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Nonlinear phase unwinding

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Abstract

We start of by studying Hardy spaces H^p and Blaschke products

$$B_n(z) = z^m \prod_{j \in J_n} \frac{\overline{a_j}}{|a_j|} \frac{z - a_j}{|1 - \overline{a_j} z|}$$

Then we look at a natural nonlinear analogue of Fourier series called the unwinding series. It is obtained through iterative Blaschke factorization and unwinds the function. This allows us to write

$$F = \gamma_1 B_1 + \gamma_2 B_1 B_2 + \gamma_3 B_1 B_2 B_3 + \dots$$

We discuss the convergence of the unwinding series in various spaces and quantify how this unwinding happens. We then show that functions with some useful characteristics are close to being Hardy space functions. This can be bettered further by adding carrier frequencies which we also investigate. Then we consider decompositions of invariant subspaces of Hardy spaces and show how these relate to the unwinding series.

Oppsummering

Oppgaven begynner med å studere Hardy rom H^p og Blaschke produkt

$$B_n(z) = z^m \prod_{j \in J_n} \frac{\overline{a_j}}{|a_j|} \frac{z - a_j}{|1 - \overline{a_j} z|}$$

Deretter ser vi på en naturlig, ikke-lineær analog av Fourier serier som kalles the unwinding series. Serien er konstruert ved hjelp av en iterativ applikasjon av Blaschke faktorisering. Det tillater oss å skrive

$$F = \gamma_1 B_1 + \gamma_2 B_1 B_2 + \gamma_3 B_1 B_2 B_3 + \dots$$

Deretter diskuterer vi konvergens av the unwinding series i forskjellige rom og kvantifiserer på forskjellige måter hvorfor dette er hjelpsomt. Så viser vi at en spesiell gruppe funksjoner er nære å være holomorfe funksjoner, or mer presist Hardy rom funksjoner. Dette kan bedres enda mer ved å legge til bærefrekvenser. Avsluttende ser vi på en dekomposisjon av invariante subrom av Hardy rom og viser hvordan disse kan relateres til the unwinding series.

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Contents

Introduction

Signal analysis is a research field which is widely regarded to have started in the 1940's and 1950's. It is a theory which we see applied in many different areas. From early on in our mathematical careers we meet simple "signals" as sines and cosines, or linear combinations of them. Theory of such types of signals are handled quite well by Fourier analysis. However, general signals are often more complicated than these. Fourier analysis is not powerful enough to handle signals where the amplitude and phase shift relative to time. Therefore we seek a more elaborate and powerful way of analysis.

In 1946 in the paper [5], Gabor proposes a way of defining a signal. He also suggests a way to construct a holomorphic signal from a real-valued signal $s(t)$. We call it $s^+(t)$ and it is defined as

$$s^+(t) = \frac{1}{2}(s(t) + i\mathcal{H}s(t))$$

where $\mathcal{H}$ is the Hilbert transform. This definition of an analytic signal allows us to express the signal as $s^+(t) = A(t)e^{i\phi(t)}$, which leads to the definitions of amplitude modulation as $A(t) \geq 0$ and instantaneous frequency as $\phi'(t)$. The question then becomes; how does one extract this information?

Our discussion starts out by considering the Hardy spaces H^p, for $1 \leq p \leq \infty$, and Blaschke products,

$$B(z) = z^m \prod_{j\in J} \frac{\overline{a_j}}{|a_j|} \frac{z - a_j}{|1 - \overline{a_j}z|}$$

The discussion on the Hardy spaces results in two particular properties of functions $f \in H^p$. Every continuous function $\tilde{f}$ on the unit circle uniquely defines a holomorphic function $f \in H^p$, and for $1 \leq p \leq \infty$ and $f \in H^p$, the Fourier coefficients $\{c_n\}$ vanish for all negative n, thus $f(t) = \sum_{n=0}^{\infty} c_n e^{int}$.

This first result is quite useful in our setting. If for example we are equipped with a signal s^+ of the mentioned form, we may extend it to a holomorphic, and actually harmonic, function in the unit disk $\{z : |z| < 1\}$.

In physical practice, the frequency of signals is nonnegative. This is also suggested through mathematical considerations. Being able to express a function as $f(t) = \sum_{n=0}^{\infty} c_n e^{int}$ makes f consist of components of nonnegative frequency. As we just mentioned, this is desirable. This also causes the Hardy space setting to seem very natural when discussing signals.

The discussion on Blaschke products basically allows us to write a function F which is holomorphic in a neighbourhood of the unit disk as $F(z) = B(z)G(z)$, where B is a Blaschke product and G is a function which does not vanish in the unit disk. Through an iterative application of this, we arrive at the unwinding series, which allows us to express

$$F = \gamma_1 B_1 + \gamma_2 B_1 B_2 + \gamma_3 B_1 B_2 B_3 + \dots$$

We show convergence in L^2 as well as some other convergence results. Then we turn to some geometric considerations in an attempt to quantify why the unwinding series is useful and easier to work with than F.

The nonnegativity of the frequencies of the components of a signal is, as mentioned, desirable. However, it is not always the case. Therefore, we discuss a group of functions, which have a very natural set of properties, called the intrinsic mode type functions. We show that their anti-holomorphic part, which happens to be the part of the function consisting of nonnegative frequencies, is small. Moreover, we show that the difference of the phase of the holomorphic part of such a function and the phase of the actual function is small in L^2. Moreover, we show how this difference can be made arbitrarily small by adding a carrier frequency e^{iNt} for large enough N. This allows us to consider any intrinsic mode type function as a function consisting only of components of nonnegative frequencies. This enables us to use mathematical tool which consider holomorphic functions also in the setting of this natural group of functions.

We also gain some insight on how white noise affects a signal. It turns out that close to the boundary of the unit disk the effect of white noise is very small.

The unwinding series is usually very hard to find, and in many cases this cannot, yet, be done, even numerically. With this in mind, we introduce a class of functions, which has exponential convergence of its Fourier coefficients, where the unwinding series is easy to find.

We finish of the thesis by discussing an analogue of the unwinding series and show convergence in H^p.

This thesis' main objective is to investigate most of the results found in [2], [3] and [4] and it will be structured in the following way: Chapter 1 contains preliminary material. The purpose of this chapter is essentially to provide an introduction to the Hardy spaces It should also give some much needed insight into the dynamics of Hardy space functions and Blaschke products which will be welcomed later on. In Chapter 2 we construct the unwinding series. We show convergence and quantify how we unwind the functions in question. In Chapter 3 we introduce the Littlewood-Paley projections and use them to discuss how intrinsic mode functions are close to being holomorphic. We gain some intuition on the stability of a signal under disturbance through white noise and how carrier frequencies can better stability. Chapter 4 shows us an alternative to the unwinding series.

Chapter 1

Preliminaries

1.1 Nontangential limits

To start out, we should get some definitions and notation out of the way. We will be denoting the open unit disc in the complex plane by

$$D = \{z \in \mathbb{C} : |z| < 1\}$$

and the unit circle by

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} = \left\{e^{it} : 0 \leq t \leq 2\pi\right\}$$

We will now develop some theory of the Hardy space H^p. Our discussion will only treat the case when $p = 2$ as well as mention some more general results. Theory for more general p is discussed in multiple text and can be found in for example [6]. The reason for this is to give a short introduction to the H^p spaces and that there is a certain niceness of this space. We will denote by

$$\frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|dt$$

the normalized Lebesgue measure $dm = \frac{dt}{2\pi}$ on the unit circle.

Definition 1. *Let f be a complex-valued function defined on D and let $\zeta \in \mathbb{T}$.*

(i) We say that f has the ***radial*** *limit L at ζ if $\lim_{r\to 1} f(r\zeta) = L$*

(ii) We say that a sequence (z_n) in D converges ***nontangentially*** *to a point $\zeta \in \mathbb{T}$ if there is angle centered at ζ and symmetric about the line connecting the origin and ζ such that the angle is less than π and all z_n are in this angle and $z_n \to \zeta$ as $n \to \infty$.*

(iii) We say that the function f has the ***nontangential*** *limit L at $\zeta \in \mathbb{T}$ if for every sequence (z_n) in D that converges nontangentially to ζ we have $\lim_{n\to\infty} f(z_n) = L$.*

There are cases of functions f in D which have radial limits, but do not have nontangential limits at almost every boundary point. Some of these are even analytic. We will investigate a way of making the two types of limits more cohesive. We will see that a certain growth restriction makes our lives easier with this in mind. To start out, here is an example of sufficient conditions where radial limits imply nontangential limits.

Theorem 1. *Let f be an analytic and bounded function in D. If f has a radial limit at a boundary point $e^{it} \in \mathbb{T}$ then f has a nontangential limit at that point and these limits coincide.*

Proof. The proof of this can be found in [8]. □

1.2 Boundary behaviour of power series

We continue our journey to the growth restrictions which was mentioned. Here is a theorem by Abel.

Theorem 2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a convergent power series in D. Assume that for $\zeta \in \mathbb{T}$ we have $\sum_{n=0}^{\infty} a_n \zeta^n = L$. Then f has the nontangential limit L at ζ.*

In proving this, we will need this result, which we will not prove here.

Lemma 3. *Let α be an angle as in Definition 1. Then there is a K_α such that the inequality $|z-1| < K_\alpha(1-|z|)$ holds for any z in this angle, where K_α depends only on this angle.*

Proof. Without loss of generality, assume $\zeta = 1$. Define $S_n = \sum_{k=n}^{\infty} a_k$. Our assumptions, then, become $S_0 = L$ and $S_n \to 0$ as $n \to \infty$. So there exists some M such that $|S_n| < M$ for all n. Now, write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} (S_n - S_{n+1}) z^n$$

The series $\sum_{n=0}^{\infty} S_n z^n$ converges pointwise in D by the ratio test due to the fact that $S_n \to 0$ implies $a_n \to 0$. Therefore, we have

$$f(z) = \sum_{n=0}^{\infty} (S_n - S_{n+1}) z^n = \sum_{n=0}^{\infty} S_n z^n - \sum_{n=0}^{\infty} S_{n+1} z^n = S_0 + \sum_{n=1}^{\infty} S_n \left(z^n - z^{n-1}\right)$$

Recall by the definition of nontangential limit, we wish to show that for any sequence (z_k) in D that converges nontangentially to 1 we have $\limsup_{k\to\infty} |f(z_k) - L| = 0$. With this in mind, let $\varepsilon > 0$ and N be such that

$|S_n| < \varepsilon$ for all $n \geq N$. Then we get

$$|f(z) - L| \leq |z-1| \sum_{n=1}^{\infty} |S_n||z^{n-1}| \leq M|z-1| \sum_{n=1}^{N} |z|^{n-1} + \varepsilon |z-1| \sum_{n=N+1}^{\infty} |z|^{n-1}$$
$$\leq MN|z-1| + \varepsilon|z-1|/(1-|z|)$$

because $S_0 = L$. Now take any sequence (z_k) in D which converges nontangentially to 1. By the previously mentioned result, we now have

$$\limsup_{k\to\infty} |f(z_k) - L| = \limsup_{k\to\infty} [MN|z_k - 1| + \varepsilon|z_k - 1|/(1 - |z_k|)] \leq \varepsilon K_\alpha$$

Because ε was arbitrary, we are done. □

Now we have the following theorem, which is by Carleson

Theorem 4. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series with square summable coefficients, that is, $\sum_{n=0}^{\infty} |a_n|^2 < \infty$. Then the series $\sum_{n=0}^{\infty} a_n \zeta^n$ converges for almost every $\zeta \in \mathbb{T}$.*

Proof. For $e^{it} = \zeta \in \mathbb{T}$ we have $f(e^{it}) = \sum_{n=0}^{\infty} a_n e^{ixn}$. If we manage to show that this is a convergent Fourier series, the sum must converge. So let us define $s_n(x) = \sum_{n=0}^{\infty} a_n e^{ixn}$. For $m > k$ we have

$$\|s_m - s_k\| = \| \sum_{n=k+1}^{m} a_n e^{inx} \| = \langle \sum_{n=k+1}^{m} a_n e^{inx}, \sum_{n=k+1}^{m} a_n e^{inx} \rangle$$
$$= \sum_{n=k+1}^{m} a_n \sum_{n=k+1}^{m} \overline{a_n} \langle e^{inx}, e^{inx} \rangle = \sum_{n=k+1}^{m} |a_n|^2$$

which is finite due to our assumption of square summability. Now we know that s_n is a Cauchy sequence, and thus, because L^2 is complete, there exist some function f such that $\lim_{n\to\infty} \|s_n - f\| = 0$. This tells us that $f(x) = \sum_{n=k+1}^{m} a_n e^{inx}$, and we are done. □

Next, we will show a variation of *Parseval's identity.*

Theorem 5. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a convergent power series in D. Then*

$$\sum_{n=0}^{\infty} |a_n|^2 = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt = \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt$$

Proof. First, we observe that

$$|f(re^{it})|^2 = f(re^{it})\overline{f(re^{it})} = \left(\sum_{n=0}^{\infty} a_n r^n e^{int} \right) \left(\sum_{n=0}^{\infty} \overline{a_n} r^n e^{-int} \right)$$

Now we combine these sums to a double sum, and see that

$$\sum_{n,m=0}^{\infty} a_n \overline{a_m} r^{n+m} e^{i(n-m)t}$$

is absolutely convergent because $r < 1$. Therefore we may write

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^2 dt = \frac{1}{2\pi}\int_0^{2\pi} \sum_{n,m=0}^{\infty} a_n \overline{a_m} r^{n+m} e^{i(n-m)t} dt \tag{1.1}$$

$$= \sum_{n,m=0}^{\infty} a_n \overline{a_m} r^{n+m} \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)t} dt \tag{1.2}$$

Now, because the value of the integral is 2π or 0 depending on whether $n = m$ or not, we are left with

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^2 dt = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

Taking the supremum of both sides as $0 < r < 1$ and observing that the integrals on the left increases with r yields the desired equalities. □

Part of the reason we said that the H^2 space was nicer compared to the general p was that we have this orthonormal basis of e^{int} which is quite useful, as we just saw. Now we have expressed the square summability condition of a power series to a growth restriction of a function. We may then express the existence of nontangential limits almost everywhere in terms of this same growth restriction.

Theorem 6. *Let f be analytic in D and suppose that*

$$\lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^2 dt < \infty$$

Then f has nontangential limits for almost every point e^{it} on the unit circle.

This is the condition we were looking for. You can now see that there is a fairly nice condition under which we know that an analytic function in the unit disc has nontangential limits. Therefore, this seems like a natural time to define the H^2 space.

$$H^2(D) = \left\{ f : \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^2 dt < \infty \right\}$$

This gives rise to the H^2 norm. Let $f \in H^2$. Then

$$\|f(z)\|_{H^2} = \|\lim_{r\to\infty} f(re^{it})\|_{L^2} = \left(\lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^2 dt\right)^{\frac{1}{2}}$$

Of course, this definition is similar for general p. We have the following definition of the H^p spaces and the H^p norms for $1 \leq p \leq \infty$

$$H^p(D) = \left\{ f : \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt < \infty \right\}$$

$$\|f(z)\|_{H^p} = \| \lim_{r\to 1} f(re^{it}) \|_p = \left(\lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{\frac{1}{p}}$$

When $p = \infty$ the H^p space simply becomes the space of bounded holomorphic functions on the unit disc and the norm

$$\|f(z)\|_{H^\infty} = \sup_{z\in D} |f(re^{it})| = \sup_{e^{it}\in \mathbb{T}} |f(re^{it})|$$

We will end this section by mentioning some of the properties of H^p spaces without going in depth as to why they are true. I would again suggest [6] if you want a more thorough disussion on the matter.

The H^p spaces are nesten in the following manner, if $1 \leq p < q \leq \infty$, then $H^p(D) \subsetneq H^q(D)$. It is also worth noting that the H^p spaces are certain subspaces of L^p which we will mention again later. Actually, they are closed subspaces. That implies that, since L^p is a Banach space, also $H^p(D)$ is a Banach space with the H^p norm. Moreover, H^p is the space of $L^p(\mathbb{T})$ functions whose negative Fourier coefficients vanish. That is, functions $f \in L^p(\mathbb{T})$ such that

$$\frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) e^{-int} dt = 0$$

which gives us $f(e^{it}) = \sum_{n=0}^{\infty} a_n e^{int}$.

By the Cauchy-Riemann equations holomorphic functions on D are harmonic as well. Harmonic functions satisfy the mean value principle. If we look at the Dirichlet problem on $\mathbb{T}$ there is a solution due to the Poisson kernel. Any such solution is unique due to the maximum modulus principle for harmonic functions. The solution is given by this convolution $u(re^{it}) = f * P_r(e^{it})$. So every continuous function on T uniquely determines a holomorphic (harmonic) function on D.

1.3 Poisson integrals

Again, we shall use the notation

$$D = \{z \in \mathbb{C} : |z| < 1\}$$

and

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$$

Recall that a complex-valued function f is said to be harmonic in D if it satisfies Laplace's equation. That is,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

It is said to be analytic in D if it satisfies the Cauchy-Riemann equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\ , \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Any analytic function f can be expressed as $f = u + iv$ where u and v are harmonic, real-valued functions. Given a u we call such a v, determined by the Cauchy-Riemann equations, the harmonic conjugate of u, and v is unique up to an additive constant. Also any real-valued function u is harmonic if and only if it is the real part of an analytic function. We are all familiar with Cauchy's integral formula and some of it's consequences for analytic functions. With this close relation between harmonic and analytic functions, it seems natural to try to find an analogue of Cauchy's integral formula for analytic functions.

First, we let u be a harmonic and real-valued function in some disc containing the closed unit disc. Let $f = u + iv$ be analytic and v be the harmonic conjugate of u. Then by Cauchy's integral formula we have

$$u(z) = \operatorname{Re} f(z) = \operatorname{Re}\left(\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{f(\zeta)}{\zeta - z}d\zeta\right) = \operatorname{Re}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{f(e^{it})e^{it}}{e^{it} - z}dt\right)$$

by a change of variables. With f being the sum of u and v we wish to get rid of v in this equation and express the integral in terms of u only. This is a step in that direction.

Lemma 7. *If u is harmonic in a disc containing the closed unit disc, then for all $z \in D$ we have*

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1 - |z|^2}{|e^{it} - z|^2}u(e^{it})dt$$

Proof. Let us start out by writing $f(e^{it}) = \sum_{n=0}^{\infty} a_n e^{int}$. Because this sum converges uniformly on the unit circle, we obtain

$$\int_0^{2\pi}\frac{f(e^{it})\overline{z}}{e^{-it} - \overline{z}}dt = \sum_{n=0}^{\infty} a_n \int_0^{2\pi}\frac{e^{int}\overline{z}}{e^{-it} - \overline{z}}dt$$

This is equal to zero for all $z \in D$ and we will use this fact later on in the proof. Another useful fact to note is that

$$\frac{e^{it}}{e^{it} - z} + \frac{\overline{z}}{e^{-it} - \overline{z}} = \frac{1 - \overline{z}e^{it} + \overline{z}e^{it} - |z|^2}{|e^{it} - z|^2} = \frac{1 - |z|^2}{|e^{it} - z|^2}$$

Now, from Cauchy's integral formula, we know have

$$
\begin{aligned}
u(z) &= \mathrm{Re}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{f(e^{it})e^{it}}{e^{it}-z}dt\right)\\
&= \mathrm{Re}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{f(e^{it})e^{it}}{e^{it}-z}+\frac{f(e^{it})\overline{z}}{e^{-it}-\overline{z}}dt\right)\\
&= \mathrm{Re}\left(\frac{1}{2\pi}\int_0^{2\pi}f(e^{it})\left(\frac{e^{it}}{e^{it}-z}+\frac{\overline{z}}{e^{-it}-\overline{z}}\right)dt\right)\\
&= \mathrm{Re}\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|e^{it}-z|^2}f(e^{it})dt\right)\\
&= \frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\left(\frac{1-|z|^2}{|e^{it}-z|^2}f(e^{it})\right)dt\\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|e^{it}-z|^2}u(e^{it})dt
\end{aligned}
$$

because $\frac{1-|z|^2}{|e^{it}-z|^2}$ is real. So we are done. □

This is what is called Poisson's kernel, $P_z(e^{it}) = \frac{1-|z|^2}{|e^{it}-z|^2}$. It is worth noting that in polar coordinates on the unit disc, it becomes $\frac{1-r^2}{1-2r\cos(\theta)+r^2}$ or $\sum_{n\in\mathbb{Z}} r^{|n|}e^{int}$.

Now that we have expressed the integral only in terms of u, the next step will be to relax the condition of harmonicity in the larger disc. We will be working with Hardy spaces where this would be an issue. The next result does this.

Corollary 8. *Let u be harmonic in D and continuous in the closed unit disc, then*

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|e^{it}-z|^2}u(e^{it})dt$$

Proof. For some $0 < r < 1$ fixed, but arbitrary, we consider first the function $u_r(z) = u(rz)$. Notice then, that u_r is harmonic in the disc of radius $1/r$ and we may apply theorem 7. We have

$$u_r(z) = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|e^{it}-z|^2}u_r(e^{it})dt$$

Note that if u_r converges uniformly to u on T we have

$$
\begin{aligned}
\lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|e^{it}-z|^2}u_r(e^{it})dt &= \frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|e^{it}-z|^2}\lim_{r\to 1}u_r(e^{it})dt\\
&= \frac{1}{2\pi}\int_0^{2\pi}\frac{1-|z|^2}{|e^{it}-z|^2}u(e^{it})dt
\end{aligned}
$$

since then the integrand converges uniformly on $\mathbb{T}$. Because we also have pointwise convergence of u_r to u, that is $\lim_{r\to 1} u_r(z) = u(z)$, we would be done. So let us now show that u_r converges uniformly to u. This becomes clear if we observe that $u(z)$ is continuous on the closed unit disc and therefore, uniformly continuous as well. Thus we have uniform convergence of $u_r(z)$ to $u(z)$ on $\mathbb{T}$. □

1.4 H^p spaces

Here we will show that functions in H^p can be identified by different functions on D depending on p. Recall first, that a Banach space X is said to be reflexive if it is linearly isometric to its bidual. An equivalent condition to this, is that every bounded sequence in X has a weakly convergent subsequence, which we will use here.

Theorem 9. *Let u be harmonic on D and suppose that for $r \in (0,1)$ there is some constant C such that*

$$\int_0^{2\pi} |u(re^{it})|^p dt < C$$

for $1 \le p < \infty$ and for $p = \infty$ we have

$$\sup_{t\in[0,2\pi]} |u(re^{it})| < C$$

Then,

(i) If $p > 1$, there exists a unique $g \in L^p(m)$ such that

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2} g(e^{it})dt$$

(ii) If $p = 1$, there exists a unique finite Borel measure μ on $\mathbb{T}$ such that

$$u(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} d\mu(\zeta)$$

Proof. First we prove existence. This part will be similar to the proof of theorem 8. Let us again define $u_r(z) = u(rz)$ for $z \in D$ and $0 < r < 1$. Our u_r are then harmonic in the disc of radius $1/r$ centered at the origin. We may apply theorem 7 and again let $r \to 1$.

For (i) we start with the case where $p < \infty$. In this case we know that L^p is reflexive. This means that there is some sequence (r_n) tending to 1 such that (u_{r_n}) converges weakly in L^p to some $g \in L^p$. Now, because the Poisson kernel

$$P_z(e^{it}) = \frac{1-|z|^2}{|e^{it}-z|^2}$$

is a bounded function with respect to t and by the definition of weak convergence in a Banach space we have

$$u_{r_n}(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2} u_{r_n}(e^{it})dt \to u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2} g(e^{it})dt$$

Now for the $p=\infty$ case, we first recall that L^∞ is a sequentially compact space. Define our u_r as before and let (r_n) be a sequence tending to 1. Then by sequential compactness, the sequence (u_{r_n}) has a subsequence which converges to some $g \in L^\infty$. Similar to the $p < \infty$ case, we obtain our result.

For (ii) we need only observe that $P(\cdot)$ is continuous on $\mathbb{T}$ and that the measures $u_r dm$ are bounded by C in total variation, and apply Alaoglu's theorem. Thus for some sequence (r_n) tending to 1 there is a weak-star convergent subsequence, $(u_{r_{n_k}} dm)$ which weak-star converges to some finite Borel measure μ on $\mathbb{T}$.

Now proving uniqueness must be shown and will conclude our proof. First we calculate

$$P_z(e^{it}) = 1 + 2\,\mathrm{Re}\,\frac{ze^{-it}}{1-ze^{-it}} = 1 + 2\,\mathrm{Re}\sum_{n=1}^{\infty} z^n e^{-int}$$

This is a uniformly convergent sequence on $[0, 2\pi]$. Because there is equality in (i) and (ii) for all $z \in D$ then g and μ are unique. This is easily seen by uniqueness of Fourier coefficients. □

Corollary 10. *Let u be harmonic and nonnegative in D. Then there exists a unique nonnegative Borel measure μ on $\mathbb{T}$ such that $\mu(\mathbb{T}) = \mu(0)$ and*

$$u(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} d\mu(\zeta)$$

Proof. A nonnegative harmonic function u on D satisfies the assumption if part *(ii)* of theorem 9 since

$$\frac{1}{2\pi}\int_0^{2\pi} |u(re^{it})|dt = \frac{1}{2\pi}\int_0^{2\pi} u(re^{it})dt = u(0)$$

Since the measure μ given by the theorem is the weak-star limit of nonnegative measures, it will be nonnegative as well. □

1.5 Blaschke products

In this section we shall discuss a way of constructing an analytic function in D which has prescribed boundary values of their modulus, $h(z)$ on the unit circle. The proofs which are omitted here, are omitted due to space and material which is required to complete the proofs not being discussed here. First, we define a Blaschke product. A Blaschke product is a function of the form

$$B(z) = z^m \prod_{j \in J} \frac{\overline{a_j}}{|a_j|} \frac{z - a_j}{1 - \overline{a_j} z}$$

where m is a nonnegative integer and $\{a_j : j \in J\}$ are zeros in D. The Blaschke products will prove crucial in the following discussion and has the convenient property that $|B(e^{it})| = 1$.

Proposition 11. *Let h be a nonnegative function on the unit circle such that*

$$\int_{\mathbb{T}} |\log h| dm < \infty$$

Then the function in D defined by

$$F(z) = exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log h(\zeta) dm(\zeta)\right)$$

has nontangential limits almost everywhere on $\mathbb{T}$. Moreover, if we denote by $F(\zeta)$ the nontangential limit of F at $\zeta \in \mathbb{T}$ (when this value exists) then $|F(\zeta)| = h(\zeta)$ almost everywhere on $\mathbb{T}$.

Proof. See [8]. □

The F constructed here will play a central role in what follows, but is not the only function which is analytic in D and whose modulus equals h almost everywhere on $\mathbb{T}$. Any finite Blaschke product B in D extends continuously to $\mathbb{T}$ and has the value 1 there, and thus FB also equals h almost everywhere on $\mathbb{T}$. We could also construct a function G which is analytic and without zeros in D and which is such that $|G|$ has the same nontangential limits as $|F|$ almost everywhere on $\mathbb{T}$. Indeed, if $g(z) = exp\left(-\frac{1+z}{1-z}\right)$ then $G = gF$ does the job. This is because g extends continuously to $\mathbb{T}\backslash\{1\}$ and $\text{Re}\,(g(\zeta)) = 0$ for all $\zeta \in \mathbb{T}\backslash\{1\}$.

Definition 2. *An analytic function F in D is called an outer function if there exists a nonnegative function h on the unit circle such that*

$$\int_{\mathbb{T}} |\log h| dm < \infty$$

and

$$F(z) = exp\left(\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log h(\zeta) dm(\zeta)\right)$$

In this case, F is the outer function whose modulus equals h almost everywhere on $\mathbb{T}$.

Proposition 12. *Let h be a nonnegative function on the unit circle such that*

$$\int_{\mathbb{T}} |\log h| dm < \infty$$

and let F_h be the outer function whose modulus equals h almost everywhere on $\mathbb{T}$. Then

$$|F_h(z)| \leq \int_{\mathbb{T}} P_z(\zeta) h(\zeta) dm(\zeta)$$

The proof is based on this small lemma,

Lemma 13. *If $u, v : [a,b] \mapsto \mathbb{R}$ are integrable functions, v is nonnegative with $\int_a^b v(x)dx = 1$ then*

$$exp\left(\int_a^b u(t)v(t)dt\right) \leq \int_a^b e^{u(t)} v(t)dt$$

Let us first prove this.

Proof. First, we divide through by $exp\left(\int_a^b u(x)v(x)dx\right)$ to obtain

$$1 \leq \int_a^b v(t) e^{\left(u(t) - \int_a^b u(x)v(x)dx\right)} dt$$

Next, because $e^y \geq y + 1$ we have

$$e^{\left(u(t) - \int_a^b u(x)v(x)dx\right)} \geq u(t) - \int_a^b u(x)v(x)dx + 1$$

where $t \in [a,b]$. Now, we simply compute, and see that

$$\int_a^b v(t) e^{\left(u(t) - \int_a^b u(x)v(x)dx\right)} dt \geq \int_a^b v(t) \left(u(t) - \int_a^b u(x)v(x)dx + 1\right) dt = 1$$

and we are done. □

Now we will move on to proving the theorem.

Proof. Let $u(t) = \log h(e^{it})$ and $v(t) = \frac{1}{2\pi} P_z(e^{it})$. Now we use the lemma, and we obtain

$$|F_h(z)| = exp\left(\int_{\mathbb{T}} P_z(\zeta) \log h(\zeta) dm(\zeta)\right) \leq \int_{\mathbb{T}} P_z(\zeta) h(\zeta) dm(\zeta)$$

Here, we also used that $\int_{\mathbb{T}} P_z dm = 1$. □

Lemma 14. *Let h be a nonnegative function on $\mathbb{T}$ such that*

$$\int_{\mathbb{T}} |\log h| dm < \infty$$

and let F_h be the outer function whose modulus equals h almost everywhere on $\mathbb{T}$. Then for $1 \le p \le \infty$, $F_h \in H^p$ if and only if $h \in L^p$, and

$$\|F_h\|_p = \|h\|_{L^p}$$

Proof. First we will show $\|F_h\|_p \ge \|h\|_{L^p}$. We have

$$\begin{aligned}
\|h\|_{L^p} &= \int_{\mathbb{T}} |h(z)|^p dm \overset{*}{=} \int_{\mathbb{T}} \liminf_{r\to 1} |F_h(rz)|^p dm \overset{**}{\le} \liminf_{r\to 1} \int_{\mathbb{T}} |F_h(rz)| dm \\
&= \frac{1}{2\pi} \sup_{r \to 1} \int_0^{2\pi} |F_h(re^{it})| dt = \|F_h\|_p
\end{aligned}$$

where $*$ is by proposition 11 and $**$ is by applying Fatou's lemma. So now we must show the reverse inequality.

Notice that $P_z(e^{it})$ is always positive and bounded by 1. Thus by proposition 12 we have

$$|F_h(z)|^p \le \int_{\mathbb{T}} P_z(\zeta) h(\zeta)^p dm(\zeta) \le \int_{\mathbb{T}} h(\zeta)^p dm(\zeta)$$

Because this is true for all $z \in D$, in particular $z = re^{it}$, by integrating both sides over $\mathbb{T}$ and taking supremum over $0 \le r < 1$ we are done. □

We have defined what it means for a function to be an element of H^p and we have defined the Blaschke product. What we have not discussed, is how these definitions play a role in a functions' zeros. It turns out that in this regard, the two definitions are equal and quite restrictive. This will dictate the direction of our discussion for a while.

Lemma 15. *If f is analytic in D and $0 < p < \infty$, then*

$$M_p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt$$

is an increasing function of $r \in [0, 1)$.

Proof. See [8]. □

Theorem 16. *Let $0 < p \le \infty$, and assume that $f \in H^p$ is not identically zero. Let $a_1, a_2, ...$ be the zeros of f in D, repeated according to the multiplicity of the zero. Then*

$$\sum_{n=1}^{\infty} (1 - |a_n|) < \infty$$

Moreover, if B is the Blaschke product with zeros $a_1, a_2, \ldots$ then $f/B \in H^p$, and

$$\|f/B\|_p = \|f\|_p$$

Proof. Let us denote by B_N

$$B_N(z) = z^m \prod_{n=m+1}^{N} \frac{-\overline{a_n}}{|a_n|} \frac{z - a_n}{1 - \overline{a_n} z}$$

Notice that f/B_N is analytic in D. Now we wish to show that for $0 < p \leq \infty$ we have $f/B_N \in H^p$ and $\|f/B_N\|_p = \|f\|_p$.

First we let $p = \infty$. Because B_N is continuous on D and $|B_N| = 1$ on $\mathbb{T}$ we know that f/B_N is a member of H^∞. Now we choose a sequence (z_n) with $|z_n| \to 1$ such that

$$\|f/B_N\|_\infty = \lim_{n\to\infty} \frac{|f(z_n)|}{|B_N(z_n)|}$$

And then, $|B_N| = 1$ on $\mathbb{T}$ and B_N is continuous, so we have

$$\|f/B_N\|_\infty = \lim_{n\to\infty} |f(z_n)| \leq \|f\|_\infty$$

By the maximum principle, $|B_N| < 1$ on D, and the reverse inequality follows. Thus, we have $\|f/B_N\|_\infty = \|f\|_\infty$.

Let now $0 < p < \infty$. Again, B_N is continuous on D and $|B_N| = 1$ on $\mathbb{T}$, so $f/B_N \in H^p$. We have

$$\|f/B_N\|_p^p = \lim_{r\to 1} M_p(r, f/B_N) \leq \limsup_{r\to 1} M_p(r, f) = \|f\|_p$$

because $|B_N| = 1$ on $\mathbb{T}$. The reverse inequality follows as before.

Notice now that (B_N) cannot converge to zero uniformly on compact subsets of D. Thus $\sum_{n=1}^{\infty}$ cannot equal ∞. Now letting $N \to \infty$ yields one inequality, and noticing that $B = z^m \prod_{n=m+1}^{\infty} \frac{-\overline{a_n}}{|a_n|} \frac{z-a_n}{1-\overline{a_n} z} < 1$ on D yields the other. □

Corollary 17. *Let (a_n) be a sequence in D such that*

$$\sum_{n=1}^{\infty} (1 - |a_n|) < \infty$$

and let B be the corresponding Blaschke product. Then B has nontangential limits of modulus 1 almost everywhere on $\mathbb{T}$.

Proof. It is clear that $|B| \leq 1$ on D, so $\|B\|_\infty \leq 1$. If we let f be any bounded, analytic function on D and apply theorem 16 to fB, we have

$$\|fB/B\|_\infty = \|f\|_\infty = \|fB\|_\infty$$

Now let $\varepsilon > 0$. Assume there is a measurable set $E \subset \mathbb{T}$ with $m(E) > 0$ such that the nontangential limits $B(\zeta)$ satisfy $|B(\zeta)| < 1 - \varepsilon$ for $\zeta \in E$. If we let f be the outer function whose modulus equals 1 a.e. on E and 1/2 a.e. on its complement, then $\|f\|_\infty = 1$. Now, because we have

$$\|fB\|_\infty = \sup_{e^{it} \in \mathbb{T}} |fB(re^{it})| < \max\{1/2, 1 - \varepsilon\}$$

there is a contradiction, so such a set E does not exist. Thus B has nontangential limits of modulus 1 a.e. on $\mathbb{T}$. □

Now we move on to applying these results to represent $f \in H^p$ in a way wich will prove useful later on. By theorem 16 we see that every $f \in H^p$ for $0 < p \leq \infty$ can be written as $f = Bg$, where B is a Blaschke product, $g \in H^p$ does not vanish in D and $\|g\|_p = \|f\|_p$. This all leads to a factorization of functions in H^p. Now we will se a series of results with this is mind.

Theorem 18. *(i) If $f \in H^p$ and $0 < p \leq \infty$, then f has nontangential limits $f(\zeta)$ almost everywhere on $\mathbb{T}$ and $\|f\|_p = \|f\|_{L^p}$.*

(ii) If $f \in H^p$ and $0 < p \leq \infty$, then its maximal nontangential function f^* belongs to L^p, and there exists $c_p > 0$ such that $\|f^*\|_{L^p} \leq c_p \|f\|_p$.

Proof. See [8] for proof. □

Theorem 19. *If $f \in H^p$, $0 < p \leq \infty$, for $0 \leq r < 1$ $f_r(z) = f(rz)$ and $z \in D$, then*

$$\lim_{r \to 1} \|f - f_r\|_p = 0$$

Proof. By theorem 18 *(i)* we have

$$\|f - f_r\|_p^p = \int_{\mathbb{T}} |f - f_r|^p dm$$

Now, we know $|f(\zeta) - f_r(\zeta)| \leq 2|f^*(\zeta)|$ for $\zeta \in \mathbb{T}$. Also, $|f - f_r|^p \in L^1$ and $|f - f_r|$ converges pointwise a.e. to zero, so we may use the dominated convergence theorem. Then

$$\lim_{r \to 1} \|f - f_r\|_p^p = \lim_{r \to 1} \int_{\mathbb{T}} |f - f_r|^p dm = \int_{\mathbb{T}} \lim_{r \to 1} |f - f_r|^p dm = 0$$

□

Corollary 20. *(i) For $1 \leq p < \infty$, H^p is a separable Banach space.*

(ii) For $0 < p < 1$, H^p is a complete separable space with respect to the metric

$$d_p(f, g) = \|f - g\|_p^p$$

Proof. For $f \in H^p$ and $0 < r < 1$ we may approximate f_r uniformly by polynomials. Polynomials with rational coefficients is a countable set and also dense in H^p. Thus H^p is a separable Banach space. □

Theorem 21. *If $f \in H^1$ then*

$$f(z) = \int_{\mathbb{T}} P_z(\zeta) f(\zeta) dm(\zeta), \quad \zeta \in D$$

Proof. Because we have, for $0 < r < 1$, $f_r \in H^\infty$, we have

$$f(rz) = \int_{\mathbb{T}} P_z(\zeta) f(r\zeta) dm(\zeta), \quad \zeta \in D$$

Now let $r \to 1$. By applying theorem 19 we obtain the statement. □

Theorem 22. *Let μ be a finite Borel measure on $\mathbb{T}$ with the property that*

$$\int_{\mathbb{T}} \zeta^n d\mu(\zeta) = 0$$

for all nonnegative integers n. Then μ is absolutely continuous with respect to m and there exists $f \in H^1$ with $f(0) = 0$ such that $\frac{d\mu}{dm} = f$.

Proof. Let u be the Poisson integral of μ. We know that

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |u(re^{it})| dt < \infty$$

Then because $P_z(\zeta)$ is real, we have for $z \in D$

$$2u(z) = 2\int_{\mathbb{T}} P_z(\zeta) d\mu(\zeta) = \int_{\mathbb{T}} P_z(\zeta) d\mu(\zeta) + \int_{\mathbb{T}} \overline{P_z}(\zeta) d\mu(\zeta)$$

If $\zeta \in \mathbb{T}$

$$\frac{\overline{\zeta} + \overline{z}}{\overline{\zeta} - \overline{z}} = \frac{1 + \overline{z}\zeta}{1 - \overline{z}\zeta} = 2\sum_{n=1}^{\infty} \overline{z}^n \zeta^n - 1$$

which is uniformly convergent on $\mathbb{T}$ with fixed $z \in D$. So by our assumtions,

$$\int_{\mathbb{T}} \frac{\overline{\zeta} + \overline{z}}{\overline{\zeta} - \overline{z}} d\mu(\zeta) = 0$$

and we are left with

$$2u(z) = \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta)$$

and $u \in H^1$. This also shows that $u(0) = 0$. Then by theorem 21 we have

$$u(z) = \int_{\mathbb{T}} P_z(\zeta) u(\zeta) dm(\zeta)$$

which proves the claim. □

Now we will refine the factorization which we have discussed to obtain a useful form. We start out with this proposition.

Proposition 23. *If $f \in H^p$, $0 < p < \infty$, then for all $z \in D$,*

$$|f(z)|^p \leq \int_{\mathbb{T}} P_z(\zeta)|f(\zeta)|^p dm(\zeta)$$

In particular,

$$|f(z)| \leq \left(\frac{1+|z|}{1-|z|}\right)^{1/p} \|f\|_p, \quad z \in D$$

Proof. See [8] for proof. □

Now we continue toward the goal.

Proposition 24. *If $f \in H^p$, $0 < p \leq \infty$ is not identically zero, then $\log|f| \in L^1(m)$. Moreover, if F is the outer function with $|F| = \log|f|$ m-almost everywhere on $\mathbb{T}$, then*

$$|f(z)| \leq |F(z)|, \quad z \in D$$

Proof. See [8] for proof.

□

The outer function F from proposition 24 will be called the outer factor of f.

Definition 3. *A bounded analytic function I in D is called inner if its nontangential limits satisfy $|I(\zeta)| = 1$, m-almost everywhere on $\mathbb{T}$.*

Clearly, inner functions are bounded by 1 in D. According to proposition 24, $I = f/F$ is inner whenever $f \in H^p$. The factorization

$$f = IF$$

will be referred to as the inner-outer factorization of f.

Proposition 25. *The inner-outer factorization of $f \in H^p$, $0 < p \leq \infty$ is unique.*

Proof. If $f = IF = JG$, with I, J inner and F, G outer then

$$\frac{I}{J} = \frac{G}{F}$$

This implies that the nontangential limits of G/F are of absolute value 1 almost everywhere on $\mathbb{T}$. Since G/F is outer it follows by definition that $G/F = 1$. □

By theorem 16, the inner factorization can be further factorized as $I = BS$ where B is a Blaschke product and S is a function which does not vanish on D.

Definition 4. *An inner function without zeros in D is called singular inner.*

Singular inner functions have special forms. Let S be such a function, then $\log S$ is harmonic and negative in D. In particular, it satisfies

$$S(z) = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |S(re^{it})| dt < \infty$$

Then, by corollary 10 there is a nonnegative Borel measure μ on $\mathbb{T}$ such that

$$-\log |S(z)| = \int_{\mathbb{T}} \frac{1-|z|^2}{|\zeta - z|^2} d\mu(\zeta)$$

This leads us to this corollary.

Corollary 26. *If S is a singular inner function then there exists $\alpha \in \mathbb{R}$ and a nonnegative finite Borel measure μ on $\mathbb{T}$ which is singular with respect to m, such that*

$$S(z) = e^{\left(i\alpha - \int_{\mathbb{T}} \frac{\zeta+z}{\zeta-z} d\mu(\zeta)\right)}$$

Proof. See [8] for proof. □

The final result is stated below and proven above.

Theorem 27. *Let $f \in H^p$, $0 < p \leq \infty$. If f is not identically zero, it can be expressed uniquely in the form*

$$f = BSF$$

where B is a Blaschke product, S is a singular inner function and F is the outer factor of f.

1.6 Significance

The theory of Hardy spaces is an important one. It turns out to be very natural in, for example, working with signal analysis. As we have just seen, the Blaschke products appear very naturally in the Hardy space setting, and are of quite some significance in its own right. They both deserve a much longer introduction than given here, but for the purpose of time and convenience of this thesis, we will stop the discussion here. It is also worth noting that the Hardy space theory we have discussed here, has a natural analogue on the upper half-plane $\mathbb{C}^+$, the lower half-plane $\mathbb{C}^-$ and the outside of the unit disk.

Chapter 2

Nonlinear phase unwinding

Signal analysis is a theory which has proven useful for quite some time and we see applied in many different areas. It is seen in the handling of pictures, sound and video, and these are just a few of the examples of areas where it has proven useful. From early on in our mathematical careers we meet simple "signals" as sines and cosines, or linear combinations of them. Theory of such types of signals are handled quite well by Fourier analysis. However, general signals are often more complicated than these. Fourier analysis is not powerful enough to handle signals where the amplitude and phase shift relative to time. Therefore, Fourier analysis is incapable of handling signals in general, and we seek a more elaborate and powerful way of analysis. Moreover, we seek a way of defining a signal which is general in some sense.

Instantaneous frequency and amplitude modulation are quantities of interest when discussing a signal. How one defines these quantities does not seem to have a definite answer. There are different ways of doing so, all of which have different strengths and weaknesses. It seems as one chooses the definition which offers strengths most useful for your purpose. Many believe there does not exist a general way of defining these quantities. A popular definition, which we will discuss briefly below, will leave some signals having instantaneous frequency, and some not. The ones which do will be called mono-components, and the ones which do not will be called multi-components. For multi-component signals one often seeks a mono-component decomposition.

In 1946 Gabor proposed his analytic signal approach in [5]. If one has a real-valued signal $s(t)$ of finite energy, where the energy of a function is defined as

$$\text{energy}(\gamma) = \int_0^{2\pi} |\gamma'(t)|^2 dt$$

then the associated analytic signal, denoted by $s^+(t)$, is defined as

$$s^+(t) = \frac{1}{2}(s(t) + i\mathcal{H}s(t))$$

where $\mathcal{H}$ is the Hilbert transform. The Hilbert transform of a function $u(t)$

of a real variable is given as

$$\mathcal{H}(u)(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{u(\tau)}{t-\tau} d\tau$$

This definition of an analytic signal allow us to express the signal as $s^+(t) = A(t)e^{i\phi(t)}$, which leads to the definitions of amplitude modulation as $A(t) \geq 0$ and instantaneous frequency as $\phi'(t)$. Representation in this way allows us to extract information about these two important quantities. It may prove difficult to do this in practice, and therefore a lot of signal analysis is focused on exactly this and proposes different ways of retrieving this information. It is worth noting that expressing a signal in this way by using the Hilbert transform causes s^+ to be holomorphic, which enables use of the machinery which comes along with that. Moreover, the analytic signals are the non-tangential boundary limits of Hardy space functions, which we have already studied to a certain extent.

2.1 Unwinding series

We now start out study of signals. We will mainly be considering functions $F : \mathbb{C} \to \mathbb{C}$ which are holomorphic in a neighbourhood of the unit disk D. The "signal" $f : \mathbb{T} \to \mathbb{C}$ is then the restriction of F to the unit circle $\mathbb{T}$, or ∂D.

To start out we will discuss a way of unravelling the oscillation of these signals, which will be done by an iterative use of Blaschke decomposition. We do this because in many respects, G is easier to work with than F. One instance of this is seen as if we restrict $F = BG$ to the unit circle, then G has a smaller winding number around the origin than F.

We consider the Blaschke decomposition $F = BG$. If F does not have any zeros in D then $G = 1$ is trivial. If F does have zeros in D, then G has no zeros in D. The way we iteratively apply Blaschke decomposition is that we consider $F(z) = B(z)\left(G(0) + (G(z) - (G(0))\right)$ as we know that $G(z) - G(0)$ has at least one root. If we continue in this fashion we see that

$$\begin{aligned}
F(z) &= B_1(z)G_1(z) = B_1(z)\left(G_1(0) + (G_1(z) - G_1(0))\right)\\
F(z) &= B_1(z)G_1(0) + B_1(z)B_2(z)G_2(z)\\
&= B_1(z)G_1(0) + B_1(z)B_2(z)\left(G_2(0) + (G_2(z) - G_2(0))\right)\\
F(z) &= B_1(z)G_1(0) + B_1(z)B_2(z)G_2(0) + B_1(z)B_2(z)B_3(z)G_3(z)\\
&\vdots\\
F(z) &= B_1(z)G_1(0) + B_1(z)B_2(z)G_2(0) + B_1(z)B_2(z)B_3(z)G_3(0) + \dots
\end{aligned}$$

This is what we will call the unwinding series of F. Although we will show convergence of this series, it is often not numerically feasible to compute. The dynamics which make this series converge has yet to be fully understood. We will return to this discussion and present a class of functions where where the B_n are easy to compute later on. For now, let us visit an example which tries to yield some insight into what makes G simpler than F.

Consider the function

$$F(e^{it}) = P_+\left(e^{-(t-\pi)^2} e^{10it}\right)$$

which is the projection of a modulated Gaussian onto holomorphic functions. In figure 2.2 we can see the curves $F(e^{it})$, $G(e^{it})$ and $B(e^{it})$.

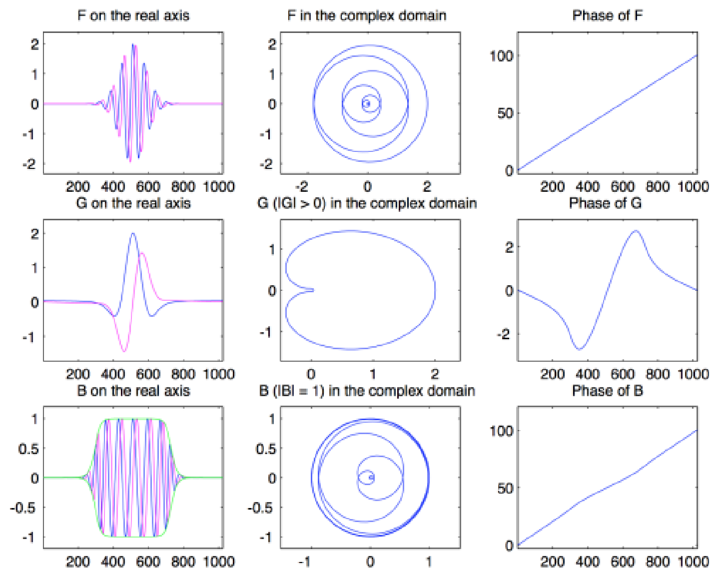

Figure 2.1: A picture taken from Michael Nahon's thesis [7] where the unwinding series first appeared. It illustrates how B captures some of the "badness" of F and leaves G considerably simpler.

2.2 Convergence

Let us discuss some convergence properties of the unwinding series. First off, we have this theorem.

Theorem 28. *The unwinding series converges in L^2 for all*

$$F(e^{it}) = \sum_{n=0}^{\infty} a_n e^{int} \quad \textit{with} \quad \sum_{n=0}^{\infty} |a_n|^2 < \infty$$

We will use this small lemma.

Lemma 29. *Any two Blaschke terms of the unwinding series* $\gamma_l z^l \left(\prod_{k=1}^{l} B_k(e^{it})\right)$ *and* $\gamma_m z^m \left(\prod_{k=1}^{m} B_k(e^{it})\right)$ *are orthogonal on* $L^2(\mathbb{T})$*.*

Proof. Let $l < m$. We compute,

$$\int_0^{2\pi} \overline{\gamma_l e^{itl} \left(\prod_{k=1}^{l} B_k(e^{it})\right)} \gamma_m e^{itm} \left(\prod_{k=1}^{m} B_k(e^{it})\right) dt = \int_0^{2\pi} \overline{\gamma_l} \gamma_m e^{it(m-l)} \left(\prod_{k=l+1}^{m} B_k(e^{it})\right) dt = 0$$

Here we used that $|B_k(e^{it})| = 1$ and that we are left with a holomorphic function.

□

Now we prove the theorem.

Proof. Because the unwinding series proceeds by factoring out a root at zero at each step, we may modify the B_n slightly to write

$$F = F(0) + \gamma_1 z B_1 + \gamma_2 z^2 B_1 B_2 + \gamma_3 z^3 B_1 B_2 B_3 + \cdots + z^n B_1 B_2 \dots B_n (G - G(0))$$

We showed in lemma 29 that any two of the Blaschke terms are orthogonal on $L^2(\mathbb{T})$. We now observe that the last term is orthogonal to all the others. This can be seen through the inner product

$$\int_0^{2\pi} \overline{\gamma_l} \gamma_n e^{it(n-l)} \left(\prod_{k=1}^{n} B_k(e^{it})\right) (G(e^{it}) - G(0)) dt = 0$$

Because of these orthogonalities, we immediately get

$$\|F\|^2_{L^2(\mathbb{T})} = \|F(0)\|^2_{L^2(\mathbb{T})} + \|\gamma_1 e^{it} B\|^2_{L^2(\mathbb{T})} + \cdots + \|e^{int} B_1 B_2 \dots B_n (G - G(0))\|^2_{L^2(\mathbb{T})}$$

Now we just need to show that the remainder term gets small. This can be done by showing that the remainder is orthogonal to all z^k where $0 \leq k \leq n-1$. We have

$$\int_0^{2\pi} \gamma_n e^{itn} \left(\prod_{k=1}^{n} B_k(e^{it})\right) (G(e^{it}) - G(0)) e^{-ikt} dt = 0$$

Thus we know that

$$\|e^{int} B_1(e^{it}) B_2(e^{it}) \dots B_n(e^{it}) (G(e^{it}) - G(0))\|^2_{L^2(\mathbb{T})} \leq \sum_{k=n}^{\infty} |a_k|^2$$

which again implies that the unwinding series converges as $n \to \infty$. □

Recall that for a function $F : \mathbb{C} \to \mathbb{C}$ we define the outer function G_1 of the Blaschke decomposition

$$F = B_1 G_1$$

Iteratively, we call G_{n+1} the outer function of

$$G_n(z) - G_n(0) = B_{n+1}(z) G_{n+1}(z)$$

We would like to find a useful space where $\|G_n\|_x \to 0$. You will see how this is done through the following statements and proofs.

In this decomposition, the zeros of F are captured by G in a specific way. Let F have the roots $\{\alpha_1, \alpha_2, \dots\}$, where for simplicity we assume none of them are on the unit circle. Then, as we know,

$$B(z) = z^m \sum_{|\alpha_i|<1} \frac{\overline{\alpha_i}}{|\alpha_i|} \frac{z - \alpha_i}{1 - \overline{\alpha_i} z}$$

where m is the multiplicity of the zero at $z = 0$ and the α_i ranges over all roots in the unit disk. Notice that G captures these roots in a different way. The roots of G are

$$\begin{aligned} \alpha_i \quad &\text{when} \quad |\alpha_i| > 1 \\ \overline{1/\alpha_i} \quad &\text{when} \quad |\alpha_i| < 1 \end{aligned}$$

Here is a preliminary result of the convergence of the unwinding series.

Theorem 30. *Let $F : \mathbb{C} \to \mathbb{C}$ be given by a polynomial of degree n. Then the formal series converges and is exact after n steps.*

Proof. Notice first, that if we do this procedure on a polynomial, we again get a polynomial. So the procedure is closed in the set of polynomials. Now, in every step we study $G_k(z) - G_k(0)$, and because it has a root at 0 we see that $B_n G_n$ will at least reduce its degree by one in each step. This proves the claim. □

As we have seen in theorem 27 the Blaschke decomposition actually factors into a Blaschke product, a singular inner function and the outer factor. At this point, it is not clear how to work with the singular inner function, so we will restrict the further scope of this chapter to functions which are holomorphic in the disk of radius $1 + \varepsilon$ for some $\varepsilon > 0$. What this does is that the Blaschke factorization really factors into a Blaschke product and an outer function. So we will keep writing the factorization as $F = BG$.

Furthermore, all functions which are holomorphic in a neighbourhood of the unit disk have a finite amount of zeros in the unit disk. This is easily seen through this small argument. If there were an infinite amount of zeros in the unit disk, there would have to be an accumulation point of them. This would cause the function to be identically zero. So for our purposes, all functions have a finite Blaschke product. This is not a serious restriction for the application of this theory as most signals can be quite well approximated by trigonometric polynomials. However, for mathematical purposes a more complete theory would be more desirable. This material is not fully developed and some aspects far from understood.

2.3 Main result

We now turn to the main result of [2]. We shall first see the result in a general form, before discussing particular cases later on. In this regard, we introduce a semi-norm and a norm. Let $0 = \gamma_0 \leq \gamma_1 \leq \dots$ be an arbitrary monotonically increasing sequence of real numbers and let X be the subspace of $H^2(\mathbb{T})$ for which

$$\left\| \sum_{n=0}^{\infty} a_n z^n |_{\partial D} \right\|_X^2 = \left\| \sum_{n=0}^{\infty} a_n e^{int} \right\|_X^2 = \sum_{n=0}^{\infty} \gamma_n |a_n|^2 < \infty$$

This norm is on the space Y and defined to be (this, too, is a semi-norm if the sequence of γ is not strictly increasing)

$$\left\| \sum_{n=0}^{\infty} a_n z^n |_{\partial D} \right\|_Y^2 = \left\| \sum_{n=0}^{\infty} a_n e^{int} \right\|_Y^2 = \sum_{n=0}^{\infty} (\gamma_{n+1} - \gamma_n) |a_n|^2 < \infty$$

As we will see, these spaces will prove useful in the stating of the next theorem and in the discussions which follows. Theorem 31 states that on these spaces, X and Y, the Blaschke factorization behaves nicely and provides another quantification of how G is nicer than F. Also, the first inequality can be equivalently phrased as follows. Given a Blaschke decomposition $F = BG$ and their Fourier series

$$F(z) = \sum_{n=0}^{\infty} f_n z^n \quad \text{and} \quad G(z) = \sum_{n=0}^{\infty} g_n z^n$$

then, for every $N \in \mathbb{N}$

$$\sum_{n \geq N} |g_n|^2 \leq \sum_{n \geq N} |f_n|^2$$

In other words, the Blaschke factorization shifts the energy to lower frequencies in a monotonous way. Now, to the result.

Theorem 31. *If $F : \mathbb{D} \to \mathbb{C}$ is holomorphic on some neighbourhood of the unit disc and has a Blaschke factorization $F = BG$, then*

$$\|G(e^{i\cdot})\|_X \leq \|F(e^{i\cdot})\|_X$$

Moreover, if $F(\alpha) = 0$ for some $\alpha \in \mathbb{D}$, we have

$$\|G(e^{i\cdot})\|_X^2 \leq \|F(e^{i\cdot})\|_X^2 - (1 - |\alpha|^2) \left\| \frac{G(e^{i\cdot})}{1 - \overline{\alpha} z} \right\|_Y^2$$

Proof. We start out by studying

$$f(z) = (z - \alpha)F(z) \quad \text{and} \quad g(z) = (1 - \overline{\alpha} z)F(z)$$

on ∂D, where $|\alpha| < 1$ is a root of f. By doing so we investigate what the consequences are of moving a root from the inside of the unit disc to the outside in this fashion. By writing $F(z) = \sum_{n=0}^{\infty} a_n z^n$ we see that

$$f(z) = (z - \alpha) \sum_{n=0}^{\infty} a_n z^n = -\alpha a_0 + \sum_{n=1}^{\infty} (a_{n-1} - \alpha a_n) z^n$$

and

$$g(z) = (1 - \overline{\alpha} z) \sum_{n=0}^{\infty} a_n z^n = a_0 + \sum_{n=1}^{\infty} (a_n - \overline{\alpha} a_{n-1}) z^n$$

Now we calculate the norms of f and g in X and subtract. We have

$$\begin{aligned} \|f(z)\|_X^2 &= \gamma_0 |\alpha|^2 |a_0|^2 + \sum_{n=1}^{\infty} \gamma_n |a_{n-1} - \alpha a_n|^2 \\ &= \gamma_0 |\alpha|^2 |a_0|^2 + \sum_{n=1}^{\infty} \gamma_n (|a_{n-1}|^2 - \overline{\alpha} a_{n-1} \overline{a_n} - \alpha \overline{a_{n-1}} a_n + |\alpha|^2 |a_n|^2) \end{aligned}$$

$$\begin{aligned} \|g(z)\|_X^2 &= \gamma_0 |a_0|^2 + \sum_{n=1}^{\infty} \gamma_n |a_n - \overline{\alpha} a_{n-1}|^2 \\ &= \gamma_0 |a_0|^2 + \sum_{n=1}^{\infty} \gamma_n (|a_n|^2 - \overline{\alpha} a_{n-1} \overline{a_n} - \alpha \overline{a_{n-1}} a_n + |\alpha|^2 |a_{n-1}|^2) \end{aligned}$$

Remembering that $\gamma_0 = 0$ we have

$$\begin{aligned} \|f(z)\|_X - \|g(z)\|_X &= (1 - |\alpha|^2) \sum_{n=1}^{\infty} \gamma_n (|a_{n-1}|^2 - |a_n|^2) \\ &= (1 - |\alpha|^2) \sum_{n=0}^{\infty} (\gamma_{n+1} - \gamma_n) |a_n|^2 \\ &= (1 - |\alpha|^2) \sum_{n=0}^{\infty} (\gamma_{n+1} - \gamma_n) |a_n|^2 \\ &= (1 - |\alpha|^2) \|F(z)\|_Y^2 \end{aligned}$$

Any $F(z)$ which is analytic in a some domain $\Omega \supset D$ will have at most finitely many zeros in D. Given F let us assume that it has the following list of roots $\{\alpha_1, \alpha_2, ..., \alpha_n\} \subset D$. By construction we know that one root is at the origin, and we assume without loss og generality that it is α_n. Now, consider the following sequence of functions

$$\begin{aligned} F(z) &= (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_{n-1})(z-\alpha_n)H(z) \\ F_1(z) &= (1-\overline{\alpha_1}z)(z-\alpha_2)\cdots(z-\alpha_{n-1})(z-\alpha_n)H(z) \\ F_2(z) &= (1-\overline{\alpha_1}z)(1-\overline{\alpha_2}z)\cdots(z-\alpha_{n-1})(z-\alpha_n)H(z) \\ &\cdots \\ F_{n-1}(z) &= (1-\overline{\alpha_1}z)(1-\overline{\alpha_2}z)\cdots(1-\overline{\alpha_{n-1}}z)(z-\alpha_n)H(z) \end{aligned}$$

where $H(z)$ is some holomorphic function not vanishing in $\overline{D}$. From our previous computation we can conclude that

$$\|F(z)\|_X \geq \|F_1(z)\|_X \geq \cdots \geq \|F_{n-1}(z)\|_X$$

This proves the first statement if we just observe that the outer function G in the Blaschke decomposition $F = BG$ must be of the form

$$G(z) = F_{n-1}(z) = (1-\overline{\alpha_1}z)(1-\overline{\alpha_2}z)\cdots(1-\overline{\alpha_{n-1}}z)(1-\overline{\alpha_n}z)H(z)$$

Now on to the second inequality. We use the fact that $a_n = 0$ is one of the roots and the definition of the norms. We get

$$\begin{aligned} \|F_{n-1}(z)\|_X^2 - \|F_n(z)\|_X^2 &= \left\|\sum_{k=0}^{\infty} \alpha_k z^k\right\|_X^2 - \left\|\sum_{k=0}^{\infty} \alpha_k z^{k+1}\right\|_X^2 \\ &= \sum_{k=0}^{\infty} \gamma_k |\alpha_k|^2 - \sum_{k=0}^{\infty} \gamma_{k+1} |\alpha_k|^2 \\ &= \sum_{k=1}^{\infty} (\gamma_{k+1} - \gamma_n)|\alpha_k|^2 \\ &= \left\|\prod_{k=1}^{n-1} (1-\overline{\alpha_n}z)H(z)\right\|_Y^2 = \|G(z)\|_Y^2 \end{aligned}$$

More generally, if there is not a root at 0, we similarly get

$$\begin{aligned} \|F_{n-1}(z)\|_X^2 - \|F_n(z)\|_X^2 &= (1-|\alpha_n|^2)\left\|\prod_{k=1}^{n-1}(1-\overline{\alpha_n}z)H(z)\right\|_Y^2 \\ &= (1-|\alpha_n|^2)\left\|\frac{G(z)}{1-\overline{\alpha_n}z}\right\|_Y^2 \end{aligned}$$

Because we investigated this restricted to the unit circle, we have proved the claim. □

An important consequence of this result is the convergence of $\|G_n\|_Y$ if F lies in X. This is seen as follows. The unwinding series is constructed by setting

$$G_n(0) - G_n(z) = B_{n+1}(z)G_{n+1}(z)$$

so by construction, if we write $F_n = G_n(z) - G_n(0)$, we know that $F_n(0) = 0$. If we apply the second part of our result, we then get

$$\|G_{n+1}\|_X^2 \leq \|F_n\|_X^2 - \|G_{n+1}\|_Y^2$$

Notice now that if we add a constant to a function the X norm does not change, because $\gamma_0 = 0$. If we now rearrange we get

$$\|G_{n+1}\|_Y^2 \leq \|G_n(z) - G_n(0)\|_X^2 - \|G_{n+1}\|_X^2 \leq \|G_n\|_X^2 - \|G_{n+1}\|_X^2$$

If we construct a telescoping series this becomes

$$\sum_{n=2}^{\infty} \|G_n\|_Y^2 \leq \|F\|_X^2$$

due to the first part of the theorem.

This of course implies that $\|G_n\|_Y \to 0$ as $n \to \infty$, and the unwinding series converges. If we, after n steps in the process, call $G_n(0) = a_n$, we have

$$F = a_1B_1 + a_2B_1B_2 + \cdots + a_{n-1}B_1 \cdots B_n + B_1 \cdots B_n(G_n - G_n(0))$$

Then remembering that $|B_i| = 1$ when $|z| = 1$ we see that

$$\|F - (a_1B_1 + a_2B_1B_2 + \cdots + a_{n-1}B_1 \cdots B_n)\|_{L^2(\partial D)} = \|G_n - G_n(0)\|_{L^2(\partial D)}$$

This equality is interesting in its own right. Moreover, it motivates studying the space X arising from $\gamma_n = n$, because in this case $Y = L^2$. The space X, in this case, is known as the Dirichlet space on the unit disk, $\mathcal{D}$. This space one the unit disk is defined as

$$\mathcal{D}(D) = \left\{ f \in H^2(D) : \sum_{n=1}^{\infty} n|a_n|^2 < \infty \right\}$$

and has the norm

$$\|f\|_{\mathcal{D}}^2 = \sum_{n=0}^{\infty} (n+1)|a_n|^2$$

This space has some particular geometric properties which gives us better inequalities than in the general X and Y cases, which we will show in a bit. First, we mention another special case. By letting $\gamma_n = n^2$, the space X becomes the Sobolev space (often denoted by H^1, but since we have denoted the Hardy spaces by H we leave in in a parenthesis) and Y becomes $\mathcal{D}(D)$. Let us denote the Sobolev space H^1 as $\mathcal{S}$. The general Sobolev spaces are of particular significance when studying differential equations and appear in physical applications. We have

$$\mathcal{S} = \left\{ f \in L^2(\mathbb{T}) : \sum_{n=-\infty}^{\infty} (1+n^2)|a_n|^2 \right\}$$

and has the norm

$$\|f\|_S^2 = \sum_{n=-\infty}^{\infty} (|n|+1)^2 |a_n|^2$$

These next few results can be generalized to various spaces, for example this next result has a completely analogous version on the upper half-plane $\mathbb{C}_+$.

Let us just mention that our procedure of unwinding these functions can be slightly more generalized. Instead of adding and subtracting $G_n(0)$ in each step, we may instead add and subtract $G_n(\alpha)$ for some $|\alpha| < 1$. In each step, $G_n(z) - G_n(\alpha)$ has a zero in D. We could even vary what we choose to be our α in every step. We stated theorem 31 in the general way, and as we see it comes with the cost of adding a factor of $1 - |\alpha|^2$.

2.4 Special case

Let us now study some specific cases which seem to have a natural geometric significance. We will be considering functions F which are holomorphic in some neighbourhood of the unit disk as maps $\gamma_F : \mathbb{T} \to \mathbb{R}^2$ where

$$\gamma_F(t) = F(e^{it})$$

From our previous discussions and from figure 2.4 one would expect the length of γ_F to be less than that of γ_G. We have been unable to prove this for a general function, but some relation between an L^2 version of length and what we call the energy of the curves has been established. Recall that the energy of a function is defined as

$$\text{energy}(\gamma) = \int_0^{2\pi} |\gamma'(t)|^2 dt$$

By a simple application of Hölder's inequality we get

$$\text{length}(\gamma)^2 = \left(\int_0^{2\pi} |\gamma'(t)| dt \right)^2 \leq 2\pi \int_0^{2\pi} |\gamma'(t)|^2 dt = 2\pi \text{energy}(\gamma)$$

So at least we can see that if the energy of a curve tends to zero, so does this L^2 length. The next statement allows us to quantify this in a more specific way.

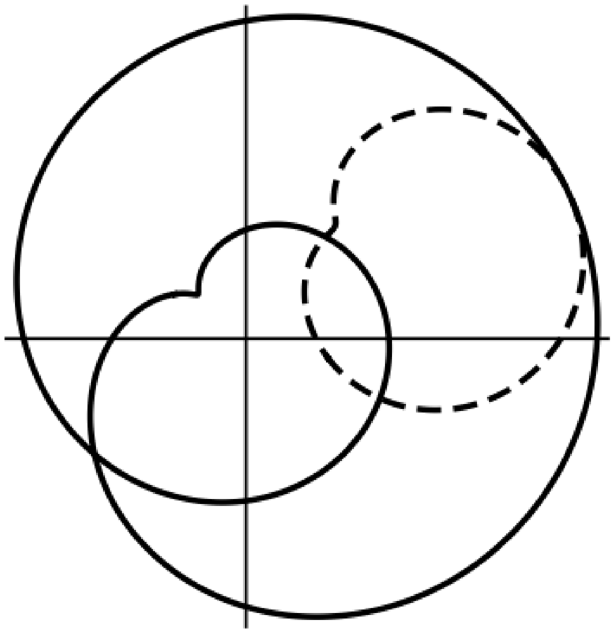

Figure 2.2: Here we see $F(e^{it})$ for a cubic polynomial, and in dashed we see the corresponding function $G(e^{it})$.

Theorem 32. *Let $F : D \to \mathbb{C}$ be holomorphic in some neighbourhood of the unit disk. Then, if $\{\alpha_j : j \in J\}$ are the roots of F in D and $F = BG$, then*

$$\int_0^{2\pi} |G'(e^{it})|^2 dt \leq \int_0^{2\pi} |F'(e^{it})|^2 dt - \frac{1}{2\pi} \int_0^{2\pi} |G(e^{it})|^2 \sum_{j \in J} \frac{1 - |\alpha_j|^2}{|e^{it} - \alpha_j|^2} dt$$

In proving this we first prove the following lemma.

Lemma 33. *Let F be holomorphic in a neighbourhood of the origin and $a \in \mathbb{C}$ with $|\alpha| < 1$. If*

$$f = (z - \alpha)F \quad \textit{and} \quad g = (1 - \overline{\alpha} z)F$$

then

$$\int_0^{2\pi} |g'(e^{it})|^2 dt \leq \int_0^{2\pi} |f'(e^{it})|^2 dt - (1 - |\alpha|^2) \int_0^{2\pi} |F(e^{it})|^2 dt$$

whenever all terms are defined and finite.

Proof. Let us start out with some simple calculations. We have

$$f' = F + (z - \alpha)F' \quad \text{and} \quad g' = -\overline{\alpha}F + (1 - \overline{\alpha}z)F'$$

This gives us the following,

$$|f'|^2 = |F|^2 + F\overline{(z-\alpha)F'} + \overline{F}(z-\alpha)F' + |z-\alpha|^2|F'|^2$$

and

$$|g'|^2 = |\alpha|^2|F|^2 - \alpha\overline{F}(1-\overline{\alpha}z)F' - \overline{\alpha}F\overline{(1-\overline{\alpha}z)F'} + |1-\overline{\alpha}z|^2|F'|^2$$

Then we have $|z-\alpha|^2 = |z|^2|1-\alpha/z|^2$ which if $|z| = 1$ gives $|z-\alpha|^2 = |1-\overline{\alpha}z|^2$. This yields

$$\int_{\partial D} |f'|^2 - |g'|^2 = (1-|\alpha|^2)\int_{\partial D} |F|^2 + \int_{\partial D} \Big(F\overline{(z-\alpha)F'} + \overline{F}(z-\alpha)F' \\ + \alpha\overline{F}(1-\overline{\alpha}z)F' + \overline{\alpha}F\overline{(1-\overline{\alpha}z)F'}\Big)$$

As we can see, we are not far from our goal. By isolating $\int_{\partial D} |g'|^2$ we see that the lemma is proven if we show

$$\int_{\partial D} F\overline{(z-\alpha)F'} + \overline{F}(z-\alpha)F' + \alpha\overline{F}(1-\overline{\alpha}z)F' + \overline{\alpha}F\overline{(1-\overline{\alpha}z)F'} \leq 0$$

This will be done by rewriting the term, and eventually using Green's theorem.

As a first step we notice that we may rewrite the integrand as

$$F\overline{F'}(\overline{(z-\alpha)} + \overline{\alpha}\overline{(1-\overline{\alpha}z)}) + \overline{F}F'((z-\alpha) + \alpha(1-\overline{\alpha}z))$$

which equals

$$(1-|\alpha|^2)(F\overline{F'}\overline{z} + \overline{F}F'z)$$

If we for a second move from the standard derivative $h'(z)$ to the angular derivative along the boundary of the unit circle $\dot{h}(e^{i\theta})$, we have

$$\frac{d}{d\theta}h(e^{i\theta}) = h'(e^{i\theta})ie^{i\theta} \quad \text{or} \quad h'(e^{i\theta}) = -ie^{-i\theta}\dot{h}(e^{i\theta})$$

Because we are working on the unit circle, we then get the following

$$F\overline{z}\overline{F'} = F\overline{z}\overline{(-i)}z\overline{\dot{F}} = iF\overline{\dot{F}}$$
$$F'\overline{z}\overline{F} = -iz\overline{z}\dot{F}\overline{F} = -i\dot{F}\overline{F}$$

Therefore, our problem is to show that

$$i\int_{\partial D} F\overline{\dot{F}} - \dot{F}\overline{F} \geq 0$$

This is where we turn to Green's theorem. First we write

$$F(e^{it}) = x(t) + iy(t)$$

We then have

$$i(F\overline{\dot{F}} - \dot{F}\overline{F}) = 2(x\dot{y} - \dot{x}y)$$

and we notice that

$$2(-y, x) \cdot (\dot{x}, \dot{y}) = 2(x\dot{y} - \dot{x}y)$$

Our problem is then reduced to integrating the vector field $2(-y(t), x(t))$ over the curve $\gamma(t) = (x(t), y(t))$. This is, by Green's theorem, equal to a nonnegative constant times an area, thus greater than or equal to 0. □

No we move on to proving the theorem.

Proof. The Dirichlet space on the unit disk is a space of functions and is contained in $H^2(D)$. Furthermore, in our case the Dirichlet norm for a function $F(z) = \sum_{n=0}^{\infty} a_n z^n$ can be written as

$$\|F(z)\|_{\mathcal{D}}^2 = \sum_{n=1}^{\infty}(n+1)|a_n|^2$$

We may again compare

$$f = (z - \alpha)F \quad \text{and} \quad g = (1 - \overline{\alpha}z)F$$

Some simple calculations yield

$$\|f(z)\|_{\mathcal{D}}^2 - \|g(z)\|_{\mathcal{D}}^2 = (1 - |\alpha|^2) \sum_{n=0}^{\infty} |a_n|^2$$

Which gives us

$$\|f(z)\|_{\mathcal{D}}^2 - \|g(z)\|_{\mathcal{D}}^2 = (1 - |\alpha|^2)\|F(z)\|_{L^2}^2 = (1 - |\alpha|^2) \left\| \frac{f(z)}{z - \alpha} \right\|_{L^2}^2$$

Now because $|F_i| = |F_j|$ and $G = F_n$ we apply this to F and G to land on

$$\|F\|_{\mathcal{D}}^2 - \|G\|_{\mathcal{D}}^2 = \frac{1}{2\pi} \int_0^{2\pi} |G(e^{it})|^2 \sum_{k=1}^{n} \frac{1 - |\alpha_k|^2}{|e^{it} - \alpha_k|^2} dt$$

□

This next result speaks to the L^∞ convergence of $G_n - G_n(0)$. We use an argument based on the uncertainty principle and projections. $a \sim b$ means either quantity can be made greater than the other, and $a \lesssim b$ means some constant times a is greater than b.

Corollary 34. *Suppose $F : D \to \mathbb{C}$ converges on some neighbourhood of the unit disk. Then the formal series converges in L^∞. Moreover,*

$$|\{n \in \mathbb{N} : \|G_n(z) - G_n(0)\|_{L^\infty(\partial D)} \geq \varepsilon\}| \lesssim \left(\int_0^{2\pi} |F'(e^{it})|^2 dt\right)^2 /\varepsilon^4$$

Lemma 35. *Let $h : \mathbb{T} \to \mathbb{R}$ be a differentiable function which changes sign. then*

$$\|h\|_{L^2}^2 \geq \frac{1}{16}\frac{\|h\|_{L^\infty}^4}{\|h'\|_{L^2}^2}$$

Proof. Without loss of generality we assume that $h(0) = 0$. Now we let choose x such that $|h(x)| = \|h\|_{L^\infty}$. Then, by the fundamental theorem of calculus and Hölder's inequality we have

$$\|h\|_{L^\infty}^2 = |h(x)|^2 = \left|\int_0^x 2h(t)h'(t)dt\right| \leq 4\left(\int_0^x |h(t)|^2 dt\right)^{1/2}\left(\int_0^x |h'(t)|^2 dt\right)^{1/2}$$

Then, since the integrands are both positive, we may extend the integrals from x to 2π, and by squaring both sides we obtain

$$\|h\|_{L^\infty}^2 \leq 16\|h\|_{L^2}^2\|h'\|_{L^2}^2$$

This proves the lemma. □

Lemma 36. *Let $\gamma : \mathbb{T} \to \mathbb{R}^2$ be a periodic curve in the plane and assume that $\gamma(0) \neq (0,0)$. Then there exists a unit vector ν with*

$$|\langle\gamma(0), \nu\rangle| \geq \frac{1}{\sqrt{2}}|\gamma(0)|$$

and

$$\int_0^{2\pi} |\gamma(t)|^2 dt \leq 6\int_0^{2\pi} |\langle\nu, \gamma(t)\rangle|^2 dt$$

Proof. The first inequality basically means that the vector ν forms an angle with the vector from the origin to $\gamma(0)$ of degree less than or equal to 45. So we want to find a vector which does this and satisfies the second inequality. We will show that such a ν exists by finding the mean of all such ν and deducing the fact from there. The following calculation will prove useful in that regard. Let $l \in \mathbb{R}^2$ and let $0 \leq \alpha \leq 2\pi$. We have

$$\begin{aligned}\int_{\alpha-\pi/4}^{\alpha+\pi/4} |\langle(\cos t, \sin t), l\rangle|^2 dt &\geq |l|^2\int_{\alpha-\pi/4}^{\alpha+\pi/4} |\langle(\cos t, \sin t), (0,1)\rangle|^2 dt \\ &\geq |l|^2\int_{-\pi/4}^{\pi/4}(\sin t)^2 dt = |l|^2\frac{\pi-2}{4}\end{aligned}$$

Now we compute the expectation as mentioned. We have

$$\begin{aligned}\frac{2}{\pi}\int_{\alpha-\pi/4}^{\alpha+\pi/4}\int_0^{2\pi}|\langle(\cos s,\sin s),\gamma(t)\rangle|^2 dtds &= \frac{2}{\pi}\int_0^{2\pi}\int_{\alpha-\pi/4}^{\alpha+\pi/4}|\langle(\cos s,\sin s),\gamma(t)\rangle|^2 dsdt \\ &\geq \frac{2}{\pi}\frac{\pi-2}{4}\int_0^{2\pi}|\gamma(t)|^2 dt\end{aligned}$$

Here we see that since the integrand is nonnegative, Fubini's theorem applies and the change of integral order is justified.

Notice that $\frac{2(\pi-2)}{4\pi}\geq\frac{1}{6}$, so the expectation of such $|\nu|=1$ satisfies

$$6\int_0^{2\pi}|\langle\nu,\gamma(t)\rangle|^2 dt\geq\int_0^{2\pi}|\gamma(t)|^2 dt$$

Because of this, there must be at least one ν which also satisfies it. This completes the argument. □

Proof. Now suppose that for some n and some $z\in\partial D$ we have

$$|G_n(z)-G_n(0)|\geq\varepsilon$$

If we show that such n are bounded by, say

$$\frac{\int_0^{2\pi}|F'(e^{it})|^2 dt}{\varepsilon^4}$$

we have completed the proof. With this aim, our first step is to identify $G_n(z)-G_n(0)$ on the unit circle with a curve $\gamma:\mathbb{T}\to\mathbb{R}$ such that $|\gamma(0)|\geq\varepsilon$. By our lemma, we then have

$$|\langle\gamma(0),\nu\rangle|\geq\frac{\varepsilon}{\sqrt{2}}$$

and

$$\int_0^{2\pi}|\gamma(t)|^2 dt\leq 6\int_0^{2\pi}|\langle\nu,\gamma(t)\rangle|^2 dt$$

We now apply the other lemma to the function

$$h(t)=\langle\gamma(t),\nu\rangle$$

$$|h'(t)|=|\langle\gamma'(t),\nu\rangle|\leq|\gamma'(t)|$$

So applying the lemma we have

$$\int_0^{2\pi}|G_n(e^{it})-G_n(0)|^2 dt\gtrsim\|h\|_{L^2}^2\gtrsim\frac{\|h\|_{l^\infty}^4}{\|h'\|_{L^2}^2}\geq\frac{\varepsilon^4}{\int_0^{2\pi}|F'(e^{it})|^2 dt}$$

We now use the fact that $|B| = 1$ on the unit disk and $G_n(z) - G_n(0) = B(z)G_{n+1}(z)$.

$$\int_0^{2\pi} |G_{n+1}(e^{it})|^2 dt = \int_0^{2\pi} |G_n(e^{it}) - G_n(0)|^2 dt \gtrsim \frac{\varepsilon^4}{\int_0^{2\pi} |F'(e^{it})|^2 dt}$$

$$\begin{aligned}\int_0^{2\pi} |G'_{n+1}(e^{it})|^2 dt &\leq \int_0^{2\pi} |G'_n(e^{it})|^2 dt - \int_0^{2\pi} |G_n(e^{it}) - G_n(0)|^2 dt \\ &\leq \int_0^{2\pi} |G'_n(e^{it})|^2 dt - c\frac{\varepsilon^4}{\int_0^{2\pi} |F(e^{it})|^2 dt}\end{aligned}$$

This gives us

$$c\frac{\varepsilon^4}{\int_0^{2\pi} |F(e^{it})|^2 dt} \leq \int_0^{2\pi} |G'_n(e^{it})|^2 dt - \int_0^{2\pi} |G'_{n+1}(e^{it})|^2 dt$$

and by creating a telescoping series, this yields

$$c\frac{n\varepsilon^4}{\int_0^{2\pi} |F(e^{it})|^2 dt} \leq \int_0^{2\pi} |F'(e^{it})|^2 dt - \int_0^{2\pi} |G'_{n+1}(e^{it})|^2 dt$$

We now have a bound for n, namely

$$n \lesssim \frac{\left(\int_0^{2\pi} |F'(e^{it}|^2 dt\right)^2}{\varepsilon^4}$$

which completes the argument. □

2.5 Winding numbers and the Dirichlet space

Here we will discuss some properties of closed curves in $\mathbb{C}$ given by $\gamma_F(t) = F(e^{it})$. Our discussion will lead to a bettering of theorem 31. A lot of the motivation for this comes from geometric considerations. The winding number of a curve in the complex plane γ about a point z_0 is defined to be

$$\text{wind}_\gamma(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - z_0}$$

We will refer to the quantity

$$\int_{\mathbb{C}} \text{wind}_\gamma(z) dz$$

as the average weighted winding number. This can be thought of as a measure of the area enclosed by the curve γ but weighted by the winding number of each respective point. Considering our previous discussions and illustrations of the simplification which takes place when looking at G instead of F we would intuitively assume that this quantity be lower for G than F. This is the question which our next result somewhat successfully attempts to answer. We obtain a nice quantification of the decrease in average weighted winding number. This quantity naturally arises if one applies Green's formula to compute the area surrounded by a simple, closed curve $\gamma : [0, 2\pi] \to \mathbb{R}^2$ oriented counter-clockwise and written as $\gamma(t) = (x(t), y(t))$ via

$$\frac{1}{2}\int_0^{2\pi} x(t)\dot{y}(t) - \dot{x}(t)y(t)dt$$

If we do the same thing, but this time with a non-simple, closed curve, we get

$$\frac{1}{2}\int_0^{2\pi} x(t)\dot{y}(t) - \dot{x}(t)y(t)dt = \int_{\mathbb{C}} \text{wind}_\gamma(z)dz$$

This is not a new phenomenon and dates back to at least 1936. We have a neat result about this, if F is holomorphic, then we have

$$\int_{\mathbb{C}} \text{wind}_{\gamma_F}(z)dz = \int_D |F'(z)|^2 dz$$

Let us use an alternate definition of the Dirichlet space which will be more useful for now. We have

$$\mathcal{D} = \left\{ f : D \to \mathbb{C} : f \quad \text{holomorphic and} \int_D |f'(z)|^2 dz < \infty \right\}$$

When equipped with the norm

$$\langle f, g\rangle_{\mathcal{D}} = \langle f, g\rangle_{H^2} + \frac{1}{\pi}\int_D f'(z)\overline{g'(z)}dz$$

this space becomes a Hilbert space, and we have briefly touched on its geometrical significance. With this space properly introduced, let us move on to the result.

Corollary 37. *Assume $F \in H^\infty$ with roots $\{\alpha_j : j \in J\}$ in D and has the Blaschke factorization $F = BG$, then*

$$\int_D |F'(z)|^2 dz = \int_D |G'(z)|^2 dz + \frac{1}{2}\int_0^{2\pi} |G(e^{it})|^2 \sum_{j\in J} \frac{1 - |\alpha_j|^2}{|e^{it} - \alpha_j|^2} dt$$

In proving this we will need some other results, and we will show them first.

Theorem 38. *Let f be a holomorphic function on D and let $f(z) = \sum_{k=1}^{\infty} a_k z^k$. Then*

$$\frac{1}{\pi}\int_D |f'(z)|^2 dA(z) = \sum_{k=1}^{\infty} k|a_k|^2$$

Proof. To start out, we do the substitution $z = re^{i\theta}$. Then we calculate,

$$\begin{aligned}\frac{1}{\pi}\int_D |f'(z)|^2 dA(z) &= \frac{1}{\pi}\int_0^1\int_0^{2\pi}\left|\sum_{k=1}^{\infty} k a_k z^{k-1}\right|^2 r d\theta dr \\ &= \frac{1}{\pi}\int_0^1\int_0^{2\pi}\left|\sum_{k=1}^{\infty} k a_k r^{k-1} e^{i\theta(k-1)}\right|^2 r d\theta dr \\ &= 2\int_0^1 \sum_{k=1}^{\infty}\left|k a_k r^{k-1}\right|^2 r d\theta dr \\ &= \sum_{k=1}^{\infty} k|a_k|^2\end{aligned}$$

Here we used Parseval's identity. □

Theorem 39. *Let D_1 and D_2 be domains, let $\phi : D_1 \to D_2$ be a conformal mapping and let $f : D_2 \to \mathbb{C}$ be a holomorphic function. Then*

$$\int_{D_1} \left|(f \circ \phi)'(z)\right|^2 dA(z) = \int_{D_2} \left|f'(w)\right| dA(w)$$

Proof. Again we start out with a substitution, namely $w = \phi(z)$. By the Jacobian we obtain $dA(w) = |\phi'(z)|^2 dA(z)$. This yields

$$\int_{D_2} |f'(w)|^2 dA(w) = \int_{D_1} |f'(\phi(z))|^2 |\phi'(z)|^2 dA(z) = \int_{D_1} |(f \circ \phi)'(z)|^2 dA(z)$$

□

The next result follows at once if we let $D_1 = D_2 = D$.

Corollary 40. *If f is a holomorphic function and ϕ is an automorphism of the unit disk, then*

$$\frac{1}{\pi}\int_D |(f \circ \phi)'(z)|^2 dA(z) = \frac{1}{\pi}\int_D |f'(z)|^2 dA(z)$$

This tells us that the energy of the curve $f(z)$ is invariant under these automorphisms. It is worth noting that the automorphisms on D are precisely the Möbius transformations of the form

$$\phi(z) = e^{i\theta}\frac{a - z}{1 - \bar{a}z}, \quad \text{where } a \in D$$

Now, we have the following result,

Theorem 41. *Let $f \in H^2$ and let B be a Blaschke product. Let $\{\alpha_j : j \in J\}$ be the roots of B in D. Then*

$$\frac{1}{\pi}\int_D |(Bf)'(z)|^2 dA(z) = \frac{1}{\pi}\int_D |f'(z)|^2 dA(z) + \sum_{j \in J}\frac{1}{2\pi}\int_0^{2\pi}\frac{1 - |\alpha_j|^2}{|e^{i\theta} - \alpha_j|^2}|f(e^{i\theta})|^2 d\theta$$

Notice now that by proving this theorem and letting the G in corollary 37 equal f we have proven the corollary. Let us do that now.

Proof. First, we suppose that $B(z) = z$. Then the only zeros of B are for $z = 0$, and by theorem 38 we have

$$\frac{1}{\pi}\int_D |(zf)'(z)|^2 dA(z) = \sum_{k=0}^{\infty}(k+1)|a_k|^2 = \frac{1}{\pi}\int_D |f'(z)|^2 dA(z) + \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2 d\theta$$

and we are done.

Now we let $B(z) = e^{i\theta}(\alpha_1 - z)/(1 - \overline{\alpha_1}z)$, where $0 < |z_1| < 1$. Using the invariance result of corollary 40 we have

$$\frac{1}{\pi}\int_D |(Bf)'(z)|^2 dA(z) = \frac{1}{\pi}\int_D |(z(f \circ B^{-1}))'(z)|^2 dA(z)$$

Then, by a change in variables we have

$$\begin{aligned}
\frac{1}{\pi}\int_D |(Bf)'(z)|^2 dA(z) &= \frac{1}{\pi}\int_D |(f \circ B^{-1})'(z)|^2 dA(z) + \frac{1}{2\pi}\int_{\mathbb{T}} |f \circ B^{-1}(\zeta)|^2 |d\zeta| \\
&= \frac{1}{\pi}\int_D |f'(z)|^2 dA(z) + \frac{1}{2\pi}\int_{\mathbb{T}} |f'(z)|^2 |B'(z)||dz| \\
&= \frac{1}{\pi}\int_D |f'(z)|^2 dA(z) + \frac{1}{2\pi}\int_{\mathbb{T}} |f(z)|^2 \frac{1 - |\alpha_1|^2}{|z - \alpha_1|^2}|dz| \\
&= \frac{1}{\pi}\int_D |f'(z)|^2 dA(z) + \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2 \frac{1 - |\alpha_1|^2}{|e^{i\theta} - \alpha_1|^2} d\theta
\end{aligned}$$

This shows the claim for such a B and the case for any finite Blaschke product follows immediately by induction.

The general case takes a little more effort. Let b_n be the product of the first n terms of the Blaschke product and let B_n be the product of the remaining terms. We know that $b_n f \to Bf$ uniformly on compact subsets of D and we have proved the claim for any finite Blaschke product,

$$\frac{1}{\pi}\int_D |(b_n f)'(z)|^2 = \frac{1}{\pi}\int_D |f'(z)|^2 + \sum_{k=1}^{n} \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2 \frac{1 - |\alpha_k|^2}{|e^{i\theta} - \alpha_k|^2} d\theta$$

Thus, we have

$$\frac{1}{\pi}\int_D |(Bf)'(z)|^2 \leq \liminf_{n\to\infty} \frac{1}{\pi}\int_D |(b_n f)'(z)|^2$$

which implies that

$$\frac{1}{\pi}\int_D |(Bf)'(z)|^2 \leq \frac{1}{\pi}\int_D |f'(z)|^2 + \sum_{k=1}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2 \frac{1 - |\alpha_k|^2}{|e^{i\theta} - \alpha_k|^2} d\theta$$

Now what remains is the reverse inequality. By writing $Bf = b_n B_n f$ we have

$$\frac{1}{\pi}\int_D |(Bf)'(z)|^2 = \frac{1}{\pi}\int_D |B_n f'(z)|^2 + \sum_{k=1}^{n}\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})B_n(e^{i\theta})|^2 \frac{1-|\alpha_k|^2}{|e^{i\theta}-\alpha_k|^2} d\theta$$
$$= \frac{1}{\pi}\int_D |B_n f'(z)|^2 + \sum_{k=1}^{n}\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2 \frac{1-|\alpha_k|^2}{|e^{i\theta}-\alpha_k|^2} d\theta$$

Then, $B_n f \to f$ uniformly on compact subsets of D, so

$$\liminf_{n\to\infty} \frac{1}{\pi}\int_D |(B_n f)'(z)|^2 \geq \frac{1}{\pi}\int_D |f'(z)|^2$$

and

$$\frac{1}{\pi}\int_D |(Bf)'(z)|^2 \geq \frac{1}{\pi}\int_D |f'(z)|^2 + \sum_{k=1}^{\infty}\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2 \frac{1-|\alpha_k|^2}{|e^{i\theta}-\alpha_k|^2} d\theta$$

This concludes the proof. □

The key process of this section is the Blaschke decomposition. If we were to attempt to numerically do this process, we would naturally introduce roundoff errors. These errors can be thought of as perturbing the roots of the functions a little. This leads us to this next small result about the pointwise stability of the Blaschke decomposition.

Theorem 42. *Suppose $F_1, F_2 : \mathbb{C} \to \mathbb{C}$ are polynomials having the same roots outside D, and the same number of roots inside D. Then the Blaschke factorizations*

$$F_1 = B_1 G_1 \quad \textit{and} \quad F_2 = B_2 G_2$$

satisfy

$$|G_1(z) - G_2(z)| = |F_1(z) - F_2(z)| \quad \textit{whenever} |z| = 1$$

Proof. We easily see that this is a pointwise statement. Also, it is invariant under multiplication by a polynomial which has its roots outside the unit disk, because then all the contributing factors would appear in both F and G. Threrefore it suffices to consider only

$$F_1(z) = \prod_{i=1}^{n}(z-\alpha_i) \quad \text{and} \quad F_2(z) = \prod_{i=1}^{n}(z-\beta_i)$$

We then have

$$B_1(z) = \prod_{i=1}^{n}\frac{z-\alpha_i}{1-\overline{\alpha_i}z} \quad \text{and} \quad B_2(z) = \prod_{i=1}^{n}\frac{z-\beta_i}{1-\overline{\beta_i}z}$$

as well as

$$G_1(z) = \prod_{i=1}^{n}(1 - \overline{\alpha_i} z) \quad \text{and} \quad G_2(z) = \prod_{i=1}^{n}(1 - \overline{\beta_i} z)$$

Now we subtract G_2 from G_1 and F_2 from F_1 to get these expressions in a more useful form. Let A_k and B_k be arbitrary choices of k elements from the set $\{1, 2, \ldots, n\}$. We have

$$G_1(z) - G_2(z) = \sum_{k=1}^{n} z^k \left(\sum_{A_k} \prod_{j \in A_k} (-\overline{\alpha_j}) - \sum_{A_k} \prod_{j \in A_k} (-\overline{\beta_j}) \right)$$

and

$$F_1(z) - F_2(z) = \sum_{k=1}^{n} z^k \left(\sum_{A_{n-k}} \prod_{j \in A_{n-k}} (-\alpha_j) - \sum_{A_{n-k}} \prod_{j \in A_{n-k}} (-\beta_j) \right)$$

From this, we immediately see that if $G_1(z) - G_2(z) = \sum_{k=1}^{n} c_k z^k$ then $F_1(z) - F_2(z) = \sum_{k=1}^{n} \overline{c_{n-k}} z^k$. Now we must show that if $|z| = 1$ these terms have equal norm. Now, let $|z| = 1$. We have

$$\begin{aligned}
|F_1(z) - F_2(z)| &= \left| \sum_{k=1}^{n} \overline{c_{n-k}} z^k \right| &&= \left| \frac{1}{z^n} \sum_{k=1}^{n} \overline{c_{n-k}} z^k \right| = \left| \sum_{k=1}^{n} \overline{c_{n-k}} z^{k-n} \right| \\
&= \left| \sum_{k=1}^{n} \overline{c_{n-k}} \; \overline{z}^{n-k} \right| &&= \left| \sum_{k=1}^{n} c_{n-k} z^{n-k} \right| = \left| \sum_{k=1}^{n} c_n z^n \right| \\
&= |G_1(z) - G_2(z)|
\end{aligned}$$

□

This statement was discovered by accident and leads to the question of how many similar statements there are that we do not know of yet. Also this goes to show how little we actually do know.

Chapter 3

Holomorphy and Carrier frequencies

On the real line, the Fourier transform of $\mathcal{H}s(t)$ is $-i\text{sign}(t)\hat{s}(t)$. This causes the Fourier transform of $s^+(t)$ to vanish for all negative values of t. In particular, we have

$$s^+(t) = \frac{1}{2\pi}\int_0^\infty \widehat{s^+}(\omega)e^{it\omega}d\omega$$

This suggests that s^+ is a "linear combination" of terms of nonnegative frequencies. By again regarding s^+ as a "linear combination", the phase derivative of $s^+(t) = A(t)e^{i\phi(t)}$ would have to satisfy $\phi'(t) \geq 0$. Unfortunately, this is not true. The main result of this chapter which is from [3] enables us to quantify how close certain functions in signal analysis is to consisting only of terms of nonnegative frequency. We will be presenting an inequality which states that the part of the function which consists of negative frequencies is small.

Nonnegativity of the frequency of signals is important in some sense and deserves a mention. Primarily this is important because this is how signals arise in physical practice. It also makes quantities like the mean of a signal make sense. This could be important and give more information about a signal, or at the very least unlock additional research topics.

Regarding the question of instantaneous frequency, there have been proposed several different approaches. Some of these include empirical mode decomposition, the sparsity approach, the approximation approach as well as short time Fourier transform, continuous wavelet transform, Chirplet transform, S-transform, the synchrosqueezing transform and more. All these methods have advantages and disadvantages. Roughly, these various methods can be split into two groups. One group tries to obtain the time-frequency representation of the signal, while the other group composes the signal into oscillatory parts before extracting the amplitude modulation and instantaneous frequency information. These oscillatory parts are often called intrinsic mode functions and we will introduce one way one might define these below.

3.1 Intrinsic mode functions

It is necessary to put some restrictions on a function to be able to present, and prove our main result. Recall that we will show that the part of our function which consists of negative frequencies is small.

Definition 5. *A periodic, continuous function $f : \mathbb{R} \to \mathbb{C}$ is said to be of intrinsic mode type with accuracy $\varepsilon > 0$ if for $f(t) = A(t)e^{i\phi(t)}$ we have*

$$
\begin{aligned}
&A \in C^1(\mathbb{T}, \mathbb{R}_+) && \phi \in C^2(\mathbb{T}, \mathbb{T}) \\
&\inf_{t\in\mathbb{R}} \phi'(t) > 0 && \sup_{t\in\mathbb{R}} \phi'(t) < \infty \\
&|A'(t)| \le \varepsilon\phi'(t) && |\phi''(t)| \le \varepsilon\phi'(t)
\end{aligned}
$$

This definition guarantees a number of things. Firstly, the function f winds counter-clockwise around the origin and its change in modulus is restricted by this angular motion. Moreover, $|\phi''(t)| \le \varepsilon\phi'(t)$ makes sure that we have a certain level of control of ϕ'. This is important to ensure that when one samples a signal with a finite amount of samples, ϕ' cannot vary to much in between these samples. This makes us able to get some understanding of ϕ'.

3.2 Main result

Now comes the main result of [3]. It states that intrinsic mode functions are close to being holomorphic. In the statement and proof we will use the Littlewood-Paley projections P_- and P_+, where P_- projects a function to its components of negative frequency, and P_+ projects a function on to its components of nonnegative frequencies. Also P_-f and P_+f are referred to as the ani-holomorphic and the holomorphic parts of the function $f = P_-f + P_+f$. Formally, if $f = \sum_{-\infty}^{\infty} a_n e^{int}$, we have

$$
P_-f = \sum_{n=1}^{\infty} a_{-n}e^{-int} \quad \text{and} \quad P_+f = \sum_{n=0}^{\infty} a_n e^{int}
$$

We then have the following result.

Theorem 43. *Let $f = A(t)e^{i\phi(t)}$ be an intrinsic mode function with accuracy $\varepsilon > 0$. Then*

$$
\|P_-f\|_{L^2}^2 \le \left(\frac{8\pi^2}{\|A\|_{L^2}^2} \frac{\|A'\|_{L^\infty}^2 + \varepsilon^2\|A\|_{L^\infty}^2}{\inf\limits_{0<t<2\pi} \phi'(t)} \right) \|f\|_{L^2}^2
$$

Proof. We know that f equals its Fourier series on the unit circle, $f(t) = \sum_{-\infty}^{\infty} a_n e^{int}$. Because the Littlewood-Paley projection P_- only captures the negative frequencies of f this means that $P_- f = \sum_{n=1}^{\infty} a_{-n} e^{-int}$. Consequently, we have

$$\|P_- f\|_{L^2(\mathbb{T})}^2 = \sum_{n=1}^{\infty} \left| \int_0^{2\pi} A(t) e^{i\phi(t)+int} dt \right|^2$$

Let us now investigate an isolated term from this sum. If we use integration by parts we have

$$\int_0^{2\pi} A(t) e^{i\phi(t)+int} dt = -\int_0^{2\pi} \frac{A'(t)\left(i\phi'(t)+in\right) - A(t) i\phi''(t)}{\left(i\phi'(t)+in\right)^2} e^{i\phi(t)+int} dt$$

by noticing that the first part equals 0. Now if we take absolute values of each side, we have

$$\begin{aligned}\left| \int_0^{2\pi} A(t) e^{i\phi(t)+int} dt \right| &\leq \int_0^{2\pi} \left| \frac{A'(t)\left(i\phi'(t)+in\right)}{\left(i\phi'(t)+in\right)^2} - \frac{A(t) i\phi''(t)}{\left(i\phi'(t)+in\right)^2} e^{i\phi(t)+int} \right| dt \\ &\leq \int_0^{2\pi} \left| \frac{A'(t)}{i\phi'(t)+in} \right| + \left| \frac{A(t)\phi''(t)}{\left(i\phi'(t)+in\right)^2} \right| dt\end{aligned}$$

If we now consider the first term, we have the following bound

$$\int_0^{2\pi} \left| \frac{A'(t)}{\phi'(t)+n} \right| \leq 2\pi \frac{\|A'\|_{L^\infty}}{\inf\limits_{0<t<2\pi} \phi'(t)+n}$$

and if we consider the second, we have this one

$$\int_0^{2\pi} \left| \frac{A(t)\phi''(t)}{\left(i\phi'(t)+in\right)^2} \right| dt \leq \int_0^{2\pi} \frac{A(t)\varepsilon\phi'(t)}{\left(\phi'(t)+n\right)^2} \leq 2\pi\varepsilon \|A\|_{L^\infty} \sup_{0<t<2\pi} \frac{\phi'(t)}{\left(\phi'(t)+n\right)^2}$$

Now we apply this to the whole sum and make use of the fact that for any nonnegative a and b the following holds true $(a+b)^2 \leq 2(a^2+b^2)$. We get

$$\begin{aligned}\|P_- f\|_{L^2(\mathbb{T})} =& \sum_{n=1}^{\infty} \left| \int_0^{2\pi} A(t) e^{i\phi(t)+int} dt \right|^2 \\ \leq& \sum_{n=1}^{\infty} \left(\int_0^{2\pi} \left| \frac{A'(t)}{\phi'(t)+n} \right| + \left| \frac{A(t)\phi''(t)}{\left(i\phi'(t)+in\right)^2} \right| dt \right)^2 \\ \leq& \sum_{n=1}^{\infty} \left(2\pi \frac{\|A'\|_{L^\infty}}{\inf\limits_{0<t<2\pi} \phi'(t)+n} + 2\pi\varepsilon\|A\|_{L^\infty} \sup_{0<t<2\pi} \frac{\phi'(t)}{\left(\phi'(t)+n\right)^2} \right)^2 \\ \leq& 2\sum_{n=1}^{\infty} \left(2\pi \frac{\|A'\|_{L^\infty}}{\inf\limits_{0<t<2\pi} \phi'(t)+n} \right)^2 + 2\sum_{n=1}^{\infty} \left(2\pi\varepsilon\|A\|_{L^\infty} \sup_{0<t<2\pi} \frac{\phi'(t)}{\left(\phi'(t)+n\right)^2} \right)^2\end{aligned}$$

Now we are getting close. By basic calculus we have

$$\sum_{n=1}^{\infty} \frac{1}{\inf\limits_{0<t<2\pi} \phi'(t) + n}^{2} = \sum_{k=\inf\limits_{0<t<2\pi} \phi'(t)+1}^{\infty} \frac{1}{k^2} \leq \int_{\inf\limits_{0<t<2\pi} \phi'(t)}^{\infty} \frac{1}{x} dx = \frac{1}{\inf\limits_{0<t<2\pi} \phi'(t)}$$

Then by noticing that

$$\sup_{0<t<2\pi} \frac{\phi'(t)}{(\phi'(t)+n)^2} \leq \sup_{0<t<2\pi} \frac{1}{\phi'(t)+n}$$

we also have

$$\sum_{n=1}^{\infty} \sup_{0<t<2\pi} \frac{\phi'(t)}{(\phi'(t)+n)^2} \leq \frac{1}{\inf\limits_{0<t<2\pi} \phi'(t)}$$

Now, we combine these inequalities to obtain

$$\|P_- f\|_{L^2(\mathbb{T})} \leq \frac{8\pi^2 \|A'\|_{L^\infty}^2}{\inf\limits_{0<t<2\pi} \phi'(t)} + \frac{8\pi^2 \varepsilon^2 \|A\|_{L^\infty}^2}{\inf\limits_{0<t<2\pi} \phi'(t)}$$

By noticing that

$$\|f\|_{L^2}^2 = \int_0^{2\pi} |A(t)e^{i\phi(t)}|^2 dt = \int_0^{2\pi} |A(t)| dt = \|A\|_{L^2}^2$$

we can then conclude that

$$\|P_- f\|_{L^2(\mathbb{T})} \leq \frac{8\pi^2}{\|A\|_{L^2}^2} \frac{\|A'\|_{L^\infty}^2 + \varepsilon^2 \|A\|_{L^\infty}^2}{\inf\limits_{0<t<2\pi} \phi'(t)} \|f\|_{L^2}^2$$

and we are done. □

It may not be immediately clear that this really does control $P_- f$. In this setting, however, ϕ' is big everywhere and A' is small. So the bound actually does guarantee that any periodic, continuous function which has small variations in amplitude compared to its counter-clockwise movement is close to being holomorphic.

Theorem 44. *Given a signal $f(t) = A(t)e^{i\phi(t)}$, we use ϕ^* to denote the phase of its holomorphic projection*

$$P_+(A(t)e^{i\phi(t)}) = |P_+(A(t)e^{i\phi(t)})| e^{i\phi^*(t)}$$

Then we can control the error

$$\|\phi(t) - \phi^*(t)\|_{L^2}^2 \leq \left(\frac{8\pi^4}{\|A\|_{L^2}^2} \frac{\|A'\|_{L^\infty}^2 + \varepsilon^2 \|A\|_{L^\infty}^2}{\inf\limits_{0<t<2\pi} \phi'(t)} \frac{1}{\inf\limits_{0<t<2\pi} A(t)^2} \right) \|f\|_{L^2}^2$$

Proof. Let us fix $0 \le t \le 2\pi$. Suppose now, that the phases $\phi(t)$ and $\phi^*(t)$ of

$$A(t)e^{i\phi(t)} \quad \text{and} \quad P_+(A(t)e^{i\phi(t)}) = |P_+(A(t)e^{i\phi(t)})|e^{i\phi^*(t)}$$

differ by some angle α. We will now show that this causes $|A(t)e^{i\phi(t)} - P_+(A(t)e^{i\phi(t)})|$ to be sufficiently big. Notice that

$$\sin(\alpha) = \frac{|A(t)e^{i\phi(t)} - P_+(A(t)e^{i\phi(t)})|}{A(t)}$$

so we have

$$|A(t)e^{i\phi(t)} - P_+(A(t)e^{i\phi(t)})| \ge \begin{cases} A(t)\sin(\alpha), & \text{if} \quad -\dfrac{\pi}{2} \le \alpha \le \dfrac{\pi}{2} \\ A(t), & \text{otherwise} \end{cases}$$

Notice now that $\alpha = |\phi(t) - \phi^*(t)|_{\mathbb{T}}$ really is less that π and positive. Taking advantage of this we get

$$\begin{aligned} |A(t)e^{i\phi(t)} - P_+(A(t)e^{i\phi(t)})| &\ge \begin{cases} A(t)\sin(|\phi(t) - \phi^*(t)|_{\mathbb{T}}), & \text{if} \quad 0 \le |\phi(t) - \phi^*(t)|_{\mathbb{T}} \le \dfrac{\pi}{2} \\ A(t), & \text{if} \quad 0 \le |\phi(t) - \phi^*(t)|_{\mathbb{T}} \le \pi \end{cases} \\ &\ge A(t)\frac{|\phi(t) - \phi^*(t)|_{\mathbb{T}}}{\pi} \\ &\ge \left(\inf_{0<t<2\pi} A(t)\right)\frac{|\phi(t) - \phi^*(t)|_{\mathbb{T}}}{\pi} \end{aligned}$$

because $\sin x > \frac{2}{\pi}x$ whenever $0 < x < \pi/2$ and the fact that $\frac{|\phi(t)-\phi^*(t)|_{\mathbb{T}}}{\pi} \le 1$. By isolating $|\phi(t) - \phi^*(t)|_{\mathbb{T}}$, squaring both sides and integrating both sides over $\mathbb{T}$ we have

$$\|\phi(t) - \phi^*(t)\|^2_{L^2(\mathbb{T})} \le \frac{\pi^2}{\inf\limits_{0<t<2\pi} A(t)^2}\|A(t)e^{i\phi(t)} - P_+(A(t)e^{i\phi(t)})\|^2_{L^2(\mathbb{T})}$$

Now we are almost there. We know that $f = P_+f + P_-f$, and thus we have

$$\|A(t)e^{i\phi(t)} - P_+(A(t)e^{i\phi(t)})\|^2_{L^2(\mathbb{T})} \le \|P_-(A(t)e^{i\phi(t)})\|^2_{L^2(\mathbb{T})}$$

Now using this and theorem 43 we get

$$\|\phi(t) - \phi^*(t)\|^2_{L^2(\mathbb{T})} \le \left(\frac{8\pi^4}{\|A\|^2_{L^2}}\frac{\|A'\|^2_{L^\infty} + \varepsilon^2\|A\|^2_{L^\infty}}{\inf\limits_{0<t<2\pi}\phi'(t)}\frac{1}{\inf\limits_{0<t<2\pi} A(t)^2}\right)\|f\|^2_{L^2}$$

which is our claim, so we are done. □

What follows is is a very important consequence of this theorem. First, add a carrier frequency e^{iNt} to our signal to obtain

$$f(t)e^{iNt} = A(t)e^{i\phi(t)}e^{iNt}$$

for some $N \in \mathbb{N}$. Then, notice that the quantity $\inf_{0<t<2\pi} \phi'(t)$ in the inequality increases by at least N and the amplitude $A(t)$ does not change at all. This guarantees that we can make this error arbitrarily small by adding a carrier frequency for large enough N. With this in mind, the following procedure may therefore be very useful.

- Suppose we are to analyze $f(t) = A(t)e^{i\phi(t)}$
- Add to f the carrier frequency to obtain $A(t)e^{i\phi(t)}e^{iNt}$
- Use the Littlewood-Paley projection onto holomorphic functions to obtain $P_+(A(t)e^{i\phi(t)}e^{iNt})$
- Find the phase $\phi^*(t)$ of the holomorphic projection $P_+(A(t)e^{i\phi(t)}e^{iNt})$
- Use $\phi^*(t) - Nt$ as an approximation of the phase $\phi(t)$

As mentioned, as we increase N, the function we obtain by adding the carrier frequency becomes closer to the subspace of holomorphic functions. This means that in theory, any intrinsic mode function can be thought of as holomorphic up to an arbitrarily small error. This may of course be used in relation to any other approach to signal analysis, as long as we define our intrinsic mode functions in this way.

This allows us to approach the subject with a pure complex analysis point of view where one requires holomorphic functions. So through these carrier signals we may treat any intrinsic mode function as holomorphic with as small error as we want. This is where our previous chapter reenters the picture. Our results from there required holomorphic input, and seem like a natural tool to use here.

3.3 Stability

Let us consider $F(z)$ and let us try to gain some insight into what happens when the signal F is exposed to white noise. First, we will need to give a brief introduction as to what white noise is.

White noise is a basic model which is used as an attempt to mimic the effect of a random processes that occur in nature. It is commonly used in signal analysis and similar fields of research. There are many ways of defining it, and they differ dependent on what the context is and what process you are attempting to mimic. Often, white noise is characterized in some sense by the Gaussian distribution,

$$\mathcal{N}(\mu, \sigma^2)$$

where μ is the mean of the distribution and σ^2 is the variance. Because white noise is a phenomena which occurs in the physical world it would be nice to have some intuition on what effects it has in our setting.

For our purpose we shall impose three conditions on white noise. The first is that it exists as a stochastic process. The second is that for all intervals $[a, b] \subset \mathbb{T} \cong [0, 2\pi]$

$$\int_a^b \Phi(t)dt = \frac{1}{2\pi}\mathcal{N}(0, b-a)$$

and the third is that for disjoint intervals the arising two random variables are independent. We will also need to recall the addition law for independent Gaussian variables,

$$a\mathcal{N}(\mu_1, \sigma_1) + b\mathcal{N}(\mu_2, \sigma_2) \sim \mathcal{N}(a\mu_1 + b\mu_2, a^2\sigma_1 + b^2\sigma_2)$$

As discussed, we will assume F to be a holomorphic signal on the boundary of the unit disk. Now we assume F to be exposed by white noise. This is done by perturbing F by our white noise function denoted by Φ. Of course, this Φ only lives on the boundary of the unit disk, $\mathbb{T}$. Denote by $\mathcal{P}\Phi$ the extension of Φ to the unit disk D by convuluting it with the Poisson kernel in the following way

$$\mathcal{P}\Phi(z) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - t)\Phi(e^{it})dt$$

Given a perturbed function,

$$(F + \mathcal{P}\Phi)(z) = B_1(z)G_1(z)$$

we have the following result, giving us some insight into what happens on the boundary of the unit disk.

Theorem 45. *We have*

$$(\mathcal{P}\Phi)(z) = \mathcal{N}\left(0, \frac{1}{2\pi} + \frac{1}{\pi}\frac{|z|^2}{1-|z|^2}\right) \qquad \text{for} \quad z \in D$$

Proof. In this proof we analyze the extension of white noise by Poisson's kernel. The Poisson extension of Φ is then

$$
\begin{aligned}
(\mathcal{P}\Phi)(z) =& \frac{1}{2\pi}\int_0^{2\pi} P_r(\phi - t)\Phi(e^{it})dt \\
=& \frac{1}{2\pi}\lim_{N\to\infty}\sum_{n=1}^{N}\int_{\frac{2\pi(n-1)}{N}}^{\frac{2\pi n}{N}} P_r(\theta - t)\Phi(e^{it})dt \\
=& \frac{1}{2\pi}\lim_{N\to\infty}\sum_{n=1}^{N} P_r\left(\theta - \frac{2\pi n}{N}\right)\mathcal{N}\left(0, \frac{2\pi}{N}\right) \\
=& \frac{1}{2\pi}\lim_{N\to\infty}\mathcal{N}\left(0, \sum_{n=1}^{N}\left(P_r\left(\theta - \frac{2\pi(n-1)}{N}\right)^2 + P_r\left(\theta - \frac{2\pi n}{N}\right)^2\right)\frac{2\pi}{2N}\right) \\
=& \frac{1}{2\pi}\mathcal{N}\left(0, \int_0^{2\pi} P_r(\theta)^2 d\theta\right)
\end{aligned}
$$

Let us calculate further. We write

$$
\begin{aligned}
\int_0^{2\pi} P_r(\theta)^2 d\theta =& \int_0^{2\pi}\left(\frac{1}{2\pi}\sum_{n\in\mathbb{Z}} r^{|n|}e^{in\theta}\right)^2 d\theta \\
=& \frac{1}{4\pi^2}\int_0^{2\pi}\left(1 + \sum_{n\neq 0} r^{|n|}e^{in\theta}\right)^2 d\theta
\end{aligned}
$$

If we were to multiply this out, we have the term 1, the terms $r^{|n|}e^{in\theta}$ and the terms $r^{|m|+|n|}e^{i(m+n)\theta}$. The second groups all integrate to zero, and the only terms from the third group which survive the integration are the ones where $m = -n$. Thus, we have

$$
\int_0^{2\pi} P_r(\theta)^2 d\theta = \frac{1}{2\pi} + \frac{2\sum_{n=1}^{\infty} r^{2n}}{2\pi} = \frac{1}{2\pi} + \frac{1}{\pi}\frac{r^2}{1-r^2}
$$

Now the claim follows easily. □

Our theorem tells us that when $|z|$ is close to 1 we have a variance which tends to ∞. This means that as $|z|$ tends to 1, the value of $\mathcal{P}\Phi$ is close to 0. So when z is close to the boundary of the unit disk, the white noise gets smaller and smaller. Thus we know that, close to the boundary of the unit circle, the perturbation is small.

It is important to note that this example is meant to gain some intuition and is not really what happens in reality. Remember, the signal F is obtained by having a real valued signal $s(t)$ and extending it to the complex valued signal $F(t) = s(t) + i\mathcal{H}s(t)$. In physical practice, it is not F being perturbed, but rather $s(t)$. The simplification done here does, however, provide some welcomed intuition as mentioned.

3.4 Explicit solvability

The convergence or explicit computation of the unwinding series has been discussed some already. There are many examples of functions where one can explicitly compute the unwinding series, however, there are certain classes of functions where explicit computation of the unwinding series is particularly nice. Here, we introduce one such class of functions where, in particular, the unwinding series coincides with the Fourier series.

Proposition 46. *Let $0 \leq n_0 < n_1 < n_2 < \ldots$ be a strictly increasing sequence of integers and*

$$F(z) = \sum_{k=0}^{\infty} a_k z^{n_k} \quad \textit{where, for all } n, \quad |a_n| > \sum_{k=n+1}^{\infty} |a_k|$$

Then the N-th term of the unwinding series is given by

$$f(0) + a_1 B_1 + a_2 B_1 B_2 + \cdots + a_N B_1 \ldots B_N = \sum_{k=0}^{N} a_k z^{n_k}$$

Proof. We will prove this by induction. The first step is obvious, because $f(0) = a_0$. Assuming the statement is true up to some $n \in \mathbb{N}$, it suffices to prove that all arising roots are in $z = 0$. In this case the Blaschke factors are just z^m for some $m \in \mathbb{N}$. Let us now show this. For all $|z| \leq 1$ we have

$$\begin{aligned}\left|\sum_{k=N+1}^{\infty} a_k z^{n_k}\right| &= \left|a_{N+1} z^{n_{N+1}} + \sum_{k=N+2}^{\infty} a_k z^{n_k}\right| \\ &= |z|^{n_{N+1}} \left|a_{N+1} + \sum_{k=N+2}^{\infty} a_k z^{n_k - n_{N+1}}\right|\end{aligned}$$

Now, notice that

$$\begin{aligned}\left|a_{N+1} + \sum_{k=N+2}^{\infty} a_k z^{n_k - n_{N+1}}\right| &\geq |a_{N+1}| - \left|\sum_{k=N+2}^{\infty} a_k z^{n_k - n_{N+1}}\right| \\ &\geq |a_{N+1}| - \sum_{k=N+2}^{\infty} |a_k| > 0\end{aligned}$$

That means that, if ever $\sum_{k=N+1}^{\infty} a_k z^{n_k} = 0$, then $z = 0$. This proves the claim. □

This is a particularly nice set of functions. As we can see, by iterating the restriction on the coefficients,

$$|a_n| > \sum_{k=n+1}^{\infty} |a_k| > 2 \sum_{k=n+2}^{\infty} |a_k| > \ldots 2 \sum_{k=n+l}^{\infty} |a_k|$$

This means that for functions with exponential decay of its Fourier coefficients $\{a_n\}$, we can easily compute its unwinding series.

Chapter 4

Invariant subspace decomposition of Hardy spaces

In this chapter we will consider orthogonal decompositions of invariant subspaces of the hardy space H^p. Also, we will show how some of these relate to a generalized version of the decomposition we already have studied. We will discuss convergence as well.

A subspace V of $H^p(\mathbb{T})$ is called invariant if it is invariant under multiplication by $e^{i\theta}$. This means that for any $v \in V$, also $ve^{i\theta} \in V$. It is known that the invariant subspaces are of the form uH^p where u is an inner function. This means that the invariant subspace determines u up to a constant $c = e^{i\theta}$. Additionally, a subspace of $H^2(\mathbb{T})$ is said to be invariant if it is stable by multiplication by the functions $e^{i\theta x}$ for all $\theta > 0$. Also in this case the invariant subspaces are given by uH^2 where u is an inner function.

Let H be the operator of orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{T})$. It results from properties of the Hilbert space that this operator extends as a bounded operator from $L^p(\mathbb{T})$ to $H^p(\mathbb{T})$ for $1 < p < \infty$.

If u is an inner function, denote by χ_u the operator of multiplication by u. Then we define the operator $H_u = \chi_u H \chi_u^{-1}$. This is the operator of orthogonal projection of $L^2(\mathbb{T})$ onto $uH^2(\mathbb{T})$. This operator extends as a bounded operator from $L^p(\mathbb{T})$ to $H^p(\mathbb{T})$ for all $p \in (1, \infty)$ with a norm which is independent of p. Thus, for all $1 < p$, there exists some constant C_p such that

$$\|H_u f\|_p \leq C_p \|f\|_p$$

4.1 A different way of phase unwinding

Denote now by H^p the space $H^p(\mathbb{T})$. Let us take a look at a slightly different way of unwinding the phase of a Hardy space function. Let $f \in H^p$ and let Q_0 be a projector on some subspace of H^p. Then we let $g_0 = Q_0 f$ and write $f = g_0 + u_1 f_1$, where u_1 is an inner function and f_1 is an outer function. For the next step, we choose a projector Q_1, not necessarily different from Q_0. Write $g_1 = Q_1 f_1$, and let $f_1 = g_1 + u_2 f_2$. Repeat this procedure to infinity, or until $f_n = 0$. This yields

$$f = g_0 + u_1 g_1 + u_1 u_2 g_2 + \cdots + g_{n-1} \prod_{k=1}^{n-1} u_j + f_n \prod_{j=1}^{n} u_j$$

This sum is orthogonal in the case where $p = 2$.

Whenever f_n never equals zero, it is natural to ask how well this sum represents f. We will denote it by

$$f^* = g_0 + \sum_{n=1}^{\infty} g_n \prod_{k=1}^{n} u_j$$

and return to this question in a bit.

The projectors Q_j of the form $\mathrm{Id} - H_v$, where v is an inner function have been used in various works. Let us take a closer look at this case.

Assume we have a sequence $\{v_n\}$ of inner functions and let $f \in H^p$ for some $p \geq 1$. Recursively, we define the three sequences of functions $\{f_n\}$, $\{g_n\}$ and $\{u_n\}$, where, for $n \geq 1$, g_n are outer functions and u_n are inner functions, in the following manner.

- Set $f_0 = f$.
- Project f_n to $v_{n+1} H^p$ by $h_n = H_{v_{n+1}} f_n$.
- If $h_0 = 0$ we stop, if not let f_{n+1} be the outer part of h_n and u_{n+1} the inner part of h_n and set $g_n = f_n - h_n = f_n - u_{n+1} f_{n+1}$.

Theorem 47. *Let $\{V_n\}$ be a decreasing sequence of invariant subspaces, with $V_0 = H^2$. Set $V_\infty = \cap V_n$ and let $\mathcal{P}_n$ denote the operator associated with the inner function defining V_n. Then, for all $p \in (1, \infty)$ and for every $f \in H^p$, one has*

$$\lim_{n \to \infty} \|\mathcal{P}_n f - \mathcal{P}_\infty f\|_p = 0$$

Proof. First, fix $p \in (1, \infty)$ $(p \neq 2)$ and $p_0 \in (1, \infty)$ such that p lies in the interval determined by p_0 and 2. Now let $g \in H^2 \cap H^{p_0}$. We then know, that there exists some $\alpha \in (0, 1)$ such that

$$\|\mathcal{P}_n g - \mathcal{P}_\infty g\|_p \leq \|\mathcal{P}_n g - \mathcal{P}_\infty g\|_2^\alpha \|\mathcal{P}_n g - \mathcal{P}_\infty g\|_{p_0}^{(1-\alpha)}$$

This is basically just an application of Hölder's inequality. If we now apply the constant C_p which we introduced, we obtain the following bound

$$\|\mathcal{P}_n g - \mathcal{P}_\infty g\|_p \leq \|\mathcal{P}_n g - \mathcal{P}_\infty g\|_2^\alpha (2C_{p_0})^{(1-\alpha)} \|g\|_{p_0}^{(1-\alpha)}$$

Taking limits now yield

$$\lim_{n \to \infty} \|\mathcal{P}_n g - \mathcal{P}_\infty g\|_p = 0$$

If $f \in H^p$, we then have, for all $g \in H^2 \cap H^{p_0}$,

$$\|\mathcal{P}_n f - \mathcal{P}_\infty f\|_p \leq \|\mathcal{P}_n g - \mathcal{P}_\infty g\|_p + \|\mathcal{P}_n(f-g) - \mathcal{P}_\infty(f-g)\|_p$$

Similarly to before, we then have

$$\lim_{n\to\infty} \|\mathcal{P}_n f - \mathcal{P}_\infty f\|_p \leq 2C_p\|f-g\|_p$$

Notice that if $p < 2$ then $H^p \subset H^{p_0}$, and if $p > 2$ then $H^p \subset H^2$. So because the last inequality holds for all $g \in H^2 \cap H^{p_0}$, we have

$$\lim_{n\to\infty} \|\mathcal{P}_n f - \mathcal{P}_\infty f\|_p = 0$$

□

We then have this, immediate corollory.

Corollary 48. *Let $Q_n = \mathcal{P}_n - \mathcal{P}_{n-1}$. Then, for all $p \in (1,\infty)$ and $f \in H^p$, the series*

$$\sum_{n=0}^{\infty} Q_n f$$

converges to $f - \mathcal{P}_\infty f$ in L^p.

A particular consequence of this corollary is that f^* converges to f in H^p.

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