NTNU
Norwegian University of
Science and Technology

Classifying Subcategories of Triangulated Categories

Johanne Haugland

Master of Science in Mathematical Sciences
Submission date: June 2018
Supervisor: Petter Andreas Bergh, IMF

Norwegian University of Science and Technology
Department of Mathematical Sciences

Abstract

The topic of this thesis is classification of subcategories of triangulated categories. We first state and prove the Hopkins-Neeman theorem, which gives a bijection between thick subcategories of the derived category of perfect complexes over a commutative noetherian ring and specialization closed subsets of the prime ideal spectrum. Next, we present Benson, Iyengar and Krause's approach to classification problems, which involves using a central ring action on a compactly generated triangulated category to define local cohomology functors. If the stratification conditions are satisfied, the notion of triangulated support yields classification of both thick and localizing subcategories. Finally, we use the BIK-approach to investigate the case of a quantum polynomial ring A in two variables. We show that a nice commutative subring of A acts centrally on $\mathbf{D}(A)$. In order to figure out if this action satisfies the stratification conditions, we consider the representation theory of certain quotients of A. The situation turns out to be more complicated than in the commutative setting, and we conclude that the central ring action satisfies the local-global principle, but not the minimality condition.

Sammendrag

Temaet for denne oppgaven er klassifisering av underkategorier av triangulerte kategorier. Først formulerer og beviser vi Hopkins-Neeman-teoremet, som gir en bijeksjon mellom tykke underkategorier av den deriverte kategorien av perfekte komplekser over en kommutativ noethersk ring og spesialiseringslukkede delmengder av primidealspekteret. Videre presenterer vi Benson, Iyengar og Krause sin tilnærming til klassifiseringsproblemer, hvilket innebærer å bruke en sentral ringvirkning på en kompaktgenerert triangulert kategori for å definere lokale kohomologifunktorer. Hvis stratifiseringsbetingelsene er tilfredsstilt, vil man gjennom begrepet triangulert støtte oppnå klassifisering av både tykke og lokaliserende underkategorier. Til slutt bruker vi BIK-tilnærmingen for å studere en kvantepolynomring A i to variabler. Vi viser at en fin kommutativ underring av A virker sentralt på $\mathbf{D}(A)$. For å finne ut om denne virkningen tilfredsstiller stratifiseringsbetingelsene, undersøker vi representasjonsteorien for visse kvotienter av A. Situasjonen viser seg å være mer komplisert enn i det kommutative tilfellet, og vi konkluderer at den sentrale ringvirkningen tilfredsstiller lokal-global-prinsippet, men ikke minimalitetsbetingelsen.

Acknowledgements

At this point, there are several persons who deserve to be mentioned. First and foremost, I want to thank my supervisor Professor Petter Andreas Bergh for all his help and support. He has always found time to discuss my questions, regardless of how trivial or non-trivial they turned out to be. Moreover, I wish to thank him for suggesting the topic of my thesis, for careful proof reading, and also for encouraging me to pursue a PhD in mathematics.

I next wish to extend my thanks to Professor Srikanth B. Iyengar at the University of Utah, whom I visited for three weeks in October and November 2017. My stay was very inspiring, and I really appreciate the way Professor Iyengar was able to meet me regularly. Our discussions were very helpful for the work on this thesis. I would also like to thank the staff at the university for their hospitality, as well as the Norwegian Mathematical Society for awarding me an Abel Scholarship which funded my travel and accommodation.

I want to thank my fellow students for creating an excellent both social and academic atmosphere. A particular thanks to Erlend Børve, Joakim Fremstad and Eivind Hjelle for countless interesting mathematical discussions during the last four years.

Finally, I would like to express my gratitude to Bjørn and my family. Your support has been of great help.

Johanne Haugland
Trondheim, May 2018

Preface

This thesis is the final part of my work to achieve a master's degree in mathematics at the Norwegian University of Science and Technology. It has been written during the academic year 2017-2018 under supervision of Professor Petter Andreas Bergh. Parts of the thesis have also been inspired by talks with Professor Srikanth B. Iyengar at the University of Utah, whom I visited for three weeks in October and November 2017.

We will discuss classification of subcategories of triangulated categories from both an original and a modern point of view. For a commutative noetherian ring R, we present results which allow us to understand thick subcategories of $\mathbf{D}^{\mathrm{b}}(\operatorname{proj} R)$ and localizing subcategories of $\mathbf{D}(R)$ in terms of the prime ideal spectrum $\operatorname{Spec} R$. This type of classification theory originates from a result of Hopkins from 1987 [10], where he gives a bijection between thick subcategories of $\mathbf{D}^{\mathrm{b}}(\operatorname{proj} R)$ and specialization closed subsets of $\operatorname{Spec} R$, as well as Neeman's clarification and extension of this result from 1992 [15]. In addition to giving an overview of established classification methods, the motivation for this thesis has been to investigate if it is possible to generalize classification results from the context of commutative noetherian rings to a quantum polynomial ring in two variables.

Throughout the thesis, we will assume the reader to have basic knowledge about homological algebra, commutative algebra and representation theory, corresponding to material covered in courses such as MA3204 Homological algebra, MA8202 Commutative algebra and MA3203 Ring theory at NTNU. The reader should also be familiar with general results for triangulated categories, for instance through [9]. We do not assume any knowledge about classification theory for thick or localizing subcategories of triangulated categories.

Let us give a short description of how the content of the thesis is organized. We will start with an overview of the conventions and notation which are used throughout the thesis. After this, the work is divided into three chapters.

Chapter 1 is the historical background for the rest of the thesis. We will here give necessary definitions and present our first notion of support. This enables us to describe a map between the collection of thick subcategories of $\mathbf{D}^{\mathrm{b}}(\operatorname{proj} R)$ and that of specialization closed subsets of $\operatorname{Spec} R$. Our main goal in Chapter 1 is to prove that this map is a bijection – a result which is known as the Hopkins-Neeman theorem.

In **Chapter 2** we give an overview of modern classification methods which have been established by Benson, Iyengar and Krause. The main idea in this approach is to use a central ring action on a compactly generated triangulated category to define local cohomology functors, which again enable us to discuss a notion of triangulated support. If the central ring action satisfies the crucial stratification conditions, we obtain classification of both thick and localizing subcategories. This chapter will primarily be based on [3] and [6].

In **Chapter 3** we investigate the case of a quantum polynomial ring A in two variables. We show that a nice commutative subring of A acts centrally on $\mathbf{D}(A)$. In order to figure out if this action satisfies the stratification conditions, we consider the representation theory of certain quotients of A. The situation turns out to be more complicated than in the commutative setting, and we conclude that the central ring action satisfies the first, but not the second stratification condition.

At the end of the thesis is an appendix which contains a list of translations of some of the mathematical terms one will encounter. Norwegian readers are strongly encouraged to look up translations for words they are not yet familiar with.

Contents

Conventions and notation

Throughout this thesis, we will use the following conventions:

- All rings have a multiplicative identity.
- Given a ring R, we let $\operatorname{Mod} R$ denote the category of left R-modules and $\operatorname{mod} R$ the category of finitely generated left R-modules. $\mathbf{K}(R)$ will be the homotopy category of chain complexes of left R-modules, and $\mathbf{D}(R)$ the derived category of $\operatorname{Mod} R$.
- $\mathbf{D}^{\mathrm{b}}(\operatorname{mod} R)$ is the full isomorphism-closed subcategory of $\mathbf{D}(R)$ given by bounded complexes of finitely generated left R-modules.
- $\mathbf{D}^{\mathrm{b}}(\operatorname{proj} R)$ is the subcategory of perfect complexes in $\mathbf{D}(R)$, i.e. the full isomorphism-closed subcategory given by bounded complexes of finitely generated projective left R-modules. The corresponding category in $\mathbf{K}(R)$ is denoted by $\mathbf{K}^{\mathrm{b}}(\operatorname{proj} R)$.
- All complexes

  $$\cdots \longrightarrow X^{n-1} \xrightarrow{d^{n-1}} X^{n} \xrightarrow{d^{n}} X^{n+1} \longrightarrow \cdots$$

  will be cohomologically graded, i.e. with increasing degree.
- Whenever we deal with a triangulated category, its translation functor will be denoted by Σ unless otherwise specified.
- We associate a module M to the stalk complex with M in degree zero. Both the module and the stalk complex will be denoted by M.

Chapter 1

The Hopkins-Neeman theorem

Classification of various – in particular thick – subcategories of triangulated categories is a research area that was initiated by Hopkins in 1987. In [10] he gave a classification of thick subcategories of the derived category of perfect complexes over a commutative ring in terms of specialization closed subsets of the prime ideal spectrum. Neeman [15] later discovered that we need one more assumption for Hopkins' result to be true, namely that the ring has to be noetherian, and he also gave a new proof. Hence, the theorem which is the main topic for this chapter, is contributed to both Hopkins and Neeman. This classification result is the historical background for the rest of the thesis.

We will give a proof of the Hopkins-Neeman theorem by applying techniques which are similar to those used in the original proofs.

1.1 Definitions and preliminary results

Before we can present the statement of the theorem, we need some definitions.

Definition 1.1.1. Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ a full subcategory of $\mathcal{T}$. We say that $\mathcal{S}$ is a *triangulated subcategory* if the following conditions hold:

(1) $\mathcal{S}$ is non-empty.

(2) $X \in \mathcal{S} \implies \Sigma^n X \in \mathcal{S}$ for all $n \in \mathbb{Z}$.

(3) For every distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

in $\mathcal{T}$ with X and Y in $\mathcal{S}$, the object Z is also in $\mathcal{S}$.

The rotation axiom together with (2) and (3) implies that we have the 2/3-property: Given a distinguished triangle as in (3) where two objects from the set $\{X, Y, Z\}$ are in $\mathcal{S}$, also the last object is in $\mathcal{S}$.

Note that triangulated subcategories are necessarily additive and closed under isomorphisms. As one would expect, a triangulated subcategory is in itself a triangulated category, where the distinguished triangles and the translation functor are inherited from the ambient category.

We will restrict our attention to certain classes of triangulated subcategories.

Definition 1.1.2. Let $\mathcal{T}$ be a triangulated category. A triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$ is called *thick* if it is closed under direct summands, i.e. if

$$X \oplus Y \in \mathcal{S} \implies X \in \mathcal{S} \text{ and } Y \in \mathcal{S}.$$

One reason why thick subcategories are important, is that they arise very naturally. For instance, one can easily show that the kernel of a triangulated functor is a thick subcategory. In some sense, thick subcategories of triangulated categories correspond to normal subgroups in group theory.

We denote the thick subcategory generated by an object X by $\operatorname{Thick}_{\mathcal{T}}(X)$. This is the smallest thick subcategory of $\mathcal{T}$ where X is an object, and can be described as the intersection of all thick subcategories containing X. Whenever it is obvious in which category the thick subcategory is generated, we will omit the lower index.

In Chapter 2 we show that the subcategory of perfect complexes in $\mathbf{D}(R)$ is precisely the thick subcategory generated by the ring, i.e. that $\mathbf{D}^{\mathrm{b}}(\operatorname{proj} R) = \operatorname{Thick}(R)$. This is Theorem 2.2.8. We will use this result already in Chapter 1.

The Hopkins-Neeman theorem classifies thick subcategories of $\mathbf{D}^{\mathrm{b}}(\operatorname{proj} R)$ for a commutative noetherian ring R in terms of certain subsets of the prime ideal spectrum of R.

Definition 1.1.3. Let R be a commutative ring and $\operatorname{Spec} R$ the collection of prime ideals in R. A subset $V \subseteq \operatorname{Spec} R$ is called *specialization closed* if

$$[\mathfrak{p} \in V,\ \mathfrak{p} \subseteq \mathfrak{q}] \implies \mathfrak{q} \in V, \quad \text{for } \mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} R.$$

The notion of a specialization closed subset could be defined more generally. A subset of a topological space is specialization closed if it contains the closure of each of its points. Hence, being specialization closed is a generalization of being closed, as a specialization closed subset is a union, not necessarily finite, of closed subsets. This agrees with the definition above, considering $\operatorname{Spec} R$ as a topological space with the Zariski topology.

In order to give a bijection between the thick subcategories of $\mathbf{D}^{\mathrm{b}}(\operatorname{proj} R)$ and the specialization closed subsets of $\operatorname{Spec} R$, we need the notion of support. In the following definition, the localization functor with respect to a prime ideal $\mathfrak{p}$ is denoted by $(-)_{\mathfrak{p}}$, and $H^*(X) = \oplus_{n\in\mathbb{Z}} H^n(X)$ is the total cohomology of a complex X.

Definition 1.1.4. Let R be a commutative ring.

(1) For $M \in \operatorname{Mod} R$, the support of M is given by

$$\operatorname{Supp} M = \{\mathfrak{p} \in \operatorname{Spec} R \mid M_{\mathfrak{p}} \neq 0\}.$$

(2) For $X \in \mathbf{D}(R)$, the support of X is given by

$$\begin{aligned}\operatorname{Supp} X &= \{\mathfrak{p} \in \operatorname{Spec} R \mid X_{\mathfrak{p}} \neq 0 \text{ in } \mathbf{D}(R)\}\\ &= \{\mathfrak{p} \in \operatorname{Spec} R \mid X_{\mathfrak{p}} \text{ is not exact}\}\\ &= \{\mathfrak{p} \in \operatorname{Spec} R \mid H^*(X_{\mathfrak{p}}) \neq 0\}.\end{aligned}$$

Let us look at some basic properties of support. Recall that the closed sets in the Zariski topology on $\operatorname{Spec} R$ are given by subsets of the form

$$\mathcal{V}(I) = \{\mathfrak{p} \in \operatorname{Spec} R \mid I \subseteq \mathfrak{p}\},$$

where $I \subseteq R$ is an ideal.

Lemma 1.1.5. *The following statements hold for a commutative noetherian ring R:*

(1) $\operatorname{Supp} X \oplus Y = \operatorname{Supp} X \cup \operatorname{Supp} Y$ *for* $X, Y \in \mathbf{D}(R)$.

(2) $\operatorname{Supp} X = \operatorname{Supp} H^*(X) = \mathcal{V}(\operatorname{ann}_R(H^*(X)))$ *for* $X \in \mathbf{D}^{\mathrm{b}}(\operatorname{mod} R)$.

(3) $\operatorname{Supp} X = \varnothing \iff H^*(X) = 0$ *for* $X \in \mathbf{D}(R)$.

(4) Given a short exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

of R-modules, one has $\operatorname{Supp} M = \operatorname{Supp} M' \cup \operatorname{Supp} M''$.

(5) Given a distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow \Sigma X$$

in $\mathbf{D}(R)$*, one has* $\operatorname{Supp} Y \subseteq \operatorname{Supp} X \cup \operatorname{Supp} Z$.

Proof. (1) follows from the fact that localization is an additive functor.

For (2), notice that $H^*(X_\mathfrak{p}) = H^*(X)_\mathfrak{p}$, since localization is exact. This gives the first equality. Recall that $M_\mathfrak{p} \neq 0 \iff \operatorname{ann}_R(M) \subseteq \mathfrak{p}$ for $M \in \operatorname{mod} R$. This applies to our situation as R is noetherian and $X \in \mathbf{D}^{\mathrm{b}}(\operatorname{mod} R)$, which yields the second inequality.

For (3), assume $\operatorname{Supp} X = \varnothing$. Notice that the first equality in (2) holds for any $X \in \mathbf{D}(R)$, so $H^*(X)_\mathfrak{p} = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$. Being zero is a local property, by [1, Prop. 3.8], so $H^*(X) = 0$. The reverse implication is obvious.

The short exact sequence

$$0 \longrightarrow M'_\mathfrak{p} \longrightarrow M_\mathfrak{p} \longrightarrow M''_\mathfrak{p} \longrightarrow 0$$

gives $\operatorname{Supp} M = \operatorname{Supp} M' \cup \operatorname{Supp} M''$, which shows (4).

An exact functor induces a triangulated functor on the derived category. Hence,

$$X_\mathfrak{p} \longrightarrow Y_\mathfrak{p} \longrightarrow Z_\mathfrak{p} \longrightarrow \Sigma X_\mathfrak{p}$$

is a distinguished triangle. If $Y_\mathfrak{p} \neq 0$, we must also have $X_\mathfrak{p} \neq 0$ or $Z_\mathfrak{p} \neq 0$, so $\operatorname{Supp} Y \subseteq \operatorname{Supp} X \cup \operatorname{Supp} Z$. This proves (5). □

Later in the thesis, we will need the following definition.

Definition 1.1.6. Let R be a commutative ring, M an R-module and $\mathfrak{p} \in \operatorname{Spec} R$.

(1) M is $\mathfrak{p}$-*local* if the natural morphism $M \to M_\mathfrak{p}$ is an isomorphism.

(2) M is $\mathfrak{p}$-*torsion* if every element of M is annihilated by a power of $\mathfrak{p}$.

These properties can be described in terms of support as in the following proposition. For a proof, combine Lemma 2.2 and Lemma 2.4 from [3].

Proposition 1.1.7. *Let R be a commutative noetherian ring, M an R-module and $\mathfrak{p} \in \operatorname{Spec} R$. The following properties hold:*

(1) M is $\mathfrak{p}$-local $\Longleftarrow$ $\operatorname{Supp} M \subseteq \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \subseteq \mathfrak{p}\}$.

(2) M is $\mathfrak{p}$-torsion $\iff$ $\operatorname{Supp} M \subseteq \mathcal{V}(\mathfrak{p})$.

If M is finitely generated, we have an equivalence also in (1). In this case, the module M is both $\mathfrak{p}$-local and $\mathfrak{p}$-torsion if and only if $\operatorname{Supp} M = \{\mathfrak{p}\}$.

In our proof of the Hopkins-Neeman theorem, the notion of Koszul complexes is needed. We will now assume R to be a commutative ring and let r be an element in R. The Koszul complex on r is the complex of R-modules given by

$$\cdots \longrightarrow 0 \longrightarrow R \xrightarrow{\ r\ } R \longrightarrow 0 \longrightarrow \cdots,$$

which is non-zero in degrees -1 and 0. The non-trivial map is multiplication by r. Following [4], we will denote the Koszul complex on r by $R/\!\!/r$. Observe that if we think of R as a stalk complex concentrated in degree zero, then $R/\!\!/r = \operatorname{Cone}(R \xrightarrow{r} R)$. Given a complex X of R-modules, the Koszul complex on r with coefficients in X is $X/\!\!/r = \operatorname{Cone}(X \xrightarrow{r} X)$. This construction can be iterated: The Koszul complex on two elements $r_1, r_2 \in R$ is

$$X/\!\!/(r_1, r_2) = (X/\!\!/r_1)/\!\!/r_2.$$

In general, for a finite set of elements $(r_1, \dots, r_t) \subseteq R$,

$$X/\!\!/(r_1, \dots, r_t) = X_t, \text{ where } X_0 = X \text{ and } X_i = X_{i-1}/\!\!/r_i \text{ for } i = 1, \dots, t.$$

It is not immediately clear that this construction is independent of the ordering of the ring elements. One can, however, show that

$$X/\!\!/(r_1, \dots, r_t) \simeq (X/\!\!/r_1) \otimes_R \cdots \otimes_R (X/\!\!/r_t).$$

For a finitely generated ideal $\mathfrak{a}$, we define the Koszul complex $X/\!\!/\mathfrak{a}$ as the Koszul complex on a finite set of generators of $\mathfrak{a}$. Note that this object may depend on our choice of generators. When it comes to support, however, there is no ambiguity, as we see from part (2) of the following proposition.

Proposition 1.1.8. *Let R be a commutative noetherian ring,* $\mathfrak{a} \subseteq R$ *an ideal and* $X \in \mathbf{D}^{\mathrm{b}}(\operatorname{mod} R)$. *The following statements hold:*

(1) $X/\!\!/\mathfrak{a} \in \operatorname{Thick}(X)$. *In particular, as* $\operatorname{Thick}(R) = \mathbf{D}^{\mathrm{b}}(\operatorname{proj} R)$*, this means that if* X *is perfect, then so is* $X/\!\!/\mathfrak{a}$.

(2) $\operatorname{Supp} X/\!\!/\mathfrak{a} = \operatorname{Supp} X \cap \mathcal{V}(\mathfrak{a})$.

(3) $\operatorname{Supp} R/\!\!/\mathfrak{a} = \mathcal{V}(\mathfrak{a})$.

Proof. As R is noetherian, the ideal $\mathfrak{a}$ will necessarily be finitely generated, say by the subset $\{a_1, \dots, a_t\}$. We clearly have $X \in \operatorname{Thick}(X)$. The 2/3-property on the distinguished triangle

$$X \xrightarrow{a_1} X \longrightarrow X/\!\!/a_1 \longrightarrow \Sigma X,$$

gives that $X/\!\!/a_1 \in \operatorname{Thick}(X)$. Iterating this process yields $X/\!\!/\mathfrak{a} \in \operatorname{Thick}(X)$, which proves (1).

For (2), we notice that by the definition of Koszul complexes and from the fact that $\mathcal{V}(a_1, \dots, a_t) = \mathcal{V}(a_1) \cap \cdots \cap \mathcal{V}(a_t)$, it is enough to show the result for one element a. From the short exact sequence

$$0 \longrightarrow X \longrightarrow X/\!\!/a \longrightarrow \Sigma X \longrightarrow 0,$$

we get a long exact sequence of cohomology

$$\cdots \to H^n(X) \xrightarrow{a} H^n(X) \xrightarrow{\phi} H^n(X/\!\!/a) \xrightarrow{\psi} H^{n+1}(X) \xrightarrow{a} H^{n+1}(X) \to \cdots .$$

We hence have the short exact sequence

$$0 \longrightarrow \operatorname{Ker}(\psi) \longrightarrow H^n(X/\!\!/a) \longrightarrow \operatorname{Im}(\psi) \longrightarrow 0.$$

Let $(0 :_{H^{n+1}(X)} a) = \{x \in H^{n+1}(X) \mid ax = 0\}$, i.e. the kernel of the map $H^{n+1}(X) \xrightarrow{a} H^{n+1}(X)$ above. Note that

$$\operatorname{Im}(\psi) = (0 :_{H^{n+1}(X)} a) = (0 :_{H^n(\Sigma X)} a).$$

From the short exact sequence

$$0 \longrightarrow \operatorname{Ker}(\phi) \longrightarrow H^n(X) \longrightarrow \operatorname{Ker}(\psi) \longrightarrow 0,$$

we also observe that

$$\operatorname{Ker}(\psi) \simeq H^n(X)/\operatorname{Ker}(\phi) = H^n(X)/aH^n(X).$$

This yields a short exact sequence of graded R-modules

$$0 \longrightarrow H^*(X)/aH^*(X) \longrightarrow H^*(X/\!\!/a) \longrightarrow (0 :_{H^*(\Sigma X)} a) \longrightarrow 0.$$

Since $\operatorname{Supp} Y = \operatorname{Supp} H^*(Y)$ for any complex Y, Lemma 1.1.5 (4) implies

$$\operatorname{Supp} X/\!\!/a = \operatorname{Supp}\,(H^*(X)/aH^*(X)) \cup \operatorname{Supp}\,(0 :_{H^*(\Sigma X)} a).$$

We will show that the right hand side is equal to $\operatorname{Supp} X \cap \mathcal{V}(a)$.

By localizing the short exact sequence

$$0 \longrightarrow (0 :_{H^*(X)} a) \longrightarrow H^*(X) \xrightarrow{\ a\ } aH^*(X) \longrightarrow 0,$$

we see that $\operatorname{Supp}\,(0 :_{H^*(X)} a) \subseteq \operatorname{Supp} X \cap \mathcal{V}(a)$. As $H^*(X) = H^*(\Sigma X)$, we get $\operatorname{Supp}\,(0 :_{H^*(\Sigma X)} a) \subseteq \operatorname{Supp} X \cap \mathcal{V}(a)$.

It remains to show that we have $\operatorname{Supp}\,(H^*(X)/aH^*(X)) = \operatorname{Supp} X \cap \mathcal{V}(a)$. Assume that

$$(H^*(X)/aH^*(X))_p = H^*(X)_{\mathfrak{p}}/aH^*(X)_{\mathfrak{p}} = 0.$$

Notice that as $X \in \mathbf{D}^{\mathrm{b}}(\operatorname{mod} R)$ and R is noetherian, the total cohomology $H^*(X)$, and hence also $H^*(X)_{\mathfrak{p}}$, is finitely generated. If $\mathfrak{p} \in \mathcal{V}(a)$, then a is in the radical of $R_\mathfrak{p}$. By Nakayama's lemma, we can deduce that $H^*(X)_\mathfrak{p} = 0$, which shows the inclusion $\operatorname{Supp}(H^*(X)/aH^*(X)) \supseteq \operatorname{Supp} X \cap \mathcal{V}(a)$. For the reverse inclusion, we have

$$\operatorname{Supp}(H^*(X)/aH^*(X)) = \mathcal{V}(\operatorname{ann}_R(H^*(X)/aH^*(X)) \subseteq \operatorname{Supp} X \cap \mathcal{V}(a).$$

The equality follows from Lemma 1.1.5 (2), while the inequality is a consequence of the fact that $\langle \operatorname{ann}_R(H^*(X)), a\rangle \subseteq \operatorname{ann}_R(H^*(X)/aH^*(X))$, where $\langle \operatorname{ann}_R(H^*(X)), a\rangle$ is the ideal generated by $\operatorname{ann}_R(H^*(X))$ and a. This proves (2).

Note that (3) follows directly from (2) as $\operatorname{Supp} R = \operatorname{Spec} R$. □

1.2 The Hopkins-Neeman theorem

We are now ready to formulate the Hopkins-Neeman theorem.

Theorem 1.2.1 (Hopkins-Neeman)**.** *Let R be a commutative noetherian ring. There are maps of sets*

$$\left\{\begin{matrix}\text{thick subcategories}\\ \text{of } \mathbf{D}^{\mathrm{b}}(\operatorname{proj} R)\end{matrix}\right\} \underset{g}{\overset{f}{\rightleftarrows}} \left\{\begin{matrix}\text{specialization closed}\\ \text{subsets of } \operatorname{Spec} R\end{matrix}\right\}$$

where

$$f(\mathcal{S}) = \cup_{X\in\mathcal{S}} \operatorname{Supp} X,$$

and $g(V)$ is the full subcategory given by

$$g(V) = \{X \in \mathbf{D}^{\mathrm{b}}(\operatorname{proj} R) \mid \operatorname{Supp} X \subseteq V\}.$$

The maps f and g are inverse isomorphisms.

As a first comment on the theorem, let us make sure that the images of f and g are contained where we claim.

Lemma 1.2.2. *Let f and g be the maps from* Theorem 1.2.1.

(1) For any subcategory $\mathcal{S} \subseteq \mathbf{D}^{\mathrm{b}}(\operatorname{proj} R)$, the subset $f(\mathcal{S}) \subseteq \operatorname{Spec} R$ is specialization closed.

(2) For any subset $V \subseteq \operatorname{Spec} R$, the subcategory $g(V) \subseteq \mathbf{D}^{\mathrm{b}}(\operatorname{proj} R)$ is thick.

Proof. By Lemma 1.1.5 (2), we know that $\operatorname{Supp} X$ is closed in $\operatorname{Spec} R$ for all $X \in \mathcal{S}$. Thus, the set $f(\mathcal{S})$ is a union of closed subsets, and hence specialization closed.

The subcategory $g(V)$ is full by definition, and non-empty as the zero object has empty support. We clearly have $\operatorname{Supp} X = \operatorname{Supp} \Sigma^n X$, so $g(V)$ is closed under arbitrary shifts. The 2/3-property follows from Lemma 1.1.5 (5), so $g(V)$ is a triangulated subcategory. Finally, Lemma 1.1.5 (1) implies thickness. □

In order to prove the Hopkins-Neeman theorem, we will need an important result by Hopkins, namely Theorem 1.2.4 below. The proof of this result is a bit technical. In order to make the presentation as clear as possible, it is postponed until the end of the chapter. We will however motivate the theorem by the following lemma.

Lemma 1.2.3. *Let R be a commutative noetherian ring. The implication*

$$\operatorname{Thick}(X) \subseteq \operatorname{Thick}(Y) \implies \operatorname{Supp} X \subseteq \operatorname{Supp} Y$$

is true for all $X, Y \in \mathbf{D}(R)$.

Proof. Consider the full subcategory given by

$$\mathcal{S} = \{Z \in \mathbf{D}(R) \mid \operatorname{Supp} Z \subseteq \operatorname{Supp} Y\}.$$

Applying Lemma 1.1.5, we see that $\mathcal{S}$ is a thick subcategory. As Y is clearly contained in $\mathcal{S}$, this implies $\operatorname{Thick}(Y) \subseteq \mathcal{S}$. By assumption, the object X is in $\operatorname{Thick}(Y)$, so $\operatorname{Supp} X \subseteq \operatorname{Supp} Y$. □

The reverse implication does not hold in general. However, Hopkins discovered that it *does* hold if we restrict to perfect complexes. This is Theorem 1.2.4.

Theorem 1.2.4. *Let R be a commutative noetherian ring. The implication*

$$\operatorname{Supp} X \subseteq \operatorname{Supp} Y \implies \operatorname{Thick}(X) \subseteq \operatorname{Thick}(Y)$$

holds for all $X, Y \in \mathbf{D}^{\mathrm{b}}(\operatorname{proj} R)$.

We are now ready to give a proof of the Hopkins-Neeman theorem.

Proof of Theorem 1.2.1. Let $\mathcal{S} \subseteq \mathbf{D}^{\mathrm{b}}(\operatorname{proj} R)$ be a thick subcategory and $V \subseteq \operatorname{Spec} R$ a specialization closed subset. Our aim is to prove the equalities $(g \circ f)(\mathcal{S}) = \mathcal{S}$ and $(f \circ g)(V) = V$.

Let $X \in \mathcal{S}$. We have

$$(g \circ f)(\mathcal{S}) = \{Y \in \mathbf{D}^{\mathrm{b}}(\operatorname{proj} R) \mid \operatorname{Supp} Y \subseteq f(\mathcal{S})\}.$$

$\operatorname{Supp} X$ is clearly contained in $f(\mathcal{S})$, so $X \in (g \circ f)(\mathcal{S})$. This shows that $\mathcal{S} \subseteq (g \circ f)(\mathcal{S})$.

For the reverse inclusion, let $X \in (g \circ f)(\mathcal{S})$. If $X = 0$, then X is clearly in $\mathcal{S}$, so we may assume $X \neq 0$. Hence, the ideal $\mathfrak{a} = \operatorname{ann}_R(H^*(X))$ is proper. Let us denote the set of minimal elements in $\mathcal{V}(\mathfrak{a})$ by $\operatorname{Min}(\mathfrak{a})$. Using a Zorn's lemma argument, one can check that $\operatorname{Min}(\mathfrak{a})$ is non-empty for any proper ideal $\mathfrak{a}$. Over a noetherian ring, the set of minimal primes over an ideal will always be finite, see for instance [7, Prop. 2.1.12]. Hence, we know that $|\operatorname{Min}(\mathfrak{a})| < \infty$, say $\operatorname{Min}(\mathfrak{a}) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$. Now,

$$\operatorname{Supp} X = \mathcal{V}(\mathfrak{a}) = \cup_{i=1}^{t} \mathcal{V}(\mathfrak{p}_i) \subseteq \cup_{Z \in \mathcal{S}} \operatorname{Supp} Z,$$

where the inclusion follows from the assumption $X \in (g \circ f)(\mathcal{S})$. This implies that for each $i = 1, \dots, t$, there exists an $Y_i \in \mathcal{S}$ such that $\mathfrak{p}_i \in \operatorname{Supp} Y_i$. As $\operatorname{Supp} Y_i$ is closed, we have $\mathcal{V}(\mathfrak{p}_i) \subseteq \operatorname{Supp} Y_i$. This gives

$$\operatorname{Supp} X = \cup_{i=1}^{t} \mathcal{V}(\mathfrak{p}_i) \subseteq \cup_{i=1}^{t} \operatorname{Supp} Y_i = \operatorname{Supp} Y,$$

where $Y = \oplus_{i=1}^{t} Y_i$. Note that because each Y_i is in $\mathcal{S}$ and the coproduct is finite, the object Y will also be in $\mathcal{S}$. By thickness of $\mathcal{S}$, this implies $\operatorname{Thick}(Y) \subseteq \mathcal{S}$. As both X and Y are perfect, we can use Theorem 1.2.4 to deduce that $X \subseteq \operatorname{Thick}(Y)$, so $X \in \mathcal{S}$. This allows us to conclude that $(g \circ f)(\mathcal{S}) = \mathcal{S}$.

We now want to show the equality $(f \circ g)(V) = V$. Let

$$\mathfrak{p} \in (f \circ g)(V) = \cup_{X \in g(V)} \operatorname{Supp} X.$$

We clearly have $\mathfrak{p} \in V$, so $(f \circ g)(V) \subseteq V$.

Let us next assume $\mathfrak{p} \in V$. Consider the Koszul complex $R/\!\!/\mathfrak{p}$. By Proposition 1.1.8, we know that $\operatorname{Supp}(R/\!\!/\mathfrak{p}) = \mathcal{V}(\mathfrak{p})$. As V is specialization closed and contains $\mathfrak{p}$, this gives $\operatorname{Supp}(R/\!\!/\mathfrak{p}) = \mathcal{V}(\mathfrak{p}) \subseteq V$. Hence, we have shown that $R/\!\!/\mathfrak{p} \in g(V)$. Using that $\mathfrak{p} \in \operatorname{Supp}(R/\!\!/\mathfrak{p})$, one obtains $\mathfrak{p} \in (f \circ g)(V)$, which implies our second equality. □

1.3 A proof of Theorem 1.2.4

As we have seen, Theorem 1.2.4 was crucial in our proof of the Hopkins-Neeman theorem. In order to prove Theorem 1.2.4, we will use a result known as the tensor-nilpotence theorem. Our approach will be based on techniques which are similar to those used in Hopkins' original proof [10], but the exhibition will also be inspired by Carlson and Iyengar's approach in [8], where they prove a similar statement in the setting of DG-modules.

In this section we will use the notion of derived functors, in particular $\mathbf{R}\operatorname{Hom}$ and $\otimes^{\mathbf{L}}$. For an introduction to derived functors, see for instance [21, Section 10.4–10.7]. The derived tensor product $X \otimes_R^{\mathbf{L}} \cdots \otimes_R^{\mathbf{L}} X$ with n factors is denoted by $X^{\otimes n}$, and $f^{\otimes n} \colon X^{\otimes n} \to Y^{\otimes n}$ is the morphism induced by $f \colon X \to Y$.

Based on [21, Section 10.4], we know that the category $\mathbf{K}^{\mathrm{b}}(\operatorname{proj} R)$ is equivalent to $\mathbf{D}^{\mathrm{b}}(\operatorname{proj} R)$. In the proofs in this section, we will often work with perfect complexes, and in such cases the equivalence above allows us to represent $\mathbf{R}\operatorname{Hom}$ and $\otimes^{\mathbf{L}}$ by Hom and $\otimes$.

Let us first look at some preliminary results.

Lemma 1.3.1. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be triangulated categories and $F\colon \mathcal{T}_1 \to \mathcal{T}_2$ a triangulated functor. Let X and Y be objects in $\mathcal{T}_1$. We have the implication*

$$X \in \operatorname{Thick}_{\mathcal{T}_1}(Y) \implies F(X) \in \operatorname{Thick}_{\mathcal{T}_2}(F(Y)).$$

Proof. Consider the full subcategory

$$\mathcal{S} = \{Z \in \mathcal{T}_1 \mid FZ \in \operatorname{Thick}_{\mathcal{T}_2}(FY)\}.$$

This is clearly a thick subcategory of $\mathcal{T}_1$. As $Y \in \mathcal{S}$, our assumption implies that $X \in \mathcal{S}$, so $FX \in \operatorname{Thick}_{\mathcal{T}_2}(FY)$. □

It is often convenient to use the description of support given in the proposition below. The proof is based on Nakayama's lemma, for details see [8, Thm. 2.4].

Proposition 1.3.2. *Let R be a commutative noetherian ring and $k(\mathfrak{p})$ the residue field of R at $\mathfrak{p}$, i.e. $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. For $X \in \mathbf{D}^{\mathrm{b}}(\operatorname{mod} R)$, we have*

$$\operatorname{Supp} X = \{\mathfrak{p} \in \operatorname{Spec} R \mid k(\mathfrak{p}) \otimes^{\mathbf{L}}_R X \neq 0\}.$$

In our next results, we make heavily use of the fact that we have an isomorphism $\operatorname{Hom}_{\mathbf{D}(R)}(R, Y) \simeq \operatorname{Hom}_{\mathbf{K}(R)}(R, Y)$ for all $Y \in \mathbf{D}(R)$, which is true as the stalk complex R is perfect [21, Cor. 10.4.7]. This enables us to represent a morphism in the derived category by a morphism in the homotopy category. For this morphism we will, slightly imprecisely, choose to use the same notation.

For a complex Y given by

$$\cdots \longrightarrow Y^{n-1} \xrightarrow{d^{n-1}} Y^n \xrightarrow{d^n} Y^{n+1} \longrightarrow \cdots$$

we say that an element $y \in Y^n$ is a cycle if $d^n(y) = 0$. Two cycles $y_1, y_2 \in Y^n$ are homologous, denoted by $y_1 \sim y_2$, if $y_1 - y_2 \in \operatorname{Im}(d^{n-1})$, i.e. if the elements represent the same equivalence class in $H^n(Y)$.

Lemma 1.3.3. *Let R be a commutative noetherian ring, Y an object in $\mathbf{D}(R)$ and $g, h \in \operatorname{Hom}_{\mathbf{D}(R)}(R, Y)$. We have the equivalence*

$$g = h \text{ in } \mathbf{D}(R) \iff g(1) \sim h(1).$$

In particular, we see that $g = 0$ in $\mathbf{D}(R)$ if and only if $g(1) \sim 0$.

Proof. Let us think of g and h as morphisms in the homotopy category. These morphisms are equal in $\mathbf{K}(R)$, and hence also in $\mathbf{D}(R)$, if and only if there exists an s as in the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, "g-h"] \arrow[dl, dashed, "s"'] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & Y^{-1} \arrow[r, "d^{-1}"] & Y^0 \arrow[r, "d^0"] & Y^1 \arrow[r] & \cdots
\end{tikzcd}$$

such that $d^{-1} \circ s = g - h$. This is equivalent to the existence of $y \in Y^{-1}$ such that $d^{-1}(y) = g(1) - h(1)$, which again is equivalent to $g(1) \sim h(1)$. $\square$

Lemma 1.3.4. *Let R be a commutative noetherian ring, $Y \in \mathbf{D}^{\mathrm{b}}(\operatorname{proj} R)$ and $f \in \operatorname{Hom}_{\mathbf{D}(R)}(R, Y)$. Let $\{I_1, \dots, I_t\}$ be a set of ideals in R such that $R/I_i \otimes_R^{\mathbf{L}} f = 0$ for $i = 1, \dots, t$. If the product $I_1 \cdots I_t = 0$, then $f^{\otimes t} = 0$.*

Proof. As $\mathbf{D}^{\mathrm{b}}(\operatorname{proj} R)$ is equivalent to $\mathbf{K}^{\mathrm{b}}(\operatorname{proj} R)$, we can represent $R/I_i \otimes_R^{\mathbf{L}} f$ by $R/I_i \otimes_R f$. We can hence assume $R/I_i \otimes_R f = 0$ in $\mathbf{D}(R)$, i.e. that there exists a complex W_i and a quasi-isomorphism $q_i \colon W_i \to R/I_i$ such that $(R/I_i \otimes_R f) \circ q_i = 0$ in $\mathbf{K}(R)$. Note that by truncation, we can assume $W^m = 0$ for $m > 0$. We hence have the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & W^{-1} \arrow[r] \arrow[d] & W^0 \arrow[r] \arrow[d, "q_i^0"] \arrow[dl, dashed, "s_i"'] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & R/I_i \arrow[r] \arrow[d, "R/I_i \otimes_R f"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & Y^{-1}/I_iY^{-1} \arrow[r, "R/I_i \otimes_R d^{-1}"] & Y^0/I_iY^0 \arrow[r, "R/I_i \otimes_R d^0"] & Y^1/I_iY^1 \arrow[r] & \cdots .
\end{tikzcd}$$

The existence of an s_i such that the triangle commutes, follows from the assumption $(R/I_i \otimes_R f) \circ q_i = 0$ in $\mathbf{K}(R)$. Notice that as q_i is a quasi-isomorphism and R/I_i a stalk complex, the map q_i^0 is surjective. By a similar argument as in the proof of our previous lemma, there must exist a cycle $y_i \in I_iY^0$ such that $f(1) \sim y_i$. This implies that

$$(f^{\otimes t})(1) = (f(1))^{\otimes t} \sim y_1 \otimes \cdots \otimes y_t \in (I_1 \cdots I_t)(Y^0 \otimes_R \cdots \otimes_R Y^0).$$

If $I_1 \cdots I_t = 0$, we will hence have $(f^{\otimes t})(1) \sim 0$, which by Lemma 1.3.3 gives $f^{\otimes t} = 0$. $\square$

We are able to strengthen the formulation in the previous lemma. A morphism f in $\mathbf{D}(R)$ is called *tensor-nilpotent* if there exists $n \geq 1$ such that $f^{\otimes n} = 0$ in $\mathbf{D}(R)$.

Lemma 1.3.5. *Let R be a commutative noetherian ring, $Y \in \mathbf{D}^{\mathrm{b}}(\operatorname{proj} R)$ and $f \in \operatorname{Hom}_{\mathbf{D}(R)}(R, Y)$. Let $\{I_1, \dots, I_t\}$ be a set of ideals in R such that $(R/I_i) \otimes_R^{\mathbf{L}} f$ is tensor-nilpotent for $i = 1, \dots, t$. If the product $I_1 \cdots I_t = 0$, then also f is tensor-nilpotent.*

Proof. As $R/I_i \otimes_R^{\mathbf{L}} f$ is tensor-nilpotent, there exists n_i such that

$$(R/I_i \otimes_R f)^{\otimes n_i} = R/I_i \otimes_R f^{\otimes n_i} = 0$$

for all $i = 1, \dots, t$. By a similar argument as above, this is equivalent to the existence of cycles $y_i \in I_i Y^0$ such that $f^{\otimes n_i} \sim y_i$. We hence have

$$\begin{aligned}(f^{\otimes(n_1+\cdots+n_t)})(1) &= (f^{\otimes n_1})(1) \otimes \cdots \otimes (f^{\otimes n_t})(1) \\ &\sim y_1 \otimes \cdots \otimes y_t \in (I_1 \cdots I_t)(Y^0 \otimes_R \cdots \otimes_R Y^0),\end{aligned}$$

so $(f^{\otimes(n_1+\cdots+n_t)})(1) \sim 0$. Consequently, Lemma 1.3.3 yields that also f is tensor-nilpotent. □

Theorem 1.3.6 (tensor-nilpotence)**.** *Let R be a commutative noetherian ring and $X, Y \in \mathbf{D}^{\mathrm{b}}(\operatorname{proj} R)$. Let $f\colon X \to Y$ be a morphism in $\mathbf{D}(R)$. If $k(\mathfrak{p}) \otimes^{\mathbf{L}}_R f = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$, there exists $n \geq 1$ such that $f^{\otimes n} = 0$ in $\mathbf{D}(R)$.*

Proof. Let $f'\colon R \to \mathbf{R}\mathrm{Hom}_R(X, Y)$ be the morphism defined by mapping 1_R to f and using the isomorphism $\mathrm{Hom}_R(X, Y) \simeq \mathbf{R}\mathrm{Hom}_R(X, Y)$. We see that $f^{\otimes n} = 0$ if and only $f'^{\otimes n} = 0$. If $k(\mathfrak{p}) \otimes^{\mathbf{L}}_R f = 0$, we also have $k(\mathfrak{p}) \otimes^{\mathbf{L}}_R f' = 0$. As $\mathbf{R}\mathrm{Hom}_R(X, Y)$ is perfect, which follows from Lemma 1.3.1, it is hence enough to show the theorem in the case $X = R$.

From now on, dimension will mean Krull dimension, denoted by $\dim R$. Our strategy is to show the theorem for finite dimensional rings, and later generalize to the infinite dimensional case. We will first see that in the finite dimensional setting, we can reduce to the case where R is an integral domain. This is done through proving the two claims below. In both claims, we let d be a non-negative integer.

Claim 1: If the theorem is true for integral domains of dimension $\leq d$, it is true for reduced rings of dimension $\leq d$.

Claim 2: If the theorem is true for reduced rings of dimension $\leq d$, it is true for all rings of dimension $\leq d$.

Proof of Claim 1: Let R be a reduced ring, i.e. a ring in which the nilradical is zero, and assume $\dim R \leq d$. We know that the nilradical can be described as the intersection of all prime ideals, see [1, Prop. 1.8]. Let $\{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$ be the set of minimal primes in R, which is finite as R is noetherian. The product $\mathfrak{p}_1 \cdots \mathfrak{p}_t$ is contained in the nilradical, and hence zero. Consider the morphism

$$R/\mathfrak{p}_i \xrightarrow{R/\mathfrak{p}_i \otimes^{\mathbf{L}}_R f} R/\mathfrak{p}_i \otimes^{\mathbf{L}}_R Y.$$

Note that $R/\mathfrak{p}_i$ is an integral domain of dimension $\leq d$ and that $\operatorname{Spec}(R/\mathfrak{p}_i)$ corresponds to $\operatorname{Spec} R \cap \mathcal{V}(\mathfrak{p}_i)$. The residue field of $R/\mathfrak{p}_i$ at a prime ideal in $\operatorname{Spec} R/\mathfrak{p}_i$ coincides with the residue field of R at the corresponding prime ideal in $\operatorname{Spec} R$. As

$$k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R/\mathfrak{p}_i} R/\mathfrak{p}_i \otimes^{\mathbf{L}}_R f = k(\mathfrak{p}) \otimes^{\mathbf{L}}_R f = 0$$

for all primes $\mathfrak{p} \in \operatorname{Spec} R/\mathfrak{p}_i$, the morphism $R/\mathfrak{p}_i \otimes^{\mathbf{L}}_R f$ is tensor-nilpotent. By Lemma 1.3.5, this means that also f is tensor nilpotent, which proves *Claim 1*.

Proof of Claim 2: This follows the same lines as the proof of *Claim 1*. Let R be a ring with $\dim R \le d$, and consider the morphism

$$R/\mathfrak{n} \xrightarrow{R/\mathfrak{n}\otimes^{\mathbf{L}}_R f} R/\mathfrak{n} \otimes^{\mathbf{L}}_R Y,$$

where $\mathfrak{n}$ denotes the nilradical of R. The ring $R/\mathfrak{n}$ is reduced of dimension $\le d$. In the same way as above, we see that $R/\mathfrak{n} \otimes^{\mathbf{L}}_R f$ is tensor-nilpotent. Combining this with the fact that $\mathfrak{n}$ is nilpotent, as R is noetherian, and using Lemma 1.3.5, yields the desired conclusion.

Our next aim is to show that the theorem is true for a finite dimensional integral domain R. This will be done by induction on $\dim R$.

If $\dim R = 0$, the zero ideal is the only prime ideal, which means that R is a field. The statement is clearly true in this case. Note that by *Claim 1* and *Claim 2*, this means that the result is true for all rings of dimension zero.

For the induction step, assume that the statement is true for all rings of dimension $\le d-1$, with $d \ge 1$. Let R be an integral domain with $\dim R = d$.

Consider the ideal

$$\operatorname{ann}(f) = \{r \in R \mid rf = 0 \text{ in } \mathbf{D}(R)\} = \{r \in R \mid (rf)(1) \sim 0\},$$

where the second equality is Lemma 1.3.3.

By assumption, the morphism $k(\mathfrak{p}) \otimes^{\mathbf{L}}_R f$ is zero for all $\mathfrak{p} \in \operatorname{Spec} R$. As R is an integral domain, the zero ideal is prime. Applying $k(\mathbf{0}) \otimes^{\mathbf{L}}_R (-)$ corresponds to localizing at $\mathbf{0}$, i.e. inverting all non-zero elements. Using similar arguments as before, we see that $k(\mathbf{0}) \otimes^{\mathbf{L}}_R f = 0$ implies the existence of non-zero $r \in R$ such that $(rf)(1) \sim 0$. This means that $\mathbf{0} \subsetneq \operatorname{ann}(f)$. If $f = 0$, the statement is obviously true, so we can assume $f \ne 0$, which gives $\operatorname{ann}(f) \subsetneq R$.

It is hence possible to pick a non-zero non-unit $r \in \operatorname{ann}(f)$. Consider the morphism

$$R/\langle r\rangle \xrightarrow{R/\langle r\rangle\otimes^{\mathbf{L}}_R f} R/\langle r\rangle \otimes^{\mathbf{L}}_R Y,$$

where $\langle r\rangle$ denotes the ideal generated by r. We have

$$k(\mathfrak{p}) \otimes^{\mathbf{L}}_{R/\langle r\rangle} R/\langle r\rangle \otimes^{\mathbf{L}}_R f = k(\mathfrak{p}) \otimes^{\mathbf{L}}_R f = 0,$$

for all $\mathfrak{p} \in \operatorname{Spec} R/\langle r\rangle$. Note that $r \ne 0$ implies $\dim R/\langle r\rangle \le d-1$, so by the induction hypothesis, the morphism $R/\langle r\rangle \otimes^{\mathbf{L}}_R f$ is tensor nilpotent. Consequently, there exists $n \ge 1$ such that $R/\langle r\rangle \otimes^{\mathbf{L}}_R f^{\otimes n} = 0$. This is equivalent to

$(R/\langle r\rangle \otimes^{\mathbf{L}}_R f^{\otimes n})(1) \sim 0$, which again implies the existence of $y \in (Y^{\otimes n})^0$ such that $f^{\otimes n}(1) \sim ry$. Hence,

$$f^{\otimes(n+1)}(1) = f^{\otimes n}(1) \otimes f(1) \sim ry \otimes f(1) = y \otimes (rf)(1) \sim y \otimes 0 = 0,$$

where the second equivalence follows from that $r \in \operatorname{ann}(f)$. We can thus conclude that $f^{\otimes(n+1)} = 0$, which finishes the induction.

Our last step is to show that the theorem is true also in the infinite dimensional case. Let R be a ring of arbitrary dimension. We have a chain of ideals

$$\operatorname{ann}(f) \subseteq \operatorname{ann}(f^{\otimes 2}) \subseteq \operatorname{ann}(f^{\otimes 3}) \subseteq \cdots .$$

As R is noetherian, this chain has to stabilize, so there exists $n \geq 1$ such that $\operatorname{ann}(f^{\otimes n}) = \operatorname{ann}(f^{\otimes i})$ for all $i \geq n$. Our aim is to show that this implies $\operatorname{ann}(f^{\otimes n}) = R$, which is equivalent to $f^{\otimes n} = 0$.

If $\operatorname{ann}(f^{\otimes n}) \subsetneq R$, there exists $\mathfrak{p} \in \operatorname{Spec} R$ such that $\operatorname{ann}(f^{\otimes n}) \subseteq \mathfrak{p}$. Consider the morphism

$$R_{\mathfrak{p}} \xrightarrow{R_{\mathfrak{p}} \otimes^{\mathbf{L}}_R f = f_{\mathfrak{p}}} R_{\mathfrak{p}} \otimes^{\mathbf{L}}_R Y = Y_{\mathfrak{p}}.$$

We know that $\dim R_{\mathfrak{p}} = \operatorname{ht}(\mathfrak{p})$, which is finite as R is noetherian. A prime ideal in $\operatorname{Spec} R_{\mathfrak{p}}$ corresponds to a prime ideal in $\operatorname{Spec} R$ contained in $\mathfrak{p}$, and the residue field of $R_{\mathfrak{p}}$ at a prime in $\operatorname{Spec} R_{\mathfrak{p}}$ coincides with the residue field of R at the corresponding prime. As before, this gives that the hypothesis in the tensor-nilpotence theorem is satisfied for $R_{\mathfrak{p}} \otimes^{\mathbf{L}}_R f$, and as $\dim R_{\mathfrak{p}} < \infty$, we can conclude that $f_{\mathfrak{p}}^{\otimes i} = 0$ for $i \gg 0$. We consequently have

$$\operatorname{ann}(f^{\otimes n})_{\mathfrak{p}} = \operatorname{ann}(f^{\otimes i})_{\mathfrak{p}} \simeq \operatorname{ann}(f_{\mathfrak{p}}^{\otimes i}) = R_{\mathfrak{p}}.$$

For the isomorphism, we are using propositions 3.7 and 3.14 from [1]. But this is a contradiction, as

$$\operatorname{ann}(f^{\otimes n}) \subseteq \mathfrak{p} \subsetneq R \implies \operatorname{ann}(f^{\otimes n})_{\mathfrak{p}} \subseteq \mathfrak{p}_{\mathfrak{p}} \subsetneq R_{\mathfrak{p}}.$$

We can hence conclude that $\operatorname{ann}(f^{\otimes n}) = R$, i.e. that $f^{\otimes n} = 0$, which finishes the proof. $\square$

We are finally ready to prove Theorem 1.2.4.

Proof of Theorem 1.2.4. We have a natural morphism $R \to \operatorname{Hom}_R(Y, Y)$, given by multiplication. As the complex Y is perfect, one obtains an isomorphism $\operatorname{Hom}_R(Y, Y) \simeq \mathbf{R}\operatorname{Hom}_R(Y, Y)$. Applying the functor $X \otimes^{\mathbf{L}}_R (-)$ gives a morphism $X \xrightarrow{g} X \otimes^{\mathbf{L}}_R \mathbf{R}\operatorname{Hom}_R(Y, Y)$. By Lemma 1.3.1, also $X \otimes^{\mathbf{L}}_R \mathbf{R}\operatorname{Hom}_R(Y, Y)$ is perfect. Extend g to a distinguished triangle

$$Z \xrightarrow{f} X \xrightarrow{g} X \otimes^{\mathbf{L}}_R \mathbf{R}\operatorname{Hom}_R(Y, Y) \longrightarrow \Sigma Z,$$

where Z is perfect by the 2/3-property.

We now wish to apply the tensor-nilpotence theorem to f. In order to do so, we must show that $k(\mathfrak{p}) \otimes^{\mathbf{L}}_R f = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$. If $\mathfrak{p} \notin \operatorname{Supp} X$, Proposition 1.3.2 implies $k(\mathfrak{p}) \otimes^{\mathbf{L}}_R X = 0$, so clearly $k(\mathfrak{p}) \otimes^{\mathbf{L}}_R f = 0$. If $\mathfrak{p} \in \operatorname{Supp} X$, then $\mathfrak{p}$ is also contained in $\operatorname{Supp} Y$ by our assumption. Let $(*)$ denote the distinguished triangle obtained by applying $k(\mathfrak{p}) \otimes^{\mathbf{L}}_R (-)$ to the triangle above. Since every monomorphism in a triangulated category splits, it is sufficient to show that $k(\mathfrak{p}) \otimes^{\mathbf{L}}_R g$ is a monomorphism.

Let us think of the triangle $(*)$ as a distinguished triangle in $\mathbf{D}(k(\mathfrak{p}))$. As $k(\mathfrak{p})$ is a field, the derived category $\mathbf{D}(k(\mathfrak{p}))$ is equivalent to the category of $\mathbb{Z}$-graded vector spaces over $k(\mathfrak{p})$. The equivalence is given by taking total cohomology, see for instance [11, Section 1.6]. To show that $k(\mathfrak{p}) \otimes^{\mathbf{L}}_R g$ is a monomorphism, it is hence enough to show that $H^*(k(\mathfrak{p}) \otimes^{\mathbf{L}}_R g)$ is injective.

Using [21, Section 10.8.2] and the fact that extension and restriction of scalars is an adjoint pair, we get the isomorphism

$$(k(\mathfrak{p}) \otimes^{\mathbf{L}}_R X) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(Y, Y) \simeq (k(\mathfrak{p}) \otimes^{\mathbf{L}}_R X) \otimes^{\mathbf{L}}_{k(\mathfrak{p})} \mathbf{R}\mathrm{End}_{k(\mathfrak{p})}(k(\mathfrak{p}) \otimes^{\mathbf{L}}_R Y).$$

By Künneth's formula, see [14, Section 17.2], the functor $H^*(-)$ will split up on the tensor product over $k(\mathfrak{p})$ above. As $\mathfrak{p} \in \operatorname{Supp} Y$, Proposition 1.3.2 gives $k(\mathfrak{p}) \otimes^{\mathbf{L}}_R Y \neq 0$. The way g is defined hence implies that $H^*(k(\mathfrak{p}) \otimes^{\mathbf{L}}_R g)$ is injective, so $k(\mathfrak{p}) \otimes^{\mathbf{L}}_R f = 0$ in $\mathbf{D}(k(\mathfrak{p}))$ and hence also in $\mathbf{D}(R)$.

By the tensor-nilpotence theorem, we now know that there exists an integer $n \geq 1$ such that $f^{\otimes n} = 0$ in $\mathbf{D}(R)$. Let us extend $f^{\otimes n}$ to the distinguished triangle

$$Z^{\otimes n} \xrightarrow{f^{\otimes n}} X^{\otimes n} \longrightarrow \operatorname{Cone}(f^{\otimes n}) \longrightarrow \Sigma Z^{\otimes n}.$$

Applying $X \otimes^{\mathbf{L}}_R (-)$, we get the distinguished triangle

$$X \otimes^{\mathbf{L}}_R Z^{\otimes n} \xrightarrow{X \otimes^{\mathbf{L}}_R f^{\otimes n}} X^{\otimes (n+1)} \longrightarrow X \otimes^{\mathbf{L}}_R \operatorname{Cone}(f^{\otimes n}) \longrightarrow X \otimes^{\mathbf{L}}_R \Sigma Z^{\otimes n}.$$

As $f^{\otimes n} = 0$, the first morphism in this triangle is also zero, so the triangle splits. Hence, the object $X^{\otimes (n+1)}$ is a direct summand in $X \otimes^{\mathbf{L}}_R \operatorname{Cone}(f^{\otimes n})$.

Our next aim is to prove that $X \otimes^{\mathbf{L}}_R \operatorname{Cone}(f^{\otimes n}) \in \operatorname{Thick}(Y)$. We will first show that $\operatorname{Cone}(f^{\otimes i}) \in \operatorname{Thick}(Y)$ for all $i \geq 1$. This is done by induction on i. We have already verified that $\operatorname{Cone}(f) \simeq X \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(Y, Y) \in \operatorname{Thick}(Y)$. Assume $\operatorname{Cone}(f^{\otimes k}) \in \operatorname{Thick}(Y)$ for all $1 \leq k \leq i$. The morphism $f^{\otimes (i+1)}$ factors as below, and we get the solid part of the diagram

$$\begin{tikzcd}
Z^{\otimes(i+1)} \arrow[r, "Z^{\otimes i}\otimes^{\mathbf{L}}_R f"] \arrow[d, equal] & Z^{\otimes i} \otimes^{\mathbf{L}}_R X \arrow[r] \arrow[d, "f^{\otimes i}\otimes^{\mathbf{L}}_R X"] & \mathrm{Cone}(Z^{\otimes i} \otimes^{\mathbf{L}}_R f) \arrow[r] \arrow[d, dashed] & \Sigma Z^{\otimes(i+1)} \arrow[d, equal] \\
Z^{\otimes(i+1)} \arrow[r, "f^{\otimes(i+1)}"] & X^{\otimes(i+1)} \arrow[r] \arrow[d] & \mathrm{Cone}(f^{\otimes(i+1)}) \arrow[r] \arrow[d, dashed] & \Sigma Z^{\otimes(i+1)} \\
& \mathrm{Cone}(f^{\otimes i} \otimes^{\mathbf{L}}_R X) \arrow[r, equal] \arrow[d] & \mathrm{Cone}(f^{\otimes i} \otimes^{\mathbf{L}}_R X) \arrow[d] & \\
& \Sigma(Z^{\otimes i} \otimes^{\mathbf{L}}_R X) \arrow[r] & \Sigma(\mathrm{Cone}(Z^{\otimes i} \otimes^{\mathbf{L}}_R f)), &
\end{tikzcd}$$

where the two top rows and the second column are distinguished triangles. By the octahedron axiom, the dashed arrows exist, and the third column is a distinguished triangle. We have $\mathrm{Cone}(Z^{\otimes i} \otimes^{\mathbf{L}}_R f) \simeq Z^{\otimes i} \otimes^{\mathbf{L}}_R \mathrm{Cone}(f)$. The latter is in $\mathrm{Thick}(Y)$ as $Z^{\otimes i}$ is perfect and $\mathrm{Cone}(f) \in \mathrm{Thick}(Y)$. The analogue argument shows that $\mathrm{Cone}(Z^{\otimes i} \otimes^{\mathbf{L}}_R f) \in \mathrm{Thick}(Y)$, so the 2/3-property gives $\mathrm{Cone}(f^{\otimes(i+1)}) \in \mathrm{Thick}(Y)$.

From the argument above, it is clear that $\mathrm{Cone}(f^{\otimes n}) \in \mathrm{Thick}(Y)$. As X is perfect, this implies $X \otimes^{\mathbf{L}}_R \mathrm{Cone}(f^{\otimes n}) \in \mathrm{Thick}(Y)$, so $X^{\otimes(n+1)} \in \mathrm{Thick}(Y)$.

We finally want to show that we can reduce the power of X and still remain in $\mathrm{Thick}(Y)$. Again using Lemma 1.3.1, we know that $X^{\otimes(n+1)} \in \mathrm{Thick}(Y)$ implies $\mathbf{R}\mathrm{Hom}_R(X, R) \otimes^{\mathbf{L}}_R X^{\otimes(n+1)} \in \mathrm{Thick}(Y)$. We have the isomorphisms

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_R(X, R) \otimes^{\mathbf{L}}_R X^{\otimes(n+1)} &\simeq \mathbf{R}\mathrm{Hom}_R(X, R \otimes^{\mathbf{L}}_R X) \otimes^{\mathbf{L}}_R X^{\otimes n} \\
&= [\mathbf{R}\mathrm{Hom}_R(X, X) \otimes^{\mathbf{L}}_R X] \otimes^{\mathbf{L}}_R X^{\otimes(n-1)}.
\end{aligned}$$

The object X is a direct summand in $\mathbf{R}\mathrm{Hom}_R(X, X) \otimes^{\mathbf{L}}_R X$. This can be seen by considering $(-) \otimes^{\mathbf{L}}_R X$ applied to the natural morphism $R \to \mathbf{R}\mathrm{Hom}_R(X, X)$ and the evaluation morphism in the other direction. These clearly compose to the identity on X. Consequently, we see that $X^{\otimes n}$ is a direct summand in $[\mathbf{R}\mathrm{Hom}_R(X, X) \otimes^{\mathbf{L}}_R X] \otimes^{\mathbf{L}}_R X^{\otimes(n-1)}$, so $X^{\otimes n} \in \mathrm{Thick}(Y)$. Iterating this process yields $X \in \mathrm{Thick}(Y)$. $\square$

Chapter 2

The BIK-approach

The result of Hopkins and Neeman inspired extensive research on classification of subcategories of triangulated categories. Some of the major contributors to this field have been Benson, Iyengar and Krause. In a series of influential papers [3, 6, 5], they have recently developed a new approach to classification problems. This has enabled them to give classification results in contexts which were earlier unknown, in particular in the setting of modules over group algebras. Their approach also gives access to different proofs of already known results.

This part of the thesis will give an overview of Benson, Iyengar and Krause's techniques, from now on called the BIK-approach.

2.1 Subcategories

The Hopkins-Neeman theorem classifies thick subcategories of $\mathbf{D}^{\mathrm{b}}(\operatorname{proj} R)$ for a commutative noetherian ring R. In the BIK-approach a slightly more general class of subcategories plays an important role.

Definition 2.1.1. Let $\mathcal{T}$ be a triangulated category admitting set-indexed coproducts. A triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$ is called *localizing* if it is closed under set-indexed coproducts.

Lemma 2.1.2. *Let $\mathcal{T}$ be a triangulated category admitting set-indexed coproducts and $\mathcal{S}$ a localizing subcategory of $\mathcal{T}$. Then $\mathcal{S}$ is thick.*

Proof. Let $X \oplus Y \in \mathcal{S}$. If ι_x and π_x are the natural morphisms associated to the biproduct, then $\iota_x \circ \pi_x$ is clearly an idempotent. As $\mathcal{S}$ is a localizing subcategory of $\mathcal{T}$, we know from Neeman [17, Prop. 1.6.8] that every idempotent in $\mathcal{S}$ splits. Hence, there is an object $Z \in \mathcal{S}$ and morphisms f and g

$$\begin{tikzcd} X \oplus Y \arrow[rr, "\iota_x \circ \pi_x"] \arrow[dr, "f"'] & & X \oplus Y \\ & Z \arrow[ur, "g"'] & \end{tikzcd}$$

such that $g \circ f = \iota_x \circ \pi_x$ and $f \circ g = id_Z$. This yields $X \simeq Z$ via the isomorphism $\pi_x \circ g$ with inverse $f \circ \iota_x$. Since $\mathcal{S}$ is a triangulated subcategory and hence closed under isomorphisms, this gives $X \in \mathcal{S}$. The same argument shows that $Y \in \mathcal{S}$, so $\mathcal{S}$ is thick. □

We will denote the localizing subcategory generated by an object X by $\mathrm{Loc}_{\mathcal{T}}(X)$. This is the smallest localizing subcategory of $\mathcal{T}$ where X is an object, and can be described as the intersection of all localizing subcategories containing X. One can, of course, also look at subcategories generated by more than one object.

When dealing with localizing subcategories, we will repeatedly, and often even without mentioning, use that the translation functor of a triangulated category commutes with arbitrary coproducts. This is true by the following more general result.

Proposition 2.1.3. *The translation functor of a triangulated category commutes with both limits and colimits.*

Proof. As usual, we denote the translation functor by Σ. Observe that (Σ, Σ^{-1}) and (Σ^{-1}, Σ) are adjoint pairs. As Σ is both left and right adjoint, it preserves both limits and colimits. □

We will also need the following lemma. Its proof follows exactly the same lines as the proof of Lemma 1.3.1 in Chapter 1, and is hence omitted.

Lemma 2.1.4. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be triangulated categories admitting set-indexed coproducts and $F \colon \mathcal{T}_1 \to \mathcal{T}_2$ a triangulated functor which commutes with set-indexed coproducts. The implication*

$$X \in \mathrm{Loc}_{\mathcal{T}_1}(Y) \implies F(X) \in \mathrm{Loc}_{\mathcal{T}_2}(F(Y))$$

holds for all objects X and Y in $\mathcal{T}_1$.

2.2 Compact objects

Definition 2.2.1. Let $\mathcal{T}$ be a triangulated category admitting set-indexed coproducts. An object X in $\mathcal{T}$ is called *compact* if the functor $\mathrm{Hom}_{\mathcal{T}}(X, -)$ commutes with set-indexed coproducts.

Example 2.2.2. Let R be a ring, and think of R as a stalk complex concentrated in degree zero. Let I be an arbitrary index set and $\{Y_i \mid i \in I\}$ a family of objects in $\mathbf{D}(R)$. We have

$$\mathrm{Hom}_{\mathbf{D}(R)}(R, \oplus_{i\in I} Y_i) \simeq \mathrm{H}^0(\oplus_{i\in I} Y_i) \simeq \oplus_{i\in I} \mathrm{H}^0(Y_i) \simeq \oplus_{i\in I} \mathrm{Hom}_{\mathbf{D}(R)}(R, Y_i),$$

so R is a compact object in $\mathbf{D}(R)$.

We denote the full subcategory consisting of all compact objects in $\mathcal{T}$ by $\mathcal{T}^c$.

Lemma 2.2.3. *Let $\mathcal{T}$ be a triangulated category admitting set-indexed coproducts. The subcategory $\mathcal{T}^c$ is thick in $\mathcal{T}$.*

Proof. This is what we will later often call a routine argument, but we will do it in detail in this case. Let I be an arbitrary index set, and $\{Y_i \mid i \in I\}$ a family of objects in $\mathcal{T}$.

The zero object is always compact, since

$$\mathrm{Hom}_{\mathcal{T}}(0, \oplus_{i\in I} Y_i) = 0 = \oplus_{i\in I} \mathrm{Hom}_{\mathcal{T}}(0, Y_i),$$

so $\mathcal{T}^c$ is certainly non-empty. It is also a full subcategory by definition. If X is compact, then $\Sigma^n X$ will also be compact for all $n \in \mathbb{Z}$ as

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{T}}(\Sigma^n X, \oplus_{i\in I} Y_i) &\simeq \mathrm{Hom}_{\mathcal{T}}(X, \Sigma^{-n}(\oplus_{i\in I} Y_i)) \\
&\simeq \mathrm{Hom}_{\mathcal{T}}(X, \oplus_{i\in I}(\Sigma^{-n} Y_i)) \\
&\simeq \oplus_{i\in I} \mathrm{Hom}_{\mathcal{T}}(X, \Sigma^{-n} Y_i) \\
&\simeq \oplus_{i\in I} \mathrm{Hom}_{\mathcal{T}}(\Sigma^n X, Y_i).
\end{aligned}$$

The third isomorphism is by compactness of X. In order to verify the 2/3-property, we use that $\mathrm{Hom}_{\mathcal{T}}(-, \oplus_{i\in I} Y_i)$ is a cohomological functor and apply the five lemma. This allows us to conclude that $\mathcal{T}^c$ is a triangulated subcategory of $\mathcal{T}$.

To see that $\mathcal{T}^c$ is thick, assume $X_1 \oplus X_2 \in \mathcal{T}^c$. We hence have

$$\mathrm{Hom}_{\mathcal{T}}(X_1 \oplus X_2, \oplus_{i\in I} Y_i) \simeq \oplus_{i\in I} \mathrm{Hom}_{\mathcal{T}}(X_1 \oplus X_2, Y_i).$$

Since Hom-functors respect finite coproducts, we get that X_1 and X_2 also have to be compact. We can thus conclude that $\mathcal{T}^c$ is a thick subcategory. □

Our theory will deal with what we call compactly generated categories.

Definition 2.2.4. Let $\mathcal{T}$ be a triangulated category admitting set-indexed coproducts. We say that $\mathcal{T}$ is *compactly generated* if there exists a subset $\{C_i\}_{i\in I} \subseteq \mathcal{T}^c$ such that $\mathrm{Loc}_{\mathcal{T}}(\{C_i\}_{i\in I}) = \mathcal{T}$.

Note in the following that whenever we assume a triangulated category to be compactly generated, we implicitly assume that it admits set-indexed coproducts.

The proposition below is a useful criterion for when a triangulated category is compactly generated. The proof is based on the Brown representability theorem. For details, see for instance [4, Prop. 1.47].

Proposition 2.2.5. *Let $\mathcal{T}$ be a triangulated category admitting set-indexed coproducts. The following are equivalent for a set of objects $\{C_i\}_{i\in I}$ in $\mathcal{T}^c$:*

(i) $\operatorname{Loc}_{\mathcal{T}}(\{C_i\}_{i\in I}) = \mathcal{T}$.

(ii) For any non-zero object $X \in \mathcal{T}$, there exists $i \in I$ and $n \in \mathbb{Z}$ such that $\operatorname{Hom}_{\mathcal{T}}(\Sigma^n C_i, X) \neq 0$.

Several categories that arise naturally turn out to be compactly generated.

Example 2.2.6. The category $\mathbf{D}(R)$ is compactly generated for any ring R. To see this, observe that if X is a non-zero object in $\mathbf{D}(R)$, there exists $n \in \mathbb{Z}$ such that $\mathrm{H}^n(X) \neq 0$. Since

$$\operatorname{Hom}_{\mathbf{D}(R)}(\Sigma^{-n}R, X) \simeq \operatorname{Hom}_{\mathbf{D}(R)}(R, \Sigma^n X) \simeq \mathrm{H}^n(X),$$

it follows from Proposition 2.2.5 that $\mathbf{D}(R) = \operatorname{Loc}_{\mathbf{D}(R)}(R)$. As R is a compact object in $\mathbf{D}(R)$, the category is compactly generated.

Given a ring R, we have a nice description of the compact objects in the derived category $\mathbf{D}(R)$. In order to see this, we will use the following theorem, which originates from Ravenel's work on stable homotopy theory [18]. A proof can be found in [16, Lemma 2.2].

Theorem 2.2.7. *Let $\mathcal{T}$ be a triangulated category admitting set-indexed coproducts. For $X, Y \in \mathcal{T}^c$, we have the implication*

$$X \in \operatorname{Loc}_{\mathcal{T}}(Y) \implies X \in \operatorname{Thick}(Y).$$

We are now ready to prove Theorem 2.2.8, which we used already in Chapter 1.

Theorem 2.2.8. *Let R be a ring and $\mathcal{T} = \mathbf{D}(R)$. There are equalities*

$$\mathcal{T}^c = \operatorname{Thick}(R) = \mathbf{D}^{\mathrm{b}}(\operatorname{proj} R).$$

Proof. Since we work with full subcategories, it is enough to show that we have inclusions of objects. As R is compact and $\mathcal{T}^c$ is thick, we have $\operatorname{Thick}(R) \subseteq \mathcal{T}^c$. The reverse inclusion follows from Example 2.2.6 and Theorem 2.2.7, and hence $\operatorname{Thick}(R) = \mathcal{T}^c$.

The subcategory $\mathbf{D}^{\mathrm{b}}(\operatorname{proj} R)$ is thick in $\mathbf{D}(R)$ – for details see for instance [20]. As R is obviously perfect, we must hence have $\operatorname{Thick}(R) \subseteq \mathbf{D}^{\mathrm{b}}(\operatorname{proj} R)$. For the reverse inclusion, notice that the stalk complex R^n is in $\operatorname{Thick}(R)$ for all $n \in \mathbb{N}$. The same is true for all finitely generated projective modules, since $\operatorname{Thick}(R)$ is closed under direct summands. By induction on the number of non-zero components in a perfect complex, using that $\operatorname{Thick}(R)$ is closed when taking cones, we see that any perfect complex will be contained in $\operatorname{Thick}(R)$. Hence, we can conclude that $\operatorname{Thick}(R) = \mathbf{D}^{\mathrm{b}}(\operatorname{proj} R)$. Combining our two equalities gives the desired characterization. □

2.3 Central ring actions

One of the main ideas in the BIK-approach is to use a central ring action to define what we call local cohomology functors. This will in turn enable us to establish a notion of support for triangulated categories.

Let us first look at what it means to have a central ring action on a triangulated category. The centre of a ring consists of the ring elements which commute with all other elements. This notion can be generalized to categories.

Definition 2.3.1. Let $\mathcal{C}$ be a category. The *centre* of $\mathcal{C}$, denoted by $Z(\mathcal{C})$, consists of all natural transformations $\eta\colon \operatorname{Id}_{\mathcal{C}} \to \operatorname{Id}_{\mathcal{C}}$.

Under the mild assumption that $\mathcal{C}$ is a preadditive category, we can see that $Z(\mathcal{C})$ is a commutative ring, at least up to set-theoretical issues. Note also that if we think of a ring as a preadditive category with one object, the categorical centre is the same as the centre of the ring.

We now want a similar notion for triangulated categories.

Definition 2.3.2. Let $\mathcal{T}$ be a triangulated category. The *graded centre* of $\mathcal{T}$ is given by
$$Z^*(\mathcal{T}) = \oplus_{n\in\mathbb{Z}} Z^n(\mathcal{T}),$$
where $Z^n(\mathcal{T})$ consists of all natural transformations $\eta\colon \operatorname{Id}_{\mathcal{T}} \to \Sigma^n$ such that $\eta\Sigma = (-1)^n \Sigma\eta$.

To be precise, note that when we write $\eta\Sigma = (-1)^n\Sigma\eta$, we really mean that $\eta_{\Sigma X} = (-1)^n \Sigma\eta_X$ for every object X in $\mathcal{T}$. We will use simplified notation as in the definition above whenever this is convenient.

Recall that a graded ring R is called *graded-commutative* if $rs = (-1)^{|r||s|} sr$ for all homogeneous elements $r, s \in R$.

Remark 2.3.3. The graded centre $Z^*(\mathcal{T})$ is a graded-commutative ring. The multiplication of homogeneous elements is given by $(\tau\eta)_X = (\Sigma^n \tau_X) \circ \eta_X$, where n is the degree of η.

Given two objects X and Y in a triangulated category $\mathcal{T}$, we can look at the graded Hom-set

$$\operatorname{Hom}^*_{\mathcal{T}}(X,Y) = \oplus_{n\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{T}}(X,\Sigma^n Y).$$

We can also consider $\operatorname{End}^*_{\mathcal{T}}(X) = \operatorname{Hom}^*_{\mathcal{T}}(X,X)$. With multiplication of homogeneous elements given by $gf = (\Sigma^{|f|}g)\circ f$, the endomorphism ring $\operatorname{End}^*_{\mathcal{T}}(X)$ becomes a graded ring. The graded Hom-sets $\operatorname{Hom}^*_{\mathcal{T}}(X,Y)$ and $\operatorname{Hom}^*_{\mathcal{T}}(Y,X)$ are right and left $\operatorname{End}^*_{\mathcal{T}}(X)$-modules, respectively.

Definition 2.3.4. Let R be a graded-commutative ring and $\mathcal{T}$ a triangulated category. We say that R *acts centrally* on $\mathcal{T}$, or that $\mathcal{T}$ is R*-linear*, if we have a homomorphism of graded rings $\phi\colon R \to Z^*(\mathcal{T})$.

A triangulated category $\mathcal{T}$ is R-linear if and only if we for each object $X \in \mathcal{T}$ have a homomorphism of graded rings $\phi_X\colon R \to \operatorname{End}^*_{\mathcal{T}}(X)$ such that if $r \in R$ is homogeneous of degree n and $f \in \operatorname{Hom}_{\mathcal{T}}(X,\Sigma^m Y)$, we have

$$f\phi_X(r) = (-1)^{nm}\phi_Y(r)f.$$

In other words, the R-module structures on $\operatorname{Hom}^*_{\mathcal{T}}(X,Y)$ induced by ϕ_X and ϕ_Y agree up to our usual sign rule.

Example 2.3.5. Let R be a commutative ring and $\mathbf{D}(R)$ its derived category. Every commutative ring is trivially a graded-commutative ring, where all elements are homogeneous of degree zero. For any X in $\mathbf{D}(R)$, we can define

$$\phi_X\colon R \to \operatorname{End}^*_{\mathbf{D}(R)}(X)$$

by multiplication. This is clearly a ring homomorphism. Since every element in R is of degree zero, we get that the necessary equation is satisfied. Hence, the derived category $\mathbf{D}(R)$ is R-linear.

The following definition gives a condition under which a central ring action is particularly well-behaved.

Definition 2.3.6. Let R be a graded-commutative ring and $\mathcal{T}$ an R-linear triangulated category. We say that $\mathcal{T}$ is *noetherian* under the action from R if $\operatorname{End}^*_{\mathcal{T}}(C)$ is a finitely generated R-module for all compact objects C in $\mathcal{T}$.

We will see that for a commutative noetherian ring R, the category $\mathbf{D}(R)$ is noetherian under the action from R. In order to prove this, we need the following basic lemma.

Lemma 2.3.7. *Let R be a noetherian ring, and consider a long exact sequence*

$$\cdots \longrightarrow M^{n-1} \xrightarrow{f^{n-1}} M^n \xrightarrow{f^n} M^{n+1} \longrightarrow \cdots$$

of R-modules. If M^{n-1} and M^{n+1} are finitely generated, then so is M^n.

Proof. Form the short exact sequence

$$0 \longrightarrow \operatorname{Ker}(f^n) \longrightarrow M^n \longrightarrow \operatorname{Im}(f^n) \longrightarrow 0.$$

The image $\operatorname{Im}(f^n)$ is a submodule of a finitely generated module over a noetherian ring, and thus finitely generated. So is $\operatorname{Ker}(f^n) = \operatorname{Im}(f^{n-1}) \simeq M^{n-1}/\operatorname{Ker}(f^{n-1})$. As the category of finitely generated modules is closed under extensions, also M^n has to be finitely generated. $\square$

Proposition 2.3.8. *The category $\mathbf{D}(R)$ is noetherian under the action from R for any graded-commutative noetherian ring R.*

Proof. By Theorem 2.2.8, we need to verify that $\operatorname{End}^*_{\mathbf{D}(R)}(C)$ is a finitely generated R-module for any C in $\operatorname{Thick}(R)$. We see that $\operatorname{End}^*_{\mathbf{D}(R)}(R) \simeq R$ is finitely generated. This turns out to be sufficient by the following argument.

Consider the full subcategory of $\mathbf{D}(R)$ given by

$$\mathcal{S}_1 = \{Y \in \mathbf{D}(R) \mid \operatorname{Hom}^*_{\mathbf{D}(R)}(R, Y) \text{ is a finitely generated } R\text{-module}\}.$$

A routine argument shows that this is a thick subcategory of $\mathbf{D}(R)$. Note that the 2/3-property follows from Lemma 2.3.7. Since $R \in \mathcal{S}_1$, we have $\operatorname{Thick}(R) \subseteq \mathcal{S}_1$, and thus $\operatorname{Hom}^*_{\mathbf{D}(R)}(R, Y)$ is finitely generated for any compact object Y. Let us now fix $Y \in \operatorname{Thick}(R)$ and consider the full subcategory

$$\mathcal{S}_2 = \{X \in \mathbf{D}(R) \mid \operatorname{Hom}^*_{\mathbf{D}(R)}(X, Y) \text{ is a finitely generated } R\text{-module}\}.$$

As before, the subcategory $\mathcal{S}_2$ turns out to be thick. Since $R \in \mathcal{S}_2$, we have $\operatorname{Thick}(R) \subseteq \mathcal{S}_2$. Hence, the R-module $\operatorname{Hom}^*_{\mathbf{D}(R)}(X, Y)$ is finitely generated for all compact objects X and Y. In particular, we can conclude that $\operatorname{End}^*_{\mathbf{D}(R)}(C)$ is finitely generated for any compact object C, so $\mathbf{D}(R)$ is noetherian under the action from R. $\square$

2.4 Local cohomology functors and support

We will now see that given a central ring action, we can construct what we call local cohomology functors. This enables us to define a notion of support which is more general than the one discussed in Chapter 1. The theory presented will be based on [3].

Throughout this section, let R be a graded-commutative noetherian ring and $\mathcal{T}$ a compactly generated R-linear triangulated category. We will now denote the set of homogeneous prime ideals in R by $\operatorname{Spec} R$. For a graded R-module M, we write $M_{\mathfrak{p}}$ for the homogeneous localization with respect to $\mathfrak{p} \in \operatorname{Spec} R$.

2.4.1 Localization functors

Definition 2.4.1. A triangulated functor $L\colon \mathcal{T} \to \mathcal{T}$ is called a *localization functor* if there exists a natural transformation $\eta\colon \operatorname{Id}_{\mathcal{T}} \to L$ such that $L\eta\colon L \to L^2$ is a natural isomorphism and $L\eta = \eta L$.

The following result gives a characterization of localization functors. For a proof, see [12, Prop. 2.4.1].

Proposition 2.4.2. *Let $L\colon \mathcal{T} \to \mathcal{T}$ be a triangulated functor. The following are equivalent:*

(i) L is a localization functor with natural transformation $\eta\colon \operatorname{Id}_{\mathcal{T}} \longrightarrow L$.

(ii) There exist a triangulated category $\mathcal{T}'$ and triangulated functors $F\colon \mathcal{T} \longrightarrow \mathcal{T}'$ and $G\colon \mathcal{T}' \longrightarrow \mathcal{T}$ such that G is fully faithful, $L = GF$ and (F, G) is an adjoint pair with unit $\eta\colon \operatorname{Id}_{\mathcal{T}} \longrightarrow GF$.

Corollary 2.4.3. *The kernel of a localization functor is a localizing subcategory.*

Proof. Let us denote our localization functor by L and the adjoint pair from Proposition 2.4.2 by (F, G). As L is a triangulated functor, the kernel of L is clearly a triangulated subcategory. To see that $\operatorname{Ker}(L)$ is localizing, notice that

$$\operatorname{Ker}(L) = \operatorname{Ker}(GF) = \operatorname{Ker}(F).$$

For the second equality, we use that G is faithful. Since F is a left adjoint functor, it commutes with colimits, so its kernel is closed under set-indexed coproducts. $\square$

We will also need the dual concept, namely that of colocalization functors.

Definition 2.4.4. A triangulated functor $\varGamma\colon \mathcal{T} \to \mathcal{T}$ is called a *colocalization functor* if there exists a natural transformation $\theta\colon \varGamma \to \operatorname{Id}_{\mathcal{T}}$ such that $\varGamma\theta\colon \varGamma^2 \to \varGamma$ is a natural isomorphism and $\varGamma\theta = \theta\varGamma$.

Note that a colocalization functor is a localization functor on the opposite category. We will see that given a localization functor, we can define a corresponding colocalization functor. In order to do so, we need Lemma 2.4.6 from below.

Definition 2.4.5. Let $L\colon \mathcal{T} \to \mathcal{T}$ be a localization functor. An object $X \in \mathcal{T}$ is called *L-acyclic* if $X \in \operatorname{Ker}(L)$ and *L-local* if $X \in \operatorname{Im}(L)$.

Lemma 2.4.6. *Let $L\colon \mathcal{T} \to \mathcal{T}$ be a localization functor. An object X is L-acyclic if and only if $\operatorname{Hom}_{\mathcal{T}}(X, Y) = 0$ for all L-local objects Y.*

Proof. Let the adjoint pair from Proposition 2.4.2 be denoted by (F, G), and recall that $L = GF$. Assume that X is L-acyclic, and let $Y \simeq L(Y')$ for some $Y' \in \mathcal{T}$. Since G is faithful, we see that $F(X) = 0$, and so

$$\begin{aligned}\operatorname{Hom}_{\mathcal{T}}(X, Y) &\simeq \operatorname{Hom}_{\mathcal{T}}(X, L(Y')) \simeq \operatorname{Hom}_{\mathcal{T}}(X, GF(Y')) \\ &\simeq \operatorname{Hom}_{\mathcal{T}}(F(X), F(Y')) = \operatorname{Hom}_{\mathcal{T}}(0, F(Y')) = 0,\end{aligned}$$

where we have used adjointness. For the reverse implication, assume that $\operatorname{Hom}_{\mathcal{T}}(X, Y) = 0$ for all L-local objects Y. Using adjointness again, we obtain

$$\operatorname{Hom}_{\mathcal{T}}(F(X), F(X)) \simeq \operatorname{Hom}_{\mathcal{T}}(X, GF(X)) \simeq \operatorname{Hom}_{\mathcal{T}}(X, L(X)) = 0,$$

so $F(X) = 0$. This implies $L(X) = 0$, so X is L-acyclic. □

We are now ready to define the colocalization functor corresponding to a localization functor $L\colon \mathcal{T} \to \mathcal{T}$ with natural transformation $\eta\colon \operatorname{Id}_{\mathcal{T}} \longrightarrow L$. Given any object $X \in \mathcal{T}$, we can complete the morphism η_X to a distinguished triangle

$$\varGamma(X) \xrightarrow{\theta_X} X \xrightarrow{\eta_X} L(X) \longrightarrow \Sigma\varGamma(X).$$

As the object $\varGamma(X)$ is unique up to isomorphism, we see that $\varGamma$ is well defined on objects. Given a morphism $f\colon X \to Y$, we define $\varGamma(f)$ as indicated by the dashed arrow

$$\begin{tikzcd} \varGamma(X) \arrow[r, "\theta_X"] \arrow[d, dashed, "\varGamma(f)"] & X \arrow[r, "\eta_X"] \arrow[d, "f"] & L(X) \arrow[r] \arrow[d, "L(f)"] & \Sigma\varGamma(X) \arrow[d] \\ \varGamma(Y) \arrow[r, "\theta_Y"] & Y \arrow[r, "\eta_Y"] & L(Y) \arrow[r] & \Sigma\varGamma(Y). \end{tikzcd}$$

The existence of $\varGamma(f)$ is clear from the axioms of a triangulated category, but we need to check that this is the *unique* morphism making the left square commute.

Note first that $\varGamma(X) \in \operatorname{Ker}(L)$. This is seen by applying L to the distinguished triangle in the first row above and using that $L\eta_X$ is an isomorphism. Let us now apply $\mathrm{Hom}_{\mathcal{T}}(\mathit{\Gamma}(X), -)$ to the distinguished triangle in the second row. Using that $\mathrm{Hom}_{\mathcal{T}}(\mathit{\Gamma}(X), \Sigma^{-1}L(Y)) = 0 = \mathrm{Hom}_{\mathcal{T}}(\mathit{\Gamma}(X), L(Y))$, which is true by Lemma 2.4.6, gives that

$$\begin{aligned} \mathrm{Hom}_{\mathcal{T}}(\mathit{\Gamma}(X), \mathit{\Gamma}(Y)) &\xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{T}}(\mathit{\Gamma}(X), Y) \\ g &\longmapsto \theta_Y \circ g \end{aligned}$$

is an isomorphism. In particular, the composition map above is injective. Returning to our commutative square, this yields that $\mathit{\Gamma}$ is well-defined also on morphisms. It is now easy to check that $\mathit{\Gamma}$ is indeed a functor.

Lemma 2.4.7. *For L and $\mathit{\Gamma}$ as above, we have equalities* $\mathrm{Ker}(L) = \mathrm{Im}(\mathit{\Gamma})$ *and* $\mathrm{Ker}(\mathit{\Gamma}) = \mathrm{Im}(L)$.

Proof. We have already seen that $\mathrm{Im}(\mathit{\Gamma}) \subseteq \mathrm{Ker}(L)$. For the reverse inclusion, notice that for $X \in \mathrm{Ker}(L)$, the morphism θ_X will be an isomorphism. The equality $\mathrm{Ker}(\mathit{\Gamma}) = \mathrm{Im}(L)$ is shown similarly. □

Proposition 2.4.8. *Let $L\colon \mathcal{T} \to \mathcal{T}$ be a localization functor and $\mathit{\Gamma}$ the corresponding functor defined as above. The following statements hold:*

(1) An object Y is L-local if and only if $\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0$ *for all L-acyclic objects X.*

(2) $\mathit{\Gamma}$ is right adjoint to the inclusion $\mathrm{Ker}(L) \hookrightarrow \mathcal{T}$.

(3) L is left adjoint to the inclusion $\mathrm{Ker}(\mathit{\Gamma}) \hookrightarrow \mathcal{T}$.

Proof. For (1), assume first that Y is L-local. For an L-acyclic object X, Lemma 2.4.6 implies that $\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0$.

Assume now that $\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0$ for all L-acyclic objects X. As $\mathit{\Gamma}(Y)$ is L-acyclic, this implies $\mathrm{Hom}_{\mathcal{T}}(\mathit{\Gamma}(Y), Y) = 0$. Hence, the distinguished triangle

$$\mathit{\Gamma}(Y) \xrightarrow{0} Y \xrightarrow{\eta_Y} L(Y) \longrightarrow \Sigma\mathit{\Gamma}(Y)$$

splits, so $L(Y) \simeq Y \oplus \Sigma\mathit{\Gamma}(Y)$. But as $\Sigma\mathit{\Gamma}(Y)$ is L-acyclic and $L(Y)$ is L-local, we have $\mathrm{Hom}_{\mathcal{T}}(\Sigma\mathit{\Gamma}(Y), L(Y)) = 0$, again by Lemma 2.4.6. The isomorphism hence implies that $\Sigma\mathit{\Gamma}(Y) = 0$, so Y is L-local.

Statement (2) is true by the same argument as the one used to show that $\mathit{\Gamma}$ is well defined on morphisms. (3) is shown analogously, but by applying a contravariant Hom-functor. □

Corollary 2.4.9. *$\mathit{\Gamma}$ is a triangulated functor.*

Proof. We know that $\varGamma$ is right adjoint to a triangulated functor, namely the inclusion $\mathrm{Ker}(L) \hookrightarrow \mathcal{T}$. By [17, Lemma 5.3.6], the functor $\varGamma$ is hence triangulated. $\square$

Notice that we can give a characterization of colocalization functors which is dual to the one given in Proposition 2.4.2. As the functor $\varGamma$ is right adjoint to the inclusion $\mathrm{Ker}(L) \hookrightarrow \mathcal{T}$, which is fully faithful, it is hence clear that $\varGamma$ is a colocalization functor with natural transformation $\theta\colon \varGamma \longrightarrow \mathrm{Id}_{\mathcal{T}}$. This gives a one-to-one correspondence between localization and colocalization functors.

2.4.2 Local cohomology functors

We are now ready to start developing one of the main tools in the BIK-approach, namely local cohomology functors. Recall that R is a graded-commutative noetherian ring and $\mathcal{T}$ a compactly generated R-linear triangulated category.

From Corollary 2.4.3, we know that the kernel of a localization functor is a localizing subcategory. It is hence natural to discuss which localizing subcategories that arise in this way.

Recall that $\operatorname{Spec} R$ denotes the set of homogeneous prime ideals in R and that $(-)_{\mathfrak{p}}$ is homogeneous localization at $\mathfrak{p} \in \operatorname{Spec} R$. The definition of closed and specialization closed subsets of $\operatorname{Spec} R$ from Chapter 1 carries over to our graded setting. Given a subset $V \subseteq \operatorname{Spec} R$, we define the full subcategory

$$\mathcal{T}_V = \{X \in \mathcal{T} \mid \mathrm{Hom}^*_{\mathcal{T}}(C, X)_{\mathfrak{p}} = 0 \text{ for all } \mathfrak{p} \in \operatorname{Spec} R \setminus V \text{ and } C \in \mathcal{T}^c\}.$$

The subcategory $\mathcal{T}_V$ is known as the subcategory of V-torsion objects in $\mathcal{T}$. This terminology is reasonable when we recall the characterization of $\mathfrak{p}$-torsion modules in terms of support given in Proposition 1.1.7.

By [3, Lemma 4.3], the subcategory $\mathcal{T}_V$ is localizing whenever V is specialization closed. In this case, there is always a localization functor $L_V\colon \mathcal{T} \to \mathcal{T}$ such that $\mathrm{Ker}(L_V) = \mathcal{T}_V$. For the construction, see [3, Prop. 4.5]. The colocalization functor corresponding to L_V is denoted by $\varGamma_V$. Given $X \in \mathcal{T}$, the object $\varGamma_V(X)$ is called the *local cohomology* of X supported on V, and $\varGamma_V$ is known as the *local cohomology functor* with respect to V.

Recall that $\mathcal{T}_V = \mathrm{Ker}(L_V) = \mathrm{Im}(\varGamma_V)$ and $\mathrm{Ker}(\varGamma_V) = \mathrm{Im}(L_V)$. The colocalization functor $\varGamma_V$ is right adjoint to the inclusion $\mathcal{T}_V \hookrightarrow \mathcal{T}$, as we saw in Proposition 2.4.8. For any object $X \in \mathcal{T}$, there is a distinguished triangle

$$\varGamma_V(X) \longrightarrow X \longrightarrow L_V(X) \longrightarrow \Sigma\varGamma_V(X).$$

Given $\mathfrak{p} \in \operatorname{Spec} R$, we define $\mathcal{Z}(\mathfrak{p})$ to be the subset

$$\mathcal{Z}(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \not\subseteq \mathfrak{p}\}.$$

Identifying $\operatorname{Spec} R_{\mathfrak{p}}$ with $\{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \subseteq \mathfrak{p}\}$, we can think of $\mathcal{Z}(\mathfrak{p})$ as $\operatorname{Spec} R \setminus \operatorname{Spec} R_{\mathfrak{p}}$. It is easy to verify that $\mathcal{Z}(\mathfrak{p})$ is specialization closed, so we have a localization functor $L_{\mathcal{Z}(\mathfrak{p})}$ with kernel $\mathcal{T}_{\mathcal{Z}(\mathfrak{p})}$. For a proof of the following property of $L_{\mathcal{Z}(\mathfrak{p})}$, see [3, Thm. 4.7].

Theorem 2.4.10. *For any $\mathfrak{p} \in \operatorname{Spec} R$ and objects $X \in \mathcal{T}$ and $C \in \mathcal{T}^{\mathrm{c}}$, we have an isomorphism*

$$\operatorname{Hom}^*_{\mathcal{T}}(C, X)_{\mathfrak{p}} \simeq \operatorname{Hom}^*_{\mathcal{T}}(C, L_{\mathcal{Z}(\mathfrak{p})}X).$$

Definition 2.4.11. Let $\mathfrak{p} \in \operatorname{Spec} R$. We define $\Gamma_{\mathfrak{p}}$ to be the functor given by the composition $\Gamma_{\mathfrak{p}} = \Gamma_{\mathcal{V}(\mathfrak{p})} \circ L_{\mathcal{Z}(\mathfrak{p})}$. The essential image of $\Gamma_{\mathfrak{p}}$ is denoted by $\Gamma_{\mathfrak{p}}\mathcal{T}$.

Note that by [3, Prop. 6.1], we could have interchanged the order in our definition, as $\Gamma_{\mathfrak{p}} = \Gamma_{V(\mathfrak{p})} L_{\mathcal{Z}(\mathfrak{p})} \simeq L_{\mathcal{Z}(\mathfrak{p})} \Gamma_{V(\mathfrak{p})}$.

We say that an object $X \in \mathcal{T}$ is $\mathfrak{p}$-*local* if $X \in \operatorname{Im}(L_{\mathcal{Z}(\mathfrak{p})})$, i.e. if X is $L_{\mathcal{Z}(\mathfrak{p})}$-local. This is a generalization of the terminology from Definition 1.1.6, because if $\mathcal{T} = \mathbf{D}(R)$ for a commutative noetherian ring R, the localization functor $(-)_{\mathfrak{p}}$ coincides with $L_{\mathcal{Z}(\mathfrak{p})}$. This is shown in [3, §9].

An object $X \in \mathcal{T}$ is $\mathfrak{p}$-*torsion* if $X \in \operatorname{Im}(\Gamma_{V(\mathfrak{p})}) = \operatorname{Ker}(L_{V(\mathfrak{p})})$, i.e. if X is in the subcategory of $V(\mathfrak{p})$-torsion objects in $\mathcal{T}$.

Using that $\Gamma_{\mathfrak{p}} = \Gamma_{V(\mathfrak{p})} L_{\mathcal{Z}(\mathfrak{p})} \simeq L_{\mathcal{Z}(\mathfrak{p})} \Gamma_{V(\mathfrak{p})}$, we see that $\Gamma_{\mathfrak{p}}\mathcal{T}$ is the subcategory of $\mathcal{T}$ consisting of objects which are both $\mathfrak{p}$-local and $\mathfrak{p}$-torsion.

2.4.3 Support

Local cohomology functors enable us to define a notion of support for R-linear compactly generated triangulated categories.

Definition 2.4.12. Let X be an object in $\mathcal{T}$. The support of X is given by

$$\operatorname{supp} X = \{\mathfrak{p} \in \operatorname{Spec} R \mid \Gamma_{\mathfrak{p}}(X) \neq 0\}.$$

This notion of support is sometimes called *triangulated support*. Note that we here use a lower case s as opposed to the notation used in Definition 1.1.4. However, by [3], we know that in the case $\mathcal{T} = \mathbf{D}^{\mathrm{b}}(\operatorname{mod} R)$ for a commutative noetherian ring R, our two notions of support coincide.

The following result is an important tool for calculating support. For a proof, see [3, Thm. 5.6].

Theorem 2.4.13. *Let $V \subseteq \operatorname{Spec} R$ be specialization closed. Given an object $X \in \mathcal{T}$, the following equalities hold:*

$$\operatorname{supp} \Gamma_V(X) = V \cap \operatorname{supp} X$$
$$\operatorname{supp} L_V(X) = (\operatorname{Spec} R \setminus V) \cap \operatorname{supp} X.$$

Theorem 2.4.13 has several important consequences.

Proposition 2.4.14. *Let $V \subseteq \operatorname{Spec} R$ be specialization closed. Given an object $X \in \mathcal{T}$, the following hold:*

(1) $\operatorname{supp} X \subseteq V \iff \varGamma_V(X) \simeq X$.

(2) $V \cap \operatorname{supp} X = \varnothing \iff X \simeq L_V(X)$.

Proof. For (1) we will argue that

$$\begin{aligned} \varGamma_V(X) \simeq X &\iff L_V(X) = 0 \\ &\iff \operatorname{supp} L_V(X) = (\operatorname{Spec} R \setminus V) \cap \operatorname{supp} X = \varnothing \\ &\iff \operatorname{supp} X \subseteq V. \end{aligned}$$

The first equivalence is clear from the distinguished triangle

$$\varGamma_V(X) \longrightarrow X \longrightarrow L_V(X) \longrightarrow \Sigma\varGamma_V(X).$$

By [3, Thm. 5.2], we know that an object is zero if and only if it has empty support. This, together with Theorem 2.4.13, gives the second equivalence, while the third one is straightforward.

Statement (2) is shown by a similar argument. □

Proposition 2.4.15. *Let $V \subseteq \operatorname{Spec} R$ be specialization closed and $\mathfrak{p} \in \operatorname{Spec} R$. Given an object $X \in \mathcal{T}$, we have*

$$\varGamma_{\mathfrak{p}}\varGamma_V(X) \simeq \begin{cases} \varGamma_{\mathfrak{p}}(X) & \mathfrak{p} \in V \\ 0 & \mathfrak{p} \notin V \end{cases} \quad \textit{and} \quad \varGamma_{\mathfrak{p}}L_V(X) \simeq \begin{cases} \varGamma_{\mathfrak{p}}(X) & \mathfrak{p} \notin V \\ 0 & \mathfrak{p} \in V. \end{cases}$$

Proof. Apply $\varGamma_{\mathfrak{p}}$ to the distinguished triangle

$$\varGamma_V(X) \longrightarrow X \longrightarrow L_V(X) \longrightarrow \Sigma\varGamma_V(X).$$

As $\operatorname{supp} \varGamma_V(X) = V \cap \operatorname{supp} X$, we must have $\varGamma_{\mathfrak{p}}\varGamma_V(X) = 0$ whenever $\mathfrak{p} \notin V$, which by the triangle yields $\varGamma_{\mathfrak{p}}L_V(X) \simeq \varGamma_{\mathfrak{p}}(X)$. The case $\mathfrak{p} \in V$ is shown analogously. □

Proposition 2.4.16. *Let $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} R$. Given an object $X \in \mathcal{T}$, we have*

$$\varGamma_{\mathfrak{p}}\varGamma_{\mathfrak{q}}(X) \simeq \begin{cases} \varGamma_{\mathfrak{p}}(X) & \mathfrak{p} = \mathfrak{q} \\ 0 & \mathfrak{p} \neq \mathfrak{q}. \end{cases}$$

Proof. By Proposition 2.4.15,

$$\varGamma_{\mathfrak{p}}\varGamma_{\mathfrak{q}}(X) = \varGamma_{\mathfrak{p}}\varGamma_{V(\mathfrak{q})}L_{\mathcal{Z}(\mathfrak{q})}(X) \simeq \begin{cases} \varGamma_{\mathfrak{p}}L_{\mathcal{Z}(\mathfrak{q})}(X) & \mathfrak{p} \in V(\mathfrak{q}) \\ 0 & \mathfrak{p} \notin V(\mathfrak{q}), \end{cases}$$

and

$$\varGamma_{\mathfrak{p}}L_{\mathcal{Z}(\mathfrak{q})}(X) \simeq \begin{cases} \varGamma_{\mathfrak{p}}(X) & \mathfrak{p} \notin \mathcal{Z}(\mathfrak{q}) \\ 0 & \mathfrak{p} \in \mathcal{Z}(\mathfrak{q}). \end{cases}$$

Thus, the only case in which $\varGamma_{\mathfrak{p}}\varGamma_{\mathfrak{q}}(X) \neq 0$ is if $\mathfrak{p} \in V(\mathfrak{q}) \cap (\operatorname{Spec} R \setminus \mathcal{Z}(\mathfrak{q})) = \{\mathfrak{q}\}$, which yields our desired result. □

Note that as $\varGamma_{\mathfrak{p}}^2 \simeq \varGamma_{\mathfrak{p}}$, an object X is in $\varGamma_{\mathfrak{p}}\mathcal{T}$ if and only if $\varGamma_{\mathfrak{p}}(X) \simeq X$. We are now ready to describe $\varGamma_{\mathfrak{p}}\mathcal{T}$ as the subcategory consisting of objects supported on $\{\mathfrak{p}\}$. Note that our definition of $\mathfrak{p}$-local and $\mathfrak{p}$-torsion modules from Chapter 1 carries over to the graded setting in the natural way.

Proposition 2.4.17. *Let X be a non-zero object in $\mathcal{T}$. The following statements are equivalent for $\mathfrak{p} \in \operatorname{Spec} R$:*

(i) $\varGamma_{\mathfrak{p}}(X) \simeq X$.

(ii) $\operatorname{supp} X = \{\mathfrak{p}\}$.

(iii) $\operatorname{Hom}^*_{\mathcal{T}}(C, X)$ *is $\mathfrak{p}$-local and $\mathfrak{p}$-torsion for all $C \in \mathcal{T}^c$.*

Proof. $(i) \implies (iii)$: Assume $\varGamma_{\mathfrak{p}}(X) \simeq X$. For any $C \in \mathcal{T}^c$, this gives

$$\begin{aligned} \operatorname{Hom}^*_{\mathcal{T}}(C, X) &\simeq \operatorname{Hom}^*_{\mathcal{T}}(C, \varGamma_{\mathfrak{p}}(X)) \\ &\simeq \operatorname{Hom}^*_{\mathcal{T}}(C, L_{\mathcal{Z}(\mathfrak{p})}\varGamma_{V(\mathfrak{p})}(X)) \\ &\simeq \operatorname{Hom}^*_{\mathcal{T}}(C, \varGamma_{V(\mathfrak{p})}(X))_{\mathfrak{p}}, \end{aligned}$$

where the last isomorphism is by Theorem 2.4.10. Consequently, we see that $\operatorname{Hom}^*_{\mathcal{T}}(C, X)$ is $\mathfrak{p}$-local.

Recall that $\operatorname{Im}(\varGamma_{V(\mathfrak{p})}) = \operatorname{Ker}(L_{V(\mathfrak{p})}) = \mathcal{T}_{V(\mathfrak{p})}$, so $\operatorname{Hom}^*_{\mathcal{T}}(C, \varGamma_{V(\mathfrak{p})}(X))$ is $\mathfrak{p}$-torsion by Proposition 1.1.7. As a localization of a $\mathfrak{p}$-torsion module is again $\mathfrak{p}$-torsion, the isomorphism above hence also gives that $\operatorname{Hom}^*_{\mathcal{T}}(C, X)$ is $\mathfrak{p}$-torsion.

$(iii) \implies (ii)$: This follows from Lemma 2.4 and [3, Thm. 5.2].

$(ii) \implies (i)$: Assume $\operatorname{supp} X = \{\mathfrak{p}\}$. By Theorem 2.4.13, this implies that $\operatorname{supp} \varGamma_{V(\mathfrak{p})}(X) = \{\mathfrak{p}\}$. Now,

$$X \simeq \varGamma_{V(\mathfrak{p})}(X) \simeq L_{\mathcal{Z}(\mathfrak{p})}\varGamma_{V(\mathfrak{p})}(X) = \varGamma_{\mathfrak{p}}(X),$$

where the isomorphisms are by Proposition 2.4.14. For the second one, notice that $\mathcal{Z}(\mathfrak{p}) \cap \{\mathfrak{p}\} = \varnothing$. □

Corollary 2.4.18. $\varGamma_{\mathfrak{p}}\mathcal{T}$ *is a localizing subcategory for all* $\mathfrak{p} \in \operatorname{Spec} R$.

2.5 Stratification and classification

Benson, Iyengar and Krause show in [6] that if a central ring action satisfies certain conditions, we can use the associated support to classify localizing, and in nice cases also thick, subcategories.

A non-zero localizing subcategory is called *minimal* if it contains no non-zero proper localizing subcategories. For a graded-commutative ring R and a compactly generated R-linear triangulated category $\mathcal{T}$, we will use the notation

$$\operatorname{supp}\mathcal{T} = \bigcup_{X\in\mathcal{T}} \operatorname{supp} X.$$

Definition 2.5.1. Let R be a graded-commutative noetherian ring and $\mathcal{T}$ a compactly generated R-linear triangulated category. We say that the action from R on $\mathcal{T}$ is *stratifying* if the following two conditions are satisfied:

(1) For any object X in $\mathcal{T}$, we have

$$\operatorname{Loc}_{\mathcal{T}}(X) = \operatorname{Loc}_{\mathcal{T}}(\{\varGamma_{\mathfrak{p}}(X) \mid \mathfrak{p} \in \operatorname{Spec} R\}).$$

(2) For any $\mathfrak{p}$ in $\operatorname{supp}\mathcal{T}$, the subcategory $\varGamma_{\mathfrak{p}}\mathcal{T}$ is a minimal localizing subcategory of $\mathcal{T}$.

These two conditions are known as the *local-global principle* and the *minimality condition*, respectively.

We are now ready to present one of the main results in the BIK-approach.

Theorem 2.5.2. *Let R be a graded-commutative noetherian ring and $\mathcal{T}$ a compactly generated R-linear triangulated category. We have maps of sets*

$$\left\{\begin{matrix}\textit{localizing}\\ \textit{subcategories of } \mathcal{T}\end{matrix}\right\} \underset{\tau}{\overset{\sigma}{\rightleftarrows}} \left\{\begin{matrix}\textit{subsets of}\\ \operatorname{supp}\mathcal{T}\end{matrix}\right\}$$

where $\sigma(\mathcal{S}) = \operatorname{supp}\mathcal{S}$ for a localizing subcategory $\mathcal{S}$, and $\tau(V)$ is the full subcategory given by $\{X \in \mathcal{T} \mid \operatorname{supp} X \subseteq V\}$ for a subset $V \subseteq \operatorname{Spec} R$.

If the action from R on $\mathcal{T}$ is stratifying, then σ and τ are inverse bijections.

Proof. Let us first check that $\tau(V)$ is a localizing subcategory for any subset $V \subseteq \operatorname{Spec} R$. Noticing that the property given in Lemma 1.1.5 (5) holds also for triangulated support, we see that $\tau(V)$ is a triangulated subcategory. By [3, Cor. 6.5], the functor $\varGamma_{\mathfrak{p}}$ commutes with set-indexed coproducts, so $\tau(V)$ is localizing.

We will next show that $\sigma\tau(V) = V$ for a subset $V \subseteq \operatorname{supp}\mathcal{T}$. Let $\mathfrak{p} \in V$. As $V \subseteq \operatorname{supp}\mathcal{T}$, there exists $X \in \mathcal{T}$ such that $\varGamma_{\mathfrak{p}}(X) \neq 0$. By Theorem 2.4.13, this implies that $\operatorname{supp} \varGamma_{\mathfrak{p}}(X) = \{\mathfrak{p}\}$. Hence, we have $\varGamma_{\mathfrak{p}}(X) \in \tau(V)$, so $\mathfrak{p} \in \sigma\tau(V)$. This gives $V \subseteq \sigma\tau(V)$. The reverse inclusion is obvious.

Our next aim is to prove that $\tau\sigma(\mathcal{S}) = \mathcal{S}$ for a localizing subcategory $\mathcal{S} \subseteq \mathcal{T}$. Observe that the inclusion $\mathcal{S} \subseteq \tau\sigma(\mathcal{S})$ follows directly from the definitions. Note also that everything we have done so far is independent of the stratification conditions. The crucial point is hence to verify that $\tau\sigma(\mathcal{S}) \subseteq \mathcal{S}$. Instead of showing this directly, we will see that it is a consequence of a more general result, namely Theorem 2.5.3 below. Note that when the minimality condition is satisfied, the maps σ and τ in the two theorems coincide. To see this, we identify a family $(\mathcal{S}(\mathfrak{p}))_{\mathfrak{p} \in \operatorname{supp} \mathcal{T}}$ with the subset $\{\mathfrak{p} \in \operatorname{Spec} R \mid \mathcal{S}(\mathfrak{p}) \neq 0\}$. □

It remains to state and prove Theorem 2.5.3.

Theorem 2.5.3. *Let R be a graded-commutative noetherian ring and $\mathcal{T}$ a compactly generated R-linear triangulated category. Assume that the R-action satisfies the local-global principle, and set $\mathcal{V} = \operatorname{supp} \mathcal{T}$. We then have inverse bijections*

$$\left\{\begin{matrix}\textit{localizing}\\ \textit{subcategories of } \mathcal{T}\end{matrix}\right\} \underset{\tau}{\overset{\sigma}{\rightleftarrows}} \left\{\begin{matrix}\textit{families } (\mathcal{S}(\mathfrak{p}))_{\mathfrak{p}\in\mathcal{V}}, \textit{ where } \mathcal{S}(\mathfrak{p})\\ \textit{is a localizing subcategory of } \varGamma_{\mathfrak{p}}\mathcal{T}\end{matrix}\right\}$$

where $\sigma(\mathcal{S}) = (\mathcal{S} \cap \varGamma_{\mathfrak{p}}\mathcal{T})_{\mathfrak{p}\in\mathcal{V}}$ for a localizing subcategory $\mathcal{S} \subseteq \mathcal{T}$ and

$$\tau((\mathcal{S}(\mathfrak{p}))_{\mathfrak{p}\in\mathcal{V}}) = \operatorname{Loc}_{\mathcal{T}}(\{\mathcal{S}(\mathfrak{p}) \mid \mathfrak{p} \in \mathcal{V}\}).$$

Proof. We will now fix a family $(\mathcal{S}(\mathfrak{p}))_{\mathfrak{p}\in\mathcal{V}}$ as described above, and let $\mathcal{S}$ denote the subcategory $\tau((\mathcal{S}(\mathfrak{p}))_{\mathfrak{p}\in\mathcal{V}}) = \operatorname{Loc}_{\mathcal{T}}(\{\mathcal{S}(\mathfrak{p}) \mid \mathfrak{p} \in \mathcal{V}\})$. Our first aim is to prove that $\sigma(\mathcal{S}) = (\mathcal{S}(\mathfrak{p}))_{\mathfrak{p}\in\mathcal{V}}$. It suffices to show that $\mathcal{S} \cap \varGamma_{\mathfrak{p}}\mathcal{T} = \mathcal{S}(\mathfrak{p})$ for all $\mathfrak{p} \in \mathcal{V}$. Using Proposition 2.4.16 together with Lemma 2.1.4, we see that $\mathcal{S} \cap \varGamma_{\mathfrak{p}}\mathcal{T} \subseteq \varGamma_{\mathfrak{p}}\mathcal{S} = \mathcal{S}(\mathfrak{p})$. The reverse inclusion is clear, as $\mathcal{S}(\mathfrak{p}) \subseteq \mathcal{S}$ and $\mathcal{S}(\mathfrak{p}) \subseteq \varGamma_{\mathfrak{p}}\mathcal{T}$.

We next want to prove that $\tau\sigma(\mathcal{S}) = \mathcal{S}$ for any localizing subcategory $\mathcal{S} \subseteq \mathcal{T}$, i.e. that $\operatorname{Loc}_{\mathcal{T}}(\{\mathcal{S} \cap \varGamma_{\mathfrak{p}} \mid \mathfrak{p} \in \mathcal{V}\}) = \mathcal{S}$. Let $X \in \mathcal{S}$. We have

$$X \in \operatorname{Loc}_{\mathcal{T}}(X) = \operatorname{Loc}_{\mathcal{T}}(\{\varGamma_{\mathfrak{p}}(X) \mid \mathfrak{p} \in V\}) \subseteq \operatorname{Loc}_{\mathcal{T}}(\{\mathcal{S} \cap \varGamma_{\mathfrak{p}}\mathcal{T} \mid \mathfrak{p} \in \mathcal{V}\}),$$

where the equality is by the local-global principle. For the inequality, notice that $\operatorname{Loc}_{\mathcal{T}}(\varGamma_{\mathfrak{p}}(X)) \subseteq \operatorname{Loc}_{\mathcal{T}}(X) \subseteq \mathcal{S}$, which gives $\varGamma_{\mathfrak{p}}(X) \in \mathcal{S} \cap \varGamma_{\mathfrak{p}}\mathcal{T}$ for all $\mathfrak{p} \in \mathcal{V}$. Hence, we know that $\mathcal{S} \subseteq \operatorname{Loc}_{\mathcal{T}}(\{\mathcal{S} \cap \varGamma_{\mathfrak{p}} \mid \mathfrak{p} \in \mathcal{V}\})$. The reverse inclusion is straightforward. □

A commutative ring is trivially a graded-commutative ring concentrated in degree 0. The action from a commutative noetherian ring R on $\mathbf{D}(R)$ is stratifying, see [4, Thm. 5.1], and we know that $\mathbf{D}(R)$ is a compactly generated R-linear triangulated category. The BIK-approach hence provides a new setup for proving that there is a bijection between localizing subcategories of $\mathbf{D}(R)$ and subsets of $\operatorname{Spec} R$, which was originally proved by Neeman [15].

So far in this chapter, we have presented classification results for localizing subcategories. Our next aim is to show that the BIK-approach also provides classification of thick subcategories, which again will enable us to recover the Hopkins-Neeman theorem from this setup. We first need some lemmas.

Recall that Theorem 1.2.4 was crucial in the proof of the Hopkins-Neeman theorem which was presented in Chapter 1. Given Theorem 2.5.2, we can deduce similar results in our more general setting.

Lemma 2.5.4. *Let R be a graded-commutative noetherian ring and $\mathcal{T}$ a compactly generated R-linear triangulated category. Assume that the action from R on $\mathcal{T}$ is stratifying, and let X and Y be objects in $\mathcal{T}$. We have the following implication:*

$$\operatorname{supp} X \subseteq \operatorname{supp} Y \implies \operatorname{Loc}_{\mathcal{T}}(X) \subseteq \operatorname{Loc}_{\mathcal{T}}(Y).$$

Proof. Let σ and τ be defined as in Theorem 2.5.2. If $\operatorname{supp} X \subseteq \operatorname{supp} Y$, then $X \in \tau\sigma(\operatorname{Loc}_{\mathcal{T}}(Y))$. But as $\operatorname{Loc}_{\mathcal{T}}(Y) = \tau\sigma(\operatorname{Loc}_{\mathcal{T}}(Y))$, this implies that $\operatorname{Loc}_{\mathcal{T}}(X) \subseteq \operatorname{Loc}_{\mathcal{T}}(Y)$. □

This can be strengthened in the context of compact objects.

Lemma 2.5.5. *Let R be a graded-commutative noetherian ring and $\mathcal{T}$ a compactly generated R-linear triangulated category. Assume that the action from R on $\mathcal{T}$ is stratifying, and let X and Y be objects in $\mathcal{T}^c$. We have the following implication:*

$$\operatorname{supp} X \subseteq \operatorname{supp} Y \implies \operatorname{Thick}(X) \subseteq \operatorname{Thick}(Y).$$

Proof. Assume $\operatorname{supp} X \subseteq \operatorname{supp} Y$. By our previous lemma, we know that $\operatorname{Loc}_{\mathcal{T}}(X) \subseteq \operatorname{Loc}_{\mathcal{T}}(Y)$. As X and Y are compact, Theorem 2.2.7 now implies $\operatorname{Thick}(X) \subseteq \operatorname{Thick}(Y)$. □

Remark 2.5.6.

(1) In our proof of the Hopkins-Neeman theorem in Chapter 1, we used that the support of a perfect complex is closed. Analogously, if our R-linear category $\mathcal{T}$ is noetherian, [4, Thm. 4.22] gives that the triangulated support of a compact object is closed in $\operatorname{Spec} R$.

(2) In order to show some of the results we have used, but not proved, Benson, Iyengar and Krause consider Koszul objects in $\mathcal{T}$. This construction corresponds to the way we defined Koszul complexes in Chapter 1. Just as we saw in Proposition 1.1.8, it turns out that every closed subset of $\operatorname{supp} \mathcal{T}$ can be realized as the support of a compact object in $\mathcal{T}$.

We are now ready to present the BIK-analogue of the Hopkins-Neeman theorem.

Theorem 2.5.7. *Let R be a graded-commutative noetherian ring and $\mathcal{T}$ a compactly generated R-linear triangulated category. Assume that $\mathcal{T}$ is noetherian under the action from R and that the stratification conditions are satisfied. We have inverse bijections*

$$\left\{\begin{matrix}\textit{thick subcategories}\\ \textit{of } \mathcal{T}^c\end{matrix}\right\} \underset{\tau}{\overset{\sigma}{\rightleftarrows}} \left\{\begin{matrix}\textit{specialization closed}\\ \textit{subsets of } \operatorname{supp}\mathcal{T}\end{matrix}\right\}$$

where $\sigma(\mathcal{S}) = \operatorname{supp}\mathcal{S}$ for a thick subcategory $\mathcal{S}$, and $\tau(V)$ is the full subcategory given by $\{X \in \mathcal{T}^c \mid \operatorname{supp} X \subseteq V\}$ for a specialization closed subset $V \subseteq \operatorname{Spec} R$.

Proof. Applying Lemma 2.5.5 and the remark above, the proof follows exactly the same lines as the proof of Theorem 1.2.1 (Hopkins-Neeman), which is given in full detail in Chapter 1. $\square$

Note that for a commutative noetherian ring R, we have $\operatorname{supp}\mathbf{D}(R) = \operatorname{Spec} R$. Recalling that $\mathbf{D}(R)$ is a noetherian category where the compact objects are given by $\mathbf{D}^{\mathrm{b}}(\operatorname{proj} R)$, we see that the Hopkins-Neeman theorem is a special case of Theorem 2.5.7.

As we have seen, we can use the BIK-approach to recover already known classification results. The real strength of the approach, however, is that it can be enlightening also in new situations. In Chapter 3 we will apply the techniques presented in this chapter to a triangulated category for which classification results are not yet developed.

Chapter 3

The case of a quantum polynomial ring

Through the theory presented so far in this thesis, we understand the localizing subcategories of the derived category of a *commutative* noetherian ring, as well as the thick subcategories of its category of perfect complexes. Our motivating question for this chapter is whether one can develop similar results for non-commutative rings. We will work with a class of rings which are very "close to" being commutative.

3.1 Notation and basic properties

We will use the notation $k[x, y]$ for the polynomial ring in two variables over a field k, and the notation $k\langle x, y\rangle$ for the free algebra on two generators over the same field. In $k[x, y]$, the variables x and y commute, i.e. $xy = yx$, while this is not the case in $k\langle x, y\rangle$. We can think of the free algebra as a non-commutative analogue of the polynomial ring. However, we want to look at a quotient of $k\langle x, y\rangle$ where xy and yx differ only by a certain coefficient.

Throughout the rest of this thesis, we will use the following notation:

- $A = k\langle x, y\rangle/(xy - qyx)$, where $q \in k$ is a primitive n-th root of unity and k is an algebraically closed field whose characteristic does not divide n. A is what we call a *quantum polynomial ring* in two variables.
- $B = Z(A)$, where $Z(A)$ denotes the centre of A.

We start by looking at some elementary, but useful, properties of A and B. Recall first that a ring with no non-trivial zero divisors is called a *domain*.

Proposition 3.1.1. *A is a domain.*

Proof. Let α and β be two non-zero elements in A. Using our relations, we order all terms such that the x's are to the left, and the y's to the right. We think of α and β as polynomials in non-commuting variables and count degree in the natural way. Look at the terms in α with largest x-degree. Among these, pick the term with largest y-degree. Do the same for β. The term in $\alpha\beta$ with largest y-degree among the terms with largest x-degree, will be the product of our two chosen terms. This product is non-zero and cannot be cancelled by any other term. We can hence conclude that $\alpha\beta$ is non-zero. □

As A is a domain, we know that cancellation laws are satisfied. This will be used in the proof of our next proposition.

Proposition 3.1.2. *The following statements hold:*

(1) B is equal to $k[x^n, y^n]$.

(2) A is a finitely generated free module over B.

(3) A is a noetherian ring.

Proof. For (1), consider at the subalgebra of A generated by x^n and y^n. Since $q^n = 1$, this subalgebra is commutative and can be identified with the polynomial ring $k[x^n, y^n]$. The generators x^n and y^n commute with every ring element in A, so $k[x^n, y^n] \subseteq B$. For the reverse inclusion, let $\alpha \in B$. We can assume $\alpha = x^i y^j$ for some $i, j \in \mathbb{N}$. Since $\alpha \in Z(A)$, it commutes with all elements in A. In particular, we have $x\alpha = \alpha x$. As

$$x\alpha = xx^iy^j = q^jx^iy^jx = q^j\alpha x,$$

and cancellation laws hold, this implies $q^j = 1$. Since q is a primitive n-th root of unity, this means that n must divide j. By the same argument, we know that n divides i, so $\alpha \in k[x^n, y^n]$.

The set $\{x^iy^j \mid 0 \leq i, j < n\}$ is a basis for A as a B-module, which yields (2).

For (3), notice that the ring $B = k[x^n, y^n]$ is noetherian by Hilbert's basis theorem. As A is finitely generated as an algebra over B, for instance by the generating set $\{1, x, x^2, \ldots, x^{n-1}, y, y^2, \ldots, y^{n-1}\}$, also A is noetherian.

□

3.2 A central ring action

In Section 2.3 we discussed what it means for a graded-commutative ring to act centrally on a triangulated category. Since B is commutative, we know from Example 2.3.5 that B acts on $\mathbf{D}(B)$. Of more interest to us, is that it also acts on $\mathbf{D}(A)$.

Proposition 3.2.1. *The category* $\mathbf{D}(A)$ *is* B*-linear.*

Proof. We think of B as a graded-commutative ring concentrated in degree zero. For any object X in $\mathbf{D}(A)$, define

$$\phi_X : B \to \operatorname{End}^*_{\mathbf{D}(A)}(X)$$

by multiplication. Note that this is possible because $B = Z(A)$, so multiplying with an element in B is an A-linear map. One can easily check that ϕ_X is a ring homomorphism. We also have the equality

$$f\phi_X(b) = \phi_Y(b)f$$

for any element b in B, object Y in $\mathbf{D}(A)$ and homogeneous morphism f in $\operatorname{Hom}^*_{\mathbf{D}(A)}(X, Y)$. Since every $b \in B$ is of degree zero, we hence have a central ring action. □

Note that this argument is valid in a more general setup. Given a ring R, any subring contained in $Z(R)$ will act centrally on $\mathbf{D}(R)$.

Proposition 3.2.2. *The category* $\mathbf{D}(A)$ *is noetherian under the action from* B.

Proof. By Theorem 2.2.8, we need to verify that $\operatorname{End}^*_{\mathbf{D}(A)}(C)$ is a finitely generated B-module for any C in $\operatorname{Thick}(A)$. As in the proof of Proposition 2.3.8, we notice that $\operatorname{End}^*_{\mathbf{D}(A)}(A) \simeq A$, which by Proposition 3.1.2 is finitely generated over B. This turns out to be sufficient by the same argument as in Proposition 2.3.8, using that B is noetherian. □

3.3 The stratification conditions

As we have seen, Benson, Iyengar and Krause's developments give an alternative approach to the proof of the Hopkins-Neeman theorem, namely through use of Theorem 2.5.7. We have shown that some of the assumptions required in order to use this theorem are satisfied also in our case; B is a commutative noetherian ring and $\mathbf{D}(A)$ a compactly generated B-linear triangulated category which is noetherian under the action from B. With this in mind, we want to investigate whether the action from B on $\mathbf{D}(A)$ could be stratifying. If this was the case, we would be able to to classify subcategories of $\mathbf{D}(A)$ in terms of $\operatorname{Spec} B$ as described in Chapter 2.

Recall from Section 2.5 that the action from a commutative noetherian ring R on a compactly generated R-linear triangulated category $\mathcal{T}$ is stratifying if the following two conditions are satisfied:

(1) *The local-global principle:* For any object X in $\mathcal{T}$, we have

$$\mathrm{Loc}_{\mathcal{T}}(X) = \mathrm{Loc}_{\mathcal{T}}(\{\varGamma_{\mathfrak{p}}(X) \mid \mathfrak{p} \in \operatorname{Spec} R\}).$$

(2) *The minimality condition:* For any $\mathfrak{p}$ in $\operatorname{supp} \mathcal{T}$, the subcategory $\varGamma_{\mathfrak{p}}\mathcal{T}$ is a minimal localizing subcategory of $\mathcal{T}$.

It is fairly easy to see that the first stratification condition is satisfied in our case, while examining the second one turns out to be more complicated. This will be one of our main objectives in the rest of the thesis.

One can give a general condition under which the local-global principle is always satisfied. This is [6, Thm. 3.4].

Theorem 3.3.1. *Let R be a graded-commutative noetherian ring acting centrally on a compactly generated triangulated category. If R has finite Krull dimension, then the local-global principle is satisfied.*

Corollary 3.3.2. *The action from B on $\mathbf{D}(A)$ satisfies the local-global principle.*

Proof. The Krull dimension of B is two, see [1, Ex. 11.7]. Our result now follows from Theorem 3.3.1. □

To investigate whether or not the second stratification condition is satisfied, it is useful to have a result which characterizes minimal localizing subcategories. The following is [6, Lemma 4.1].

Lemma 3.3.3. *Let $\mathcal{T}$ be a compactly generated triangulated category and $\mathcal{S} \subseteq \mathcal{T}$ a non-zero localizing subcategory. We have that $\mathcal{S}$ is a minimal localizing subcategory if and only if $\mathrm{Hom}^*_{\mathcal{T}}(X, Y) \neq 0$ for all non-zero objects X and Y in $\mathcal{S}$.*

Proof. Assume that $\mathcal{S}$ is minimal, and fix a non-zero object Y in $\mathcal{S}$. Consider the full subcategory

$$\mathcal{U} = \{Z \in \mathcal{T} \mid \mathrm{Hom}^*_{\mathcal{T}}(Z, Y) = 0\}.$$

This is a localizing subcategory. If X is a non-zero object in $\mathcal{S}$, the minimality assumption implies that $\mathrm{Loc}_{\mathcal{T}}(X) = \mathrm{Loc}_{\mathcal{T}}(Y) = \mathcal{S}$. If X was in $\mathcal{U}$, we would hence also have Y in $\mathcal{U}$, which is impossible.

For the other direction, assume that $\mathcal{S}$ is not a minimal localizing subcategory. This means that there exists a non-zero object $X \in \mathcal{S}$ such that $\mathrm{Loc}_{\mathcal{T}}(X)$ is strictly contained in $\mathcal{S}$. As $\mathcal{T}$ is compactly generated, there is a localization functor $L\colon \mathcal{T} \to \mathcal{T}$ such that $\mathrm{Ker}(L) = \mathrm{Loc}_{\mathcal{T}}(X)$, see [6, Lemma 2.1]. Let $\varGamma$ denote the corresponding colocalization functor. Pick an object $W \in \mathcal{S} \setminus \mathrm{Loc}_{\mathcal{T}}(X)$, and consider the distinguished triangle

$$\Gamma(W) \longrightarrow W \longrightarrow L(W) \longrightarrow \Sigma\Gamma(W).$$

As $\Gamma(W) \in \mathrm{Im}(\Gamma) = \mathrm{Ker}(L) = \mathrm{Loc}_{\mathcal{T}}(X)$ and $W \notin \mathrm{Loc}_{\mathcal{T}}(X)$, the first morphism in the triangle is not an isomorphism. Hence, the object $L(W)$ is non-zero. Since both $\Gamma(W)$ and W are in $\mathcal{S}$, so is $L(W)$ by the 2/3-property. It remains to notice that $\mathrm{Hom}^*_{\mathcal{T}}(X, L(W)) = 0$. This is true by Lemma 2.4.6, as X is L-acyclic and $L(W)$ is L-local. □

3.3.1 A description of $\Gamma_\mathfrak{p}\,\mathbf{D}(A)$

Given a prime ideal $\mathfrak{p} \in \operatorname{Spec} B$, we know immediately that $\Gamma_\mathfrak{p}\,\mathbf{D}(A)$ is a non-zero localizing subcategory of $\mathbf{D}(A)$. We can hence use Lemma 3.3.3 to check the minimality condition. In order to do so, we will first need another description of the subcategory $\Gamma_\mathfrak{p}\,\mathbf{D}(A)$.

In Benson, Iyengar and Krause's proof of the fact that the action from a commutative noetherian ring R stratifies $\mathbf{D}(R)$, they use that for $\mathfrak{p} \in \operatorname{Spec} R$, the subcategory $\Gamma_\mathfrak{p}\,\mathbf{D}(R)$ is generated by the residue field $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$, i.e. that $\Gamma_\mathfrak{p}\,\mathbf{D}(R) = \mathrm{Loc}_{\mathbf{D}(R)}(k(\mathfrak{p}))$. Inspired by this, we make the following definition.

Definition 3.3.4. Let $\mathfrak{p} \in \operatorname{Spec} B$. We define $A(\mathfrak{p})$ to be the k-algebra $A \otimes_B k(\mathfrak{p})$, where $k(\mathfrak{p}) = B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p}$ denotes the residue field of B at $\mathfrak{p}$.

Note that

$$A(\mathfrak{p}) = A \otimes_B k(\mathfrak{p}) \simeq A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \simeq (A/\mathfrak{p}A)_\mathfrak{p}.$$

The next result will be essential in our further work.

Proposition 3.3.5. *We have the equality* $\Gamma_\mathfrak{p}\,\mathbf{D}(A) = \mathrm{Loc}_{\mathbf{D}(A)}(A(\mathfrak{p}))$ *for all prime ideals* $\mathfrak{p} \in \operatorname{Spec} B$.

Proof. Let us first show that $\mathrm{Loc}_{\mathbf{D}(A)}(A(\mathfrak{p})) \subseteq \Gamma_\mathfrak{p}\,\mathbf{D}(A)$. As $\Gamma_\mathfrak{p}\,\mathbf{D}(A)$ is a localizing subcategory, it is enough to show that $A(\mathfrak{p}) \in \Gamma_\mathfrak{p}\,\mathbf{D}(A)$. Notice that $\mathrm{Hom}^*_{\mathbf{D}(A)}(A, A(\mathfrak{p})) \simeq A(\mathfrak{p})$, which is clearly $\mathfrak{p}$-local and $\mathfrak{p}$-torsion. By Theorem 2.2.8, the stalk complex A generates all the compact objects in $\mathbf{D}(A)$, which implies that part (iii) of Proposition 2.4.17 is satisfied. Hence, we know that $A(\mathfrak{p}) \in \Gamma_\mathfrak{p}\,\mathbf{D}(A)$.

Our next aim is to prove that $\Gamma_\mathfrak{p}\,\mathbf{D}(A) \subseteq \mathrm{Loc}_{\mathbf{D}(A)}(A(\mathfrak{p}))$. Recall the definition of Koszul complexes from Chapter 1, and set $A[\mathfrak{p}] = (A/\!\!/\mathfrak{p})_\mathfrak{p}$. We have

$$\Gamma_\mathfrak{p}\,\mathbf{D}(A) = \mathrm{Loc}_{\mathbf{D}(A)}(\Gamma_\mathfrak{p}(A)) = \mathrm{Loc}_{\mathbf{D}(A)}(A[\mathfrak{p}]).$$

The second equality is by [6, Prop. 2.11(2)]. For the first one, observe that $\varGamma_{\mathfrak{p}}(A) \in \varGamma_{\mathfrak{p}}\,\mathbf{D}(A)$, so $\operatorname{Loc}_{\mathbf{D}(A)}(\varGamma_{\mathfrak{p}}(A)) \subseteq \varGamma_{\mathfrak{p}}\,\mathbf{D}(A)$. For the reverse inclusion, assume $X \in \varGamma_{\mathfrak{p}}\,\mathbf{D}(A)$. As $\varGamma_{\mathfrak{p}}^2 \simeq \varGamma_{\mathfrak{p}}$, this gives $X \simeq \varGamma_{\mathfrak{p}}(X)$. From Lemma 2.1.4 we have the implication

$$X \in \operatorname{Loc}_{\mathbf{D}(A)}(A) \implies \varGamma_{\mathfrak{p}}(X) \in \operatorname{Loc}_{\mathbf{D}(A)}(\varGamma_{\mathfrak{p}}(A)),$$

and thus $X \in \operatorname{Loc}_{\mathbf{D}(A)}(\varGamma_{\mathfrak{p}}(A))$.

To prove that $\varGamma_{\mathfrak{p}}\,\mathbf{D}(A) \subseteq \operatorname{Loc}_{\mathbf{D}(A)}(A(\mathfrak{p}))$, it is hence enough to show that $A[\mathfrak{p}] \in \operatorname{Loc}_{\mathbf{D}(A)}(A(\mathfrak{p}))$. The ring $B_{\mathfrak{p}}$ is a noetherian local ring of finite global dimension, and hence a regular local ring by Serre's homological characterization [13, Thm. 19.2]. Hence, the ideal $\mathfrak{p}B_{\mathfrak{p}} \subseteq B_{\mathfrak{p}}$ is generated by a regular sequence, see for instance [19]. If we use this sequence to compute the Koszul complex $B_{\mathfrak{p}}/\!\!/\mathfrak{p}B_{\mathfrak{p}}$, we get a free resolution of $k(\mathfrak{p}) = B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$, see [21, Cor. 4.5.5]. Consequently, we have $B_{\mathfrak{p}}/\!\!/\mathfrak{p}B_{\mathfrak{p}} \simeq k(\mathfrak{p})$ in $\mathbf{D}(B)$. By [4, Lemma 3.11], this implies that $B_{\mathfrak{p}}/\!\!/\mathfrak{p}B_{\mathfrak{p}} \in \operatorname{Loc}_{\mathbf{D}(B)}(k(\mathfrak{p}))$ regardless of which generating set of the ideal we use to compute the Koszul complex. As A is free as module over B, the functor $A \otimes_B (-)\colon \operatorname{Mod} B \to \operatorname{Mod} A$ is exact and can be extended to a triangulated functor on the derived categories. Note that this functor preserves colimits, so Lemma 2.1.4 yields that

$$A \otimes_B B_{\mathfrak{p}}/\!\!/\mathfrak{p}B_{\mathfrak{p}} \in \operatorname{Loc}_{\mathbf{D}(A)}(A \otimes_B k(\mathfrak{p})) = \operatorname{Loc}_{\mathbf{D}(A)}(A(\mathfrak{p}))$$

for any choice of generating set of the ideal $\mathfrak{p}B_{\mathfrak{p}}$.

If we compute our Koszul complexes with respect to a fixed generating set $(x_1, \ldots, x_t)$ for $\mathfrak{p} \subseteq B$, we have

$$A[\mathfrak{p}] = (A/\!\!/\mathfrak{p})_{\mathfrak{p}} \simeq A/\!\!/\mathfrak{p} \otimes_B B_{\mathfrak{p}} \simeq A \otimes_B B/\!\!/\mathfrak{p} \otimes B_{\mathfrak{p}} \simeq A \otimes_B (B/\!\!/\mathfrak{p})_{\mathfrak{p}},$$

and by exactness of localization, we get that $(B/\!\!/\mathfrak{p})_{\mathfrak{p}} \simeq B_{\mathfrak{p}}/\!\!/\mathfrak{p}$. Using $(\frac{x_1}{1} \ldots \frac{x_t}{1})$ as generating set for $\mathfrak{p}B_{\mathfrak{p}}$ gives the isomorphism $B_{\mathfrak{p}}/\!\!/\mathfrak{p} \simeq B_{\mathfrak{p}}/\!\!/\mathfrak{p}B_{\mathfrak{p}}$. Combining this, we see that $A[\mathfrak{p}] \simeq A \otimes_B B_{\mathfrak{p}}/\!\!/\mathfrak{p}B_{\mathfrak{p}}$, so $A[\mathfrak{p}] \in \operatorname{Loc}_{\mathbf{D}(A)}(A(\mathfrak{p}))$. □

Notice that $A(\mathfrak{p})$ is finite dimensional as a vector space over the field $k(\mathfrak{p})$, as A is finitely generated as a B-module. Hence, there are only finitely many non-isomorphic simple $A(\mathfrak{p})$-modules.

From now on, we will let $\{S_i(\mathfrak{p})\}_{i=1}^{t_{\mathfrak{p}}}$ denote the isomorphism classes of simple $A(\mathfrak{p})$-modules. Note that whenever we talk about these as the *only* simple $A(\mathfrak{p})$-modules, we will always mean up to isomorphism. We can use these simple modules to understand the subcategory $\varGamma_{\mathfrak{p}}\,\mathbf{D}(A)$. Our next proposition shows that the set $\{S_i(\mathfrak{p})\}_{i=1}^{t_{\mathfrak{p}}}$ generates $\varGamma_{\mathfrak{p}}\,\mathbf{D}(A)$ as a localizing subcategory of $\mathbf{D}(A)$.

Proposition 3.3.6. *We have the equality* $\varGamma_{\mathfrak{p}}\,\mathbf{D}(A) = \operatorname{Loc}_{\mathbf{D}(A)}(\oplus_{i=1}^{t_{\mathfrak{p}}} S_i(\mathfrak{p}))$ *for all prime ideals* $\mathfrak{p} \in \operatorname{Spec} B$.

Proof. By Proposition 3.3.5, it suffices to show that we have the equality $\mathrm{Loc}_{\mathbf{D}(A)}(A(\mathfrak{p})) = \mathrm{Loc}_{\mathbf{D}(A)}(\oplus_{i=1}^{t_\mathfrak{p}} S_i(\mathfrak{p}))$. Notice that $S_i(\mathfrak{p}) \in \mathrm{Loc}_{\mathbf{D}(A(\mathfrak{p}))}(A(\mathfrak{p}))$ for every $i \in \{1, \dots, t_\mathfrak{p}\}$. Restriction of scalars hence yields $S_i(\mathfrak{p}) \in \mathrm{Loc}_{\mathbf{D}(A)}(A(\mathfrak{p}))$, which implies the inclusion $\mathrm{Loc}_{\mathbf{D}(A)}(\oplus_{i=1}^{t_\mathfrak{p}} S_i(\mathfrak{p})) \subseteq \mathrm{Loc}_{\mathbf{D}(A)}(A(\mathfrak{p}))$.

As $A(\mathfrak{p})$ is a finite-dimensional algebra over a field, it has finite length as a module over itself. Hence, the 2/3-property implies that $A(\mathfrak{p}) \in \mathrm{Loc}_{\mathbf{D}(A(\mathfrak{p}))}(\oplus_{i=1}^{t_\mathfrak{p}} S_i(\mathfrak{p}))$, which by restriction of scalars gives the reverse inclusion. □

In order to learn more about $\Gamma_\mathfrak{p}\,\mathbf{D}(A)$ and to determine if the second stratification condition is satisfied, we need to understand the simple $A(\mathfrak{p})$-modules. With this in mind, we take a closer look at the representation theory of $A(\mathfrak{p})$, which is also interesting for its own sake. We will focus on the case where $\mathfrak{p}$ is a maximal ideal in B.

3.3.2 The representation theory of $A(\mathfrak{p})$ for $\mathfrak{p}$ maximal

In this section we will assume the reader to know basic representation theory for finite dimensional algebras, and we will use some results without proofs. See for instance [2] for the theoretical framework for this discussion.

For a maximal ideal $\mathfrak{p} \subseteq B$, we have $k(\mathfrak{p}) \simeq B/\mathfrak{p}$. We hence get

$$A(\mathfrak{p}) = A \otimes_B k(\mathfrak{p}) \simeq A \otimes_B B/\mathfrak{p} \simeq A/\mathfrak{p}A.$$

Our field k is algebraically closed, so by Hilbert's Nullstellensatz, see for instance [1], we know that the maximal ideals of B are given by

$$\{(x^n - \alpha, y^n - \beta) \mid \alpha, \beta \in k\}.$$

We have thus proved the following lemma.

Lemma 3.3.7. *Let $\mathfrak{p}$ be a maximal ideal in B. We then have*

$$A(\mathfrak{p}) \simeq k\langle x, y\rangle/(x^n - \alpha, xy - qyx, y^n - \beta)$$

for some α, β in k.

We want to understand the simple $A(\mathfrak{p})$-modules $\{S_i(\mathfrak{p})\}_{i=1}^{t_\mathfrak{p}}$ in this case. Notice that $A(\mathfrak{p})$ is a finite dimensional k-algebra. Since k is algebraically closed, this means that $A(\mathfrak{p})$ is Morita equivalent to the path algebra of a quiver modulo an admissible ideal. Any simple module is one-dimensional over k, corresponding to a representation with a one-dimensional vector space for one vertex and zero everywhere else.

We will look at three different possibilities for the maximal ideal $\mathfrak{p} = (x^n - \alpha, y^n - \beta)$, depending on whether α and β are zero. This will cover the representation theory of $A(\mathfrak{p})$ for all maximal ideals $\mathfrak{p}$ in B.

Case 1: $\alpha = \beta = 0$

Let us first consider the case where $\alpha = \beta = 0$, i.e. where

$$A(\mathfrak{p}) \simeq k\langle x, y\rangle/(x^n, xy - qyx, y^n).$$

We can immediately see that this is the path algebra of the quiver

$$\begin{tikzcd} 1 \arrow[loop left, "y"] \arrow[loop right, "x"] \end{tikzcd}$$

modulo the relations specified above.

Since this quiver has only one vertex, we know that there is only one simple $A(\mathfrak{p})$-module for this choice of $\mathfrak{p}$.

Case 2: $\alpha \neq 0, \beta \neq 0$

Let us next look at the case where both α and β are non-zero. In this situation $A(\mathfrak{p})$ turns out to be isomorphic to a matrix ring. For the proof of this fact, we will need the following basic lemma. Recall that q is a primitive n-th root of unity.

Lemma 3.3.8. *The equality* $\sum_{i=0}^{n-1} q^{il} = 0$ *holds for any* $l \in \{1, \ldots, n-1\}$.

Proof. Denote the sum above by S. One can easily check that $q^l S = S$. If $S \neq 0$, cancellation would give $q^l = 1$, but this is impossible as $l \leq n-1$ and q is a primitive n-th root of unity. Hence, we must have $S = 0$. $\square$

Proposition 3.3.9. *Let* $\mathfrak{p} = (x^n - \alpha, y^n - \beta) \subseteq B$ *with* $\alpha \neq 0$ *and* $\beta \neq 0$. *We then have an isomorphism* $A(\mathfrak{p}) \simeq M_n(k)$, *where* $M_n(k)$ *denotes the full matrix ring of* $n \times n$*-matrices over the field* k.

Proof. As k is algebraically closed, the polynomials $x^n - \alpha$ and $y^n - \beta$ split into linear factors over k. Let a be a root of $x^n - \alpha$ and b a root of $x^n - \beta$. Our polynomials factor as

$$\begin{aligned} x^n - \alpha &= (x-a)(x-qa)\cdots(x-q^{n-1}a) \\ x^n - \beta &= (x-b)(x-qb)\cdots(x-q^{n-1}b). \end{aligned}$$

Consider the k-algebra homomorphism $\phi\colon A \longrightarrow M_n(k)$ defined on the generators x and y by

$$\phi(x) = \begin{bmatrix} 0 & a & 0 & \ldots & 0 \\ \vdots & 0 & a & & 0 \\ & \vdots & \ddots & \ddots & \vdots \\ 0 & & \ldots & 0 & a \\ a & 0 & \ldots & & 0 \end{bmatrix} \quad \text{and} \quad \phi(y) = \begin{bmatrix} b & 0 & \ldots & 0 \\ 0 & qb & & \vdots \\ \vdots & & \ddots & \\ 0 & \ldots & 0 & q^{n-1}b \end{bmatrix}.$$

Notice that ϕ is well defined, as $\phi(xy - qyx) = 0$.

To check that ϕ is surjective, we will consider the generating set for $M_n(k)$ as k-vector space given by matrices with exactly one non-zero entry. If all these matrices are in $\mathrm{Im}(\phi)$, then ϕ will be surjective. Since $\alpha \neq 0$, we have $a \neq 0$, so we can look at the matrix $\phi(a^{-1}x)$. Multiplying with this matrix from left and right allows us to change the order of rows and columns. To see that ϕ is surjective, it is thus enough to show that *one* of the generators mentioned above is in $\mathrm{Im}(\phi)$.

Since $\beta \neq 0$, we have that $b \neq 0$. We can hence consider the matrix

$$\phi(b^{-1}y) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & q & & \vdots \\ \vdots & & \ddots & \\ 0 & \dots & 0 & q^{n-1} \end{bmatrix},$$

which we will denote by M. Clearly, the matrix M^i is in $\mathrm{Im}(\phi)$ for all i, and the same is true for the sum $\sum_{i=0}^{n-1} M^i$. By Lemma 3.3.8, the only non-zero entry in the matrix given by this sum, will be the upper left corner. This entry is non-zero, as $\mathrm{char}(k)$ does not divide n. Hence, the morphism ϕ is surjective.

Let us now show that $\mathrm{Ker}(\phi) = \mathfrak{p}A = (x^n - \alpha, y^n - \beta)$, by which we now mean the two-sided ideal generated by $x^n - \alpha$ and $y^n - \beta$ in A.

Straightforward computations show that $(x^n - \alpha, y^n - \beta) \subseteq \mathrm{Ker}(\phi)$. For the reverse inclusion, look at an element $\gamma \in \mathrm{Ker}(\phi)$. Let $\tilde{\gamma}$ be the element obtained from γ by reducing modulo $(x^n - \alpha)$, replacing each x^n by α. As $\gamma - \tilde{\gamma} \in (x^n - \alpha)$, it is sufficient to show that $\tilde{\gamma} \in (y^n - \beta)$.

Our polynomial $\tilde{\gamma}$ can be written as a finite sum

$$\tilde{\gamma} = \sum_{i=0}^{n-1} \sum_j c_{ij} x^i y^j, \quad \text{with } c_{ij} \in k.$$

Since $\gamma \in \mathrm{Ker}(\phi)$ and $\phi(x^n) = \phi(\alpha)$, we have $\tilde{\gamma} \in \mathrm{Ker}(\phi)$. In what positions the matrix $\phi(c_{ij} x^i y^j)$ may have non-zero entries, depends only on i, since the matrix $\phi(y^j)$ is diagonal. For different values of i, the non-zero entries will never be in the same positions, as $i \in \{0, \dots, n-1\}$. Notice that the non-zero entries will be of the form $c_{ij} q^{lj} b^j a^i$ for $l \in \{0, \dots, n-1\}$, where l depends on which column the entry appears in. Hence, we see that $\phi(\tilde{\gamma}) = 0$ implies

$$\sum_j c_{ij} q^{lj} b^j = 0 \text{ for every } i, l \in \{0, \dots, n-1\}.$$

This gives that $y = q^l b$ is a root in $\tilde{\gamma}$ for every $l \in \{0, \dots, n-1\}$, which means that $(y^n - \beta)$ is a factor in $\tilde{\gamma}$. Consequently, we have shown that $\tilde{\gamma} \in (y^n - \beta)$. This proves that $\mathrm{Ker}(\phi) = \mathfrak{p}A$, which yields $A(\mathfrak{p}) = A/\mathfrak{p}A \simeq M_n(k)$. $\square$

$M_n(k)$ is Morita equivalent to k via the functor sending a k-vector space V to the $M_n(k)$-module V^n, where the action on V^n as a column vector is given by left matrix multiplication. There is clearly only one simple k-module, so the column vector k^n is the only simple $M_n(k)$-module. As $A(\mathfrak{p}) \simeq M_n(k)$, there is hence only one isomorphism class of simple $A(\mathfrak{p})$-modules. To find non-isomorphic simple modules, we need to move on to the last case.

Case 3: $\alpha = 0, \beta \neq 0$

It remains to consider the case where one of α and β is zero and the other one non-zero. We can without loss of generality assume $\alpha = 0$. By the following lemma, it will also be enough to consider the case $\beta = 1$.

Lemma 3.3.10. *Let $\beta \neq 0$. We then have an isomorphism*

$$k\langle x, y\rangle/(x^n, xy - qyx, y^n - \beta) \;\simeq\; k\langle x, y\rangle/(x^n, xy - qyx, y^n - 1).$$

Proof. Let $b \in k$ be a root in $(y^n - \beta)$, which exists since k is algebraically closed. As $\beta \neq 0$, we know that $b \neq 0$. Define the following k-algebra homomorphism:

$$\begin{aligned} \phi\colon k\langle x, y\rangle/(x^n, xy - qyx, y^n - \beta) &\longrightarrow k\langle x, y\rangle/(x^n, xy - qyx, y^n - 1) \\ x &\longmapsto x \\ y &\longmapsto by. \end{aligned}$$

We can easily check that the necessary relations are satisfied, i.e. that ϕ is well-defined. This is an isomorphism, with inverse given by sending $y \longmapsto b^{-1}y$. $\square$

Based on this, we will restrict our attention to the case where $\alpha = 0$ and $\beta = 1$, i.e. we consider only the ideal $\mathfrak{p} = (x^n, y^n - 1)$.

Let us first find the Jacobson radical of $A(\mathfrak{p})$. Notice that as $A(\mathfrak{p})$ is a finite dimensional algebra over k, it is an artinian ring. The ideal in $A(\mathfrak{p})$ generated by x is clearly nilpotent. The quotient is

$$A(\mathfrak{p})/(x) \simeq k\langle x, y\rangle/(x, y^n - 1) \simeq k[y]/(y^n - 1) \simeq \oplus_{i=1}^{n} k[y]/(y - q^i), \quad (*)$$

where the last isomorphism follows from the Chinese remainder theorem. This means that $A(\mathfrak{p})/(x)$ is semisimple, so by [2, Prop. 3.3], the Jacobson radical of $A(\mathfrak{p})$ is equal to (x). We will from now on denote this ideal by $\mathfrak{r}$.

The $A(\mathfrak{p})$-modules $k[y]/(y - q^i)$ and $k[y]/(y - q^j)$ are non-isomorphic for $i \neq j$ with $i, j \in \{1, \ldots, n\}$. Hence, the algebra $A(\mathfrak{p})$ is what we call basic. Every finite dimensional basic k-algebra over an algebraically closed field is isomorphic to a path algebra modulo an admissible ideal, see for instance [2, Thm. 3.1.9]. We will follow the construction in the proof of this statement to find the quiver Q and the admissible ideal I such that $A(\mathfrak{p}) \simeq kQ/I$.

Our first aim is to find the quiver $Q = \{Q_0, Q_1\}$, where Q_0 denotes the set of vertices and Q_1 the set of arrows. By $(*)$, we know that $A(\mathfrak{p})/\mathfrak{r} \simeq k^n$, which implies that Q has n vertices, say $Q_0 = \{1, \dots, n\}$.

In k^n we have a complete set of primitive orthogonal idempotents given by $\{f_i\}_{i=1}^n$, where f_i has 1 in position i and 0 otherwise. Using the isomorphism $A(\mathfrak{p})/\mathfrak{r} \simeq k^n$ given by the Chinese remainder theorem, we get a complete set of primitive orthogonal idempotents $\{\bar{v}_i\}_{i=1}^n$ in $A(\mathfrak{p})/\mathfrak{r}$ as well. These are explicitly represented by the elements

$$v_i = (\prod_{\substack{j=1 \\ j\neq i}}^{n} (q^i - q^j))^{-1} \prod_{\substack{j=1 \\ j\neq i}}^{n} (y - q^j)$$

in $A(\mathfrak{p})$. Notice that if $i, j \in \{1, \dots, n\}$, we have $q^i \neq q^j$ for $i \neq j$, so the first product is non-zero and hence invertible. This factor is just a coefficient from k, and it can often be omitted when doing calculations. As $\mathfrak{r} = (x)$ and v_i contains no x-term, the set $\{v_i\}_{i=1}^n$ is a complete set of primitive orthogonal idempotents in $A(\mathfrak{p})$.

Let us now look at the k-vector spaces $v_j\mathfrak{r}/\mathfrak{r}^2 v_i$ for $i, j \in \{1, \dots, n\}$. The dimension of $v_j\mathfrak{r}/\mathfrak{r}^2 v_i$ as k-vector space will give us the number of arrows from vertex i to vertex j in the quiver Q. In order to calculate these dimensions, we will need the following lemma.

Lemma 3.3.11. *$yv_i = q^i v_i$ for $i \in \{1, \dots, n\}$.*

Proof. Using polynomial division, we observe that

$$\prod_{\substack{j=1 \\ j\neq i}}^{n} (y - q^j) = y^{n-1} + q^i y^{n-2} + \cdots + q^{(n-2)i} y + q^{(n-1)i}.$$

Multiplying this expression with y gives the same as multiplying with q^i, and the result follows. □

Proposition 3.3.12. *For $s, t \in \{1, \dots, n\}$, we have*

$$\dim_k(v_t\mathfrak{r}/\mathfrak{r}^2 v_s) = \begin{cases} 1 & \text{if} \quad s \equiv t+1 \pmod n \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Notice that $\{xy^j \mid 0 \leq j \leq n-1\}$ is a k-basis for $\mathfrak{r}/\mathfrak{r}^2$. Let us first assume that $s \not\equiv t+1 \pmod n$. We will show that $v_t x y^j v_s = 0$ for all j. Since y commutes with v_i for every $i \in \{1, \dots, n\}$, it suffices to check that $v_t x v_s = 0$. To see this, notice that

$$\prod_{\substack{j=1\\ j\neq t}}^{n}(y-q^j)x\prod_{\substack{j=1\\ j\neq s}}^{n}(y-q^j) = \prod_{\substack{j=1\\ j\neq t}}^{n}(y-q^j)\prod_{\substack{j=1\\ j\neq s}}^{n}(qy-q^j)x.$$

As j runs over all values except s in the last product, and we have assumed $s \not\equiv t+1 \pmod n$, one can factor out $(y-q^t)$. Combining this with the first product yields that (y^n-1) is a factor in $v_t x v_s$. Consequently, the element $v_t x v_s = 0$ for $s \not\equiv t+1 \pmod n$.

Let us now assume that $s \equiv t+1 \pmod n$. We will first show that $\dim_k(v_t \mathfrak{r}/\mathfrak{r}^2 v_s) \geq 1$, and for this it is enough to check that $v_t x v_s$ is non-zero. Assume to the contrary that $v_t x v_s = 0$. This would imply that

$$\prod_{\substack{j=1\\ j\neq t}}^{n}(y-q^j)\prod_{\substack{j=1\\ j\neq s}}^{n}(qy-q^j)$$

is equal to zero, which only happens if $y = q^t$ is a root in the second product. But this is impossible, since $s \equiv t+1 \pmod n$.

To see that $\dim_k(v_t \mathfrak{r}/\mathfrak{r}^2 v_s) \leq 1$, we show that $v_t x v_s$ is actually a basis. This is true as

$$v_t x y^i v_s = v_t x (q^s)^i v_s = q^{si} v_t x v_s,$$

where the first equality follows from Lemma 3.3.11. □

Let Q be the quiver

$$\begin{tikzcd}
& 1 \arrow[dr, bend left, "\alpha_n"] & \\
2 \arrow[ur, bend left, "\alpha_1"] & & n \arrow[dl, bend left, "\alpha_{n-1}"] \\
& \cdots \arrow[ul, bend left, "\alpha_2"] &
\end{tikzcd}$$

and kQ the corresponding path algebra over k. Let e_i denote the trivial path corresponding to vertex i for $i \in \{1, \dots, n\}$. Note that α_i is the arrow which ends in vertex i. By our work so far, we know that we have a surjective k-algebra homomorphism

$$\begin{aligned}\phi\colon kQ &\longrightarrow A(\mathfrak{p})\\ e_i &\longmapsto v_i\\ \alpha_i &\longmapsto v_i x v_{i+1}.\end{aligned}$$

Our next aim is to find the kernel of ϕ.

Proposition 3.3.13. *Let $(kQ_+)^n$ denote the subspace of kQ with basis all paths of length n or more. For the homomorphism ϕ as defined above, we have* $\mathrm{Ker}(\phi) = (kQ_+)^n$.

Proof. Clearly, the subspace $(kQ_+)^n$ is an ideal in kQ. Let $p \in (kQ_+)^n$ be a path of length n or more. As each arrow α_i is sent to $v_i x v_{i+1}$, the factor x will appear at least n times in $\phi(p)$. Collecting the x-terms and using the relation $x^n = 0$ hence shows that $\phi(p) = 0$. This yields the inclusion $(kQ_+)^n \subseteq \mathrm{Ker}(\phi)$.

Let us now assume p to be a path of length l, with $l < n$. Writing indices modulo n, we can express our path as $p = \alpha_i \alpha_{i+1} \dots \alpha_{i+l-1}$, where i is the end vertex of p. This gives

$$\begin{aligned}\phi(p) &= \phi(\alpha_i)\phi(\alpha_{i+1}) \dots \phi(\alpha_{i+l-1}) \\ &= v_i x v_{i+1} x v_{i+2} \cdots v_{i+l-1} x v_{i+l}.\end{aligned}$$

Collecting x-terms and omitting the coefficients in the idempotents, we see that $p \in \mathrm{Ker}(\phi)$ would imply that

$$\prod_{\substack{j=1 \\ j \neq i}}^{n} (y - q^j) \prod_{\substack{j=1 \\ j \neq i+1}}^{n} (qy - q^j) \cdots \prod_{\substack{j=1 \\ j \neq i+l}}^{n} (q^l y - q^j)$$

was equal to zero. But this is impossible by a similar argument as applied earlier, namely by noticing that $y = q^i$ will not be a root. This shows that if a path is in $\mathrm{Ker}(\phi)$, its length has to be at least n.

Let $c_1 p_1 + \cdots + c_r p_r \in \mathrm{Ker}(\phi)$ be a linear combination of distinct paths with coefficients from k. We want to show that each term in the sum then has to be in the kernel, which will prove the inclusion $\mathrm{Ker}(\phi) \subseteq (kQ_+)^n$. As the length of a path p_i determines the numbers of x-factors in $\phi(p_i)$, we can assume that all the paths $\{p_1, \dots, p_r\}$ have length s. It is also enough to consider the case $s \in \{0, \dots, n-1\}$, as we have already seen that paths of length n or more are sent to 0. The claim now follows by noticing that for distinct paths $\{p_i\}_{i=1}^r$ of length $0 \leq s < n$, the elements $\{\phi(p_i)\}_{i=1}^r$ are linearly independent over k. □

Our work so far proves the following proposition.

Proposition 3.3.14. *Let $\mathfrak{p}$ be the ideal $(x^n, y^n - 1) \subseteq B$ and Q the quiver defined above. We then have an isomorphism*

$$A(\mathfrak{p}) \simeq kQ/(kQ_+)^n.$$

Corollary 3.3.15. *Let $\mathfrak{p}$ be the ideal $(x^n, y^n - 1) \subseteq B$. There are n isomorphism classes of simple $A(\mathfrak{p})$-modules, where*

$$S_i(\mathfrak{p}) = A(\mathfrak{p})/(x, y - q^i) \simeq k[y]/(y - q^i)$$

for $i \in \{1, \dots, n\}$.

Proof. The $A(\mathfrak{p})$-module $S_i(\mathfrak{p})$, as defined above, has k-dimension equal to one, and is hence clearly simple. As the quiver Q has n vertices, there are exactly n isomorphism classes of simple $A(\mathfrak{p})$-modules, and since $S_i(\mathfrak{p})$ is not isomorphic to $S_j(\mathfrak{p})$ for $i, j \in \{1, \dots, n\}$ with $i \neq j$, these modules represent all the isomorphism classes. $\square$

We know that for our ideal $\mathfrak{p} = (x^n, y^n - 1)$ in B, the subcategory $\varGamma_{\mathfrak{p}} \mathbf{D}(A)$ can be described as $\mathrm{Loc}_{\mathbf{D}(A)}(\oplus_{i=1}^n S_i(\mathfrak{p}))$. This description finally enables us to show that $\varGamma_{\mathfrak{p}} \mathbf{D}(A)$ is in general *not* a minimal localizing subcategory, which again proves that the action from B on $\mathbf{D}(A)$ is *not* stratifying. In order to see this, we need to calculate $\mathrm{Hom}^*_{\mathbf{D}(A)}(S_i(\mathfrak{p}), S_j(\mathfrak{p}))$.

Proposition 3.3.16. *Let $\mathfrak{p}$ be the ideal $(x^n, y^n - 1) \subseteq B$ and $\{S_i(\mathfrak{p})\}_{i=1}^n$ the simple $A(\mathfrak{p})$-modules as described above. Then*

$$\begin{aligned} &\mathrm{Hom}^*_{\mathbf{D}(A)}(S_i(\mathfrak{p}), S_j(\mathfrak{p})) \neq 0 \quad \textit{for } i = j \textit{ and } i \equiv j + 1 \pmod{n} \\ &\mathrm{Hom}^*_{\mathbf{D}(A)}(S_i(\mathfrak{p}), S_j(\mathfrak{p})) = 0 \quad \textit{otherwise.} \end{aligned}$$

Proof. Note that as an A-module, we have $S_i(\mathfrak{p}) \simeq A/(x, y - q^i)$ and that $(x, y - q^i)$ is a two-sided ideal. We will first argue that

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] & A \arrow[rr, "{\begin{bmatrix} -qy+q^i & x \end{bmatrix}}"] \arrow[dr, dashed, two heads] & & A^2 \arrow[rr, "{\begin{bmatrix} x \\ y-q^i \end{bmatrix}}"] \arrow[dr, dashed, two heads, "\pi"] & & A \arrow[r, two heads] & S_i(\mathfrak{p}) \arrow[r] & 0 \\ & & & A\begin{bmatrix} -qy+q^i & x \end{bmatrix} \arrow[ur, dashed, tail] & & (x, y-q^i) \arrow[ur, dashed, tail] & & & \end{tikzcd}$$

is a projective resolution of $S_i(\mathfrak{p})$ as an A-module. Note that as we work with left modules, we will think of the elements in A^2 as row vectors. The A-linear maps are given by multiplication with the matrices from the right.

Let us prove that $A\begin{bmatrix} -qy + q^i & x \end{bmatrix} = \mathrm{Ker}(\pi)$. By straightforward computation,

$$\pi(\begin{bmatrix} -qy + q^i & x \end{bmatrix}) = \begin{bmatrix} -qy + q^i & x \end{bmatrix} \begin{bmatrix} x \\ y - q^i \end{bmatrix} = 0,$$

so $A\begin{bmatrix} -qy + q^i & x \end{bmatrix} \subseteq \mathrm{Ker}(\pi)$. For the reverse inclusion, let us consider an element $\begin{bmatrix} \gamma & \delta \end{bmatrix} \in \mathrm{Ker}(\pi)$. This means that $\gamma x + \delta(y - q^i) = 0$ in A. As usual, we think of elements in A as polynomials in two non-commuting variables, ordered such that x-factors are to the left. Assume that δ has a term of the form cy^j for some non-zero $c \in k$. We can then write $\delta = \delta' + cy^j$, where $\delta' \in A$ has no y^j-term. This gives

$$\begin{aligned}\gamma x + \delta(y - q^i) &= \gamma x + (\delta' + cy^j)(y - q^i)\\ &= \gamma x + \delta' y - q^i\delta' + cy^{j+1} - cq^iy^j.\end{aligned}$$

If this expression is zero, the term cq^iy^j has to be cancelled out. For this to be possible, the polynomial δ' must have a y-term of degree $j-1$. Iterating this argument, we can reduce the degree of our y-term, which finally means that the polynomial δ must have a constant term. This clearly contradicts the assumption that $\begin{bmatrix}\gamma & \delta\end{bmatrix} \in \operatorname{Ker}(\pi)$, as our equation would have a constant term which is not cancelled out. We can hence conclude that δ has no term cy^j, i.e. that δ is of the form $\delta = \tilde{\delta}x$ for some $\tilde{\delta} \in A$.

Consequently,

$$\gamma x + \tilde{\delta}x(y - q^i) = (\gamma + \tilde{\delta}(qy - q^i))x = 0.$$

As x is non-zero and A a domain, this implies that $\gamma = -\tilde{\delta}(qy - q^i)$. Hence, $y = q^{i-1}$ is a root in γ, so $\gamma = \tilde{\gamma}(y - q^{i-1})$ for some $\tilde{\gamma} \in A$.

Using the equalities $\gamma = \tilde{\gamma}(y - q^{i-1})$ and $\delta = \tilde{\delta}x$, we get

$$\tilde{\gamma}(y - q^{i-1})x + \tilde{\delta}x(y - q^i) = (\tilde{\gamma}(y - q^{i-1}) + q\tilde{\delta}(y - q^{i-1}))x = 0,$$

which implies that $\tilde{\gamma} = -q\tilde{\delta}$. Hence,

$$\begin{bmatrix}\gamma & \delta\end{bmatrix} = \begin{bmatrix}-q\tilde{\delta}(y - q^{i-1}) & \tilde{\delta}x\end{bmatrix} = \tilde{\delta}\begin{bmatrix}-qy + q^i & x\end{bmatrix},$$

so $\begin{bmatrix}\gamma & \delta\end{bmatrix} \in A\begin{bmatrix}-qy + q^i & x\end{bmatrix}$, and we have proved our equality.

To see that we have a projective resolution, it remains to show that the map

$$\begin{aligned}A &\longrightarrow A\begin{bmatrix}-qy + q^i & x\end{bmatrix}\\ 1 &\longmapsto \begin{bmatrix}-qy + q^i & x\end{bmatrix}\end{aligned}$$

is an isomorphism of A-modules. This is straightforward; surjectivity is clear, while injectivity follows from the fact that A has no non-trivial zero divisors.

We are now ready to calculate $\operatorname{Hom}^*_{\mathbf{D}(A)}(S_i(\mathfrak{p}), S_j(\mathfrak{p}))$. Observe that the statement is clearly true in the case $i = j$, as $S_i(\mathfrak{p}) \neq 0$ in $\mathbf{D}(A)$. Let P_i denote the projective resolution of $S_i(\mathfrak{p})$ as described above. We know that

$$\begin{aligned}\operatorname{Hom}^*_{\mathbf{D}(A)}(S_i(\mathfrak{p}), S_j(\mathfrak{p})) &= \oplus_{t\in\mathbb{Z}} \operatorname{Hom}_{\mathbf{D}(A)}(S_i(\mathfrak{p}), \Sigma^t S_j(\mathfrak{p}))\\ &\simeq \oplus_{t\in\mathbb{Z}} \operatorname{Hom}_{\mathbf{K}(A)}(P_i, \Sigma^t S_j(\mathfrak{p})).\end{aligned}$$

To show that the statement is true in the case $i \equiv j+1 \pmod{n}$, it hence suffices to verify that $\operatorname{Hom}_{\mathbf{K}(A)}(P_{j+1}, \Sigma^2 S_j(\mathfrak{p})) \neq 0$.

Using that $S_j(\mathfrak{p}) \simeq A/(x, y-q^j)$, a morphism f in $\operatorname{Hom}_{\mathbf{K}(A)}(P_{j+1}, \Sigma^2 S_j(\mathfrak{p}))$ is represented by a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r, "{\begin{bmatrix} -qy+q^{j+1} & x \end{bmatrix}}"] \arrow[d, "f"] & A^2 \arrow[r, "{\begin{bmatrix} x \\ y-q^{j+1} \end{bmatrix}}"] \arrow[d] \arrow[dl, dashed, "h"] & A \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & A/(x,y-q^j) \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots .
\end{tikzcd}$$

Any map $f\colon A \longrightarrow A/(x, y-q^j)$ is determined by $f(1) = [\lambda]$ for some $\lambda \in A$. If $f = 0$ in $\mathbf{K}(A)$, there must exist a morphism h as indicated by the dashed arrow such that the triangle commutes. In particular, we must have

$$f(1) = (-qy + q^{j+1})h(\begin{bmatrix} 1 & 0 \end{bmatrix}) + xh(\begin{bmatrix} 0 & 1 \end{bmatrix}) \quad \text{in} \quad A/(x, y-q^j).$$

But the right hand side is already zero in $A/(x, y-q^j)$. Hence, any choice of $\lambda \notin (x, y-q^j)$ gives a non-zero morphism f in $\operatorname{Hom}_{\mathbf{K}(A)}(P_{j+1}, S_j(\mathfrak{p}))$, which proves the statement in the case $i = j+1 \pmod{n}$.

Let us now move on to the last case and assume $i \neq j$ and $i \not\equiv j+1 \pmod{n}$. Notice that as $S_j(\mathfrak{p})$ is a stalk complex, we have

$$\operatorname{Hom}^*_{\mathbf{D}(A)}(S_i(\mathfrak{p}), S_j(\mathfrak{p})) \simeq \oplus_{t=0}^2 \operatorname{Hom}_{\mathbf{K}(A)}(P_i, \Sigma^t S_j(\mathfrak{p})).$$

We must hence show that $\operatorname{Hom}_{\mathbf{K}(A)}(P_i, \Sigma^t S_j(\mathfrak{p})) = 0$ for $t \in \{0, 1, 2\}$.

Let us first look at the case $t = 2$. Consider the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r, "{\begin{bmatrix} -qy+q^{i} & x \end{bmatrix}}"] \arrow[d, "f"] & A^2 \arrow[r, "{\begin{bmatrix} x \\ y-q^{i} \end{bmatrix}}"] \arrow[d] \arrow[dl, dashed, "h"] & A \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & A/(x,y-q^j) \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots .
\end{tikzcd}$$

Let $f(1) = [\lambda]$ for some $\lambda \in A$. Notice that as $i \not\equiv j+1 \pmod{n}$, the element $(-q^{j+1} + q^i)$ is non-zero in k. We can hence define $h\colon A^2 \to A/(x, y-q^j)$ by sending $\begin{bmatrix} 1 & 0 \end{bmatrix}$ to the residue class of $(-q^{j+1} + q^i)^{-1}\lambda$ and $\begin{bmatrix} 0 & 1 \end{bmatrix}$ to an arbitrary element. Now,

$$\lambda - (-qy + q^i)(-q^{j+1} + q^i)^{-1}\lambda \in (x, y-q^j)$$

as $y = q^j$ is a root. This implies that f is null-homotopic, and hence that $\operatorname{Hom}_{\mathbf{K}(A)}(P_i, \Sigma^2 S_j(\mathfrak{p})) = 0$.

In the case $t = 1$, we get the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & A \arrow[r, "{\begin{bmatrix} -qy+q^i & x \end{bmatrix}}"] \arrow[d] & A^2 \arrow[r, "{\begin{bmatrix} x \\ y-q^i \end{bmatrix}}"] \arrow[d, "f"] & A \arrow[r] \arrow[d] \arrow[dl, dashed, "h"] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & A/(x, y-q^j) \arrow[r] & 0 \arrow[r] & \cdots .
\end{tikzcd}$$

A map $f\colon A^2 \longrightarrow A/(x, y-q^j)$ is determined by $f(\begin{bmatrix} 1 & 0 \end{bmatrix}) = [\lambda_1]$ and $f(\begin{bmatrix} 0 & 1 \end{bmatrix}) = [\lambda_2]$ for some $\lambda_1, \lambda_2 \in A$. In order to make the diagram commute, we must assume $(-qy+q^i)\lambda_1 \in (x, y-q^j)$. This is the case if and only if $\lambda_1 \in (x, y-q^j)$, which is seen by using the assumption $i \not\equiv j+1 \pmod n$. Noticing that $(q^j - q^i) \neq 0$ in k, as $i \neq j$, we can now define the homotopy h by sending 1 to $(q^j - q^i)^{-1}\lambda_2$. Just as in the previous case, this gives that $\operatorname{Hom}_{\mathbf{K}(A)}(P_i, \Sigma S_j(\mathfrak{p})) = 0$.

For the last case, namely $t = 0$, look at the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & A \arrow[r, "{\begin{bmatrix} -qy+q^i & x \end{bmatrix}}"] \arrow[d] & A^2 \arrow[r, "{\begin{bmatrix} x \\ y-q^i \end{bmatrix}}"] \arrow[d] & A \arrow[r] \arrow[d, "f"] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & A/(x, y-q^j) \arrow[r] & \cdots .
\end{tikzcd}$$

As before, we let $f(1) = [\lambda]$ for some $\lambda \in A$. To make the diagram commute, we must have $(y-q^i)\lambda \in (x, y-q^j)$. Using that $i \neq j$, we can check that this is equivalent to $\lambda \in (x, y-q^j)$. Hence, also $\operatorname{Hom}_{\mathbf{K}(A)}(P_i, S_j(\mathfrak{p})) = 0$, and we can conclude that $\operatorname{Hom}^*_{\mathbf{D}(A)}(S_i(\mathfrak{p}), S_j(\mathfrak{p})) = 0$ for $i \neq j$ and $i \not\equiv j+1 \pmod n$. $\square$

3.3.3 The minimality condition

We are finally able to conclude that the action from B on $\mathbf{D}(A)$ is generally not stratifying. Recall that $A = k\langle x, y\rangle/(xy - qyx)$, where $q \in k$ is a primitive n-th root of unity, and $B = Z(A) = k[x^n, y^n]$.

Theorem 3.3.17. *The action from B on $\mathbf{D}(A)$ does not satisfy the minimality condition for $n \geq 3$.*

Proof. We need to show that there exists a $\mathfrak{p} \in \operatorname{Spec} B$ for which $\varGamma_{\mathfrak{p}} \mathbf{D}(A)$ is not a minimal localizing subcategory of $\mathbf{D}(A)$. We know from Example 2.2.6 that $\mathbf{D}(A)$ is a compactly generated triangulated category, so by Lemma 3.3.3 it suffices to show that $\operatorname{Hom}^*_{\mathbf{D}(A)}(X, Y) = 0$ for some non-zero objects X and Y in $\varGamma_{\mathfrak{p}} \mathbf{D}(A)$.

Let $\mathfrak{p}$ be the ideal $(x^n, y^n - 1) \subseteq B$. We have seen, in Corollary 3.3.15, that $A(\mathfrak{p})$ has n simple modules, with $S_i(\mathfrak{p}) \simeq A/(x, y - q^i)$ for $i \in \{1, \dots, n\}$. By Proposition 3.3.6, we can describe $\varGamma_{\mathfrak{p}} \mathbf{D}(A)$ as $\operatorname{Loc}_{\mathbf{D}(A)}(\oplus_{i=1}^n S_i(\mathfrak{p}))$. Let $X = S_i(\mathfrak{p})$ and $Y = S_j(\mathfrak{p})$ for some $i, j \in \{1, \dots, n\}$ with $i \neq j$ and $i \neq j + 1 \pmod{n}$. Clearly, these are non-zero objects in $\operatorname{Loc}_{\mathbf{D}(A)}(\oplus_{i=1}^n S_i(\mathfrak{p}))$. By Proposition 3.3.16, we have $\operatorname{Hom}^*_{\mathbf{D}(A)}(S_i(\mathfrak{p}), S_j(\mathfrak{p})) = 0$. We can hence conclude that $\varGamma_{\mathfrak{p}} \mathbf{D}(A)$ is not a minimal localizing subcategory, which means that the second stratification condition is not satisfied. □

Even though the action from B on $\mathbf{D}(A)$ turned out not to satisfy the minimality condition, we are still able to say something about the structure of the localizing subcategories of $\mathbf{D}(A)$. We will end this section by restating Theorem 2.5.3 from Chapter 2 in our context. Note that we have shown that all the necessary assumptions are satisfied.

Theorem 3.3.18. *Let $\mathcal{V} = \operatorname{Spec} B$. We have inverse bijections*

$$\left\{\begin{array}{l}\textit{localizing}\\ \textit{subcategories of } \mathbf{D}(A)\end{array}\right\} \underset{\tau}{\overset{\sigma}{\rightleftarrows}} \left\{\begin{array}{l}\textit{families } (\mathcal{S}(\mathfrak{p}))_{\mathfrak{p}\in\mathcal{V}}\textit{, where } \mathcal{S}(\mathfrak{p})\\ \textit{is a localizing subcategory of } \varGamma_{\mathfrak{p}} \mathbf{D}(A)\end{array}\right\}$$

where $\sigma(\mathcal{S}) = (\mathcal{S} \cap \varGamma_{\mathfrak{p}} \mathbf{D}(A))_{\mathfrak{p}\in\mathcal{V}}$ for a localizing subcategory $\mathcal{S} \subseteq \mathbf{D}(A)$ and

$$\tau((\mathcal{S}(\mathfrak{p}))_{\mathfrak{p}\in\mathcal{V}}) = \operatorname{Loc}_{\mathbf{D}(A)}(\{\mathcal{S}(\mathfrak{p}) \mid \mathfrak{p} \in \mathcal{V}\}).$$

Appendix: Norwegian translations

Here is a list of suggested translations for terminology used in the thesis.

English	Norwegian (bokmål)
basic algebra	basal algebra
central ring action	sentral ringvirkning
colocalization functor	kolokaliseringsfunktor
compactly generated	kompaktgenerert
distinguished triangle	distingvert triangel
Hopkins-Neeman theorem	Hopkins-Neeman-teoremet
Koszul complex	koszulkompleks
local cohomology functor	lokal kohomologifunktor
local-global principle	lokal-global-prinsippet
localization functor	lokaliseringsfunktor
localizing subcategory	lokaliserende underkategori
minimality condition	minimalitetsbetingelsen
perfect complex	perfekt kompleks
quantum polynomial ring	kvantepolynomring
quasi-isomorphism	kvasiisomorfi
residue field	restklassekropp
specialization closed subset	spesialiseringslukket delmengde
stalk complex	stilkkompleks
stratification	stratifisering
stratification condition	stratifiseringsbetingelse
support	støtte
tensor-nilpotence theorem	tensor-nilpotens-teoremet
thick subcategory	tykk underkategori
triangulated category	triangulert kategori
triangulated subcategory	triangulert underkategori
Zariski topology	zariskitopologien

Bibliography

[1] Michael F. Atiyah and Ian G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969, pp. ix+128.

[2] Maurice Auslander, Idun Reiten and Sverre O. Smalø. *Representation theory of Artin algebras*. Vol. 36. Cambridge Studies in Advanced Mathematics. Corrected reprint of the 1995 original. Cambridge University Press, Cambridge, 1997, pp. xiv+425. ISBN: 0-521-41134-3; 0-521-59923-7.

[3] David J. Benson, Srikanth B. Iyengar and Henning Krause. 'Local cohomology and support for triangulated categories'. In: *Ann. Sci. Éc. Norm. Supér. (4)* 41.4 (2008), pp. 573–619. ISSN: 0012-9593. URL: `https://doi.org/10.24033/asens.2076`.

[4] David J. Benson, Srikanth B. Iyengar and Henning Krause. *Representations of finite groups: local cohomology and support*. Vol. 43. Oberwolfach Seminars. Birkhäuser/Springer Basel AG, Basel, 2012, pp. x+105. ISBN: 978-3-0348-0259-8. URL: `https://doi.org/10.1007/978-3-0348-0260-4`.

[5] David J. Benson, Srikanth B. Iyengar and Henning Krause. 'Stratifying modular representations of finite groups'. In: *Ann. of Math. (2)* 174.3 (2011), pp. 1643–1684. ISSN: 0003-486X. DOI: `10.4007/annals.2011.174.3.6`. URL: `https://doi.org/10.4007/annals.2011.174.3.6`.

[6] David J. Benson, Srikanth B. Iyengar and Henning Krause. 'Stratifying triangulated categories'. In: *J. Topol.* 4.3 (2011), pp. 641–666. ISSN: 1753-8416. URL: `https://doi.org/10.1112/jtopol/jtr017`.

[7] Siegfried Bosch. *Algebraic geometry and commutative algebra*. Universitext. Springer, London, 2013, pp. x+504. ISBN: 978-1-4471-4828-9; 978-1-4471-4829-6. URL: `https://doi.org/10.1007/978-1-4471-4829-6`.

[8] Jon F. Carlson and Srikanth B. Iyengar. 'Thick subcategories of the bounded derived category of a finite group'. In: *Trans. Amer. Math. Soc.* 367.4 (2015), pp. 2703–2717. ISSN: 0002-9947. URL: https://doi.org/10.1090/S0002-9947-2014-06121-7.

[9] Thorsten Holm and Peter Jørgensen. 'Triangulated categories: definitions, properties, and examples'. In: *Triangulated categories*. Vol. 375. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2010, pp. 1–51. DOI: 10.1017/CBO9781139107075.002. URL: https://doi.org/10.1017/CBO9781139107075.002.

[10] Michael J. Hopkins. 'Global methods in homotopy theory'. In: *Homotopy theory (Durham, 1985)*. Vol. 117. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 1987, pp. 73–96.

[11] Henning Krause. 'Derived categories, resolutions, and Brown representability'. In: *Interactions between homotopy theory and algebra*. Vol. 436. Contemp. Math. Amer. Math. Soc., Providence, RI, 2007, pp. 101–139. URL: https://doi.org/10.1090/conm/436/08405.

[12] Henning Krause. 'Localization theory for triangulated categories'. In: *Triangulated categories*. Vol. 375. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2010, pp. 161–235. DOI: 10.1017/CBO9781139107075.005. URL: https://doi.org/10.1017/CBO9781139107075.005.

[13] Hideyuki Matsumura. *Commutative ring theory*. Vol. 8. Cambridge Studies in Advanced Mathematics. Translated from the Japanese by M. Reid. Cambridge University Press, Cambridge, 1986, pp. xiv+320. ISBN: 0-521-25916-9.

[14] Jon Peter May. *A concise course in algebraic topology*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999, pp. x+243. ISBN: 0-226-51182-0; 0-226-51183-9.

[15] Amnon Neeman. 'The chromatic tower for $D(R)$'. In: *Topology* 31.3 (1992). With an appendix by Marcel Bökstedt, pp. 519–532. ISSN: 0040-9383. URL: https://doi.org/10.1016/0040-9383(92)90047-L.

[16] Amnon Neeman. 'The connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel'. In: *Ann. Sci. École Norm. Sup. (4)* 25.5 (1992), pp. 547–566. ISSN: 0012-9593. URL: http://www.numdam.org/item?id=ASENS_1992_4_25_5_547_0.

[17] Amnon Neeman. *Triangulated categories*. Vol. 148. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2001, pp. viii+449. ISBN: 0-691-08685-0; 0-691-08686-9. URL: https://doi.org/10.1515/9781400837212.

[18] Douglas C. Ravenel. 'Localization with respect to certain periodic homology theories'. In: *Amer. J. Math.* 106.2 (1984), pp. 351–414. ISSN: 0002-9327. URL: https://doi.org/10.2307/2374308.

[19] The Stacks Project Authors. *Stacks Project, Tag 00NN*. http://stacks.math.columbia.edu. 2018.

[20] The Stacks Project Authors. *Stacks Project, Tag 066S*. http://stacks.math.columbia.edu. 2018.

[21] Charles A. Weibel. *An introduction to homological algebra*. Vol. 38. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994, pp. xiv+450. ISBN: 0-521-43500-5; 0-521-55987-1. URL: https://doi.org/10.1017/CBO9781139644136.