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A Bosonic Coherent State Path Integral Representation for Spin Systems based on a Projection Operator Implementation of the Schwinger boson number constraint

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Abstract

The path integral single spin partition function in the basis of boson coherent states, for a general normal ordered two-mode *Schwinger boson* Hamiltonian, has been computed. Within the initial phase of the calculational process, by a specific approach, the *Schwinger boson* constraint by means of the *projection operator* has been implemented. After, the case of the *Zeeman Hamiltonian* has been taken on. Using the expression for the partition function obtained for the general case, the appropriate single spin partition function has been produced. Additionally, partition functions have been computed for a few specific spin quantum numbers and then compared with those calculated by the straightforward means, which brought about great confidence in the *projection operator* implementation; *both results match perfectly.* Finally, the *Heisenberg Model* has been tackled, and the appropriate expressions for the partition functions have been computed. Their validity has not been verified for any specific cases, but the Zeeman Hamiltonian case dealt with earlier appears to be a strong indicator that these are indeed the correct expressions. Many open questions remain, however, for future research to address. All in all, it has been discovered that using the projection operator as a means to enforce the *Schwinger boson* constraint, veritably works for a *Zeeman Hamiltonian* problem for specific spin quantum numbers.

Contents

Introduction

The *path integral* formalism of quantization deals with an *ensemble of paths*, rather than with *wave functions* and constitutes an alternative to *canonical quantization* in quantizing a classical theory[1]. In this thesis, *boson coherent state path integrals* are specifically studied and worked with . The unique mathematical properties of *bosonic coherent states*, and the calculational techniques involved in the path integral procedure, are incisively applied throughout this work.

In the context of studying spin models such as the *quantum spin liquid*, the *Schwinger boson* representation of the spin operators, with the *Schwinger boson* constraint implemented on the physical space[2], tends to be applied in order to deal with the highly quantum states which are rotationally symmetric[3]. One approach that has been used for such an implementation is by means of the *Dirac delta function*. **As the crux of this thesis, however, the *projection operator* method is employed instead**, and where that leads to is explored deeply and widely.

In order for logical lucidity to be maintained, this thesis is structured in the following fashion:

- A group of established principles of physics that are of special relevance, are reviewed in the initial stage.
- The heart of the thesis is embedded in *Chapters 3 and 4*.
- Tedious calculational details and other trivial matters are relegated to the Appendix.

This thesis is judged to be able to be read without much struggle by an individual whose education in physics matches that at the Master's level and above. The mathematical rigor certainly does not fulfill the crystalline and rigid standard demanded of by a pure mathematician. Nevertheless, a theoretical physicist should be fairly satisfied by it; such is the author's conviction. It is hoped that the reader is shed, by some arbitrary extent, comprehensive light on this novel approach to enforcing the *Schwinger boson* number constraint, ultimately.

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Chapter 1

Some Foundational Principles

1.1 From Stationary Action to Lagrangian Mechanics

1.1.1 Hamilton's Principle

Hamilton's Principle[1] is stated as follows:

"The motion of a mechanical system from time t_1 to time t_2 is such that the line integral

$$\boxed{S = \int_{t_1}^{t_2} L dt}$$

where $L = T - V$, has a ***stationary*** *value. Conversely, for this value, the system traces out its* ***empirical*** *path of motion".*

T and V represent the *kinetic energy* and the *potential energy* of the system, respectively. The quantity S is referred to as the *action*. But what does it mean by *stationary*? Let us say S is stationary. This means that **S along its particular path has the *same* value to within first-order infinitesimals as that along all paths that differ from it by *infinitesimal displacements***. In other words, Hamilton's principle proclaims that the motion is such that the *variation* of S for fixed t_1 and t_2 vanishes:

$$\boxed{\delta S = \delta \int_{t_1}^{t_2} L(q_1, ..., q_n, \dot{q}_1, ..., \dot{q}_n, t) dt = 0}$$

[1] This tends to be loosely referred to as the *Principle of Least Action*. The material in this section (*From Stationary Action to Lagrangian Mechanics*) is based upon the chapter *Variational Principles and Lagrange's Equations* of *Herbert Goldstein*'s book *Classical Mechanics*.[4]

for some particular set of paths(which differ from the stationary path infinitesimally).

1.1.2 The Fundamental Problem of The Calculus of Variations

The following one-dimensional problem is called *The Fundamental Problem of The Calculus of Variations*:

There exists a function $f(y, \dot{y}, x)$ defined on a path $y = y(x)$ between two values x_1 and x_2, where $\dot{y}$ is the derivative of y with respect to x:

"*What is the path $y(x)$ such that*

$$S = \int_{x_1}^{x_2} f(y, \dot{y}, x)dx \tag{1.1}$$

is stationary?"

In this problem, we take into account only paths which satisfy $y(x_1) = y_1$ and $y(x_2) = y_2$. Additionally, we consider the fulfillment of two conditions. These conditions are that S must have a stationary value for the correct path relative to **any neighboring path**, and the variation must be zero relative to **some particular set of neighboring paths**. Let us label this particular set by an infinitesimal parameter α, and characterize them by $y(x, \alpha)$, with $y(x, 0)$ depicting the stationary path.

As an example, one set of varied paths could be[2]

$$y(x, \alpha) = y(x, 0) + \alpha\eta(x) \tag{1.2}$$

where $\eta(x)$ vanishes at $x = x_1$ and $x = x_2$.

For any such family of curves defined by the parameter α, S in (1.1) becomes a *functional*:

$$S[\alpha] = \int_{x_1}^{x_2} f\left[y(x, \alpha), \dot{y}(x, \alpha), x\right]dx \tag{1.3}$$

The stationary condition for this is

$$\left(\frac{dS}{d\alpha}\right)_{\alpha=0} = 0 \tag{1.4}$$

[2]We assume that both the correct path $y(x)$ and the auxiliary function $\eta(x)$ are continuous and non-singular between x_1 and x_2, with continuous first and second derivatives in the same interval.

Now, by differentiating under the integral sign and then performing partial integration one gets

$$\frac{dS}{d\alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} \right) \frac{\partial y}{\partial \alpha} dx \tag{1.5}$$

Therefore, the condition (1.4) becomes equivalent to

$$\int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} \right) \left(\frac{\partial y}{\partial \alpha} \right)_{\alpha=0} dx = 0 \tag{1.6}$$

Now, the *fundamental lemma* of the variational calculus states:

"*If*

$$\boxed{\int_{x_1}^{x_2} M(x)\eta(x)dx = 0}$$

for all arbitrary $\eta(x)$ *continuous through the second derivative, then* $M(x)$ *must identically vanish in the interval* (x_1, x_2)"

Thus, it follows that S is stationary only if

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}} \right) = 0 \tag{1.7}$$

Now, the differential quantity

$$\boxed{\left(\frac{dy}{d\alpha} \right)_{\alpha=0} d\alpha \equiv \delta y}$$

represents the infinitesimal deviation of the varied path from the stationary path $y(x)$ at the point x. Analogously, the infinitesimal variation of S about the stationary path is

$$\boxed{\left(\frac{dS}{d\alpha} \right)_{\alpha=0} d\alpha \equiv \delta S}$$

Hence, the fact that S is stationary can be written

$$\delta S = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} \right] \delta y dx = 0 \tag{1.8}$$

which requires that $y(x)$ fulfil the differential equation (1.7).

1.1.3 Lagrangian Mechanics as An Outcome of Further Generalization

The fundamental problem of the calculus of variations can be generalized further to include the following cases:

- f is a function of many independent variables y_i, and their derivatives $\dot{y}_i$
- f is a function of much higher derivatives of y.
- Several x_i parameters exist
- Variations exist in which the end points are not held fixed

One non-trivial consequence of the first of the above cases, in connection with Hamilton's principle, is the derivation of the equations of motion of Lagrangian mechanics, an alternative formulation of classical mechanics to the prevailing Newtonian formulation. Realize that the entirety of Lagrangian mechanics can be derived from the rather innocent idea of the *stationary action*.

1.2 Path Integral Quantum Mechanics

1.2.1 The Probability Amplitude

Imagine a particle moving from some initial position (at some time) to some final position (at some other time). In quantum mechanics, there exists an entity called the *probability amplitude*, which is a complex quantity that can be used to calculate the likelihood that a particle will arrive at some final position from some initial position. Furthermore, there exists the notion of the *propagator*, $K(b,a)$, which contains some very specific information about going from one point a to another point b. In fact, $K(b,a)$ is the sum, over **all** of the paths that link the two end points a and b, of a numerical contribution from each path.[3]

Contrary to how only the path of stationary action is important in describing the motion of classical mechanical systems, all paths make a contribution to the kernel, which describes quantum mechanical motion. Each contribution is equal in magnitude, but possesses dissimilar *phase*. The phase of the contribution for a given path is the action S for that path, in units of the quantum of action, $\hbar$.

The probability $P(b,a)$ to go from a point x_a at time t_a to the point x_b at time t_b is

$$\boxed{P(b,a) = |K(b,a)|^2}$$

where $K(b,a)$ is the sum, of the probability amplitudes for each path.

[3] The material in this section (*Path Integral Quantum Mechanics*) is based upon the chapter *The Quantum-mechanical Law of Motion* of *Richard P.Feynman and Albert R.Hibb*'s book *Quantum Mechanics and Path Integrals*.[5]

Each probability amplitude contribution has a phase proportional to the action S:

$$\boxed{\phi[x(t)] = A \exp\left(\frac{i}{\hbar} S[x(t)]\right)}$$

where A is a normalization constant.

Now, let us look at $K(b, a)$ more deeply. **What does it really mean to add up the probability amplitudes for each path?**

1.2.2 Summing Over Paths

Consider an ordinary Riemann integral. One could then say that the area A under a curve is proportional to the sum of all its ordinates. Let us take a subset of **all** ordinates such as those spaced at equal intervals of h, and add them up. Then, we can write the following mathematical relationship:

$$\boxed{A \sim \sum_i f(x_i)}$$

where the summation is carried out over the finite set of points x_i.

Next, let us define A as the limit of this sum as the subset of points (and thus the subset of ordinates), gradually approaches infinity. To acquire a limit to this process, we must specify some normalizing factor which should depend on h. For the Riemann integral, this factor is just h itself. Hence, the limit exists now:

$$\boxed{A = lim_{h \to 0}\left[h \sum_i f(x_i)\right]}$$

By a similar reasoning, we can understand the sum over **all** paths.

First, we choose a subset of all paths by dividing the independent variable time into steps of width ϵ. Through this, we get a set of values t_i spaced an interval ϵ apart between the values t_a and t_b. At each time t_i we select some special point x_i. Now, a path is constructed by connecting all the selected points with **straight** lines. Having done that, it is possible to define a sum over all paths constructed in this manner by taking a multiple integral over all values of x_i for i between 1 and

$N-1$, where

$$
\begin{aligned}
N\epsilon &= t_b - t_a \\
\epsilon &= t_{i+1} - t_i \\
t_0 &= t_a \\
t_N &= t_b \\
x_0 &= x_a \\
x_N &= x_b
\end{aligned}
$$

The mathematical relationship we obtain, as a result, is:

$$
\boxed{K(b,a) \sim \int ... \int \int \phi[x(t)] dx_1 dx_2 ... dx_{N-1}}
$$

We do not integrate over x_0 or x_N because these are the fixed end points x_a and x_b. We can obtain a sample that resembles the complete set of all possible paths between a and b more, by making ϵ smaller. However, just as in the case of the Riemann integral, we cannot directly work on finding the limit of this process because the limit does not yet exist. As with the former case, we must provide some normalizing factor which we expect will depend upon ϵ.

There are many ways to define a subset of all the paths between a and b, and each of them could incorporate different non-stringent, convenient and fairly effective artifices in order to deal with the mathematical 'awkwardness' which could arise in it. One may say the emergence of these inelegant complications is in some sense an inevitable outcome of the mathematically flexible and lenient nature of the calculational details surrounding the idea of summing over paths.

The concept of the sum over all paths, on the other hand, is valid despite the existence of the aforementioned mathematical 'awkwardness'. One may casually write the sum over all paths as

$$
\boxed{K(b,a) = \int_a^b \exp\left(\frac{i}{\hbar} S[b,a]\right) Dx(t)}
$$

which is called a *path integral.*

1.3 Creation and Annihilation Operators of Bosons

In describing many-particle states, creation and annihilation operators can be very helpful. They generate the entire Hilbert space by their action on a single reference state, and provide a basis for the algebra of operators of the Hilbert space.[4]

[4]The material in this section (*Creation and Annihilation Operators of Bosons*) and the next (*Coherent States*) is based upon the chapter *Second Quantization and Coherent States* of *John W.Negele and Henri Orland*'s book *Quantum Many-Particle Systems.*[6]

For each single-particle state $|\lambda\rangle$ of the single-particle space H, one defines a boson or fermion creation operator $a_\lambda^\dagger$ by its action on any symmetrized or anti-symmetrized state $|\lambda_1...\lambda_N\rangle$ of the boson Hilbert space or the fermion Hilbert space as follows:

$$a_\lambda^\dagger|\lambda_1...\lambda_N\rangle \equiv |\lambda\lambda_1...\lambda_N\rangle \tag{1.9}$$

For an orthonormal basis $\{|\mu_i\rangle\}$:

$$\boxed{a_\mu^\dagger|\mu_1...\mu_N\rangle = \sqrt{n_\mu + 1}|\mu\mu_1...\mu_N\rangle}$$

where n_μ is the occupation number of the state $|\mu\rangle$ in $|\mu_1...\mu_N\rangle$.

Physically, the operator $a_\mu^\dagger$ adds a particle in state $|\mu\rangle$ to the state on which it operates, and symmetrizes or anti-symmetrizes the new state.

The vacuum state, denoted $|0\rangle$, represents a state with zero particles. $a_\mu^\dagger$ acting on the vacuum $|0\rangle$ creates a particle in state $|\mu\rangle$:

$$a_\mu^\dagger|0\rangle = |\mu\rangle \tag{1.10}$$

A general state $|\phi\rangle$ of the Fock space is a linear combination of states with any number of particles. Any basis vector $|\mu_1...\mu_N\rangle$ may be generated by repeated action of the creation operators in the vacuum $|0\rangle$. The symmetry or anti-symmetry properties of the many-particle states impose commutation or anti-commutation relations between the creation operators.

For bosons, the creation operators commute:

$$[a_\lambda^\dagger, a_\mu^\dagger] = 0 \tag{1.11}$$

The annihilation operators a_λ are defined as the adjoints of the creation operators $a_\lambda^\dagger$. Their commutation relations are:

$$[a_\lambda, a_\mu] = 0 \tag{1.12}$$

When acting on the vacuum:

$$a_\lambda|0\rangle = 0 \tag{1.13}$$

which implies

$$\langle 0|a_\lambda^\dagger = 0 \tag{1.14}$$

Also, the bosons satisfy the following commutation relation:

$$[a_\lambda, a_\mu^\dagger] = \delta_{\lambda\mu} \tag{1.15}$$

1.4 Coherent States

One useful basis of the Fock space is the basis of *coherent states.* Although this is not an orthonormal basis, it spans the entire Fock space. Just as the position states $|\boldsymbol{r}\rangle$ are defined as the eigenstates of $\hat{\boldsymbol{r}}$, the coherent states are defined as *eigenstates of the annihilation operators*:

$$\boxed{a_\alpha|\phi\rangle = \phi_\alpha|\phi\rangle}$$

1.4.1 Boson Coherent States

A boson coherent state in occupation number representation is:

$$|\phi\rangle = \sum_{n_{\alpha_1}, n_{\alpha_2}, ..., n_{\alpha_p}} \phi_{n_{\alpha_1} n_{\alpha_2} ... n_{\alpha_p}} ... |n_{\alpha_1} n_{\alpha_2} ... n_{\alpha_p} ...\rangle \tag{1.16}$$

where $|n_{\alpha_1} n_{\alpha_2} ... n_{\alpha_p} ...\rangle$ denotes a normalized symmetrized state with n_{α_1} particles in state $|\alpha_1\rangle$, n_{α_2} particles in state $|\alpha_2\rangle$,... and $\{|\alpha_i\rangle\}$ is an orthonormal basis.

Using the definition of coherent states and the quantum harmonic oscillator, one obtains the following results:

$$\begin{aligned}
|\phi\rangle &= \sum_{n_{\alpha_1}, n_{\alpha_2}, ..., n_{\alpha_p}} \frac{(\phi_{\alpha_1} a^\dagger_{\alpha_1})^{n_{\alpha_1}}}{n_{\alpha_1}!} \frac{(\phi_{\alpha_2} a^\dagger_{\alpha_2})^{n_{\alpha_2}}}{n_{\alpha_2}!} ... \frac{(\phi_{\alpha_p} a^\dagger_{\alpha_p})^{n_{\alpha_p}}}{n_{\alpha_p}!} |0\rangle \\
&= \exp\left[\sum_\alpha \phi_\alpha a^\dagger_\alpha\right] |0\rangle
\end{aligned}$$

The overlap of two coherent states is given by

$$\langle\phi|\phi^{'}\rangle = \exp\left[\sum_\alpha \bar{\phi}_\alpha \phi^{'}_\alpha\right] \tag{1.17}$$

A crucial property of the coherent states is their *overcompleteness* in the Fock space, that is, the fact that any vector of the Fock space can be expanded in terms of coherent states. This is expressed by the closure relation:

$$\boxed{\int \prod_\alpha \frac{d\bar{\phi}_\alpha d\phi_\alpha}{2\pi i} \exp\left[-\sum_\alpha \bar{\phi}_\alpha \phi_\alpha\right] |\phi\rangle\langle\phi| = \mathbf{1}}$$

where $\mathbf{1}$ is the unit operator in the Fock space.

The measure is:

$$\begin{aligned}\frac{d\bar{\phi}_\alpha d\phi_\alpha}{2\pi i} &= \frac{1}{2\pi i}|J|dx_\alpha dy_\alpha \\ &= \frac{1}{2\pi i}(2i)dx_\alpha dy_\alpha \\ &= \frac{dx_\alpha dy_\alpha}{\pi} \\ &= \frac{d(Re\phi_\alpha)d(Im\phi_\alpha)}{\pi}\end{aligned}$$

and the integration extends over all values of $Re\phi_\alpha$ and $Im\phi_\alpha$.

Chapter 2

Coherent State Path Integrals

2.1 Propagator In The Basis of Boson Coherent States For A General Single Mode Hamiltonian

The following is the expression for the *propagator* or the *total probability amplitude* of a system, in the basis of boson coherent states, moving from some initial state $|z_0\rangle$ to some final state $|z_N\rangle$:

$$\boxed{U(z_N, z_0, t) = \langle z_N|U(t)|z_0\rangle}$$

where $U(t) = \exp\left(\frac{-it\hat{H}}{\hbar}\right)$ is the *time-evolution operator*.

The Hamiltonian is assumed to be for a general single mode:

$$\hat{H} = \hat{H}(a^\dagger, a) \tag{2.1}$$

Furthermore, it is assumed to be *normal-ordered*.

This will now be computed in the *path-integral representation*. Firstly, the time-evolution operator is broken apart as follows:

$$\begin{aligned} U(z_N, z_0, t) &= \langle z_N| \exp(-it\hat{H}(a^\dagger, a)/\hbar)|z_0\rangle \\ &= \langle z_N|[\exp(-i\epsilon\hat{H}(a^\dagger, a)/\hbar)]^N|z_0\rangle; \qquad \epsilon = t/N \\ &= \langle z_N|U^N(\epsilon)|z_0\rangle \end{aligned}$$

At this point, a set of coherent states $\{|z_j\rangle\}$ is inserted. This is done by inserting the coherent state closure relation[7] at each interval of time t_j as follows:

$$U(z_N, z_0, t) = \langle z_N|U(\epsilon)\int \frac{dz_{N-1}d\bar{z_{N-1}}}{2\pi i}\exp(-z_{N-1}\bar{z_{N-1}})|z_{N-1}\rangle$$
$$\langle z_{N-1}|U(\epsilon)\int \frac{dz_{N-2}d\bar{z_{N-2}}}{2\pi i}\exp(-z_{N-2}\bar{z_{N-2}}|z_{N-2}\rangle$$

$$\langle z_{N-2}|U(\epsilon)\int \frac{dz_{N-3}d\bar{z_{N-3}}}{2\pi i}\exp(-z_{N-3}\bar{z_{N-3}}|z_{N-3}\rangle$$
$$...$$
$$\langle z_2|U(\epsilon)\int \frac{dz_1 d\bar{z_1}}{2\pi i}\exp(-z_1\bar{z_1})|z_1\rangle$$
$$\langle z_1|U(\epsilon)|z_0\rangle$$

$$= \int \prod_{i=1}^{N-1} \frac{dz_i d\bar{z_i}}{2\pi i}\exp(-\sum_{k=1}^{N-1} z_k\bar{z_k})\langle z_N|U(\epsilon)|z_{N-1}\rangle$$
$$\langle z_{N-1}|U(\epsilon)|z_{N-2}\rangle\langle z_{N-2}|U(\epsilon)|z_{N-3}\rangle$$
$$...$$
$$\langle z_2|U(\epsilon)|z_1\rangle\langle z_1|U(\epsilon)|z_0\rangle$$

It is known that

$$\boxed{U(\epsilon) \to 1 - \frac{iH(a^\dagger, a)\epsilon}{\hbar}}$$

as $\epsilon \to 0$

Therefore,

$$lim_{\epsilon\to 0}U(z_N, z_0, t) = \int \prod_{i=1}^{N-1} \frac{dz_i d\bar{z_i}}{2\pi i}\exp(-\sum_{k=1}^{N-1} z_k\bar{z_k})$$
$$\langle z_N|z_{N-1}\rangle[1 - \frac{iH(\bar{z_N}, z_{N-1})\epsilon}{\hbar}]$$
$$\langle z_{N-1}|z_{N-2}\rangle[1 - \frac{iH(\bar{z_{N-1}}, z_{N-2})\epsilon}{\hbar}]$$
$$...$$
$$\langle z_1|z_0\rangle[1 - \frac{iH(\bar{z_1}, z_0)\epsilon}{\hbar}]$$

Maintaining $\epsilon \to 0$, and reverting $1 - \frac{iH(a^\dagger,a)\epsilon}{\hbar}$ back to its exponential form:

$$\begin{aligned} lim_{\epsilon\to 0}U(z_N, z_0, t) = \int \prod_{i=1}^{N-1} \frac{dz_i d\bar{z_i}}{2\pi i} \exp(-\sum_{k=1}^{N-1} z_k \bar{z_k}) \\ \langle z_N|z_{N-1}\rangle \exp(-\frac{iH(\bar{z_N}, z_{N-1})\epsilon}{\hbar}) \\ \langle z_N|z_{N-1}\rangle \exp(-\frac{iH(\bar{z_{N-1}}, z_{N-2})\epsilon}{\hbar}) \\ ... \\ \langle z_1|z_0\rangle \exp(-\frac{iH(\bar{z_1}, z_0)\epsilon}{\hbar}) \end{aligned}$$

Now

$$\langle z_N|z_{N-1}\rangle \quad = \quad \langle 0|\exp(\bar{z_N}a)\exp(z_{N-1}a^\dagger)|0\rangle \tag{2.2}$$

The identity

$$\boxed{\exp(A)\exp(B) = \exp(B)\exp(A)\exp([A,B])}$$

is valid if [A,B] commutes with A and B.

Using this identity, the following is obtained:

$$\begin{aligned} \langle z_N|z_{N-1}\rangle &= \langle 0|\exp(z_{N-1}a^\dagger)\exp(\bar{z_N}a)|0\rangle \exp(\bar{z_N}, z_{N-1}) \\ &= \exp(\bar{z_N}z_{N-1}) \end{aligned}$$

Furthermore,

$$\begin{aligned} lim_{\epsilon\to 0}U(z_N, z_0, t) = \int \prod_{i=1}^{N-1} \frac{dz_i d\bar{z_i}}{2\pi i} \exp(-\sum_{k=1}^{N-1} z_k \bar{z_k}) \\ \exp(\bar{z_N}z_{N-1}) \exp(-\frac{iH(\bar{z_N}, z_{N-1})\epsilon}{\hbar}) \\ \exp(\bar{z_{N-1}}z_{N-2}) \exp(-\frac{iH(\bar{z_{N-1}}, z_{N-2})\epsilon}{\hbar}) \\ ... \\ \exp(\bar{z_2}z_1) \exp(-\frac{iH(\bar{z_2}, z_1)\epsilon}{\hbar}) \\ \exp(\bar{z_1}z_0) \exp(-\frac{iH(\bar{z_1}, z_0)\epsilon}{\hbar}) \end{aligned}$$

This result is now examined in two parts[8]:

The first part:

$$\exp[-\frac{iH(\bar{z_N}, z_{N-1})\epsilon}{\hbar} - \frac{iH(\bar{z_{N-1}}, z_{N-2})\epsilon}{\hbar} - ... - \frac{iH(\bar{z_1}, z_0)\epsilon}{\hbar}]$$
$$= \exp[\frac{-i}{\hbar}\epsilon(H(\bar{z_N}, z_{N-1}) + H(\bar{z_{N-1}}, z_{N-2}) + ... + H(\bar{z_1}, z_0))]$$
$$= \exp[-\frac{i}{\hbar}\epsilon(\sum_{k=1}^{N-1} H(\bar{z}_{k+1}, z_k) + H(\bar{z_1}, z_0))]$$

The second part:

$$\exp[-\sum_{k=1}^{N-1} z_k\bar{z_k} + \sum_{k=1}^{N} \bar{z_k} z_{k-1}]$$
$$= \exp[(\bar{z_N} - \bar{z_{N-1}})z_{N-1} + (\bar{z_{N-1}} - \bar{z_{N-2}})z_{N-2} + (\bar{z_{N-2}} - \bar{z_{N-3}})z_{N-3} + ... + (\bar{z_2} - \bar{z_1})z_1 + \bar{z_1}z_0]$$
$$= \exp[\sum_{k=1}^{N-1}(\bar{z}_{k+1} - \bar{z_k})z_k + \bar{z_1}z_0]$$

Therefore,

$$\boxed{lim_{\epsilon\to 0}U(z_N, z_0, t) = \int\left[\prod_{k=1}^{N-1}\frac{dz_k d\bar{z_k}}{2\pi i}\right]\exp\left[\frac{i}{\hbar}S\right]}$$

where

$$\boxed{S = -i\hbar\sum_{k=1}^{N-1}\left[\frac{-i\epsilon}{\hbar}H(\bar{z}_{k+1}, z_k) + (\bar{z}_{k+1} - \bar{z_k})z_k\right] - i\hbar\left[\frac{-i\epsilon}{\hbar}H(\bar{z_1}, z_0) + \bar{z_1}z_0\right]}$$

2.2 Partition Function In The Basis of Boson Coherent States For A General Single Mode Hamiltonian

Similar to the previous context, consider a system governed by a general single mode Hamiltonian, which is also *normal ordered* $\hat{H}(a^\dagger, a)$. Inserting two boson coherent state closure relations in the occupation number representation of the partition function:

$$
\begin{aligned}
Z &= Tr[\exp(-\beta\hat{H})] \\
&= \sum_n \langle n|\exp(-\beta\hat{H})|n\rangle \\
&= \sum_n \langle n| \int \frac{d\bar{z}_1 dz_1}{2\pi i}\exp(-\bar{z}_1 z_1)|z_1\rangle\langle z_1|\exp(-\beta\hat{H}) \int \frac{d\bar{z}_2 dz_2}{2\pi i}\exp(-\bar{z}_2 z_2)|z_2\rangle\langle z_2|n\rangle \\
&= \int \frac{d\bar{z}_1 dz_1}{2\pi i}\exp(-\bar{z}_1 z_1) \int \frac{d\bar{z}_2 dz_2}{2\pi i}\exp(-\bar{z}_2 z_2)\sum_n \langle n|z_1\rangle\langle z_1|\exp(-\beta\hat{H})|z_2\rangle\langle z_2|n\rangle \\
&= \int \frac{d\bar{z}_1 dz_1}{2\pi i}\exp(-\bar{z}_1 z_1) \int \frac{d\bar{z}_2 dz_2}{2\pi i}\exp(-\bar{z}_2 z_2)\sum_n \langle z_2|n\rangle\langle n|z_1\rangle\langle z_1|\exp(-\beta\hat{H})|z_2\rangle \\
&= \int \frac{d\bar{z}_1 dz_1}{2\pi i}\exp(-\bar{z}_1 z_1) \int \frac{d\bar{z}_2 dz_2}{2\pi i}\exp(-\bar{z}_2 z_2)\langle z_2|\sum_n |n\rangle\langle n|z_1\rangle\langle z_1|\exp(-\beta\hat{H})|z_2\rangle \\
&= \int \frac{d\bar{z}_1 dz_1}{2\pi i}\exp(-\bar{z}_1 z_1) \int \frac{d\bar{z}_2 dz_2}{2\pi i}\exp(-\bar{z}_2 z_2)\langle z_2|\mathbf{1}|z_1\rangle\langle z_1|\exp(-\beta\hat{H})|z_2\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \int \frac{d\bar{z}_1 dz_1}{2\pi i}\exp(-\bar{z}_1 z_1) \int \frac{d\bar{z}_2 dz_2}{2\pi i}\exp(-\bar{z}_2 z_2)\langle z_2|z_1\rangle\langle z_1|\exp(-\beta\hat{H})|z_2\rangle \\
&= \int \frac{d\bar{z}_2 dz_2}{2\pi i}\exp(-\bar{z}_2 z_2)\langle z_2| \int \frac{d\bar{z}_1 dz_1}{2\pi i}\exp(-\bar{z}_1 z_1)|z_1\rangle\langle z_1|\exp(-\beta\hat{H})|z_2\rangle \\
&= \int \frac{d\bar{z}_2 dz_2}{2\pi i}\exp(-\bar{z}_2 z_2)\langle z_2|\mathbf{1}\exp(-\beta\hat{H})|z_2\rangle \\
&= \int \frac{d\bar{z}_2 dz_2}{2\pi i}\exp(-\bar{z}_2 z_2)\langle z_2|\exp(-\beta\hat{H})|z_2\rangle
\end{aligned}
$$

Now let $z_2 = z$:

$$Z = \int \frac{d\bar{z}dz}{2\pi i}\exp(-\bar{z}z)\langle z|\exp(-\beta\hat{H})|z\rangle \tag{2.3}$$

Observe that the term $\langle z|\exp(-\beta\hat{H})|z\rangle$ appears quite similar to the propagator $\langle z_N|\exp(-\frac{itH(\hat{a^\dagger},a)}{\hbar})|z_0\rangle$. Therefore, exploiting some of the results derived in the earlier section seems reasonable. In order to make this work, the following periodic boundary conditions are first imposed:

$$
\begin{aligned}
\bar{z} &= \bar{z}_N = \bar{z}_0 \\
z &= z_N = z_0
\end{aligned}
$$

and $\frac{i\epsilon}{\hbar}$ is replaced with $\Delta\beta$.

Thus:

$$
\begin{aligned}
\langle z|\exp(-\beta\hat{H})|z\rangle = &\int\prod_{i=1}^{N-1}(\frac{dz_i d\bar{z}_i}{2\pi i})\exp[-\Delta\beta(H(\bar{z},z_{N-1})+H(\bar{z}_{N-1},z_{N-2})+...+H(\bar{z_1},z))] \\
&\exp[(\bar{z}-\bar{z}_{N-1})z_{N-1}+(\bar{z}_{N-1}-\bar{z}_{N-2})z_{N-2}+...+(\bar{z_2}-\bar{z_1})z_1+\bar{z_1}z] \\
= &\int\prod_{i=1}^{N-1}(\frac{dz_i d\bar{z}_i}{2\pi i})\exp[-\Delta\beta(H(\bar{z},z_{N-1})+H(\bar{z_1},z)+\sum_{k=1}^{N-2}H(\bar{z}_{k+1},z_k)] \\
&\exp[\bar{z_1}z+(\bar{z}-\bar{z}_{N-1})z_{N-1}+\sum_{k=1}^{N-2}(\bar{z}_{k+1}-\bar{z_k})z_k]
\end{aligned}
$$

Combining this with the other term, the partition function can be written as

$$
\boxed{Z=\int\left[\prod_{k=1}^{N}\frac{dz_k d\bar{z}_k}{2\pi i}\right]\exp(S)}
$$

where

$$
\boxed{S=\sum_{k=1}^{N}\left[(\bar{z}_{k+1}-\bar{z_k})z_k-\Delta\beta H(\bar{z}_{k+1},z_k)\right]}
$$

2.2.1 Partition Function In The Basis of Boson Coherent States For The Single Mode Quantum Harmonic Oscillator Hamiltonian

Consider now the single mode quantum harmonic oscillator Hamiltonian

$$
\hat{H}=\hat{H}(a^\dagger,a)=\hbar\omega(a^\dagger a+\frac{1}{2}) \tag{2.4}
$$

Then,

$$
\begin{aligned}
\langle z|\exp(-\beta\hat{H})|z\rangle = &\int\prod_{i=1}^{N-1}(\frac{dz_i d\bar{z}_i}{2\pi i}) \\
&\exp[-\Delta\beta(H(\bar{z},z_{N-1})+H(z_{\bar{N}-1},z_{N-2})+...+H(\bar{z_1},z))] \\
&\exp[(\bar{z}-\bar{z}_{N-1})z_{N-1}+(\bar{z}_{N-1}-\bar{z}_{N-2})z_{N-2}+...+(\bar{z_2}-\bar{z_1})z_1+\bar{z_1}z]
\end{aligned}
$$

$$= \int \prod_{i=1}^{N-1} (\frac{dz_i d\bar{z}_i}{2\pi i})$$

$$\exp[-\Delta\beta\hbar\omega(\bar{z}z_{N-1} + \frac{1}{2} + \sum_{r=2}^{N-1}(\bar{z_r}z_{r-1} + \frac{1}{2}) + \bar{z_1}z + \frac{1}{2})]$$

$$\exp[\bar{z_1}z + \sum_{r=1}^{N-2}(\bar{z_{r+1}} - \bar{z_r})z_r + (\bar{z} - \bar{z_{N-1}})z_{N-1}]$$

$$= \exp(-\Delta\beta\hbar\omega\frac{N}{2}) \int \prod_{i=1}^{N-1} (\frac{dz_i d\bar{z}_i}{2\pi i}) \exp[-d\beta\hbar\omega(\bar{z}z_{N-1} + \sum_{r=2}^{N-1}(\bar{z_r}z_{r-1}) + \bar{z_1}z)]$$

$$\exp[\bar{z_1}z + \sum_{r=1}^{N-2}(\bar{z_{r+1}} - \bar{z_r})z_r + (\bar{z} - \bar{z_{N-1}})z_{N-1}]$$

$$= \exp(-\frac{\beta\hbar\omega}{2}) \int \prod_{i=1}^{N-1} (\frac{d\bar{z}_i dz_i}{2\pi i}) \exp[\bar{z_1}z + \sum_{r=1}^{N-1}(\bar{z_{r+1}} - \bar{z_r})z_r - d\beta\hbar\omega\sum_{r=1}^{N}\bar{z}z_{r-1}]$$

Combining with the other term, the partition function becomes

$$Z = \int \prod_{i=1}^{N} (\frac{d\bar{z_i}dz_i}{2\pi i}) \exp(\frac{-\beta\hbar\omega}{2}) \exp[\sum_{r=1}^{N}(\bar{z_{r+1}}z_r - \bar{z_r}z_r - \Delta\beta\hbar\omega \bar{z_{r+1}}z_r)]$$

$$= \int \prod_{i=1}^{N} (\frac{d\bar{z_i}dz_i}{2\pi i}) \exp(\frac{-\beta\hbar\omega}{2}) \exp[-\sum_{r=1}^{N}(\bar{z_r}z_r + \bar{z_{r+1}}z_r(\Delta\beta\hbar\omega - 1))]$$

Modifying the argument of the exponential[9]:

$$-\sum_{r=1}^{N}(\bar{z_r}z_r + \bar{z_{r+1}}z_r(\Delta\beta\hbar\omega - 1))$$

$$= -\sum_{r,p}^{N}\bar{z_r}A_{rp}z_p$$

where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & -1+\Delta\beta\hbar\omega \\ -1+\Delta\beta\hbar\omega & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1+\Delta\beta\hbar\omega & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & -1+\Delta\beta\hbar\omega & 1 \end{bmatrix}$$

is an N by N matrix.

Hence, the argument of the exponential may be rewritten as

$$-\sum_{r,p}^{N} \bar{z}_r A_{rp} z_p = -\bar{\boldsymbol{z}}\boldsymbol{A}\boldsymbol{z} \tag{2.5}$$

where

$$\bar{\boldsymbol{z}} = (\bar{z}_1, \bar{z}_2, \bar{z}_3, ..., \bar{z}_{N-1}, \bar{z}_N) \tag{2.6}$$

$$\boldsymbol{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \ldots \\ z_{N-1} \\ z_N \end{bmatrix} \tag{2.7}$$

This implies

$$Z = \exp(-\frac{\beta\hbar\omega}{2}) \int (\prod_{i=1}^{N} \frac{d\bar{z}_i dz_i}{2\pi i}) \exp(-\bar{\boldsymbol{z}}\boldsymbol{A}\boldsymbol{z})$$

Now

$$\begin{aligned} -\bar{\boldsymbol{z}}\boldsymbol{A}\boldsymbol{z} &= -\boldsymbol{z}^{\dagger}\boldsymbol{A}\boldsymbol{z} \\ &= -\boldsymbol{z}^{\dagger}\boldsymbol{U}\boldsymbol{U}^{\dagger}\boldsymbol{A}\boldsymbol{U}\boldsymbol{U}^{\dagger}\boldsymbol{z} \\ &= -\tilde{\boldsymbol{z}}^{\dagger}\boldsymbol{A}^{\boldsymbol{D}}\tilde{\boldsymbol{z}} \\ &= -\sum_{r,p}^{N} \tilde{z}_r^{\dagger} A_{rp}^{D} \tilde{z}_p \\ &= -\sum_{r=1}^{N} A_{rr}^{D} \tilde{z}_r^{\dagger} \tilde{z}_r \\ &= -\sum_{r=1}^{N} A_{rr}^{D} (\tilde{x_r}^2 + \tilde{y_r}^2) \end{aligned}$$

Note the presence of the following:

$$\boldsymbol{z}^{\dagger}\boldsymbol{U} = \tilde{\boldsymbol{z}}^{\dagger}$$
$$\boldsymbol{U}^{\dagger}\boldsymbol{z} = \tilde{\boldsymbol{z}}$$

Calculating the Jacobian matrix for the new variables $\tilde{z}^{\dagger}$ and $\tilde{z}$:

$$\begin{aligned} J &= \begin{bmatrix} \frac{\partial \tilde{\bar{z}}}{\partial \bar{z}} & 0 \\ 0 & \frac{\partial \tilde{z}}{\partial z} \end{bmatrix} \\ &= \begin{bmatrix} U & 0 \\ 0 & U^{\dagger} \end{bmatrix} \end{aligned}$$

Thus, $|J|$ is $UU^{\dagger} = 1$.

Finally, since

$$\frac{dzd\bar{z}}{2\pi i} = \frac{dxdy}{\pi}$$

This also means

$$\frac{d\tilde{z}d\bar{\tilde{z}}}{2\pi i} = \frac{d\tilde{x}d\tilde{y}}{\pi}$$

Therefore,

$$\begin{aligned} Z &= \exp(-\frac{\beta\hbar\omega}{2}) \int (\prod_{i=1}^{N} \frac{\tilde{dx}_i \tilde{dy}_i}{\pi}) \exp(-\sum_{r=1}^{N} A^D_{rr}(\tilde{x}^2_r + \tilde{y}^2_r)) \\ &= \exp(-\frac{\beta\hbar\omega}{2}) \prod_{r=1}^{N} \frac{1}{\pi} \sqrt{\frac{\pi}{A^D_{rr}}} \sqrt{\frac{\pi}{A^D_{rr}}} \\ &= \exp(-\frac{\beta\hbar\omega}{2}) \prod_{r=1}^{N} \frac{1}{A^D_{rr}} \\ &= \exp(-\frac{\beta\hbar\omega}{2}) \frac{1}{det[\boldsymbol{A}^D]} \end{aligned}$$

Now,

$$\begin{aligned} det\boldsymbol{A^D} &= det[\boldsymbol{U}^{\dagger}\boldsymbol{A}\boldsymbol{U}] \\ &= det[\boldsymbol{U}^{\dagger}\boldsymbol{U}]\boldsymbol{det}[\boldsymbol{A}] \\ &= det[\boldsymbol{A}] \end{aligned}$$

$det[\boldsymbol{A}]$ is calculated using the cofactor expansion of $\boldsymbol{A}$ (all the terms turn out to be zero, apart from the first and the last term), as well as the fact that *the determinant of any triangular matrix is the product of its diagonal entries.*

Hence,

$$\begin{aligned} det\boldsymbol{A^D} &= 1 + (-1)^{N-1}(\Delta\beta\hbar\omega - 1)^N \\ &= 1 - (1 - \Delta\beta\hbar\omega)^N \\ &= 1 - (1 - \frac{\beta\hbar\omega}{N})^N \\ &\longrightarrow 1 - \exp(-\beta\hbar\omega) \qquad\qquad (N \longrightarrow \infty) \end{aligned}$$

Finally, the following expression for Z is obtained:

$$\boxed{Z = \frac{\exp(-\frac{\beta\hbar\omega}{2})}{1 - \exp(-\beta\hbar\omega)}}$$

This is indeed the correct result.

Chapter 3

Single Spin Partition Function In The Basis of Boson Coherent States For A General Two-Mode *Schwinger boson* Hamiltonian And The *Schwinger boson* Constraint Implementation

3.1 The *Schwinger boson* representation of Spin Operators

Give the two second quantized operators $a_\uparrow$ and $a_\downarrow$(or $a_\uparrow^\dagger$ and $a_\downarrow^\dagger$), the name *Schwinger bosons*[1]. Then, the *Schwinger boson representation* of spin operators is written as follows:

$$\hat{S}_x + i\hat{S}_y = a_\uparrow^\dagger a_\downarrow \tag{3.1}$$

$$\hat{S}_x - i\hat{S}_y = a_\downarrow^\dagger a_\uparrow \tag{3.2}$$

[1] The material in this section (*The Schwinger Boson Representation of Spin Operators*) is based upon the chapter *Spin Representations* of *Assa Auerbach*'s book *Interacting Electrons and Quantum Magnetism.*[2]

$$\hat{S_z} = \frac{1}{2}(a_\uparrow^\dagger a_\uparrow - a_\downarrow^\dagger a_\downarrow) \tag{3.3}$$

and which are subject to the constraint

$$\boxed{a_\uparrow^\dagger a_\uparrow + a_\downarrow^\dagger a_\downarrow = 2S}$$

It can be stated that the spin magnitude S defines the physical subspace:

$$\{|n_\uparrow, n_\downarrow\rangle : n_\uparrow + n_\downarrow = 2S\} \tag{3.4}$$

Using (3.1), (3.2) and (3.3) the following expressions for $\hat{S}_x$ and $\hat{S}_y$ can easily be derived:

$$\hat{S_x} = \frac{1}{2}(a_\uparrow^\dagger a_\downarrow + a_\downarrow^\dagger a_\uparrow) \qquad \hat{S_y} = \frac{1}{2i}(a_\uparrow^\dagger a_\downarrow - a_\downarrow^\dagger a_\uparrow)$$

3.2 Implementation of the *Schwinger Boson* constraint

3.2.1 The *Dirac delta function* Implementation

The *Dirac delta function* can be heuristically characterized as follows:

$$\delta(x) = \begin{cases} +\infty, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}$$

Now, the inverse *Fourier transform* of 1 is the *Dirac Delta Function*:

$$\delta(x) = \int_{-\infty}^{\infty} \exp\big[i2\pi x\lambda\big]d\lambda \tag{3.5}$$

The *Schwinger boson* constraint for two bosons of species "up" and "down", can be written as:

$$a_\uparrow^\dagger a_\uparrow + a_\downarrow^\dagger a_\downarrow = 2S \tag{3.6}$$

Now, re-writing the *Dirac delta function* in terms of the *Schwinger boson* constraint:

$$\delta(a_\uparrow^\dagger a_\uparrow + a_\downarrow^\dagger a_\downarrow - 2S) = \begin{cases} +\infty, & \text{if } a_\uparrow^\dagger a_\uparrow + a_\downarrow^\dagger a_\downarrow = 2S \\ 0, & \text{otherwise} \end{cases}$$

Similarly,

$$\delta(a_\uparrow^\dagger a_\uparrow + a_\downarrow^\dagger a_\downarrow - 2S) = \int_{-\infty}^{\infty} \exp\big[i2\pi(a_\uparrow^\dagger a_\uparrow + a_\downarrow^\dagger a_\downarrow - 2S)\lambda\big]d\lambda \tag{3.7}$$

The integral representation of the *Schwinger boson* constraint, which is rooted in the above integral representation of the *Dirac delta function*, can be written as[2]:

$$\int D\lambda \exp\Bigg[-i\epsilon\sum_{i=\uparrow,\downarrow}\lambda_i(\tau)(a_i^\dagger a_i - S)\Bigg] \tag{3.8}$$

Inserting the above for each time step, in the coherent state path integral calculation, one can derive the expression for the partition function that has the following form[2]:

$$Z = \int_{-\infty}^{\infty} D\lambda \int D^2\boldsymbol{z} \exp\left\{ -\int_0^\beta d\tau \Big[\sum_i \bar{z}_i \partial_\tau z_i + H + i \sum_i \lambda_i(\tau)(\bar{z}_i z_i - S) \Big] \right\} \quad (3.9)$$

3.2.2 The Projection Operator Implementation

The ground state *projection operator* for the quantum harmonic oscillator can be written as[10]:

$$\boxed{|0\rangle\langle 0| =: \exp[-a^\dagger a] :}$$

where the colons indicate *normal ordering.*

Now, the *projection operator* onto the physical space, on which it is imposed the *Schwinger boson* constraint, can be written as:

$$\begin{aligned} P &= \sum_{n_\uparrow=0}^{2S} |n_\uparrow, n_\downarrow\rangle\langle n_\uparrow, n_\downarrow| \\ &= \sum_{n_\uparrow=0}^{2S} |n_\uparrow, 2S - n_\uparrow\rangle\langle n_\uparrow, 2S - n_\uparrow| \\ &= \sum_{n=0}^{2S} |n, 2S - n\rangle\langle n, 2S - n| \end{aligned}$$

Now $|n, 2S - n\rangle\langle n, 2S - n|$ can be re-written as follows:

$$\begin{aligned} |n, 2S - n\rangle\langle n, 2S - n| &= \frac{1}{n!(2S-n)!}(a_\uparrow^\dagger)^n (a_\downarrow^\dagger)^{2S-n} |0,0\rangle\langle 0,0| (a_\uparrow)^n (a_\downarrow)^{2S-n} \\ &= \frac{1}{n!(2S-n)!}(a_\uparrow^\dagger)^n (a_\downarrow^\dagger)^{2S-n} : \exp[-a_\downarrow^\dagger a_\downarrow - a_\uparrow^\dagger a_\uparrow] : (a_\uparrow)^n (a_\downarrow)^{2S-n} \\ &= \frac{1}{n!(2S-n)!}(a_\uparrow^\dagger)^n (a_\downarrow^\dagger)^{2S-n} : \exp[-\hat{n}_\downarrow - \hat{n}_\uparrow] : (a_\uparrow)^n (a_\downarrow)^{2S-n} \end{aligned}$$

Therefore, P can be re-written as:

$$\boxed{P = \sum_{n=0}^{2S} \frac{1}{n!(2S-n)!}(a_\uparrow^\dagger)^n (a_\downarrow^\dagger)^{2S-n} : \exp[-\hat{n}_\downarrow - \hat{n}_\uparrow] : (a_\uparrow)^n (a_\downarrow)^{2S-n}}$$

Now, a single spin partition function in the basis of boson coherent states will be computed. Since a single spin can be represented by the *Schwinger bosons*, the Hamiltonian has the following form:

$$\hat{H} = \hat{H}(a_\uparrow^\dagger, a_\uparrow, a_\downarrow^\dagger, a_\downarrow) \quad (3.10)$$

Similar to what was done previously, two boson coherent state closure relations are inserted in the occupation number representation of the partition function, and the resultant expression is manipulated accordingly. Furthermore, the *Schwinger boson* constraint is enforced using the *projection operator*; this is done by multiplying the *projection operator* with the exponential term in the trace expression of Z:

$$
\begin{aligned}
Z &= Tr\Bigg[\exp\big[-\beta\hat{H}(a_\uparrow^\dagger, a_\uparrow, a_\downarrow^\dagger, a_\downarrow)\big]P\Bigg] \\
&= \sum_{n_\uparrow, n_\downarrow} \langle n_\uparrow, n_\downarrow|\exp(-\beta\hat{H})P|n_\uparrow, n_\downarrow\rangle \\
&= \sum_{n_\uparrow, n_\downarrow} \langle n_\uparrow, n_\downarrow| \int \prod_{k=\uparrow,\downarrow} \frac{d\bar{z}_k dz_k}{2\pi i} \exp\big[-\sum_{r=\uparrow,\downarrow} \bar{z}_r z_r\big]|z\rangle\langle z|\exp(-\beta\hat{H})P \\
&\int \prod_{k=\uparrow,\downarrow} \frac{d\bar{w}_k dw_k}{2\pi i} \exp\big[-\sum_{r=\uparrow,\downarrow} \bar{w}_r w_r\big]|w\rangle\langle w|n_\uparrow, n_\downarrow\rangle \\
&= \int \prod_{k=\uparrow,\downarrow} \frac{d\bar{z}_k dz_k}{2\pi i} \exp\big[-\sum_{r=\uparrow,\downarrow} \bar{z}_r z_r\big] \int \prod_{k=\uparrow,\downarrow} \frac{d\bar{w}_k dw_k}{2\pi i} \exp\big[-\sum_{r=\uparrow,\downarrow} \bar{w}_r w_r\big] \\
&\sum_{n_\uparrow, n_\downarrow} \langle n_\uparrow, n_\downarrow|z\rangle\langle z|\exp(-\beta\hat{H})P|w\rangle\langle w|n_\uparrow, n_{n_\downarrow}\rangle \\
&= \int \prod_{k=\uparrow,\downarrow} \frac{d\bar{z}_k dz_k}{2\pi i} \exp\big[-\sum_{r=\uparrow,\downarrow} \bar{z}_r z_r\big] \int \prod_{k=\uparrow,\downarrow} \frac{d\bar{w}_k dw_k}{2\pi i} \exp\big[-\sum_{r=\uparrow,\downarrow} \bar{w}_r w_r\big] \\
&\sum_{n_\uparrow, n_\downarrow} \langle w|n_\uparrow, n_{n_\downarrow}\rangle\langle n_\uparrow, n_\downarrow|z\rangle\langle z|\exp(-\beta\hat{H})P|w\rangle \\
&= \int \prod_{k=\uparrow,\downarrow} \frac{d\bar{z}_k dz_k}{2\pi i} \exp\big[-\sum_{r=\uparrow,\downarrow} \bar{z}_r z_r\big] \int \prod_{k=\uparrow,\downarrow} \frac{d\bar{w}_k dw_k}{2\pi i} \exp\big[-\sum_{r=\uparrow,\downarrow} \bar{w}_r w_r\big] \\
&\langle w| \sum_{n_\uparrow, n_\downarrow} |n_\uparrow, n_{n_\downarrow}\rangle\langle n_\uparrow, n_\downarrow|z\rangle\langle z|\exp(-\beta\hat{H})P|w\rangle \\
&= \int \prod_{k=\uparrow,\downarrow} \frac{d\bar{z}_k dz_k}{2\pi i} \exp\big[-\sum_{r=\uparrow,\downarrow} \bar{z}_r z_r\big] \int \prod_{k=\uparrow,\downarrow} \frac{d\bar{w}_k dw_k}{2\pi i} \exp\big[-\sum_{r=\uparrow,\downarrow} \bar{w}_r w_r\big] \\
&\langle w|\mathbf{1}|z\rangle\langle z|\exp(-\beta\hat{H})P|w\rangle \\
&= \int \prod_{k=\uparrow,\downarrow} \frac{d\bar{z}_k dz_k}{2\pi i} \exp\big[-\sum_{r=\uparrow,\downarrow} \bar{z}_r z_r\big] \int \prod_{k=\uparrow,\downarrow} \frac{d\bar{w}_k dw_k}{2\pi i} \exp\big[-\sum_{r=\uparrow,\downarrow} \bar{w}_r w_r\big] \\
&\langle w|z\rangle\langle z|\exp(-\beta\hat{H})P|w\rangle
\end{aligned}
$$

$$= \int \prod_{k=\uparrow,\downarrow} \frac{d\bar{w}_k dw_k}{2\pi i} \exp\big[-\sum_{r=\uparrow,\downarrow} \bar{w}_r w_r\big] \langle w| \int \prod_{k=\uparrow,\downarrow} \frac{d\bar{z}_k dz_k}{2\pi i} \exp\big[-\sum_{r=\uparrow,\downarrow} \bar{z}_r z_r\big] |z\rangle$$
$$\langle z| \exp(-\beta \hat{H}) P |w\rangle$$
$$= \int \prod_{k=\uparrow,\downarrow} \frac{d\bar{w}_k dw_k}{2\pi i} \exp\big[-\sum_{r=\uparrow,\downarrow} \bar{w}_r w_r\big] \langle w| \mathbf{1} \exp(-\beta \hat{H}) P |w\rangle$$

$$= \int \prod_{k=\uparrow,\downarrow} \frac{d\bar{w}_k dw_k}{2\pi i} \exp\big[-\sum_{r=\uparrow,\downarrow} \bar{w}_r w_r\big] \langle w| \exp(-\beta \hat{H}) P |w\rangle$$

Letting w = z:

$$Z = \int \prod_{k=\uparrow,\downarrow} \frac{d\bar{z}_k dz_k}{2\pi i} \exp\big[-\sum_{r=\uparrow,\downarrow} \bar{z}_r z_r\big] \langle z| \exp(-\beta \hat{H}) P |z\rangle \tag{3.11}$$

Now,

$$\langle z| \exp(-\beta \hat{H}) |z\rangle \equiv \langle z| U(\beta) |z\rangle = \langle z| U^N(\triangle\beta) |z\rangle$$

Inserting the coherent state closure relations and the projection operators for each time interval as follows:

$$Z = \int \prod_{k=\uparrow,\downarrow} \frac{d\bar{z}_k dz_k}{2\pi i} \exp\big[-\sum_{r=\uparrow,\downarrow} \bar{z}_r z_r\big]$$
$$\langle z| U(\triangle\beta) \hat{I} P \hat{I}' U(\triangle\beta) \hat{I} P \hat{I}' ... U(\triangle\beta) \hat{I} P \hat{I}' U(\triangle\beta) \hat{I} P |z\rangle$$

where

$$\hat{I} = \int \prod_{k=\uparrow,\downarrow} \frac{d\bar{w}_k dw_k}{2\pi i} \exp\big[-\sum_{r=\uparrow,\downarrow} \bar{w}_r w_r\big] |w\rangle\langle w| \tag{3.12}$$

$$\hat{I}' = \int \prod_{k=\uparrow,\downarrow} \frac{d\bar{z}_k dz_k}{2\pi i} \exp\big[-\sum_{r=\uparrow,\downarrow} \bar{z}_r z_r\big] |z\rangle\langle z| \tag{3.13}$$

Thus:

$$Z = \int \prod_{k=\uparrow,\downarrow} \frac{d\bar{z}_k dz_k}{2\pi i} \exp\big[-\sum_{r=\uparrow,\downarrow} \bar{z}_r z_r\big]$$

$$\int \prod_{k=\uparrow,\downarrow} \frac{d\bar{z}_k{}^{N-1} dz_k^{N-1}}{2\pi i} \exp\big[-\sum_{r=\uparrow,\downarrow} \bar{z}_r{}^{N-1} z_r^{N-1}\big]\langle z|U(\triangle\beta)\hat{I}P|z^{N-1}\rangle ...$$

$$= \int \prod_{k=\uparrow,\downarrow} \frac{d\bar{z}_k dz_k}{2\pi i} \exp\big[-\sum_{r=\uparrow,\downarrow} \bar{z}_r z_r\big]$$

$$\int \prod_{l=1}^{N-1} \Big[\prod_{k=\uparrow,\downarrow} \frac{d\bar{z}_k{}^l dz_k^l}{2\pi i}\Big] \exp\Big[-\sum_{m=1}^{N-1}\big(\sum_{r=\uparrow,\downarrow} \bar{z}_r{}^m z_r^m\big)\Big]$$

$$\langle z|U(\triangle\beta)\hat{I}P|z^{N-1}\rangle\langle z^{N-1}|U(\triangle\beta)\hat{I}P|z^{N-2}\rangle$$

$$...$$

$$\langle z^2|U(\triangle\beta)\hat{I}P|z^1\rangle\langle z^1|U(\triangle\beta)\hat{I}P|z\rangle$$

$$= \int \prod_{k=\uparrow,\downarrow} \frac{d\bar{z}_k dz_k}{2\pi i} \exp\big[-\sum_{r=\uparrow,\downarrow} \bar{z}_r z_r\big]$$

$$\int \prod_{l=1}^{N-1} \Big[\prod_{k=\uparrow,\downarrow} \frac{d\bar{z}_k{}^l dz_k^l}{2\pi i}\Big] \exp\Big[-\sum_{m=1}^{N-1}\big(\sum_{r=\uparrow,\downarrow} \bar{z}_r{}^m z_r^m\big)\Big]$$

$$\int \prod_{f=0}^{N-1} \Big[\prod_{k=\uparrow,\downarrow} \frac{d\bar{w}_k{}^f dw_k^f}{2\pi i}\Big] \exp\Big[-\sum_{j=0}^{N-1}\big(\sum_{r=\uparrow,\downarrow} \bar{w}_r{}^j w_r^j\big)\Big]$$

$$\langle z|U(\triangle\beta)|w^{N-1}\rangle\langle w^{N-1}|P|z^{N-1}\rangle$$

$$\langle z^{N-1}|U(\triangle\beta)|w^{N-2}\rangle\langle w^{N-2}|P|z^{N-2}\rangle$$

$$...$$

$$\langle z^2|U(\triangle\beta)|w^1\rangle\langle w^1|P|z^1\rangle$$

$$\langle z^1|U(\triangle\beta)|w\rangle\langle w|P|z\rangle$$

$$= \int \prod_{k=\uparrow,\downarrow} \frac{d\bar{z}_k dz_k}{2\pi i} \exp\big[-\sum_{r=\uparrow,\downarrow} \bar{z}_r z_r\big]$$

$$\int \prod_{l=1}^{N-1} \Big[\prod_{k=\uparrow,\downarrow} \frac{d\bar{z}_k{}^l dz_k^l}{2\pi i}\Big] \exp\Big[-\sum_{m=1}^{N-1}\big(\sum_{r=\uparrow,\downarrow} \bar{z}_r{}^m z_r^m\big)\Big]$$

$$\int \prod_{f=0}^{N-1} \Big[\prod_{k=\uparrow,\downarrow} \frac{d\bar{w}_k^f dw_k^f}{2\pi i}\Big] \exp\Big[-\sum_{j=0}^{N-1}\big(\sum_{r=\uparrow,\downarrow} \bar{w}_r{}^j w_r^j\big)\Big]$$

$$\prod_{j=0}^{N-1}\Big[\langle z^{j+1}|U(\triangle\beta)|w^j\rangle\langle w^j|P|z^j\rangle\Big]$$

Note the application of the periodic boundary condition:

$$z^N = z^0 = z,$$

and the substitution of w with w_0.

From the previous calculations, it is known that:

$$\langle z^{j+1}|U(\triangle\beta)|w^j\rangle = \langle z^{j+1}|w^j\rangle \exp\Big[-\triangle\beta H(\bar{z}_\uparrow^{j+1}, w_\uparrow^j, \bar{z}_\downarrow^{j+1}, w_\downarrow^j)\Big] \tag{3.14}$$

Computing $\langle z^{j+1}|w^j\rangle^2$:

$$\langle z^{j+1}|w^j\rangle = \exp\Big[\sum_{l=\uparrow,\downarrow} \bar{z}_l^{j+1} w_l^j\Big] \tag{3.15}$$

Thus

$$\langle z^{j+1}|U(\triangle\beta)|w^j\rangle = \exp\Big[\sum_{l=\uparrow,\downarrow} \bar{z}_l^{j+1} w_l^j\Big] \exp\Big[-\triangle\beta H(\bar{z}_\uparrow^{j+1}, w_\uparrow^j, \bar{z}_\downarrow^{j+1}, w_\downarrow^j)\Big] \tag{3.16}$$

Now, computing $\langle w^j|P|z^j\rangle^3$:

$$\langle w^j|P|z^j\rangle = \sum_{n=0}^{2S} \frac{1}{n!(2S-n)!} (\bar{w}_j^\uparrow)^n (\bar{w}_j^\downarrow)^{2S-n} (z_j^\uparrow)^n (z_j^\downarrow)^{2S-n} \langle w^j| : \exp\left[-(\hat{n_\uparrow}+\hat{n_\downarrow})\right] : |z^j\rangle \tag{3.17}$$

Compute $\langle w^j| : \exp\big[-(\hat{n_\uparrow}+\hat{n_\downarrow})\big] : |z^j\rangle$:

$$\begin{aligned}\langle w^j| : \exp\big[-(\hat{n_\uparrow}+\hat{n_\downarrow})\big] : |z^j\rangle &= \langle w^j|\sum_{p=0}^{\infty} \frac{(-1)^p : (\hat{n_\uparrow}+\hat{n_\downarrow})^p :}{p!}|z^j\rangle \\ &= \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \langle w^j| : (a_\uparrow^\dagger a_\dagger + a_\downarrow^\dagger a_\downarrow)^p : |z^j\rangle\end{aligned}$$

Compute $\langle w^j| : (a_\uparrow^\dagger a_\dagger + a_\downarrow^\dagger a_\downarrow)^p : |z^j\rangle^4$:

$$\langle w^j| : (a_\uparrow^\dagger a_\dagger + a_\downarrow^\dagger a_\downarrow)^p : |z^j\rangle = \Big[\bar{w}_\uparrow^j z_\uparrow^j + \bar{w}_\downarrow^j z_\downarrow^j\Big]^p \exp\Big[\sum_{r=\uparrow,\downarrow} \bar{w}_r^j z_r^j\Big] \tag{3.18}$$

Thus:

$$\langle w^j| : \exp\big[-(\hat{n_\uparrow}+\hat{n_\downarrow})\big] : |z^j\rangle = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!} \Big[\bar{w}_\uparrow^j z_\uparrow^j + \bar{w}_\downarrow^j z_\downarrow^j\Big]^p \exp\Big[\sum_{r=\uparrow,\downarrow} \bar{w}_j^r z_j^r\Big] \tag{3.19}$$

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[2]The calculational details are in the Appendix.
[3]The calculational details are in the Appendix
[4]The calculational details are in the Appendix

This implies:

$$\begin{aligned}\langle w^j|P|z^j\rangle &= \sum_{n=0}^{2S} \frac{1}{n!(2S-n)!}(\bar{w}_j^{\uparrow})^n(\bar{w}_j^{\downarrow})^{2S-n}(z_j^{\uparrow})^n(z_j^{\downarrow})^{2S-n} \\ &\sum_{p=0}^{\infty}\frac{(-1)^p}{p!}\Big[\bar{w}_{\uparrow}^{j}z_{\uparrow}^{j}+\bar{w}_{\downarrow}^{j}z_{\downarrow}^{j}\Big]^p \exp\Big[\sum_{r=\uparrow,\downarrow}\bar{w}_j^r z_j^r\Big] \\ &= \sum_{n=0}^{2S} \frac{1}{n!(2S-n)!}(\bar{w}_j^{\uparrow})^n(\bar{w}_j^{\downarrow})^{2S-n}(z_j^{\uparrow})^n(z_j^{\downarrow})^{2S-n} \\ &= \frac{1}{(2S)!}\Big[\bar{w}_j^{\uparrow}z_j^{\uparrow}+\bar{w}_j^{\downarrow}z_j^{\downarrow}\Big]^{2S}\end{aligned}$$

Finally:

$$\begin{aligned}Z = &\int \prod_{k=\uparrow,\downarrow}\frac{d\bar{z}_k dz_k}{2\pi i}\exp\big[-\sum_{r=\uparrow,\downarrow}\bar{z}_r z_r\big] \\ &\int\prod_{l=1}^{N-1}\big[\prod_{k=\uparrow,\downarrow}\frac{d\bar{z}_k{}^l dz_k^l}{2\pi i}\big]\exp\Big[-\sum_{m=1}^{N-1}\big(\sum_{r=\uparrow,\downarrow}\bar{z}_r{}^m z_r^m\big)\Big] \\ &\int\prod_{f=0}^{N-1}\big[\prod_{k=\uparrow,\downarrow}\frac{d\bar{w}_k{}^f dw_k^f}{2\pi i}\big]\exp\Big[-\sum_{j=0}^{N-1}\big(\sum_{r=\uparrow,\downarrow}\bar{w}_r{}^j w_r^j\big)\Big] \\ &\prod_{j=0}^{N-1}\left[\exp\Big[\sum_{r=\uparrow,\downarrow}\bar{z}_{j+1}^r w_j^r\Big]\exp\Big[-\triangle\beta H(\bar{z}_{j+1}^{\uparrow},w_j^{\uparrow},\bar{z}_{j+1}^{\downarrow},w_j^{\downarrow})\Big]\frac{1}{(2S)!}\Big[\bar{w}_j^{\uparrow}z_j^{\uparrow}+\bar{w}_j^{\downarrow}z_j^{\downarrow}\Big]^{2S}\right]\end{aligned}$$

$$\begin{aligned}= &\int \prod_{k=\uparrow,\downarrow}\frac{d\bar{z}_k dz_k}{2\pi i}\exp\big[-\sum_{r=\uparrow,\downarrow}\bar{z}_r z_r\big] \\ &\int\prod_{l=1}^{N-1}\big[\prod_{k=\uparrow,\downarrow}\frac{d\bar{z}_k{}^l dz_k^l}{2\pi i}\big]\int\prod_{f=0}^{N-1}\big[\prod_{k=\uparrow,\downarrow}\frac{d\bar{w}_k{}^f dw_k^f}{2\pi i}\big]\big[\frac{1}{(2S)!}\big]^N \\ &\exp\left[\sum_{j=1}^{N-1}\Big(\sum_{r=\uparrow,\downarrow}(\bar{z}_{j+1}^r w_j^r-\bar{z}_j^r z_j^r-\bar{w}_j^r w_j^r)-\triangle\beta H+[2S]ln[\bar{w}_j^{\uparrow}z_j^{\uparrow}+\bar{w}_j^{\downarrow}z_j^{\downarrow}]\Big)\right. \\ &\left.+\Big(\sum_{i=\uparrow,\downarrow}(\bar{z}_1^i w^i-\bar{w}^i w^i)-\triangle\beta H+[2S]ln[\bar{w}^{\uparrow}z^{\uparrow}+\bar{w}^{\downarrow}z^{\downarrow}]\Big)\right]\end{aligned}$$

Thus:

$$\boxed{Z=\Big[\frac{1}{(2S)!}\Big]^N\int\prod_{l=1}^{N}\big[\prod_{k=\uparrow,\downarrow}\frac{d\bar{z}_k{}^l dz_k^l}{2\pi i}\big]\int\prod_{f=1}^{N}\big[\prod_{k=\uparrow,\downarrow}\frac{d\bar{w}_k^f dw_k^f}{2\pi i}\big]\exp\Big[-A\Big]}$$

where

$$A = \sum_{j=1}^{N} \Big[\sum_{r=\uparrow,\downarrow} (-\bar{z}^r_{j+1} w^r_j + \bar{z}^r_j z^r_j + \bar{w}^r_j w^r_j) + \triangle\beta H - [2S] ln(\bar{w}^\uparrow_j z^\uparrow_j + \bar{w}^\downarrow_j z^\downarrow_j) \Big]$$

The logarithmic terms in the action will be eliminated as follows. Firstly, taking apart $\exp\Big[-A\Big]$:

$$\begin{aligned}
\exp\Big[-A\Big] &= \exp\Big[-\sum_{j=1}^{N} A'_j\Big] \exp\Big[\sum_{j=1}^{N} ln(\sum_{r=\uparrow,\downarrow} \bar{w}^r_j z^r_j)^{2S}\Big] \\
&= \exp\Big[-\sum_{j=1}^{N} A'_j\Big] \prod_{j=1}^{N} exp\Big[ln(\sum_{r=\uparrow,\downarrow} \bar{w}^r_j z^r_j)^{2S}\Big] \\
&= \exp\Big[-\sum_{j=1}^{N} A'_j\Big] \prod_{j=1}^{N} \Big[\sum_{r=\uparrow,\downarrow} \bar{w}^r_j z^r_j\Big]^{2S} \\
&= \exp\Big[-\sum_{j=1}^{N} A'_j\Big] \prod_{j=1}^{N} \Big[B_j\Big]^{2S} \\
&= \exp\Big[\sum_{j=1}^{N} \Big(\sum_{r=\uparrow,\downarrow} (\bar{z}^r_{j+1} w^r_j - \bar{z}^r_j z^r_j - \bar{w}^r_j w^r_j) - \triangle\beta H(\bar{z}^\uparrow_{j+1}, w^\uparrow_j, \bar{z}^\downarrow_{j+1}, w^\downarrow_j)\Big)\Big] \prod_{j=1}^{N} B_j^{2S} \\
&= \prod_{j=1}^{N} \Big[\exp\Big(\sum_{r=\uparrow,\downarrow} (\bar{z}^r_{j+1} w^r_j - \bar{z}^r_j z^r_j - \bar{w}^r_j w^r_j) - \triangle\beta H(\bar{z}^\uparrow_{j+1}, w^\uparrow_j, \bar{z}^\downarrow_{j+1}, w^\downarrow_j)\Big) B_j^{2S}\Big] \\
&= \prod_{j=1}^{N} \Big[B_j^{2S} \exp(-A'_j)\Big]
\end{aligned}$$

Now let

$$A''_j \equiv A'_j - \eta B_j$$

This implies

$$\exp(-A''_j) = \exp(-A'_j + \eta B_j)$$

Note that

$$\begin{aligned}
\frac{\partial^{2S}}{\partial \eta^{2S}} \exp(-A''_j) &= \frac{\partial^{2S}}{\partial \eta^{2S}} \exp(-A'_j + \eta B_j) \\
&= B_j^{2S} \exp(-A'_j + \eta B_j)
\end{aligned}$$

Hence,

$$\frac{\partial^{2S}}{\partial \eta^{2S}} \exp(-A''_j)|_{\eta=0} = B_j^{2S} \exp(-A'_j)$$

With this, $\exp\Big[-A\Big]$ can be re-written as:

$$\begin{aligned}
\exp\Big[-A\Big] &= \prod_{j=1}^{N} \Big[\frac{\partial^{2S}}{\partial \eta^{2S}} \exp(-A''_j)|_{\eta=0}\Big] \\
&= \prod_{j=1}^{N} \Big[\frac{\partial^{2S}}{\partial \eta^{2S}} \exp(-A'_j + \eta B_j)|_{\eta=0}\Big] \\
&= \prod_{j=1}^{N} \frac{\partial^{2S}}{\partial \eta_j^{2S}} \exp\Big[\sum_{j=1}^{N}(-A'_j + \eta_j B_j)|_{\eta_j=0})\Big] \\
&= \prod_{j=1}^{N} \frac{\partial^{2S}}{\partial \eta_j^{2S}} \exp\Big[\sum_{j=1}^{N}\Big(\sum_{r=\uparrow,\downarrow}(\bar{z}_{j+1}^r w_j^r - \bar{z}_j^r z_j^r - \bar{w}_j^r w_j^r + \eta_j \bar{w}_j^r z_j^r|_{\eta=0}) \\
&+ \triangle\beta H(\bar{z}_{j+1}^{\uparrow}, w_j^{\uparrow}, \bar{z}_{j+1}^{\downarrow}, w_j^{\downarrow})\Big)\Big]
\end{aligned}$$

Thus, the partition function can be re-written as:

$$\begin{aligned}
Z &= \Big[\frac{1}{(2S)!}\Big]^N \int \prod_{l=1}^{N}\Big[\prod_{k=\uparrow,\downarrow} \frac{d\bar{z}_k{}^l dz_k^l}{2\pi i}\Big] \int \prod_{f=1}^{N}\Big[\prod_{k=\uparrow,\downarrow} \frac{d\bar{w}_k^f dw_k^f}{2\pi i}\Big] \\
&\prod_{j=1}^{N} \frac{\partial^{2S}}{\partial \eta_j^{2S}} \exp\Big[\sum_{j=1}^{N}\Big(\sum_{r=\uparrow,\downarrow}(\bar{z}_{j+1}^r w_j^r - \bar{z}_j^r z_j^r - \bar{w}_j^r w_j^r + \eta_j \bar{w}_j^r z_j^r|_{\eta=0}) \\
&+ \triangle\beta H(\bar{z}_{j+1}^{\uparrow}, w_j^{\uparrow}, \bar{z}_{j+1}^{\downarrow}, w_j^{\downarrow})\Big)\Big] \\
&= \Big[\frac{1}{(2S)!}\Big]^N \Big[\prod_{j=1}^{N} \frac{\partial^{2S}}{\partial \eta_j^{2S}}\Big] \int \prod_{l=1}^{N}\Big[\prod_{k=\uparrow,\downarrow} \frac{d\bar{z}_k{}^l dz_k^l}{2\pi i}\Big] \int \prod_{f=1}^{N}\Big[\prod_{k=\uparrow,\downarrow} \frac{d\bar{w}_k^f dw_k^f}{2\pi i}\Big] \\
&\exp\Big[\sum_{j=1}^{N}\Big(\sum_{r=\uparrow,\downarrow}(\bar{z}_{j+1}^r w_j^r - \bar{z}_j^r z_j^r - \bar{w}_j^r w_j^r + \eta_j \bar{w}_j^r z_j^r|_{\eta=0}) \\
&+ \triangle\beta H(\bar{z}_{j+1}^{\uparrow}, w_j^{\uparrow}, \bar{z}_{j+1}^{\downarrow}, w_j^{\downarrow})\Big)\Big]
\end{aligned}$$

Relabeling the variables as follows

$$u_1, u_2, ..., u_{2N-1}, u_{2N} \equiv z_1, w_1 ..., z_N, w_N$$

the following expression is obtained:

$$\boxed{Z = \Big[\frac{1}{(2S)!}\Big]^N \Big[\prod_{j=1}^{N} \frac{\partial^{2S}}{\partial \eta_j^{2S}}\Big] \int \prod_{l=1}^{2N} \Big[\prod_{k=\uparrow,\downarrow} \frac{d\bar{u}_k{}^l du_k^l}{2\pi i}\Big] \exp\Big[G\Big]}$$

where

$$\boxed{\begin{aligned} G = \sum_{j=1}^{N} \Bigg[& \sum_{r=\uparrow,\downarrow} (\bar{u}^r_{2j+1} u^r_{2j} - \bar{u}^r_{2j-1} u^r_{2j-1} - \bar{u}^r_{2j} u^r_{2j} + \eta_j \bar{u}^r_{2j} u^r_{2j-1}|_{\eta_j=0}) \\ & + \triangle\beta H(\bar{u}^\uparrow_{2j+1}, u^\uparrow_{2j}, \bar{u}^\downarrow_{2j+1}, u^\downarrow_{2j}) \Bigg] \end{aligned}}$$

3.3 Single Spin Partition Function In The Basis of Boson Coherent States For A Zeeman Hamiltonian And The Projection Operator Implementation of the *Schwinger boson* Constraint

Consider the following Zeeman spin hamiltonian

$$\boxed{\hat{H} = -B\hat{S}_z}$$

In the Schwinger-boson representation,

$$\boxed{\hat{H} = -\frac{B\hbar}{2}(a^\dagger_\uparrow a_\uparrow - a^\dagger_\downarrow a_\downarrow)}$$

Thus,

$$\begin{aligned} Z = & \Big[\frac{1}{(2S)!}\Big]^N \Big[\prod_{j=1}^{N} \frac{\partial^{2S}}{\partial \eta_j^{2S}}\Big] \int \prod_{l=1}^{2N} \Big[\prod_{k=\uparrow,\downarrow} \frac{d\bar{u}_k{}^l du_k^l}{2\pi i}\Big] \\ & \exp\Big[\sum_{j=1}^{N} \Big(\sum_{r=\uparrow,\downarrow} (\bar{u}^r_{2j+1} u^r_{2j} - \bar{u}^r_{2j-1} u^r_{2j-1} - \bar{u}^r_{2j} u^r_{2j} + \eta_j \bar{u}^r_{2j} u^r_{2j-1}|_{\eta_j=0}) \\ & - \triangle\beta \frac{B\hbar}{2}(\bar{u}^\uparrow_{2j+1} u^\uparrow_{2j} - \bar{u}^\downarrow_{2j+1} u^\downarrow_{2j}) \Big)\Big] \end{aligned}$$

$$= \Big[\frac{1}{(2S)!}\Big]^N \Big[\prod_{j=1}^{N} \frac{\partial^{2S}}{\partial \eta_j^{2S}}\Big] \int \prod_{l=1}^{2N} \Big[\prod_{k=\uparrow,\downarrow} \frac{d\bar{u}_k{}^l du_k^l}{2\pi i}\Big]$$

$$\exp\Big[\sum_{j=1}^{N} \Big(\bar{u}_{2j+1}^{\uparrow} u_{2j}^{\uparrow} - \bar{u}_{2j-1}^{\uparrow} u_{2j-1}^{\uparrow} - \bar{u}_{2j}^{\uparrow} u_{2j}^{\uparrow} + \eta_j \bar{u}_{2j}^{\uparrow} u_{2j-1}^{\uparrow}|_{\eta_j=0} - \triangle\beta\frac{B\hbar}{2}\bar{u}_{2j+1}^{\uparrow} u_{2j}^{\uparrow}$$

$$+ \bar{u}_{2j+1}^{\downarrow} u_{2j}^{\downarrow} - \bar{u}_{2j-1}^{\downarrow} u_{2j-1}^{\downarrow} - \bar{u}_{2j}^{\downarrow} u_{2j}^{\downarrow} + \eta_j \bar{u}_{2j}^{\downarrow} u_{2j-1}^{\downarrow}|_{\eta_j=0} + \triangle\beta\frac{B\hbar}{2}\bar{u}_{2j+1}^{\downarrow} u_{2j}^{\downarrow}\Big)\Big]$$

$$= \Big[\frac{1}{(2S)!}\Big]^N \Big[\prod_{j=1}^{N} \frac{\partial^{2S}}{\partial \eta_j^{2S}}\Big] \int \prod_{l=1}^{2N} \Big[\prod_{k=\uparrow,\downarrow} \frac{d\bar{u}_k{}^l du_k^l}{2\pi i}\Big]$$

$$\exp\Big[\sum_{j=1}^{N} \Big(\bar{u}_{2j+1}^{\uparrow} u_{2j}^{\uparrow}(1 - \triangle\beta\frac{B\hbar}{2}) - \bar{u}_{2j-1}^{\uparrow} u_{2j-1}^{\uparrow} - \bar{u}_{2j}^{\uparrow} u_{2j}^{\uparrow} + \eta_j \bar{u}_{2j}^{\uparrow} u_{2j-1}^{\uparrow}|_{\eta_j=0}$$

$$+ \bar{u}_{2j+1}^{\downarrow} u_{2j}^{\downarrow}(1 + \triangle\beta\frac{B\hbar}{2}) - \bar{u}_{2j-1}^{\downarrow} u_{2j-1}^{\downarrow} - \bar{u}_{2j}^{\downarrow} u_{2j}^{\downarrow} + \eta_j \bar{u}_{2j}^{\downarrow} u_{2j-1}^{\downarrow}|_{\eta_j=0}\Big)\Big]$$

$$= \Big[\frac{1}{(2S)!}\Big]^N \Big[\prod_{j=1}^{N} \frac{\partial^{2S}}{\partial \eta_j^{2S}}\Big] \int \prod_{l=1}^{2N} \Big[\prod_{k=\uparrow,\downarrow} \frac{d\bar{u}_k{}^l du_k^l}{2\pi i}\Big] \exp\Big[-\bar{\boldsymbol{u}}^{\uparrow}\ \boldsymbol{S}\boldsymbol{u}^{\uparrow}\ -\bar{\boldsymbol{u}}^{\downarrow}\boldsymbol{Q}\boldsymbol{u}^{\downarrow}\Big]\ |_{n_j=0}$$

where

$$\bar{\boldsymbol{u}}^{\boldsymbol{r}} = (\bar{u}_1^r, \bar{u}_2^r, \bar{u}_3^r, ..., \bar{u}_{2N-1}^r, \bar{u}_{2N}^r), \boldsymbol{r} = r =\uparrow,\downarrow$$

$$\boldsymbol{u}^{\boldsymbol{r}} = \begin{bmatrix} u_1^r \\ u_2^r \\ u_3^r \\ ... \\ u_{2N-1}^r \\ u_{2N}^r \end{bmatrix}, \boldsymbol{r} = r =\uparrow,\downarrow$$

and

$$\boldsymbol{S} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & ... & 0 & -1+\triangle\beta\frac{B\hbar}{2} \\ -\eta_1 & 1 & 0 & 0 & 0 & 0 & ... & 0 & 0 \\ 0 & -1+\triangle\beta\frac{B\hbar}{2} & 1 & 0 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & -\eta_2 & 1 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & -1+\triangle\beta\frac{B\hbar}{2} & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & 0 & -\eta_3 & 1 & ... & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1+\triangle\beta\frac{B\hbar}{2} & ... & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & ... & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & ... & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & ... & -\eta_N & 1 \end{pmatrix}$$

$$\boldsymbol{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & ... & 0 & -1-\triangle\beta\frac{B\hbar}{2} \\ -\eta_1 & 1 & 0 & 0 & 0 & 0 & ... & 0 & 0 \\ 0 & -1-\triangle\beta\frac{B\hbar}{2} & 1 & 0 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & -\eta_2 & 1 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & -1-\triangle\beta\frac{B\hbar}{2} & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & 0 & -\eta_3 & 1 & ... & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1-\triangle\beta\frac{B\hbar}{2} & ... & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & ... & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & ... & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & ... & -\eta_N & 1 \end{pmatrix}$$

$$Z = \Big[\frac{1}{(2S)!}\Big]^N \Big[\prod_{j=1}^{N} \frac{\partial^{2S}}{\partial \eta_j^{2S}}\Big] \int \prod_{l=1}^{2N} \frac{d\bar{u}_\uparrow^l du_\uparrow^l}{2\pi i} \exp(-\bar{\boldsymbol{u}}^\uparrow \ \boldsymbol{S}\boldsymbol{u}^\uparrow \ ) \mid_{\eta_j=0} \int \prod_{l=1}^{2N} \frac{d\bar{u}_\downarrow^l du_\downarrow^l}{2\pi i} \exp(-\bar{\boldsymbol{u}}^\downarrow \boldsymbol{Q}\boldsymbol{u}^\downarrow) \mid_{\eta_j=0}$$

Now,

$$\begin{aligned} -\bar{\boldsymbol{u}}^\uparrow \ \boldsymbol{S}\boldsymbol{u}^\uparrow &= -\boldsymbol{u}_\dagger{}^\uparrow \ \boldsymbol{S}\boldsymbol{u}^\uparrow \\ &= -\boldsymbol{u}_\dagger{}^\uparrow \ \boldsymbol{U}\boldsymbol{U}^\dagger \boldsymbol{S}\boldsymbol{U}\boldsymbol{U}^\dagger \boldsymbol{u}^\uparrow \\ &= -\tilde{\boldsymbol{u}}_\dagger{}^\uparrow \ \boldsymbol{S}^{\boldsymbol{D}} \tilde{\boldsymbol{u}}^\uparrow \\ &= -\sum_{r,p}^{2N} \tilde{u}_r^{\uparrow,\dagger} S_{rp}^D \tilde{u}_p^\uparrow \\ &= -\sum_{r=1}^{2N} S_{rr}^D \tilde{u}_r^{\uparrow,\dagger} \tilde{u}_r^\uparrow \\ &= -\sum_{r=1}^{2N} S_{rr}^D (\tilde{x}_{r,\uparrow}^2 + \tilde{y}_{r,\uparrow}^2) \end{aligned}$$

Similarly,

$$-\bar{\boldsymbol{u}}^{\downarrow}\boldsymbol{Q}\boldsymbol{u}^{\downarrow} = -\sum_{r=1}^{2N} Q^D_{rr}(\tilde{x}^2_{r,\downarrow} + \tilde{y}^2_{r,\downarrow})$$

Therefore,

$$Z = \Big[\frac{1}{(2S)!}\Big]^N \Big[\prod_{j=1}^{N} \frac{\partial^{2S}}{\partial \eta_j^{2S}}\Big] \int \prod_{l=1}^{2N} \frac{d\bar{u}^l_{\uparrow} du^l_{\uparrow}}{2\pi i} \exp(-\sum_{r=1}^{2N} S^D_{rr}(\tilde{x}^2_{r,\uparrow} + \tilde{y}^2_{r,\uparrow})) \mid_{\eta_j=0}$$

$$\int \prod_{l=1}^{2N} \frac{d\bar{u}^l_{\downarrow} du^l_{\downarrow}}{2\pi i} \exp(-\sum_{r=1}^{2N} Q^D_{rr}(\tilde{x}^2_{r,\downarrow} + \tilde{y}^2_{r,\downarrow})) \mid_{\eta_j=0}$$

$$= \Big[\frac{1}{(2S)!}\Big]^N \Big[\prod_{j=1}^{N} \frac{\partial^{2S}}{\partial \eta_j^{2S}}\Big] \int \prod_{l=1}^{2N} \frac{d\bar{x}^l_{\uparrow} dy^l_{\uparrow}}{\pi} \exp(-\sum_{r=1}^{2N} S^D_{rr}(\tilde{x}^2_{r,\uparrow} + \tilde{y}^2_{r,\uparrow})) \mid_{\eta_j=0}$$

$$\int \prod_{l=1}^{2N} \frac{d\bar{x}^l_{\downarrow} dy^l_{\downarrow}}{\pi} \exp(-\sum_{r=1}^{2N} Q^D_{rr}(\tilde{x}^2_{r,\downarrow} + \tilde{y}^2_{r,\downarrow})) \mid_{\eta_j=0}$$

$$= \Big[\frac{1}{(2S)!}\Big]^N \Big[\prod_{j=1}^{N} \frac{\partial^{2S}}{\partial \eta_j^{2S}}\Big] \prod_{r=1}^{2N} \left[\frac{1}{\pi}\sqrt{\frac{\pi}{S^D_{rr}}}\sqrt{\frac{\pi}{S^D_{rr}}}\frac{1}{\pi}\sqrt{\frac{\pi}{Q^D_{rr}}}\sqrt{\frac{\pi}{Q^D_{rr}}}\right] \mid_{\eta_j=0}$$

$$= \Big[\frac{1}{(2S)!}\Big]^N \Big[\prod_{j=1}^{N} \frac{\partial^{2S}}{\partial \eta_j^{2S}}\Big] \prod_{r=1}^{2N} \frac{1}{\pi^2}\frac{\pi^2}{S^D_{rr}Q^D_{rr}} \mid_{\eta_j=0}$$

$$= \Big[\frac{1}{(2S)!}\Big]^N \Big[\prod_{j=1}^{N} \frac{\partial^{2S}}{\partial \eta_j^{2S}}\Big] \prod_{r=1}^{2N} \frac{1}{S^D_{rr}Q^D_{rr}} \mid_{\eta_j=0}$$

$$= \Big[\frac{1}{(2S)!}\Big]^N \Big[\prod_{j=1}^{N} \frac{\partial^{2S}}{\partial \eta_j^{2S}}\Big] \frac{1}{det(\boldsymbol{S^D})det(\boldsymbol{Q^D})} \mid_{\eta_j=0}$$

$$= \Big[\frac{1}{(2S)!}\Big]^N \Big[\prod_{j=1}^{N} \frac{\partial^{2S}}{\partial \eta_j^{2S}}\Big] \frac{1}{det(\boldsymbol{S})det(\boldsymbol{Q})} \mid_{\eta_j=0}$$

$$= \Big[\frac{1}{(2S)!}\Big]^N \Big[\prod_{j=1}^{N} \frac{\partial^{2S}}{\partial \eta_j^{2S}}\Big]$$

$$\frac{1}{\Big[1-(\eta_1\eta_2...\eta_N)(1-\frac{\beta B\hbar}{2N})^N\Big]\Big[1-(\eta_1,\eta_2,...,\eta_N)(1+\frac{\beta B\hbar}{2N})^N\Big]} \mid_{n_j=0}$$

$$= \Big[\frac{1}{(2S)!}\Big]^N \Big[\prod_{j=1}^{N} \frac{\partial^{2S}}{\partial \eta_j^{2S}}\Big]$$

$$\frac{1}{1-(\eta_1\eta_2...\eta_N)\Big[(1-\frac{\beta B\hbar}{2N})^N+(1+\frac{\beta B\hbar}{2N})^N\Big]+(\eta_1\eta_2...\eta_N)^2\Big[(1-\frac{\beta B\hbar}{2N})(1+\frac{\beta B\hbar}{2N})\Big]^N} \mid_{\eta_j=0}$$

$$Z = \Big[\frac{1}{(2S)!}\Big]^N \Big[\prod_{j=1}^{N} \frac{\partial^{2S}}{\partial \eta_j^{2S}}\Big] \Bigg[1 - (\eta_1\eta_2...\eta_N)\Big[(1-\frac{\beta B\hbar}{2N})^N + (1+\frac{\beta B\hbar}{2N})^N\Big] + (\eta_1\eta_2...\eta_N)^2\Big[(1-\frac{\beta B\hbar}{2N})(1+\frac{\beta B\hbar}{2N})\Big]^N\Bigg]^{-1} |_{\eta_j=0}$$

3.3.1 Z For Specific Spin Quantum Numbers

i) S=0

Using the Zeeman partition function result obtained above:

$$Z = \Bigg[1 - (\eta_1\eta_2...\eta_N)\Big[(1-\frac{\beta B\hbar}{2N})^N + (1+\frac{\beta B\hbar}{2N})^N\Big] \tag{3.20}$$

$$+ (\eta_1\eta_2...\eta_N)^2\Big[(1-\frac{\beta B\hbar}{2N})(1+\frac{\beta B\hbar}{2N})\Big]^N\Bigg]^{-1} |_{\eta_j=0} \tag{3.21}$$

$$= 1 \tag{3.22}$$

Alternatively, using the regular formula for the partition function:

$$Z = \sum_{S_z=0} \exp(\beta B S_z) \tag{3.23}$$

$$= 1 \tag{3.24}$$

The results match.

The same procedure is applied for the other spin quantum number cases:

ii) S=1/2

$$Z = \prod_{j=1}^{N} \frac{\partial}{\partial \eta_j} \Bigg[1 - (\eta_1\eta_2...\eta_N)\Big[(1-\frac{\beta B\hbar}{2N})^N + (1+\frac{\beta B\hbar}{2N})^N\Big] + (\eta_1\eta_2...\eta_N)^2\Big[(1-\frac{\beta B\hbar}{2N})(1+\frac{\beta B\hbar}{2N})\Big]^N\Bigg]^{-1} |_{\eta_j=0}$$

$$= \prod_{j=2}^{N} \frac{\partial}{\partial \eta_j}(-1)$$

$$\Bigg[1 - (\eta_1\eta_2...\eta_N)\Big[(1 - \frac{\beta B\hbar}{2N})^N + (1 + \frac{\beta B\hbar}{2N})^N\Big]$$

$$+ (\eta_1\eta_2...\eta_N)^2\Big[(1 - \frac{\beta B\hbar}{2N})(1 + \frac{\beta B\hbar}{2N})\Big]^N\Bigg]^{-2} |_{\eta_j=0}$$

$$\Bigg[-(\eta_2...\eta_N)\Big[(1 - \frac{\beta B\hbar}{2N})^N + (1 + \frac{\beta B\hbar}{2N})^N\Big]$$

$$+ 2(\eta_1\eta_2...\eta_N)\Big[(1 - \frac{\beta B\hbar}{2N})(1 + \frac{\beta B\hbar}{2N})\Big]^N(\eta_2...\eta_N)\Bigg] |_{\eta_j=0}$$

$$= \prod_{j=2}^{N} \frac{\partial}{\partial \eta_j}(\eta_2...\eta_N)\Big[(1 - \frac{\beta B\hbar}{2N})^N + (1 + \frac{\beta B\hbar}{2N})^N\Big] |_{\eta_j=0}$$

$$\boxed{Z = (1 - \frac{\beta B\hbar}{2N})^N + (1 + \frac{\beta B\hbar}{2N})^N}$$

$$= \exp(\frac{-\beta B\hbar}{2}) + \exp(\frac{\beta B\hbar}{2}) \qquad (N \to \infty)$$

Alternatively,

$$Z = \sum_{S_z = \frac{\hbar}{2}, \frac{-\hbar}{2}} \exp(\beta B S_z)$$

$$= \exp(\frac{\beta B\hbar}{2}) + \exp(\frac{-\beta B\hbar}{2})$$

iii)S = 1

$$Z = \Big[\frac{1}{2^N}\Big]\Big[\prod_{j=1}^{N}\frac{\partial^2}{\partial\eta_j^2}\Big]$$

$$\Bigg[1-(\eta_1\eta_2...\eta_N)\Big[(1-\frac{\beta B\hbar}{2N})^N+(1+\frac{\beta B\hbar}{2N})^N\Big]$$

$$+(\eta_1\eta_2...\eta_N)^2\Big[(1-\frac{\beta B\hbar}{2N})(1+\frac{\beta B\hbar}{2N})\Big]^N\Bigg]^{-1}|_{\eta_j=0}$$

$$= \Big[\frac{1}{2^N}\Big]\Big[\prod_{j=2}^{N}\frac{\partial^2}{\partial\eta_j^2}\Big]\frac{\partial}{\partial\eta_1}$$

$$\Bigg[1-(\eta_1\eta_2...\eta_N)\Big[(1-\frac{\beta B\hbar}{2N})^N+(1+\frac{\beta B\hbar}{2N})^N\Big]$$

$$+(\eta_1\eta_2...\eta_N)^2\Big[(1-\frac{\beta B\hbar}{2N})(1+\frac{\beta B\hbar}{2N})\Big]^N\Bigg]^{-2}|_{n_j=0}$$

$$\Bigg[(\eta_2...\eta_N)\Big[(1-\frac{\beta B\hbar}{2N})^N+(1+\frac{\beta B\hbar}{2N})^N\Big]$$

$$-2(\eta_1\eta_2...\eta_N)(\eta_2...\eta_N)\Big[(1-\frac{\beta B\hbar}{2N})(1+\frac{\beta B\hbar}{2N})\Big]^N\Bigg]|_{\eta_j=0}$$

$$= \Big[\frac{1}{2^N}\Big]\Big[\prod_{j=2}^{N}\frac{\partial^2}{\partial\eta_j^2}\Big]\frac{\partial}{\partial\eta_1}\Bigg(\Bigg[1-(\eta_1\eta_2...\eta_N)\Big[(1-\frac{\beta B\hbar}{2N})^N+(1+\frac{\beta B\hbar}{2N})^N\Big]$$

$$+(\eta_1\eta_2...\eta_N)^2\Big[(1-\frac{\beta B\hbar}{2N})(1+\frac{\beta B\hbar}{2N})\Big]^N\Bigg]^{-2}(\eta_2...\eta_N)\Big[(1-\frac{\beta B\hbar}{2N})^N+(1+\frac{\beta B\hbar}{2N})^N\Big]$$

$$-\Bigg[1-(\eta_1\eta_2...\eta_N)\Big[(1-\frac{\beta B\hbar}{2N})^N+(1+\frac{\beta B\hbar}{2N})^N\Big]$$

$$+(\eta_1\eta_2...\eta_N)^2\Big[(1-\frac{\beta B\hbar}{2N})(1+\frac{\beta B\hbar}{2N})\Big]^N\Bigg]^{-2}2(\eta_1\eta_2...\eta_N)(\eta_2...\eta_N)\Big[(1-\frac{\beta B\hbar}{2N})$$

$$(1+\frac{\beta B\hbar}{2N})\Big]^N\Bigg)|_{\eta_j=0}$$

$$= \Big[\frac{1}{2^N}\Big]\Big[\prod_{j=2}^{N}\frac{\partial^2}{\partial\eta_j^2}\Big]\Bigg((\eta_2...\eta_N)\Big[(1-\frac{\beta B\hbar}{2N})^N+(1+\frac{\beta B\hbar}{2N})^N\Big](-2)\Big[1-(\eta_1,\eta_2...\eta_N)\Big[(1-\frac{\beta B\hbar}{2N})^N$$

$$+(1+\frac{\beta B\hbar}{2N})^N\Big]+(\eta_1\eta_2...\eta_N)^2\Big[(1-\frac{\beta B\hbar}{2N})(1+\frac{\beta B\hbar}{2N})\Big]^N\Big]^{-3}$$

$$\Big[-(\eta_2...\eta_N)\Big[(1-\frac{\beta B\hbar}{2N})^N+(1+\frac{\beta B\hbar}{2N})^N\Big]$$

$$+2(\eta_1\eta_2...\eta_N)(\eta_2...\eta_N)$$

$$\Big[(1-\frac{\beta B\hbar}{2N})(1+\frac{\beta B\hbar}{2N})\Big]^N\Big]$$

$$-\Big[1-(\eta_1\eta_2...\eta_N)\Big[(1-\frac{\beta B\hbar}{2N})^N+(1+\frac{\beta B\hbar}{2N})^N\Big](\eta_1\eta_2...\eta_N)^2\Big[(1-\frac{\beta B\hbar}{2N})(1+\frac{\beta B\hbar}{2N})\Big]^N\Big]^{-2}$$

$$2(\eta_2...\eta_N)^2\Big[(1-\frac{\beta B\hbar}{2N})(1+\frac{\beta B\hbar}{2N})\Big]^N$$

$$+4(\eta_1\eta_2...\eta_N)(\eta_2...\eta_N)\Big[(1-\frac{\beta B\hbar}{2N})(1+\frac{\beta B\hbar}{2N})\Big]^N$$

$$\Big[1-(\eta_1\eta_2...\eta_N)\Big[(1-\frac{\beta B\hbar}{2N})^N+(1+\frac{\beta B\hbar}{2N})^N\Big]$$

$$+(\eta_1\eta_2...\eta_N)^2\Big[(1-\frac{\beta B\hbar}{2N})(1+\frac{\beta B\hbar}{2N})\Big]^N\Big]^{-3}\Big[-(\eta_2...\eta_N)\Big[(1-\frac{\beta B\hbar}{2N})^N+(1+\frac{\beta B\hbar}{2N})^N\Big]$$

$$+2(\eta_1\eta_2...\eta_N)(\eta_2...\eta_N)\Big[(1-\frac{\beta B\hbar}{2N})(1+\frac{\beta B\hbar}{2N})\Big]^N\Big]\Bigg)\ |_{\eta_j=0}$$

$$= \Big[\frac{1}{2^N}\Big]\Big[\prod_{j=2}^{N}\frac{\partial^2}{\partial\eta_j^2}\Big]\Bigg((\eta_2...\eta_N)\Big[(1-\frac{\beta B\hbar}{2N})^N+(1+\frac{\beta B\hbar}{2N})^N\Big](-2)$$

$$\Big[-(\eta_2...\eta_N)\Big[(1-\frac{\beta B\hbar}{2N})^N+(1+\frac{\beta B\hbar}{2N})^N\Big]\Big]-2(\eta_2...\eta_N)^2\Big[(1-\frac{\beta B\hbar}{2N})(1+\frac{\beta B\hbar}{2N})\Big]^N$$

$$= \Big[\frac{2}{2^N}\Big]\Big[\prod_{j=2}^{N}\frac{\partial^2}{\partial\eta_j^2}\Big](\eta_2...\eta_N)^2\Big[\Big[(1-\frac{\beta B\hbar}{2N})^N+(1+\frac{\beta B\hbar}{2N})^N\Big]^2-\Big[(1-\frac{\beta B\hbar}{2N})(1+\frac{\beta B\hbar}{2N})\Big]^N\Big]$$

$$\begin{aligned}
&= \frac{2^N}{2^N}\Big[\Big[(1-\frac{\beta B\hbar}{2N})^N+(1+\frac{\beta B\hbar}{2N})^N\Big]^2-\Big[(1-\frac{\beta B\hbar}{2N})(1+\frac{\beta B\hbar}{2N})\Big]^N\Big]\\
&= \Big[(1-\frac{\beta B\hbar}{2N})^N+(1+\frac{\beta B\hbar}{2N})^N\Big]^2-\Big[(1-\frac{\beta B\hbar}{2N})(1+\frac{\beta B\hbar}{2N})\Big]^N
\end{aligned}$$

$$\boxed{Z = (1 - \frac{\beta B\hbar}{2N})^{2N} + (1 + \frac{\beta B\hbar}{2N})^{2N} + \Big[(1 - \frac{\beta B\hbar}{2N})(1 + \frac{\beta B\hbar}{2N})\Big]^{N}}$$

$$= \exp(-\beta B\hbar) + \exp(\beta B\hbar) + 1 \qquad (N \to \infty)$$

Alternatively,

$$Z = \sum_{S_z=-\hbar,0,\hbar} \exp(\beta B S_z)$$
$$= \exp(-\beta B\hbar) + \exp(\beta B\hbar) + 1$$

iv)S = 3/2

$$Z = \frac{1}{6^N}\Big[\prod_{j=1}^{N} \frac{\partial^3}{\partial \eta_j^3}\Big]\Big[1 - (\eta_1\eta_2...\eta_N)\clubsuit + (\eta_1\eta_2...\eta_N)^2\diamondsuit\Big]^{-1}|_{\eta_j=0}$$

where

$$\clubsuit = (1 - \frac{\beta B\hbar}{2N})^N + (1 + \frac{\beta B\hbar}{2N})^N$$
$$\diamondsuit = \Big[(1 - \frac{\beta B\hbar}{2N})(1 + \frac{\beta B\hbar}{2N})\Big]^N$$

$$Z = \frac{1}{6^N}\Big[\prod_{j=2}^{N} \frac{\partial^3}{\partial \eta_j^3}\Big]\frac{\partial^2}{\partial \eta_1^2}$$

$$\Big[(\eta_2...\eta_N)\clubsuit - 2(\eta_1\eta_2...\eta_N)\diamondsuit(\eta_2...\eta_N)\Big]\Big)\ |_{\eta_j=0}$$

$$= \frac{1}{6^N}\Big[\prod_{j=2}^{N} \frac{\partial^3}{\partial \eta_j^3}\Big]\frac{\partial}{\partial \eta_1}\Bigg(-2\diamondsuit(\eta_2...\eta_N)^2\Big[1 - (\eta_1\eta_2...\eta_N)\clubsuit + (\eta_1\eta_2...\eta_N)^2\diamondsuit\Big]^{-2}$$

$$+ 2\Big[1 - (\eta_1\eta_2...\eta_N)\clubsuit + (\eta_1\eta_2...\eta_N)^2\diamondsuit\Big]^{-3}\Big[(\eta_2...\eta_N)\clubsuit - 2(\eta_1\eta_2...\eta_N)\diamondsuit(\eta_2...\eta_N)\Big]^2\Bigg)$$

$$|_{\eta_j=0}$$

$$= \frac{1}{6^N}\Big[\prod_{j=2}^{N} \frac{\partial^3}{\partial \eta_j^3}\Big]\Bigg(-12\clubsuit(\eta_2...\eta_N)^3\diamondsuit + 6(\eta_2...\eta_N)^3\clubsuit^3\Bigg)\ |_{\eta_j=0}$$

$$= \frac{6}{6^N}\Big[\prod_{j=2}^{N} \frac{\partial^3}{\partial \eta_j^3}\Big](\eta_2...\eta_N)^3\Bigg(-2\diamondsuit\clubsuit + \clubsuit^3\Bigg)\ |_{\eta_j=0}$$

$$= -2\diamondsuit\clubsuit + \clubsuit^3$$

$$= -2\Big[(1 - \frac{\beta B\hbar}{2N})(1 + \frac{\beta B\hbar}{2N})\Big]^N\Big[(1 - \frac{\beta B\hbar}{2N})^N + (1 + \frac{\beta B\hbar}{2N})^N\Big] + \Big[(1 - \frac{\beta B\hbar}{2N})^N + (1 + \frac{\beta B\hbar}{2N})^N\Big]^3$$

$$\boxed{\begin{aligned} Z = &-2\Big[(1-\frac{\beta B\hbar}{2N})(1+\frac{\beta B\hbar}{}2N)\Big]^N\Big[(1-\frac{\beta B\hbar}{2N})^N+(1+\frac{\beta B\hbar}{2N})^N\Big] \\ &+(1-\frac{\beta B\hbar}{2N})^{3N}+(1+\frac{\beta B\hbar}{2N})^{3N}+3(1-\frac{\beta B\hbar}{2N})^N+3(1+\frac{\beta B\hbar}{2N})^N \end{aligned}}$$

$$= \exp(\frac{-3\beta B\hbar}{2}) + \exp(\frac{3\beta B\hbar}{2}) + \exp(\frac{-\beta B\hbar}{2}) + \exp(\frac{\beta B\hbar}{2}) \qquad (N \to \infty)$$

Alternatively,

$$\begin{aligned} Z &= \sum_{S_z=\frac{-3\hbar}{2},\frac{-\hbar}{2},\frac{\hbar}{2},\frac{3\hbar}{2}} \exp(\beta B S_z) \\ &= \exp(\frac{-3\beta B\hbar}{2}) + \exp(\frac{3\beta B\hbar}{2}) + \exp(\frac{-\beta B\hbar}{2}) + \exp(\frac{\beta B\hbar}{2}) \end{aligned}$$

Chapter 4

Partition Function for the Heisenberg Model In The Basis of Boson Coherent States

4.0.1 Two Spins

Consider the *Heisenberg Model* for a pair of spins $\hat{\boldsymbol{S}}_{\mathbf{1}}$ and $\hat{\boldsymbol{S}}_{\mathbf{2}}$. The Hamiltonian can be written as follows:

$$\begin{aligned}\hat{H} &= -J\hat{\boldsymbol{S}}_{\mathbf{1}}\hat{\boldsymbol{S}}_{\mathbf{2}} \\ &= -J(S_1^x, S_1^y, S_1^z)(S_2^x, S_2^y, S_2^z) \\ &= -J[S_1^x S_2^x + S_1^y S_2^y + S_1^z S_2^z]\end{aligned}$$

In the Schwinger-boson representation:

$$\begin{aligned}S_1^x S_2^x &= \frac{\hbar}{2}(a_{\uparrow,1}^\dagger a_{\downarrow,1} + a_{\downarrow,1}^\dagger a_{\uparrow,1})\frac{\hbar}{2}(a_{\uparrow,2}^\dagger a_{\downarrow,2} + a_{\downarrow,2}^\dagger a_{\uparrow,2}) \\ &= \frac{\hbar^2}{4}(a_{\uparrow,1}^\dagger a_{\downarrow,1} a_{\uparrow,2}^\dagger a_{\downarrow,2} + a_{\uparrow,1}^\dagger a_{\downarrow,1} a_{\downarrow,2}^\dagger a_{\uparrow,2} + a_{\downarrow,1}^\dagger a_{\uparrow,1} a_{\uparrow,2}^\dagger a_{\downarrow,2} + a_{\downarrow,1}^\dagger a_{\uparrow,1} a_{\downarrow,2}^\dagger a_{\uparrow,2})\end{aligned}$$

$$\begin{aligned}S_1^y S_2^y &= \frac{\hbar}{2i}(a_{\uparrow,1}^\dagger a_{\downarrow,1} - a_{\downarrow,1}^\dagger a_{\uparrow,1})\frac{\hbar}{2i}(a_{\uparrow,2}^\dagger a_{\downarrow,2} - a_{\downarrow,2}^\dagger a_{\uparrow,2}) \\ &= -\frac{\hbar^2}{4}(a_{\uparrow,1}^\dagger a_{\downarrow,1} a_{\uparrow,2}^\dagger a_{\downarrow,2} - a_{\uparrow,1}^\dagger a_{\downarrow,1} a_{\downarrow,2}^\dagger a_{\uparrow,2} - a_{\downarrow,1}^\dagger a_{\uparrow,1} a_{\uparrow,2}^\dagger a_{\downarrow,2} + a_{\downarrow,1}^\dagger a_{\uparrow,1} a_{\downarrow,2}^\dagger a_{\uparrow,2})\end{aligned}$$

$$S_1^z S_2^z = \frac{\hbar}{2}(a^\dagger_{\uparrow,1}a_{\uparrow,1} - a^\dagger_{\downarrow,1}a_{\downarrow,1})\frac{\hbar}{2}(a^\dagger_{\uparrow,2}a_{\uparrow,2} - a^\dagger_{\downarrow,2}a_{\downarrow,2})$$
$$= \frac{\hbar^2}{4}(a^\dagger_{\uparrow,1}a_{\uparrow,1}a^\dagger_{\uparrow,2}a_{\uparrow,2} - a^\dagger_{\uparrow,1}a_{\uparrow,1}a^\dagger_{\downarrow,2}a_{\downarrow,2} - a^\dagger_{\downarrow,1}a_{\downarrow,1}a^\dagger_{\uparrow,2}a_{\uparrow,2} + a^\dagger_{\downarrow1}a_{\downarrow,1}a^\dagger_{\downarrow,2}a_{\downarrow,2})$$

Thus[1],

$$\hat{\boldsymbol{S}}_{\mathbf{1}}\hat{\boldsymbol{S}}_{\mathbf{2}} = \frac{\hbar^2}{4}(2a^\dagger_{\uparrow,1}a_{\downarrow,1}a^\dagger_{\downarrow,2}a_{\uparrow,2} + 2a^\dagger_{\downarrow,1}a_{\uparrow,1}a^\dagger_{\uparrow,2}a_{\downarrow,2} + a^\dagger_{\uparrow,1}a_{\uparrow,1}a^\dagger_{\uparrow,2}a_{\uparrow,2}$$
$$- a^\dagger_{\uparrow,1}a_{\uparrow,1}a^\dagger_{\downarrow,2}a_{\downarrow,2} - a^\dagger_{\downarrow,1}a_{\downarrow,1}a^\dagger_{\uparrow,2}a_{\uparrow,2} + a^\dagger_{\downarrow1}a_{\downarrow,1}a^\dagger_{\downarrow,2}a_{\downarrow,2})$$

As done before, two boson coherent state closure relations are inserted in the occupation number representation of the partition function, and the resultant expression is manipulated accordingly. Also, by the same method used previously, the *Schwinger boson* constraint is enforced using the *projection operator*:

$$Z = Tr\Bigg[\exp\big[-\beta\hat{H}(a^\dagger_{\uparrow,1}, a_{\uparrow,1}, a^\dagger_{\downarrow,1}, a_{\downarrow,1}, a^\dagger_{\uparrow,2}, a_{\uparrow,2}, a^\dagger_{\downarrow,2}, a_{\downarrow,2})\big]P\Bigg]$$
$$= \sum_{n_{\uparrow,1},n_{\downarrow,1},n_{\uparrow,2},n_{\downarrow,2}} \langle n_{\uparrow,1}, n_{\downarrow,1}, n_{\uparrow,2}, n_{\downarrow,2}|\exp(-\beta\hat{H})P|n_{\uparrow,1}, n_{\downarrow,1}, n_{\uparrow,2}, n_{\downarrow,2}\rangle$$
$$= \sum_{n_{\uparrow,1},n_{\downarrow,1},n_{\uparrow,2},n_{\downarrow,2}} \langle n_{\uparrow,1}, n_{\downarrow,1}, n_{\uparrow,2}, n_{\downarrow,2}|\int\prod_{k=(1,\uparrow)}^{(2,\downarrow)}\frac{d\bar{z}_k dz_k}{2\pi i}\exp\big[-\sum_{r=(1,\uparrow)}^{(2,\downarrow)}\bar{z}_r z_r\big]|z\rangle\langle z|\exp(-\beta\hat{H})P$$
$$\int\prod_{k=(1,\uparrow)}^{(2,\downarrow)}\frac{d\bar{w}_k dw_k}{2\pi i}\exp\big[-\sum_{r=(1,\uparrow)}^{(2,\downarrow)}\bar{w}_r w_r\big]|w\rangle\langle w|n_{\uparrow,1}, n_{\downarrow,1}, n_{\uparrow,2}, n_{\downarrow,2}\rangle$$
$$= \int\prod_{k=(1,\uparrow)}^{(2,\downarrow)}\frac{d\bar{z}_k dz_k}{2\pi i}\exp\big[-\sum_{r=(1,\uparrow)}^{(2,\downarrow)}\bar{z}_r z_r\big]\int\prod_{k=(1,\uparrow)}^{(2,\downarrow)}\frac{d\bar{w}_k dw_k}{2\pi i}\exp\big[-\sum_{r=(1,\uparrow)}^{(2,\downarrow)}\bar{w}_r w_r\big]$$
$$\sum_{n_{\uparrow,1},n_{\downarrow,1},n_{\uparrow,2},n_{\downarrow,2}} \langle n_{\uparrow,1}, n_{\downarrow,1}, n_{\uparrow,2}, n_{\downarrow,2}|z\rangle\langle z|\exp(-\beta\hat{H})P|w\rangle\langle w|n_{\uparrow,1}, n_{\downarrow,1}, n_{\uparrow,2}, n_{\downarrow,2}\rangle$$
$$= \int\prod_{k=(1,\uparrow)}^{(2,\downarrow)}\frac{d\bar{z}_k dz_k}{2\pi i}\exp\big[-\sum_{r=(1,\uparrow)}^{(2,\downarrow)}\bar{z}_r z_r\big]\int\prod_{k=(1,\uparrow)}^{(2,\downarrow)}\frac{d\bar{w}_k dw_k}{2\pi i}\exp\big[-\sum_{r=(1,\uparrow)}^{(2,\downarrow)}\bar{w}_r w_r\big]$$
$$\sum_{n_{\uparrow,1},n_{\downarrow,1},n_{\uparrow,2},n_{\downarrow,2}} \langle w|n_{\uparrow,1}, n_{\downarrow,1}, n_{\uparrow,2}, n_{\downarrow,2}\rangle\langle n_{\uparrow,1}, n_{\downarrow,1}, n_{\uparrow,2}, n_{\downarrow,2}|z\rangle\langle z|\exp(-\beta\hat{H})P|w\rangle$$

[1]The calculational details are in the Appendix.

$$= \int \prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{z}_k dz_k}{2\pi i} \exp\big[-\sum_{r=(1,\uparrow)}^{(2,\downarrow)} \bar{z}_r z_r\big] \int \prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{w}_k dw_k}{2\pi i} \exp\big[-\sum_{r=(1,\uparrow)}^{(2,\downarrow)} \bar{w}_r w_r\big]$$

$$\langle w| \sum_{n_{\uparrow,1},n_{\downarrow,1},n_{\uparrow,2},n_{\downarrow,2}} |n_{\uparrow,1}, n_{\downarrow,1}, n_{\uparrow,2}, n_{\downarrow,2}\rangle\langle n_{\uparrow,1}, n_{\downarrow,1}, n_{\uparrow,2}, n_{\downarrow,2}|z\rangle\langle z| \exp(-\beta\hat{H})P|w\rangle$$

$$= \int \prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{z}_k dz_k}{2\pi i} \exp\big[-\sum_{r=(1,\uparrow)}^{(2,\downarrow)} \bar{z}_r z_r\big] \int \prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{w}_k dw_k}{2\pi i} \exp\big[-\sum_{r=(1,\uparrow)}^{(2,\downarrow)} \bar{w}_r w_r\big]$$

$$\langle w|\mathbf{1}|z\rangle\langle z| \exp(-\beta\hat{H})P|w\rangle$$

$$= \int \prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{z}_k dz_k}{2\pi i} \exp\big[-\sum_{r=(1,\uparrow)}^{(2,\downarrow)} \bar{z}_r z_r\big] \int \prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{w}_k dw_k}{2\pi i} \exp\big[-\sum_{r=(1,\uparrow)}^{(2,\downarrow)} \bar{w}_r w_r\big]\langle w|z\rangle\langle z| \exp(-\beta\hat{H})$$

$$P|w\rangle$$

$$= \int \prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{w}_k dw_k}{2\pi i} \exp\big[-\sum_{r=(1,\uparrow)}^{(2,\downarrow)} \bar{w}_r w_r\big]\langle w| \int \prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{z}_k dz_k}{2\pi i} \exp\big[-\sum_{r=(1,\uparrow)}^{(2,\downarrow)} \bar{z}_r z_r\big]|z\rangle\langle z|$$

$$\exp(-\beta\hat{H})P|w\rangle$$

$$= \int \prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{w}_k dw_k}{2\pi i} \exp\big[-\sum_{r=(1,\uparrow)}^{(2,\downarrow)} \bar{w}_r w_r\big]\langle w|\mathbf{1} \exp(-\beta\hat{H})P|w\rangle$$

$$\equiv \int \prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{z}_k dz_k}{2\pi i} \exp\big[-\sum_{r=(1,\uparrow)}^{(2,\downarrow)} \bar{z}_r z_r\big]\langle z| \exp(-\beta\hat{H})P|z\rangle$$

Now,

$$\begin{aligned} &\langle z| \exp(-\beta\hat{H})P|z\rangle \\ &= \langle z|U(\beta)P|z\rangle \\ &= \langle z|U^N(\triangle\beta)P|z\rangle \end{aligned}$$

Inserting the coherent state closure relations and the projection operators as follows:

$$Z = \int \prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{z}_k dz_k}{2\pi i} \exp\big[-\sum_{r=(1,\uparrow)}^{(2,\downarrow)} \bar{z}_r z_r\big]$$
$$\langle z|U(\triangle\beta)\hat{I}P\hat{I}'U(\triangle\beta)\hat{I}P\hat{I}'...U(\triangle\beta)\hat{I}P\hat{I}'U(\triangle\beta)\hat{I}P|z\rangle$$

where

$$\hat{I} = \int \prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{w}_k dw_k}{2\pi i} \exp\big[-\sum_{r=(1,\uparrow)}^{(2,\downarrow)} \bar{w}_r w_r\big]|w\rangle\langle w| \tag{4.1}$$

$$\hat{I}' = \int \prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{z}_k dz_k}{2\pi i} \exp\big[-\sum_{r=(1,\uparrow)}^{(2,\downarrow)} \bar{z}_r z_r\big] |z\rangle\langle z| \tag{4.2}$$

$$P = \sum_{n_{\uparrow,1}=0}^{2S} \sum_{n_{\uparrow,2}=0}^{2S} \frac{1}{n_{\uparrow,1}!(2S-n_{\uparrow,1})!} \frac{1}{n_{\uparrow,2}!(2S-n_{\uparrow,2})!} \tag{4.3}$$

$$(a^\dagger_{\uparrow,1})^{n_{\uparrow,1}} (a^\dagger_{\downarrow,1})^{2S-n_{\uparrow,1}} (a^\dagger_{\uparrow,2})^{n_{\uparrow,2}} (a^\dagger_{\downarrow,2})^{2S-n_{\uparrow,2}}$$

$$: \exp\big[-(\hat{n}_{\uparrow,1}+\hat{n}_{\downarrow,1}) - (\hat{n}_{\uparrow,2}+\hat{n}_{\downarrow,2})\big] : a_{\uparrow,1}^{n_{\uparrow,1}} a_{\downarrow,1}^{2S-n_{\uparrow,1}} a_{\uparrow,1}^{n_{\uparrow,2}} a_{\downarrow,2}^{2S-n_{\uparrow,2}}$$

Akin to the calculations done previously:

$$Z = \int \prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{z}_k dz_k}{2\pi i} \exp\big[-\sum_{r=(1,\uparrow)}^{(2,\downarrow)} \bar{z}_r z_r\big]$$

$$\langle z|U(\triangle\beta)\hat{I}P\hat{I}'U(\triangle\beta)\hat{I}P\hat{I}'...U(\triangle\beta)\hat{I}P\hat{I}'U(\triangle\beta)\hat{I}P|z\rangle$$

$$= \int \prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{z}_k dz_k}{2\pi i} \exp\big[-\sum_{r=(1,\uparrow)}^{(2,\downarrow)} \bar{z}_r z_r\big]$$

$$\int \prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{z}_k^{\,N-1} dz_k^{N-1}}{2\pi i} \exp\big[-\sum_{r=(1,\uparrow)}^{(2,\downarrow)} \bar{z}_r^{\,N-1} z_r^{N-1}\big] \langle z|U(\triangle\beta)\hat{I}P|z^{N-1}\rangle...$$

$$= \int \prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{z}_k dz_k}{2\pi i} \exp\big[-\sum_{r=(1,\uparrow)}^{(2,\downarrow)} \bar{z}_r z_r\big]$$

$$\int \prod_{l=1}^{N-1} \big[\prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{z}_k^{\,l} dz_k^l}{2\pi i}\big] \exp\Big[-\sum_{m=1}^{N-1}\big(\sum_{r=(1,\uparrow)}^{(2,\downarrow)} \bar{z}_r^{\,m} z_r^m\big)\Big]$$

$$\langle z|U(\triangle\beta)\hat{I}P|z^{N-1}\rangle\langle z^{N-1}|U(\triangle\beta)\hat{I}P|z^{N-2}\rangle$$

...

$$\langle z^2|U(\triangle\beta)\hat{I}P|z^1\rangle\langle z^1|U(\triangle\beta)\hat{I}P|z\rangle$$

$$= \int \prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{z}_k dz_k}{2\pi i} \exp\big[-\sum_{r=(1,\uparrow)}^{(2,\downarrow)} \bar{z}_r z_r\big]$$
$$\int \prod_{l=1}^{N-1}\Big[\prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{z}_k{}^l dz_k^l}{2\pi i}\Big] \exp\Big[-\sum_{m=1}^{N-1}(\sum_{r=(1,\uparrow)}^{(2,\downarrow)} \bar{z}_r{}^m z_r^m)\Big]$$
$$\int \prod_{f=0}^{N-1}\Big[\prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{w}_k{}^f dw_k^f}{2\pi i}\Big] \exp\Big[-\sum_{j=0}^{N-1}(\sum_{r=(1,\uparrow)}^{(2,\downarrow)} \bar{w}_r{}^j w_r^j)\Big]$$
$$\langle z|U(\triangle\beta)|w^{N-1}\rangle\langle w^{N-1}|P|z^{N-1}\rangle$$
$$\langle z^{N-1}|U(\triangle\beta)|w^{N-2}\rangle\langle w^{N-2}|P|z^{N-2}\rangle$$
$$...$$
$$\langle z^2|U(\triangle\beta)|w^1\rangle\langle w^1|P|z^1\rangle$$
$$\langle z^1|U(\triangle\beta)|w\rangle\langle w|P|z\rangle$$

$$= \int \prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{z}_k dz_k}{2\pi i} \exp\big[-\sum_{r=(1,\uparrow)}^{(2,\downarrow)} \bar{z}_r z_r\big]$$
$$\int \prod_{l=1}^{N-1}\Big[\prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{z}_k{}^l dz_k^l}{2\pi i}\Big] \exp\Big[-\sum_{m=1}^{N-1}(\sum_{r=(1,\uparrow)}^{(2,\downarrow)} \bar{z}_r{}^m z_r^m)\Big]$$
$$\int \prod_{f=0}^{N-1}\Big[\prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{w}_k^f dw_k^f}{2\pi i}\Big] \exp\Big[-\sum_{j=0}^{N-1}(\sum_{r=(1,\uparrow)}^{(2,\downarrow)} \bar{w}_r{}^j w_r^j)\Big]$$
$$\prod_{j=0}^{N-1}\Big[\langle z^{j+1}|U(\triangle\beta)|w^j\rangle\langle w^j|P|z^j\rangle\Big]$$

The following periodic boundary condition is applied:

$$z^N = z^0 = z,$$

and w is substituted with w_0.

Now computing $\langle z^{j+1}|U(\triangle\beta)|w^j\rangle$:

$$\langle z^{j+1}|U(\triangle\beta)|w^j\rangle = \langle z^{j+1}|w^j\rangle \exp\Big[-\triangle\beta H(\bar{z}_{\uparrow,1}^{j+1}, w_{\uparrow,1}^j, \bar{z}_{\downarrow,1}^{j+1}, w_{\downarrow,1}^j, \bar{z}_{\uparrow,2}^{j+1}, w_{\uparrow,2}^j, \bar{z}_{\downarrow,2}^{j+1}, w_{\downarrow,2}^j)\Big]$$

It is known that,

$$\langle z^{j+1}|w^j\rangle = \exp\Big[\sum_{l=(1,\uparrow)}^{(2,\downarrow)} \bar{z}_l^{j+1} w_l^j\Big]$$

Thus,

$$\langle z^{j+1}|U(\triangle\beta)|w^j\rangle = \exp\Big[\sum_{l=(1,\uparrow)}^{(2,\downarrow)} \bar{z}_l^{j+1} w_l^j\Big] \exp\Big[-\triangle\beta H(\bar{z}_{\uparrow,1}^{j+1}, w_{\uparrow,1}^j, \bar{z}_{\downarrow,1}^{j+1}, w_{\downarrow,1}^j, \bar{z}_{\uparrow,2}^{j+1}, w_{\uparrow,2}^j, \bar{z}_{\downarrow,2}^{j+1}, w_{\downarrow,2}^j)\Big]$$

Also, computing $\langle w^j|P|z^j\rangle$[2]:

$$\langle w^j|P|z^j\rangle = \sum_{n_{\uparrow,1}=0}^{2S}\sum_{n_{\uparrow,2}=0}^{2S} \frac{1}{n_{\uparrow,1}!(2S-n_{\uparrow,1})!}\frac{1}{n_{\uparrow,2}!(2S-n_{\uparrow,2})!} (\bar{w}_{\uparrow,1}^j)^{n_{\uparrow,1}}(\bar{w}_{\downarrow,1}^j)^{2S-n_{\uparrow,1}}(\bar{w}_{\uparrow,2}^j)^{n_{\uparrow,2}}(\bar{w}_{\downarrow,2}^j)^{2S-n_{\uparrow,2}}(z_{\uparrow,1}^j)^{n_{\uparrow,1}}(z_{\downarrow,1}^j)^{2S-n_{\uparrow,1}}(z_{\uparrow,2}^j)^{n_{\uparrow,2}}(z_{\downarrow,2}^j)^{2S-n_{\uparrow,2}} \langle w^j| : \exp\big[-(\hat{n}_{\uparrow,1}+\hat{n}_{\downarrow,1})-(\hat{n}_{\uparrow,2}+\hat{n}_{\downarrow,2})\big] : |z^j\rangle$$

Now computing[3]

$$\langle w^j| : \exp\big[-(\hat{n}_{\uparrow,1}+\hat{n}_{\downarrow,1})-(\hat{n}_{\uparrow,2}+\hat{n}_{\downarrow,2})\big] : |z^j\rangle$$

$$\langle w^j| : \exp\big[-(\hat{n}_{\uparrow,1}+\hat{n}_{\downarrow,1})-(\hat{n}_{\uparrow,2}+\hat{n}_{\downarrow,2})\big] : |z^j\rangle = (\bar{w}_{\uparrow,1}z_{\uparrow,1}+\bar{w}_{\downarrow,1}z_{\downarrow,1}+\bar{w}_{\uparrow,2}z_{\uparrow,2}+\bar{w}_{\downarrow,2}z_{\downarrow,2})^p \exp\Big[\sum_{l=(1,\uparrow)}^{(2,\downarrow)} \bar{w}_l^j z_l^j\Big]$$

Thus,

$$\begin{aligned}\langle w^j| : \exp\big[-(\hat{n_{\uparrow,1}}+\hat{n_{\downarrow,1}})-(\hat{n_{\uparrow,2}}+\hat{n_{\downarrow,2}})\big] : |z^j\rangle &= \sum_{p=0}^{\infty}\frac{(-1)^p}{p!}(\bar{w}_{\uparrow,1}z_{\uparrow,1}+\bar{w}_{\downarrow,1}z_{\downarrow,1} \\ &\quad + \bar{w}_{\uparrow,2}z_{\uparrow,2}+\bar{w}_{\downarrow,2}z_{\downarrow,2})^p \exp\Big[\sum_{l=(1,\uparrow)}^{(2,\downarrow)} \bar{w}_l^j z_l^j\Big] \\ &= \exp\Big[-\sum_{l=(1,\uparrow)}^{(2,\downarrow)} \bar{w}_l^j z_l^j\Big]\exp\Big[\sum_{l=(1,\uparrow)}^{(2,\downarrow)} \bar{w}_l^j z_l^j\Big] \\ &= 1\end{aligned}$$

This implies,

$$\begin{aligned}\langle w^j|P|z^j\rangle &= \sum_{n_{\uparrow,1}=0}^{2S}\sum_{n_{\uparrow,2}=0}^{2S} \frac{1}{n_{\uparrow,1}!(2S-n_{\uparrow,1})!}\frac{1}{n_{\uparrow,2}!(2S-n_{\uparrow,2})!} \\ &\quad (\bar{w}_{\uparrow,1}^j)^{n_{\uparrow,1}}(\bar{w}_{\downarrow,1}^j)^{2S-n_{\uparrow,1}}(\bar{w}_{\uparrow,2}^j)^{n_{\uparrow,2}}(\bar{w}_{\downarrow,2}^j)^{2S-n_{\uparrow,2}}(z_{\uparrow,1}^j)^{n_{\uparrow,1}}(z_{\downarrow,1}^j)^{2S-n_{\uparrow,1}}(z_{\uparrow,2}^j)^{n_{\uparrow,2}}(z_{\downarrow,2}^j)^{2S-n_{\uparrow,2}} \\ &= \frac{1}{(2S)!}\frac{1}{(2S)!}\Big[\bar{w}_{\uparrow,1}^j z_{\uparrow,1}^j + \bar{w}_{\downarrow,1}^j z_{\downarrow,1}^j\Big]^{2S}\Big[\bar{w}_{\uparrow,2}^j z_{\uparrow,2}^j + \bar{w}_{\downarrow,2}^j z_{\downarrow,2}^j\Big]^{2S}\end{aligned}$$

[2]The calculational details are in the Appendix.
[3]The calculational details are in the Appendix.

Finally,

$$Z = \int \prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{z}_k dz_k}{2\pi i} \exp\big[-\sum_{r=(1,\uparrow)}^{(2,\downarrow)} \bar{z}_r z_r\big]$$

$$\int \prod_{l=1}^{N-1}\Big[\prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{z}_k{}^l dz_k^l}{2\pi i}\Big] \exp\Big[-\sum_{m=1}^{N-1}\big(\sum_{r=(1,\uparrow)}^{(2,\downarrow)} \bar{z}_r{}^m z_r^m\big)\Big]$$

$$\int \prod_{f=0}^{N-1}\Big[\prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{w}_k^f dw_k^f}{2\pi i}\Big] \exp\Big[-\sum_{j=0}^{N-1}\big(\sum_{r=(1,\uparrow)}^{(2,\downarrow)} \bar{w}_r{}^j w_r^j\big)\Big]$$

$$\prod_{j=0}^{N-1}\Bigg[\exp\Big[\sum_{l=(1,\uparrow)}^{(2,\downarrow)} \bar{z}_l^{j+1} w_l^j\Big] \exp\Big[-\triangle\beta H(\bar{z}_{\uparrow,1}^{j+1}, w_{\uparrow,1}^j, \bar{z}_{\downarrow,1}^{j+1}, w_{\downarrow,1}^j, \bar{z}_{\uparrow,2}^{j+1}, w_{\uparrow,2}^j, \bar{z}_{\downarrow,2}^{j+1}, w_{\downarrow,2}^j)\Big]$$

$$\frac{1}{(2S)!}\frac{1}{(2S)!}\Big[\bar{w}_{\uparrow,1}^j z_{\uparrow,1}^j + \bar{w}_{\downarrow,1}^j z_{\downarrow,1}^j\Big]^{2S}\Big[\bar{w}_{\uparrow,2}^j z_{\uparrow,2}^j + \bar{w}_{\downarrow,2}^j z_{\downarrow,2}^j\Big]^{2S}\Bigg]$$

$$= \int \prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{z}_k dz_k}{2\pi i} \exp\big[-\sum_{r=(1,\uparrow)}^{(2,\downarrow)} \bar{z}_r z_r\big]$$

$$\int \prod_{l=1}^{N-1}\Big[\prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{z}_k{}^l dz_k^l}{2\pi i}\Big] \int \prod_{f=0}^{N-1}\Big[\prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{w}_k^f dw_k^f}{2\pi i}\Big]\Big[\frac{1}{(2S)!}\Big]^N\Big[\frac{1}{(2S)!}\Big]^N$$

$$\exp\Bigg[\sum_{j=1}^{N-1}\Big(\sum_{r=(1,\uparrow)}^{(2,\downarrow)} (\bar{z}_r^{j+1} w_r^j - \bar{z}_r^j z_r^j - \bar{w}_r^j w_r^j) - \triangle\beta H + [2S]ln[\bar{w}_{\uparrow,1}^j z_{\uparrow,1}^j + \bar{w}_{\downarrow,1}^j z_{\downarrow,1}^j]$$

$$+ [2S]ln[\bar{w}_{\uparrow,2}^j z_{\uparrow,2}^j + \bar{w}_{\downarrow,2}^j z_{\downarrow,2}^j]\Big) + \Big(\sum_{r=(1,\uparrow)}^{(2,\downarrow)} (\bar{z}_i^1 w_i - \bar{w}_i w_i) - \triangle\beta H + [2S]ln[\bar{w}_{\uparrow,1} z_{\uparrow,1} + \bar{w}_{\downarrow,1} z_{\downarrow,1}]$$

$$+ [2S]ln[\bar{w}_{\uparrow,2} z_{\uparrow,2} + \bar{w}_{\downarrow,2} z_{\downarrow,2}]\Big)\Bigg]$$

Thus,

$$\boxed{Z = \Big[\frac{1}{(2S)!}\Big]^{2N} \int \prod_{l=1}^{N}\Big[\prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{z}_k{}^l dz_k^l}{2\pi i}\Big] \int \prod_{f=1}^{N}\Big[\prod_{k=(1,\uparrow)}^{(2,\downarrow)} \frac{d\bar{w}_k^f dw_k^f}{2\pi i}\Big] \exp\Big[-A\Big]}$$

where

$$A = \sum_{j=1}^{N} \Big[\Big(- \sum_{r=(1,\uparrow)}^{(2,\downarrow)} (\bar{z}_r^{j+1} w_r^j - \bar{z}_r^j z_r^j - \bar{w}_r^j w_r^j) + \Delta\beta H(\bar{z}_{\uparrow,1}^{j+1}, w_{\uparrow,1}^j, \bar{z}_{\downarrow,1}^{j+1}, w_{\downarrow,1}^j, \bar{z}_{\uparrow,2}^{j+1}, w_{\uparrow,2}^j, \bar{z}_{\downarrow,2}^{j+1}, w_{\downarrow,2}^j)$$
$$- [2S] ln[\bar{w}_{\uparrow,1}^j z_{\uparrow,1}^j + \bar{w}_{\downarrow,1}^j z_{\downarrow,1}^j] - [2S] ln[\bar{w}_{\uparrow,2}^j z_{\uparrow,2}^j + \bar{w}_{\downarrow,2}^j z_{\downarrow,2}^j] \Big) \Big]$$
$$= \sum_{j=1}^{N} \Big[\Big(- \sum_{r=(1,\uparrow)}^{(2,\downarrow)} (\bar{z}_r^{j+1} w_r^j - \bar{z}_r^j z_r^j - \bar{w}_r^j w_r^j) - \Delta\beta J \Big[\frac{\hbar^2}{4} (2\bar{z}_{\uparrow,1}^{j+1} w_{\downarrow,1}^j \bar{z}_{\downarrow,2}^{j+1} w_{\uparrow,2}^j + 2\bar{z}_{\downarrow,1}^{j+1} w_{\uparrow,1}^j \bar{z}_{\uparrow,2}^{j+1} w_{\downarrow,2}^j$$
$$+ \bar{z}_{\uparrow,1}^{j+1} w_{\uparrow,1}^j \bar{z}_{\uparrow,2}^{j+1} w_{\uparrow,2}^j - \bar{z}_{\uparrow,1}^{j+1} w_{\uparrow,1}^j \bar{z}_{\downarrow,2}^{j+1} w_{\downarrow,2}^j - \bar{z}_{\downarrow,1}^{j+1} w_{\downarrow,1}^j \bar{z}_{\uparrow,2}^{j+1} w_{\uparrow,2}^j + \bar{z}_{\downarrow,1}^{j+1} w_{\downarrow,1}^j \bar{z}_{\downarrow,2}^{j+1} w_{\downarrow,2}^j) \Big]$$
$$- [2S] ln[\bar{w}_{\uparrow,1}^j z_{\uparrow,1}^j + \bar{w}_{\downarrow,1}^j z_{\downarrow,1}^j] - [2S] ln[\bar{w}_{\uparrow,2}^j z_{\uparrow,2}^j + \bar{w}_{\downarrow,2}^j z_{\downarrow,2}^j] \Big) \Big]$$

Or,

$$Z = \Big[\frac{1}{(2S)!} \Big]^{2N} \int \prod_{l=1}^{N} \Big[\prod_{p=1}^{2} \prod_{\sigma=\uparrow,\downarrow} \frac{d\bar{z}_{p,\sigma}^l dz_{p,\sigma}^l}{2\pi i} \Big] \int \prod_{f=1}^{N} \Big[\prod_{p=1}^{2} \prod_{\sigma=\uparrow,\downarrow} \frac{d\bar{w}_{p,\sigma}^f dw_{p,\sigma}^f}{2\pi i} \Big] \exp[-A]$$

where

$$A = \sum_{j=1}^{N} \Big[- \sum_{p=1}^{2} \sum_{\sigma=\uparrow,\downarrow} (\bar{z}_{p,\sigma}^{j+1} w_{p,\sigma}^j - \bar{z}_{p,\sigma}^j z_{p,\sigma}^j - \bar{w}_{p,\sigma}^j w_{p,\sigma}^j) - \Delta\beta J \Big[\frac{\hbar^2}{4} (2\bar{z}_{\uparrow,1}^{j+1} w_{\downarrow,1}^j \bar{z}_{\downarrow,2}^{j+1} w_{\uparrow,2}^j$$
$$+ 2\bar{z}_{\downarrow,1}^{j+1} w_{\uparrow,1}^j \bar{z}_{\uparrow,2}^{j+1} w_{\downarrow,2}^j + \bar{z}_{\uparrow,1}^{j+1} w_{\uparrow,1}^j \bar{z}_{\uparrow,2}^{j+1} w_{\uparrow,2}^j - \bar{z}_{\uparrow,1}^{j+1} w_{\uparrow,1}^j \bar{z}_{\downarrow,2}^{j+1} w_{\downarrow,2}^j - \bar{z}_{\downarrow,1}^{j+1} w_{\downarrow,1}^j \bar{z}_{\uparrow,2}^{j+1} w_{\uparrow,2}^j$$
$$+ \bar{z}_{\downarrow,1}^{j+1} w_{\downarrow,1}^j \bar{z}_{\downarrow,2}^{j+1} w_{\downarrow,2}^j) \Big] - \sum_{p=1}^{2} [2S] ln \Big[\sum_{\sigma=\uparrow,\downarrow} \bar{w}_{p,\sigma}^j z_{p,\sigma}^j \Big] \Big]$$

4.0.2 M Spins

Now, the general *Heisenberg Model* for M spins is taken on:

$$\hat{H} = - \sum_{r,q}^{M} J_{rq} \boldsymbol{S_r} \boldsymbol{S_q} \qquad (r \neq q)$$
$$= - \sum_{r,q}^{M} J_{rq} \frac{\hbar^2}{4} (2a_{\uparrow,r}^\dagger a_{\downarrow,r} a_{\downarrow,q}^\dagger a_{\uparrow,q} + 2a_{\downarrow,r}^\dagger a_{\uparrow,r} a_{\uparrow,q}^\dagger a_{\downarrow,q} + a_{\uparrow,r}^\dagger a_{\uparrow,r} a_{\uparrow,q}^\dagger a_{\uparrow,q}$$
$$- a_{\uparrow,r}^\dagger a_{\uparrow,r} a_{\downarrow,q}^\dagger a_{\downarrow,q} - a_{\downarrow,r}^\dagger a_{\downarrow,r} a_{\uparrow,q}^\dagger a_{\uparrow,q} + a_{\downarrow,r}^\dagger a_{\downarrow,r} a_{\downarrow,q}^\dagger a_{\downarrow,q}) \qquad (r \neq q)$$

By comparison with the previous result:

$$Z = \Big[\frac{1}{(2S)!}\Big]^{2N} \int \prod_{l=1}^{N} \Big[\prod_{p=1}^{M} \prod_{\sigma=\uparrow,\downarrow} \frac{d\bar{z}_{p,\sigma}^{l} dz_{p,\sigma}^{l}}{2\pi i}\Big] \int \prod_{f=1}^{N} \Big[\prod_{p=1}^{M} \prod_{\sigma=\uparrow,\downarrow} \frac{d\bar{w}_{p,\sigma}^{f} dw_{p,\sigma}^{f}}{2\pi i}\Big] \exp[-A]$$

where

$$A = \sum_{j=1}^{N} \Bigg[-\sum_{p=1}^{M} \sum_{\sigma=\uparrow,\downarrow} (\bar{z}_{p,\sigma}^{j+1} w_{p,\sigma}^{j} - \bar{z}_{p,\sigma}^{j} z_{p,\sigma}^{j} - \bar{w}_{p,\sigma}^{j} w_{p,\sigma}^{j}) - \triangle\beta \Big[\sum_{r,q}^{M} J_{rq} \frac{\hbar^2}{4} (2\bar{z}_{\uparrow,r}^{j+1} w_{\downarrow,r}^{j} \bar{z}_{\downarrow,q}^{j+1} w_{\uparrow,q}^{j}$$
$$+ 2\bar{z}_{\downarrow,r}^{j+1} w_{\uparrow,r}^{j} \bar{z}_{\uparrow,q}^{j+1} w_{\downarrow,q}^{j} + \bar{z}_{\uparrow,r}^{j+1} w_{\uparrow,r}^{j} \bar{z}_{\uparrow,q}^{j+1} w_{\uparrow,q}^{j} - \bar{z}_{\uparrow,r}^{j+1} w_{\uparrow,r}^{j} \bar{z}_{\downarrow,q}^{j+1} w_{\downarrow,q}^{j} - \bar{z}_{\downarrow,r}^{j+1} w_{\downarrow,r}^{j} \bar{z}_{\uparrow,q}^{j+1} w_{\uparrow,q}^{j}$$
$$+ \bar{z}_{\downarrow,r}^{j+1} w_{\downarrow,r}^{j} \bar{z}_{\downarrow,q}^{j+1} w_{\downarrow,q}^{j})\Big] - \sum_{p=1}^{M} [2S] ln\Big[\sum_{\sigma=\uparrow,\downarrow} \bar{w}_{p,\sigma}^{j} z_{p,\sigma}^{j}\Big]\Bigg]$$

Chapter 5

Conclusion

The path integral single spin partition function in the basis of boson coherent states, for a general normal ordered two-mode *Schwinger boson* Hamiltonian, has been successfully computed. Within the initial phase of the calculational process, by a specific approach, the *Schwinger boson* constraint has been implemented by means of the *projection operator*. Now, the first expression obtained for the partition function had logarithmic terms in the *action*. They have been eliminated by an impromptu manipulation of the exponential term, which eventually led to a new expression for the partition function, with the action now devoid of any logarithmic terms.

After, and most crucially, the case of the *Zeeman Hamiltonian* has been taken on. Using the expression for the partition function for the general Hamiltonian, the appropriate single spin partition function has been produced. In addition, partition functions have been computed for a few specific spin quantum numbers. This has turned out to generate a great boost of confidence in the *projection operator* implementation in enforcing the *Schwinger boson* constraint; the expressions that have been obtained match perfectly with those computed by the regular means.

Finally, attention has been lodged in tackling the *Heisenberg Model*, and the appropriate expressions for the partition functions have been computed. Their validity has not been verified for any specific cases, but the Zeeman Hamiltonian case dealt with earlier appears to be a strong indicator that these are indeed the correct expressions. Furthermore, it is worth noting that logarithmic terms appear prominently in the action term here as well.

Many open questions still remain, however, for future research to address. Here are a few:

- What is the nature of the non-continuum expressions for the partition functions obtained for the different spin quantum numbers for the Zeeman Hamiltonian? Why do they exhibit so much complexity as the spin quantum numbers are increased in size? Is there a simple relational formula that could encapsulate them for all S?

- How do we interpret the logarithmic terms in the action? Is there any special meaning attached to them?
- Is the expression derived for the *Heisenberg Model* correct? How could we verify this?

In conclusion, it is determined that using the projection operator as a means to enforce the *Schwinger boson* constraint truly works for a *Zeeman Hamiltonian* problem, for specific spin quantum numbers. Furthermore, the promising results acquired for that problem, is judged to be cause for optimism about the future research and development of the *projection operator* method of enforcing constraints, and its related fields.

Chapter 6

Appendix

6.1 Proof of the Coherent State Closure Relation

The boson coherent state closure relation

$$\boxed{\int \prod_{\alpha} \frac{d\bar{\phi}_\alpha d\phi_\alpha}{2\pi i} \exp\left[-\sum_{\alpha} \bar{\phi}_\alpha \phi_\alpha \right] |\phi\rangle\langle\phi| = \mathbf{1}}$$

may be proved as follows:

$$\begin{aligned}
\int \prod_{\alpha} \frac{d\bar{\phi}_\alpha d\phi_\alpha}{2\pi i} \exp\left[-\sum_{\alpha} \bar{\phi}_\alpha \phi_\alpha \right] |\phi\rangle\langle\phi| &= \int \prod_{\alpha}^{N} \frac{dx_\alpha dy_\alpha}{\pi} \exp\left[-\sum_{\alpha} \bar{\phi}_\alpha \phi_\alpha \right] |\phi\rangle\langle\phi| \\
&= \int \prod_{\alpha}^{N} \frac{dx_\alpha dy_\alpha}{\pi} \exp\left[-\sum_{\alpha} (x_\alpha^2 + y_\alpha^2) \right] |\phi\rangle\langle\phi| \\
&= \left[\int \exp(-x^2) dx \right]^N \left[\int \exp(-y^2) dy \right]^N \left[\frac{1}{\pi} \right]^N |\phi\rangle\langle\phi| \\
&= (\sqrt{\pi})^N (\sqrt{\pi})^N \left[\frac{1}{\pi} \right]^N |\phi\rangle\langle\phi| \\
&= |\phi\rangle\langle\phi|
\end{aligned}$$

$$= \sum_{n_{\alpha_1}, n_{\alpha_2}, ..., n_{\alpha_p}} \phi_{n_{\alpha_1} n_{\alpha_2} ... n_{\alpha_p}} ... \bar{\phi}_{n_{\alpha_1} n_{\alpha_2} ... n_{\alpha_p}} ...$$

$$|n_{\alpha_1} n_{\alpha_2} ... n_{\alpha_p} ...\rangle\langle n_{\alpha_1} n_{\alpha_2} ... n_{\alpha_p} ...|$$

$$= \sum_{n_{\alpha_1}, n_{\alpha_2}, ..., n_{\alpha_p}} |n_{\alpha_1} n_{\alpha_2} ... n_{\alpha_p} ...\rangle\langle n_{\alpha_1} n_{\alpha_2} ... n_{\alpha_p} ...|$$

$$\equiv \mathbf{1}$$

6.2 The Commutation Relations of The Spin Operators

Using the Schwinger Boson Representation, the commutation relations of the spin operators are derived as follows:

$$\begin{aligned}
[\hat{S}_x, \hat{S}_y] &= \hat{S}_x\hat{S}_y - \hat{S}_y\hat{S}_x \\
&= \hbar^2/4i[(a^\dagger b + b^\dagger a)(a^\dagger b - b^\dagger a) - (a^\dagger b - b^\dagger a)(a^\dagger b + b^\dagger a)] \\
&= -\hbar^2 i/2(b^\dagger a a^\dagger b - a^\dagger b b^\dagger a) \\
&= -\hbar^2 i/2(b^\dagger b + a^\dagger b^\dagger b a - a^\dagger b b^\dagger a) \\
&= \hbar^2 i/2(a^\dagger a - b^\dagger b) \\
&= i\hbar\hat{S}_z \\
\\
&\Rightarrow [\hat{S}_y, \hat{S}_x] = -i\hbar\hat{S}_z
\end{aligned}$$

$$\begin{aligned}
[\hat{S}_x, \hat{S}_z] &= \hat{S}_x\hat{S}_z - \hat{S}_z\hat{S}_x \\
&= \hbar^2/4[(a^\dagger b + b^\dagger a)(a^\dagger a - b^\dagger b) - (a^\dagger a - b^\dagger b)(a^\dagger b + b^\dagger a)] \\
&= \hbar^2/4(-2a^\dagger b + 2b^\dagger a) \\
&= -i\hbar\hat{S}_y \\
\\
&\Rightarrow [\hat{S}_z, \hat{S}_x] = i\hbar\hat{S}_y
\end{aligned}$$

$$\begin{aligned}
[\hat{S}_y, \hat{S}_z] &= \hat{S}_y\hat{S}_z - \hat{S}_z\hat{S}_y \\
&= \hbar^2/4i[(a^\dagger b - b^\dagger a)(a^\dagger a - b^\dagger b) - (a^\dagger a - b^\dagger b)(a^\dagger b - b^\dagger a)] \\
&= \hbar^2/4(-2a^\dagger b + 2b^\dagger a) \\
&= i\hbar\hat{S}_x \\
\\
&\Rightarrow [\hat{S}_z, \hat{S}_y] = -i\hbar\hat{S}_x
\end{aligned}$$

6.3 Continuum Notations

Towards the end of the section *Single Boson Probability Amplitude In The Basis of Coherent States*, the two discrete parts:

$$\boxed{\exp[-\frac{i}{\hbar}\epsilon(\sum_{k=1}^{N-1} H(\bar{z}_{k+1}, z_k) + H(\bar{z_1}, z_0))]}$$

and

$$\exp[\sum_{k=1}^{N-1}(\bar{z}_{k+1}-\bar{z_k})z_k+\bar{z_1}z_0]$$

may be expressed in the *continuum notation* as follows:

$$\exp[-\frac{i}{\hbar}\epsilon(\sum_{k=1}^{N-1}H(\bar{z}_{k+1},z_k)+H(\bar{z_1},z_0))]$$
$$\longrightarrow \exp[-\frac{i}{\hbar}\int_0^t H(\bar{z(t)},z(t))dt]$$

$$\exp[\sum_{k=1}^{N-1}(\bar{z}_{k+1}-\bar{z_k})z_k+\bar{z_1}z_0]$$
$$\longrightarrow \exp[\frac{i}{\hbar}(\int_0^t(-i\hbar)\frac{d\bar{z}}{dt}zdt)+\bar{z(0)}z(0))]$$

For the second part, the sum in the argument of the exponential can be rearranged to give an alternative representation:

$$\exp[\frac{i}{\hbar}(\int_0^t(i\hbar)\frac{dz}{dt}\bar{z}dt)+z(t)\bar{z}(t)] \tag{6.1}$$

Averaging the arguments in both representations, one gets:

$$\exp[\frac{\bar{z(t)}z(t)+\bar{z(0)}z(0)}{2}+\frac{i}{\hbar}\int_0^t(\frac{i\hbar}{2}(\bar{z}\frac{dz}{dt}-\frac{d\bar{z}}{dt}z)-H(\bar{z},z))dt] \tag{6.2}$$

After rearranging using partial integration, one gets the following representation of the probability amplitude:

$$lim_{\epsilon\to 0}U(z_N,z_0,t)=\int\left[\prod_{k=1}^{N-1}\frac{dz_kd\bar{z_k}}{2\pi i}\right]\exp[\frac{i}{\hbar}S]$$

where

$$S=-i\hbar\bar{z(t)}z(t)+\int_0^t\left[i\hbar\bar{z}\frac{dz}{dt}-H(\bar{z},z)\right]dt$$

Additionally, towards the end of section *Single Boson Partition Function In The Basis of Coherent States*, the two parts

$$\exp[-\Delta\beta(H(\bar{z},z_{N-1})+H(\bar{z_1},z)+\sum_{k=1}^{N-2}H(\bar{z}_{k+1},z_k)]$$

and

$$\boxed{\exp[\bar{z_1} z + (\bar{z} - \bar{z}_{N-1}) z_{N-1} + \sum_{k=1}^{N-2} (\bar{z}_{k+1} - \bar{z_k}) z_k]}$$

combined, may be written in the continuum notation as:

$$\begin{aligned}
& \exp[-\Delta\beta(H(\bar{z}, z_{N-1}) + H(\bar{z_1}, z) + \sum_{k=1}^{N-2} H(\bar{z}_{k+1}, z_k)] \\
& \exp[\bar{z_1} z + (\bar{z} - \bar{z}_{N-1}) z_{N-1} + \sum_{k=1}^{N-2} (\bar{z}_{k+1} - \bar{z_k}) z_k] \\
\longrightarrow & \qquad \exp[-\int_0^\beta H(z\bar{(t)} z(t)) dt] \\
& \qquad \exp[\frac{i}{\hbar}(\int_0^\beta (-i\hbar) \frac{d\bar{z}}{dt} z dt) + \bar{z}(0) z(0)]
\end{aligned}$$

As done previously, the second term above can be rearranged to give an alternative representation:

$$\exp[\frac{i}{\hbar}(\int_0^\beta (i\hbar) \frac{dz}{dt} \bar{z} dt) + \bar{z}(t) z(t)]$$

Averaging the arguments in both representations:

$$\exp[\frac{z\bar{(t)} z(t) + z\bar{(0)} z(0)}{2} + \frac{i}{\hbar} \int_0^\beta (\frac{i\hbar}{2}(\bar{z}\frac{dz}{dt} - \frac{d\bar{z}}{dt} z) - \frac{\hbar}{i} H(\bar{z}, z)) dt]$$

and reearranging using partial integration, we get

$$\boxed{Z = \int \left[\prod_{k=1}^{N} \frac{d\bar{z_k} dz_k}{2\pi i} \right] \exp[S]}$$

where

$$\boxed{S = \int_0^\beta \left[i\hbar H(z\bar{(t)}, z(t)) + \frac{d\bar{z}}{dt} z \right] dt}$$

6.4 The Computation of $\langle z^{j+1}|w^j\rangle$

$$
\begin{aligned}
\langle z^{j+1}|w^j\rangle &= \langle 0,0|\exp(\bar{z}_\uparrow^{j+1}a_\uparrow)\exp(\bar{z}_\downarrow^{j+1}a_\downarrow)\exp(w_\uparrow^j a_\uparrow^\dagger)\exp(w_\downarrow^j a_\downarrow^\dagger)|0,0\rangle \\
&= \langle 0,0|\exp(\bar{z}_\uparrow^{j+1}a_\uparrow)\exp(w_\uparrow^j a_\uparrow^\dagger)\exp(\bar{z}_\downarrow^{j+1}a_\downarrow)\exp(w_\downarrow^j a_\downarrow^\dagger)|0,0\rangle \\
&= \exp(\bar{z}_\uparrow^{j+1}w_\uparrow^j)\langle 0,0|\exp(w_\uparrow^j a_\uparrow^\dagger)\exp(\bar{z}_\uparrow^{j+1}a_\uparrow)\exp(\bar{z}_\downarrow^{j+1}a_\downarrow)\exp(w_\downarrow^j a_\downarrow^\dagger)|0,0\rangle \\
&= \exp(\bar{z}_\uparrow^{j+1}w_\uparrow^j)\exp(\bar{z}_\downarrow^{j+1}w_\downarrow^j)\langle 0,0|\exp(w_\uparrow^j a_\uparrow^\dagger)\exp(\bar{z}_\uparrow^{j+1}a_\uparrow)\exp(w_\downarrow^j a_\downarrow^\dagger)\exp(\bar{z}_\downarrow^{j+1}a_\downarrow)|0,0\rangle \\
&= \exp(\bar{z}_\uparrow^{j+1}w_\uparrow^j)\exp(\bar{z}_\downarrow^{j+1}w_\downarrow^j)\langle 0,0|\exp(w_\uparrow^j a_\uparrow^\dagger)\exp(w_\downarrow^j a_\downarrow^\dagger)\exp(\bar{z}_\uparrow^{j+1}a_\uparrow)\exp(\bar{z}_\downarrow^{j+1}a_\downarrow)|0,0\rangle \\
&= \exp(\bar{z}_\uparrow^{j+1}w_\uparrow^j)\exp(\bar{z}_\downarrow^{j+1}w_\downarrow^j)\langle 0,0|0,0\rangle \\
&= \exp(\bar{z}_\uparrow^{j+1}w_\uparrow^j)\exp(\bar{z}_\downarrow^{j+1}w_\downarrow^j) \\
&= \exp\Big[\sum_{l=\uparrow,\downarrow}\bar{z}_l^{j+1}w_l^j\Big]
\end{aligned}
$$

6.5 The Computation of $\langle w^j|P|z^j\rangle$

$$
\begin{aligned}
\langle w^j|P|z^j\rangle &= \langle w^j|\sum_{n=0}^{2S}\frac{1}{n!(2S-n)!}(a_\uparrow^\dagger)^n(a_\downarrow^\dagger)^{2S-n} : \exp\big[-(\hat{n_\uparrow}+\hat{n_\downarrow})\big] : a_\uparrow^n a_\downarrow^{2S-n}|z^j\rangle \\
&= \sum_{n=0}^{2S}\frac{1}{n!(2S-n)!}\langle w^j|(a_\uparrow^\dagger)^n(a_\downarrow^\dagger)^{2S-n} : \exp\big[-(\hat{n_\uparrow}+\hat{n_\downarrow})\big] : a_\uparrow^n a_\downarrow^{2S-n}|z^j\rangle \\
&= \sum_{n=0}^{2S}\frac{1}{n!(2S-n)!}\langle w^j|(\bar{w}_j^\uparrow)^n(\bar{w}_j^\downarrow)^{2S-n} : \exp\big[-(\hat{n_\uparrow}+\hat{n_\downarrow})\big] : (z_j^\uparrow)^n(z_j^\downarrow)^{2S-n}|z^j\rangle \\
&= \sum_{n=0}^{2S}\frac{1}{n!(2S-n)!}(\bar{w}_j^\uparrow)^n(\bar{w}_j^\downarrow)^{2S-n}(z_j^\uparrow)^n(z_j^\downarrow)^{2S-n}\langle w^j| : \exp\big[-(\hat{n_\uparrow}+\hat{n_\downarrow})\big] : |z^j\rangle
\end{aligned}
$$

6.6 The Computation of $\langle w^j | : (a^\dagger_\uparrow a_\dagger + a^\dagger_\downarrow a_\downarrow)^p : | z^j \rangle$

$$
\begin{aligned}
\langle w^j | : (a^\dagger_\uparrow a_\dagger + a^\dagger_\downarrow a_\downarrow)^p : | z^j \rangle &= \langle w^j | \sum_{k=0}^{p} \frac{p!}{k!(p-k)!} : (a^\dagger_\uparrow, a_\uparrow)^{p-k} (a^\dagger_\downarrow a_\downarrow)^k : | z^j \rangle \\
&= \sum_{k=0}^{p} \frac{p!}{k!(p-k)!} \langle w^j | (a^\dagger_\uparrow)^{p-k} (a^\dagger_\downarrow)^k (a_\uparrow)^{p-k} (a_\downarrow)^k | z^j \rangle \\
&= \sum_{k=0}^{p} \frac{p!}{k!(p-k)!} \langle w^j | (\bar{w}^j_\uparrow)^{p-k} (\bar{w}^j_\downarrow)^k (z^j_\uparrow)^{p-k} (z^j_\downarrow)^k | z^j \rangle \\
&= \sum_{k=0}^{p} \frac{p!}{k!(p-k)!} (\bar{w}^j_\uparrow)^{p-k} (\bar{w}^j_\downarrow)^k (z^j_\uparrow)^{p-k} (z^j_\downarrow)^k \langle w^j | z^j \rangle \\
&= \sum_{k=0}^{p} \frac{p!}{k!(p-k)!} (\bar{w}^j_\uparrow)^{p-k} (\bar{w}^j_\downarrow)^k (z^j_\uparrow)^{p-k} (z^j_\downarrow)^k \exp\Big[\sum_{r=\uparrow,\downarrow} \bar{w}^j_r z^j_r \Big] \\
&= \Big[\bar{w}^j_\uparrow z^j_\uparrow + \bar{w}^j_\downarrow z^j_\downarrow \Big]^p \exp\Big[\sum_{r=\uparrow,\downarrow} \bar{w}^j_r z^j_r \Big]
\end{aligned}
$$

6.7 The Computation of $\hat{\boldsymbol{S}}_\mathbf{1} \hat{\boldsymbol{S}}_\mathbf{2}$

$$
\begin{aligned}
\hat{\boldsymbol{S}}_\mathbf{1} \hat{\boldsymbol{S}}_\mathbf{2} &= \frac{\hbar^2}{4} (a^\dagger_{\uparrow,1} a_{\downarrow,1} a^\dagger_{\uparrow,2} a_{\downarrow,2} + a^\dagger_{\uparrow,1} a_{\downarrow,1} a^\dagger_{\downarrow,2} a_{\uparrow,2} + a^\dagger_{\downarrow,1} a_{\uparrow,1} a^\dagger_{\uparrow,2} a_{\downarrow,2} + a^\dagger_{\downarrow,1} a_{\uparrow,1} a^\dagger_{\downarrow,2} a_{\uparrow,2}) \\
&- \frac{\hbar^2}{4} (a^\dagger_{\uparrow,1} a_{\downarrow,1} a^\dagger_{\uparrow,2} a_{\downarrow,2} - a^\dagger_{\uparrow,1} a_{\downarrow,1} a^\dagger_{\downarrow,2} a_{\uparrow,2} - a^\dagger_{\downarrow,1} a_{\uparrow,1} a^\dagger_{\uparrow,2} a_{\downarrow,2} + a^\dagger_{\downarrow,1} a_{\uparrow,1} a^\dagger_{\downarrow,2} a_{\uparrow,2}) \\
&+ \frac{\hbar^2}{4} (a^\dagger_{\uparrow,1} a_{\uparrow,1} a^\dagger_{\uparrow,2} a_{\uparrow,2} - a^\dagger_{\uparrow,1} a_{\uparrow,1} a^\dagger_{\downarrow,2} a_{\downarrow,2} - a^\dagger_{\downarrow,1} a_{\downarrow,1} a^\dagger_{\uparrow,2} a_{\uparrow,2} + a^\dagger_{\downarrow 1} a_{\downarrow,1} a^\dagger_{\downarrow,2} a_{\downarrow,2}) \\
&= \frac{\hbar^2}{4} (a^\dagger_{\uparrow,1} a_{\downarrow,1} a^\dagger_{\uparrow,2} a_{\downarrow,2} + a^\dagger_{\uparrow,1} a_{\downarrow,1} a^\dagger_{\downarrow,2} a_{\uparrow,2} + a^\dagger_{\downarrow,1} a_{\uparrow,1} a^\dagger_{\uparrow,2} a_{\downarrow,2} + a^\dagger_{\downarrow,1} a_{\uparrow,1} a^\dagger_{\downarrow,2} a_{\uparrow,2} \\
&- a^\dagger_{\uparrow,1} a_{\downarrow,1} a^\dagger_{\uparrow,2} a_{\downarrow,2} + a^\dagger_{\uparrow,1} a_{\downarrow,1} a^\dagger_{\downarrow,2} a_{\uparrow,2} + a^\dagger_{\downarrow,1} a_{\uparrow,1} a^\dagger_{\uparrow,2} a_{\downarrow,2} - a^\dagger_{\downarrow,1} a_{\uparrow,1} a^\dagger_{\downarrow,2} a_{\uparrow,2} \\
&+ a^\dagger_{\uparrow,1} a_{\uparrow,1} a^\dagger_{\uparrow,2} a_{\uparrow,2} - a^\dagger_{\uparrow,1} a_{\uparrow,1} a^\dagger_{\downarrow,2} a_{\downarrow,2} - a^\dagger_{\downarrow,1} a_{\downarrow,1} a^\dagger_{\uparrow,2} a_{\uparrow,2} + a^\dagger_{\downarrow 1} a_{\downarrow,1} a^\dagger_{\downarrow,2} a_{\downarrow,2}) \\
&= \frac{\hbar^2}{4} (2 a^\dagger_{\uparrow,1} a_{\downarrow,1} a^\dagger_{\downarrow,2} a_{\uparrow,2} + 2 a^\dagger_{\downarrow,1} a_{\uparrow,1} a^\dagger_{\uparrow,2} a_{\downarrow,2} + a^\dagger_{\uparrow,1} a_{\uparrow,1} a^\dagger_{\uparrow,2} a_{\uparrow,2} \\
&- a^\dagger_{\uparrow,1} a_{\uparrow,1} a^\dagger_{\downarrow,2} a_{\downarrow,2} - a^\dagger_{\downarrow,1} a_{\downarrow,1} a^\dagger_{\uparrow,2} a_{\uparrow,2} + a^\dagger_{\downarrow 1} a_{\downarrow,1} a^\dagger_{\downarrow,2} a_{\downarrow,2})
\end{aligned}
$$

6.8 The Computation of $\langle w^j|P|z^j\rangle$

$$
\begin{aligned}
\langle w^j|P|z^j\rangle = \langle w^j| &\sum_{n_{\uparrow,1}=0}^{2S}\sum_{n_{\uparrow,2}=0}^{2S}\frac{1}{n_{\uparrow,1}!(2S-n_{\uparrow,1})!}\frac{1}{n_{\uparrow,2}!(2S-n_{\uparrow,2})!}\\
&(a^\dagger_{\uparrow,1})^{n_{\uparrow,1}}(a^\dagger_{\downarrow,1})^{2S-n_{\uparrow,1}}(a^\dagger_{\uparrow,2})^{n_{\uparrow,2}}(a^\dagger_{\downarrow,2})^{2S-n_{\uparrow,2}}\\
&:\exp\big[-(\hat n_{\uparrow,1}+\hat n_{\downarrow,1})-(\hat n_{\uparrow,2}+\hat n_{\downarrow,2})\big]:\\
&a_{\uparrow,1}^{n_{\uparrow,1}}a_{\downarrow,1}^{2S-n_{\uparrow,1}}a_{\uparrow,1}^{n_{\uparrow,2}}a_{\downarrow,2}^{2S-n_{\uparrow,2}}|z^j\rangle\\
&=\sum_{n_{\uparrow,1}=0}^{2S}\sum_{n_{\uparrow,2}=0}^{2S}\frac{1}{n_{\uparrow,1}!(2S-n_{\uparrow,1})!}\frac{1}{n_{\uparrow,2}!(2S-n_{\uparrow,2})!}\langle w^j|\\
&(a^\dagger_{\uparrow,1})^{n_{\uparrow,1}}(a^\dagger_{\downarrow,1})^{2S-n_{\uparrow,1}}(a^\dagger_{\uparrow,2})^{n_{\uparrow,2}}(a^\dagger_{\downarrow,2})^{2S-n_{\uparrow,2}}\\
&:\exp\big[-(\hat n_{\uparrow,1}+\hat n_{\downarrow,1})-(\hat n_{\uparrow,2}+\hat n_{\downarrow,2})\big]:\\
&a_{\uparrow,1}^{n_{\uparrow,1}}a_{\downarrow,1}^{2S-n_{\uparrow,1}}a_{\uparrow,1}^{n_{\uparrow,2}}a_{\downarrow,2}^{2S-n_{\uparrow,2}}|z^j\rangle\\
&=\sum_{n_{\uparrow,1}=0}^{2S}\sum_{n_{\uparrow,2}=0}^{2S}\frac{1}{n_{\uparrow,1}!(2S-n_{\uparrow,1})!}\frac{1}{n_{\uparrow,2}!(2S-n_{\uparrow,2})!}\langle w^j|(\bar w^j_{\uparrow,1})^{n_{\uparrow,1}}(\bar w^j_{\downarrow,1})^{2S-n_{\uparrow,1}}\\
&(\bar w^j_{\uparrow,2})^{n_{\uparrow,2}}(\bar w^j_{\downarrow,2})^{2S-n_{\uparrow,2}}:\exp\big[-(\hat n_{\uparrow,1}+\hat n_{\downarrow,1})-(\hat n_{\uparrow,2}+\hat n_{\downarrow,2})\big]:\\
&(z^j_{\uparrow,1})^{n_{\uparrow,1}}(z^j_{\downarrow,1})^{2S-n_{\uparrow,1}}(z^j_{\uparrow,2})^{n_{\uparrow,2}}(z^j_{\downarrow,2})^{2S-n_{\uparrow,2}}|z^j\rangle\\
&=\sum_{n_{\uparrow,1}=0}^{2S}\sum_{n_{\uparrow,2}=0}^{2S}\frac{1}{n_{\uparrow,1}!(2S-n_{\uparrow,1})!}\frac{1}{n_{\uparrow,2}!(2S-n_{\uparrow,2})!}\\
&(\bar w^j_{\uparrow,1})^{n_{\uparrow,1}}(\bar w^j_{\downarrow,1})^{2S-n_{\uparrow,1}}(\bar w^j_{\uparrow,2})^{n_{\uparrow,2}}(\bar w^j_{\downarrow,2})^{2S-n_{\uparrow,2}}(z^j_{\uparrow,1})^{n_{\uparrow,1}}(z^j_{\downarrow,1})^{2S-n_{\uparrow,1}}(z^j_{\uparrow,2})^{n_{\uparrow,2}}(z^j_{\downarrow,2})^{2S-n_{\uparrow,2}}\\
&\langle w^j|:\exp\big[-(\hat n_{\uparrow,1}+\hat n_{\downarrow,1})-(\hat n_{\uparrow,2}+\hat n_{\downarrow,2})\big]:|z^j\rangle
\end{aligned}
$$

6.9 The Computation of $\langle w^j| : (a^\dagger_{\uparrow,1}a_{\uparrow,1} + a^\dagger_{\downarrow,1}a_{\downarrow,1} + a^\dagger_{\uparrow,2}a_{\uparrow,2} + a^\dagger_{\downarrow,2}a_{\downarrow,2})^p : |z^j\rangle$

$$\langle w^j| : (a^\dagger_{\uparrow,1}a_{\uparrow,1} + a^\dagger_{\downarrow,1}a_{\downarrow,1} + a^\dagger_{\uparrow,2}a_{\uparrow,2} + a^\dagger_{\downarrow,2}a_{\downarrow,2})^p : |z^j\rangle$$

$$= \langle w^j| \sum_{k=0}^{p} \frac{p!}{k!(p-k)!} : (a^\dagger_{\uparrow,1}a_{\uparrow,1} + a^\dagger_{\downarrow,1}a_{\downarrow,1})^{p-k}(a^\dagger_{\uparrow,2}a_{\uparrow,2} + a^\dagger_{\downarrow,2}a_{\downarrow,2})^k : |z^j\rangle$$

$$= \sum_{k=0}^{p} \frac{p!}{k!(p-k)!} \langle w^j| : (a^\dagger_{\uparrow,1}a_{\uparrow,1} + a^\dagger_{\downarrow,1}a_{\downarrow,1})^{p-k}(a^\dagger_{\uparrow,2}a_{\uparrow,2} + a^\dagger_{\downarrow,2}a_{\downarrow,2})^k : |z^j\rangle$$

$$= \sum_{k=0}^{p} \frac{p!}{k!(p-k)!} \langle w^j| \sum_{r=0}^{p-k} \frac{(p-k)!}{r!(p-k-r)!} (a^\dagger_{\uparrow,1}a_{\uparrow,1})^{p-k-r}(a^\dagger_{\downarrow,1}a_{\downarrow,1})^r$$

$$\sum_{g=0}^{k} \frac{k!}{g!(k-g)!} (a^\dagger_{\uparrow,2}a_{\uparrow,2})^{k-g}(a^\dagger_{\downarrow,2}a_{\downarrow,2})^g |z^j\rangle$$

$$= \sum_{k=0}^{p} \frac{p!}{k!(p-k)!} \sum_{r=0}^{p-k} \frac{(p-k)!}{r!(p-k-r)!} \sum_{g=0}^{k} \frac{k!}{g!(k-g)!} \langle w^j|(a^\dagger_{\uparrow,1}a_{\uparrow,1})^{p-k-r}(a^\dagger_{\downarrow,1}a_{\downarrow,1})^r$$

$$(a^\dagger_{\uparrow,2}a_{\uparrow,2})^{k-g}(a^\dagger_{\downarrow,2}a_{\downarrow,2})^g |z^j\rangle$$

$$= \sum_{k=0}^{p} \frac{p!}{k!(p-k)!} \sum_{r=0}^{p-k} \frac{(p-k)!}{r!(p-k-r)!} \sum_{g=0}^{k} \frac{k!}{g!(k-g)!} \langle w^j|(\bar{w}_{\uparrow,1}z_{\uparrow,1})^{p-k-r}(\bar{w}_{\downarrow,1}z_{\downarrow,1})^r$$

$$(\bar{w}_{\uparrow,2}z_{\uparrow,2})^{k-g}(\bar{w}_{\downarrow,2}z_{\downarrow,2})^g |z^j\rangle$$

$$= \sum_{k=0}^{p} \frac{p!}{k!(p-k)!} \sum_{r=0}^{p-k} \frac{(p-k)!}{r!(p-k-r)!} \sum_{g=0}^{k} \frac{k!}{g!(k-g)!} (\bar{w}_{\uparrow,1}z_{\uparrow,1})^{p-k-r}(\bar{w}_{\downarrow,1}z_{\downarrow,1})^r$$

$$(\bar{w}_{\uparrow,2}z_{\uparrow,2})^{k-g}(\bar{w}_{\downarrow,2}z_{\downarrow,2})^g \langle w^j|z^j\rangle$$

$$\sum_{k=0}^{p} \frac{p!}{k!(p-k)!} \sum_{r=0}^{p-k} \frac{(p-k)!}{r!(p-k-r)!} \sum_{g=0}^{k} \frac{k!}{g!(k-g)!} (\bar{w}_{\uparrow,1}z_{\uparrow,1})^{p-k-r}(\bar{w}_{\downarrow,1}z_{\downarrow,1})^r$$

$$(\bar{w}_{\uparrow,2}z_{\uparrow,2})^{k-g}(\bar{w}_{\downarrow,2}z_{\downarrow,2})^g \exp\Big[\sum_{l=(1,\uparrow)}^{(2,\downarrow)} \bar{w}^j_l z^j_l\Big]$$

$$= \sum_{k=0}^{p} \frac{p!}{k!(p-k)!} (\bar{w}_{\uparrow,1}z_{\uparrow,1} + \bar{w}_{\downarrow,1}z_{\downarrow,1})^{p-k}(\bar{w}_{\uparrow,2}z_{\uparrow,2} + \bar{w}_{\downarrow,2}z_{\downarrow,2})^k \exp\Big[\sum_{l=(1,\uparrow)}^{(2,\downarrow)} \bar{w}^j_l z^j_l\Big]$$

$$= (\bar{w}_{\uparrow,1}z_{\uparrow,1} + \bar{w}_{\downarrow,1}z_{\downarrow,1} + \bar{w}_{\uparrow,2}z_{\uparrow,2} + \bar{w}_{\downarrow,2}z_{\downarrow,2})^p \exp\Big[\sum_{l=(1,\uparrow)}^{(2,\downarrow)} \bar{w}^j_l z^j_l\Big]$$

References

[1] Harald JW Müller-Kirsten. *Introduction to Quantum Mechanics: Schrö dinger Equation and Path Integral*. World Scientific Publishing Company, 2006.

[2] Assa Auerbach. *Interacting electrons and quantum magnetism*. Springer Science & Business Media, 2012.

[3] Gregoire Misguich. Quantum spin liquids. *Exact Methods in Low-dimensional Statistical Physics and Quantum Computing: Lecture Notes of the Les Houches Summer School: Volume 89, July 2008*, 89:431, 2010.

[4] Herbert Goldstein. *Classical mechanics*. Pearson Education India, 2011.

[5] Richard P Feynman, Albert R Hibbs, and Daniel F Styer. *Quantum mechanics and path integrals*. Courier Corporation, 2010.

[6] John W Negele. *Quantum many-particle systems*. CRC Press, 2018.

[7] Eduardo Fradkin. Coherent state path integral quantization of quantum field theory. http://eduardo.physics.illinois.edu/phys582/582-chapter8.pdf.

[8] Ramamurti Shankar. *Principles of quantum mechanics*. Springer Science & Business Media, 2012. Chapter 21: *Path Integrals: Part II*.

[9] Julian Simmendinger. Path-integrals for bosons. https://www.itp3.uni-stuttgart.de/downloads/talks/03.pdf.

[10] J.O. Fjærestad. Projection operators. (Unpublished Note).