eb}^e is the position of the origin of the body frame, chosen to coincide with the coordinate center of the IMU, relative to the ECEF origin, resolved in ECEF coordinates, v_{eb}^e is the corresponding velocity, b_a is the accelerometer bias, t_{ef} and $t_{e\beta}$ are the clock errors relative to GPST of the front and back receivers, respectively, and t_{df} and $t_{d\beta}$ are the clock drift rates. N_{dd} is a vector of float estimates of the integer ambiguities of double differenced carrier phase observables. This will also be written short as

$$\boldsymbol{x} = [\bar{\boldsymbol{x}}^{\top} \quad \boldsymbol{N}_{dd}^{\top}]^{\top}. \tag{5.17}$$

The clock errors are chosen to be in meters (scaled from time in seconds with the speed of light, *c*) to improve matrix conditioning, keeping the different states closer in order of magnitude, and because the clock error appears in the pseudorange measurements as a distance bias. The position and velocity are estimated in the ECEF frame since we will be using GPS, and this is the frame in which the satellite position and velocities are calculated. The estimated attitude is chosen to be between body and ECEF frames for the same reason. The biases are estimated as increments as modelled in Section 5.1.1. Gravity is assumed constant and equal to the standard

$$g^n = [0 \quad 0 \quad 9.80665]^{\top}[\frac{m}{s^2}].$$
 (5.18)

The true state is split into nominal and error components. The attitude is the special case here, as the composition of nominal and error part is performed using the quaternion product (E.10), and the error is estimated as a different parametrization than the nominal quaternion. The nominal state vector which will be propagated outside the filter is

$$\hat{\boldsymbol{x}} = [(\hat{\boldsymbol{q}}_b^e)^\top \quad \hat{\boldsymbol{b}}_g^\top \quad (\hat{\boldsymbol{p}}_{eb}^e)^\top \quad (\hat{\boldsymbol{v}}_{eb}^e)^\top \quad \hat{\boldsymbol{b}}_a^\top \quad \hat{\boldsymbol{t}}_{\varepsilon f} \quad \hat{\boldsymbol{t}}_{\varepsilon \beta} \quad \hat{\boldsymbol{t}}_{df} \quad \hat{\boldsymbol{t}}_{d\beta} \quad \hat{\boldsymbol{N}}_{dd}^\top]^\top, \quad (5.19)$$

while the error state vector is chosen as

$$\delta \boldsymbol{x} = [\boldsymbol{a}^{\top} \quad \delta \boldsymbol{b}_{g}^{\top} \quad (\delta \boldsymbol{p}_{eb}^{e})^{\top} \quad (\delta \boldsymbol{v}_{eb}^{e})^{\top} \quad \delta \boldsymbol{b}_{a}^{\top} \quad \delta t_{\varepsilon f} \quad \delta t_{\varepsilon \beta} \quad \delta t_{df} \quad \delta t_{d\beta} \quad \delta \boldsymbol{N}_{dd}^{\top}]^{\top}.$$
(5.20)

The true values are related to the nominal and error parts as

$$\boldsymbol{q}_{b}^{e} = \hat{\boldsymbol{q}}_{b}^{e} \otimes \delta \boldsymbol{q}(\boldsymbol{a}) \tag{5.21}$$

$$\boldsymbol{b}_g = \boldsymbol{b}_g + \delta \boldsymbol{b}_g \tag{5.22}$$

$$p_{eb}^{e} = \hat{p}_{eb}^{e} + \delta p_{eb}^{e}$$
(5.23)

$$\boldsymbol{v}_{eb}^{\epsilon} = \hat{\boldsymbol{v}}_{eb}^{\epsilon} + \delta \boldsymbol{v}_{eb}^{\epsilon} \tag{5.24}$$

$$\boldsymbol{b}_a = \boldsymbol{b}_a + \delta \boldsymbol{b}_a \tag{5.25}$$

$$t_{\varepsilon f} = t_{\varepsilon f} + \delta t_{\varepsilon f} \tag{5.26}$$

$$t_{\varepsilon\beta} = t_{\varepsilon\beta} + \delta t_{\varepsilon\beta} \tag{5.27}$$

 $(\Box 20)$

$$t_{df} = t_{df} + \delta t_{df} \tag{5.28}$$

$$t_{d\beta} = t_{d\beta} + \delta t_{d\beta} \tag{5.29}$$

$$\boldsymbol{N}_{dd} = \boldsymbol{N}_{dd} + \delta \boldsymbol{N}_{dd}. \tag{5.30}$$

or we can define the operator \oplus to be the same composition as above,

,

$$\boldsymbol{x} = \hat{\boldsymbol{x}} \oplus \delta \boldsymbol{x}. \tag{5.31}$$

5.1.1 IMU output model

The STIM300 IMU (Sensonor, 2018) which will be used for the testing of the algorithm is configured to output incremental angle and incremental velocity. The internal measurements of angular rate and specific force are taken at a rate of 2 kHz, while the output rate is 250 Hz. This means that eight evenly spaced measurements are taken for each output. For the discrete dynamics this can be approximated by the following:

$$\Delta \boldsymbol{\theta}_{\text{IMU}} \approx \int_{t_k}^{t_{k+1}} (\boldsymbol{\omega}_{ib}^b + \boldsymbol{w}_g + \boldsymbol{b}_g) dt$$
 (5.32)

$$\Delta \boldsymbol{v}_{\mathrm{IMU}} \approx \int_{t_k}^{t_{k+1}} (\boldsymbol{f}_{ib}^b + \boldsymbol{w}_a + \boldsymbol{b}_a) dt$$
 (5.33)

where *i* is the ECI frame described in Appendix D, which is assumed intertial. Note that the velocity increments here are not the actual change in velocity in ECEF, since the gravity term means that the output is the velocity increments relative to free fall.

In the cases where expected values of the angular rate or acceleration values are needed in the continous system matrices or measurement models in the MEKF, the relations

$$\hat{\boldsymbol{\omega}}_{ib}^{b} \approx \frac{\Delta \boldsymbol{\theta}_{\rm IMU}}{\Delta t} - \hat{\boldsymbol{b}}_{g} \tag{5.34}$$

$$\hat{\boldsymbol{f}}_{ib}^{b} \approx \frac{\Delta \boldsymbol{v}_{\mathrm{IMU}}}{\Delta t} - \hat{\boldsymbol{b}}_{a} \tag{5.35}$$

will be used.

5.1.2 True state model

The continous time dynamics are modelled using angular rates and specific force even though the actual IMU outputs, modelled in 5.1.1, are increments which are suitable for discrete time.

Attitude kinematics

The continous time gyro measurement is modelled as

$$\boldsymbol{\omega}_{\rm IMU} = \boldsymbol{\omega}_{ib}^b + \boldsymbol{b}_g + \boldsymbol{w}_g \tag{5.36}$$

where the model (2.1) has been used and the measurement frame has been rotated to the body frame as described in Section 2.5.1. w_g is continous white Gaussian noise, $w_g \sim \mathcal{N}(0, \sigma_g^2)$. The angular rate of the body frame relative to ECEF is

$$\boldsymbol{\omega}_{eb}^{b} = \boldsymbol{\omega}_{ib}^{b} - \boldsymbol{\omega}_{ie}^{b} \tag{5.37}$$

with ω_{ie}^{b} being the angular rate of the earth relative to inertial space, decomposed in the body frame. Solving (5.36) for ω_{ib}^{b} and inserting this in (5.37) gives

$$\boldsymbol{\omega}_{eb}^{b} = \boldsymbol{\omega}_{\text{IMU}} - \boldsymbol{b}_{g} - \boldsymbol{w}_{g} - \mathbf{R}(\boldsymbol{q}_{b}^{e})^{\top} \boldsymbol{\omega}_{ie}^{e}$$
(5.38)

From (2.10) we have

$$\dot{\boldsymbol{q}}_{b}^{e} = \frac{1}{2} \boldsymbol{q}_{b}^{e} \otimes \boldsymbol{\omega}_{eb}^{b}$$
(5.39)

$$= \frac{1}{2}\boldsymbol{q}_{b}^{e} \otimes (\boldsymbol{\omega}_{\mathrm{IMU}} - \boldsymbol{b}_{g} - \boldsymbol{w}_{g} - \mathbf{R}(\boldsymbol{q}_{b}^{e})^{\top}\boldsymbol{\omega}_{ie}^{e})$$
(5.40)

$$= \frac{1}{2}\boldsymbol{q}_{b}^{e} \otimes (\boldsymbol{\omega}_{\mathrm{IMU}} - \boldsymbol{b}_{g} - \boldsymbol{w}_{g}) - \frac{1}{2}\boldsymbol{\omega}_{ie}^{e} \otimes \boldsymbol{q}_{b}^{e}$$
(5.41)

Translational kinematics

The continous time accelerometer measurement is modelled as

$$f_{\rm IMU}^b = f_{ib}^b + b_a + w_a, (5.42)$$

where (2.5) has been rotated to the body frame, and it has been assumed that there is no lever arm. w_a is Gaussian white noise, $w_a \sim \mathcal{N}(0, \sigma_a^2)$. Using (2.26) with (2.2) we have

$$\dot{v}_{eb}^{e} = f_{ib}^{e} + g^{e} - 2\mathbf{S}(\omega_{ie}^{e})v_{eb}^{e}.$$
(5.43)

Solving (5.42) for f_{ib}^b and inserting this gives

$$\dot{\boldsymbol{v}}_{eb}^{e} = \mathbf{R}(\boldsymbol{q}_{b}^{e})(f_{\mathrm{IMU}}^{b} - \boldsymbol{b}_{a} - \boldsymbol{w}_{a}) + \boldsymbol{g}^{e} - 2\mathbf{S}(\boldsymbol{\omega}_{ie}^{e})\boldsymbol{v}_{eb}^{e}.$$
 (5.44)

IMU biases

The gyro and accelerometer biases are commonly modelled as either Wiener or first-order Gauss-Markov processes. One thing to consider here is that all biases are not necessarily observable at all times, i.e. when the UAV is standing still on the ground. The first order Gauss-Markov process is a stable process, where insufficiently exciting UAV movement will lead to the unobservable bias states reducing towards zero.

$$\dot{\boldsymbol{b}}_g = -\frac{1}{T_g} \boldsymbol{b}_g + \boldsymbol{w}_{b_g}, \qquad (5.45)$$

$$\dot{\boldsymbol{b}}_{a} = -\frac{1}{T_{a}}\boldsymbol{b}_{a} + \boldsymbol{w}_{b_{a}}, \tag{5.46}$$

$$\boldsymbol{w}_{b_g} \sim \mathcal{N}(0, \sigma_{b_g}^2), \tag{5.47}$$

$$\boldsymbol{w}_{b_a} \sim \mathcal{N}(0, \sigma_{b_a}^2). \tag{5.48}$$

Float ambiguities

The states for the carrier phase ambiguities can be choose to as either single differenced ambiguities, as done by RTKLIB, or double differenced ambiguities, as done by goGPS (Realini and Reguzzoni, 2013). The advantage of using single differenced ambiguities is that no cumbersome handling of the values is necessary if the reference changes. However, double differenced values were used here, and the handling of changes in satellites used is explained in Section 5.3. Here the size of the ambiguity vector is kept minimal as the satellites where lock is lost is removed from the vector.

The integer ambiguities present in the double differenced carrier phase are constant as long as both receivers maintain lock on the received signals. The float estimates can in theory also be considered constant, but it can be beneficial to model it as a Wiener process with a small amount of driving noise,

$$\dot{N}_{dd} = w_N. \tag{5.49}$$

$$\boldsymbol{w}_N \sim \mathcal{N}(0, \, \sigma_N^2) \tag{5.50}$$

Errors in the baseline measurement and unmodelled placement of the antenna phase center relative to the physical center of the antenna

Clock errors

Each receiver has a temperature compensated crystal oscillator (TCXO) as its local clock reference (U-blox, 2016). Since all the raw observables output by the receivers are affected by offset and drift of the local receiver time relative to GPST, the local clock errors must be estimated. While the clock offset can be observed from

pseudoranges, and the drift rate can be observed from the Doppler frequency, this is only the case when four or more satellites are tracked. Having good clock error estimate allows us to use GPS measurements even in situations where less than four satellites are visible, for a reasonable amount of time. Errors in the local time translate into ranging errors that are the same for all satellites for each receiver. The drift rate, or other frequency depdentent errors impacts the measurement of the received signal frequency from each satellite relative to the nominal L1 frequency (1575.42 MHz). In addition, the U-blox receivers schedule their measurement outputs according to their own local time, which means that a difference in the clock offset of the two receivers relative to GPST leads to measurements that are not taken at the same time for both receivers. This is especially important for the double differenced carrier phase, where it commonly is assumed that measurements are taken at the same instant.

A two-state random clock error model is commonly used for receivers (Van Dierendonck et al., 1984; Galleani, 2008; Farrell, 2008) where the drift rate (oscillator frequency) is assumed to random walk over time, and the clock error (oscillator phase) is the sum of the integral of the drift rate, and a separate random walk process. For the front and back receivers this gives in total four states

$$\dot{t}_{\varepsilon f} = t_{df} + w_{t_{\varepsilon f}},\tag{5.51}$$

$$\dot{t}_{\varepsilon\beta} = t_{d\beta} + w_{t_{\varepsilon\beta}},\tag{5.52}$$

$$t_{df} = w_{t_{df}}, \tag{5.53}$$

$$t_{d\beta} = w_{t_{d\beta}}, \tag{5.54}$$

where the driving noises are modelled as white Gaussian random processes:

$$\begin{split} & w_{t_{\varepsilon,f}} \sim \mathcal{N}(0, \, \sigma_{t_{\varepsilon f}}^2) & w_{t_{\varepsilon,\beta}} \sim \mathcal{N}(0, \, \sigma_{t_{\varepsilon \beta}}^2) \\ & w_{t_{d,f}} \sim \mathcal{N}(0, \, \sigma_{t_{df}}^2) & w_{t_{d,b}} \sim \mathcal{N}(0, \, \sigma_{t_{d\beta}}^2). \end{split}$$

For each receiver this can be written as the linear system

$$\begin{bmatrix} \dot{t}_{\varepsilon} \\ \dot{t}_{d} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} t_{\varepsilon} \\ t_{d} \end{bmatrix} + \begin{bmatrix} w_{t_{\varepsilon}} \\ w_{t_{d}} \end{bmatrix}$$
(5.55)

with covariance matrix

$$\mathbf{Q} = \begin{bmatrix} \sigma_{t_e}^2 & 0\\ 0 & \sigma_{t_d}^2 \end{bmatrix}.$$
(5.56)

This can also be expanded to a three-state model including the frequency drift rate, but this might be more beneficial for higher quality clocks, such as the atomic time standards onboard the GNSS satellites. The clock error parameters in the GPS ephemeris are these three errors (bias, frequency error and frequency drift) at a reference time, which can be used to determine the errors relevant for pseudorange and Doppler frequency observables in the validity interval of the message by extrapolation.

5.1.3 Nominal state model

The nominal state model is simply the expected value of the true model, which is the true model with all noises set to their expected value, in this case zero. This gives a nominal state chosen such that no error state prediction is required in the MEKF, since the expected value of the errors is always zero before a measurement update.

$$\hat{\boldsymbol{\omega}}_{eb}^{b} = \boldsymbol{\omega}_{\mathrm{IMU}} - \hat{\boldsymbol{b}}_{g} - \mathbf{R}(\hat{\boldsymbol{q}}_{b}^{e})^{\mathsf{T}} \boldsymbol{\omega}_{ie}^{e}$$
(5.57)

$$\dot{\hat{\boldsymbol{q}}}_{b}^{e} = \frac{1}{2} \hat{\boldsymbol{q}}_{b}^{e} \otimes (\boldsymbol{\omega}_{\mathrm{IMU}} - \hat{\boldsymbol{b}}_{g}) - \frac{1}{2} \boldsymbol{\omega}_{ie}^{e} \otimes \hat{\boldsymbol{q}}_{b}^{e}$$
(5.58)

$$\dot{\hat{\boldsymbol{b}}}_g = -\frac{1}{T_g} \hat{\boldsymbol{b}}_g \tag{5.59}$$

$$\dot{\hat{p}}^e_{eb} = \hat{v}^e_{eb} \tag{5.60}$$

$$\dot{\hat{\boldsymbol{v}}}_{eb}^{e} = \mathbf{R}(\hat{\boldsymbol{q}}_{b}^{e})(\boldsymbol{f}_{\mathrm{IMU}}^{b} - \hat{\boldsymbol{b}}_{a}) + \boldsymbol{g}^{e} - 2\mathbf{S}(\boldsymbol{\omega}_{ie}^{e})\hat{\boldsymbol{v}}_{eb}^{e}$$
(5.61)

$$\dot{\hat{\boldsymbol{b}}}_a = -\frac{1}{T_a}\hat{\boldsymbol{b}}_a \tag{5.62}$$

$$\dot{\hat{t}}_{\varepsilon f} = \hat{t}_{df} \tag{5.63}$$

$$\dot{\hat{t}}_{\varepsilon\beta} = \hat{t}_{d\beta} \tag{5.64}$$

$$\hat{t}_{df} = 0 \tag{5.65}$$

$$\dot{\tilde{t}}_{d\beta} = 0 \tag{5.66}$$

$$\hat{\boldsymbol{N}}_{dd} = \boldsymbol{0} \tag{5.67}$$

5.1.4 Discretized nominal state model

Since the nominal state is to be propagated on a computer, we need to discretize the dynamics. When implementing these as updates directly on the nominal values without storing the values from the previous timestep until all updates are finished, the order of the updates matter. The sample time Δt is determined by the output rate of the IMU, which in this case is 250 Hz. In the discretized model it must also be taken into account that the IMU output values are increments, as modelled in Section 5.1.1. For the position-velocity part of the dynamics we can use

$$\hat{p}_{eb}^{e} \leftarrow \hat{p}_{eb}^{e} + \hat{v}_{eb}^{e} \Delta t + \frac{\Delta t}{2} (\mathbf{R}(\hat{q}_{b}^{e})(\Delta v_{\mathrm{IMU},f} - \hat{b}_{a}\Delta t) + g^{e} \Delta t - 2\mathbf{S}(\omega_{ie}^{e})\hat{v}_{eb}^{e}\Delta t) \quad (5.68)$$

$$\hat{\boldsymbol{v}}_{eb}^{e} \leftarrow \hat{\boldsymbol{v}}_{eb}^{e} + \mathbf{R}(\hat{\boldsymbol{q}}_{b}^{e})(\Delta \boldsymbol{v}_{\mathrm{IMU},f} - \hat{\boldsymbol{b}}_{a}\Delta t) + \boldsymbol{g}^{e}\Delta t - 2\mathbf{S}(\omega_{ie}^{e})\hat{\boldsymbol{v}}_{eb}^{e}\Delta t.$$
(5.69)

While we could use

$$\hat{q}_{b,k+1}^{e} = \hat{q}_{b,k}^{e} + \delta t \hat{q}_{b,k}^{e}$$
(5.70)

for infinitesimal time increments δt , implementing this with a finitely small increment Δt will require normalization after every time update, as the unit norm

constraint is not maintaned with this method, as shown in Section 4.3. Another method is to create a valid quaternion from the angle increment sensed by the gyro during time interval using (E.13),

$$\boldsymbol{q}(\Delta\boldsymbol{\theta}_{\mathrm{IMU}} - \hat{\boldsymbol{b}}_{g}\Delta t) = \begin{bmatrix} \cos\left(\frac{\|\Delta\boldsymbol{\theta}_{\mathrm{IMU}} - \hat{\boldsymbol{b}}_{g}\Delta t\|}{2}\right) \\ \frac{\Delta\boldsymbol{\theta}_{\mathrm{IMU}} - \hat{\boldsymbol{b}}_{g}\Delta t}{\|\Delta\boldsymbol{\theta}_{\mathrm{IMU}} - \hat{\boldsymbol{b}}_{g}\Delta t\|} \sin\left(\frac{\|\Delta\boldsymbol{\theta}_{\mathrm{IMU}} - \hat{\boldsymbol{b}}_{g}\Delta t\|}{2}\right) \end{bmatrix}$$
(5.71)

and the equivalent for the Earth rotation during the sample interval. This gives the time update

$$\hat{\boldsymbol{q}}_{b}^{e} \leftarrow \hat{\boldsymbol{q}}_{b}^{e} \otimes \boldsymbol{q}(\Delta \boldsymbol{\theta}_{\mathrm{IMU}} - \hat{\boldsymbol{b}}_{g} \Delta t) - \boldsymbol{q}(\boldsymbol{\omega}_{ie}^{e} \Delta t) \otimes \hat{\boldsymbol{q}}_{b}^{e}$$
(5.72)

which maintains the unit norm constraint. The float ambiguities and the clock drift rates are not expected to change and thus have the trivial updates

$$\hat{N}_{dd} \leftarrow \hat{N}_{dd} \tag{5.73}$$

$$\hat{t}_{df} \leftarrow \hat{t}_{df} \tag{5.74}$$

$$\hat{t}_{d\beta} \leftarrow \hat{t}_{d\beta}. \tag{5.75}$$

For the biases and the clock errors we can use simple Euler integration,

$$\hat{\boldsymbol{b}}_{g} \leftarrow \left(1 - \frac{\Delta t}{T_{g}}\right) \hat{\boldsymbol{b}}_{g} \tag{5.76}$$

$$\hat{\boldsymbol{b}}_a \leftarrow \left(1 - \frac{\Delta t}{T_a}\right) \hat{\boldsymbol{b}}_a$$
 (5.77)

$$\hat{t}_{\varepsilon f} \leftarrow \hat{t}_{\varepsilon f} + \hat{t}_{df} \Delta t \tag{5.78}$$

$$\hat{t}_{\varepsilon\beta} \leftarrow \hat{t}_{\varepsilon\beta} + \hat{t}_{d\beta}\Delta t. \tag{5.79}$$

5.1.5 Error-state model

Attitude

The dynamics of the Gibbs vector parametrization of the attitude error will now be derived. The definition (5.14) is used as the starting point, using the quotient rule for differentiation:

$$\dot{a} = 2\frac{d}{dt}\frac{\delta\epsilon}{\delta\eta} = 2\frac{\dot{\delta\epsilon}\delta\eta - \delta\epsilon\dot{\delta\eta}}{\delta\eta^2}$$
(5.80)

We first find the derivatives $\dot{\delta \epsilon}$ and $\dot{\delta \eta}$. Using (5.21) the error can be written as

$$\delta \boldsymbol{q}(\boldsymbol{a}) = (\hat{\boldsymbol{q}}_{b}^{e})^{-1} \otimes \boldsymbol{q}_{b}^{e}$$
(5.81)

Differentiating this gives

$$\dot{\delta q}(a) = (\dot{\hat{q}}_b^e)^{-1} \otimes q_b^e + (\hat{q}_b^e)^{-1} \otimes \dot{q}_b^e$$
(5.82)

Knowing $\hat{q}_b^e \otimes (\hat{q}_b^e)^{-1}$ to be the unit quaternion which is constant, it has the derivative

$$\dot{\hat{q}}_{b}^{e} \otimes (\hat{q}_{b}^{e})^{-1} + \hat{q}_{b}^{e} \otimes (\dot{\hat{q}}_{b}^{e})^{-1} = \mathbf{0},$$
(5.83)

which can be solved for $(\dot{\hat{q}}_{b}^{e})^{-1}$:

$$(\dot{\boldsymbol{q}}_b^e)^{-1} = -(\hat{\boldsymbol{q}}_b^e)^{-1} \otimes \dot{\boldsymbol{q}}_b^e \otimes (\hat{\boldsymbol{q}}_b^e)^{-1}$$
(5.84)

Combining this with (5.39), we get

$$\dot{\delta q}(a) = -(\hat{q}_b^e)^{-1} \otimes \dot{\hat{q}}_b^e \otimes (\hat{q}_b^e)^{-1} \otimes q_b^e + (\hat{q}_b^e)^{-1} \otimes \frac{1}{2} q_b^e \otimes \omega_{eb}^b$$
(5.85)

Using (5.81) and substituting the nominal form of (5.39), $\dot{\hat{q}}_b^e = \frac{1}{2} \hat{q}_b^e \otimes \hat{\omega}_{eb}^b$:

$$\dot{\delta q}(a) = \frac{1}{2} \delta q(a) \otimes \omega_{eb}^{b} - (\hat{q}_{b}^{e})^{-1} \otimes \frac{1}{2} \hat{q}_{b}^{e} \otimes \hat{\omega}_{eb}^{b} \otimes \delta q(a)$$
(5.86)

which reduces to

$$\dot{\delta q}(a) = \frac{1}{2} \delta q(a) \otimes \omega_{eb}^b - \frac{1}{2} \hat{\omega}_{eb}^b \otimes \delta q(a)$$
(5.87)

Using this along with the expressions for $\delta \epsilon$ and $\delta \eta$ from (5.15) in the original (5.80) results in

$$\dot{a} = \frac{4 + \|a\|_2^2}{2} \left(\left(\frac{1}{2} \delta q(a) \otimes \omega_{eb}^b - \frac{1}{2} \hat{\omega}_{eb}^b \otimes \delta q(a) \right)_{\epsilon} \frac{2}{\sqrt{4 + \|a\|_2^2}} - \frac{a}{\sqrt{4 + \|a\|_2^2}} \left(\frac{1}{2} \delta q(a) \otimes \omega_{eb}^b - \frac{1}{2} \hat{\omega}_{eb}^b \otimes \delta q(a) \right)_{\eta} \right).$$
(5.88)

where the subscripts ϵ and η represents the vector and scalar parts of the quaternion. Then substituting in (5.15) for $\delta q(a)$:

$$\dot{a} = \frac{4 + \|a\|_{2}^{2}}{2} \left(\left(\frac{1}{2} \frac{1}{\sqrt{4 + \|a\|_{2}^{2}}} \begin{bmatrix} 2\\ a \end{bmatrix} \otimes \omega_{eb}^{b} - \frac{1}{2} \hat{\omega}_{eb}^{b} \otimes \frac{1}{\sqrt{4 + \|a\|_{2}^{2}}} \begin{bmatrix} 2\\ a \end{bmatrix} \right)_{e} \frac{2}{\sqrt{4 + \|a\|_{2}^{2}}} - \frac{a}{\sqrt{4 + \|a\|_{2}^{2}}} \left(\frac{1}{2} \frac{1}{\sqrt{4 + \|a\|_{2}^{2}}} \begin{bmatrix} 2\\ a \end{bmatrix} \otimes \omega_{eb}^{b} - \frac{1}{2} \hat{\omega}_{eb}^{b} \otimes \frac{1}{\sqrt{4 + \|a\|_{2}^{2}}} \begin{bmatrix} 2\\ a \end{bmatrix} \right)_{\eta} \right)$$
(5.89)

The normalizing factors from (5.15) can now be cancelled,

$$\dot{a} = \left(\frac{1}{2} \begin{bmatrix} 2\\ a \end{bmatrix} \otimes \omega_{eb}^{b} - \frac{1}{2} \hat{\omega}_{eb}^{b} \otimes \begin{bmatrix} 2\\ a \end{bmatrix}\right)_{\epsilon} - \frac{a}{2} \left(\frac{1}{2} \begin{bmatrix} 2\\ a \end{bmatrix} \otimes \omega_{eb}^{b} - \frac{1}{2} \hat{\omega}_{eb}^{b} \otimes \begin{bmatrix} 2\\ a \end{bmatrix}\right)_{\eta}$$
(5.90)

The scalar and vector parts are now found by using the definition of the quaternion product (E.10)

$$\dot{a} = \frac{1}{2} (2\omega_{eb}^b + a \times \omega_{eb}^b) - \frac{1}{2} (2\hat{\omega}_{eb}^b + \hat{\omega}_{eb}^b \times a) - \frac{a}{2} \left(\frac{1}{2} (-a \cdot \omega_{eb}^b) - \frac{1}{2} (-\hat{\omega}_{eb}^b \cdot a) \right)$$
(5.91)

This can be rearranged as

$$\dot{a} = \omega_{eb}^{b} - \hat{\omega}_{eb}^{b} + \frac{1}{2}a \times \omega_{eb}^{b} - \frac{1}{2}\hat{\omega}_{eb}^{b} \times a + \frac{1}{4}(a \cdot \omega_{eb}^{b} - \hat{\omega}_{eb}^{b} \cdot a)a.$$
(5.92)

We then rewrite the differential equation such that the true angular rate ω_{eb}^b only appears in differences $\omega_{eb}^b - \hat{\omega}_{eb}^b$, which can be modelled using the gyro bias and attitude errors:

$$\dot{a} = (\omega_{eb}^b - \hat{\omega}_{eb}^b) - \hat{\omega}_{eb}^b \times a - \frac{1}{2}(\omega_{eb}^b - \hat{\omega}_{eb}^b) \times a + \frac{1}{4}((\omega_{eb}^b - \hat{\omega}_{eb}^b) \cdot a)a \qquad (5.93)$$

Writing the difference as $\Delta \omega = \omega_{eb}^b - \hat{\omega}_{eb}^b$ we can shorten this to

$$\dot{a} = \Delta \omega - \hat{\omega}^{b}_{eb} \times a - \frac{1}{2} \Delta \omega \times a + \frac{1}{4} (\Delta \omega \cdot a) a$$
(5.94)

Using the models for the true and nominal angular rates (5.38) and (5.57), this difference can be written as

$$\Delta \boldsymbol{\omega} = \underbrace{\hat{\boldsymbol{b}}_g - \boldsymbol{b}_g}_{-\delta \boldsymbol{b}_g} - \boldsymbol{w}_g + \mathbf{R} (\hat{\boldsymbol{q}}_b^e)^\top \boldsymbol{\omega}_{ie}^e - \mathbf{R} (\boldsymbol{q}_b^e)^\top \boldsymbol{\omega}_{ie}^e.$$
(5.95)

The true rotation matrix $\mathbf{R}(\boldsymbol{q}_b^e)$ can be written as the product of rotation matrices of the nominal and error attitude components:

$$\mathbf{R}(\boldsymbol{q}_{b}^{e}) = \mathbf{R}(\hat{\boldsymbol{q}}_{b}^{e})\mathbf{R}(\boldsymbol{a})$$
(5.96)

Using the formula $\mathbf{R}(q) = \mathbf{I}_{3\times 3} + 2\eta \mathbf{S}(\epsilon) + 2\mathbf{S}^2(\epsilon)$ from Fossen (2011, p. 28) and (5.15), the attitude error gives the rotation matrix

$$\mathbf{R}(a) = \mathbf{I}_{3\times 3} + \frac{4}{4 + \|a\|^2} \mathbf{S}(a) + \frac{2}{4 + \|a\|^2} \mathbf{S}^2(a)$$
(5.97)

$$\mathbf{R}(\boldsymbol{q}_{b}^{e}) = \mathbf{R}(\hat{\boldsymbol{q}}_{b}^{e}) + \frac{4}{4 + \|\boldsymbol{a}\|^{2}} \mathbf{R}(\hat{\boldsymbol{q}}_{b}^{e}) \mathbf{S}(\boldsymbol{a}) + \frac{2}{4 + \|\boldsymbol{a}\|^{2}} \mathbf{R}(\hat{\boldsymbol{q}}_{b}^{e}) \mathbf{S}^{2}(\boldsymbol{a})$$
(5.98)

Substituting this into (5.95):

$$\Delta \boldsymbol{\omega} = -\delta \boldsymbol{b}_g - \boldsymbol{w}_g - \left(\frac{4}{4 + \|\boldsymbol{a}\|^2} \mathbf{R}(\hat{\boldsymbol{q}}_b^e) \mathbf{S}(\boldsymbol{a}) + \frac{2}{4 + \|\boldsymbol{a}\|^2} \mathbf{R}(\hat{\boldsymbol{q}}_b^e) \mathbf{S}^2(\boldsymbol{a})\right)^\top \boldsymbol{\omega}_{ie}^e.$$
(5.99)

Translation

From (5.23), (2.18) and (5.60) we have that the position error is simply the integral of the velocity error,

$$\dot{\delta p}_{eb}^e = \delta v_{eb}^e. \tag{5.100}$$

For the velocity error (5.44), (5.61), (5.24) and (5.25) are used to get

$$\dot{\delta v}_{eb}^{e} = \mathbf{R}(q_{b}^{e})(f_{\rm IMU}^{b} - \hat{b}_{a} - \delta b_{a} - w_{a}) + g^{e} - 2\mathbf{S}(\omega_{ie}^{e})v_{eb}^{e} - (\mathbf{R}(\hat{q}_{b}^{e})(f_{\rm IMU}^{b} - \hat{b}_{a}) + g^{e} - 2\mathbf{S}(\omega_{ie}^{e})\hat{v}_{eb}^{e}),$$
(5.101)

which can be rearranged as

$$\dot{\delta v}_{eb}^{e} = -2\mathbf{S}(\boldsymbol{\omega}_{ie}^{e})\delta v_{eb}^{e} + \delta g^{e} + (\mathbf{R}(\boldsymbol{q}_{b}^{e}) - \mathbf{R}(\hat{\boldsymbol{q}}_{b}^{e}))(\boldsymbol{f}_{imu}^{b} - \hat{\boldsymbol{b}}_{a}) - \mathbf{R}(\boldsymbol{q}_{b}^{e})(\delta \boldsymbol{b}_{a} + \boldsymbol{w}_{a}).$$
(5.102)

Using (5.98)

Biases

The bias errors satisfy the differential equations

$$\dot{\delta \boldsymbol{b}}_{g} = -\frac{1}{T_{g}} \delta \boldsymbol{b}_{g} + \boldsymbol{w}_{b_{g}}, \qquad (5.104)$$

$$\dot{\delta \boldsymbol{b}}_a = -\frac{1}{T_a} \delta \boldsymbol{b}_a + \boldsymbol{w}_{b_a}. \tag{5.105}$$

Clock errors

The clock error and drift rate errors satisfy

$$\delta t_{\varepsilon f} = \delta t_{df} + w_{t_{\varepsilon f}}, \tag{5.106}$$

$$\delta t_{\varepsilon\beta} = \delta t_{d\beta} + w_{t_{\varepsilon\beta}}, \tag{5.107}$$

$$\dot{\delta t}_{df} = w_{t_{df}},\tag{5.108}$$

$$\dot{\delta t}_{d\beta} = w_{t_{d\beta}}.\tag{5.109}$$

Ambiguities

The double differenced ambiguity errors are modelled as

$$\delta \dot{\boldsymbol{N}}_{dd} = \boldsymbol{w}_{\boldsymbol{N}}.\tag{5.110}$$

5.1.6 Linearized model for covariance propagation

Because the expected value of the error state is always zero between measurement corrections, and measurement corrections are chosen to always be followed by a error injection and reset, we can always linearize at $\delta x = 0$. For the linearization we can use the first order approximation of (5.97), which is

$$\mathbf{R}(a) \approx \mathbf{I}_{3\times 3} + \mathbf{S}(a) \tag{5.111}$$

Attitude

Differentiating (5.94) with respect to *a* results in

$$\frac{\partial}{\partial a}\dot{a} = -\mathbf{S}(\mathbf{R}(\hat{q}_b^e)^\top \omega_{ie}^e) - \mathbf{S}(\hat{\omega}_{eb}^b) - \frac{1}{2}\mathbf{S}(\Delta \omega) + \frac{1}{4}\frac{\partial}{\partial a}((\Delta \omega \cdot a) \cdot a), \quad (5.112)$$

where the first term is an error component caused by the error in the transformation of Earth rotation to the body frame. The two last terms cancel when the expected value $\delta x = 0$ is inserted,

$$E\left[\frac{\partial}{\partial a}\dot{a}\right] = -\mathbf{S}(\mathbf{R}(\hat{q}_{b}^{e})^{\mathsf{T}}\boldsymbol{\omega}_{ie}^{e}) - \mathbf{S}(\hat{\boldsymbol{\omega}}_{eb}^{b}).$$
(5.113)

For the gyro bias we have

$$\frac{\partial}{\partial \delta \boldsymbol{b}_{g}} \dot{\boldsymbol{a}} = -\mathbf{I}_{3\times 3} + \frac{1}{2} \mathbf{S}(\boldsymbol{a}) + \frac{1}{4} \frac{\partial}{\partial \delta \boldsymbol{b}_{g}} ((\Delta \boldsymbol{\omega} \cdot \boldsymbol{a}) \cdot \boldsymbol{a}), \qquad (5.114)$$

which is reduced to

$$E\left[\frac{\partial}{\partial \delta \boldsymbol{b}_{g}}\dot{\boldsymbol{a}}\right] = -\mathbf{I}_{3\times 3}.$$
(5.115)

The derivative with respect to the noise is the same

$$\frac{\partial}{\partial w_g} \dot{a} = -\mathbf{I}_{3\times 3} + \frac{1}{2} \mathbf{S}(a) + \frac{1}{4} \frac{\partial}{\partial w_g} ((\Delta \omega \cdot a) \cdot a), \qquad (5.116)$$

$$E\left[\frac{\partial}{\partial w_g}\dot{a}\right] = -\mathbf{I}_{3\times 3}.$$
 (5.117)

Derivatives with respect to the other states are all zero.

Translation

The derivative of position error with respect to velocity error is trivial,

$$\frac{\partial}{\partial \delta \boldsymbol{v}_{eb}^{e}} \delta \dot{\boldsymbol{p}}_{eb}^{e} = \mathbf{I}_{3\times 3}.$$
(5.118)

Using the approximation (5.111), the differential equation for the velocity error can be approximated by

$$\dot{\delta v}_{eb}^{e} \approx -2\mathbf{S}(\boldsymbol{\omega}_{ie}^{e})\delta v_{eb}^{e} + \delta g^{e} + \mathbf{R}(\hat{\boldsymbol{q}}_{b}^{e})\mathbf{S}(\boldsymbol{a})(\boldsymbol{f}_{\mathrm{IMU}}^{b} - \hat{\boldsymbol{b}}_{a}) - \mathbf{R}(\hat{\boldsymbol{q}}_{b}^{e})(\mathbf{I}_{3\times3} + \mathbf{S}(\boldsymbol{a}))(\delta \boldsymbol{b}_{a} + \boldsymbol{w}_{a}).$$
(5.119)

which is equivalent to first order for small δx . Differentiating this with respect to the attitude error gives

$$\frac{\partial}{\partial a}\delta \dot{\boldsymbol{v}}_{eb}^{e} = -\mathbf{R}(\hat{\boldsymbol{q}}_{b}^{e})\mathbf{S}(\boldsymbol{f}_{\mathrm{IMU}} - \hat{\boldsymbol{b}}_{a}) + \mathbf{R}(\hat{\boldsymbol{q}}_{b}^{e})\mathbf{S}(\delta \boldsymbol{b}_{a} + \boldsymbol{w}_{a})\boldsymbol{a}, \qquad (5.120)$$

with expected value

$$E\left[\frac{\partial}{\partial a}\delta \dot{\boldsymbol{v}}_{eb}^{e}\right] = -\mathbf{R}(\hat{\boldsymbol{q}}_{b}^{e})\mathbf{S}(\boldsymbol{f}_{\mathrm{IMU}} - \hat{\boldsymbol{b}}_{a}).$$
(5.121)

$$\frac{\partial}{\partial \delta v_{eb}^{e}} \delta v_{eb}^{i} = -2\mathbf{S}(\boldsymbol{\omega}_{ie}^{e})$$
(5.122)

The derivative of (5.103) with respect to the accelerometer bias error can be found by simple inspection

$$\frac{\partial}{\partial \delta \boldsymbol{b}_{a}} \delta \boldsymbol{v}_{eb}^{\dot{e}} = -\left(\mathbf{R}(\hat{\boldsymbol{q}}_{b}^{e}) + \frac{4}{4 + \|\boldsymbol{a}\|^{2}} \mathbf{R}(\hat{\boldsymbol{q}}_{b}^{e}) \mathbf{S}(\boldsymbol{a}) + \frac{2}{4 + \|\boldsymbol{a}\|^{2}} \mathbf{R}(\hat{\boldsymbol{q}}_{b}^{e}) \mathbf{S}^{2}(\boldsymbol{a})\right)$$
(5.123)

$$E\left[\frac{\partial}{\partial\delta\boldsymbol{b}_{a}}\delta\boldsymbol{v}_{eb}^{\dot{e}}\right] = -\mathbf{R}(\hat{\boldsymbol{q}}_{b}^{e})$$
(5.124)

The result for the noise is the same

$$E\left[\frac{\partial}{\partial \boldsymbol{w}_{a}}\delta\boldsymbol{v}_{eb}^{\dot{e}}\right] = -\mathbf{R}(\hat{\boldsymbol{q}}_{b}^{e}), \qquad (5.125)$$

but the rotation matrix can be replaced by the identity matrix if the accelerometer noise is isotropic, which is assumed here.

Biases

From (5.104) and (5.105) we get

$$\frac{\partial}{\partial \delta \boldsymbol{b}_g} \delta \dot{\boldsymbol{b}}_g = -\frac{1}{T_g}$$
(5.126)

$$\frac{\partial}{\partial \delta \boldsymbol{b}_a} \delta \dot{\boldsymbol{b}}_a = -\frac{1}{T_a} \tag{5.127}$$

$$\frac{\partial}{\partial \boldsymbol{w}_{bg}} \delta \dot{\boldsymbol{b}}_{g} = \mathbf{I}_{3\times 3} \tag{5.128}$$

$$\frac{\partial}{\partial w_{ba}} \dot{\delta b}_a = \mathbf{I}_{3\times 3}. \tag{5.129}$$

Clock errors and ambiguities

The models for errors in clock bias and drift rate, and the models for the ambiguit errors, are already linear and values for the system matrices can be found by simple inspection.

Matrices

The matrices will vary in size as the number of tracked satellites change. The upper left part of the linear state dynamics matrix, with size 19×19 , is

$$\bar{\mathbf{F}} = \begin{bmatrix} -\mathbf{S}(\mathbf{R}(\hat{q}_{b}^{e})^{\top} \omega_{ie}^{e}) - \mathbf{S}(\hat{\omega}_{eb}^{o}) & -\mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & -\frac{1}{T_{g}}\mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{1}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} \\ -\mathbf{R}(\hat{q}_{b}^{e})\mathbf{S}(\hat{f}_{ib}^{b}) & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & -\mathbf{2S}(\omega_{ie}^{e}) & -\mathbf{R}(\hat{q}_{b}^{e}) & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & -\frac{1}{T_{d}}\mathbf{I}_{3\times3} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} &$$

and the total matrix is

$$\mathbf{F} = \begin{bmatrix} \bar{\mathbf{F}} & \mathbf{0}_{19 \times (k-1)} \\ \mathbf{0}_{(k-1) \times 19} & \mathbf{0}_{(k-1) \times (k-1)} \end{bmatrix} \in \mathbb{R}^{(19+k-1) \times (19+k-1)}.$$
 (5.131)

The upper left part of the process noise input matrix becomes

$$\bar{\mathbf{G}} = \begin{bmatrix} -\mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & -\mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \in \mathbb{R}^{19\times16}, \quad (5.132)$$

such that the complete noise input matrix end up as

$$\mathbf{G} = \begin{bmatrix} \bar{\mathbf{G}} & \mathbf{0}_{19 \times (k-1)} \\ \mathbf{0}_{\times (k-1) \times 16} & \mathbf{I}_{(k-1) \times (k-1)} \end{bmatrix} \in \mathbb{R}^{(19+k-1) \times (16+k-1)}.$$
 (5.133)

The linear process model is then

$$\dot{\delta x} = \mathbf{F} \delta x + \mathbf{G} w, \qquad (5.134)$$

where the noise vector is

$$\boldsymbol{w} = [\boldsymbol{w}_{g}^{\top} \quad \boldsymbol{w}_{bg}^{\top} \quad \boldsymbol{w}_{a}^{\top} \quad \boldsymbol{w}_{ba}^{\top} \quad \boldsymbol{w}_{t_{\varepsilon f}} \quad \boldsymbol{w}_{t_{\varepsilon \beta}} \quad \boldsymbol{w}_{t_{df}} \quad \boldsymbol{w}_{t_{d\beta}} \quad \boldsymbol{w}_{N}^{\top}]^{\top}.$$
(5.135)

5.1.7 Process noise

Clocks

A conventional way to characterize the noises or errors present in oscillator outputs is by using Allan variance (Allan, 1966). The Allan variance is also called the twosample variance, and is a special case of the M-sample variance. The advantage of the Allan variance is that unlike standard deviation / classical variance which diverges as the number of samples increases for some common noise processes such as random walk (Vig, 2008), the Allan variance does converge. The Allan variance is a function of the observation time interval τ between two samples and looks at the average squared change of the normalized frequency deviation $y = \frac{\Delta f}{f_0}$ between samples,

$$\sigma_y^2(\tau) = \frac{1}{2} < (y_{k+1} - y_k)^2 >, \tag{5.136}$$

where <> denotes time average of ideally an infinite number of sample pairs. In reality this is done for a limited number of *m* sample pairs, but gives good results for large *m*, i.e. $m \ge 100$ (Vig, 2008),

$$\sigma_y^2(\tau,m) = \frac{1}{m} \sum_{j=1}^m \left(\frac{1}{2} (y_{k+1} - y_k)^2 \right)_j,$$
(5.137)

where j represents each sample pair used. The Allan variance, or Allan deviation which is the square root of the Allan variance, is commonly plotted in log-log plots as a function of the time interval τ . An example of Allan deviation plots used to characterize noise in the STIM300 IMU can be found in Appendix G. Creating a plot like this experimentally for an oscillator requires a higly accuracy reference frequency which can be assumed constant. In a Allan deviation plot distinct asymptotic regions can be identified, and the following coefficients can be used to describe the regions for different oscillators (Allan, 1966):

- *h*₂ white phase noise
- *h*₁ flicker phase noise
- *h*⁰ white frequency noise
- *h*₋₁ flicker frequency noise
- *h*₋₂ random walk frequency noise

The first two of these are normally overlooked when finding the covariance parameters for a KF system model. Typical values of the last three of these parameters for different clock types can be found in Table 5.2. The Allan deviation at a time such as $\tau = 0.1$ s or $\tau = 1$ s is commonly found in oscillator datasheets, but in the case of U-Blox receivers no such information is provided by the manufacturer. Because
Hwang, 2012).				
Timing standard	h_0	h_{-1}	h_{-2}	

Table 5.2: Typical Allan variance coefficients of timing standards (Brown and

Timing standard	h_0	h_{-1}	h_{-2}
TCXO (low quality)	2×10^{-19}	7×10^{-21}	2×10^{-20}
TCXO (high quality)	2×10^{-21}	1×10^{-22}	3×10^{-24}
Ovenized crystal oscillator (OCXO)	2×10^{-25}	7×10^{-36}	6×10^{-25}
Rubidium	2×10^{-22}	4.5×10^{-26}	1×10^{-30}
Cesium	2×10^{-22}	5×10^{-27}	1.5×10^{-33}

the receiver is relatively inexpensive, the parameters for the low quality TCXO will be assumed.

Because Allan variance is calculated using discrete samples with varying sample interval, finding continous variance values for the clock model from Allan variance parameters is done by first finding the discrete form of the covariance matrix (5.56). This takes the form (Farrell, 2008; Brown and Hwang, 2012)

$$\mathbf{Q}_{d} = \begin{bmatrix} \sigma_{t_{\varepsilon}}^{2} \Delta t + \sigma_{t_{d}}^{2} \frac{\Delta t^{3}}{3} & \sigma_{t_{d}}^{2} \frac{\Delta t^{2}}{2} \\ \sigma_{t_{d}}^{2} \frac{\Delta t^{2}}{2} & \sigma_{t_{d}}^{2} \Delta t \end{bmatrix}.$$
(5.138)

Van Dierendonck et al. (1984) shows that the variance of the clock phase in the two state model is

$$\frac{h_0}{2}\Delta t + 2h_{-1}\Delta t^2 + \frac{2}{3}\pi^2 h_{-2}\Delta t^3.$$
(5.139)

If the flicker frequency noise term $2h_{-1}\Delta t^2$ is to be included, an exact fit of this expression to $\sigma_{t_e}^2 \Delta t + \sigma_{t_d}^2 \frac{\Delta t^3}{3}$ is impossible because flicker noise is not a rational process (Van Dierendonck et al., 1984). One option is to just ignore this term, which leads to the results

$$\sigma_{t_{\varepsilon}}^2 = \frac{h_0}{2},\tag{5.140}$$

$$\sigma_{t_d}^2 = 2\pi^2 h_{-2}.\tag{5.141}$$

Omitting the flicker noise term gives a modelled noise that has a too low variance for some sampling intervals. One approximate way to handle this is to simple increase both $\sigma_{t_{\epsilon}}^2$ and $\sigma_{t_d}^2$ Brown and Hwang (2012). Another solution to this issue is to fit the variances to (5.139) using a least squares solution for two chosen time intervals, as done in Farrell (2008). Because the Allan variance parameters used are approximate anyway, and not based on manufacturer specifications, using the simple method of ignoring the frequency flicker noise will be used as the starting point, with manual tuning used if appropriate.

STIM300 IMU

The standard deviation of the white Gaussian noise in the gyro and accelerometer models can be found from values in the STIM300 datasheet, and have been included in Table 5.3. The Allan variance plots where these values can be identified are also included in Appendix G. The random walk parameters can be used to find the

Table 5.3: Noise parameters from the STIM300 datasheet (Sensonor, 2018)

Gyro angle random walk	$0.15^{\circ}/\sqrt{h}$
Accelerometer velocity random walk	$0.07\frac{m/s}{s}$
Accelerometer bias instability	0.05mg

variance of the white noise in the gyro and accelerometer measurments. The continous time white noise standard deviation is found as the Allan deviation at the sample interval 1Hz, where the error is dominated by white noise. The gyro angle random walk value in the datasheet is found as the value of the white noise asymptote with slope -1/2 (seen as the straight line on the left of the plots) at $\tau = 3600$ s. The value at $\tau = 1$ s is found by division by $\sqrt{3600} = 60$. Scaling this to radians, this gives the continous time gyro standard deviation

$$\sigma_g = 2.5 \times 10^{-3} \frac{\circ'/s}{\sqrt{\text{Hz}}} = 4.36 \times 10^{-5} \frac{\text{rad/s}}{\sqrt{\text{Hz}}}.$$
(5.142)

The value from the datasheet can be compared with the noise visible in Figure 2.1 by multiplying with the square root of the sample rate, which in this case is 250Hz. This gives a standard deviation of $0.0395^{\circ}/s$, which when expressed as increment per samle by dividing by the sample rate is $1.58 \times 10^{-4\circ}$. This appears to agree with the noise amplitude in the figure.

The value for the continous accelerometer standard deviation found the same way is

$$\sigma_a = 1.167 \times 10^{-3} \frac{\text{m/s}^2}{\sqrt{\text{Hz}}}$$
(5.143)

The bias instability values in the datasheet are the Allan deviations at the lowest point in the Allan deviation curve. This is basically the noise levels at the optimal averaging time. These cannot directly be used to find the parameters for the bias models. Random walk parameters depend on the section of the Allan deviation plot with slope ¹/₂ to the right of the bias instability region, which is outside the range plotted in the datasheet. Manual tuning is therefore used for the parameters for the Gauss-Markov bias models.

Carrier phase ambiguities

The variance σ_N^2 was initially set to 0 and increased by manual tuning.

Continous time covariance matrix

	$[\sigma_g^2 \mathbf{I}_3]$	0	0	0	0	0	0	0	0]	
	0	$\sigma_{bg}^2 \mathbf{I}_3$	0	0	0	0	0	0	0	
	0	Ő	$\sigma_a^2 \mathbf{I}_3$	0	0	0	0	0	0	
	0	0	0	$\sigma_{ha}^2 \mathbf{I}_3$	0	0	0	0	0	
Q =	0	0	0	Ő	$\sigma_{t_{\varepsilon f}}^2$	0	0	0	0	(5.144)
	0	0	0	0	0	$\sigma_{t_{\varepsilon\beta}}^2$	0	0	0	
	0	0	0	0	0	0	$\sigma_{t_{df}}^2$	0	0	
	0	0	0	0	0	0	0	$\sigma_{t_{d\beta}}^2$	0	
	0	0	0	0	0	0	0	0	$\sigma_N^2 \mathbf{I}_{(k-1)}$	

5.1.8 Discrete time error-state model

The MEKF is to be implemented on a computer and the system dynamics thus need to be discretized. The inertial measurements are available at a fixed rate of 250 Hz, which determines the discretization timesteps that should be used, $\Delta t = 4$ ms. Because the expected error state is always zero except between a measurement correction and the injection of the error in to the nominal state, followed by a reset, there is no point in actually implementing the propagation of the error state. The covariance of the error is however propagated, using

$$\mathbf{P}_{k}^{-} = \mathbf{\Phi}_{k} \mathbf{P}_{k-1} \mathbf{\Phi}_{k} + \mathbf{Q}_{k}. \tag{5.145}$$

Both the discrete transformation matrix and dicrete covariance matrix can be found using the method of Van Loan (1978). We first construct the matrix

$$\mathbf{M} = \begin{bmatrix} -\mathbf{F}_k & \mathbf{G}\mathbf{Q}\mathbf{G}^\top\\ \mathbf{0} & \mathbf{F}_k^\top \end{bmatrix}$$
(5.146)

and then take the matrix exponential of this, resulting in

$$e^{\mathbf{M}\Delta t} = \begin{bmatrix} \cdots & \mathbf{\Phi}_k^{-1} \mathbf{Q}_k \\ \mathbf{0} & \mathbf{\Phi}_k^{\top} \end{bmatrix}.$$
 (5.147)

The transition matrix is then simply the transpose of the lower right matrix, and the discrete covariance matrix is the transition matrix multiplied by the upper right matrix,

$$\mathbf{Q}_k = (\mathbf{\Phi}_k^{\mathsf{T}})^{\mathsf{T}} \mathbf{\Phi}_k^{-1} \mathbf{Q}_k.$$
(5.148)

The Matlab function expm is used to find the matrix exponential.

5.2 Measurement models

For all the measurements we use the antenna positions

$$\boldsymbol{p}_{eb,f}^{e} = \boldsymbol{p}_{eb}^{e} + \mathbf{R}(\boldsymbol{q}_{b}^{e})\Delta\boldsymbol{r}_{f}^{b}$$
(5.149)

$$\boldsymbol{p}_{eb,\beta}^{e} = \boldsymbol{p}_{eb}^{e} + \mathbf{R}(\boldsymbol{q}_{b}^{e})\Delta r_{\beta}^{b}$$
(5.150)

with p_{eb}^{e} being the position of the IMU and Δr_{f}^{b} and Δr_{β}^{b} being the body-frame lever arms from the IMU to each GPS receiver antenna. In the following p_{es}^{e} will refer to $p_{es}^{e,rx}(t_{tx})$ from Section 3.4.1, which is the satellite position at time of signal transmission, in the ECEF frame of the time of signal reception, and v_{es}^{e} is the corresponding velocity. For expressions which are valid for both receivers the explicit reference to f or β will be omitted.

5.2.1 Pseudorange

For the pseudorange observable the model (3.2) is used. Rewriting this with the receiver clock error t_{ε} and the satellite clock error from (3.32), we have

$$P = \rho + t_{\varepsilon} + \epsilon_P + I + T - c\Delta t_{SV}.$$
(5.151)

The position we want to estimate is however not the position of each antenna, but the position of the IMU. This means that the lever arms from the IMU to each antenna must be included in the model. Since the lever arms are known in the body frame, this introduces a dependency on the attitude in the model. With the lever arm included the true range is

$$\rho = \|\boldsymbol{p}_{\text{sat}}^e - (\boldsymbol{p}_{eb}^e + \mathbf{R}(\boldsymbol{q}_b^e)\Delta r)\|_2$$
(5.152)

Inserting this gives into the pseudorange model gives

$$P = \|\boldsymbol{p}_{\text{sat}}^{e} - (\boldsymbol{p}^{e} + \mathbf{R}(\boldsymbol{q}_{b}^{e})\Delta\boldsymbol{r})\|_{2} + t_{\varepsilon} + \epsilon_{P} + I + T - c\Delta t_{SV}.$$
(5.153)

By using the simple expression for the derivative of the Euclidean norm

$$\frac{\partial}{\partial x} \|x\|_2 = \frac{\partial}{\partial x} \sqrt{x^2 + y^2 + z^2} = \frac{1}{2\sqrt{x^2 + y^2 + z^2}} 2x^\top = \frac{x^\top}{\|x\|_2} = \left(\frac{x}{\|x\|_2}\right)^\top, \quad (5.154)$$

we find the derivatives with respect to errors in attitude, position and clock bias, using the first order approximation (5.111). With start with the relationship with the attitude error:

$$\frac{\partial}{\partial a}P = \frac{\partial}{\partial a} \|\boldsymbol{p}_{sat}^{e} - \boldsymbol{p}_{eb}^{e} - \mathbf{R}(\boldsymbol{q}_{b}^{e})\Delta \boldsymbol{r}^{b}\|_{2}$$

$$= \left(\frac{\boldsymbol{p}_{sat}^{e} - \boldsymbol{p}_{eb}^{e} - \mathbf{R}(\boldsymbol{q}_{b}^{e})\Delta \boldsymbol{r}^{b}}{\|\boldsymbol{p}_{sat}^{e} - \boldsymbol{p}_{eb}^{e} - \mathbf{R}(\boldsymbol{q}_{b}^{e})\Delta \boldsymbol{r}^{b}\|_{2}}\right)^{\mathsf{T}} \frac{\partial}{\partial a} (-\mathbf{R}(\hat{\boldsymbol{q}}_{b}^{e})(\mathbf{I} + \mathbf{S}(a))\Delta \boldsymbol{r}^{b})$$

$$= \left(-\mathbf{p}_{sat}^{e} - \mathbf{p}_{eb}^{e} - \mathbf{R}(\boldsymbol{q}_{b}^{e})\Delta \boldsymbol{r}^{b}\right)^{\mathsf{T}} \frac{\partial}{\partial a} (\mathbf{R}(\hat{\boldsymbol{q}}_{b}^{e})\mathbf{S}(\Delta \boldsymbol{r}^{b})\boldsymbol{q})$$
(5.155)

$$= \left(\frac{r_{\text{sat}} - r_{eb}^{e} - \mathbf{R}(q_{b}^{e}) \Delta r^{b}}{\|\boldsymbol{p}_{\text{sat}}^{e} - \boldsymbol{p}_{eb}^{e} - \mathbf{R}(q_{b}^{e}) \Delta r^{b}\|_{2}}\right) - \frac{\partial}{\partial a} (\mathbf{R}(\hat{q}_{b}^{e}) \mathbf{S}(\Delta r^{b}) \boldsymbol{a})$$
(5.155)

$$= \left(\frac{\boldsymbol{p}_{\text{sat}}^{e} - \boldsymbol{p}_{eb}^{e} - \mathbf{R}(\boldsymbol{q}_{b}^{e})\Delta \boldsymbol{r}^{b}}{\|\boldsymbol{p}_{\text{sat}}^{e} - \boldsymbol{p}_{eb}^{e} - \mathbf{R}(\boldsymbol{q}_{b}^{e})\Delta \boldsymbol{r}^{b}\|_{2}}\right) \quad \mathbf{R}(\hat{\boldsymbol{q}}_{b}^{e})\mathbf{S}(\Delta \boldsymbol{r}^{b})$$
(5.156)

$$= l^{e^{\top}} \mathbf{R}(\hat{\boldsymbol{q}}_b^e) \mathbf{S}(\Delta \boldsymbol{r}^b)$$
(5.157)

The most important relationship is to the position error. In this thesis the pseudoranges are the only measurements which contribute to observability of absolute position errors,

$$\frac{\partial}{\partial \delta \boldsymbol{p}_{eb}^{e}} P = \frac{\partial}{\partial \delta \boldsymbol{p}_{eb}^{e}} \| \boldsymbol{p}_{sat}^{e} - \hat{\boldsymbol{p}}_{eb}^{e} - \delta \boldsymbol{p}_{eb}^{e} - \mathbf{R}(\boldsymbol{q}_{b}^{e})\Delta \boldsymbol{r} \|_{2}$$
$$= -\left(\frac{\boldsymbol{p}_{sat}^{e} - \boldsymbol{p}_{eb}^{e} - \mathbf{R}(\boldsymbol{q}_{b}^{e})\Delta \boldsymbol{r}}{\|\boldsymbol{p}_{sat}^{e} - \boldsymbol{p}_{eb}^{e} - \mathbf{R}(\boldsymbol{q}_{b}^{e})\Delta \boldsymbol{r} \|_{2}}\right)^{\mathsf{T}} = -\boldsymbol{l}^{e\mathsf{T}}.$$
(5.158)

The relationship with the receiver clock error is trivial,

$$\frac{\partial}{\partial \delta t_{\varepsilon}} P = 1. \tag{5.159}$$

The expected value of the two first are simply

$$E\left[\frac{\partial}{\partial a}P\right] = \hat{\boldsymbol{l}}^{e^{\top}}\mathbf{R}(\hat{\boldsymbol{q}}_{b}^{e})\mathbf{S}(\Delta \boldsymbol{r})$$
(5.160)

$$E\left[\frac{\partial}{\partial\delta p_{eb}^{e}}P\right] = -\hat{l}^{e^{\top}}.$$
(5.161)

Combining these gives the following measurement matrices for the front and back receivers, respectively:

5.2.2 Doppler frequency

For the Doppler frequency the Δf is read from the receivers, but scaled to unit of meters so the measurement used by the filter is $\lambda \Delta f$. The time derivatives of the atmosheric errors are assumed negligible. Because of the lever arms, the velocity of the front antenna is

$$\boldsymbol{v}_{eb,f}^{e} = \boldsymbol{v}_{eb}^{e} + \mathbf{R}(\boldsymbol{q}_{b}^{e})\mathbf{S}(\boldsymbol{\omega}_{eb}^{b})\Delta \boldsymbol{r}_{f}^{b}$$
(5.164)

and the equivalent for the back receiver. The means that the relative velocity between the antenna and the satellite depends on the angular rate of the UAV. By using equations (3.20), (3.12) and (3.11), the model can be written as

$$\lambda \Delta f = \boldsymbol{l}^{e^{\top}} (\boldsymbol{v}^{e}_{eb} + \mathbf{R}(\boldsymbol{q}^{e}_{b})\mathbf{S}(\boldsymbol{\omega}^{b}_{eb})\Delta \boldsymbol{r}^{b} + \mathbf{S}(\boldsymbol{\omega}^{e}_{ie})\boldsymbol{p}^{e}_{eb} - \boldsymbol{v}^{e}_{es} - \mathbf{S}(\boldsymbol{\omega}^{e}_{ie})\boldsymbol{p}^{e}_{es}) - \boldsymbol{t}_{d} + c\Delta \dot{\boldsymbol{t}}_{SV} + \boldsymbol{\epsilon}_{\Delta f}.$$
(5.165)

When now creating the KF measurement matrix, an approximation will be made. The LOS vector in the above expression depends on the UAV position, and attitude due to the lever arms. This means that Doppler measurements can be used to correct the position. This is the concept used by Doppler positioning, where other sources of velocity information is used, and the Doppler measurements corrects the position, by basically finding the position where the LOS vectors best fit the position and Doppler measurements. In this case this will not be used, and the LOS vector will be approximated as independent of attitude and position errors. It can then be treated a constant when taking the derivatives, making the expressions a lot simpler. Differentiating the model with respect to errors in attitude, gyro bias, velocity and clock drift rates we get

$$\frac{\partial}{\partial a} (\lambda \Delta f) \approx \frac{\partial}{\partial a} l^{e^{\top}} \mathbf{R}(\hat{q}^{e}_{b}) (\mathbf{I} + \mathbf{S}(a)) \mathbf{S}(\boldsymbol{\omega}^{b}_{eb}) \Delta r^{b},$$

$$= \frac{\partial}{\partial a} l^{e^{\top}} \mathbf{R}(\hat{q}^{e}_{b}) \mathbf{S}(a) \mathbf{S}(\boldsymbol{\omega}^{b}_{eb}) \Delta r^{b},$$

$$= -\frac{\partial}{\partial a} l^{e^{\top}} \mathbf{R}(\hat{q}^{e}_{b}) \mathbf{S}(\mathbf{S}(\boldsymbol{\omega}^{b}_{eb}) \Delta r^{b}) a,$$

$$= -l^{e^{\top}} \mathbf{R}(\hat{q}^{e}_{b}) \mathbf{S}(\mathbf{S}(\boldsymbol{\omega}^{b}_{eb}) \Delta r^{b}),$$
(5.166)

with respect the attitude error and

$$\frac{\partial}{\partial \delta \boldsymbol{b}_{g}} (\lambda \Delta f) = \frac{\partial}{\partial \delta \boldsymbol{b}_{g}} \boldsymbol{l}^{e^{\top}} \mathbf{R}(\boldsymbol{q}_{b}^{e}) \mathbf{S}(\boldsymbol{\omega}_{IMU} - \delta \boldsymbol{b}_{g} - \boldsymbol{\hat{b}}_{g} - \boldsymbol{w}_{g}) \Delta \boldsymbol{r}^{b},$$

$$= -\frac{\partial}{\partial \boldsymbol{b}_{g}} \boldsymbol{l}^{e^{\top}} \mathbf{R}(\boldsymbol{q}_{b}^{e}) \mathbf{S}(\boldsymbol{b}_{g}) \Delta \boldsymbol{r}^{b},$$

$$= \frac{\partial}{\partial \boldsymbol{b}_{g}} \boldsymbol{l}^{e^{\top}} \mathbf{R}(\boldsymbol{q}_{b}^{e}) \mathbf{S}(\Delta \boldsymbol{r}^{b}) \boldsymbol{b}_{g},$$

$$= \boldsymbol{l}^{e^{\top}} \mathbf{R}(\boldsymbol{q}_{b}^{e}) \mathbf{S}(\Delta \boldsymbol{r}^{b}),$$
(5.167)

with resepect to the gyro bias error, and finally,

$$\frac{\partial}{\partial \delta \boldsymbol{v}_{eb}^{e}} (\lambda \Delta f) = \boldsymbol{l}^{e\top}, \qquad (5.168)$$

$$\frac{\partial}{\partial \delta t_d} (\lambda \Delta f) = -1, \tag{5.169}$$

with respect to the velocity and clock drift rate error. Combining the expected value of these gives the following measurement matrices for the front and back

receivers, respectively,

$$\mathbf{H}_{f} = \begin{bmatrix}
-\hat{l}_{f,1}^{e^{\top}} \mathbf{R}(\hat{q}_{b}^{e}) \mathbf{S}(\mathbf{S}(\hat{\omega}_{eb}^{b}) \Delta r_{f}^{b}) \quad \hat{l}_{f,1}^{e^{\top}} \mathbf{R}(\hat{q}_{b}^{e}) \mathbf{S}(\Delta r_{f}^{b}) \quad \mathbf{0}_{1\times 3} \quad \hat{l}_{f,1}^{e^{\top}} \quad \mathbf{0}_{1\times 3} \quad 0 \quad -1 \quad 0 \quad \mathbf{0}_{1\times(k-1)} \\
-\hat{l}_{f,2}^{e^{\top}} \mathbf{R}(\hat{q}_{b}^{e}) \mathbf{S}(\mathbf{S}(\hat{\omega}_{eb}^{b}) \Delta r_{f}^{b}) \quad \hat{l}_{f,2}^{e^{\top}} \mathbf{R}(\hat{q}_{b}^{e}) \mathbf{S}(\Delta r_{f}^{b}) \quad \mathbf{0}_{1\times 3} \quad \hat{l}_{f,2}^{e^{\top}} \quad \mathbf{0}_{1\times 3} \quad 0 \quad -1 \quad 0 \quad \mathbf{0}_{1\times(k-1)} \\
\vdots \\
-\hat{l}_{f,k}^{e^{\top}} \mathbf{R}(\hat{q}_{b}^{e}) \mathbf{S}(\mathbf{S}(\hat{\omega}_{eb}^{b}) \Delta r_{f}^{b}) \quad \hat{l}_{f,k}^{e^{\top}} \mathbf{R}(\hat{q}_{b}^{e}) \mathbf{S}(\Delta r_{f}^{b}) \quad \mathbf{0}_{1\times 3} \quad \hat{l}_{f,k}^{e^{\top}} \quad \mathbf{0}_{1\times 3} \quad 0 \quad 0 \quad -1 \quad 0 \quad \mathbf{0}_{1\times(k-1)} \\
\end{bmatrix} \in \mathbb{R}^{k \times (19+k-1)}, \\
\mathbf{H}_{\beta} = \begin{bmatrix}
-\hat{l}_{\beta,1}^{e^{\top}} \mathbf{R}(\hat{q}_{b}^{e}) \mathbf{S}(\mathbf{S}(\hat{\omega}_{eb}^{b}) \Delta r_{\beta}^{b}) \quad \hat{l}_{\beta,1}^{e^{\top}} \mathbf{R}(\hat{q}_{b}^{e}) \mathbf{S}(\Delta r_{\beta}^{b}) \quad \mathbf{0}_{1\times 3} \quad \hat{l}_{\beta,1}^{e^{\top}} \quad \mathbf{0}_{1\times 3} \quad 0 \quad 0 \quad -1 \quad \mathbf{0}_{1\times(k-1)} \\
-\hat{l}_{\beta,2}^{e^{\top}} \mathbf{R}(\hat{q}_{b}^{e}) \mathbf{S}(\mathbf{S}(\hat{\omega}_{eb}^{b}) \Delta r_{\beta}^{b}) \quad \hat{l}_{\beta,2}^{e^{\top}} \mathbf{R}(\hat{q}_{b}^{e}) \mathbf{S}(\Delta r_{\beta}^{b}) \quad \mathbf{0}_{1\times 3} \quad \hat{l}_{\beta,2}^{e^{\top}} \quad \mathbf{0}_{1\times 3} \quad 0 \quad 0 \quad -1 \quad \mathbf{0}_{1\times(k-1)} \\
-\hat{l}_{\beta,2}^{e^{\top}} \mathbf{R}(\hat{q}_{b}^{e}) \mathbf{S}(\mathbf{S}(\hat{\omega}_{eb}^{b}) \Delta r_{\beta}^{b}) \quad \hat{l}_{\beta,2}^{e^{\top}} \mathbf{R}(\hat{q}_{b}^{e}) \mathbf{S}(\Delta r_{\beta}^{b}) \quad \mathbf{0}_{1\times 3} \quad \hat{l}_{\beta,2}^{e^{\top}} \quad \mathbf{0}_{1\times 3} \quad 0 \quad 0 \quad -1 \quad \mathbf{0}_{1\times(k-1)} \\
-\hat{l}_{\beta,k}^{e^{\top}} \mathbf{R}(\hat{q}_{b}^{e}) \mathbf{S}(\mathbf{S}(\hat{\omega}_{eb}^{b}) \Delta r_{\beta}^{b}) \quad \hat{l}_{\beta,k}^{e^{\top}} \mathbf{R}(\hat{q}_{b}^{e}) \mathbf{S}(\Delta r_{\beta}^{b}) \quad \mathbf{0}_{1\times 3} \quad \hat{l}_{\beta,k}^{e^{\top}} \quad \mathbf{0}_{1\times 3} \quad 0 \quad 0 \quad -1 \quad \mathbf{0}_{1\times(k-1)} \\
-\hat{l}_{\beta,k}^{e^{\top}} \mathbf{R}(\hat{q}_{b}^{e}) \mathbf{S}(\mathbf{S}(\hat{\omega}_{eb}^{b}) \Delta r_{\beta}^{b}) \quad \hat{l}_{\beta,k}^{e^{\top}} \mathbf{R}(\hat{q}_{b}^{e^{\top}}) \mathbf{S}(\Delta r_{\beta}^{b}) \quad \mathbf{0}_{1\times 3} \quad \hat{l}_{\beta,k}^{e^{\top}} \quad \mathbf{0}_{1\times 3} \quad 0 \quad 0 \quad -1 \quad \mathbf{0}_{1\times(k-1)} \\
\end{bmatrix} \in \mathbb{R}^{k \times (19+k-1)}. \quad (5.171)$$

5.2.3 Double differenced carrier phase

Using the model (3.47), a vector of DDCP measurement formed as

$$\boldsymbol{\phi}_{dd} = \mathbf{A}(\boldsymbol{\phi}_{\beta} - \boldsymbol{\phi}_{f}) \tag{5.172}$$

can be used as correction in a KF with the nonlinear model

$$\begin{vmatrix} \phi_{dd,1} \\ \phi_{dd,2} \\ \vdots \\ \phi_{dd,k-1} \end{vmatrix} = \mathbf{L}_{dd}^{e^{\top}} \mathbf{R}(\boldsymbol{q}_{b}^{e}) \boldsymbol{b}^{b} - \boldsymbol{N}_{dd} + \boldsymbol{\epsilon}_{\phi_{dd}}$$
(5.173)

This model is valid when the measurement vectors ϕ_{β} and ϕ_{f} are sampled at the same time. The model is then quite simple, because of the assumptions of equal LOS vectors for both antennas. However, this is not the case when using independent receivers, using their own clock to determine when a new measurement is to be output. Thus the model needs to be modified.

Extrapolating carrier phase measurements to a common epoch

While the U-blox receivers do not steer their internal clock to align with GPST, they can slew the clock in increments of 1ms in order to keep it *approximately* aligned with GPST (U-blox 8/M8 Receiver Descr., 2017). When this occurs, a step will appear in the clock error, which should be accounted for. A measurement where the clock has been slewed will have a receiver flag set, indicating this.

It must also be taken into consideration that the receivers do not necessarily schedule each measurement at the exact same time according to their own local clock. Thus the difference in the reported measurement time must be used in addition to the estimated clock errors. Using the Taylor series, a measurement of the back receiver at the measurement time of the front receiver can be written as

$$\phi_{\beta}(t_f) = \phi_b(t_{\beta}) + (t_f - t_{\beta})\dot{\phi}_{\beta}(t_{\beta}) + (t_f - t_{\beta})^2 \frac{\phi_{\beta}(t_{\beta})}{2} + \cdots$$
 (5.174)

While it is possible to use the measured Doppler frequency directly to extrapolate using the first order approximation

$$\phi_{\beta}(t_f) \approx \phi_{\beta}(t_{\beta}) + (t_f - t_b)\phi_{\beta}(t_{\beta})$$
(5.175)

•• • •

the noise in the Doppler measurement will propagate to the extrapolated carrier phase. Since we have already modelled the Doppler measurement, we can use the predicted Doppler using the smooth states of the INS instead.

Like ϕ can be modelled using the velocity of the satellite and receiver, ϕ can be modelled using satellite and receiver acceleration, making it possible to use the second order approximation

$$\phi_{\beta}(t_f) \approx \phi_{\beta}(t_{\beta}) + (t_f - t_{\beta})\dot{\phi}_{\beta}(t_{\beta}) + (t_f - t_{\beta})^2 \frac{\dot{\phi}_{\beta}(t_{\beta})}{2}, \qquad (5.176)$$

but only a linear approximation will be used in this thesis. The difference in measurement time is the combination of difference in clock errors and the difference in the measurement time reported by each receiver, which is found in the rcvTow field of the UBX-RXM-RAWX message (U-blox 8/M8 Receiver Descr., 2017),

$$(t_f - t_\beta) = \frac{t_{\varepsilon\beta} - t_{\varepsilon f}}{c} + t_{mf} - t_{m\beta}.$$
(5.177)

The measurement matrix will be constructed first for the fixed part of the state vector, \bar{x} , for single differenced measurements, with the correspoding double differenced measurement matrix then simply calculated as

$$\mathbf{H}_{dd,\bar{x}} = \mathbf{A}\mathbf{H}_{sd,\bar{x}}.\tag{5.178}$$

The complete double differenced measurement matrix including the double differenced ambiguities is then

$$\mathbf{H}_{dd} = \begin{bmatrix} \mathbf{H}_{dd,\bar{\mathbf{x}}} & -\mathbf{I}_{(k-1)\times(k-1)} \end{bmatrix} \in \mathbb{R}^{(k-1)\times(19+k-1)}.$$
(5.179)

Using the linear time extrapolation, the single differenced measurement model for a single satellite becomes

$$\phi_{\beta} + (t_f - t_{\beta})\dot{\phi}_{\beta} - \phi_f = \frac{1}{\lambda} \boldsymbol{l}^{e^{\top}} \mathbf{R}(\boldsymbol{q}_b^e) \boldsymbol{b}^b - N_{sd}.$$
(5.180)

Substituting in the expression for the carrier phase derivative using (5.165) and (3.11) gives

$$\phi_{\beta} - \phi_{f} = \frac{1}{\lambda} \boldsymbol{l}^{e^{\top}} \mathbf{R}(\boldsymbol{q}_{b}^{e}) \boldsymbol{b}^{b} - N_{sd} + (t_{f} - t_{\beta}) \frac{1}{\lambda} (\boldsymbol{l}^{e^{\top}} (\boldsymbol{v}_{eb}^{e} + \mathbf{R}(\boldsymbol{q}_{b}^{e}) \mathbf{S}(\boldsymbol{\omega}_{eb}^{b}) \Delta \boldsymbol{r}_{\beta}^{b} + \mathbf{S}(\boldsymbol{\omega}_{ie}^{e}) \boldsymbol{p}_{eb}^{e} - \boldsymbol{v}_{es}^{e} - \mathbf{S}(\boldsymbol{\omega}_{ie}^{e}) \boldsymbol{p}_{es}^{e}) - t_{d\beta} + c \Delta \dot{t}_{SV}) + \boldsymbol{\epsilon}_{\phi_{sd}}.$$
(5.181)

Note that because we will double difference the model later, terms that are the same for all satellites will cancel, in this case this only applies to the drift rate $t_{d\beta}$, which will therefore be ignored in the following derivatives. The dependency of the model on position errors through the Earth rotation rate will also be overlooked as these terms are negligible due to the slow Earth-rotation rate. The derivative with respect to the attitude error is

$$\frac{\partial}{\partial a}(\phi_{\beta} - \phi_{f}) = -\frac{1}{\lambda} \boldsymbol{l}^{e^{\top}} \mathbf{R}(\boldsymbol{q}_{b}^{e}) \mathbf{S}(\boldsymbol{b}^{b}) - \frac{t_{f} - t_{\beta}}{\lambda} \boldsymbol{l}^{e^{\top}} \mathbf{R}(\boldsymbol{q}_{b}^{e}) \mathbf{S}(\mathbf{S}(\boldsymbol{\omega}_{eb}^{b}) \Delta \boldsymbol{r}_{\beta}).$$
(5.182)

The angular rate ω_{eb}^{b} , given by equation (5.38), makes the model dependent on the gyro bias. Differentiating with respect the gyro bias error gives

$$\frac{\partial}{\partial \delta \boldsymbol{b}_{g}}(\phi_{\beta} - \phi_{f}) = \frac{t_{f} - t_{\beta}}{\lambda} \boldsymbol{l}^{e^{\top}} \mathbf{R}(\boldsymbol{q}_{b}^{e}) \mathbf{S}(\Delta r_{\beta}).$$
(5.183)

Differentiating with respect to the clock biases give

$$\frac{\partial}{\partial \delta \boldsymbol{t}_{\varepsilon f}} (\phi_{\beta} - \phi_{f}) = -\frac{1}{c\lambda} (\boldsymbol{l}^{e^{\top}} (\boldsymbol{v}_{eb}^{e} + \mathbf{R}(\boldsymbol{q}_{b}^{e}) \mathbf{S}(\boldsymbol{\omega}_{eb}^{b}) \Delta \boldsymbol{r}_{\beta} + \mathbf{S}(\boldsymbol{\omega}_{ie}^{e}) \boldsymbol{p}_{eb}^{e}) - \boldsymbol{v}_{es}^{e} - \mathbf{S}(\boldsymbol{\omega}_{ie}^{e}) \boldsymbol{p}_{es}^{e}) - \boldsymbol{t}_{d\beta} + c\Delta \dot{\boldsymbol{t}}_{SV}),$$
(5.184)

$$\frac{\partial}{\partial \delta t_{\varepsilon\beta}} (\phi_{\beta} - \phi_{f}) = -\frac{\partial}{\partial \delta t_{\varepsilon f}}.$$
(5.185)

With respect to the velocity we have

$$\frac{\partial}{\partial \delta \boldsymbol{v}_{eb}^{e}}(\phi_{\beta} - \phi_{f}) = (t_{f} - t_{\beta})\frac{1}{\lambda}\boldsymbol{l}^{e^{\top}}.$$
(5.186)

Writing $\Delta \phi = \phi_{\beta} - \phi_f$ and using expected values, these can be combined to the matrix

$$\mathbf{H}_{sd,\bar{x}} = \begin{bmatrix} \frac{\partial}{\partial a} \Delta \phi_1 & \frac{\partial}{\partial \delta b_g} \Delta \phi_1 & \mathbf{0}_{1\times 3} & \frac{\partial}{\partial \delta v_{eb}^e} \Delta \phi_1 & \mathbf{0}_{1\times 3} & \frac{\partial}{\partial \delta t_{ef}} \Delta \phi_1 & \frac{\partial}{\partial \delta t_{ef}} \Delta \phi_1 & 0 & 0\\ \frac{\partial}{\partial a} \Delta \phi_2 & \frac{\partial}{\partial \delta b_g} \Delta \phi_2 & \mathbf{0}_{1\times 3} & \frac{\partial}{\partial \delta v_{eb}^e} \Delta \phi_2 & \mathbf{0}_{1\times 3} & \frac{\partial}{\partial \delta t_{ef}} \Delta \phi_2 & \frac{\partial}{\partial \delta t_{ef}} \Delta \phi_2 & 0 & 0\\ & \vdots & & & \\ \frac{\partial}{\partial a} \Delta \phi_k & \frac{\partial}{\partial \delta b_g} \Delta \phi_k & \mathbf{0}_{1\times 3} & \frac{\partial}{\partial \delta v_{eb}^e} \Delta \phi_k & \mathbf{0}_{1\times 3} & \frac{\partial}{\partial \delta t_{ef}} \Delta \phi_k & \frac{\partial}{\partial \delta t_{ef}} \Delta \phi_k & 0 & 0 \end{bmatrix} \in \mathbb{R}^{k \times 19}$$

$$(5.187)$$

and the complete measurement matrix, including the ambiguity states, is constructed using (5.178) and (5.179).

5.2.4 Zero angular velocity update

When the UAV is known to be standing stationary on the ground the gyro biases can be observed by correcting with the known $\omega_{eb}^{b} = 0$. This can be used as a part

of the initialization. Using the gyro measurement, ω_{eb}^{b} is be modelled as (5.38). Differentiating this with respect to the attitude error and gyro bias we have

$$\frac{\partial}{\partial a}\omega_{eb}^{b} = -\frac{\partial}{\partial a}\mathbf{R}(\boldsymbol{q}_{b}^{e})^{\top}\omega_{ie}^{e}
= -\frac{\partial}{\partial a}(\mathbf{I} + \mathbf{S}(a)^{\top})\mathbf{R}(\hat{\boldsymbol{q}}_{b}^{e})^{\top}\omega_{ie}^{e}
= \frac{\partial}{\partial a}\mathbf{S}(a)\mathbf{R}(\hat{\boldsymbol{q}}_{b}^{e})^{\top}\omega_{ie}^{e}
= -\mathbf{S}(\mathbf{R}(\hat{\boldsymbol{q}}_{b}^{e})^{\top}\omega_{ie}^{e}).$$
(5.188)

The model is already linear with respect to the gyro bias, so the derivative is simply

$$\frac{\partial}{\partial \delta \boldsymbol{b}_{g}}\boldsymbol{\omega}_{eb}^{b} = -\mathbf{I}_{3\times 3}.$$
(5.189)

This results in the measurement matrix

$$\mathbf{H} = \begin{bmatrix} -\mathbf{S}(\mathbf{R}(\hat{\boldsymbol{q}}_{b}^{e})^{\top}\boldsymbol{\omega}_{ie}^{e}) & -\mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times4} & \mathbf{0}_{3\times k-1} \end{bmatrix} \in \mathbb{R}^{3 \times 19 + k - 1}.$$
(5.190)

5.2.5 Measurement noise

The U-Blox receiver estimates the standard deviation of the measured observables, providing this along with the measurements in the UBX-RXM-RAWX message (U-blox 8/M8 Receiver Descr., 2017). These values are used as a starting point for the measurement noise covariance matrices. For the pseudorange and Doppler frequency observables the measurement matrices are independently constructed for the front and back receiver as

$$\mathbf{R}_P = \operatorname{diag}(\sigma_{P1}^2, \sigma_{P2}^2, \cdots, \sigma_{Pk}^2), \qquad (5.191)$$

$$\mathbf{R}_{\Delta f} = \operatorname{diag}(\sigma_{\Delta f1}^2, \sigma_{\Delta f2}^2, \cdots, \sigma_{\Delta fk}^2), \qquad (5.192)$$

while the matrix for DDCP depends on the estimates standard deviations from both receivers, and the satellite differencing matrix **A**,

$$\mathbf{R}_{\text{DDCP}} = \mathbf{A} \cdot \text{diag}(\sigma_{\phi 1, f}^2 + \sigma_{\phi 1, \beta}^2, \sigma_{\phi 2, f}^2 + \sigma_{\phi 2, \beta}^2, \cdots, \sigma_{\phi k, f}^2 + \sigma_{\phi k, \beta}^2) \cdot \mathbf{A}^{\top}.$$
 (5.193)

 $diag(\cdot)$ is a matrix with the given diagonal and all other elements set to 0.

5.3 Handling changes in tracked satellites for carrier phase differencing

Over the length of the UAV flight, the satellites that can be used for DDCP interferometry changes, both due to satellites disappearing below and rising above the

5.3. HANDLING CHANGES IN TRACKED SATELLITES FOR CARRIER PHASE DIFFERENCING 93

chosen elevation mask (the elevation angle below which satellites are not used), because of cycle slips and the receiver losing track of the satellites momentarily so no measurements are available. Losing satellite lock at only one of the receivers is enough to prevent its use for the interferometry. This does of course not apply to the use of pseudorange or Doppler frequency corrections, as each receiver is then used independently. When working with double differenced measurements and related ambiguity estimates, handling changes in available satellites is not as easy as just excluding a measurement. For instance, we might lose the reference satellite. The ambiguity for a satellite that is lost is no longer valid can thus be removed, and when we start using a satellite its ambiguity should be initialized to the best possible estimate. Every epoch the following is done:

- 1. Find the satellites that can be used for interferometry. This includes the satellites above the elevation mask tracked by both receivers where the receiver carrier phase and half cycle validity flags are set.
- 2. Determine the satellites that are no longer tracked, and new tracked satellites.
- 3. Determine this epoch's reference satellite, which is chosen as the one with the highest elevation that was also available at the previous epoch (unless no satellites were available then, in that case ignore this requirement). Because we are estimating double differenced albiguities, chosing the reference as a new satellites would invalidate all apriori estimates, this should thus be avoided.
- 4. Create the differencing matrix $\mathbf{A} \in \mathbb{R}^{k-1 \times k}$ (for k usable satellites) on the form of (3.37), where the column for the reference satellite is all -1, and the remaining colums forms an identity matrix of size $k \times k$.
- 5. If there are any changes in the usable SVs (except if no satellites were available last epoch), do the following:
 - a) Create an ambiguity transformation matrix **T**. As an example illustrating this transformation, assume that SVs 1,2,3,4,5,6,7 are available at epoch 1, with 7 chosen as the reference. At epoch 2, SVs 1,2,3,5,6,7,8 are available with 6 chosen as the reference. Thus SV4 was lost and SV8 was gained, and the reference changed from 7 to 6. The double differenced ambiguity vectors are then

$$\boldsymbol{N}_{\text{old}} = \begin{bmatrix} N_1 - N_7 \\ N_2 - N_7 \\ N_3 - N_7 \\ N_4 - N_7 \\ N_5 - N_7 \\ N_6 - N_7 \end{bmatrix} \qquad \qquad \boldsymbol{N}_{\text{new}} = \begin{bmatrix} N_1 - N_6 \\ N_2 - N_6 \\ N_3 - N_6 \\ N_5 - N_6 \\ N_7 - N_6 \\ N_8 - N_6 \end{bmatrix}.$$
(5.194)

The new vector can be written as a linear transformation of the old,

$$N_{\text{new}} = \begin{bmatrix} (N_1 - N_7) - (N_6 - N_7) \\ (N_2 - N_7) - (N_6 - N_7) \\ (N_3 - N_7) - (N_6 - N_7) \\ (N_5 - N_7) - (N_6 - N_7) \\ -(N_6 - N_7) \\ \text{Unknown value} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ * & * & * & * & * & * \end{bmatrix} N_{\text{old}}.$$
(5.195)

where * means that we don't care since the new value is unknown and must be initialized anyway. A matrix satisfying this can be made as follows:

- i. Create an empty matrix of size $(k 1) \times (k_{prev} 1)$, where k is the number of satellites available at the currect epoch and k_{prev} is the same for the previous epoch.
- ii. For every satellite which is not the old or new reference, place a 1 at the intersection of the column corresponding to its index in the old ambiguity vector, and the row corresponding to its index in the new one.
- iii. If the reference changed, fill the column corresponding to the index of the new reference in the old ambiguity vector with -1s.
- b) Transform the nominal ambiguities

$$\hat{N}_{dd} = \mathbf{T}\hat{N}_{dd}.$$
(5.196)

c) Transform the covariance matrix

$$\mathbf{P} = \begin{bmatrix} \mathbf{I}_{19 \times 19} & \mathbf{0}_{19 \times (k-1)} \\ \mathbf{0}_{(k-1) \times 19} & \mathbf{T} \end{bmatrix} \mathbf{P} \begin{bmatrix} \mathbf{I}_{19 \times 19} & \mathbf{0}_{19 \times (k-1)} \\ \mathbf{0}_{(k-1) \times 19} & \mathbf{T} \end{bmatrix}^{\top}$$
(5.197)

- d) Resize the error state vector to length 19 + k 1. This contains all zeros before a measurement correction.
- e) Initialize the new ambiguities in the nominal state. For this we can use either (3.52) or (3.54), but since most of the times the initialization is needed we actually have a good estimate of the attitude, (3.54) should be the better option. In this non-ideal case however we also want to take the difference in measurement time into account, so the estimate becomes

$$\hat{N}_{dd,\text{new}} = \frac{\rho_{dd,\text{new}}}{\lambda} - (\Delta\phi_{\text{new}} + (\widehat{t_f - t_\beta})\hat{\phi}_{\beta,\text{new}} - \Delta\phi_{\text{ref}} - (\widehat{t_f - t_\beta})\hat{\phi}_{\beta,\text{ref}})$$
(5.198)

$$\hat{N}_{dd,\text{new}} = \frac{\rho_{dd,\text{new}}}{\lambda} - \phi_{dd} - (\widehat{t_f - t_\beta})(\hat{\phi}_{\beta,\text{new}} - \hat{\phi}_{\beta,\text{ref}})$$
(5.199)

$$\hat{N}_{dd,\text{new}} = \frac{1}{\lambda} (\boldsymbol{l}_{\text{new}}^e - \boldsymbol{l}_{\text{ref}}^e)^\top \mathbf{R}(\boldsymbol{q}_b^e) \boldsymbol{b}^b - \phi_{dd} - (\widehat{\boldsymbol{t}_f - \boldsymbol{t}_\beta}) (\hat{\phi}_{\beta,\text{new}} - \hat{\phi}_{\beta,\text{ref}})$$
(5.200)

where the carrier phase rates use the estimates from the Doppler frequency model.

f) Initialize the covariance of the new ambiguities. For this simply set the diagonal element for the new value to a constant, here chosen as a standard deviation of 2 carrier cycles.

5.4 Nominal state initialization

5.4.1 Single epoch least squares estimation

The position, velocity and the clock errors for both receivers are initialized by using measurements from the first epoch available with an iterative LS method, causing the Kalman filter to converge faster. Since the front antenna is close to the body frame origin, we ignore the lever arms and assume $p = p_f$. Thus the measurements from the front receiver initializes the position, velocity and front clock errors, while the measurements from the back receiver only initializes the back clock errors. Initializing the receiver in the Earth center with zero velocity, $p_{eb}^e = 0$, $v_{eb}^e = 0$, $t_e = 0$, $t_e = 0$, $t_d = 0$, we run the following for each receiver independently, with $x = [p_{eb}^e \ t_e \ v_{eb}^e \ t_d]^{\mathsf{T}}$, until convergence:

- 1. Calculate the satellite position at the time of transmission, rotated to the ECEF frame at the time of reception, and the satellite clock errors and drift rates.
- 2. At the estimated position, create the ECEF LOS vectors and form the matrix

$$\mathbf{G} = \begin{bmatrix} -l_1^{\top} & 1 & \mathbf{0}_{1\times 3} & 0 \\ & \vdots & & \\ -l_k^{\top} & 1 & \mathbf{0}_{1\times 3} & 0 \\ \mathbf{0}_{1\times 3} & 0 & l_1^{\top} & -1 \\ & \vdots & & \\ \mathbf{0}_{1\times 3} & 0 & l_k^{\top} & -1 \end{bmatrix}$$
(5.201)

- 3. Estimate the ionospheric delay using the Klobuchar model and the tropospheric delay using the NATO STANAG model.
- 4. Form the pseudorange measurement residual as

$$\Delta P_i = P_{\text{measured}} - \|\boldsymbol{p}_{eb}^e - \boldsymbol{p}_{es}^e\| + t_{\varepsilon} - \Delta t_{SV} + I + T$$
(5.202)

for each satellite *i*, and combine to the vector $\Delta \mathbf{P} = [\Delta P_1, \Delta P_2, ... \Delta P_m]^{\top}$

5. Form the Doppler requency measurement residual

$$\Delta f_{\text{res},i} = \Delta f_{\text{measured}} - \boldsymbol{l}^{e^{\top}} (\boldsymbol{v}_{eb}^{e} + \mathbf{S}(\boldsymbol{\omega}_{ie}^{e}) \boldsymbol{p}_{eb}^{e} - \boldsymbol{v}_{es}^{e} - \mathbf{S}(\boldsymbol{\omega}_{ie}^{e}) \boldsymbol{p}_{es}^{e}) + t_{d} - \Delta \dot{t}_{SV} \quad (5.203)$$
for each satellite, and combine to the vector Δf_{res}

6. Calculate the least squares estimate of the state error using the left Moore-Penrose pseudoinverse (generalized inverse if weighting is used):

$$\Delta \boldsymbol{x} = (\mathbf{G}^{\top}\mathbf{G})^{-1}\mathbf{G}^{\top} \begin{bmatrix} \Delta \boldsymbol{P} \\ \Delta \boldsymbol{f}_{\text{res}} \end{bmatrix}$$
(5.204)

7. Update the estimate: $\hat{x} \leftarrow \hat{x} + \Delta x$

Note that some of the pseudorange corrections made here, such as the recalculation of the transmission time based on estimated clock error, can be omitted if a lower computational footprint is desired. The KF only needs a few iterations to compensate for the error caused by this simplification.

5.4.2 Attitude initialization

The attitude initialization is depentent on the movements of the UAV. For a stationary vehicle where the distance between receiver antennas is sufficiently greater than the noise present in the pseudorange antenna positions, the difference in these positiont should provide initial values for the heading. Averaging the measurements should improve this estimate. Pitch and roll for a stationary IMU can be found by using accelerometer leveling, as explained in Section 2.8.

If the filter is started while the fixed-wing UAV is already flying, another initialization method is to assume that the sideslip is small such that heading and course can be approximated as equal. We can then use the LS-velocity estimate to initialize the heading and pitch angles.

$$\theta = \operatorname{atan2}(-v_z^n, \sqrt{(v_x^n)^2 + (v_y^n)^2})$$
(5.205)

$$\psi = \operatorname{atan2}(v_y^n, v_x^n) \tag{5.206}$$

This will however not be tested in this thesis.

5.4.3 Initial covariance values

The covariance matrix was initialized as a diagonal matrix. The initial standard deviation for pitch and roll was set at 3°, but a few different values were used depending on the initial heading error. When using antenna positions for initialization a standard deviation of 45° was used. Bias values were chose the same for all axes, 7×10^{-6} radians per increment for the gyro and $2 \times 10^{-4} \frac{\text{m}}{\text{s}}$ per sample for the accelerometer. Position and velocity standard deviations were set at 8m and $0.2\frac{\text{m}}{\text{s}}$, and for clock errors and drift rates 0.1ms and 4×10^{-8} (both then multiplied by the speed of light *c*).

5.5 Integer ambiguity resolution and fixed output

The Matlab implementation of the LAMBDA algorithm of Verhagen et al. (2012) is used to fix the ambiguity estimates to integer values. Because the integer output of the LAMBDA algorithm is not guaranteed to be correct, with the error rate depending on how strict the integer acceptance test is tuned, feeding back corrections that depend on the LAMBDA integer output to the nominal state can potentially destabilize the filter. Tuning the acceptance test to give a lower error rate also leads to a lower rate of integer fixes being available, and a tradeoff must be made. The fixed values will therefore be used in a separate correction on the filter output, while the nominal state only depends on the float ambiguity estimates. While this is the same as done by RTKLIB with the "continous" integer ambiguity resolution option (Takasu and Yasuda, 2013, RTKLIB Manual E.7(5)), the integration with INS means that the "fixed" output depending on LAMBDA will need to be predicted forward using IMU measurements alongside the "float" nominal state, such that the output is always dependent on fixed ambiguities if available. Only when the next DDCP correction has been made and LAMBDA is ran again can the predicted "fixed" estimates be replaced.

The output correction done after the DDCP measurement correction assumes that an integer results that have passed the acceptance test can be assumed to be the true integers. The vector \hat{N}_{fixed} is then treated as a measurement of the ambiguity vector with no uncertainty, $\mathbf{R}_{\hat{N}_{\text{fixed}}} = \mathbf{0}$. This has the simple measurement matrix $\mathbf{H} = [\mathbf{0}_{(k-1)\times 19} \quad \mathbf{I}_{(k-1)\times (k-1)}]$. The Kalman gain matrix is then

$$\mathbf{K} = \mathbf{P}\mathbf{H}^{\mathsf{T}}(\mathbf{H}\mathbf{P}\mathbf{H}^{\mathsf{T}})^{-1},\tag{5.207}$$

and the fixed nominal state estimate is constructed as

$$\hat{x}_{\text{fixed}} = \hat{x} + \mathbf{K}(\hat{N}_{\text{fixed}} - \mathbf{H}\hat{x}).$$
(5.208)

5.6 Cycle slips and half cycle errors

Like in Sollie (2017) the receiver flags of the U-Blox UBX-RXM-RAWX message (U-blox 8/M8 Receiver Descr., 2017) will be used to exclude measurements in the event of cycle slips and following half cycle errors. While the "carrier phase valid" flag is checked and values with this set to zero are excluded, this does not provide information about cycle slips. The effect of a cycle slip on the receiver however is that the half cycle ambiguity is unresolved, meaning that the "half cycle valid flag" can be used to exclude measurements where a cycle clip is occuring, and where a potential half cycle offset occurs. The values of these bits for a cycle slip and half cycle error can be seen in Figure 5.1.



Figure 5.1: The figure shows the DDCP (corrected to the same measurement time) between SV 1 and 32, the value predicted from PPK positions and the error between them. The red/green colored bars shows the value of bits reported by the U-blox receivers, green represents the value 1 and red the value 0. Plot taken from Sollie (2017).

5.7 IMU-GNSS measurement synchronization and delay handling

The measurements from both the IMU and each GNSS receiver are received by the *SenTiBoard* (sensor timing board) developed by the NTNU UAV lab, previously known as the SyncBoard (Albrektsen and Johansen, 2017), which also receives synchronization signals from each sensor. The SenTiBoard is connected to the onboard computer where the measurement data and timing information is logged. The SenTiBoard has a clock running at 100MHz, incrementing a 32 bit counter. This reaches its maximum value $2^{32} - 1$ and wraps around about every 42.9 seconds. Both U-Blox receivers have pulse per second (PPS) signal outputs which triggers at the top of each GPST second, staying high for 0.1s. The rising edge of the PPS signals (which should trigger at the same time for both receivers because clock errors are accounted for) give a reference point which makes it possible to find the time of IMU measurements. Using the estimated receiver clock errors

and the measurent time reported by each receiver, the absolute time of the GPS measurements can be also be found.

5.7.1 IMU measurement time

The STIM300 has an ouput synchronization signal called time of validity (TOV) which is kept at a high logic level as default, but is pulled low when a new measurement is to be transmitted (Sensonor, 2018). This can be used to find the time at which the measurement was actually taken to within microseconds. According to the datasheet, the time from a measurement is taken until the TOV output is pulled low is given by

$$\Delta t_{\rm imu} = t_{\rm tov} - t_{\rm measurement} = \text{Group delay} + 0.5\text{ms} + t_{\rm tov_dl}.$$
 (5.209)

where the group delay is the delay caused by the internal low-pass filter, the 0.5ms delay is a single internal sample interval (the STIM300 does internal sampling at 2kHz) and t_{tov_dl} is the delay from the internal time-tick to the TOV signal being active, which is stated to be nominally 1.2µs with a maximum of 6µs. With the low pass filter -3dB frequency set to the highest option, 262Hz (which was done in the experimental testing in this thesis), the datasheet states a group delay of 1.5ms for the gyro and 6.5ms for the accelerometer (Sensonor, 2018). The difference in group delay for the different sensors means that gyro and accelerometer measurements received simultaneously from the IMU represent dynamics experienced by the sensors 5ms offset in time.

The time t_{tov} can be found within a GPST second by using the PPS signal. An unfortunate configuration bug made the SenTiBoard trigger on the falling edge of the PPS signal, but this can easily be taken into account. The signals are illustrated in Figure 5.2. With the counter values tov_{imu} and pps_{eps}, the relation is

$$\frac{\text{tov}_{\text{imu}} - \text{pps}_{\text{gps}}}{10^8 \text{Hz}} + 0.1\text{s} - \Delta t_{\text{imu}} = t_{\text{measurement}} \pmod{1\text{s}}.$$
 (5.210)

A plot of the fractional measurement comparing IMU and GPS measurements are shown in Figure 5.3.

By using the absolute measurement time of the last received GPS measurement and the fractional time of the IMU measurement, the absolute time of the IMU measurment can be found.

5.7.2 MEKF measurement timing

Because of the difference in group delay for the gyro and accelerometer measurements, and the 250Hz IMU measurement rate which is too low for predicting the pose of the vehicle over the few milliseconds of difference in GPS measurement time, without interpolation, the only delay that was taken into consideration was



Figure 5.2: Time sync. The IMU rate is shown reduced in the figure, in reality it is running at 250Hz, and the IMU TOV signal is pulled low for a shorter time than the internal sample period of the IMU, which is 0.5ms.



Figure 5.3: Difference in logging delay of IMU and GNSS measurements. The fractional time of the IMU and GPS measurement times is plotted as a function of the logging time according to the onboard computer. While this information is ambiguous, the reasonable interpretation is that the GPS measurements are received by the onboard computer about 70ms after the IMU measurements.

the large ~ 70ms delay between the IMU and GPS measurement times. Except for the correction for measurement time in the DDCP model, the pseudorange and Doppler frequency measurements taken by the two receivers will be assumed to be taken at the same time.

6

Experimental Setup

6.1 UAV Flight



Figure 6.1: Cruiser Mini being prepared for launch at Raudstein.

The UAV used for data collection is the ET-Air Slovakia Cruiser Mini. It is controlled by a Pixhawk 2.1 flight, also known as *The Cube* (ArduPilot Dev Team, 2018b), controller running the ArduPlane flight stack. The flight controller estimates its position using a separate GNSS receiver from the pair used by the proposed algorithm. State estimates from the flight controller are logged for comparison to the results from the implemented algorithm. The measurements from both the IMU and each GNSS receiver are received by the prevously mentioned *SenTiBoard* which also receives synchronization signals from each sensor. The SenTiBoard is connected to the onboard Odroid computer using USB, where the measurement data and timing information is logged.

The UAV was launched from Raudstein, Agdenes, Norway on September 27th 2017 just after 12.00 UTC. The flight was approximately one hour long. The flight path is shown in Figure 6.2.



Figure 6.2: Flight path. The position was logged at 1 Hz from the UBX-NAV-POSECEF message of the front receiver (U-blox 8/M8 Receiver Descr., 2017). The visualiation was created using the kml-toolbox of Rafael-aero (2018) and Google Earth Pro Google (2018).

6.2 Inertial measurement unit

A Sensonor STIM300 IMU was part of the payload, outputting angular and velocity increments at a rate of 250 Hz. This is a high quality MEMS IMU. Noise parameters are given in Table 5.3, and plots of Allan deviation, also called root Allan variace, is found in Appendix G.

6.3 GNSS equipment

Two U-blox NEO-M8T GNSS receiver modules mounted on InCase PIN series UBLOX NEO-M8T reference boards were connected to the SenTiBoard. The receivers are capable of outputting raw GNSS observables and navigation messages, and are connected to Harxon HX-CH3602A helical antennas. Raw legacy L1 GPS observables and received GPS navigation messages (LNAV) frames were received by the SenTiBoard and transmitted to the Odroid for logging. While the receivers are capable of using multiple GNSS constellations at once, only GPS was used for the experiment. Additional systems can improve perormance by increasing the amount of SVs in view, but requires reading ephemeris parameters in different formats, using different satellite position calculation algorithms due to differences in the orbital parameters, and keeping track of multiple time references. Legacy GLONASS also uses frequency division multiple access (FDMA), not CDMA, making carrier phase processing more complicated (Wanninger, 2012).

6.4 Coordinate frames and baseline

The body frame of the UAV is defined to have its origin at the location of the accelerometer. This means that no accelerometer lever arm needs to be considered. The location of the GNSS receiver antennas relative to the location of the IMU were measured physically, giving the following lever arms:

$$\Delta \boldsymbol{r}_{f}^{b} = \begin{bmatrix} 0.005\\0\\-0.123 \end{bmatrix} \tag{6.1}$$

$$\Delta \boldsymbol{r}_{\beta}^{b} = \begin{bmatrix} -0.695\\0\\-0.09 \end{bmatrix} \tag{6.2}$$

The baseline from the back receiver to the front receiver, which is of interest for the use of DDCP, resulting from this is

$$\boldsymbol{b}^{b} = \begin{bmatrix} 0.70\\0\\-0.033 \end{bmatrix}.$$
 (6.3)

The IMU was mounted with a 90 degree yaw rotation compared to the UAV body frame, so the rotation matrix used to transform the measurements to the body-frame was

$$\mathbf{R}_{m}^{b} = \mathbf{R}_{z} \left(-\frac{\pi}{2} \right) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$
 (6.4)

6.5 Reference values using post processing

A stationary U-blox LEA-M8T GNSS receiver with a NovAtel GPS-700 series antenna was placed near the launch site, and raw GPS observables were logged using the base station computer. The raw data from the base station and each antenna on the UAV was used for PPK positioning of each antenna using RTKLIB (Takasu, 2017). Moving-base PPK estimates where one antenna was used as the reference and no base station was used was also performed. These results were only used for comparison with the results from the MEKF. While the PPK positions are very precise, the accuracy depends on how well the location of the base station is known. Here the position of the base station was assumed to be the average of the "single" position calculated using pseudoranges. Unlike with the implemented algorithm, both GPS, GLONASS and EGNOS corrections were used in RTKLIB to calculate the reference signals for comparison.

6.6 Algorithm implementation

The MEKF was implented in Matlab, running on a desktop computer. Some minor functions from the MSS GNC Toolbox (Fossen and Perez, 2004) were used. The Matlab implementation of the LAMBDA algorithm from Verhagen et al. (2012) was used for integer ambiguity resolution, using the ratio test with $\mu = 0.5$ to accept integer results.

Part III

Results

Results and Discussion

This chapter presents the results from running the proposed MEKF, described in Chapter 5, with the logged measurements from the UAV flight described in Section 6.1. To compare the effects of the different measurement corrections, a few different configurations are tested:

- 1. Only pseudorange from the front receiver.
- 2. Pseudorange and Doppler from both receivers.
- 3. All corrections, including DDCP were interger ambiguities are fixed using the LAMBDA algorithm described in Section 3.7.4.
- 4. Only Doppler from both receivers. Without direct observation of the position this is only dead reckoning, and position drift over time can be expected, but how much?

Most focus will be on configuration 3. Position and velocity are instantaneously observable for all configurations, except from 4. These states are also easy to initialize accurately, so testing several initial conditions for these states are not the most interesting. Since the IMU biases can be expected to be small, their estimates are initialized to zero, but the gyro bias is observed by using the measurements from Section 5.2.4 for the first few seconds when the UAV is known to be stationary. The most interesting initial condition is the attitude, especially the heading, which is also the main motivation behind using two GNSS antennas and receivers to aid the INS. Therefore, the MEKF is tested with different initial heading estimates, from a correct estimate to an initial 120 degree error, which is worse than the initial value achievable by the single-epoch least squares initialization from Section 5.4.1 for this baseline length.

PPK positioning using RTKLIB (Takasu, 2017) provides reference positions for both antennas for comparison to the MEKF estimates. Because the baseline is not excactly parallel to the body frame x-axis, heading and pitch angles cannot be computed directly from these without knowing the roll angle, without errors being introduced. Heading and pitch values will be computed under the assumption that the baseline is parallel to the body x-axis, and a small error can be expected in the pitch when roll is zero. The heading calculated from this will be accurate in the absence of roll. Most of the heading evaluation was done when the UAV roll was small.

Note that as explained in Section 3.8, the reference position from PPK positioning is not known to be accurate in an absolute sense, although it should be close as it uses GPS, GLONASS and SBAS corrections. RTKLIB does not provide velocity output directly, so numerical differentiation of the PPK position was used to obtain velocities, in addition to velocity estimates from the Pixhawk autopilot, for comparison.

In the following results the nominal attitude quaternion estimate is converted to Euler angles and presented as such, since this is an more intuitive attitude interpretation to process as a reader.

7.1 Final process and measurement noise variances

The variances of the clock model and the white noise part of the IMU models presented in Section 5.1.7 were found to work fine and kept unchanged. The ambiguity noise was increased to a final value of $\sigma_N = 8 \times 10^{-3}$. Values for the IMU bias models ended up at a correlation time of T = 1000s for both the accelerometer and gyro, and the standard deviation of the driving noise were chosen as $\sigma_{bg} = 8 \times 10^{-7}$ and $\sigma_{ba} = 1.5 \times 10^{-5}$.

Because of the strong time-correlated noise in the pseudoranges, the variance assumed by the MEKF for these in (5.191) were increased by a factor of 10 over the receiver estimates. Doppler frequency measurement variances were used directly as estimated by the receivers. The standard deviation of the DDCP in (5.193) were set 10 times higher than the receiver's estimates in order to reduce visible noise in the INS's estimates.

7.2 Position and velocity

This section presents and discusses results for the estimated position and velocity. The position estimate while the UAV is on the ground, prior to launch, is plotted in Figure 7.1. Estimated positions of the IMU for three of the configurations, and the front and back GPS antennas for configuration 3, are shown along with the autopilot (AP) position and the PPK position estimates for both antennas from RTKLIB. The logging starts when the UAV is in the lower part of the figure and after about 16s the vehicle is lifted onto the launch ramp, where it sits until about 200s. This movement is visible for all three position sources, but the AP and MEKF in configuration 1 definitively performs the worst. The MEKF estimate appear to have an eastward bias of a little over 1m compared to the PPK, but is consistent with the AP estimate. The visible distance between the front and back receivers is a little under 0.7m, which is the horizontal component of the baseline (6.3). The time-correlated errors seen in the MEKF estimate using configuration 3 is caused by the time-correlated errors in the pseudorange, as mentioned in Section 3.4.9. By increasing the measurement noise value (2.19) of the pseudorange used in the MEKF further, this effect can be reduced. The estimates obtained by initializing the position using pseudoranges, but running the INS using only Doppler aiding, shown in black, do not exhibit this error. Without the use of Doppler frequency measurements, e.g. in configuration 1, increasing the modelled pseudorange noise



Figure 7.1: Lat/Lon position plot for the first 200 seconds, up until the UAV launches.

too much will lead to reduced observability of the accelerometer biases, which can cause the drift visible in green in Figure 7.3 to increase. A local NED plot also showing the altitude is shown in Figure 7.2. It can be seen that the MEKF estimates the altitude as too high (negative in NED coordinates), while the AP estimates it as too low. A larger spread in the vertical direction is not surprising since the satellite geometry makes estimates associated with the vertical more sensitive to errors along the receiver to satellite LOS vectors.

The position estimation error, when the PPK position is considered the reference, for the entire flight is plotted in Figure 7.3. The addition of Doppler frequency corrections significantly reduces the noise present in configuration 1. The Doppler corrections alone are sufficient to prevent significant drift in the North and East directions, but drifts around 15m in altitude. A drift in the altitude error of about 4m is visible also when all GNSS corrections are used (from about -2m at the beginning to 2m at the end), indicating that there are not only errors associated with the Doppler measurement, but also the pseudorange. Considering that the Doppler-only test is essentially dead reckoning and no absolute position corrections are made after initialization, this shows that the Doppler frequency



Figure 7.2: Position plot in local NED frame for the first 200 seconds, up until the UAV launches.

measurement model is quite accurate for horizontal velocity. When all corrections are used, the eastward position bias is present through the entire flight. Mean and root mean square (RMS) position errors compared to PPK after the initial convergence is shown in Table 7.1. These results are significantly better than one can expect from GPS's standard positioning service which seldom gives better results than a 3-5 m uncertainty horizontally (Misra and Enge, 2012).

Direction	Mean	RMS
North	0.176m	1.385m
East	1.511m	1.853m
Down	1.125m	1.819m

Table 7.1: NED position errors relative PPK (configuration 3).

A plot of the estimated velocity when the UAV lies on the ramp prior to launch is seen in Figure 7.4. The velocity appears to be quite accurate, but in the North and East directions the AP provides imprecise estimates compared to the output of the MEKF. While the AP estimates also are based on an aided INS (ArduPilot Dev Team, 2018a), is has a significantly less expensive IMU (ArduPilot Dev Team,



Figure 7.3: Position error compared to PPK positions in NED frame for the entire flight. The use of Doppler corrections for velocity reduces the position noise significantly. Doppler corrections by itself does give some altitude drift, but very little horizontal drift.

2018b), which can be a source of the velocity noise. The AP also uses a different type of GNSS antenna, placed on the inside of the fuselage, which can increase noise in the GNSS signal tracking by reducing the signal strength of the received signal. The MEKF in configuration 1 appears to oscillate slightly, which can be caused by too large variances in \mathbf{R}_P noise as mentioned concerning the position presented in Figure 7.1. High measurement noise values for the pseudorange can be beneficial when other measurements contributing to observability of the accelerometer biases are used, but with pseudorange measurements alone, a sufficient weighing of the measurements is necessary to prevent errors caused by the biases. Doppler frequency measurements significantly reduces this. An enlarged plot of the East velocity when all corrections are applied is found in Figure 7.5. The East and North velocity errors are in order of a few centimetres per second, which is the accuracy expected by measuring Doppler frequency shifts in Gaglione (2015).

The magnitude of the vertical velocity errors are larger for both the AP and the MEKF in configuration 3, which just like for the position is not surprising because of the satellite geometry. It should be possible to achieve even better velocity results by using time differenced carrier phase (TDCP), because of the fractional phase measurement in (3.13), which makes it more accurate. Accuracies of a few millimetres per seconds is obtainable using this method according to Gaglione (2015).



Figure 7.4: Velocity estimates when the UAV is on the ramp before launch, comparing use of pseudorange from a single receiver (configuration 1), and all corrections used in the MEKF (configuration 3). Pixhawk velocity estimates are used as reference, but are significantly more noisy than the complete MEKF. A closeup of the east velocity using configuration 3 is found in Figure 7.5.

The velocity of the UAV for a segment of flight is plotted in Figure 7.6. A small mismatch in the timing of the AP output compared to the MEKF results can be seen. The AP estimates are logged and timestamped by the onboard computer with an unknown delay, which is likely the source of this. The mean and RMS NED velocity errors compared to time differenced PPK position estimates are shown in Table 7.2. The RMS errors for the flight is larger than expected for the high precision shown on the ground, but can be caused by a timing misalignment



Figure 7.5: Closeup of the East component of the velocity estimate before launch.

of the MEKF and PPK values used to calculate the error. The mean velocity error should be small regardless of the MEKF configuration as long as pseudorange is included, and mostly depends on errors in the estimated difference between start and end points of the flight path.

Direction	Mean	RMS
North	0.00201 ^m /s	0.112 ^m /s
East	-0.00153m/s	0.070 m/s
Down	0.01207m/s	0.087 m/s

Table 7.2: NED velocity errors relative PPK (configuration 3).



Figure 7.6: Velocity estimates in air.

7.3 Attitude

This section presents and discusses results for attitude with different initial conditions, for both static conditions on the ground and dynamic conditions in the air. The Euler angle estimates in the case when heading is initialized using the LS antenna positions from the first measurements are shown in Figure 7.7. The initial heading error for this case is about 35 degrees. While the MEKF using all corrections appear to have an erroneous integer fix at the very beginning, this does not destabilize the filter as the fixed estimates are independent of the nominal state. While the MEKF using only pseudorange corrections struggles, the others quickly approach the reference value when a small amount of excitation, in the form of the UAV being lifted onto the launch ramp, occurs.

The pitch estimates from the AP are very similar to those produced in all MEKF configurations, although the difference compared to the PPK was expected to be larger based on the angle the baseline has on the UAV. With the measured baseline (6.3) from Section 6.4, the pitch angle of the baseline in the body frame is

$$\theta_{h^b} = \operatorname{atan2}(0.033 \mathrm{m}, 0.7 \mathrm{m}) \approx 2.7^{\circ}$$
 (7.1)

which is not observed in Figure 7.7. Uncertaincies in the measurement of the antenna lever arms, or the definition of the frame in which they are measured, can be one source of this error.

The roll estimate exibits a similar trend as the pitch signal from the AP, which uses accelerometer leveling, until around 21s, when something happens that make them output estimates with one degree offset. This error is not corrected as the roll error is not observable for the MEKF in the static case, because roll is rotation about the body frame x-axis, which is close to parallel to the baseline vector between the GNSS receivers. Including accelerometer leveling corrections for static conditions, or adding a third GNSS antenna to obtain complete observability of the attitude, would solve this problem.

The MEKF's heading estimate while the UAV was on the launch ramp can be seen in Figure 7.8. While the MEKF in configuration 2 yields decent heading estimates, the usage of carrier phase in configuration 3 yields the best results relative the PPK-calculated reference. Once the UAV is launched the observability of the heading enables good performance using all MEKF configurations as shown in Figure 7.9, where MEKF configurations 2-4 are nearly identical and configuration 1 is also close. The AP estimate is off by around 4 degrees, most likely caused by a misalignment in how the Pixhawk flight controller is mounted to the UAV. For the AP a small error in the heading is not really a problem, as mentioned in Section 1.1, the UAV maintains the desired path by feedback control where the course is more important than the heading.

Initializing the heading with no error (and an initial standard deviation of 5 degrees in the MEKF covariance matrix) gives the results shown in Figure 7.10. The MEKF in configuration 3 is the only estimate closely following the PPK reference, until the UAV launches. The estimated heading when initialized with a very large of 120 degrees with an initial standard deviation of 150 degrees is shown in Figure 7.11. The MEKF using only pseudorange corrections did some abrupt corrections in the very beginning, but when both receivers with Doppler measurements was used, the initial error is quickly reduced. Decent attitude estimates are obtained in configurations 2 and 3 as soon as the UAV is lifted onto the ramp. Comparing this with the result for a 80 degree initial error in Sollie (2017), where the heading had not reached agreement with the reference even after 10 minutes of flight, the use of an aided INS shows a significant improvement compared to solely using dual-GNSS to estimate heading. When the UAV is launched, the heading estimate using



Figure 7.7: Plot of Euler angles, initialized using accelerometer leveling and LS positions for both receivers.

only pseudoranges immediately approach the reference, showing that additional measurements are most useful in situations with low dynamics.

Figure 7.12 shows a section of in-flight pitch estimates. The different MEKF configurations give nearly identical estimates, but does not shown the expected difference compared to the PPK-derived pitch in this part of the flight. One pos-







Figure 7.9: Plot of heading in air.

sible explanation of this is that the DDCP pitch corrections are not weighted as much as the pitch corrections from the position and velocity errors observed using pseudoranges and Doppler frequency, due to high measurement noise assumptions mentioned in 7.1. The low weighting of the DDCP pitch combined with possible errors in \mathbf{R}_m^b (rotation matrix from the IMU's mounting frame to BODY) or the assumed lever arms, could yield the results shown. The PPK-derived pitch is less precise than the estimates based on aided INSs, from both the AP and the MEKF, with several abrupt spikes. The pitch estimates from the MEKF and AP



Figure 7.10: Heading intialized with a very small error.



Figure 7.11: Heading intialized with a very large error of 120 degrees.

do not agree at all parts of the section shown. Comparing the points where these errors occur with the acceleration logged by the AP, shown in Figure 7.13, may suggest that an error source is accelerometer leveling (explained in Section 2.8) performed by the AP, where accelerations causes errors in the estimated attitude if the accelerometer measurements are assumed to be dominated by gravity. The roll angle estimates for the same part of the flight, for MEKF configurations 1-3 and the AP, are plotted in Figure 7.14. The acceleration-dependend error seen for pitch is not as visible in plot of the roll angle, except for in the 315-320s interval when the


Figure 7.13: Comparison of MEKF-AP pitch error and AP acceleration estimates. The accelerations are measured in the AP measurement frame, which is close in alignment to the body frame.

acceleration is the largest. The different MEKF configurations give nearly identical estimates. Mean and RMS errors of the Euler angle estimates relative to the AP and the PPK-derived pitch and yaw (heading) are shown in Tables 7.3 and 7.4. No comparison of the roll estimates are made for PPK because the roll angle cannot be calculated using a single baseline parallell to the body x-axis. The mean errors relative the AP indicate that the STIM300 IMU and the AP are mounted to the UAV with similar pitch and roll, but with an offset in yaw. The yaw RMS error is mainly caused by the large mean offset, with a small spread. The yaw mounting error will also contribute to the observed error in pitch depending on the roll angle, but is



Figure 7.14: Plot of roll angle in air

likely not the source of the error alone. With the maximum roll angle of 40° seen in Figure 7.14, the 2° error in yaw should give a pitch error of $\sin(40^{\circ}) \cdot 2^{\circ} \approx 1.3^{\circ}$, less than the observed error in Figure 7.13.

The mean PPK-relative pitch error is about two degrees off the expected offset due to the pitch angle of the baseline. As mentioned, this can be caused by inaccurate measurement of the antenna lever arms and baseline, and there can be errors in the assumed relationship between the body frame, where the lever arms are measured, and the IMU measurement frame. The RMS values for both pitch and yaw are largely caused by the mean offset.

Angle	Mean	RMS
Roll	0.033°	0.540°
Pitch	-0.067°	0.766°
Yaw	-2.060°	2.983°

Table 7.3: Attitude errors relative AP (configuration 3).

The UAV used in this thesis is small and has dynamics that were sufficiently exciting for observability of the attitude in the flight test performed. For a larger aircraft, flying in straight lines for long time periods, the dynamics in flight might not be sufficient for observability of the heading with only a single GNSS antenna. A dual-antenna setup can give improved estimates over a magnetic compass in such situations. The same would be the case for a hovering rotary-wing aircraft.

Angle	Mean	RMS	
Pitch	-0.888°	1.164°	
Yaw	-0.327°	0.672°	

Table 7.4: Attitude errors relative PPK (configuration 3).

It should also be mentioned that the reference signals derived from PPK positioning with RTKLIB are not always accurate due to cases where either only a float ambiguity result is available, or incorrect integer ambiguities are accepted. Figure 7.15 shows the heading estimate of the MEKF compared to the PPK-derived reference for a different part of the flight. Several abrupt errors are visible. The rate of these events increase as the distance from the UAV to the base station antenna increases. An attempt has been made to remove most of these errors before calculating the mean and RMS yaw and pitch errors with the PPK-derived values as reference, but some still remain. The settings for integer ambiguity resolution, including the acceptance threshold ratio, are user configurable and default settings with the "Fix and hold" method (Takasu and Yasuda, 2013) were used for the results used in this thesis. RTKLIB also has the option to perform "Moving-Base" estimation where one of the receivers onboard the UAV is used as the reference, similar to the method used in the MEKF in this thesis, but these estimates were noisy, as can be seen in Figure 7.16.



Figure 7.15: Errors in yaw reference from RTKLIB.

Comparing the results obtained here with the Vectornav VN-300 mentioned in the introduction (Section 1.1), the product brief (Vectornav, 2016) states that the dual-antenna heading is used for static conditions, while the INS provides the heading in dynamics conditions. In the results obtained from the MEKF, it is clear that the main benefit of the dual-antenna setup is the performance when the vehicle dynamics are insufficiently exciting for observability of the heading. Vectornav states an accuracy of 0.3 degrees for the dual-antenna heading. If the problems related to accurate calibration of the IMU mounting angles and antenna positions



Figure 7.16: Comparison of pitch derived from RTKLIB with moving and fixed reference.

are solved, this accuracy should be obtainable even with the low-cost GNSS setup used in this thesis. The VN-300 also includes a magnetometer, which is very useful for fast initialization of the system, removing the need for any dynamics for heading convergence.

Commercially available systems most likely use single receivers capable of using multiple antennas, such that a single clock can be used. This makes it easy to perform measurements at the same time for both antennas, and also makes it possible to use only single differencing, as the clock errors in (3.34) cancel. The increased processing complexity necessary to handle clock errors and measurement times is the main downside of using independent receivers.

7.4 Number of tracked satellites

Figure 7.17 shows the number of satellites tracked by both receivers, and the number of common satellites usable for forming DDCP. These are satellites where none of the receivers are experiencing cycle slipping, and have resolved the half cycle ambiguity explained in Section 3.4.7. It is interesting to see that the front receiver momentarily loses track of satellites more often than the back receiver. The satellite count for each receiver does not take the receiver flags into account, and therefore counts all satellites where raw data messages were available. The problem is not only that the front receiver more often experiences cycle slips or such, but that it loses track of SVs completely. As shown in Figure 6.1, the back antenna is mounted on the thin tail boom, further away from the UAV powertrain than the front antenna. The powertrain, consisting of motors and speed controllers for these, might be a source of electromagnetic interference affecting the front antenna more. The front antenna is also mounted between the wings of the UAV, and it is possible that signals can bounce off the wing surface causing multipath interference. Minor differences between the two antennas or antenna cables used (length, connection



Figure 7.17: Plot of number of satellites available.

quality), and the attitude with which the antennas are mounted to UAV fuselage, can also be factors explaining the differences in number of available satellites seen in Figure 7.17.

The points in the bottom plot where the number of satellites available for DDCP is smaller than the minimum available satellites of both receivers is due to that flags in the receivers indicated invalid carrier phase data or an unresolved half cycle ambiguity, in most cases caused by cycle slips.

7.5 Receiver clock errors

The estimated clock biases and drift rates when all corrections are used are plotted for both receivers in Figure 7.18. The front receiver drifts considerably more than the back receiver, but both have a similar trend in the drift rate, with a small peak near the beginning. These are not caused by transients in the MEKF, as estimating these values using the LS method (explained in Section 5.4.1) for each set of measurements independently, gives the same result. The frequency accuracy of a TCXO is expected to be within ± 1 ppm, equal to a maximum drift rate of 10^{-6} , over the temperature operating range (Vig, 2008). The estimated drift rates of both receivers are within this range, and the difference can simply be a result of manufacturing tolerances. One possible explanation for the transient behavior of the clock errors in the beginning is a sensitivity to thermal transients (Vig, 2008, p. 4-40). The frequency deviation in a TCXOs can depend on the rate of change of the temperature, because the temperature compensation circuit does not necessarily



Figure 7.18: Clock error estimates using all corrections.

handle fast temperature changes well. Both receivers are placed inside the UAV fuselage, and experience similar temperatures. Heating from the electronics inside the fuselage while the UAV is prepared for launch, and a reduction in temperature due to cooling by air flowing over the fuselage surface after launch, could lead to these results. Estimating the clock errors for a different dataset recorded on the same day showed a similar transient in the beginning, indicating that this is a likely explanation.

The difference in measurement time for pairs of measurements from both receivers, $t_f - t_\beta$, is plotted in Figure 7.19. As can be seen about halfway through the dataset, a step of 1ms occurs. This is not caused by a correction of the clock of any of the receivers, as can be seen in Figure 7.18, but is caused by a change in the measurement times according to the front receiver. When the step occurs, measurements change from occuring at x.x01000... s to occuring at x.x02000... s . This is a bit surprising as the U-blox 8/M8 Receiver Descr., 2017 states that the receiver clocks are kept approximately aligned to GPST by applying 1ms corrections to the clocks. Instead it chooses to move the local measurement time, moving the true measurement times closer in alignment to GPST.



Figure 7.19: Difference in measurement time between receivers, $t_f - t_\beta$.

7.6 IMU biases

The gyro and accelerometer biases estimated when using all corrections are plotted in Figure 7.20. The repeating pattern in the biases which match the pattern flown by the UAV indicates that there are residual errors which are treated as part of the biases by the MEKF, as it is clear that the true sensor biases do not act this way. While the accelerometer bias does not have a range specified for it by the manufacturer, the gyro bias does. The datasheet (Sensonor, 2018) states a range of $\pm 250^{\circ}$ /h, which the z-axis gyro bias does not satisfy. This is also visible in Figure 2.1. The STIM300 unit used here, in the flight this thesis is based upon, is an engineering sample offered at a lower cost for research use because it does not necessarily meet all specifications under all operating conditions, which might explain this. A comparison of the x-axis accelerometer bias estimate and the north component of the estimated velocity is shown in Figure 7.21. This seems to indicate that the source of the error is transformed from a frame such as NED to the body-frame, ending up in the bias estimate. An assumption made in the estimator was that the gravity vector g^n was purely vertical as in (2.36). In reality this is a significant simplification, and local variations on Earth can lead to a gravity vector with a horizontal component realtive to the reference ellipsoid. According to the Wolfram Alpha Local Acceleration of Gravity widget (Wolfram Alpha, 2018), which uses a EGM2008 12th order model, the local gravity at the town of Brekstad near the area of the UAV flight is

$$g^n = [-0.02639 - 0.00916 \ 9.85573]^{\top} \left[\frac{\mathrm{m}}{\mathrm{s}^2}\right].$$
 (7.2)

Running the MEKF with this gravity model gives the result shown in Figure 7.22. The repeating patterns seen in both the accelerometer and gyro biases are significantly reduced. This shows the importance of using an accurate gravity model. In this case the aiding sensors helped make these gravity components observable as biases, but if the UAV were to loose GNSS signal coverage the bias errors would quickly lead to drift in the estimated PVA.

-0.2

3500

3000



Figure 7.21: Comparison of estimated north velocity and y-axis accelerometer bias.

Time [s]

2000

2500

1500

1000

Residual errors in the biases, which are still visible as repeating patterns especially in the gyro biases in Figure 7.22, can be errors in the measured antenna lever arms and mounting of the IMU, and errors in timing, such as the assumption of identical measurement times for the pseudorange and Doppler frequency measurements from the two receivers in Section 5.7.2 not being perfectly satisfied.

-20

0

500



Figure 7.22: Estimated gyro and accelerometer biases with improved gravity model.

Part IV

Closing Remarks

Closing Remarks

8

8.1 Conclusion

This thesis presented the derivation and implementation of a multiplicative extended Kalman filter (MEKF) working as an INS aided by a dual-receiver GNSS setup. The MEKF-based INS estimates the position, velocity and attitude of a fixed-wing UAV. Pseudorange, Doppler frequency and double differenced carrier phase measurements were used to initialize the position, velocity and heading estimates of the MEKF, and to correct the errors caused by IMU biases and the errors in the initial attitude.

Necessary background information for the use of an IMU and raw GNSS measurements and some recent work performed in the field of UAV navigation was presented, which served as a basis for deriving the MEKF.

The results show that use of carrier phase interferometry can significantly improve the accuracy of a heading estimate for a UAV at rest. However, for a UAV in flight the benefit of attitude aiding, using dual GNSS and carrier phase is smaller. This is due to the in-air dynamics of the vehicle being sufficient for observability of attitude errors without this setup. This method could be more useful in aircraft flying for long periods in straight lines. It can also be useful for hovering aircraft, i.e. helicopters or multicopters, where the dynamics are insufficiently exciting for observability of the vehicle heading.

8.2 Future work

The algorithm in this thesis was implemented on a desktop computer, using logged measurements to do PVA estimation offline. If this algorithm is to be usable for guidance, navigation and control (GNC) for a UAV in-flight, it should be implemented to run in real-time, using measurements directly from the sensors.

Uncertancies in the mounting of the IMU and the measurement of the antenna lever arms and baseline are error sources in the implemented INS. A method for calibration, finding the correct rotation matrix between the IMU measurement frame and the UAV body frame, and accurate antenna positions, should be developed and implemented.

As seen in the results for the bias estimates, accurate modeling of gravity is important if the INS is to handle periods without aiding. The MEKF can be extended to also estimate the gravity.

For improved position accuracy, the atmospheric models should be replaced by corrections from SBAS services. The algorithm can also be extended to use multiple constellations, and the modernized GPS signals (some of which are already being transmitted by several satellites, although not declared operational (List of positioning satellites, 2018)). Carrier-smoothed pseudorange might give better results than the combination of pseudorange and Doppler frequency measurements. Comparing the two would be interesting.

An extension of the interferometric measurement correction used in the MEKF in this thesis would be to add one or more antennas and receivers to achieve full observability of the attitude even in static conditions. This again would result in a fully observerable accelerometer bias, which is beneficial. Part V

Appendices

GPS Navigation Message Parameters



Table A.1: GPS navigation message parameters. *MSBs **LSBs. The table is an expanded version of the one in Sollie (2017), with the data being compiled from the GPS Interface Specification (2015). When a subframe is received its number can be found in the handover word (HOW), bits 20-22 of word 2. The pages of subframes 4 and 5 are identified by the Data ID (bits 1-2 of word 3) and satellite ID (bits 3-8 of word 3). IODC and IODE are not parameters used in navigation calculations, but are used to ensure that all subframes received contain data from the same parameter set. If IODE in frame 2-5 differs from the 8 LSBs of IODC in frame 1, a data set cutover has occurred and we have to wait for the next frame for new data.

Parameter	Subframe	Words	Bits (number)	Scale factor	Units	Two's
						complement
SV health	1	3	17-22 (6)			
a_{f2}	1	9	1-8 (8)	2-55	s/s ²	\checkmark
a_{f1}	1	9	9-24 (16)	2^{-43}	s/s	\checkmark
a_{f0}	1	10	1-22 (22)	2 ⁻³¹	S	\checkmark
toc	1	8	9-24 (16)	2^{4}	S	
T_{GD}	1	7	17-24 (8)	2^{-31}	S	\checkmark
IODC	1	3*, 8**	23-24*, 1-8** (10)			
$\overline{M_0}$	2	4*, 5**	17-24*, 1-24** (32)	$2^{-31}\pi$	rad	\checkmark
Δn	2	4	1-16 (16)	$2^{-43}\pi$	rad/s	\checkmark
е	2	6*,7**	17-24, 1-24** (32)	2^{-33}		
\sqrt{A}	2	8*, 9**	17-24*, 1-24** (32)	2^{-19}	\sqrt{m}	
C_{uc}	2	6	1-16 (16)	2^{-29}	rad	\checkmark
C_{us}	2	8	1-16 (16)	2^{-29}	rad	\checkmark
C_{rs}	2	3	9-24 (16)	2^{-5}	m	\checkmark
toe	2	10	1-16 (16)	2^{4}	s	\checkmark
IODE	2	3	1-8 (8)			
$\overline{\Omega_0}$	3	3*, 4**	17-24*, 1-24** (32)	$2^{-31}\pi$	rad	\checkmark
i_0	3	5*, 6**	17-24*, 1-24** (32)	$2^{-31}\pi$	rad	\checkmark
ŵ	3	7*, 8**	17-24*, 1-24** (32)	$2^{-31}\pi$	rad	\checkmark
Ω	3	9	1-24 (24)	$2^{-43}\pi$	rad/s	\checkmark
i	3	10	9-22 (14)	$2^{-43}\pi$	rad/s	\checkmark
C_{rc}	3	7	1-16 (16)	2^{-5}	m	\checkmark
C_{ic}	3	3	1-16 (16)	2^{-29}	rad	\checkmark
C_{is}	3	5	1-16 (16)	2^{-29}	rad	\checkmark
IODE	3	10	1-8 (8)			
α_0	4 (page 18)	3	9-16 (8)	2 ⁻³⁰	s	\checkmark
α_1	4 (page 18)	3	17-24 (8)	2^{-27}	s/semi-circle	\checkmark
α_2	4 (page 18)	4	1-8 (8)	2^{-24}	s/semi-circle ²	\checkmark
α_3	4 (page 18)	4	9-16 (8)	2^{-24}	s/semi-circle ³	\checkmark
β_0	4 (page 18)	4	17-24 (8)	2^{11}	S	\checkmark
β_1	4 (page 18)	5	1-8 (8)	2^{14}	s/semi-circle	\checkmark
β_2	4 (page 18)	5	9-16 (8)	2^{16}	s/semi-circle ²	\checkmark
β ₃	4 (page 18)	5	17-24 (8)	2 ¹⁶	s/semi-circle ³	\checkmark

Reading the GPS Navigation Message Data

B.1 Extracting the GPS clock correction and ephemeris parameters from the binary navigation message

This explanation is taken from Sollie (2017).

Knowing the location of the required parameters from Appendix A, we can extract each of them by using the following approach for each subframe as they are received:

Notation: left bitshift operator «, right bitshift operator », binary AND operator &, word number i = w_i

- 1. Get the subframe number as $(w_2 \gg 8)$ & 0x7F. If subframe 1 is received IODC should be stored for use when the other frames are received later. If receiving subframes 2-5, check that IODE is equal to the 8 MSBs of IODC. If they are not, abort and wait for the next time subframe 1 is received.
- 2. If all bits are in a single word, in general bits with index *a* to *b*: get the bits as $(w \gg 30 b) \& (2^{b-a+1} 1)$
- 3. If the data is split in multiple words, bits a_{MSB} to b_{MSB} and a_{LSB} to b_{LSB} : read each part as above, then combine by using (MSB $\ll (b_{LSB}-a_{LSB}+1))$ & LSB
- 4. If the data is signed, using two's complement, we have two cases depending on the supported data
 - a) If the data has length directly supported on the system (8, 16 and 32 bit for Matlab), typecast/reinterpret cast it directly to signed integer
 - b) If the values has another length not directly supported on the system, like 14 or 24 bits, use sign bit extension to a directly supported size before casting.
- 5. If the parameter is real valued, cast it from integer to floating point
- 6. Scale the value if required

Data example

Table B.1: Raw data example: this is a navigation message frame from SV 32. The first row for each subframe contains words 1-5, and the second row words 6-10. Each word is 30 bits long. Subframe 4 in this message contains page 18. Subframe 5 is not used in this thesis and has been omitted.

Subframe			Data		
1	583249946	2560862528	990118131	355496234	2898942025
	2250544257	2675163324	5432742	2151675019	795633500
2	583249946	2560871120	4226612	207154234	2356540217
	2155872307	778487565	2221926472	59150479	317038344
3	583249946	2560879448	1073472966	2653755253	3220834771
	31014352	114025968	370611996	1072322635	8214883
4	583249946	2560888000	503545989	3221195966	2168422146
	3221225185	3221225450	3218664463	77742543	75497712

Table B.2: Extracted parameters from the data in table B.1.

Parameter	Value	Parameter	Value
a_{f2}	0s/s ²	C_{uc}	9.5367×10^{-7} rad
a_{f1}	-5.2296×10^{-12} s/s	C_{us}	8.4620×10^{-6} rad
a_{f0}	-5.0588×10^{-4} s	C_{rs}	15.75m
t _{oc}	309600s	C_{rc}	217.4688m
t_{oe}	309600s	C_{ic}	-3.1665×10^{-8} rad
T_{GD}	$9.3132 \times 10^{-10} s$	C_{is}	-4.4703×10^{-8} rad
M_0	–1.9587rad	Ω_0	–2.5655rad
Δn	$4.5155 imes 10^{-09}$ rad/s	i_0	0.9579rad
е	0.0014	ω	–2.5686rad
\sqrt{A}	$5.1538 \times 10^3 \sqrt{m}$	Ω	-7.9200×10^{-9} rad/s
i	-2.4251×10^{-10} rad/s	SV health	0 (all OK)
α_0	1.3039×10^{-8} s	β_0	102400s
α_1	$1.4901 \times 10^{-8} \frac{s}{semi-circle}$	β_1	65536 <u>semi-circle</u>
α_2	$-5.9605 \times 10^{-8} \frac{10^{-8} \text{ semi-circle}^2}{\text{semi-circle}^2}$	β_2	$-196608 \frac{s}{semi-circle^2}$
α ₃	$-1.1921 \times 10^{-7} \frac{\text{schurchere}}{\text{semi-circle}^3}$	β_3	$-262144 \frac{\text{schilds}^{\text{schilds}}}{\text{semi-circle}^3}$

GPS Satellite Position Calculation Algorithm

The following algorithm for calculation of the satellite positions in found in the GPS Interface Specification (2015). The exact constants used by the ground segment are $\pi = 3.1415926535898$, $c = 299792458 \frac{\text{m}}{\text{s}}$, $\mu = 3.986005 \times 10^{14} \frac{\text{m}^3}{\text{s}^2}$ and $\dot{\Omega}_e = 7.2921151467 \times 10^{-5} \frac{\text{rad}}{\text{s}}$, which should then also be used in user calculations.

$$A = \sqrt{A}^2 \tag{C.1}$$

$$n_0 = \sqrt{\frac{\mu}{A^3}} \tag{C.2}$$

$$t_k = t - t_{oe} \tag{C.3}$$

$$n = n_0 + \Delta n \tag{C.4}$$

$$M_k = M_0 + nt_k \tag{C.5}$$

$$M_k = E_k - e\sin E_k \tag{C.6}$$

$$\Phi_k = \nu_k + \omega \tag{C.7}$$

$$\delta u_k = C_{us} \sin 2\Phi_k + C_{uc} \cos 2\Phi_k \tag{C.8}$$

$$\delta r_k = C_{rs} \sin 2\Phi_k + C_{rc} \cos 2\Phi_k \tag{C.9}$$

$$\delta i_k = C_{is} \sin 2\Phi_k + C_{ic} \cos 2\Phi_k \tag{C.10}$$

$$u_k = \Phi_k + \delta u_k \tag{C.11}$$

$$r_k = A(1 - e\cos E_k) + \delta r_k \tag{C.12}$$

$$i_k = i_0 = \delta i_k + (IDOT)t_k \tag{C.13}$$

$$x'_k = r_k \cos u_k \tag{C.14}$$

$$y'_k = r_k \sin u_k \tag{C.15}$$

$$\Omega_k = \Omega_0 + (\dot{\Omega} - \dot{\Omega}_e)t_k - \dot{\Omega}_e t_{oe}$$
(C.16)

$$x_k = x'_k \cos \Omega_k - u'_k \cos i_k \sin \Omega_k \tag{C.18}$$

$$y_k = x'_k \sin \Omega_k + y'_k \cos i_k \cos \Omega_k \tag{C.19}$$

$$z_k = y'_k \sin i_k \tag{C.20}$$

Reference frames





Figure D.1: NED and ECEF frames

D.1 ECEF

The Earth-Centered Earth-Fixed (ECEF) frame, $\{e\}$, illustrated in Figure D.1, has its origin as the center of mass of the Earth, with its z^e -axis pointing towards the North Pole and the x^e -axis pointing towards the intersection of the Prime Meridian and the Equator. The y^e -axis completes a right-handed coordinate frame.

D.2 NED

The North-East-Down (NED), $\{n\}$, frame is a local frame with the z-axis pointing into the Earth, normal to the reference ellipsoid. The x-axis is tangent to the reference ellipsoid and points in the direction of true north, while the y-axis points towards east. The frame is singular at the poles. The frame is illustrated in Figure D.1, but note that z-axis does not actually point directly at the center of mass of the Earth unless the NED frame origin is at the Equator or either pole.

D.3 ECI

Like the ECEF frame, the Earth-Centered Inertial (ECI), $\{i\}$, frame has its origin at the center of mass of the Earth and z^i -axis pointing towards the North Pole. The frame does however not rotate with the Earth, and the x^i -axis points towards the intersection of the Equator and the ecliptic, which is the orbital plane of the Earth around the Sun, in the direction of the vernal equinox. The y^i -axis completes a right-handed coordinate frame.

D.4 Body

The body-frame, $\{b\}$, shown in Figure D.2, has the coordinate center chosen at the location of the IMU in the UAV fuselage. The axes are fixed relative to the airframe. The x^b -axis points forward, the y^b -axis points to the right and the z^b -axis points down.



Figure D.2: BODY frame

D.5 Sensor

Because the sensor does no have to be mounted aligned with the axes of the bodyframe, it has its own frame, $\{s\}$. The STIM300 IMU used in this thesis has the axes shown in Figure D.3.

The rotation matrix relating the STIM300 mounted in the UAV and the bodyframe are related with the rotation matrix

$$\mathbf{R}_{m}^{b} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$
 (D.1)



Figure D.3: Measurement frame of STIM300. The figure is taken from the STIM300 datasheet (Sensonor, 2018).

E

Attitude Parametrizations and related math

E.1 Euler angles

The Euler angles is a very intuitive attitude representation, but is not free of singularities. Many conventions exist for the order of the rotations, but perhaps the most popular today, and the one used here, is the ZYX convention where rotations are performed in the order given by (E.15),

$$\boldsymbol{\Theta}_{nb} = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix}$$
(E.1)

where ϕ , θ and ψ are the roll, pitch and yaw (heading) angles (Fossen, 2011).

E.2 Rotation matrix

A rotation matrix is a 3 × 3 matrix satisfying (Egeland and Gravdahl, 2002)

$$\mathbf{R}^{\mathsf{T}}\mathbf{R} = \mathbf{R}\mathbf{R}^{\mathsf{T}} = \mathbf{I}_{3\times3} \tag{E.2}$$

$$det(\mathbf{R}) = 1. \tag{E.3}$$

The rotations around the coordinate axes, called the simple rotations, are defined as

$$\mathbf{R}_{x}(\phi) = \begin{bmatrix} 1 & 0 & 0\\ 0 & \cos\phi & -\sin\phi\\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$
(E.4)

$$\mathbf{R}_{y}(\theta) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$
(E.5)

$$\mathbf{R}_{z}(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0\\ \sin\psi & \cos\psi & 0\\ 0 & 0 & 1 \end{bmatrix}.$$
 (E.6)

E.3 Quaternion

The unit quaternion is a vector in four-dimentional Euclidean space E4, constrained to lie on the surface of the unit sphere S3, thus having length 1. Quaternions are globally nonsingular, but provide double coverage of the rotation group with q and

-q representing the same rotation (Egeland and Gravdahl, 2002). A quaternion can be written as the complex number

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$$q = \eta + \epsilon_1 i + \epsilon_2 j + \epsilon_3 k, \tag{E.7}$$

written in vector form as

$$\boldsymbol{q} = \begin{bmatrix} \boldsymbol{\eta} \\ \boldsymbol{\epsilon} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\eta} \\ \boldsymbol{\epsilon}_1 \\ \boldsymbol{\epsilon}_2 \\ \boldsymbol{\epsilon}_3 \end{bmatrix}.$$
(E.8)

The sum of two quaternions is the sum of each component,

$$\boldsymbol{p} \pm \boldsymbol{q} = \begin{bmatrix} p_{\eta} \\ \boldsymbol{p}_{\varepsilon} \end{bmatrix} \pm \begin{bmatrix} q_{\eta} \\ \boldsymbol{q}_{\varepsilon} \end{bmatrix} = \begin{bmatrix} p_{\eta} \pm q_{\eta} \\ \boldsymbol{p}_{\varepsilon} \pm \boldsymbol{q}_{\varepsilon} \end{bmatrix}, \quad (E.9)$$

but this is not used much for unit quaternions as it violates the unit norm. The combined rotation of two quaternions is given by the quaternion product

$$\boldsymbol{p} \otimes \boldsymbol{q} = \begin{bmatrix} p_{\eta}q_{\eta} - \boldsymbol{p}_{\epsilon}^{\mathsf{T}}\boldsymbol{q}_{\epsilon} \\ p_{\eta}\boldsymbol{q}_{\epsilon} + q_{\eta}\boldsymbol{p}_{\epsilon} + \boldsymbol{p}_{\epsilon} \times \boldsymbol{q}_{\epsilon} \end{bmatrix} = \begin{bmatrix} p_{\eta}q_{\eta} - p_{\epsilon_{1}}q_{\epsilon_{1}} - p_{\epsilon_{2}}q_{\epsilon_{2}} - p_{\epsilon_{3}}q_{\epsilon_{3}} \\ p_{\eta}q_{\epsilon_{1}} + p_{\epsilon_{1}}q_{\eta} + p_{\epsilon_{2}}q_{\epsilon_{3}} - p_{\epsilon_{3}}q_{\epsilon_{2}} \\ p_{\eta}q_{\epsilon_{2}} - p_{\epsilon_{1}}q_{\epsilon_{3}} + p_{\epsilon_{2}}q_{\eta} + p_{\epsilon_{3}}q_{\epsilon_{1}} \\ p_{\eta}q_{\epsilon_{3}} + p_{\epsilon_{1}}q_{\epsilon_{2}} - p_{\epsilon_{2}}q_{\epsilon_{1}} + p_{\epsilon_{3}}q_{\eta} \end{bmatrix}, \quad (E.10)$$

where the unit constrained is maintained. In this thesis the notation

$$\boldsymbol{p} \otimes \boldsymbol{\omega} = \boldsymbol{p} \otimes \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{\omega} \end{bmatrix}$$
(E.11)

is used for products of quaternions and three dimensional vectors, where $\omega \in \mathbb{R}^3$. Vectors can be rotated using quaternions as

$$\mathbf{R}_{a}^{b}\boldsymbol{v}^{a} = \boldsymbol{q}_{a}^{b} \otimes \boldsymbol{v} \otimes (\boldsymbol{q}_{a}^{b})^{*}, \qquad (E.12)$$

where q^* is the quaternion conjugate $q^* = \begin{bmatrix} \eta \\ -\epsilon \end{bmatrix}$. A rotation vector θ describing the rotation of an angle $\|\theta\|$ around the axis $\frac{\theta}{\|\theta\|}$ can be converted to a quaternion as (Egeland and Gravdahl, 2002)

$$q(\boldsymbol{\theta}) = \begin{bmatrix} \cos(\frac{\|\boldsymbol{\theta}\|}{2}) \\ \frac{\boldsymbol{\theta}}{\|\boldsymbol{\theta}\|} \sin(\frac{\|\boldsymbol{\theta}\|}{2}) \end{bmatrix}.$$
 (E.13)

E.4 Conversions

Quaternion to rotation matrix:

$$\mathbf{R}(q) = \begin{bmatrix} q_{\eta}^{2} + q_{\epsilon_{1}}^{2} - q_{\epsilon_{2}}^{2} - q_{\epsilon_{3}}^{2} & 2(q_{\epsilon_{1}}q_{\epsilon_{2}} - q_{\eta}q_{\epsilon_{3}}) & 2(q_{\epsilon_{1}}q_{\epsilon_{3}} + q_{\eta}q_{\epsilon_{2}}) \\ 2(q_{\epsilon_{1}}q_{\epsilon_{2}} + q_{\eta}q_{\epsilon_{3}}) & q_{\eta}^{2} - q_{\epsilon_{1}}^{2} + q_{\epsilon_{2}}^{2} - q_{\epsilon_{3}}^{2} & 2(q_{\epsilon_{2}}q_{\epsilon_{3}} - q_{\eta}q_{\epsilon_{1}}) \\ 2(q_{\epsilon_{1}}q_{\epsilon_{3}} - q_{\eta}q_{\epsilon_{2}}) & 2(q_{\epsilon_{2}}q_{\epsilon_{3}} + q_{\eta}q_{\epsilon_{1}}) & q_{\eta}^{2} - q_{\epsilon_{1}}^{2} - q_{\epsilon_{2}}^{2} + q_{\epsilon_{3}}^{2} \end{bmatrix}$$
(E.14)

Euler angles to rotation matrix (Egeland and Gravdahl, 2002):

$$\mathbf{R}_{b}^{n}(\mathbf{\Theta}_{nb}) = \mathbf{R}_{z}(\psi)\mathbf{R}_{y}(\theta)\mathbf{R}_{x}(\phi) = \begin{bmatrix} c\psi c\theta & -s\psi c\phi + c\psi s\theta s\phi & s\psi s\phi + c\psi c\phi s\theta \\ s\psi c\theta & c\psi c\phi + s\phi s\theta s\psi & -c\psi s\phi + s\theta s\psi c\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix}$$
(E.15)

Euler angles to quaternion:

$$\boldsymbol{q}_{b}^{n}(\boldsymbol{\Theta}_{nb}) = \begin{bmatrix} \cos \psi/2 \\ 0 \\ 0 \\ \sin \psi/2 \end{bmatrix} \begin{bmatrix} \cos \theta/2 \\ 0 \\ \sin \theta/2 \\ 0 \end{bmatrix} \begin{bmatrix} \cos \phi/2 \\ \sin \phi/2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos \phi/2 \cos \theta/2 \cos \psi/2 + \sin \phi/2 \sin \theta/2 \sin \psi/2 \\ \sin \phi/2 \cos \theta/2 \cos \psi/2 - \cos \phi/2 \sin \theta/2 \sin \psi/2 \\ \cos \phi/2 \sin \theta/2 \cos \psi/2 + \sin \phi/2 \cos \theta/2 \sin \psi/2 \\ \cos \phi/2 \cos \theta/2 \sin \psi/2 - \sin \phi/2 \sin \theta/2 \cos \psi/2 \\ \end{bmatrix}$$
(E.16)

E.5 Skew-symmetric matrices

The skew symmetric matrix form of a vector *a* is defined as (Fossen, 2011)

$$\mathbf{S}(a) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix},$$
 (E.17)

and is also called the cross product matrix because the following holds

$$a \times b = \mathbf{S}(a)b = -b \times a = -\mathbf{S}(b)a. \tag{E.18}$$

A property of a skew symmetric matrices is that its transpose is equal to a change in sign of all elements,

$$\mathbf{S}(a) = -\mathbf{S}^{\top}(a). \tag{E.19}$$

Klobuchar ionospheric delay model

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The parameters included in the GPS navigation message used by the model are: α_0 , α_1 , α_2 , α_3 , β_0 , β_1 , β_2 , β_3 . For each tracked satellite the receiver calculates the satellite elevation, E, and the azimuth, A. The user latitude and longitude, ϕ_u and λ_u , and the GPS time are also used. These can be found with sufficient accuracy initially without atmospheric models. The estimated ionospheric delay is then found as follows:

$$F = 1.0 + 16.0(0.53 - E)^3$$
(F.1)

$$\psi = \frac{0.0137}{E + 0.11} - 0.022 \tag{F.2}$$

$$\phi_i = \phi_u + \psi \cos A \tag{F.3}$$

if this gives
$$|\phi_i| \le 0.416$$
. (F.4)

If
$$\phi_i > 0.416$$
: $\phi_i = 0.416$, if $\phi_i < -0.416$: $\phi_i = -0.416$ (F.5)

$$\lambda_i = \lambda_u + \frac{\psi \sin A}{\cos \phi_i} \tag{F.6}$$

$$t = 4.32 \times 10^4 \lambda_i + \text{GPS time} \tag{F.7}$$

$$\phi_m = \phi_i + 0.064 \cos(\lambda_i - 1.617) \tag{F.8}$$

$$PER = \max(\beta_0 + \beta_1 \phi_m + \beta_2 \phi_m^2 + \beta_3 \phi_m^3, 72000)$$
(F.9)

$$x = \frac{2\pi(t - 50400)}{PER}$$
(F.10)

$$AMP = \max(\alpha_0 + \alpha_1 \phi_m + \alpha_2 \phi_m^2 + \alpha_3 \phi_m^3, 0)$$
(F.11)

$$T_{iono} = \begin{cases} F(5 \times 10^{-9} + AMP(1 - \frac{x^2}{2} + \frac{x^4}{24})), & \text{if } |x| < 1.57\\ F \cdot 5 \times 10^{-9}, & \text{otherwise} \end{cases}$$
(F.12)

G

STIM300 Allan variance plots

These plots are taken from the STIM300 datasheet (Sensonor, 2018).



Figure G.1: Allan deviation plot for the gyroscope of the STIM300 IMU



Figure G.2: Allan deviation plot for the accelerometer of the STIM300 IMU

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