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## Singular equivalence and the (Fg) condition



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### ABSTRACT

We show that singular equivalences of Morita type with level between finite-dimensional Gorenstein algebras over a field preserve the (Fg) condition.

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### 1. Introduction

Throughout the paper, we let  $k$  be a fixed field.

Support varieties for modules over a group algebra  $kG$  were introduced by J.F. Carlson in [4], using the group cohomology ring  $H^*(G, k)$ . Later, Snashall and Solberg [18] defined support varieties for modules over an arbitrary finite-dimensional  $k$ -algebra  $\Lambda$ , using the Hochschild cohomology ring  $HH^*(\Lambda)$ .

We say that a finite-dimensional  $k$ -algebra  $\Lambda$  satisfies the **(Fg)** condition if the Hochschild cohomology ring  $HH^*(\Lambda)$  of  $\Lambda$  is Noetherian and the Yoneda algebra  $\text{Ext}_\Lambda^*(M, M)$  is a finitely generated  $HH^*(\Lambda)$ -module for every finitely generated  $\Lambda$ -module  $M$  (for more details, see the definition in Section 7). It was shown in [9] that many of the results for support varieties over a group algebra also hold for support varieties over a selfinjective algebra which satisfies the **(Fg)** condition. We can thus think of the **(Fg)** condition as a criterion for deciding whether a given algebra has a nice theory of support varieties.

It is therefore interesting to investigate whether the **(Fg)** condition holds for various algebras, and to find out which relations between algebras preserve the **(Fg)** condition. This question has been considered in [16] for algebras whose module categories are related by a recollement of abelian categories, and in [12] for derived equivalence of algebras. In this paper, we consider singular equivalence of algebras.

The *singularity category*  $D_{\text{sg}}(\Lambda)$  of a  $k$ -algebra  $\Lambda$ , introduced by Buchweitz in [3], is defined as the Verdier quotient

$$D_{\text{sg}}(\Lambda) = D^b(\text{mod } \Lambda) / \text{perf}(\Lambda)$$

of the bounded derived category  $D^b(\text{mod } \Lambda)$  by the subcategory of perfect complexes. This is a triangulated category. We say that two  $k$ -algebras  $\Lambda$  and  $\Sigma$  are *singularly equivalent* if there exists a triangle equivalence  $f: D_{\text{sg}}(\Lambda) \rightarrow D_{\text{sg}}(\Sigma)$  between their singularity categories, and the functor  $f$  is then called a *singular equivalence* between the algebras  $\Lambda$  and  $\Sigma$ .

The purpose of this paper is to investigate to what extent singular equivalences preserve the **(Fg)** condition. Since arbitrary singular equivalences are hard to work with and do not necessarily have nice properties, we restrict our attention to special classes of singular equivalences.

A *singular equivalence of Morita type* (introduced by Chen and Sun in [7]) between  $k$ -algebras  $\Lambda$  and  $\Sigma$  is a singular equivalence

$$D_{\text{sg}}(\Lambda) \xrightarrow{N \otimes_\Lambda -} D_{\text{sg}}(\Sigma)$$

which is induced by a tensor functor  $N \otimes_\Lambda -$ , where  $N$  is a  $\Sigma$ - $\Lambda$  bimodule subject to some technical requirements. Wang [19] has introduced a generalized version of singular equivalence of Morita type called *singular equivalence of Morita type with level*. We recall the definitions of these two types of singular equivalences in Section 2. The question we

want to answer in this paper is: Do singular equivalences of Morita type with level preserve the **(Fg)** condition?

All algebras that satisfy the **(Fg)** condition are Gorenstein algebras (see [Theorem 7.2](#)), and singular equivalences of Morita type with level do not preserve Gorensteinness. Moreover, even if one of the algebras involved in a singular equivalence of Morita type with level satisfies the **(Fg)** condition, the other algebra does not need to be a Gorenstein algebra (see [Example 7.5](#)). This means that the **(Fg)** condition is in general not preserved under singular equivalence of Morita type with level.

However, we can consider the question of whether it is only when one of the algebras is non-Gorenstein that such counterexamples arise. In other words, if we require all our algebras to be Gorenstein, is it then true that singular equivalences of Morita type with level preserve the **(Fg)** condition? The main result of this paper, [Theorem 7.4](#), answers this question affirmatively: A singular equivalence of Morita type with level between finite-dimensional Gorenstein algebras over a field preserves the **(Fg)** condition. As a consequence of this, we obtain a similar statement for *stable* equivalence of Morita type ([Corollary 7.6](#)), where we do not need the assumption of Gorensteinness.

A result similar to the main result of this paper has been shown independently, and in a different way, in a recent preprint by Yiping Chen.

The content of the paper is structured as follows.

In [Section 2](#), we state the definitions of singular equivalence of Morita type and singular equivalence of Morita type with level, and look at some easily derived consequences.

In [Section 3](#), we begin to look at what more we can deduce from a singular equivalence of Morita type with level when the assumption of Gorensteinness is added. We recall the well-known result stating that the singularity category of a Gorenstein algebra is equivalent to the stable category of maximal Cohen–Macaulay modules. This implies that a singular equivalence

$$f: D_{\text{sg}}(\Lambda) \xrightarrow{\simeq} D_{\text{sg}}(\Sigma)$$

between Gorenstein algebras  $\Lambda$  and  $\Sigma$  gives an equivalence

$$g: \underline{\text{CM}}(\Lambda) \xrightarrow{\simeq} \underline{\text{CM}}(\Sigma)$$

between their stable categories of maximal Cohen–Macaulay modules. We show that if the singular equivalence  $f$  is of Morita type with level, and thus induced by a tensor functor, then the equivalence  $g$  is induced by the same tensor functor.

In [Section 4](#), we consider certain maps of the form

$$\text{Ext}_{\Lambda}^n(U, V) \rightarrow \text{Ext}_{\Lambda}^n(\Omega_{\Lambda}^i(U), \Omega_{\Lambda}^i(V)),$$

which we call rotation maps. We show that these maps are isomorphisms if the algebra  $\Lambda$  is Gorenstein and  $n > \text{id}_{\Lambda} \Lambda$ . This means that in extension groups of sufficiently high degree over a Gorenstein algebra, we can replace both modules by syzygies. This result is used in the following three sections.

In Section 5, we show that if we have a singular equivalence of Morita type with level

$$D_{\text{sg}}(\Lambda) \xrightarrow[N \otimes_{\Lambda} -]{\cong} D_{\text{sg}}(\Sigma)$$

between two Gorenstein algebras  $\Lambda$  and  $\Sigma$ , then we have isomorphisms

$$\text{Ext}_{\Lambda}^n(A, B) \xrightarrow[N \otimes_{\Lambda} -]{\cong} \text{Ext}_{\Sigma}^n(N \otimes_{\Lambda} A, N \otimes_{\Lambda} B) \quad (\text{for } A \text{ and } B \text{ in } \text{mod } \Lambda) \quad (1.1)$$

between extension groups over  $\Lambda$  and extension groups over  $\Sigma$ , in all sufficiently large degrees  $n$ . In the terminology of [16], this implies that a tensor functor inducing a singular equivalence of Morita type with level between Gorenstein algebras is an eventually homological isomorphism. The proof of this result builds on the result about stable categories of Cohen–Macaulay modules from Section 3.

In Section 6, we show that a singular equivalence of Morita type with level between Gorenstein algebras preserves Hochschild cohomology in almost all degrees. That is, if two Gorenstein algebras  $\Lambda$  and  $\Sigma$  are singularly equivalent of Morita type with level, then there are isomorphisms

$$\text{HH}^n(\Lambda) \cong \text{HH}^n(\Sigma) \quad (1.2)$$

for almost all  $n$ , and these isomorphisms respect the ring structure of the Hochschild cohomology.

In Section 7, we show the main result of the paper: A singular equivalence of Morita type with level between finite-dimensional Gorenstein algebras over a field preserves the **(Fg)** condition. The main ingredients in the proof of this result are the isomorphism (1.1) of extension groups from Section 5 and the isomorphism (1.2) of Hochschild cohomology groups from Section 6.

## 2. Singular equivalences of Morita type with level

In this section, we recall the definitions we need regarding singular equivalences. We begin with the concept of singularity categories.

**Definition 2.1.** Let  $\Lambda$  be a  $k$ -algebra. The *singularity category*  $D_{\text{sg}}(\Lambda)$  of  $\Lambda$  is a triangulated category defined as the Verdier quotient

$$D_{\text{sg}}(\Lambda) = D^{\text{b}}(\text{mod } \Lambda) / \text{perf}(\Lambda)$$

of the bounded derived category  $D^{\text{b}}(\text{mod } \Lambda)$  by the subcategory of perfect complexes. We say that two algebras  $\Lambda$  and  $\Sigma$  are *singularly equivalent* if their singularity categories  $D_{\text{sg}}(\Lambda)$  and  $D_{\text{sg}}(\Sigma)$  are equivalent as triangulated categories. A triangle equivalence between  $D_{\text{sg}}(\Lambda)$  and  $D_{\text{sg}}(\Sigma)$  is called a *singular equivalence* between the algebras  $\Lambda$  and  $\Sigma$ .

The singularity category of an algebra was first defined by Buchweitz in [3, Definition 1.2.2]. In his definition, the singularity category is called the *stabilized derived category*, and it is denoted by  $\underline{\mathbf{D}}^b(\Lambda)$ . Later, Orlov [15] used the same construction in algebraic geometry to define the *triangulated category of singularities* of a scheme  $X$ , denoted  $\mathbf{D}_{\text{sg}}(X)$ . We follow the recent convention of using Orlov's terminology and notation for algebras as well. The term *singular equivalence* was introduced by Chen [6].

Analogously to the special type of stable equivalences called *stable equivalences of Morita type*, Chen and Sun have defined a special type of singular equivalences called *singular equivalences of Morita type* in their preprint [7]. This concept was further explored by Zhou and Zimmermann in [20].

**Definition 2.2.** Let  $\Lambda$  and  $\Sigma$  be finite-dimensional  $k$ -algebras, and let  $M$  be a  $\Lambda$ - $\Sigma$  bimodule and  $N$  a  $\Sigma$ - $\Lambda$  bimodule. We say that  $M$  and  $N$  induce a *singular equivalence of Morita type* between  $\Lambda$  and  $\Sigma$  (and that  $\Lambda$  and  $\Sigma$  are *singularly equivalent of Morita type*) if the following conditions are satisfied:

1.  $M$  is finitely generated and projective as a left  $\Lambda$ -module and as a right  $\Sigma$ -module.
2.  $N$  is finitely generated and projective as a left  $\Sigma$ -module and as a right  $\Lambda$ -module.
3. There is a finitely generated  $\Lambda^e$ -module  $X$  with finite projective dimension such that  $M \otimes_{\Sigma} N \cong \Lambda \oplus X$  as  $\Lambda^e$ -modules.
4. There is a finitely generated  $\Sigma^e$ -module  $Y$  with finite projective dimension such that  $N \otimes_{\Lambda} M \cong \Sigma \oplus Y$  as  $\Sigma^e$ -modules.

Notice that the definition is precisely the same as the definition of stable equivalence of Morita type, except that the modules  $X$  and  $Y$  are not necessarily projective, but only have finite projective dimension. Thus stable equivalences of Morita type occur as a special case of singular equivalences of Morita type.

The following proposition describes how a singular equivalence of Morita type is a singular equivalence, thus justifying the name.

**Proposition 2.3.** (See [20, Proposition 2.3].) Let  ${}_{\Lambda}M_{\Sigma}$  and  ${}_{\Sigma}N_{\Lambda}$  be bimodules which induce a singular equivalence of Morita type between two  $k$ -algebras  $\Lambda$  and  $\Sigma$ . Then the functors

$$N \otimes_{\Lambda} - : \mathbf{D}_{\text{sg}}(\Lambda) \rightarrow \mathbf{D}_{\text{sg}}(\Sigma) \quad \text{and} \quad M \otimes_{\Sigma} - : \mathbf{D}_{\text{sg}}(\Sigma) \rightarrow \mathbf{D}_{\text{sg}}(\Lambda)$$

are equivalences of triangulated categories, and they are quasi-inverses of each other.

Inspired by the notion of singular equivalence of Morita type, Wang [19] has defined a more general type of singular equivalence called *singular equivalence of Morita type with level*.

**Definition 2.4.** Let  $\Lambda$  and  $\Sigma$  be finite-dimensional  $k$ -algebras, and let  $M$  be a  $\Lambda$ - $\Sigma$  bimodule and  $N$  a  $\Sigma$ - $\Lambda$  bimodule. Let  $l$  be a nonnegative integer. We say that  $M$  and  $N$

induce a *singular equivalence of Morita type with level  $l$*  between  $\Lambda$  and  $\Sigma$  (and that  $\Lambda$  and  $\Sigma$  are *singularly equivalent of Morita type with level  $l$* ) if the following conditions are satisfied:

1.  $M$  is finitely generated and projective as a left  $\Lambda$ -module and as a right  $\Sigma$ -module.
2.  $N$  is finitely generated and projective as a left  $\Sigma$ -module and as a right  $\Lambda$ -module.
3. There is an isomorphism  $M \otimes_{\Sigma} N \cong \Omega_{\Lambda^e}^l(\Lambda)$  in the stable category  $\underline{\text{mod}} \Lambda^e$ .
4. There is an isomorphism  $N \otimes_{\Lambda} M \cong \Omega_{\Sigma^e}^l(\Sigma)$  in the stable category  $\underline{\text{mod}} \Sigma^e$ .

Just as in the case of singular equivalence of Morita type, the conditions in the definition of singular equivalence of Morita type with level are designed to ensure that the functors  $N \otimes_{\Lambda} -$  and  $M \otimes_{\Sigma} -$  induce singular equivalences.

**Proposition 2.5.** (See [19, Remark 2.2].) *Let  ${}_{\Lambda}M_{\Sigma}$  and  ${}_{\Sigma}N_{\Lambda}$  be bimodules which induce a singular equivalence of Morita type with level  $l$  between two  $k$ -algebras  $\Lambda$  and  $\Sigma$ . Then the functors*

$$N \otimes_{\Lambda} -: D_{\text{sg}}(\Lambda) \rightarrow D_{\text{sg}}(\Sigma) \quad \text{and} \quad M \otimes_{\Sigma} -: D_{\text{sg}}(\Sigma) \rightarrow D_{\text{sg}}(\Lambda)$$

are equivalences of triangulated categories. The compositions

$$M \otimes_{\Sigma} N \otimes_{\Lambda} -: D_{\text{sg}}(\Lambda) \rightarrow D_{\text{sg}}(\Lambda) \quad \text{and} \quad N \otimes_{\Lambda} M \otimes_{\Sigma} -: D_{\text{sg}}(\Sigma) \rightarrow D_{\text{sg}}(\Sigma)$$

are isomorphic to the shift functor  $[-l]$  on the respective categories  $D_{\text{sg}}(\Lambda)$  and  $D_{\text{sg}}(\Sigma)$ .

We now show that the notion of singular equivalence of Morita type with level generalizes the notion of singular equivalence of Morita type, in the sense that any equivalence of the latter type is also of the former type. This is mentioned without proof in [19].

**Proposition 2.6.** *Let  $\Lambda$  and  $\Sigma$  be finite-dimensional  $k$ -algebras. If a functor  $f: D_{\text{sg}}(\Lambda) \rightarrow D_{\text{sg}}(\Sigma)$  is a singular equivalence of Morita type, then it is also a singular equivalence of Morita type with level.*

**Proof.** Let  $M, N, X$  and  $Y$  be bimodules satisfying the requirements of a singular equivalence of Morita type, such that  $f = (N \otimes_{\Lambda} -)$ . Let  $l = \max\{\text{pd}_{\Lambda^e} X, \text{pd}_{\Sigma^e} Y\}$ . Let  $M'$  be an  $l$ -th syzygy of  $M$  as  $\Lambda$ - $\Sigma$ -bimodule, and let

$$0 \rightarrow M' \rightarrow P_{l-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0 \tag{2.1}$$

be the beginning of a projective resolution of  $M$ . We show that the bimodules  $M'$  and  $N$  induce a singular equivalence of Morita type with level  $l$ .

If we consider the bimodules in sequence (2.1) as one-sided modules (left  $\Lambda$ -modules or right  $\Sigma$ -modules), then  $M$  and the modules  $P_0, \dots, P_{l-1}$  are projective, and thus  $M'$

must be projective as well. Thus condition (1) in [Definition 2.4](#) is satisfied. Condition (2) in [Definition 2.4](#) is trivially satisfied, since it is the same as condition (2) in [Definition 2.2](#).

Tensoring sequence (2.1) with  $N$  gives the sequence

$$0 \rightarrow M' \otimes_{\Sigma} N \rightarrow P_{l-1} \otimes_{\Sigma} N \rightarrow \cdots \rightarrow P_0 \otimes_{\Sigma} N \rightarrow M \otimes_{\Sigma} N \rightarrow 0.$$

This sequence is exact since  $N$  is projective as left  $\Sigma$ -module, and the modules  $P_i \otimes_{\Sigma} N$  are projective  $\Lambda^e$ -modules since  $N$  is projective as right  $\Lambda$ -module. The  $\Lambda^e$ -module  $M' \otimes_{\Sigma} N$  is therefore an  $l$ -th syzygy of  $M \otimes_{\Sigma} N$ . Since  $M \otimes_{\Sigma} N$  is isomorphic to  $\Lambda \oplus X$  and the projective dimension of  $X$  is at most  $l$ , this means that  $M' \otimes_{\Sigma} N$  is an  $l$ -th syzygy of  $\Lambda$  as  $\Lambda^e$ -module. Similarly, we can show that  $N \otimes_{\Lambda} M'$  is an  $l$ -th syzygy of  $\Sigma$  as  $\Sigma^e$ -module. This means that conditions (3) and (4) in [Definition 2.4](#) are satisfied.  $\square$

In the rest of the paper we work with singular equivalences of Morita type with level. By the above proposition, all results where we assume such an equivalence are also applicable to singular equivalences of Morita type.

As seen above, if  ${}_{\Lambda}M_{\Sigma}$  and  ${}_{\Sigma}N_{\Lambda}$  are bimodules which induce a singular equivalence of Morita type with level, then the functors  $N \otimes_{\Lambda} -$  and  $M \otimes_{\Sigma} -$  are equivalences between the singularity categories of  $\Lambda$  and  $\Sigma$ . We end this section by examining some properties of these tensor functors when viewed as functors between the module categories  $\text{mod } \Lambda$  and  $\text{mod } \Sigma$ .

**Lemma 2.7.** *Let  ${}_{\Lambda}M_{\Sigma}$  and  ${}_{\Sigma}N_{\Lambda}$  be bimodules which induce a singular equivalence of Morita type with level between two  $k$ -algebras  $\Lambda$  and  $\Sigma$ . Then the functors*

$$N \otimes_{\Lambda} - : \text{mod } \Lambda \rightarrow \text{mod } \Sigma \quad \text{and} \quad M \otimes_{\Sigma} - : \text{mod } \Sigma \rightarrow \text{mod } \Lambda$$

*are exact and take projective modules to projective modules. In particular, this means that they take projective resolutions to projective resolutions.*

**Proof.** Consider the functor  $N \otimes_{\Lambda} -$ . This functor is exact since  $N$  is projective as right  $\Lambda$ -module, and it takes projective modules to projective modules since  $N$  is projective as left  $\Sigma$ -module.  $\square$

Let  $\Lambda, \Sigma, M$  and  $N$  be as in the above lemma. Since the functor

$$N \otimes_{\Lambda} - : \text{mod } \Lambda \rightarrow \text{mod } \Sigma$$

is exact, it induces homomorphisms of extension groups. By abuse of notation, we denote these maps by  $N \otimes_{\Lambda} -$  as well. More precisely, for  $\Lambda$ -modules  $U$  and  $V$  and an integer  $n \geq 0$ , we define a map

$$N \otimes_{\Lambda} - : \text{Ext}_{\Lambda}^n(U, V) \rightarrow \text{Ext}_{\Sigma}^n(N \otimes_{\Lambda} U, N \otimes_{\Lambda} V). \tag{2.2}$$

For  $n = 0$ , the map  $N \otimes_{\Lambda} -$  simply sends a homomorphism  $f: U \rightarrow V$  to the homomorphism  $N \otimes_{\Lambda} f: N \otimes_{\Lambda} U \rightarrow N \otimes_{\Lambda} V$ . For  $n > 0$ , the map  $N \otimes_{\Lambda} -$  sends the element represented by the extension

$$0 \rightarrow V \rightarrow E_n \rightarrow \cdots \rightarrow E_1 \rightarrow U \rightarrow 0$$

to the element represented by the extension

$$0 \rightarrow N \otimes_{\Lambda} V \rightarrow N \otimes_{\Lambda} E_n \rightarrow \cdots \rightarrow N \otimes_{\Lambda} E_1 \rightarrow N \otimes_{\Lambda} U \rightarrow 0$$

obtained by applying the functor  $N \otimes_{\Lambda} -$  to all objects and maps.

The maps (2.2) play an important role later in the paper. In Section 5, we show that if  $\Lambda$  and  $\Sigma$  are Gorenstein algebras, then these maps are isomorphisms for almost all  $n$ . This fact is used in the proof of the main theorem (Theorem 7.4).

### 3. Gorenstein algebras and maximal Cohen–Macaulay modules

So far, we have considered the situation of two  $k$ -algebras  $\Lambda$  and  $\Sigma$ , together with bimodules  ${}_{\Lambda}M_{\Sigma}$  and  ${}_{\Sigma}N_{\Lambda}$  inducing a singular equivalence of Morita type with level between  $\Lambda$  and  $\Sigma$ . From now on, we restrict our attention to the special case where both  $\Lambda$  and  $\Sigma$  are Gorenstein algebras. In this section, we prove our first result under this assumption, namely Proposition 3.7, which states that the tensor functors  $N \otimes_{\Lambda} -$  and  $M \otimes_{\Sigma} -$  induce triangle equivalences between the stable categories of maximal Cohen–Macaulay modules over  $\Lambda$  and  $\Sigma$ .

We begin by recalling the definition of Gorenstein algebras.

**Definition 3.1.** A  $k$ -algebra  $\Lambda$  is a *Gorenstein algebra* if the injective dimension of  $\Lambda$  as a left  $\Lambda$ -module is finite and the injective dimension of  $\Lambda$  as a right  $\Lambda$ -module is finite:

$$\text{id}_{\Lambda}({}_{\Lambda}\Lambda) < \infty \quad \text{and} \quad \text{id}_{\Lambda^{\text{op}}}(\Lambda_{\Lambda}) < \infty.$$

If  $\Lambda$  is a Gorenstein algebra, then  $\text{id}_{\Lambda}({}_{\Lambda}\Lambda)$  and  $\text{id}_{\Lambda^{\text{op}}}(\Lambda_{\Lambda})$  are the same number, and this number is called the *Gorenstein dimension* of  $\Lambda$ . In later sections, we need the following result about Gorenstein algebras.

**Lemma 3.2.** (See [2, Lemma 2.1].) *If  $\Lambda$  is a Gorenstein  $k$ -algebra with Gorenstein dimension  $d$ , then its enveloping algebra  $\Lambda^e$  is a Gorenstein algebra with Gorenstein dimension at most  $2d$ .*

We continue by recalling the definition of maximal Cohen–Macaulay modules.

**Definition 3.3.** Let  $\Lambda$  be a  $k$ -algebra. A finitely generated  $\Lambda$ -module  $C$  is a *maximal Cohen–Macaulay module* if  $\text{Ext}_{\Lambda}^n(C, \Lambda) = 0$  for every positive integer  $n$ . We denote the



subcategory of  $\text{mod } \Lambda$  consisting of all maximal Cohen–Macaulay modules by  $\text{CM}(\Lambda)$ , and the corresponding stable category modulo projectives by  $\underline{\text{CM}}(\Lambda)$ .

In the following lemma, we recall some characterizations of maximal Cohen–Macaulay modules over Gorenstein algebras.

**Lemma 3.4.** *Let  $\Lambda$  be a finite-dimensional Gorenstein  $k$ -algebra and  $C$  a finitely generated  $\Lambda$ -module. The following are equivalent.*

1.  $C$  is a maximal Cohen–Macaulay module.
2.  $C$  has a projective coresolution. That is, there exists an exact sequence

$$0 \rightarrow C \rightarrow P_{-1} \rightarrow P_{-2} \rightarrow \dots$$

where every  $P_i$  is a projective  $\Lambda$ -module.

3. For every  $n > 0$ , there is a  $\Lambda$ -module  $A$  such that  $C$  is an  $n$ -th syzygy of  $A$ .
4. For some  $n \geq \text{id}_\Lambda \Lambda$ , there is a  $\Lambda$ -module  $A$  such that  $C$  is an  $n$ -th syzygy of  $A$ .

**Proof.** We only need to show that statement (1) implies statement (2); the implications (2)  $\implies$  (3)  $\implies$  (4) are obvious, and the implication (4)  $\implies$  (1) follows directly from Definition 3.3.

We use Theorem 5.4 (b) from [1]. We first describe the notation used in [1] for certain subcategories of a module category.

For a  $\Lambda$ -module  $T$  with the property that  $\text{Ext}_\Lambda^i(T, T) = 0$  for every  $i > 0$ , we define the subcategories  ${}^\perp T$  and  $\mathcal{X}_T$  of  $\text{mod } \Lambda$ . The category  ${}^\perp T$  is the subcategory of  $\text{mod } \Lambda$  consisting of all modules  $A$  such that  $\text{Ext}_\Lambda^i(A, T) = 0$  for every  $i > 0$ . The category  $\mathcal{X}_T$  is the subcategory of  ${}^\perp T$  consisting of all modules  $A$  such that there is an exact sequence

$$0 \rightarrow A \rightarrow T_0 \xrightarrow{f_0} T_1 \xrightarrow{f_1} T_2 \xrightarrow{f_2} \dots$$

where  $T_i$  is in  $\text{add } T$  and  $\text{im } f_i$  is in  ${}^\perp T$  for every  $i \geq 0$ .

Theorem 5.4 (b) in [1] says that if  $T$  is a cotilting module, then the categories  ${}^\perp T$  and  $\mathcal{X}_T$  are equal.

Now consider the case  $T = \Lambda$ . Since  $\Lambda$  is a Gorenstein algebra, it is a cotilting module, and then by the above we have  ${}^\perp \Lambda = \mathcal{X}_\Lambda$ . Furthermore,  ${}^\perp \Lambda$  is the category  $\text{CM}(\Lambda)$  of maximal Cohen–Macaulay modules. Therefore, every maximal Cohen–Macaulay module is in the category  $\mathcal{X}_\Lambda$ , and thus it has a resolution of the form

$$0 \rightarrow C \rightarrow P_{-1} \rightarrow P_{-2} \rightarrow \dots$$

where every  $P_i$  is a projective  $\Lambda$ -module.  $\square$

We now recall the theorem by Buchweitz which provides the connection we need between singularity categories and stable categories of maximal Cohen–Macaulay modules.

**Theorem 3.5.** (See [3, Theorem 4.4.1].) *Let  $\Lambda$  be a finite-dimensional Gorenstein algebra. Then there is an equivalence of triangulated categories*

$$\underline{\mathbf{CM}}(\Lambda) \xrightarrow{\cong} \mathbf{D}_{\text{sg}}(\Lambda)$$

*given by sending every object in  $\underline{\mathbf{CM}}(\Lambda)$  to a stalk complex concentrated in degree 0.*

A direct consequence of Theorem 3.5 is that if two finite-dimensional Gorenstein algebras  $\Lambda$  and  $\Sigma$  are singularly equivalent, then the categories  $\underline{\mathbf{CM}}(\Lambda)$  and  $\underline{\mathbf{CM}}(\Sigma)$  are triangle equivalent. If the algebras are not only singularly equivalent, but singularly equivalent of Morita type (with level), then there are tensor functors  $N \otimes_{\Lambda} -$  and  $M \otimes_{\Sigma} -$  that induce equivalences between the singularity categories  $\mathbf{D}_{\text{sg}}(\Lambda)$  and  $\mathbf{D}_{\text{sg}}(\Sigma)$ . What we aim to prove now is that these tensor functors also induce equivalences between the stable categories  $\underline{\mathbf{CM}}(\Lambda)$  and  $\underline{\mathbf{CM}}(\Sigma)$  of maximal Cohen–Macaulay modules. We first show that these functors preserve the property of being a maximal Cohen–Macaulay module.

**Lemma 3.6.** *Let  ${}_{\Lambda}M_{\Sigma}$  and  ${}_{\Sigma}N_{\Lambda}$  be bimodules which induce a singular equivalence of Morita type with level between two finite-dimensional Gorenstein  $k$ -algebras  $\Lambda$  and  $\Sigma$ . Then the functors*

$$N \otimes_{\Lambda} - : \text{mod } \Lambda \rightarrow \text{mod } \Sigma \quad \text{and} \quad M \otimes_{\Sigma} - : \text{mod } \Sigma \rightarrow \text{mod } \Lambda$$

*send maximal Cohen–Macaulay modules to maximal Cohen–Macaulay modules.*

**Proof.** Let  $n = \max\{\text{id}_{\Lambda} \Lambda, \text{id}_{\Sigma} \Sigma\}$  (this is finite since the algebras  $\Lambda$  and  $\Sigma$  are Gorenstein). Let  $C$  be a maximal Cohen–Macaulay module over  $\Lambda$ . Then by Lemma 3.4, there is a  $\Lambda$ -module  $A$  such that  $C$  is an  $n$ -th syzygy of  $A$ . By Lemma 2.7, the  $\Sigma$ -module  $N \otimes_{\Lambda} C$  is an  $n$ -th syzygy of  $N \otimes_{\Lambda} A$ , and therefore by Lemma 3.4 it is a maximal Cohen–Macaulay module.  $\square$

Finally, we are ready to prove the main result of this section.

**Proposition 3.7.** *Let  ${}_{\Lambda}M_{\Sigma}$  and  ${}_{\Sigma}N_{\Lambda}$  be bimodules which induce a singular equivalence of Morita type with level between two finite-dimensional Gorenstein  $k$ -algebras  $\Lambda$  and  $\Sigma$ . Then the functors*

$$N \otimes_{\Lambda} - : \underline{\mathbf{CM}}(\Lambda) \rightarrow \underline{\mathbf{CM}}(\Sigma) \quad \text{and} \quad M \otimes_{\Sigma} - : \underline{\mathbf{CM}}(\Sigma) \rightarrow \underline{\mathbf{CM}}(\Lambda)$$

*are equivalences of triangulated categories.*

**Proof.** We first check that  $N \otimes_{\Lambda} -$  actually gives a functor from  $\underline{\mathbf{CM}}(\Lambda)$  to  $\underline{\mathbf{CM}}(\Sigma)$ . We know from Lemma 3.6 that it gives a functor from  $\mathbf{CM}(\Lambda)$  to  $\mathbf{CM}(\Sigma)$ . By Lemma 2.7, we see that if  $f$  is a map of  $\Lambda$ -modules that factors through a projective module, then the

map  $N \otimes_{\Lambda} f$  also factors through a projective module. Thus  $N \otimes_{\Lambda} -$  gives a well-defined functor from  $\underline{\text{CM}}(\Lambda)$  to  $\underline{\text{CM}}(\Sigma)$ .

Consider the diagram

$$\begin{array}{ccc} \underline{\text{CM}}(\Lambda) & \xrightarrow{N \otimes_{\Lambda} -} & \underline{\text{CM}}(\Sigma) \\ \downarrow \simeq & & \downarrow \simeq \\ \text{D}_{\text{sg}}(\Lambda) & \xrightarrow[\simeq]{N \otimes_{\Lambda} -} & \text{D}_{\text{sg}}(\Sigma) \end{array}$$

of categories and functors, where the vertical functors are the equivalences from [Theorem 3.5](#), and the functor  $N \otimes_{\Lambda} -$  in the bottom row is an equivalence by [Proposition 2.5](#). The diagram commutes, and therefore the functor  $N \otimes_{\Lambda} -$  in the top row is also an equivalence.  $\square$

#### 4. Rotations of extensions

If  $U$  and  $V$  are modules over an algebra  $\Lambda$ , then dimension shift gives isomorphisms  $\text{Ext}_{\Lambda}^n(U, V) \cong \text{Ext}_{\Lambda}^{n-i}(\Omega_{\Lambda}^i(U), V)$  for integers  $n$  and  $i$  with  $n > i > 0$ . If the algebra  $\Lambda$  is Gorenstein, then all projective  $\Lambda$ -modules have finite injective dimension. This means that for sufficiently large  $n$  (more precisely,  $n > \text{id}_{\Lambda} \Lambda$ ), we can use projective resolutions to do dimension shifting in the second argument of  $\text{Ext}$  as well. That is, we have isomorphisms  $\text{Ext}_{\Lambda}^n(U, V) \cong \text{Ext}_{\Lambda}^{n+i}(U, \Omega_{\Lambda}^i(V))$ . By dimension shifting in both arguments, we then get isomorphisms

$$\text{Ext}_{\Lambda}^n(U, V) \cong \text{Ext}_{\Lambda}^n(\Omega_{\Lambda}^i(U), \Omega_{\Lambda}^i(V)),$$

where we stay in the same degree  $n$ , but replace both arguments to  $\text{Ext}$  by their  $i$ -th syzygies. In this section, we describe such isomorphisms, which we call rotation maps, and which are going to be used several times in later sections.

For defining the rotation maps, we do not need to assume that we are working over a Gorenstein algebra. This however means that the maps are not necessarily isomorphisms. We first define the maps in a general setting, and then in [Lemma 4.2](#) describe the conditions we need for ensuring that they are isomorphisms.

**Definition 4.1.** Let  $\Lambda$  be a finite-dimensional  $k$ -algebra, and let  $U$  and  $V$  be finitely generated  $\Lambda$ -modules. Choose projective resolutions  $\pi: \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow U \rightarrow 0$  and  $\tau: \cdots \rightarrow Q_1 \rightarrow Q_0 \rightarrow V \rightarrow 0$  of the modules  $U$  and  $V$ . Let  $i$  and  $n$  be integers with  $i < n$ , and let

$$\begin{aligned} \pi_i: 0 \rightarrow \Omega_{\Lambda}^i(U) \rightarrow P_{i-1} \rightarrow \cdots \rightarrow P_0 \rightarrow U \rightarrow 0, \\ \tau_i: 0 \rightarrow \Omega_{\Lambda}^i(V) \rightarrow Q_{i-1} \rightarrow \cdots \rightarrow Q_0 \rightarrow V \rightarrow 0 \end{aligned}$$

be truncations of the chosen projective resolutions. We define the *i*-th rotation of the extension group  $\text{Ext}_\Lambda^n(U, V)$  with respect to the resolutions  $\pi$  and  $\tau$  to be the map

$$\begin{array}{ccc} \rho_i: \text{Ext}_\Lambda^n(U, V) & \xrightarrow{\hspace{10em}} & \text{Ext}_\Lambda^n(\Omega_\Lambda^i(U), \Omega_\Lambda^i(V)) \\ & \searrow^{(\tau_i)_*} & \nearrow^{(\pi_i^*)^{-1}} \\ & \text{Ext}_\Lambda^{n+i}(U, \Omega_\Lambda^i(V)) & \end{array}$$

given by  $\rho_i = (\pi_i^*)^{-1}(\tau_i)_*$ .

Consider the situation in the above definition. If the algebra  $\Lambda$  is Gorenstein and  $n > \text{id}_\Lambda \Lambda$ , then for each of the projective modules  $Q_j$ , we have  $\text{id}_\Lambda Q_j \leq \text{id}_\Lambda \Lambda < n$ , and thus the map  $(\tau_i)_*$  is an isomorphism. This gives the following result.

**Lemma 4.2.** *Let  $\Lambda$  be a finite-dimensional Gorenstein  $k$ -algebra, and let  $U$  and  $V$  be finitely generated  $\Lambda$ -modules. For every  $n > \text{id}_\Lambda \Lambda$  and every  $i < n$ , the *i*-th rotation*

$$\rho_i: \text{Ext}_\Lambda^n(U, V) \rightarrow \text{Ext}_\Lambda^n(\Omega_\Lambda^i(U), \Omega_\Lambda^i(V))$$

(with respect to any projective resolutions of  $U$  and  $V$ ) is an isomorphism.

If we look at a rotation map of an extension group  $\text{Ext}_\Lambda^n(U, U)$  with the same module in both arguments, then the action of the map can be viewed as a concrete “rotation” of the extensions, as we will now see. Let  $\pi: \dots \rightarrow P_1 \rightarrow P_0 \rightarrow U \rightarrow 0$  be a projective resolution of  $U$ , and consider the *i*-th rotation map

$$\rho_i: \text{Ext}_\Lambda^n(U, U) \rightarrow \text{Ext}_\Lambda^n(\Omega_\Lambda^i(U), \Omega_\Lambda^i(U))$$

with respect to the resolution  $\pi$ . Every element of  $\text{Ext}_\Lambda^n(U, U)$  can be represented by an exact sequence of the form

$$0 \rightarrow U \rightarrow E \rightarrow P_{n-2} \rightarrow \dots \rightarrow P_i \xrightarrow{\hspace{1em}} P_{i-1} \rightarrow \dots \rightarrow P_0 \rightarrow U \rightarrow 0$$

$\searrow \hspace{1em} \nearrow$   
 $\Omega_\Lambda^i(U)$

Applying the map  $\rho_i$  to the element represented by this sequence produces the element represented by the following sequence:

$$0 \rightarrow \Omega_\Lambda^i(U) \rightarrow P_{i-1} \rightarrow \dots \rightarrow P_0 \xrightarrow{\hspace{1em}} E \rightarrow P_{n-2} \rightarrow \dots \rightarrow P_i \rightarrow \Omega_\Lambda^i(U) \rightarrow 0$$

$\searrow \hspace{1em} \nearrow$   
 $U$

We have thus rotated the sequence by removing an *i*-fold sequence from the right side and moving it to the left side.

## 5. Isomorphisms between extension groups

In this section, we show that if  $\Lambda$  and  $\Sigma$  are Gorenstein algebras which are singularly equivalent of Morita type with level, then we have isomorphisms between extension groups over  $\Lambda$  and extension groups over  $\Sigma$  in sufficiently high degrees. More precisely, if  ${}_{\Lambda}M_{\Sigma}$  and  ${}_{\Sigma}N_{\Lambda}$  are bimodules which induce a singular equivalence of Morita type with level between the algebras  $\Lambda$  and  $\Sigma$ , then the functor  $N \otimes_{\Lambda} -$  induces an isomorphism

$$\text{Ext}_{\Lambda}^n(A, B) \cong \text{Ext}_{\Sigma}^n(N \otimes_{\Lambda} A, N \otimes_{\Lambda} B) \quad (5.1)$$

for every  $n \geq \max\{\text{id}_{\Lambda} \Lambda, \text{id}_{\Sigma} \Sigma\}$  and for any  $\Lambda$ -modules  $A$  and  $B$ . This is stated as [Proposition 5.4](#).

To prove this result, we use maximal Cohen–Macaulay modules and the results from [Section 3](#), as well as the rotation maps from [Section 4](#). By [Proposition 3.7](#), we know that in the setting described above, we have isomorphisms

$$\underline{\text{Hom}}_{\Lambda}(C, C') \cong \underline{\text{Hom}}_{\Sigma}(N \otimes_{\Lambda} C, N \otimes_{\Lambda} C') \quad (5.2)$$

between stable Hom groups over  $\Lambda$  and  $\Sigma$  for maximal Cohen–Macaulay  $\Lambda$ -modules  $C$  and  $C'$ . [Lemma 5.2](#) below relates stable Hom groups to extension groups. Using this and isomorphism [\(5.2\)](#), we show ([Proposition 5.3](#)) that there are isomorphisms

$$\text{Ext}_{\Lambda}^n(C, C') \cong \text{Ext}_{\Sigma}^n(N \otimes_{\Lambda} C, N \otimes_{\Lambda} C')$$

for all maximal Cohen–Macaulay modules  $C$  and  $C'$  and every positive integer  $n$ . Finally, to arrive at isomorphism [\(5.1\)](#) for any  $\Lambda$ -modules  $A$  and  $B$  in [Proposition 5.4](#), we use [Proposition 5.3](#) together with two facts about Gorenstein algebras from earlier sections: all syzygies of sufficiently high degree are maximal Cohen–Macaulay modules, and by using a rotation map, we can replace the modules  $A$  and  $B$  by their syzygies.

We begin this section by showing, in the following two lemmas, how extension groups between maximal Cohen–Macaulay modules can be described as stable Hom groups. If  $C$  and  $C'$  are maximal Cohen–Macaulay modules over an algebra  $\Lambda$ , then we get ([Lemma 5.2](#)) an isomorphism

$$\text{Ext}_{\Lambda}^n(C, C') \cong \underline{\text{Hom}}_{\Lambda}(K_n, C')$$

for every positive integer  $n$ , with  $K_n$  an  $n$ -th syzygy of  $C$ .

In fact, it turns out that the conditions on  $C$  and  $C'$  can be relaxed somewhat. Recall that  $C$  being a maximal Cohen–Macaulay module means that  $\text{Ext}_{\Lambda}^i(C, \Lambda) = 0$  for every positive integer  $i$ . To get the above isomorphism in degree  $n$ , it is sufficient to assume that  $\text{Ext}_{\Lambda}^n(C, \Lambda) = 0$ , and we do not need to put any assumptions on the module  $C'$ . We use this weaker assumption in the lemmas.

The following notation is used in the two lemmas. Given two modules  $A$  and  $B$  over an algebra  $\Lambda$ , we write  $\mathcal{P}_\Lambda(A, B) \subseteq \text{Hom}_\Lambda(A, B)$  for the subspace of  $\text{Hom}_\Lambda(A, B)$  consisting of morphisms that factor through a projective module; then the stable Hom group is  $\underline{\text{Hom}}_\Lambda(A, B) = \text{Hom}_\Lambda(A, B) / \mathcal{P}_\Lambda(A, B)$ .

In the first lemma, we consider the special case  $n = 1$ .

**Lemma 5.1.** *Let  $\Lambda$  be a finite-dimensional  $k$ -algebra, and let  $A$  and  $C$  be finitely generated  $\Lambda$ -modules such that  $\text{Ext}_\Lambda^1(C, \Lambda) = 0$ . Let*

$$\eta: 0 \rightarrow K \xrightarrow{\alpha} P \xrightarrow{\beta} C \rightarrow 0$$

*be a short exact sequence of  $\Lambda$ -modules with  $P$  projective. Then the sequence*

$$0 \rightarrow \mathcal{P}_\Lambda(K, A) \hookrightarrow \text{Hom}_\Lambda(K, A) \xrightarrow{\eta^*} \text{Ext}_\Lambda^1(C, A) \rightarrow 0$$

*of  $k$ -vector spaces is exact.*

**Proof.** By applying the functor  $\text{Hom}_\Lambda(-, A)$  to the sequence  $\eta$ , we get the exact sequence

$$0 \rightarrow \text{Hom}_\Lambda(C, A) \xrightarrow{\beta^*} \text{Hom}_\Lambda(P, A) \xrightarrow{\alpha^*} \text{Hom}_\Lambda(K, A) \xrightarrow{\eta^*} \text{Ext}_\Lambda^1(C, A) \rightarrow 0.$$

From this we obtain the short exact sequence

$$0 \rightarrow \text{im } \alpha^* \hookrightarrow \text{Hom}_\Lambda(K, A) \xrightarrow{\eta^*} \text{Ext}_\Lambda^1(C, A) \rightarrow 0.$$

Now we only need to show that  $\text{im } \alpha^* = \mathcal{P}_\Lambda(K, A)$ . If a homomorphism  $f: K \rightarrow A$  lies in  $\text{im } \alpha^*$ , then it factors through the map  $\alpha: K \rightarrow P$ , and since the module  $P$  is projective, this means that  $f$  lies in  $\mathcal{P}_\Lambda(K, A)$ . We thus have  $\text{im } \alpha^* \subseteq \mathcal{P}_\Lambda(K, A)$ .

For the opposite inclusion, let  $Q$  be a projective  $\Lambda$ -module. Since we have assumed that  $\text{Ext}_\Lambda^1(C, \Lambda) = 0$ , we also have  $\text{Ext}_\Lambda^1(C, Q) = 0$ . Then from the long exact sequence obtained by applying the functor  $\text{Hom}_\Lambda(-, Q)$  to the short exact sequence  $\eta$ , we see that every homomorphism  $g: K \rightarrow Q$  factors through the homomorphism  $\alpha: K \rightarrow P$ . Thus every homomorphism which starts in  $K$  and factors through some projective module, also factors through  $\alpha$ , and we get  $\mathcal{P}_\Lambda(K, A) \subseteq \text{im } \alpha^*$ .  $\square$

Now we continue to extension groups in arbitrary degree by using the above lemma and dimension shifting.

**Lemma 5.2.** *Let  $\Lambda$  be a finite-dimensional  $k$ -algebra, let  $A$  and  $C$  be finitely generated  $\Lambda$ -modules, and let  $n$  be a positive integer. Assume that  $\text{Ext}_\Lambda^n(C, \Lambda) = 0$ . Let*

$$\pi_n: 0 \rightarrow K_n \rightarrow P_{n-1} \rightarrow P_{n-2} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow C \rightarrow 0$$

be the beginning of a projective resolution of  $C$  with  $K_n$  as the  $n$ -th syzygy. Then the sequence

$$0 \rightarrow \mathcal{P}_\Lambda(K_n, A) \hookrightarrow \text{Hom}_\Lambda(K_n, A) \xrightarrow{\pi_n^*} \text{Ext}_\Lambda^n(C, A) \rightarrow 0$$

of  $k$ -vector spaces is exact, and thus the map  $\pi_n^*$  induces an isomorphism

$$\overline{\pi_n^*}: \underline{\text{Hom}}_\Lambda(K_n, A) \xrightarrow{\cong} \text{Ext}_\Lambda^n(C, A).$$

**Proof.** Decompose the sequence  $\pi_n$  into two exact sequences

$$\eta: 0 \rightarrow K_n \rightarrow P_{n-1} \rightarrow K_{n-1} \rightarrow 0$$

$$\text{and } \pi_{n-1}: 0 \rightarrow K_{n-1} \rightarrow P_{n-2} \rightarrow P_{n-3} \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow C \rightarrow 0,$$

such that  $\pi_n = \eta \circ \pi_{n-1}$ . By dimension shifting, we have an isomorphism

$$\pi_{n-1}^*: \text{Ext}_\Lambda^1(K_{n-1}, A) \xrightarrow{\cong} \text{Ext}_\Lambda^n(C, A).$$

We observe that  $\pi_n^* = \pi_{n-1}^* \circ \eta^*$ , so the following diagram is commutative.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{P}_\Lambda(K_n, A) & \hookrightarrow & \text{Hom}_\Lambda(K_n, A) & \xrightarrow{\eta^*} & \text{Ext}_\Lambda^1(K_{n-1}, A) \longrightarrow 0 \\ & & \parallel & & \parallel & & \downarrow \cong \\ 0 & \longrightarrow & \mathcal{P}_\Lambda(K_n, A) & \hookrightarrow & \text{Hom}_\Lambda(K_n, A) & \xrightarrow{\pi_n^*} & \text{Ext}_\Lambda^n(C, A) \longrightarrow 0 \end{array}$$

By [Lemma 5.1](#), the top row of this diagram is exact. Since all the vertical maps are isomorphisms, the bottom row is also exact.  $\square$

We now show that we get the isomorphisms we want between extension groups in the special case where the involved modules are maximal Cohen–Macaulay modules. In this case, we get isomorphisms between extension groups in all positive degrees, while in the general case which is considered afterwards ([Proposition 5.4](#)), we only get isomorphisms in almost all degrees.

**Proposition 5.3.** *Let  $\Lambda$  and  $\Sigma$  be finite-dimensional Gorenstein algebras which are singularly equivalent of Morita type with level, and let  ${}_\Lambda M_\Sigma$  and  ${}_\Sigma N_\Lambda$  be bimodules which induce a singular equivalence of Morita type with level between  $\Lambda$  and  $\Sigma$ . Let  $C$  and  $C'$  be maximal Cohen–Macaulay modules over  $\Lambda$ . Then for every positive integer  $n$ , the map*

$$\text{Ext}_\Lambda^n(C, C') \xrightarrow{N \otimes_\Lambda -} \text{Ext}_\Sigma^n(N \otimes_\Lambda C, N \otimes_\Lambda C')$$

is an isomorphism.

**Proof.** The idea is to translate the two Ext groups to stable Hom groups by using [Lemma 5.2](#), and then use the equivalence of stable categories of Cohen–Macaulay modules from [Proposition 3.7](#).

Let

$$\pi_n : 0 \rightarrow K_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow C \rightarrow 0$$

be the beginning of a projective resolution of  $C$  with  $K_n$  as  $n$ -th syzygy. By [Lemma 2.7](#), the sequence  $N \otimes_{\Lambda} \pi_n$ , which is obtained by applying the functor  $N \otimes_{\Lambda} -$  to all objects and maps in  $\pi_n$ , is the beginning of a projective resolution of the  $\Sigma$ -module  $N \otimes_{\Lambda} C$ , with  $N \otimes_{\Lambda} K_n$  as the  $n$ -th syzygy.

Since  $C$  and  $C'$  are maximal Cohen–Macaulay modules, we deduce that  $N \otimes_{\Lambda} C, N \otimes_{\Lambda} C', K_n$  and  $N \otimes_{\Lambda} K_n$  are also maximal Cohen–Macaulay modules, by using [Lemma 3.4](#) and [Lemma 3.6](#). We form the following commutative diagram of  $k$ -vector spaces.

$$\begin{array}{ccc} \underline{\text{Hom}}_{\Lambda}(K_n, C') & \xrightarrow[\cong]{N \otimes_{\Lambda} -} & \underline{\text{Hom}}_{\Sigma}(N \otimes_{\Lambda} K_n, N \otimes_{\Lambda} C') \\ \pi_n^* \downarrow \cong & & \cong \downarrow (N \otimes_{\Lambda} \pi_n)^* \\ \text{Ext}_{\Lambda}^n(C, C') & \xrightarrow{N \otimes_{\Lambda} -} & \text{Ext}_{\Sigma}^n(N \otimes_{\Lambda} C, N \otimes_{\Lambda} C') \end{array}$$

The vertical maps are isomorphisms by [Lemma 5.2](#), and the map in the top row is an isomorphism by [Proposition 3.7](#). Therefore the map in the bottom row is also an isomorphism, and this concludes the proof.  $\square$

Finally, we come to the main result of this section, where we show that if two Gorenstein algebras  $\Lambda$  and  $\Sigma$  are singularly equivalent of Morita type with level, then for every extension group (of sufficiently high degree) over  $\Lambda$ , there is an isomorphic extension group over  $\Sigma$ .

**Proposition 5.4.** *Let  $\Lambda$  and  $\Sigma$  be finite-dimensional Gorenstein  $k$ -algebras which are singularly equivalent of Morita type with level, and let  ${}_{\Lambda}M_{\Sigma}$  and  ${}_{\Sigma}N_{\Lambda}$  be bimodules which induce a singular equivalence of Morita type with level between  $\Lambda$  and  $\Sigma$ . Let*

$$d = \max\{\text{id}_{\Lambda} \Lambda, \text{id}_{\Sigma} \Sigma\}$$

*be the maximum of the injective dimensions of  $\Lambda$  and  $\Sigma$ . Then for every integer  $n > d$ , we have  $k$ -vector space isomorphisms*

$$\begin{array}{ll} \text{Ext}_{\Lambda}^n(A, B) \xrightarrow[\cong]{N \otimes_{\Lambda} -} \text{Ext}_{\Sigma}^n(N \otimes_{\Lambda} A, N \otimes_{\Lambda} B) & \text{for } \Lambda\text{-modules } A \text{ and } B, \\ \text{Ext}_{\Sigma}^n(A', B') \xrightarrow[\cong]{M \otimes_{\Sigma} -} \text{Ext}_{\Lambda}^n(M \otimes_{\Sigma} A', M \otimes_{\Sigma} B') & \text{for } \Sigma\text{-modules } A' \text{ and } B'. \end{array}$$



**Proof.** Let  $A$  and  $B$  be  $\Lambda$ -modules, and let

$$\pi : \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow A \rightarrow 0 \quad \text{and} \quad \tau : \cdots \rightarrow Q_1 \rightarrow Q_0 \rightarrow B \rightarrow 0$$

be projective resolutions. Then by [Lemma 2.7](#), the sequences  $N \otimes_{\Lambda} \pi$  and  $N \otimes_{\Lambda} \tau$  are projective resolutions of the  $\Sigma$ -modules  $N \otimes_{\Lambda} A$  and  $N \otimes_{\Lambda} B$ . We form the following commutative diagram, where  $\rho_d$  is the  $d$ -th rotation map with respect to the resolutions  $\pi$  and  $\tau$ , and  $\rho'_d$  the  $d$ -th rotation map with respect to the resolutions  $N \otimes_{\Lambda} \pi$  and  $N \otimes_{\Lambda} \tau$ . These maps are isomorphisms by [Lemma 4.2](#).

$$\begin{array}{ccc} \text{Ext}_{\Lambda}^n(A, B) & \xrightarrow{N \otimes_{\Lambda} -} & \text{Ext}_{\Sigma}^n(N \otimes_{\Lambda} A, N \otimes_{\Lambda} B) \\ \rho_d \downarrow \cong & & \cong \downarrow \rho'_d \\ \text{Ext}_{\Lambda}^n(\Omega_{\Lambda}^d(A), \Omega_{\Lambda}^d(B)) & \xrightarrow[\cong]{N \otimes_{\Lambda} -} & \text{Ext}_{\Sigma}^n(N \otimes_{\Lambda} \Omega_{\Lambda}^d(A), N \otimes_{\Lambda} \Omega_{\Lambda}^d(B)) \end{array}$$

By [Lemma 3.4](#), the syzygies  $\Omega_{\Lambda}^d(A)$  and  $\Omega_{\Lambda}^d(B)$  are maximal Cohen–Macaulay modules, and then by [Proposition 5.3](#), the map  $N \otimes_{\Lambda} -$  in the bottom row is an isomorphism. It follows that the map  $N \otimes_{\Lambda} -$  in the top row is an isomorphism. This gives the first of the two isomorphisms we want. The second isomorphism follows by symmetry.  $\square$

### 6. Hochschild cohomology rings

In this section, we define the Hochschild cohomology ring  $\text{HH}^*(\Lambda)$  of an algebra  $\Lambda$ , and we show that if two Gorenstein  $k$ -algebras are singularly equivalent of Morita type with level, then their Hochschild cohomology rings are isomorphic in almost all degrees.

We first introduce some notation for rings of extensions. If  $\Lambda$  is a  $k$ -algebra and  $A$  a  $\Lambda$ -module, then we define

$$\mathcal{E}_{\Lambda}^*(A) = \text{Ext}_{\Lambda}^*(A, A) = \bigoplus_{n \geq 0} \text{Ext}_{\Lambda}^n(A, A).$$

That is,  $\mathcal{E}_{\Lambda}^*(A)$  denotes the graded  $k$ -algebra which is the direct sum of all extension groups of  $A$  by itself, with multiplication given by Yoneda product.

We are interested in the “asymptotic” behaviour of such graded rings of extensions; that is, we want to find isomorphisms which hold in all degrees above some finite bound. Given an extension ring  $\mathcal{E}_{\Lambda}^*(A)$ , we therefore consider the graded ideals of the form

$$\mathcal{E}_{\Lambda}^{>d}(A) = \bigoplus_{n > d} \text{Ext}_{\Lambda}^n(A, A)$$

for some integer  $d$ . Such an ideal is a graded nonunital  $k$ -algebra. When we want to say that two extension rings  $\mathcal{E}_{\Lambda}^*(A)$  and  $\mathcal{E}_{\Sigma}^*(B)$  (for  $k$ -algebras  $\Lambda$  and  $\Sigma$  with modules

${}_{\Lambda}A$  and  ${}_{\Sigma}B$ ) are the same in almost all degrees, the appropriate kind of isomorphism to look for, in order to preserve all the relevant structure, is an isomorphism of graded nonunital  $k$ -algebras between  $\mathcal{E}_{\Lambda}^{>d}(A)$  and  $\mathcal{E}_{\Sigma}^{>d}(B)$ , for some integer  $d$ . This section is mainly concerned with finding isomorphisms of this type.

We define the Hochschild cohomology of an algebra as the extension ring of the algebra over its enveloping algebra.

**Definition 6.1.** Let  $\Lambda$  be a finite-dimensional  $k$ -algebra. The *Hochschild cohomology ring* of  $\Lambda$  is the extension ring  $\text{HH}^*(\Lambda) = \mathcal{E}_{\Lambda^e}^*(\Lambda)$ .

Hochschild cohomology was first defined by G. Hochschild in [11]. The original definition uses the bar resolution. We follow the definition in [5], where Hochschild cohomology is given by extension groups. Since we have assumed that  $k$  is a field, this definition is equivalent to the original one. More generally, the two definitions are equivalent whenever  $\Lambda$  is projective over  $k$  (see [5, IX, §6]).

We now turn to the problem of showing that singular equivalences of Morita type with level between Gorenstein algebras preserve Hochschild cohomology in almost all degrees. We need the following diagram lemma, known as the “ $3 \times 3$  splice”.

**Lemma 6.2.** (See [14, Lemma VIII.3.1].) *Let  $R$  be a ring, and let*

$$\begin{array}{ccccc}
 & \eta' : & \eta : & \eta'' : & \\
 \eta_A : & A' \longrightarrow & A \longrightarrow & A'' & \\
 & \downarrow & \downarrow & \downarrow & \\
 \eta_B : & B' \longrightarrow & B \longrightarrow & B'' & \\
 & \downarrow & \downarrow & \downarrow & \\
 \eta_C : & C' \longrightarrow & C \longrightarrow & C'' & 
 \end{array}$$

*be a commutative diagram of  $R$ -modules, where the three rows  $\eta_A$ ,  $\eta_B$  and  $\eta_C$ , as well as the three columns  $\eta'$ ,  $\eta$  and  $\eta''$ , are short exact sequences. Then the elements in the extension group  $\text{Ext}_R^1(C'', A')$  represented by the composition  $\eta_A \circ \eta''$  and by the composition  $\eta' \circ \eta_C$  are the additive inverses of each other:*

$$[\eta_A \circ \eta''] = -[\eta' \circ \eta_C].$$

If two bimodules  ${}_{\Lambda}M_{\Sigma}$  and  ${}_{\Sigma}N_{\Lambda}$  induce a singular equivalence of Morita type with level between algebras  $\Lambda$  and  $\Sigma$ , then the  $\Lambda^e$ -module  $M \otimes_{\Sigma} N$  is a syzygy of  $\Lambda$ . In the following lemma, we use Lemma 6.2 to show that under certain assumptions, the tensor functors  $(M \otimes_{\Sigma} N) \otimes_{\Lambda} -$  and  $- \otimes_{\Lambda} (M \otimes_{\Sigma} N)$  induce isomorphisms of Ext groups in almost all degrees. This is afterwards used in the proof of Theorem 6.4.

**Lemma 6.3.** *Let  $\Lambda$  be a finite-dimensional Gorenstein  $k$ -algebra, and let  $U$  be a  $\Lambda^e$ -module which is projective as a left  $\Lambda$ -module and as a right  $\Lambda$ -module. Let  $d \geq 2 \cdot \text{id}_\Lambda \Lambda$ . Let  $K$  be an  $i$ -th syzygy of  $\Lambda$  as  $\Lambda^e$ -module, for some  $i < d$ . Then the maps*

$$K \otimes_\Lambda - : \mathcal{E}_{\Lambda^e}^{>d}(U) \rightarrow \mathcal{E}_{\Lambda^e}^{>d}(K \otimes_\Lambda U) \quad \text{and} \quad - \otimes_\Lambda K : \mathcal{E}_{\Lambda^e}^{>d}(U) \rightarrow \mathcal{E}_{\Lambda^e}^{>d}(U \otimes_\Lambda K)$$

are isomorphisms of graded nonunital  $k$ -algebras.

**Proof.** We show that the map  $K \otimes_\Lambda -$  is an isomorphism; the proof for  $- \otimes_\Lambda K$  is similar. Let

$$\pi : \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow \Lambda \rightarrow 0$$

be a projective resolution of  $\Lambda$  as  $\Lambda^e$ -module, with  $K$  as the  $i$ -th syzygy, and let

$$\sigma : \cdots \rightarrow P_1 \otimes_\Lambda U \rightarrow P_0 \otimes_\Lambda U \rightarrow U \rightarrow 0$$

be the result of applying the functor  $- \otimes_\Lambda U$  to the sequence  $\pi$  and identifying  $\Lambda \otimes_\Lambda U$  with  $U$  in the last term. This sequence is exact since  $U$  is projective as left module, and every  $P_j \otimes_\Lambda U$  is projective since  $U$  is projective as right module. Thus,  $\sigma$  is a projective resolution of  $U$ , and  $K \otimes_\Lambda U$  is an  $i$ -th syzygy of  $U$ .

By Lemma 3.2, the enveloping algebra  $\Lambda^e$  of  $\Lambda$  is Gorenstein, and we have  $\text{id}_{\Lambda^e} \Lambda^e \leq 2 \cdot \text{id}_\Lambda \Lambda \leq d$ . Then by Lemma 4.2, the  $i$ -th rotation map

$$\rho_i : \mathcal{E}_{\Lambda^e}^{>d}(U) \rightarrow \mathcal{E}_{\Lambda^e}^{>d}(K \otimes_\Lambda U)$$

(with respect to the resolution  $\sigma$ ) is an isomorphism of graded nonunital  $k$ -algebras. We show that the map  $K \otimes_\Lambda -$  is an isomorphism by showing that it is equal to the map  $\rho_i$ , up to sign. More precisely, we show that for any homogeneous element  $[\eta] \in \mathcal{E}_{\Lambda^e}^{>d}(U)$  of degree  $n > d$ , we have

$$K \otimes_\Lambda [\eta] = (-1)^{in} \cdot \rho_i([\eta]).$$

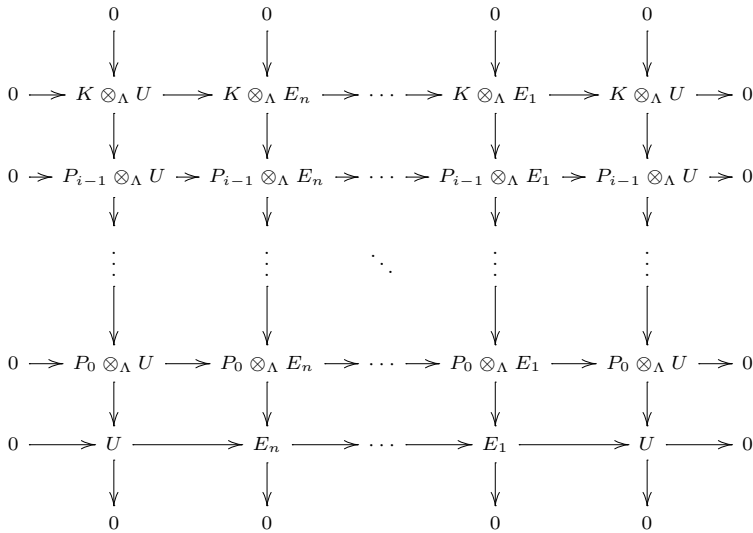
Let  $[\eta] \in \mathcal{E}_{\Lambda^e}^{>d}(U)$  be a homogeneous element of degree  $n > d$  represented by an exact sequence

$$\eta : 0 \rightarrow U \rightarrow E_n \rightarrow \cdots \rightarrow E_1 \rightarrow U \rightarrow 0.$$

We can assume without loss of generality that all the modules  $E_j$  are projective as left  $\Lambda$ -modules and as right  $\Lambda$ -modules. Let

$$\pi_i : 0 \rightarrow K \rightarrow P_{i-1} \rightarrow \cdots \rightarrow P_0 \rightarrow \Lambda \rightarrow 0$$

$$\sigma_i : 0 \rightarrow K \otimes_\Lambda U \rightarrow P_{i-1} \otimes_\Lambda U \rightarrow \cdots \rightarrow P_0 \otimes_\Lambda U \rightarrow U \rightarrow 0$$



**Fig. 1.** Commutative diagram used in the proof of [Lemma 6.3](#).

be truncations of the projective resolutions  $\pi$  and  $\sigma$ . We construct the commutative diagram in [Fig. 1](#) by tensoring  $\pi_i$  with  $\eta$  over  $\Lambda$  and identifying  $\Lambda \otimes_{\Lambda} -$  with the identity in the last row. The rows and columns of the diagram are exact sequences.

The bottom row in the diagram is the sequence  $\eta$ , the top row is the sequence  $K \otimes_{\Lambda} \eta$ , and the first and the last column are both equal to the sequence  $\sigma_i$ . By using [Lemma 6.2](#) repeatedly, we get the equality

$$[(K \otimes_{\Lambda} \eta) \circ \sigma_i] = (-1)^{in} [\sigma_i \circ \eta]$$

in the extension group  $\text{Ext}_{\Lambda^e}^{n+i}(U, K \otimes_{\Lambda} U)$ . By the definition of the rotation map  $\rho_i$  (see [Definition 4.1](#)), we then get

$$K \otimes_{\Lambda} [\eta] = [K \otimes_{\Lambda} \eta] = (-1)^{in} \cdot \rho_i([\eta]).$$

Since the map  $\rho_i$  is an isomorphism, this means that the map  $K \otimes_{\Lambda} -$  is an isomorphism as well.  $\square$

We now show that a singular equivalence of Morita type with level between Gorenstein  $k$ -algebras preserves the Hochschild cohomology in almost all degrees. A weaker form of this result, stating that a singular equivalence of Morita type preserves Hochschild cohomology groups in almost all degrees (but not necessarily the ring structure of the cohomology), appears in [\[20, Remark 4.3\]](#).

**Theorem 6.4.** *Let  $\Lambda$  and  $\Sigma$  be finite-dimensional Gorenstein  $k$ -algebras which are singularly equivalent of Morita type with level. Then we have the following.*

1. The Hochschild cohomology rings  $\mathrm{HH}^*(\Lambda)$  and  $\mathrm{HH}^*(\Sigma)$  are isomorphic in almost all degrees, with isomorphisms that respect the ring structure.
2. Let  ${}_{\Lambda}M_{\Sigma}$  and  ${}_{\Sigma}N_{\Lambda}$  be bimodules which induce a singular equivalence of Morita type with level  $l \geq 1$  (see Remark 6.5) between  $\Lambda$  and  $\Sigma$ , and let  $d = \max\{l, 2 \cdot \mathrm{id}_{\Lambda} \Lambda, 2 \cdot \mathrm{id}_{\Sigma} \Sigma\}$ . Then there are isomorphisms

$$\begin{array}{ccc}
 \mathrm{HH}^{>d}(\Lambda) & \xrightarrow[\cong]{\rho_l} & \mathcal{E}_{\Lambda^e}^{>d}(M \otimes_{\Sigma} \Sigma \otimes_{\Sigma} N) \\
 N \otimes_{\Lambda} - \otimes_{\Lambda} M \downarrow \cong & & \cong \uparrow M \otimes_{\Sigma} - \otimes_{\Sigma} N \\
 \mathcal{E}_{\Sigma^e}^{>d}(N \otimes_{\Lambda} \Lambda \otimes_{\Lambda} M) & \xleftarrow[\rho'_l]{\cong} & \mathrm{HH}^{>d}(\Sigma)
 \end{array}$$

of graded nonunital  $k$ -algebras, where the maps  $\rho_l$  and  $\rho'_l$  are rotation maps.

**Proof.** We show part (2). Part (1) then follows directly.

Since  $M$  and  $N$  induce a singular equivalence of Morita type with level  $l$ , the module  $M \otimes_{\Sigma} \Sigma \otimes_{\Sigma} N \cong M \otimes_{\Sigma} N$  is an  $l$ -th syzygy of  $\Lambda$  as a  $\Lambda^e$ -module. Let

$$\rho_l: \mathrm{HH}^{>d}(\Lambda) \rightarrow \mathcal{E}_{\Lambda^e}^{>d}(M \otimes_{\Sigma} \Sigma \otimes_{\Sigma} N)$$

be the  $l$ -th rotation map with respect to a projective resolution of  $\Lambda$  with  $M \otimes_{\Sigma} \Sigma \otimes_{\Sigma} N$  as the  $l$ -th syzygy. By Lemma 3.2, the enveloping algebras  $\Lambda^e$  and  $\Sigma^e$  are Gorenstein algebras, and we have  $\mathrm{id}_{\Lambda^e} \Lambda^e \leq 2 \cdot \mathrm{id}_{\Lambda} \Lambda$  and  $\mathrm{id}_{\Sigma^e} \Sigma^e \leq 2 \cdot \mathrm{id}_{\Sigma} \Sigma$ . By Lemma 4.2, the rotation map  $\rho_l$  is an isomorphism, since

$$\max\{l, \mathrm{id}_{\Lambda^e} \Lambda^e, \mathrm{id}_{\Sigma^e} \Sigma^e\} \leq \max\{l, 2 \cdot \mathrm{id}_{\Lambda} \Lambda, 2 \cdot \mathrm{id}_{\Sigma} \Sigma\} = d.$$

We can similarly define the rotation map  $\rho'_l$  and show that it is an isomorphism.

We now show that the maps  $N \otimes_{\Lambda} - \otimes_{\Lambda} M$  and  $M \otimes_{\Sigma} - \otimes_{\Sigma} N$  are isomorphisms. For any  $n > d$ , we can make the following diagram:

$$\begin{array}{ccc}
 \mathrm{HH}^n(\Lambda) & \xrightarrow[\cong]{\rho_l} & \mathcal{E}_{\Lambda^e}^n(M \otimes_{\Sigma} \Sigma \otimes_{\Sigma} N) \\
 N \otimes_{\Lambda} - \otimes_{\Lambda} M \downarrow & & \uparrow M \otimes_{\Sigma} - \otimes_{\Sigma} N \\
 \mathcal{E}_{\Sigma^e}^n(N \otimes_{\Lambda} \Lambda \otimes_{\Lambda} M) & \xleftarrow[\rho'_l]{\cong} & \mathrm{HH}^n(\Sigma)
 \end{array} \tag{6.1}$$

Consider the map  $N \otimes_{\Lambda} - \otimes_{\Lambda} M$  in this diagram. We construct the following commutative diagram with this map at the top:

$$\begin{array}{ccc}
 \mathrm{HH}^n(\Lambda) & \xrightarrow{N \otimes_{\Lambda} - \otimes_{\Lambda} M} & \mathcal{E}_{\Sigma^e}^n(N \otimes_{\Lambda} \Lambda \otimes_{\Lambda} M) \\
 \downarrow \cong & & \downarrow M \otimes_{\Sigma} - \otimes_{\Sigma} N \\
 (M \otimes_{\Sigma} N) \otimes_{\Lambda} - & \xrightarrow[-\otimes_{\Lambda}(M \otimes_{\Sigma} N)]{\cong} & \mathcal{E}_{\Lambda^e}^n(M \otimes_{\Sigma} N \otimes_{\Lambda} \Lambda \otimes_{\Lambda} M \otimes_{\Sigma} N)
 \end{array}$$

By Lemma 6.3, the maps  $(M \otimes_{\Sigma} N) \otimes_{\Lambda} -$  and  $-\otimes_{\Lambda}(M \otimes_{\Sigma} N)$  in this diagram are isomorphisms, since  $M \otimes_{\Sigma} N$  is an  $l$ -th syzygy of  $\Lambda$  as  $\Lambda^e$ -module. Therefore, the map  $N \otimes_{\Lambda} - \otimes_{\Lambda} M$  in diagram (6.1) is a monomorphism. By a similar argument, the map  $M \otimes_{\Sigma} - \otimes_{\Sigma} N$  in diagram (6.1) is a monomorphism. Since  $\mathrm{HH}^n(\Lambda)$  and  $\mathrm{HH}^n(\Sigma)$  are finite-dimensional over  $k$ , it follows that these monomorphisms must be isomorphisms.  $\square$

**Remark 6.5.** In Theorem 6.4 (2), we assumed that the level  $l$  is positive. The reason for this is that if we had allowed  $l = 0$ , then we could not have made the rotation maps  $\rho_l$  and  $\rho'_l$ . This assumption does not strongly affect the applicability of the theorem, since any equivalence with level 0 implies the existence of an equivalence with level 1. In general, if two bimodules  ${}_{\Lambda}M_{\Sigma}$  and  ${}_{\Sigma}N_{\Lambda}$  induce a singular equivalence of Morita type with level  $l$  between algebras  $\Lambda$  and  $\Sigma$ , then the bimodules  $\Omega_{\Lambda \otimes_k \Sigma^{\mathrm{op}}}^1(M)$  and  $N$  induce a singular equivalence of level  $l + 1$  between  $\Lambda$  and  $\Sigma$ .

### 7. Finite generation

Support varieties for modules over artin algebras were defined by Snashall and Solberg in [18], using the Hochschild cohomology ring. In [9], Erdmann, Holloway, Snashall, Solberg and Taillefer defined two finite generation conditions **Fg1** and **Fg2** for the Hochschild cohomology ring of an algebra. These conditions ensure that the support varieties for modules over the given algebra have good properties. In [10], these conditions were reformulated as a new condition called **(Fg)** which is equivalent to the combination of **Fg1** and **Fg2**. We use the definition from [10].

In this section, we describe the finite generation condition **(Fg)**. We then show the main result of this paper (Theorem 7.4): A singular equivalence of Morita type with level between finite-dimensional Gorenstein  $k$ -algebras preserves the **(Fg)** condition.

In order to define the **(Fg)** condition, we first describe a way to view extension rings over an algebra as modules over the Hochschild cohomology ring. Let  $\Lambda$  be a finite-dimensional  $k$ -algebra and  $A$  a  $\Lambda$ -module. We define a graded ring homomorphism

$$\varphi_A: \mathrm{HH}^*(\Lambda) \rightarrow \mathcal{E}_{\Lambda}^*(A)$$

as follows. A homogeneous element of  $\mathrm{HH}^*(\Lambda)$  can be represented by an exact sequence

$$\eta: 0 \rightarrow \Lambda \rightarrow E \rightarrow P_n \rightarrow \cdots \rightarrow P_0 \rightarrow \Lambda \rightarrow 0$$

of  $\Lambda^e$ -modules, where each  $P_i$  is projective. Viewed as a sequence of right  $\Lambda$ -modules, this sequence splits. The complex

$$\eta \otimes_{\Lambda} A: 0 \rightarrow \Lambda \otimes_{\Lambda} A \rightarrow E \otimes_{\Lambda} A \rightarrow P_n \otimes_{\Lambda} A \rightarrow \cdots \rightarrow P_0 \otimes_{\Lambda} A \rightarrow \Lambda \otimes_{\Lambda} A \rightarrow 0$$

is therefore an exact sequence. By composition with the isomorphism  $\mu_A: \Lambda \otimes_{\Lambda} A \rightarrow A$  and its inverse, we get an extension

$$\mu_A \circ (\eta \otimes_{\Lambda} A) \circ \mu_A^{-1}: 0 \rightarrow A \rightarrow E \otimes_{\Lambda} A \rightarrow P_n \otimes_{\Lambda} A \rightarrow \cdots \rightarrow P_0 \otimes_{\Lambda} A \rightarrow A \rightarrow 0$$

of  $A$  by itself, and thus a representative of a homogeneous element in the extension ring  $\mathcal{E}_{\Lambda}^*(A)$ . The map  $\varphi_A$  is defined by the action

$$\varphi_A([\eta]) = [\mu_A \circ (\eta \otimes_{\Lambda} A) \circ \mu_A^{-1}]$$

on homogeneous elements. By the map  $\varphi_A$ , the graded ring  $\mathcal{E}_{\Lambda}^*(A)$  becomes a graded  $\text{HH}^*(\Lambda)$ -module.

**Definition 7.1.** Let  $\Lambda$  be a finite-dimensional  $k$ -algebra. We say that  $\Lambda$  satisfies the **(Fg)** condition if the following holds.

1. The ring  $\text{HH}^*(\Lambda)$  is Noetherian.
2. The  $\text{HH}^*(\Lambda)$ -module  $\mathcal{E}_{\Lambda}^*(\Lambda/\text{rad } \Lambda)$  is finitely generated. (The module structure is given by the map  $\varphi_{\Lambda/\text{rad } \Lambda}$ , as described above.)

By [17, Proposition 5.7], the **(Fg)** condition as defined here is equivalent to the combination of the conditions **Fg1** and **Fg2** defined in [9].

The following result describes why Gorenstein algebras are important in connection with the **(Fg)** condition.

**Theorem 7.2.** (See [9, Theorem 1.5 (a)].) *If an algebra satisfies the **(Fg)** condition, then it is a Gorenstein algebra.*

Our aim is to show that if two Gorenstein  $k$ -algebras are singularly equivalent of Morita type with level, then the **(Fg)** condition holds for one of the algebras if and only if it holds for the other. We use the following result, which describes a relation between two algebras ensuring that **(Fg)** for one of the algebras implies **(Fg)** for the other.

**Proposition 7.3.** *Let  $\Lambda$  and  $\Sigma$  be finite-dimensional  $k$ -algebras. Let  $A = \Lambda/\text{rad } \Lambda$ , and assume that we have a commutative diagram*

$$\begin{array}{ccc}
 \mathrm{HH}^{>d}(\Lambda) & \xrightarrow{\varphi_A} & \mathcal{E}_\Lambda^{>d}(A) \\
 \cong \downarrow f & & g \downarrow \cong \\
 \mathrm{HH}^{>d}(\Sigma) & \xrightarrow{\varphi_B} & \mathcal{E}_\Sigma^{>d}(B)
 \end{array} \tag{7.1}$$

of graded nonunital  $k$ -algebras, for some  $\Sigma$ -module  $B$  and some positive integer  $d$ , where the vertical maps  $f$  and  $g$  are isomorphisms. Assume that  $\Sigma$  satisfies the **(Fg)** condition. Then  $\Lambda$  also satisfies **(Fg)**.

**Proof.** This follows from Proposition 6.3 in [16].  $\square$

We are now ready to prove the main result of this paper.

**Theorem 7.4.** *Let  $\Lambda$  and  $\Sigma$  be finite-dimensional Gorenstein algebras over the field  $k$ . Assume that  $\Lambda$  and  $\Sigma$  are singularly equivalent of Morita type with level. Then  $\Lambda$  satisfies **(Fg)** if and only if  $\Sigma$  satisfies **(Fg)**.*

**Proof.** We show that if  $\Sigma$  satisfies **(Fg)**, then  $\Lambda$  satisfies **(Fg)**. The opposite implication then follows by symmetry. Let  ${}_\Lambda M_\Sigma$  and  ${}_\Sigma N_\Lambda$  be bimodules which induce a singular equivalence of Morita type with level  $l \geq 1$  (see Remark 6.5) between  $\Lambda$  and  $\Sigma$ . Let  $d = \max\{l, 2 \cdot \mathrm{id}_\Lambda \Lambda, 2 \cdot \mathrm{id}_\Sigma \Sigma\}$ . Let  $A$  be the  $\Lambda$ -module  $\Lambda/\mathrm{rad} \Lambda$ .

The  $\Lambda^e$ -module  $M \otimes_\Sigma \Sigma \otimes_\Sigma N \cong M \otimes_\Sigma N$  is an  $l$ -th syzygy of  $\Lambda$  as  $\Lambda^e$ -module. Let  $\pi$  be a projective resolution of  $\Lambda$  with  $M \otimes_\Sigma \Sigma \otimes_\Sigma N$  as the  $l$ -th syzygy. Then the complex  $\pi \otimes_\Lambda A$  is a projective resolution of  $\Lambda \otimes_\Lambda A$ , with  $M \otimes_\Sigma \Sigma \otimes_\Sigma N \otimes_\Lambda A$  as the  $l$ -th syzygy. We construct the commutative diagram in Fig. 2, where the maps  $\rho_l$  and  $\rho'_l$  are the  $l$ -th rotation maps with respect to the resolutions  $\pi$  and  $\pi \otimes_\Lambda A$ , respectively. These maps are isomorphisms by Lemma 4.2. The map  $M \otimes_\Sigma - \otimes_\Sigma N$  in the diagram is an isomorphism by Theorem 6.4, and the map  $M \otimes_\Sigma -$  is an isomorphism by Proposition 5.4. The isomorphisms  $f$  and  $g$  are defined to be the appropriate compositions of the other isomorphisms in the diagram. By Proposition 7.3, this diagram shows that if the algebra  $\Sigma$  satisfies **(Fg)**, then  $\Lambda$  also satisfies **(Fg)**.  $\square$

We now show that the assumption of both algebras being Gorenstein is necessary in the above theorem. Example 5.5 in [16] contains two singularly equivalent algebras where one algebra satisfies **(Fg)** and the other is not Gorenstein. We use the same algebras, and show that there exists a singular equivalence of Morita type with level between them.

**Example 7.5.** Let  $\Lambda = kQ/\langle \rho \rangle$  and  $\Sigma = kR/\langle \sigma \rangle$  be  $k$ -algebras given by the following quivers and relations:





We show that the bimodules  $M$  and  $N$  induce a singular equivalence of Morita type with level 1 between the algebras  $\Lambda$  and  $\Sigma$ . We first check that these bimodules satisfy the first two conditions in [Definition 2.4](#). Considering only the left or right structure of  $M$  and  $N$ , we have the following four isomorphisms:

$$\begin{aligned} \Lambda M &\cong \Lambda & N_\Lambda &\cong e_2 \Lambda \\ \Sigma N &\cong \Sigma^2 & M_\Sigma &\cong \Sigma \end{aligned}$$

Thus the bimodules  $M$  and  $N$  are projective when viewed as one-sided (left or right) modules.

To check the last two conditions in [Definition 2.4](#), we compute the tensor products  $M \otimes_\Sigma N$  and  $N \otimes_\Lambda M$  as representations of quivers, and check that they are syzygies of  $\Lambda$  and  $\Sigma$ , respectively. The enveloping algebra  $\Lambda^e$  has the following quiver and relations:

$$Q \times Q^{op} : \begin{array}{ccc} & \alpha \times 1^{op} & \alpha \times 2^{op} \\ & \curvearrowright & \curvearrowright \\ 1 \times \alpha^{op} & \left( 1 \times 1^{op} \right) & \leftarrow 1 \times 2^{op} \\ & \downarrow \beta \times 1^{op} & \downarrow \beta \times 2^{op} \\ 2 \times \alpha^{op} & \left( 2 \times 1^{op} \right) & \leftarrow 2 \times 2^{op} \end{array} \quad \left\{ \begin{array}{l} (\alpha \times 1^{op})^2, (\beta \times 1^{op})(\alpha \times 1^{op}), \\ (\alpha \times 2^{op})^2, (\beta \times 2^{op})(\alpha \times 2^{op}), \\ (1 \times \alpha^{op})^2, (1 \times \alpha^{op})(1 \times \beta^{op}), \\ (2 \times \alpha^{op})^2, (2 \times \alpha^{op})(2 \times \beta^{op}), \\ (\alpha \times 1^{op})(1 \times \alpha^{op}) - (1 \times \alpha^{op})(\alpha \times 1^{op}), \\ (\alpha \times 1^{op})(1 \times \beta^{op}) - (1 \times \beta^{op})(\alpha \times 2^{op}), \\ (\beta \times 1^{op})(1 \times \alpha^{op}) - (2 \times \alpha^{op})(\beta \times 1^{op}), \\ (\beta \times 1^{op})(1 \times \beta^{op}) - (2 \times \beta^{op})(\beta \times 2^{op}) \end{array} \right\}$$

The tensor product  $M \otimes_\Sigma N$  is the  $\Lambda^e$ -module given by the following representation over  $Q \times Q^{op}$ :

$$M \otimes_\Sigma N : \begin{array}{ccc} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \\ & \curvearrowright & \curvearrowright \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \left( k^2 \right) & \leftarrow 0 \\ & \downarrow & \downarrow \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \left( k^2 \right) & \leftarrow 0 \end{array}$$

The algebra  $\Lambda$  considered as a  $\Lambda^e$ -module has the following representation over  $Q \times Q^{op}$ :

$$\Lambda : \begin{array}{ccc} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \\ & \curvearrowright & \curvearrowright \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & \left( k^2 \right) & \leftarrow 0 \\ & \downarrow & \downarrow \\ (1 \ 0) & \downarrow & \downarrow \\ 0 & \left( k \right) & \leftarrow k \\ & & \underset{1}{\leftarrow} \end{array}$$

There is an exact sequence

$$0 \rightarrow M \otimes_{\Sigma} N \rightarrow \Lambda^e e_{1 \times 1^{op}} \oplus \Lambda^e e_{2 \times 2^{op}} \rightarrow \Lambda \rightarrow 0$$

of  $\Lambda^e$ -modules, and thus  $M \otimes_{\Sigma} N$  is a first syzygy of  $\Lambda$ .

The enveloping algebra  $\Sigma^e$  has the following quiver and relations:

$$R \times R^{op}: \quad \begin{array}{c} 3 \times \gamma^{op} \\ \left( \begin{array}{c} \mathbf{3} \times \mathbf{3}^{op} \\ \curvearrowright \\ \gamma \times 3^{op} \end{array} \right) \end{array} \quad \left\{ (\gamma \times 3^{op})(3 \times \gamma^{op})^2, (3 \times \gamma^{op})^2, (3 \times \gamma^{op})(\gamma \times 3^{op}) \right\}$$

The algebra  $\Sigma$  considered as a  $\Sigma^e$ -module has the following representation over  $R \times R^{op}$ :

$$\Sigma: \quad \begin{array}{c} \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) \left( \begin{array}{c} k^2 \\ \curvearrowright \\ \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) \end{array} \right) \end{array}$$

Its minimal projective resolution is

$$\dots \rightarrow \Sigma^e \rightarrow \Sigma^e \rightarrow \Sigma \rightarrow 0,$$

with  $\Sigma$  itself as every syzygy. The tensor product  $N \otimes_{\Lambda} M$  is isomorphic to  $\Sigma$  as  $\Sigma^e$ -module; in particular, it is a first syzygy of  $\Sigma$ .

We have now shown that the bimodules  $M$  and  $N$  induce a singular equivalence of Morita type with level 1 between the algebras  $\Lambda$  and  $\Sigma$ . The algebra  $\Sigma$  satisfies the **(Fg)** condition, but  $\Lambda$  does not, and is not even a Gorenstein algebra. This shows that the assumption of both algebras being Gorenstein can not be removed in [Theorem 7.4](#).

For stable equivalences of Morita type (which are singular equivalences of Morita type with level 0), we can, under some conditions, remove the assumption of Gorensteinness.

**Corollary 7.6.** *Let  ${}_{\Lambda}M_{\Sigma}$  and  ${}_{\Sigma}N_{\Lambda}$  be indecomposable bimodules that induce a stable equivalence of Morita type between two finite-dimensional  $k$ -algebras  $\Lambda$  and  $\Sigma$ . Assume that  $\Lambda$  and  $\Sigma$  have no semisimple blocks and that  $\Lambda/\text{rad } \Lambda$  and  $\Sigma/\text{rad } \Sigma$  are separable. Then  $\Lambda$  satisfies **(Fg)** if and only if  $\Sigma$  satisfies **(Fg)**.*

**Proof.** By [\[8, Corollary 3.1 \(2\)\]](#), the assumptions in the statement of the result imply that  $(M \otimes_{\Sigma} -, N \otimes_{\Lambda} -)$  and  $(N \otimes_{\Lambda} -, M \otimes_{\Sigma} -)$  are adjoint pairs. Then, by [\[13, Corollary 4.6\]](#), it follows that  $\Lambda$  is a Gorenstein algebra if and only if  $\Sigma$  is a Gorenstein algebra. The result now follows from [Theorem 7.4](#).  $\square$

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## References

- [1] Maurice Auslander, Idun Reiten, Applications of contravariantly finite subcategories, *Adv. Math.* 86 (1) (1991) 111–152.
- [2] Petter Andreas Bergh, David A. Jorgensen, Tate-Hochschild homology and cohomology of Frobenius algebras, *J. Noncommut. Geom.* 7 (4) (2013) 907–937.
- [3] Ragnar-Olaf Buchweitz, Maximal Cohen–Macaulay modules and Tate-cohomology over Gorenstein rings, 1987, unpublished manuscript.
- [4] Jon F. Carlson, The complexity and varieties of modules, in: *Integral Representations and Applications*, Oberwolfach, 1980, in: *Lecture Notes in Math.*, vol. 882, Springer, Berlin–New York, 1981, pp. 415–422.
- [5] Henri Cartan, Samuel Eilenberg, *Homological Algebra*, Princeton University Press, Princeton, NJ, 1956.
- [6] Xiao-Wu Chen, Singular equivalences induced by homological epimorphisms, *Proc. Amer. Math. Soc.* 142 (8) (2014) 2633–2640.
- [7] Xiao-Wu Chen, Longgang Sun, Singular equivalence of Morita type, 2012, unpublished manuscript.
- [8] Alex S. Dugas, Roberto Martínez-Villa, A note on stable equivalences of Morita type, *J. Pure Appl. Algebra* 208 (2) (2007) 421–433.
- [9] Karin Erdmann, Miles Holloway, Rachel Taillefer, Nicole Snashall, Øyvind Solberg, Support varieties for selfinjective algebras, *K-Theory* 33 (1) (2004) 67–87.
- [10] Karin Erdmann, Øyvind Solberg, Radical cube zero weakly symmetric algebras and support varieties, *J. Pure Appl. Algebra* 215 (2) (2011) 185–200.
- [11] G. Hochschild, On the cohomology groups of an associative algebra, *Ann. of Math. (2)* 46 (1945) 58–67.
- [12] J. Külshammer, C. Psaroudakis, Ø. Skartsætherhagen, Derived invariance of support varieties, arXiv: 1502.04880, 2015.
- [13] Yuming Liu, Changchang Xi, Constructions of stable equivalences of Morita type for finite-dimensional algebras. III, *J. Lond. Math. Soc. (2)* 76 (3) (2007) 567–585.
- [14] Saunders Mac Lane, *Homology*, *Classics Math.*, Springer-Verlag, Berlin, 1995, reprint of the 1975 edition.
- [15] D.O. Orlov, Triangulated categories of singularities and D-branes in Landau–Ginzburg models, *Tr. Mat. Inst. Steklova* 246 (2004) 240–262, English translation in *Proc. Steklov Inst. Math.* 246 (2004) 227–248.
- [16] Chrysostomos Psaroudakis, Øystein Skartsætherhagen, Øyvind Solberg, Gorenstein categories, singular equivalences and finite generation of cohomology rings in recollements, *Trans. Amer. Math. Soc. Ser. B* 1 (2014) 45–95.
- [17] Øyvind Solberg, Support varieties for modules and complexes, in: *Trends in Representation Theory of Algebras and Related Topics*, in: *Contemp. Math.*, vol. 406, Amer. Math. Soc., Providence, RI, 2006, pp. 239–270.
- [18] Nicole Snashall, Øyvind Solberg, Support varieties and Hochschild cohomology rings, *Proc. Lond. Math. Soc. (3)* 88 (3) (2004) 705–732.
- [19] Zhengfang Wang, Singular equivalence of Morita type with level, *J. Algebra* 439 (2015) 245–269.
- [20] Guodong Zhou, Alexander Zimmermann, On singular equivalences of Morita type, *J. Algebra* 385 (2013) 64–79.