CANTOR MINIMAL SYSTEMS AND AF-EQUIVALENCE RELATIONS

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CONTENTS

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INTRODUCTION

This thesis consists of three papers, each can be read independently of the others. The papers are:

I :	AF equivalence relations associated to locally finite groups.
II:	Induced subsystems associated to a Cantor minimal system.
III :	Dynamical Choquet simplices and Cantor minimal systems.

In this introduction chapter we give an overview of the theory of dynamical systems that is relevant for this thesis. Most results will be presented without proofs.

Roughly, one can divide dynamical systems into topological and measurable, and our focus will mainly be on the former. A *topological dynamical system* is a pair (X, T), where X is a compact, second countable Hausdorff space (equivalently, X is compact and metrizable), and $T : X \to X$ is a homeomorphism. In this thesis X will (mostly) be a Cantor set and $T : X \to X$ will be a minimal homeomorphism (see definitions below). We will state the results with this framework in mind, although many of the results hold more generally.

The theory of *measurable dynamical systems* (or *ergodic theory*) is the study of measure-preserving systems $(X, T, \mathcal{B}, \mu)$, where $(X, \mathcal{B}, \mu)$ is a Lebesgue space, i.e. $\mathcal{B}$ is a (standard) σ-algebra of measurable subsets of X, and μ is a probability measure on X. The map $T : X \to X$ is measurable, and $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{B}$.

There are strong links and a lot of parallel properties and results in ergodic theory and topological dynamics, cf. [9] and [11].

1. Cantor minimal systems

A *Cantor set* is a totally disconnected compact metric space, where there are no isolated points. One can choose a basis for the topology consisting of clopen (i.e both open and closed) sets. It is a famous result by Cantor and Hausdorff, saying that any two Cantor sets are homeomorphic. One classical way to realize the Cantor set, is to start with the closed unit interval $C_1 = I = [0, 1]$, remove the open middle third to obtain $C_2 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, remove the open middle third of each of these intervals again to obtain $C_3 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$, and continue this proses infinitely many times. What remains will be the Cantor set.

Let $T : X \to X$ be a homeomorphism on a space X, and define the *orbit* (or T-orbit) $[x]_T$ of a point $x \in X$ to be the set $[x]_T = \{y \in X \mid y = T^n x \text{ for some } n \in \mathbb{Z}\}$. If all orbits are dense in X, we say that the homeomorphism T is *minimal*. There are several equivalent definitions of minimality:

Proposition 1.1. *Let (X, T) be a dynamical system. The following are equivalent:*

(i) *T is minimal.*
(ii) *X has no non-trivial closed T-invariant subsets.*
(iii) *If U is a non-empty open subset of X, then $\cup_{-\infty}^{\infty} T^{-1}(U) = X$.*

By a *Cantor minimal system* (X, T) we understand a dynamical system where X is the Cantor set and $T : X \to X$ is a minimal homeomorphism. In the theory of dynamical systems the class of Cantor minimal systems have a "universal property", in the sense that given any dynamical system (Y, S), there exists a Cantor minimal system (X,T) having (Y,S) as a factor, i.e. there exists a continuous map $F : X \to Y$ such that $F(Tx) = S(F(x))$ for all $x \in X$. This illustrates in a striking way that the family of Cantor minimal systems is vast.

Definition 1.2. Two Cantor minimal systems (X,T) and (Y,S) are *conjugate* (respectively *flip conjugate*) if there exists a homeomorphism $F : X \to Y$ such that $F(Tx) = S(F(x))$ (respectively $F(Tx) = S(F(x))$ or $F(Tx) = S^{-1}(F(x))$) for all $x \in X$. We write $(X,T) \simeq (Y,S)$ for two conjugate Cantor minimal systems.

To classify Cantor minimal systems up to conjugacy, or flip conjugacy, would seem like a hopeless task in light of the comment preceding Definition 1.2. On the other hand, motivated by C^*-algebra theory, via the so-called crossed product construction, new invariants for Cantor minimal systems were found, which were computable and of (ordered) K-theoretic nature. These invariants are complete invariants for orbit equivalence and strong orbit equivalence, respectively (cf. Definition 1.3 and Definition 1.4). We remark however, that within each strong orbit equivalence class (and even more so within each orbit equivalence class) there are an abundance (in fact, uncountably many) of non-flip conjugate Cantor minimal systems.

Definition 1.3. Two Cantor minimal systems (X,T) and (Y,S) are *orbit equivalent* if there exists a homeomorphism $F : X \to Y$ sending orbits to orbits, i.e. $F([x]_T) = [F(x)]_S$ for all $x \in X$. We call F an *orbit map.*

Orbit equivalence is an equivalence relation on the set of Cantor minimal systems. One can show that an orbit map $F : X \to Y$ maps the set of T-invariant probability measures onto the set of S-invariant probability measures. Two conjugate (or flip conjugate) systems are easily seen to be orbit equivalent.

To two Cantor minimal systems (X,T) and (Y,S) that are orbit equivalent we associate two maps (so-called orbit cocycles), $n : X \to \mathbb{Z}$ and $m : X \to \mathbb{Z}$, such that for every $x \in X$ we have $F(Tx) = S^{n(x)}(F(x))$ and $F(T^{m(x)}) = S(F(x))$. If one of the functions $n(x)$, $m(x)$ are continuous, then M. Boyle [8, Thm. 2.4] proved that the systems are either conjugate or flip conjugate.

Definition 1.4. Two Cantor minimal systems (X,T) and (Y,S) are *strongly orbit equivalent* if each of the maps $n(x)$ and $m(x)$ defined above, has at most one point of discontinuity.

According to Boyle's theorem, the strong orbit equivalence condition is the mildest weakening possible beyond flip conjugacy. As alluded to above, we will see in Section 3 that there exist complete invariants for both orbit equivalence and strong orbit equivalence of Cantor minimal systems that are computable.

There is also a notion of *weak orbit equivalence*, introduced by Glasner and Weiss in [10]. Let $FG(X,T)$ denote the group of all self homeomorphisms α of X, for which there exists a function $n : X \to \mathbb{Z}$, such that $\alpha(x) = T^{n(x)}(x)$ for all $x \in X$. The function $n(x)$ is called the *time change corresponding to* α, and the group $FG(X,T)$ is called the *full group* associated to (X,T).

Definition 1.5. Two Cantor minimal systems (X,T) and (Y,S) are said to be *weakly orbit equivalent* if there exist a homeomorphism $\alpha \in FG(X,T)$ such that the system (X,α) admits (Y,S) as a factor, and a homeomorphism $\beta \in FG(Y,S)$ such that the system (Y,β) admits (X,T) as a factor.

Some of the results obtained in the classification of weak orbit equivalent Cantor minimal systems will be presented later, after we have established some more notation and definitions. We start by introducing a concrete way to obtain a very useful model for a given Cantor minimal system.

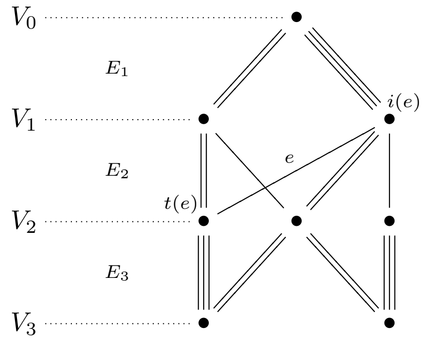

FIGURE 1. An example of the first four levels of a Bratteli diagram.

2. BRATTELI DIAGRAMS

Definition 2.1. A Bratteli diagram (V, E) consists of a vertex set V, an edge set E and two maps, $t, i : E \to V$ such that

(i) The vertex set can be written as a disjoint union of finite sets; $V = \cup_{n=0}^{\infty} V_n$, and $V_0 = \{v_0\}$ is a one-point set.
(ii) The edge set can be written as a disjoint union of finite sets; $E = \cup_{n=1}^{\infty} E_n$.
(iii) The *range* (or *terminal*) map t satisfies $t(E_n) \subset V_n$ and $t^{-1}(v) \neq \emptyset$ for all $v \in V \setminus V_0$.
(iv) The *source* (or *initial*) map i satisfies $i(E_n) \subset V_{n-1}$ and $i^{-1}(v) \neq \emptyset$ for all $v \in V$.

Originally, Bratteli diagrams were introduced for the purpose of investigating inductive limits of finite dimensional C^*-algebras, but later they have showed to be a very useful and important tool also in the study of dynamical systems.

A Bratteli diagram is often presented as a (downward) directed graph, where the nodes at level n is in one-to-one correspondence with the vertices in V_n, and where the edges between two consecutive levels n and $n+1$, are in one-to-one correspondence with the edges in E_n, in such a way that the range and source maps can be read from the graph. See Figure 1 for an example. If for each vertex set V_n, one chooses a linear order on the vertices, the diagram can be coded in a sequence of matrices, called *incidence matrices*, $\{A_n\}_{n=1}^{\infty} = \{(a_{ij}^n)\}_{n=1}^{\infty}$, where the entry a_{ij}^n gives the number of edges connecting $v_i \in V_n = \{v_1, v_2, \cdots, v_{l_n}\}$ and $w_j \in V_{n-1} = \{w_1, w_2, \cdots, w_{l_{n-1}}\}$, i.e. a_{ij} is the number of edges having range v_i and source w_j.

We say that $(e_1, e_2, \cdots)$ is an infinite *path* of the diagram (V, E) if $e_n \in E_n$ and $t(e_n) = i(e_{n+1})$ for all $n > 0$. One defines a finite path $(e_l, e_{l+1}, \cdots, e_k)$ between levels l and k, $k > l$, similarly. We denote by $X_{(V,E)}$ the set of all infinite paths starting at $v_0 \in V_0$.

There is a natural notion of isomorphism between Bratteli diagrams (V, E) and (W, F), namely a pair of bijections, one between the vertex sets V and W, and one between the edge sets E and F, preserving the gradings and intertwining the respective source and range maps. Also, starting with a Bratteli diagram (V, E), one can create a new Bratteli diagram (W, F), by telescoping to a sequence $0 = t_0 < t_1 < t_2 < \cdots$ of natural numbers, i.e. one sets $W_k = V_{n_{t_k}}$ and $F_k = \{$the set of all finite paths between level n_{k-1} and level n_k in $(V, E)\}$. The range and source maps of (W, F) are defined the obvious way.

We define the equivalence relation $\sim$ on the set of Bratteli diagrams to be the equivalence relation generated by isomorphism and telescopings, and note that between equivalent Bratteli diagrams there is a natural isomorphism between the path spaces.

Remark 2.2. *One can show (cf.* [7, Lemma 4.13]*) that two Bratteli diagrams* (V,E) *and* (W,F)*, are equivalent if and only if there exists a third diagram* $(\widetilde{V},\widetilde{E})$*, such that telescoping* $(\widetilde{V},\widetilde{E})$ *to even levels yields a telescope of* (V,E)*, while telescoping* $(\widetilde{V},\widetilde{E})$ *to odd levels yields a telescope of* (W,F)*. We will refer to the diagram* $(\widetilde{V},\widetilde{E})$ *as an* aggregate diagram *for the equivalence between* (V,E) *and* (W,F)*.*

The path space $X_{(V,E)}$ of a Bratteli diagram (V,E) is a totally disconnected, compact metric space, where one possible metric is given by $d(x,y)=\frac{1}{n+1}$, where $x=(e_1,e_2,\cdots)$, $y=(f_1,f_2,\cdots)\in X_{(V,E)}$, and $n=\sup\{k\,|\,e_1=f_1,\cdots,e_k=f_k\}$. As a clopen basis for the topology one can take the *cylinder sets*, which we now define. Let $x=(e_1,e_2,\cdots)$ be an infinite path in $X_{(V,E)}$, and define the k'th cylinder set $C_k(x)$ associated to x, by $C_k(x)=\{(f_1,f_2,\cdots)\in X_{(V,E)}\,|\,e_1=f_1,e_2=f_2,\cdots,e_k=f_k\}$, i.e. $C_k(x)$ consists of all paths that agree with x at the first k edges. By a *simple* Bratteli diagram (V,E) we will understand a Bratteli diagram such that for given n and any $v\in V_n$, there exists a level $m>n$ such that v is connected to all vertices in V_m by at least one (finite) path. For a simple Bratteli diagram it is not hard to show that there are no isolated points, and hence the path space is a Cantor set. (We will always assume our Bratteli diagrams (V,E) are non-trivial, i.e. $X_{(V,E)}$ is an infinite set.)

In order to define a minimal homeomorphism on the path space, we introduce the notion of *ordered Bratteli diagrams*. Let (V,E) be a Bratteli diagram. For $v\in V_n$, give a linear order to the set of edges in $t^{-1}(v)$. Doing this for all vertices, we say that (V,E) is an ordered Bratteli diagram, and we denote it by $(V,E,\geq)$. We can now give a partial order to the set of paths, using the lexicographic order, i.e. given two paths $x=(e_1,e_2,\cdots)$ and $y=(f_1,f_2,\cdots)$ in $X_{(V,E)}$, we say that $x>y$ if there exists k such that $e_n=f_n$ for all $n>k$, and $e_k>f_k$ in the linear order given to $t^{-1}(t(e_k))(=t^{-1}(t(f_k)))$. Note that we only compare paths that are *cofinal*, i.e. paths that follow each other from some vertex level on.

Definition 2.3. Let $(V,E,\geq)$ be an ordered Bratteli diagram. We say that $(V,E,\geq)$ is *properly ordered* if (V,E) is simple, and the order is such that there are exactly one path, denoted x_{max}, for which all edges are maximal, and exactly one path, denoted x_{min}, for which all edges are minimal.

For a properly ordered Bratteli diagram $(V,E,\geq)$, we define the *Vershik map* $T_{(V,E)}:X_{(V,E)}\to X_{(V,E)}$ to be the map sending a non-maximal path to its successor in the lexicographic order, while the unique maximal path is mapped to the unique minimal path.

Proposition 2.4. [12, section 3] *Let* $(V,E,\geq)$ *be a properly ordered Bratteli diagram. Then* $(X_{(V,E)},T_{(V,E)})$ *is a Cantor minimal system.*

The Cantor minimal system $(X_{(V,E)},T_{(V,E)})$ obtained from $(V,E,\geq)$ is called the *Bratteli-Vershik system* associated to $(V,E,\geq)$.

One would also like to go the other way, i.e. start with a Cantor minimal system (X,T) and create a Bratteli diagram representing it, i.e. find a properly ordered Bratteli diagram $(V,E,\geq)$ such that (X,T) is conjugate to $(X_{(V,E)},T_{(V,E)})$. In [12, section 4] this is done in detail. We only give a short overview here.

Let (X,T,x_0) be a Cantor minimal system, where $x_0\in X$ is a fixed point. We call (X,T,x_0) a *pointed Cantor minimal system*. Let U be a clopen set in X, and define *the return time map* $\lambda : U \to \mathbb{N}$, by $\lambda(x) = \inf\{n > 0 \,|\, T^n x \in U\}$. Because U is clopen, λ is continuous, and because U is compact, the range of λ is a finite subset of $\mathbb{N}$, say $\{h_1, h_2, \cdots h_s\}$. Define $U_j = \{x \in U \,|\, \lambda(x) = h_j\}$, $j = 1, \cdots s$. Then $U = \cup_{j=1}^{s} U_j$ is a clopen partition of U. Also, using minimality, we get that $X = \cup_{j=1}^{s} \cup_{i=0}^{h_j-1} T^i(U_j)$ is a clopen partition of X. We refer to such a partition as the *Kakutani-Rohlin* partition of X associated to T, with basis set U. The set $C_j = \cup_{i=0}^{h_j-1} T^i(U_j)$ is called a *tower* of the partition (of height h_j), while $T^i(U_j)$ is the i'th *floor* of the j'th tower. U_j is referred to as the ground floor. The floors can be considered as ordered by the powers of T^i.

Starting with a decreasing sequence of clopen neighborhoods U_n of x_0, such that $\cap_{n=1}^{\infty} U_n = \{x_0\}$, and a sequence of clopen partitions whose union generates the topology of X, one can inductively construct a nested sequence $\{\mathcal{P}_n\}_{n=1}^{\infty}$ of Kakutani-Rohlin partitions of X generating the topology. Also, the construction is such that the towers of the $(n+1)$'th partition traverse the towers of the n'th partition, meaning that the ground floor of the tower $C_j^{(n+1)}$ of the $(n+1)$'th partition will be contained in the ground floor of a tower $C_k^{(n)}$ of the n'th partition. Hence the h_k first floors of $C_j^{(n+1)}$ – where h_k is the height of $C_k^{(n)}$ – will be contained in $C_k^{(n)}$. Further, if the height of $C_j^{(n+1)}$ is greater than h_k, then the $(h_k + 1)$ floor of $C_j^{(n+1)}$ will be contained in the ground floor of some other (or possible the same) tower $C_{k'}^{(n)}$ of the n'th partition, and so $C_j^{(n+1)}$ will traverse this tower in the same manner. Because the floors are labeled by T^i, one can say that the tower $C_j^{(n+1)}$ traverses the towers of the n'th partition in a specific order.

Now, assume $\{C_1^{(n)}, C_2^{(n)}, \cdots, C_{s_n}^{(n)}\}$ are the towers of the n'th partition $\mathcal{P}_n$, and assume $\{C_1^{(n+1)}, C_2^{(n+1)}, \cdots, C_{s_{n+1}}^{(n+1)}\}$ are the towers of the $(n+1)$'th partition $\mathcal{P}_{n+1}$. Let $(V, E, \geq)$ be the ordered Bratteli diagram we get by setting $V_n = \{v_1^{(n)}, v_2^{(n)}, \cdots, v_{s_n}^{(n)}\}$, $V_{n+1} = \{v_1^{(n+1)}, v_2^{(n+1)}, \cdots, v_{s_{n+1}}^{(n+1)}\}$, i.e. we let the vertices be in one-to-one correspondence with the towers, and where we have an edge between $v_k^{(n)}$ and $v_l^{(n+1)}$ iff the tower $C_l^{(n+1)}$ traverses the tower $C_k^{(n)}$. We order the edges according to the order in which the towers traverse each other. It turns out that the diagram will be properly ordered.

In [12, Theorem 4.4.] it is proved that the equivalence class of $(V, E, \geq)$ constructed this way does not depend of the choice of Kakutani-Rohlin partitions for (X, T, x_0), and we have the following important *model theorem for Cantor minimal systems*:

Theorem 2.5. [12, Theorem 4.5] *Let (X, T, x_0) be a pointed Cantor minimal system. Let $(V, E, \geq)$ be the properly ordered Bratteli diagram constructed as explained above. Then (X, T, x_0) is (pointedly) conjugated to $(X_{(V,E)}, T_{(V,E)}, x_{min})$, i.e. $(X, T) \simeq (X_{(V,E)}, T_{(V,E)})$ by a map sending x_0 to x_{min}.*

Theorem 2.6. [12, Theorem 4.7] *There is a bijective correspondence between equivalence classes of properly ordered Bratteli diagrams and pointed topological conjugacy classes of minimal systems.*

We end this section by proving a transitivity result for Cantor minimal systems, the proof of which we have not found in the literature.

Lemma 2.7. *Let (V, E) be a a simple Bratteli diagram, and let $x = (e_1, e_2, \cdots)$ and $y = (f_1, f_2, \cdots)$ be two infinite paths in $X_{(V,E)}$, such that x is not cofinal to y. There exists a proper ordering of (V, E) such that x and y becomes the unique maximal and minimal paths, respectively.*

Proof. Because the diagram is simple, we can, by telescoping, assume that x and y have no edges in common and that there is full connection between any two consecutive vertex levels. We show how to order the edges in E_n. Let $v_x^{(n)}, w_x^{(n+1)}, v_y^{(n)}$ and $w_y^{(n+1)}$ be the vertices of x and y at level n and $n+1$, respectively. For the vertex $w_x^{(n+1)}$, we order the incoming edges such that the edge e_n of x becomes maximal, and such that one of the edges connecting $w_x^{(n+1)}$ to $v_y^{(n)}$ becomes minimal. For the vertex $w_y^{(n+1)}$, we order the edges such that edge f_n of y becomes minimal, and one of the edges connecting $w_y^{(n+1)}$ to $v_x^{(n)}$ becomes maximal. For any other vertex $w_k^{(n+1)}$ at level $n+1$, we order the incoming edges such that the minimal edge connects $w_k^{(n+1)}$ to $v_y^{(n)}$, and the maximal edge connects $w_k^{(n+1)}$ to $v_x^{(n)}$. It is easy to verify that in this order, x becomes the unique maximal path and y becomes the unique minimal path. □

Lemma 2.8. *Let X be a Cantor set and let x and y two (not necessarily distinct) points in X. There exists a homeomorphism $h : X \to X$ such than $h(x) = y$.*

Proof. If $x = y$, then we can choose h to be the identity map. The Cantor set X is homeomorphic to the path space $X_{(V,E)}$ of any simple Bratteli diagram (V, E). Choose one such representation $X_{(V,E)}$ for X. If x and y are not cofinal, choose a proper ordering of (V, E) such that x becomes the unique maximal path, and y becomes the unique minimal path. Let h be the Vershik transformation according to this order. Then $h(x) = y$. On the other hand, if x and y are cofinal, let $T_{(V,E)}$ be the Vershik transformation associated to any proper ordering of (V, E). If $x < y$ in this order, a finite iteration, say m iterations, of $T_{(V,E)}$ will map x to y, so we let $h = T^m_{(V,E)}$ in order to get $h(x) = y$. If $x > y$, then a finite iteration of T^{-1} will do the trick. □

Remark 2.9. *The homeomorphism in Lemma 2.8 is not necessarily minimal.*

We can easily generalize the above lemma to the following: Given two *pointed* Cantor sets (X, x) and (Y, y), there exists a homeomorphism $h : X \to Y$ such that $h(x) = y$. To see this, just let $\phi : X \to Y$ be a homeomorphism, and find $h_1 : X \to X$ such that $h_1(x) = \phi^{-1}(y)$. Then $h = \phi \circ h_1$ will be a homeomorphism mapping x to y.

We also have the following extension:

Lemma 2.10. *Given $(X, x_1, x_2, \dots, x_n)$ and $(Y, y_1, y_2, \dots, y_n)$, where X and Y are Cantor sets, and $x_1, \dots, x_n \in X$ and $y_1, \dots, y_n \in Y$ are distinct points. There exists a homeomorphism $h : X \to Y$ such that $h(x_i) = y_i$, $i = 1, \dots, n$.*

Proof. Find a clopen partition $\{X_1, \dots, X_n\}$ of X such that $x_i \in X_i$, and a clopen partition $\{Y_1, \dots, Y_n\}$ of Y such that $y_i \in Y_i$. Then (X_i, x_i), (Y_i, y_i) are (pointed) Cantor sets so, for $i = 1, \dots n$, we can find $h_i : X_i \to Y_i$ such that $h_i(x_i) = y_i$. Define $h : X \to Y$ by $h(x) = h_i(x)$ if $x \in X_i$. □

Proposition 2.11. *Let (X, T) be a Cantor minimal system and let $A = \{x_1, \dots, x_n\} \subset X$ and $B = \{y_1, \dots, y_n\} \subset X$ be two sets of distinct points, such that $A \cap B = \emptyset$. There exists a Cantor minimal system (X, S), such that $Sx_i = y_i$, $i = 1, \dots, n$, and such that $(X, S) \simeq (X, T)$.*

Proof. Choose n distinct points $z_1, \cdots, z_n$ of X, such that $\{z_i\}_{i=1}^n \cap \{Tz_i\}_{i=1}^n = \emptyset$. Apply Lemma 2.10 to get a homeomorphism $h : X \to X$ such that $h(z_i) = x_i$ and $h(Tz_i) = y_i$, $i = 1, \dots, n$. Let $S : X \to X$ be defined by $Sx = (h \circ T \circ h^{-1})x$. Then $Sx_i = (h \circ T \circ h^{-1})x_i = (h \circ T)z_i = y_i$, $i = 1, \dots, n$, and clearly $(X, S) \simeq (X, T)$. □

3. Dimension groups

In this section we will introduce the main invariants in the classification theory of Cantor minimal systems.

By an *ordered* group we shall mean a countable abelian group G together with a subset G^+, called the *positive cone*, so that

(i) $G^+ + G^+ \subset G^+$,
(ii) $G^+ - G^+ = G$, and
(iii) $G^+ \cap (-G^+) = \{0\}$.

We will denote such a group by (G, G^+). We shall write $a \leq b$ (resp. $a < b$) if $b - a \in G^+$ (resp. $b - a \in G^+ \setminus \{0\}$). We say that the ordered group (G, G^+) is *unperforated* if $a \in G$ and $na \in G^+$ for some $n \in \mathbb{N}$ implies $a \in G^+$. It is easily seen that an unperforated ordered group is torsion free.

Definition 3.1. A *dimension group* is an ordered group (G, G^+) isomorphic to the direct limit of a sequence

$$\mathbb{Z}^{r_0} \xrightarrow{\phi_1} \mathbb{Z}^{r_1} \xrightarrow{\phi_2} \mathbb{Z}^{r_2} \xrightarrow{\phi_3} \cdots$$

where the r_n's are positive integers, the $\mathbb{Z}^{r_n}$'s are assumed to have standard order, i.e. $\mathbb{Z}^{r_n +} = \{(z_1, \cdots, z_{r_n}) \in \mathbb{Z}^{r_n} \mid z_i \geq 0 \text{ for } i = 1, \ldots, r_n\}$, and where the ϕ_i's are positive group homomorphisms inducing the order.

One can also define a dimension group abstractly:

Definition 3.2. A dimension group is an unperforated ordered group (G, G^+) with the Riesz interpolation property, i.e. given $g_1, g_2, h_1, h_2 \in G$ with $g_i \leq h_j$, there exists an element $f \in G$ such that $g_i \leq f \leq h_j$ $(i, j = 1, 2)$.

In [3], Elliott proved that any dimension group satisfies the Riesz interpolation property, and in [2, Theorem 2.2], Effros, Handelman and Shen showed that the two definitions are in fact equivalent.

By an *order unit* u of a dimension group (G, G^+), we mean an element $u \in G^+$, such that for any $g \in G$ there exists an $n \in \mathbb{Z}$ such that $nu \geq g$. A dimension group (G, G^+) is *simple* if every element $u \in G^+ \setminus \{0\}$ is an order unit. Fixing an order unit u, we denote by $\operatorname{Inf} G$ the *infinitesimal subgroup* of G consisting of all elements $g \in G$ such that $-\frac{p}{q}u \leq g \leq \frac{p}{q}u$ for every $\frac{p}{q} \in \mathbb{Q}$. Here $\frac{p}{q}a \leq b$ means $pa \leq qb$. The subgroup $\operatorname{Inf} G$ does not depend on the order unit if G is a simple dimension group, and $G/\operatorname{Inf} G$ is again a simple dimension group.

We now define dimension groups associated to Bratteli diagrams and Cantor minimal systems.

For a properly ordered Bratteli diagram $(V, E, \geq)$, consider the sequence

$$\mathbb{Z}^{|V_0|} \xrightarrow{A_1} \mathbb{Z}^{|V_1|} \xrightarrow{A_2} \mathbb{Z}^{|V_2|} \xrightarrow{A_3} \cdots,$$

where $|V_n|$ is the number of vertices at level n, and the A_n's are the incidence matrices defining the diagram. We let $K_0(V, E)$ denote the dimension group we get by taking the direct limit of this sequence. The image of $1 \in \mathbb{Z}^{|V_0|} (= \mathbb{Z})$ is the canonical order unit, denoted by $\mathbf{1}$. The dimension group is simple if and only if the Bratteli diagram is simple. It is not hard to show that if $(V, E) \sim (W, F)$, then $(K_0(V, E), K_0(V, E)^+, \mathbf{1}_{(V,E)})$ and $(K_0(W, F), K_0(W, F)^+, \mathbf{1}_{(W,F)})$ are isomorphic as ordered groups with distinguished order units, i.e. there exists a group isomorphism $\psi : K_0(V, E) \to K_0(W, F)$ such that $\psi(K_0(V, E)^+) = K_0(W, F)^+$ and $\psi(\mathbf{1}_{(V,E)}) = \mathbf{1}_{(W,F)}$. We write $K_0(V, E) \simeq K_0(W, F)$ if this is the case.

If (X, T) is a Cantor minimal system, let $C(X, \mathbb{Z})$ denote the set of continuous functions on X taking integer values, and let $C(X, \mathbb{Z})^+ = \{f \in C(X, \mathbb{Z}) \mid f \geq 0\}$ denote the positive functions in $C(X, \mathbb{Z})$. Define the *coboundary* $\partial_T C(X, \mathbb{Z})$ of

$C(X,T)$, by $\partial_T C(X,\mathbb{Z}) = \{f - f \circ T^{-1} \mid f \in C(C,\mathbb{Z})\}$. We let $K^0(X,T)$ denote the quotient group $C(X,\mathbb{Z})/\partial_T C(X,\mathbb{Z})$. In [12] it is shown that $(K^0(X,T), K^0(X,T)^+, [1])$ is a simple dimension group with order unit [1], where $K^0(X,T)^+$ is the image of $C(X,T)^+$ in $K^0(X,T)$, and [1] denotes the image of the constant function $1 \in C(X,\mathbb{Z})$. (We will sometimes write $K^0(X,T)$ for $(K^0(X,T), K^0(X,T)^+, [1])$.)

We have the following theorem, saying that all dimension groups can be realized by Cantor minimal systems:

Theorem 3.3. [12, Cor.6.3] *Let (G, G^+, u) be a simple dimension group. Then there exists a Cantor minimal system (X,T) such that $K^0(X,T)$ is order isomorphic to G with distinguished order units, i.e. there exists a group isomorphism $\phi : G \to K^0(X,T)$ such that $\phi(G^+) = K^0(X,T)^+$ and $\phi(u) = [1]$.*

The next theorem relates the dimension group associated to (V,E), where $(V,E,\geq)$ is a properly ordered Bratteli diagram, and the dimension group of the associated Cantor minimal system $(X_{(V,E)}, T_{(V,E)})$:

Theorem 3.4. [12, Theorem 5.4] *Let $(V,E,\geq)$ be a properly ordered Bratteli diagram. If $(X_{(V,E)}, T_{(V,E)})$ is the Bratteli-Vershik system associated to $(V,E,\geq)$, then $K_0(V,E)$ and $K^0(X_{(V,E)}, T_{(V,E)})$ are isomorphic as ordered groups with distinguished order units.*

For orbit equivalent and strong orbit equivalent Cantor minimal systems, dimension groups serve as complete invariants, as the following two theorems state:

Theorem 3.5. [8, Thm. 2.1] *Let (X,T) and (Y,S) be Cantor minimal systems. The following are equivalent:*
- (i) *(X,T) and (Y,S) are strong orbit equivalent.*
- (ii) *$K^0(X,T) \simeq K^0(Y,S)$ by a map preserving order units.*

Theorem 3.6. [8, Thm. 2.2] *Let (X,T) and (Y,S) be two Cantor minimal systems. The following are equivalent:*
- (i) *(X,T) and (Y,S) are orbit equivalent.*
- (ii) *$K^0(X,T)/Inf\, K^0(X,T) \simeq K^0(Y,S)/Inf\, K^0(Y,S)$ by a map preserving order units.*

For weak orbit equivalent systems, one has a similar result. First we define weak isomorphism between dimension groups.

Definition 3.7. For two simple dimension groups (G, G^+, u) and (H, H^+, v) with (distinguished) order units u and v, G and H are said to be *weakly isomorphic* if there exists an order and order unit preserving homomorphisms from G into H and from H into G.

Theorem 3.8. [10, Theorem 2.3] *Let (X,T) and (Y,S) be Cantor minimal systems. The following are equivalent:*
- (i) *(X,T) and (Y,S) are weakly orbit equivalent.*
- (ii) *$K^0(X,T)/Inf\, K^0(X,T)$ and $K^0(Y,S)/Inf\, K^0(Y,S)$ are weakly isomorphic as simple ordered dimension groups with order units.*

We include one last result concerning weak orbit equivalence.

Definition 3.9. Two Cantor minimal systems (X,T) and (Y,S) are called *weakly isomorphic* if each is a factor of the other.

Theorem 3.10. [10, Proposition 3.2] *Let (X,T) and (Y,S) be two weakly isomorphic Cantor minimal systems. Then (X,T) and (Y,S) are also weakly orbit equivalent.*

4. Equivalence relations on Cantor minimal systems and Bratteli diagrams

In this section we still consider Cantor minimal systems and Bratteli diagrams, and we will now see how they can be used to construct equivalence relations on the Cantor set. We start with some general theory of equivalence relations.

Consider a countable equivalence relation R on a compact and metrizable Hausdorff space X. So each equivalence class $[x]_R = \{y \in X \mid (x,y) \in R\}$ is at most countable. We say that the equivalence relation is *minimal* if $[x]_R$ is dense in X for all $x \in X$. R has a natural groupoid structure, where the product of composable pair (x,y), $(y,z) \in R$ is defined by $(x,y)(y,z) = (x,z) \in R$, and the inverse of $(x,y) \in R$ is the element $(y,x) \in R$. The set $\Delta_X = \{(x,x) \mid x \in X\}$ is called the diagonal of R, and it coincides with the unit space of the groupoid. If we give R a locally compact and metrizable topology $\mathcal{T}$, such that the inverse map is a homeomorphism and the product map is continuous in the product topology, then $(R,\mathcal{T})$ is a locally compact (principal) groupoid (cf. [16]). We will be interested in a special class of topological equivalence relations, namely *étale* equivalence relations.

Definition 4.1. A countable equivalence relation R on a locally compact metric space X, with a topology $\mathcal{T}$ making it a locally compact groupoid, is an *étale equivalence relation* if the *range map* $r : R \to X$, defined by $r(x,y) = x$ for $(x,y) \in R$, is a *local homeomorphism*, i.e. for all $(x,y) \in R$ there exists an open neighborhood $U^{(x,y)}$ of (x,y), such that $r(U^{(x,y)})$ is open in X, and $r : U^{(x,y)} \to r(U^{(x,y)})$ is a homeomorphism.

We note that Δ_X can be identified with the underlying space X in a natural way, namely by the map $(x,x) \to x$, so when R is an étale equivalence relation, Δ_X is homeomorphic to X. We will use the acronym CEER to denote compact étale equivalence relations $(R,\mathcal{T})$.

We remark that the étale topology $\mathcal{T}$ on $R \subset X \times X$, only rarely coincides with the relative topology on R, considered as a subset of $X \times X$ with the product topology. The étale topology is usually finer than the relative topology. If $(R,\mathcal{T})$ is a CEER on X, then $\mathcal{T}$ coincides with the relative topology. Moreover, in this case R is a closed subset of $X \times X$, and it is uniformly finite, i.e. there is a natural number N such that the number of elements in $[x]_R$ is at most N (cf. [7, Prop. 3.2]).

Definition 4.2. An equivalence relation $(R,\mathcal{T})$ on X is an *AF-equivalence relation* if it is the inductive limit of an ascending sequence $(R_n,\mathcal{T}_n)$ of CEER's on X, where R_n is an open subequivalence relation of R_{n+1} for every n, i.e. $R_n \subset R_{n+1}$ and $R_n \in \mathcal{T}_{n+1}$, and $R = \cup_{n=1}^{\infty} R_n$. The set $U \subset R$ is in $\mathcal{T}$ iff $U \cap R_n \in \mathcal{T}_n$ for all n.

We note that every AF-equivalence relation is étale.

Definition 4.3. Two equivalence relations $(R_1,\mathcal{T}_1)$ and $(R_2,\mathcal{T}_2)$ on X and Y, respectively, are *isomorphic*, denoted $(R_1,\mathcal{T}_1) \simeq (R_2,\mathcal{T}_2)$, if there exists a homeomorphism $F : X \to Y$ such that

(i) $(x,y) \in R_1$ if and only if $(F(x),F(y)) \in R_2$, and
(ii) $F \times F : (R_1,\mathcal{T}_1) \to (R_2,\mathcal{T}_2)$ is a homeomorphism.

If only condition (i) is satisfied, we say that $(R_1,\mathcal{T}_1)$ is *orbit equivalent* to $(R_2,\mathcal{T}_2)$. (In the case that R_1 and R_2 are the equivalence relations associated to (countable) group actions, orbit equivalence of R_1 and R_2 coincides with orbit equivalence of the two group actions.)

We now introduce equivalence relations associated to Cantor minimal systems and Bratteli diagrams, and present some of the results concerning those.

Let (X,T) be a Cantor minimal system. Define an equivalence relation $R_T \subset X \times X$ by $R_T = \{(x, T^n x) \,|\, x \in X,\ n \in \mathbb{Z}\}$, i.e. the equivalence class of x is the T-orbit of x. We can topologize R_T by transferring the product topology of $X \times \mathbb{Z}$ (where $\mathbb{Z}$ has the discrete topology) to R_T, using the bijective map $(x,n) \to (x, T^n x)$. It is easy to verify that in this topology, $(R_T, \mathcal{T}_T)$ becomes an étale equivalence relation.

In [7] it is proved that if (X,T) and (Y,S) are two Cantor minimal systems, then $R_T \simeq R_S$ if and only if (X,T) and (Y,S) are conjugate or flip conjugate. We also have the following result:

Theorem 4.4. [8, Thm. 2.3] *A minimal AF-equivalence relation $(R,\mathcal{T})$ is orbit equivalent to $(R_T,\mathcal{T}_T)$ for some Cantor minimal system (X,T). Conversely, if (X,T) is a Cantor minimal system, then $(R_T,\mathcal{T}_T)$ is orbit equivalent to some minimal AF-equivalence relation $(R,\mathcal{T})$.*

For a Bratteli diagram (V,E) with associated path space $X_{(V,E)}$, we define the *tail equivalence relation* on $X_{(V,E)}$ as follows: Let $x = (e_1, e_2, \cdots)$ be an infinite path in $X_{(V,E)}$. Then the infinite path $y = (f_1, f_2, \cdots) \in X_{(V,E)}$ is (tail) equivalent to x if there exists $N \in \mathbb{N}$ such that $e_k = f_k$ for all $k > N$, i.e. x and y follow the same set of edges from level N on. We denote the tail equivalence relation on $X_{(V,E)}$ by $AF(V,E)$. It turns out that, topologized properly, $AF(V,E)$ is an AF-equivalence relation. (It will be convenient to use the same notation, $AF(V,E)$, for this AF-equivalence relation.) To see this, let $(R_n, \mathcal{T}_n)$ be the compact, étale equivalence relation on $X_{(V,E)} \times X_{(V,E)}$, where two paths are equivalent iff they are cofinal from level n, and where $\mathcal{T}_n$ is the relative topology from $X_{(V,E)} \times X_{(V,E)}$. Then $AF(V,E)$ is the inductive limit of $\{R_n\}_{n=1}^{\infty}$, and is given the inductive limit topology $\mathcal{T}$ (i.e. $U \in \mathcal{T}$ if and only if $U \cap R_n \in \mathcal{T}_n$ for all n).

If two Bratteli diagrams (V,E) and (W,F) are equivalent, it follows easily from Remark 2.2 that $AF(V,E) \simeq AF(W,F)$. In fact, we have the following:

Theorem 4.5. [7, Lemma 4.13] *Let (V,E) and (W,F) be two Bratteli diagrams. The following are equivalent:*

(i) $(V,E) \sim (W,F)$;
(ii) $AF(V,E) \simeq AF(W,F)$;
(iii) $K_0(V,E) \simeq K_0(W,F)$ *as ordered groups with distinguished order units.*

AF-equivalence relations associated to Bratteli diagrams serve as the prototype of all AF-equivalence relations, which the following theorem states precisely:

Theorem 4.6. [7, Thm. 3.9] *Let $(R,\mathcal{T})$ be an AF-equivalence relation on the Cantor set X. Then there exists a Bratteli diagram (V,E) such that $(R,\mathcal{T})$ is isomorphic to the AF-equivalence relation $AF(V,E)$ associated to (V,E). Furthermore, (V,E) is simple if and only if $(R,\mathcal{T})$ is minimal.*

One would like to say something about the equivalence relation $AF(V,E)$ associated to a properly ordered Bratteli diagram $(V,E,\geq)$ and the equivalence relation $R_{T_{(V,E)}}$ associated to the Cantor minimal system $(X_{(V,E)}, T_{(V,E)})$. Since the Vershik map $T_{(V,E)} : X_{(V,E)} \to X_{(V,E)}$ maps a non-maximal path to its successor, it should be clear that for a path $x \in X_{(V,E)}$, not cofinal with neither the maximal nor the minimal path, the equivalence class $[x]_{T_{(V,E)}}$ of x in $R_{T_{(V,E)}}$ coincide with the tail equivalence class of x in $AF(V,E)$. But for the maximal and the minimal paths this is not so. In fact, in $R_{T_{(V,E)}}$ the maximal and the minimal paths are equivalent. So $AF(V,E)$ is a subequivalence relation of $R_{T_{(V,E)}}$ with the property that by gluing together two distinct equivalence classes in $AF(V,E)$, we get $R_{T_{(V,E)}}$. This suggests that we should split an orbit of a Cantor minimal system in order to get an AF-equivalence relation. Indeed, let (X,T) be a Cantor minimal system and let x_0 be a point in X. Define $R_{\{x_0\}}$ to be the subequivalence relation of R_T we get by splitting the orbit of x_0 in its forward and backward orbits, keeping all other orbits unchanged. That is,

$$[x]_{R_{\{x_0\}}} = \begin{cases} [x]_T & \text{if } x \notin [x_0]_T, \\ \{T^n x_0 \,|\, n \geq 1\} & \text{if } x = T^k x_0 \text{ for some } k \geq 1, \\ \{T^n x_0 \,|\, n \leq 0\} & \text{if } x = T^k x_0 \text{ for some } k \leq 0. \end{cases}$$

Then $R_{\{x_0\}}$ is an open subequivalence relation of R_T, and equipped with the relative topology $\mathcal{T}_{\{x_0\}}$ from $(R_T, \mathcal{T}_T)$, it is an AF-equivalence relation on X. We have the following:

Theorem 4.7. [7, Thm. 4.3] *Let (X,T,x_0) be a (pointed) Cantor minimal system, and let $(V,E,\geq)$ be a properly ordered Bratteli diagram such that (X,T,x_0) is (pointedly) conjugate to $(X_{(V,E)}, T_{(V,E)}, x_{max})$. Then $R_{\{x_0\}} \simeq AF(V,E)$.*

5. General group actions on the Cantor set

Definition 5.1. Let G be a countable group and let X be the Cantor set. We say that $\alpha : G \to Homeo(X)$ is an *action of G on X* if $\alpha_{gh} = \alpha_g \circ \alpha_h$ for all $g,h \in G$. We will use the notation gx for $\alpha_g(x)$, $g \in G, x \in X$, and let $[x]_G = \{gx \,|\, g \in G\}$ denote the orbit of x under the action of G. Furthermore, we let (X,G) denote a group action of G on X.

Let (X,T) be a Cantor minimal system. Let $G = \mathbb{Z}$ and define $\alpha_n = T^n$. Then clearly $(X,T) = (X,G)$, so Cantor minimal systems are only a special case of general group actions on the Cantor set.

A group action (X,G) is *minimal* if every orbit is dense in X. The action is *free* if $gx = x$ for some $x \in X$, implies that g is the identity element of G. We note that a Cantor minimal system (X,T) always defines a free action.

Similarly as we did for a Cantor minimal system, one can define an equivalence relation on X associated to a general group action (X,G), by $R_G = \{(x,gx) \,|\, x \in X,\ g \in G\}$, i.e. the equivalence classes coincide with the orbits. If the action is free, this can be topologized by transferring the product topology of $X \times G$, where G is assumed to have the discrete topology, by the bijective map $(x,g) \to (x,gx)$, and the resulting equivalence relation $(R_G, \mathcal{T}_G)$ will be an étale equivalence relation. There is also a way to define an étale topology on R_G when the action is not free, provided the *fix points sets*, $\text{fix}(g) = \{x \in X \,|\, gx = x\}$, are clopen for all $g \in G$. We will not describe this here, but refer to the paper *I* in this thesis for details.

As an aside we remark that *all* étale equivalence relations on the Cantor set comes from group actions, in the following sense:

Theorem 5.2. [7, Prop. 2.3] *and* [13, Cor.4.2] *Let $(R, \mathcal{T})$ be an étale equivalence relation on the Cantor set X. There exists a countable group G of homeomorphisms of X so that $R = R_G$, where $R_G = \{(x,gx) \,|\, x \in X,\ g \in G\}$. However, in general the action of G can not be chosen to be free.*

A special class of group actions are those of locally finite groups.

Definition 5.3. A countable group G is *locally finite* if it can be written as the union of a sequence $G_0 \subset G_1 \subset \cdots \subset G = \cup_{n=0}^{\infty} G_n$ of finite groups, i.e. each G_n is a finite group.

In [7] it is proved that locally finite groups are closely related to AF-equivalence relations:

Theorem 5.4. [7, Thm. 3.8] *Let G be a countable group acting minimally and freely on the Cantor set X, and let $(R_G, \mathcal{T}_G)$ be the associated étale equivalence relation. Then $(R_G, \mathcal{T}_G)$ is an AF-equivalence relation if and only if G is locally finite.*

Furthermore, in [14] it is showed that $\mathbb{Z}$-actions serve as the prototype of all minimal actions of locally finite groups on the Cantor set, again under the assumption that fix-point sets are clopen – which certainly is true for free actions.

Theorem 5.5. [14, Cor. 2.3.1] *If G is a locally finite group acting minimally on the Cantor set X with clopen fix-point sets, then the dynamical system (X, G) is orbit equivalent to a Cantor minimal system (Y, T), i.e. it is orbit equivalent to a minimal $\mathbb{Z}$-action on the Cantor set Y.*

We end this section by presenting some recent remarkable results on the orbit structure of $\mathbb{Z}^d$-actions. We shall need the useful concept of affability:

Definition 5.6. We say that the equivalence relation R on X is *affable* (AF-able) it is orbit equivalent to an AF-equivalence relation.

We note that if an equivalence relation R (with no topology) is order equivalent to an AF-equivalence relation $(\widetilde{R}, \widetilde{\mathcal{T}})$ via the map F, then $F \times F$ can be used to transfer the topology on $\widetilde{R}$ to a topology on R, making R an AF-equivalence relation. It is clear that for a Cantor minimal system (X, T), the equivalence relation R_T is affable, being orbit equivalent to $AF(V, E)$ for some simple Bratteli diagram (V, E).

Again there is an ordered group serving as an invariant for orbit equivalence for general group actions on the Cantor set. Let (X, G) be a free and minimal group action, and let R_G be the associated étale equivalence relation on X. As before, we let $C(X, \mathbb{Z})$ denote the continuous functions on X taking integer values. Define $M(X, R_G)$ to be the set of R_G-invariant probability measures on X, i.e $\mu \in M(X, R_G)$ if μ is a probability measure on X, and for all Borel subsets $E, F \subset X$, for which there exists a Borel bijection $f : E \to F$ contained in R_G, we have $\mu(E) = \mu(F)$. (This can be shown to be equivalent to $\mu(gE) = \mu(E)$ for all $g \in G$ and all Borel sets $E \subset X$, and so $M(X, R_G)$ coincides with the set of G-invariant probability measures.) Let $B_m(X, R_G)$ be the subgroup of $C(X, \mathbb{Z})$ of all functions f such that $\int_X f d\mu = 0$ for all $\mu \in M(X, R_G)$, and define the quotient group $D_m(X, R_G) = C(X, \mathbb{Z})/B_m(X, R_G)$. Taking $D_m(X, R_G)^+$ to be the image of $C(X, \mathbb{Z})^+ = \{f \in C(X, \mathbb{Z}) \mid f \geq 0\}$ under the quotient map, and [1] to be the image of the constant function 1, we have that $(D_m(X, R_G), D_m(X, R_G)^+, [1])$ is an ordered group. We note that, in general, if $(R, \mathcal{T})$ is an étale equivalence relation on X, then the set of R-invariant probability measures, $M(X, R)$, can be defined in a similar way as for R_G. In fact, one can show that $M(X, R)$ coincides with the set of G-invariant probability measures, where G is the group in Theorem 5.2.

Theorem 5.7. [4, Thm. 1.2] *Let X be a Cantor set and let R be a minimal AF-equivalence relation on X. The group, with positive cone and distinguished order unit, $(D_m(X, R), D_m(X, R)^+, [1])$, is a complete invariant for orbit equivalence.*

The range of $D_m(X, R)$ for minimal AF-equivalence relations $(R, \mathcal{T})$, is the set of simple dimension groups with no non-trivial infinitesimal elements. Moreover, all such dimension groups can also be realized as the K_0-group of a minimal $\mathbb{Z}$-action. In [6], Giordano, Matui, Putnam and Skau have proved the following two theorems:

Theorem 5.8. [6, Thm. 2.4] *Let $\mathbb{Z}^d$, $d \geq 1$, act minimally and freely on the Cantor set X. Then the associated equivalence relation is affable. In particular, the $\mathbb{Z}^d$-action is orbit equivalent to a $\mathbb{Z}$-action.*

Theorem 5.9. [6, Thm. 2.5] *Let (X, R) and (X', R') be two minimal equivalence relations on Cantor sets which are either AF-equivalence relations or arise from minimal and free actions of the group $\mathbb{Z}^d$, $d \geq 1$. Then they are orbit equivalent if and only if $(D_m(X, R), D_m(X, R)^+, [1]) \simeq (D_m(X', R'), D_m(X', R')^+, [1'])$ as ordered groups with distinguished order units.*

6. Simple dimension groups and Choquet simplices

We summarize briefly the connection between Choquet simplices and simple dimension groups, and refer to [1] and [2] for details.

Definition 6.1. A (metrizable) Choquet simplex is a convex and compact subset K of a locally convex Hausdorff space E, with the property that for each $x \in K$ there exists a unique probability measure on K, supported on the extreme points $\partial_e K$ of K, such that $a(x) = \int_K a\, d\mu$ for every continuous *affine* function $a : K \to \mathbb{R}$ (i.e. $a(\lambda y + (1-\lambda)z) = \lambda a(y) + (1-\lambda)a(z)$ for $y,\, z \in K$, $0 \leq \lambda \leq 1$). We let $\mathrm{Aff}(K)$ denote the continuous affine functions on K.

Definition 6.2. Let (G, G^+, u) be a simple dimension group with distinguished order unit u. The *state space* $S_u(G)$ of (G, G^+, u) is the set of positive homomorphism $s : G \to \mathbb{R}$ (so $s(G^+) \subset \mathbb{R}_+$), such that $s(u) = 1$.

Proposition 6.3. [1, Thm. 4.4] *The state space $S_u(G)$ of the simple dimension group (G, G^+, u) with distinguished order unit u, is a Choquet simplex in the locally convex space $\mathbb{R}^G$ (with the product topology).*

Theorem 6.4. [1, Cor. 4.2 and Thm. 4.4] *Let (G, G^+, u) be a simple dimension group with distinguished order unit u, and with state space $K = S_u(G)$. The map $\Theta : G \to Aff(K)$, defined by $\Theta(g)(p) = p(g)$ for $p \in Aff(K)$, determines the order on G in the sense that $G^+ = \{g \in G \,|\, \Theta(g)(p) > 0 \text{ for all } p \in K\} \cup \{0\}$. Also, $g \in ker(\Theta)$ if and only if $g \in Inf(G)$. Furthermore $\Theta(G)$ is dense in $Aff(K)$ (in the uniform topology).*

Corollary 6.5. [1, Cor. 4.6] *All simple dimension groups (G, G^+) with trivial infinitesimal subgroups occur as countable dense additive subgroups of $Aff(K)$, for some Choquet simplices K. (The order on $Aff(K)$ is the strict one, i.e. $p < q$ if $p(x) < q(x)$ for all $x \in K$.)*

Remark 6.6. *If (X, G) is a minimal dynamical system, i.e. G is a countably group that acts minimally as homeomorphisms on X, then the set $M(X, G)$ of G-invariant probability measures is a Choquet simplex.*

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I. AF EQUIVALENCE RELATIONS ASSOCIATED TO LOCALLY FINITE GROUPS

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AF EQUIVALENCE RELATIONS ASSOCIATED TO LOCALLY FINITE GROUPS

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Abstract. In [5, Theorem 3.8], Giordano, Putnam and Skau showed that the étale equivalence relation associated to a (countable) group acting minimally and freely on the Cantor set is AF if and only if the group is locally finite. In this paper we answer what types of AF equivalence relations arise for such actions, linking them to Bratteli diagrams with the so-called equal path number property. Furthermore, we explore more general actions of locally finite groups, and we give a new and more transparent proof of a result by Krieger [9, Theorem 3.5 and Corollary 3.6] in the process.

1. Preliminaries and basic definitions

We start by giving some basic definitions that are relevant for this paper, notably étale equivalence relation, AF equivalence relation, Bratteli diagram and tail equivalence. At the same time we introduce notation that will be used in the sequel. As a general reference we refer to [5].

Let X be a compact, second countable (hence metrizable) Hausdorff space, and consider a countable equivalence relation $R \subset X \times X$ on X, i.e. we require all equivalence classes $[x]_R = \{y \in X \mid (x, y) \in R\}$ to be countable. We will mostly be interested in the case when X is a Cantor set, i.e. X is a metrizable, totally disconnected compact space without isolated points. (It is a classical result that all Cantor sets are homeomorphic, so we may talk about *the* Cantor set.) The equivalence relation is *minimal* if for all $x \in X$ the equivalence class $[x]_R$ of x is dense in X. We put a *groupoid structure* on R by defining the inverse of $(x, y) \in R$ to be $(x, y)^{-1} = (y, x) \in R$, and the product of two *composable* pairs $(x, y), (y, z) \in R$ to be $(x, y)(y, z) = (x, z)$. We assume R has a locally compact, Hausdorff, second countable (hence metrizable) topology $\mathcal{T}$, making the product of composable pairs (given the relative topology from $R \times R$ (with product topology)) a continuous map, and also making the inverse map on R a homeomorphism. So $(R, \mathcal{T})$ is a locally compact (principal) groupoid.

Definition 1.1. (*Étale equivalence relation)* Let $(R, \mathcal{T})$ be a locally compact groupoid, where R is a countable equivalence relation on a locally compact metric space X. We say that $(R, \mathcal{T})$ is *étale* if the range map $r : R \to X$, defined by $r((x, y)) = x$ for $(x, y) \in R$, is a local homeomorphism, i.e. for any $(x, y) \in R$ there exists an open neighborhood $U^{(x,y)} \in \mathcal{T}$ of (x, y) such that $r(U^{(x,y)})$ is open in X and r is a homeomorphism from $U^{(x,y)}$ to $r(U^{(x,y)})$. For short we will sometimes write R for $(R, \mathcal{T})$ when the topology $\mathcal{T}$ is understood from the context.

Remark 1.2. It follows easily from this definition that the source map $s : R \to X$, defined by $s(x, y) = y$, is a local homeomorphism as well. Also, the diagonal

$\Delta = \{(x,x) \,|\, x \in X\}$ is an open set homeomorphic to X by (restriction of) the map r, and so we may identify Δ with X, which we will do whenever convenient. Only rarely is $\mathcal{T}$ the relative topology from $X \times X$. In general, $\mathcal{T}$ is finer than the relative topology.

Two étale equivalence relations $(R_1, \mathcal{T}_1)$ on X and $(R_2, \mathcal{T}_2)$ on Y are *isomorphic* if there exists a homeomorphism $F : X \to Y$ such that

(i) $(x,y) \in R_1 \Leftrightarrow (F(x), F(y)) \in R_2$, and
(ii) $F \times F : (R_1, \mathcal{T}_1) \to (R_2, \mathcal{T}_2)$ is a homeomorphism.

We write $(R_1, \mathcal{T}_1) \simeq (R_2, \mathcal{T}_2)$ for two isomorphic equivalence relations, or just $R_1 \simeq R_2$.

An étale equivalence relation $(R, \mathcal{T})$ is *compact* if $R \setminus \Delta$ is a compact set, where Δ is the diagonal of R. If X itself is compact, this is the same as requiring R to be compact. We will use the acronym *CEER* to denote a compact étale equivalence relation. In [5, Proposition 3.2] it is shown that if $(R, \mathcal{T})$ is a CEER on X, then $\mathcal{T}$ is the relative topology from $X \times X$.

Definition 1.3. *(AF equivalence relation)* Let X be a compact zero-dimensional space, i.e. X is a (compact) metrizable space with a countable basis consisting of clopen (i.e. closed and open) sets. (Equivalently, X is a totally disconnected (compact) metrizable space.) An equivalence relation $(R, \mathcal{T})$ on X is *AF* (approximately finite-dimensional) if it is the inductive limit of a sequence $\{(R_n, \mathcal{T}_n)\}_{n=0}^{\infty}$ of CEERs on X, where R_n is an open subequivalence relation of R_{n+1}, and $(R, \mathcal{T})$ is given the inductive limit topology, i.e. $U \in \mathcal{T}$ if and only if $U \cap R_n \in \mathcal{T}_n$ for all n. In particular, R_n is open in R for all n.

Remark 1.4. The condition that R_n is open in R_{n+1} is superfluous – it can be shown that it is automatically satisfied (cf. comment after Definition 3.7. of [5].) It is easily seen that an AF equivalence relation is étale.

The concept of AF equivalence relation is closely related to the concept of a Bratteli diagram. Bratteli diagrams were introduced in the early 1970's by Bratteli in [1] in order to study inductive limits of finite-dimensional C^*-algebras, the so-called AF-algebras. Subsequently, Elliott [3], motivated by these diagrams, defined what he called *dimension groups*, and realized the K-theoretic underpinning of this. Later Bratteli diagrams have turned out to be important tools also for studying topological dynamical systems and equivalence relations. We give the definition of a Bratteli diagram and then explain how we can relate this to an AF equivalence relation. We refer to [4, Section 3] for details on Bratteli diagrams.

Definition 1.5. *(Bratteli diagram)* A (standard) Bratteli diagram (V, E) consists of a vertex set V, an edge set E and two maps $t, i : E \to V$, the *range* (or *terminal*) and *source* (or *initial*) maps, respectively, satisfying the following conditions:

(i) The vertex set is a disjoint union of finite, non-empty sets, $V = \bigcup_{n=0}^{\infty} V_n$, where V_0 is a one-point set, $V_0 = \{v_0\}$.
(ii) The edge set is a disjoint union of finite, non-empty sets, $E = \bigcup_{n=1}^{\infty} E_n$.
(iii) $t(E_i) \subset V_i$ for all $i \geq 1$, and $t^{-1}(v) \neq \emptyset$ for all $v \in V \setminus V_0$.
(iv) $i(E_i) \subset V_{i-1}$ for all $i \geq 1$, and $i^{-1}(v) \neq \emptyset$ for all $v \in V$.

We often draw the diagram as a downward directed graph, referring to V_n as the vertices at level n and E_n as the edges between level $(n-1)$ and n. See Figure 1 for an example.

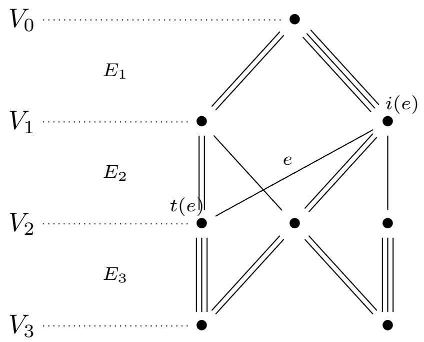

FIGURE 1. An example of the first four levels of a Bratteli diagram.

The diagram can be coded by a sequence of matrices $\{A_n\}_{n=1}^{\infty}$, where $A_n = (a_{ij})_{i,j}$ is the incidence matrix between vertex levels $(n-1)$ and n, where a_{ij} is the number of edges between the i'th vertex at level n and the j'th vertex at level $(n-1)$, the vertices being ordered from left to right.

Given a Bratteli diagram (V,E), the *path space* $X_{(V,E)}$ consists of all infinite paths of (V,E), where a path is an (infinite) sequence of edges $x = \{e_i\}_{i=1}^{\infty}$ such that $e_n \in E_n$ and $i(e_n) = t(e_{n-1})$ for all $n > 1$. If we truncate to k, i.e. we consider $z = \{e_i\}_{i=1}^{k}$, we call z a finite path (of length k). We will also consider (finite) paths not starting at v_0. The diagram is *simple* if for any n and any vertex $v \in V_n$ there exists $m > n$ such that v is connected to each vertex at level V_m by a finite path. One can put a metric on the path space, by defining the distance between two infinite paths to be $\frac{1}{n+1}$ if they agree on the n first edges and differ on the $(n+1)$'th edge. The path space $X_{(V,E)}$ is a compact, zero-dimensional space, and so, in particular, $X_{(V,E)}$ is totally disconnected. It is easy to see that when (V,E) is a simple Bratteli diagram, then $X_{(V,E)}$ is a Cantor set, since there are no isolated points. (We will always assume that (V,E) is non-trivial, i.e. $X_{(V,E)}$ is an infinite set.) As a clopen (countable) basis for the topology we can take the family of *cylinder sets* $\{C_k(x)\}$: For any finite path $x = \{e_i\}_{i=1}^{k}$ starting at $v_0 \in V_0$, we denote by $C_k(x)$ the set of all infinite paths in $X_{(V,E)}$ having $e_1, e_2, \ldots, e_k$ as their first k edges.

A *telescoping* of a Bratteli diagram (V,E) to levels $\{k_i\}_{i=0}^{\infty}$, where $0 = k_0 < k_1 < k_2 < \cdots$ is an increasing sequence of positive integers, is the Bratteli diagram (V',E'), were $V'_n = V_{k_n}$ and $E'_n = \{\text{all (finite) paths between } V_{k_{n-1}} \text{ and } V_{k_n}\}$. The range and source maps of finite paths are defined in the obvious way. So in the process of telescoping we delete some of the vertex levels, and the new edges are concatenations of the old edges. There is a natural homeomorphic map $\Pi : X_{(V,E)} \to X_{(V',E')}$.

There is an obvious notion of isomorphism between Bratteli diagrams (V,E) and (V',E'); namely there exists a pair of bijections between V and V', and between E and E' preserving the gradings and intertwining the respective source and range maps. We let $\sim$ denote the equivalence relation on Bratteli diagrams generated by isomorphism and telescoping. It is not hard to show that $(V,E) \sim (V',E')$ iff there exists an ("aggregate") Bratteli diagram (W,F) such that by telescoping (W,F)

to even and odd levels, respectively, one gets telescopings of (V,E) and (V',E'), respectively. Since the term "equivalence" will be omnipresent in this paper, we will henceforth - by abuse of language - say that (V,E) is *isomorphic* to (V',E') if $(V,E) \sim (V',E')$.

We introduce an equivalence relation on the path space $X_{(V,E)}$. Two infinite paths $x = \{e_i\}_{i=1}^{\infty}$ and $y = \{f_i\}_{i=1}^{\infty}$ are *tail-equivalent* if they coincide from some level on; i.e. there exists N such that for all $n > N$ we have $e_n = f_n$. It is straightforward to verify that this is an equivalence relation. In fact, this becomes an AF equivalence relation, by introducing a natural topology which we now describe. For $n = 0, 1, 2, \ldots$, let $R_n = \{(x,y) \in X_{(V,E)} \times X_{(V,E)} \,|\, e_i = f_i \text{ for all } i > n\}$, and let R_n be given the relative topology $\mathcal{T}_n$ as a subset of $X_{(V,E)} \times X_{(V,E)}$ (with the product topology). Then R_n is closed in $X_{(V,E)} \times X_{(V,E)}$, and so, in particular, compact. Also $(R_n, \mathcal{T}_n)$ is an étale equivalence relation. In fact, let $(x,y) \in R_n$ and take $U^{(x,y)} = (C_n(x) \times C_n(y)) \cap R_n$. Then $r : U^{(x,y)} \to C_n(x)(= r(U^{(x,y)}))$ is a homeomorphism. (Continuity of product and inverse maps is simple to show.) One sees easily that R_n is an open subset of R_{n+1}, and so the inductive limit $(R, \mathcal{T})$ of the sequence $\{(R_n, \mathcal{T}_n)\}_{n=0}^{\infty}$ is an AF equivalence relation. Clearly R is tail equivalence. Given the Bratteli diagram (V,E), let $AF(V,E)$ denote the AF equivalence relation on $X_{(V,E)}$ as described above.

We have the following theorem which characterizes AF equivalence relations in terms of Bratteli diagrams [5, Theorem 3.9.].

Theorem 1.6. *Let $(R, \mathcal{T})$ be an AF equivalence relation on the compact and zero-dimensional space X. There exists a Bratteli diagram (V,E) such that $(R,\mathcal{T})$ is isomorphic to the AF equivalence relation $AF(V,E)$ associated to (V,E). The diagram is simple if and only if $(R,\mathcal{T})$ is minimal.*

Bratteli diagrams are closely connected to *dimension groups*. Abstractly defined, a dimension group is an ordered abelian group (G, G^+), where $G^+ (\subset G)$ is the positive cone, satisfying the following conditions:

(i) $G^+ + G^+ \subset G^+$
(ii) $G^+ - G^+ = G$
(iii) $G^+ \cap (-G^+) = \{0\}$
(iv) if $g \in G$ and $ng \in G^+$ for some $n \in \mathbb{N}$, then $g \in G^+$ (i.e. G is unperforated)
(v) G has the Riesz interpolation property, i.e. if $g_1, g_2 \leq h_1, h_2$, there exists $g \in G$ such that $g_1, g_2 \leq g \leq h_1, h_2$ (where $s \leq t$ denotes $t - s \in G^+$).

An element $u \in G^+$ is an *order unit* if for every $g \in G$ there exists $n \in \mathbb{N}$ such that $g \leq nu$. If every element of $G^+ \setminus \{0\}$ is an order unit, we call (G, G^+) a *simple* dimension group.

We associate to a Bratteli diagram (V,E) a dimension group, which we will denote by $K_0(V,E)$, and it is defined to be the inductive limit of the sequence

$$\mathbb{Z} = \mathbb{Z}^{|V_0|} \xrightarrow{A_1} \mathbb{Z}^{|V_1|} \xrightarrow{A_2} \mathbb{Z}^{|V_2|} \xrightarrow{A_3} \cdots,$$

where A_n is the incidence matrix between level $(n-1)$ and n of the Bratteli diagram. Each $\mathbb{Z}^{|V_n|}$ has standard order. The natural maps determined by the matrices A_n are positive, and the group $K_0(V,E)$ is given the induced order. The canonical order unit is the element corresponding to $1 \in \mathbb{Z}^{|V_0|}(= \mathbb{Z})$.

An order isomorphism ϕ between two ordered groups with order units (G_1, G_1^+, u_1) and (G_2, G_2^+, u_2) is a group isomorphism $\phi : G_1 \to G_2$ such that $\phi(G_1^+) = G_2^+$ and

$\phi(u_1) = u_2$. One can show [2, Theorem 3.1] that any dimension group (G, G^+, u) is order-isomorphic to $K_0(V, E)$ for some Bratteli diagram (V, E). Furthermore, (G, G^+) is simple iff (V, E) is a simple Bratteli diagram.

It is a well known fact that two Bratteli diagrams (V, E) and (V', E') are isomorphic if and only if $K_0(V, E)$ is order-isomorphic to $K_0(V', E')$, by a map sending the canonical order unit of one to the canonical order unit of the other (c.f.[3]).

We recall the following result which links Bratteli diagrams, AF equivalence relations and dimension groups.

Theorem 1.7. [5, Lemma 4.13]. *If $(R_i, \mathcal{T}_i) \simeq AF(V_i, E_i)$, $i = 1, 2$, then the following are equivalent:*

(i) $(R_1, \mathcal{T}_1) \simeq (R_2, \mathcal{T}_2)$
(ii) $(V_1, E_1) \sim (V_2, E_2)$
(iii) $K_0(V_1, E_1) \simeq K_0(V_2, E_2)$

Remark 1.8. Let $(R, \mathcal{T})$ be an AF equivalence relation such that $R \simeq AF(V, E)$. By Theorem 1.7 the dimension group $K_0(V, E)$ is a complete isomorphism invariant for $(R, \mathcal{T})$, and we will denote it by $K_0(R, \mathcal{T})$ (or $K_0(R)$ for short).

2. Étale equivalence relations associated to group actions

By an *action* of a (countable, discrete) group G on a (compact) topological space X we shall mean an injective map $\alpha : G \to Homeo(X)$ such that $\alpha_g \circ \alpha_h = \alpha_{gh}$ for all $g, h \in G$. Thus we may consider G embedded as a subgroup of $Homeo(X)$. We will use the notation gx instead of $\alpha_g(x)$, and let (X, G) denote the dynamical system associated to the action α. The action is *free* if $gx = x$ for some $x \in X$, $g \in G$, implies that $g = id$, and it is *minimal* if for every $x \in X$, the G-orbit $[x]_G = \{gx\}_{g \in G}$ is dense in X.

A *conjugacy* of two dynamical systems (X, G) and (Y, H) is a pair of maps $\Theta : X \to Y$ and $\gamma : G \to H$, where Θ is a homeomorphism and γ is a group isomorphism, such that the diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\Theta"'] & X \arrow[d, "\Theta"] \\ Y \arrow[r, "\gamma(g)"] & Y \end{tikzcd}$$

commutes for all $g \in G$. We will write $(X, G) \simeq (Y, H)$.

Given an action of G on X, we define the associated equivalence relation R_G by

$$R_G = \{(x, gx) \mid x \in X, g \in G\} \subset X \times X.$$

If $fix(g) = \{x \in X \mid gx = x\}$ is an open set (hence it is clopen) for all $g \in G$, then we can define an étale topology $\mathcal{T}_G$ on R_G by choosing as a basis the family consisting of local graphs. Specifically, if $(x, gx) \in R_G$ for some $x \in X$, $g \in G$, a local basis at (x, gx) consists of the sets $graph(g|_U) = \{(y, gy) \mid y \in U\}$, where $U \subset X$ is an open neighborhood of x in X. It is routine to verify that this defines an étale topology $\mathcal{T}_G$ on R_G. We omit the details. In the particular case that the action is *free*, i.e. $fix(g) = \emptyset$ if $g \neq id$, then the topology $\mathcal{T}_G$ that we have described coincides with the topology we get by transferring the product topology of $X \times G$ to R_G via the bijection $(x, g) \to (x, gx)$. We observe that R_G is compact if and only if G is a finite group.

In particular, we will be interested in the equivalence relation associated to a *locally finite* group. A group G is locally finite if it can be written as a union of an increasing sequence of finite groups, i.e. $G_0 \subset G_1 \subset G_2 \subset \cdots \subset G = \cup_{n=0}^{\infty} G_n$, each G_n finite. The following result is the starting point of our paper, and we therefore present its simple proof.

Theorem 2.1. [5, Theorem 3.8]. *Let G be a countable group acting minimally and freely on a compact, zero-dimensional space X, and let $(R_G, \mathcal{T})$ be the associated étale equivalence relation. Then $(R_G, \mathcal{T})$ is an AF equivalence relation if and only if G is locally finite.*

Proof. Assume G is locally finite, and let $G_0 = \{id\} \subset G_1 \subset G_2 \subset \cdots \subset G = \cup_{n=0}^{\infty} G_n$, where each G_n is a finite group. Then $\{(R_{G_n}, \mathcal{T}_n)\}_{n=0}^{\infty}$ is an increasing sequence of compact étale equivalence relations (CEERs) on X, and it is easily seen that $(R_G, \mathcal{T})$ is the inductive limit of this sequence. Hence $(R_G, \mathcal{T})$ is an AF equivalence relation.

Conversely, assume $(R_G, \mathcal{T})$ is an AF equivalence relation, and write $R = \cup_{n=0}^{\infty} R_n$, each R_n being CEER and open in R, where $R_0 \subset R_1 \subset R_2 \subset \cdots \subset R$. Let $F \subset G$ be a finite set. In order to show that G is locally finite, it suffices to show that the subgroup generated by F is finite. Now $X \times F$ is compact, so by the identification $(x, g) \leftrightarrow (x, gx)$ it is contained in some R_n. Define $H = \{g \in G \mid X \times \{g\} \subset R_n\}$. Clearly $F \subset H$, and since R_n is an equivalence relation, this implies that H is a subgroup of G. In fact, if $h_1, h_2 \in H$, then both (x, h_2x) and $(h_2x, h_1(h_2x))$ are in R_n for all $x \in X$, and so by reflexivity $(x, (h_1h_2)x) \in R_n$ for all $x \in X$. Hence $h_1h_2 \in H$. Similarly it follows that $h \in H$ implies $h^{-1} \in H$. Since R_n is compact, H must be finite, and so the subgroup generated by F is finite.

□

Combining Theorem 1.6 and Theorem 2.1 we get that the étale equivalence relation $R_G = \{(x, gx) \mid x \in X, g \in G\}$ associated to a locally finite group acting freely and minimally on a compact, zero-dimensional space, is isomorphic to $AF(V, E)$ for some Bratteli diagram (V, E).

Before we study this connection in more detail, we give a few more definitions concerning groups acting on (path spaces of) Bratteli diagrams.

Let (V, E) be a Bratteli diagram. We let $\widetilde{H}_n$ denote the group acting on the path space $X_{(V,E)}$ by permuting paths that are cofinal from level n. Specifically, if $h \in \widetilde{H}_n$ and $x = (e_i)_{i=1}^{\infty} \in X_{(V,E)}$, then $h(x) = (f_i)_{i=1}^{\infty}$, where $f_k = e_k$ for $k > n$ and $t(f_n) = t(e_n)$. We may also say that $\widetilde{H}_n$ permutes the cylinder sets meeting at the same vertex at level n, and we assume that all possible permutations occur. The way $\widetilde{H}_n$ is embedded in $\widetilde{H}_{n+1}$ is the obvious one, namely that if $h \in \widetilde{H}_n$ maps the cylinder set $C_n(x)$, $x = (e_i)_{i=1}^{n}$, to the cylinder set $C_n(y)$, $y = (f_i)_{i=1}^{n}$, where $t(e_n) = t(f_n)$, then h acts, considered as an element of $\widetilde{H}_{n+1}$, respecting the subpartition of $C_n(x)$ and $C_n(y)$ that occurs at level $(n+1)$, i.e. h fixes the (n+1)'th edge of the cylinder sets defined at level $(n+1)$.

Definition 2.2. The *full AF-group* associated to a Bratteli diagram (V, E) is defined to be the locally finite group $\widetilde{H}(V, E) = \cup_{n=0}^{\infty} \widetilde{H}_n$, where $\widetilde{H}_0 = \{id\}$ and $\widetilde{H}_n$ acts on the path space $X_{(V,E)}$ as described above. We will call a subgroup H of the full AF-group $\widetilde{H}(V, E)$ an *AF-group* (associated to (V, E)), and the action of a group element will be called an *AF-action*. We say that H acts *transitively* if $H_n = H \cap \widetilde{H}_n$ acts transitively on each of the collections of cylinder sets which meet at the same vertex at level n, for $n = 1, 2, \ldots$. Clearly $H = \cup_{n=0}^{\infty} H_n$, where $H_n = H \cap \widetilde{H}_n$.

Remark 2.3. Observe that when $H = \cup_{n=0}^{\infty} H_n$ is an AF-group, the fix-point set $fix(h) = \{x \in X_{(V,E)} \mid hx = x\}$, is a clopen set for each $h \in H$. Also, if (V, E) is a simple Bratteli diagram and H is an AF-group acting transitively on $X_{(V,E)}$, then the action is minimal.

Definition 2.4. A group G acting as homeomorphism on a (compact, metrizable) zero-dimensional space X is *ample* (we also say, G acts *amply*) if it satisfies the following: If $X = \sqcup_{i=1}^{m} A_i = \sqcup_{i=1}^{m} B_i$ are two (finite) clopen partitions of X having the same cardinality, such that for each $i = 1, \ldots, m$, there exists $g_i \in G$ such that $g_i(A_i) = B_i$, then there exists $g \in G$ such that $g|_{A_i} = g_i|_{A_i}$, $i = 1, \ldots, m$, where $g|_{A_i}$ denotes the restriction of g to A_i. (The symbol $\sqcup$ is used for disjoint union.)

Remark 2.5. Observe that ampleness is preserved under conjugation. The full AF-group $\widetilde{H}(V, E) = \cup_{n=0}^{\infty} \widetilde{H}_n$ associated to a Bratteli diagram (V, E) is ample. In fact, assume we have two clopen partitions $X_{(V,E)} = \sqcup_{i=1}^{m} A_i$ and $X_{(V,E)} = \sqcup_{i=1}^{m} B_i$ and group elements h_i such that $h_i(A_i) = B_i$ $i = 1, 2, \ldots, m$. Then there is a an n such that $h_1, \ldots h_m \in \widetilde{H}_n$. So we may assume that all partition elements are cylinder sets of length n. If $h_i(A_i) = B_i$, then A_i and B_i terminate at the same vertex at level n, by definition of the AF-action. Since the full AF-group consists of all possible permutations on each vertex level, there exists a single element $h \in \widetilde{H}_n$ such that $h(A_i) = B_i$ for all $i = 1, \ldots, m$. It is easily seen that a transitive AF-group can not be ample unless it is the full AF-group. Observe that if we telescope the Bratteli diagram (V, E) and get the new Bratteli diagram (V', E'), then $(X_{(V,E)}, \widetilde{H}(V, E))$ is conjugate to $(X_{(V',E')}, \widetilde{H}(V', E'))$ via the obvious map $\Pi : X_{(V,E)} \to X_{(V',E')}$.

We introduce the following terminology: Let $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ and $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_m\}$ be two (finite) coverings of the (compact, metrizable) zero-dimensional space X, i.e. $X = \bigcup_{i=1}^{n} P_i = \bigcup_{j=1}^{m} Q_j$. We say that $\mathcal{P}$ is a *refinement* of $\mathcal{Q}$ (notation $\mathcal{Q} \prec \mathcal{P}$) if each $Q_i \in \mathcal{Q}$ is a union of P_j's in $\mathcal{P}$. In general, if $\mathcal{P}$ and $\mathcal{Q}$ are two coverings, we let $\mathcal{P} \vee \mathcal{Q}$ denote the smallest refinement of $\mathcal{P}$ and $\mathcal{Q}$, i.e. $\mathcal{P} \vee \mathcal{Q} = \{P_i \cap Q_j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$. In particular, this applies to clopen partitions of X.

3. A Bratteli diagram model for R_G

In this section we construct a Bratteli diagram representation of (X, G), where $G = \cup_{n=0}^{\infty} G_n$ is a locally finite group acting on a compact, metrizable, zero-dimensional space X. We will need the fix-point set $fix(g) = \{x \in X \mid gx = x\}$ to be clopen for each $g \in G$. An important special case of this is when the action is free, i.e. $fix(g) = \emptyset$ if $g \neq id$. Specifically, we prove that under these assumptions (X, G) is conjugate to the action of an AF-group associated to the constructed Bratteli diagram (V, E). If G acts amply the AF-group in question will be the full AF-group. We will use these results to relate the étale equivalence relation R_G associated to the G-action to the dynamical system (X, G).

We state the main theorem of this section.

Theorem 3.1. *Let $G = \cup_{n=0}^{\infty} G_n$ be a locally finite, countable group acting on a compact, metrizable, zero-dimensional space X such that the fix-point sets $fix(g) = \{x \in X \mid gx = x\}$ are clopen for all $g \in G$. There exists a Bratteli diagram (V, E) and a homeomorphism $\Pi : X \to X_{(V,E)}$ such that if we let $H = \{\Pi \circ g \circ \Pi^{-1} \mid g \in G\}$, then H is an AF-group acting transitively on $X_{(V,E)}$, and (X, G) is conjugate to $(X_{(V,E)}, H)$ via the map Π. Furthermore, if G acts amply then H is the full AF-group associated to (V, E). Finally, the Bratteli diagram (V, E) is simple if and only if the G-action is minimal.*

The next lemma is the key result for the construction of the Bratteli diagram (V, E), and hence for the proof of the theorem. The proof of the lemma is essentially contained in [9], but we have chosen to present a simple alternative proof that can be found in [8].

Lemma 3.2. *Let $G = \cup_{n=0}^{\infty} G_n$ be a locally finite group, where $\{id\} = G_0 \subset G_1 \subset G_2 \subset \cdots \subset G$, the G_n's being finite groups. Let G act on the compact, metrizable, zero-dimensional space X such that for each $g \in G$ the set fix$(g) = \{x \in X \mid gx = x\}$ is clopen. There exists a nested sequence of (finite) clopen partitions $\{X\} = \mathcal{P}_0 \prec \mathcal{P}_1 \prec \mathcal{P}_2 \prec \cdots$ of X such that:*

(i) *If $g \in G_n$ and $P \in \mathcal{P}_n$, then $gP \in \mathcal{P}_n$.*
(ii) *If $g \in G_n$, $P \in \mathcal{P}_n$ and $gP \cap P \neq \emptyset$, then $g|_P = id|_P$.*
(iii) *The sequence $\{\mathcal{P}_n\}_{n=0}^{\infty}$ generates the topology on X.*

Proof. We claim that it is sufficient to show that for any clopen, finite partition $\mathcal{Q}$ of X and any finite group G acting on X with clopen fix-point sets, we can construct a clopen partition $\mathcal{P}$ of X such that $\mathcal{P}$ is a refinement of $\mathcal{Q}$, and $\mathcal{P}$ satisfies condition (i) and (ii) of the lemma. In fact, starting with a sequence $\{\mathcal{Q}_n\}_{n=0}^{\infty}$, $\mathcal{Q}_0 = \{X\}$, of clopen partitions generating the topology of X, we inductively define $\mathcal{Q}'_n = \mathcal{P}_{n-1} \vee \mathcal{Q}_n$, where we set $\mathcal{P}_0 = \{X\}$, and apply the stated claim to $\mathcal{Q}'_n$ and G_n to obtain $\mathcal{P}_n$. Then clearly the sequence $\{\mathcal{P}_n\}_{n=0}^{\infty}$ satisfies the conditions of the lemma. So let G and $\mathcal{Q}$ be given as above. Let x be a point in X, and consider the finite set $\{gx\}_{g \in G}$. Enumerate the distinct points of this set as $x_1, x_2, \ldots, x_l$. Choose a disjoint family of clopen sets $\{U_i\}_{i=1}^{l}$ such that for each $i \in \{1, 2, \ldots, l\}$, $x_i \in U_i$ and U_i is contained in a set i $\mathcal{Q}$. Let $V_i = U_i \cap \bigcap_{\substack{g \in G \\ x_i \in fix(g)}} fix(g)$, and observe that if $gx_i = x_i$ for some g, then $g|_{V_i} = id|_{V_i}$, since $V_i \subset fix(g)$. Note that $x_i \in V_i$, and that $\{V_i\}_{i=1}^{l}$ is a disjoint family of clopen sets such that each V_i is contained in some set in $\mathcal{Q}$. For $i \in \{1, 2, \ldots, l\}$, we define the set W_i to be $W_i = \bigcap_{\substack{g \in G \\ x_i = gx_j}} g(V_j)$. We observe that $W_i \subset V_i$, since $gx_i = x_i$ for $g = id$, and so $\{W_i\}_{i=1}^{l}$ is a disjoint family of clopen sets, and each W_i is contained in some set in $\mathcal{Q}$. The following holds:

(a) Given $h \in G$ such that $hx_i = x_k$. Then
$$h(W_i) = \bigcap_{\substack{g \in G \\ x_i = gx_j}} (hg)(V_j) = \bigcap_{\substack{g \in G \\ x_k = (hg)x_j}} (hg)(V_j) = \bigcap_{\substack{g' \in G \\ x_k = g'x_j}} g'(V_j) = W_k.$$
(b) Assume $h(W_i) \cap W_i \neq \emptyset$. By (a), $h(W_i) \cap W_i = W_k \cap W_i$ for some k, so we must have $i = k$. Hence $h(W_i) = W_i$ and $hx_i = x_i$, and so $h|_{W_i} = id|_{W_i}$ (since $W_i \subset V_i$).

Hence $\mathcal{P}_x = \{W_i\}_{i=1}^l$ satisfies condition (i) and (ii) of the lemma, and each W_i is contained in some set in $\mathcal{Q}$. By compactness of X we can find a finite number of points, say $y_1, y_2, \dots, y_N$, in X such that $\mathcal{P}_{y_1}, \mathcal{P}_{y_2}, \dots, \mathcal{P}_{y_N}$ together cover X. Setting $\mathcal{P} = \vee_{j=1}^N \mathcal{P}_{y_j}$ we obtain the desired partition refining $\mathcal{Q}$ and satisfying (i) and (ii). $\square$

Next we describe the construction of the Bratteli diagram associated to the sequence of clopen partitions $\{\mathcal{P}_n\}_{n=0}^\infty$ of X, satisfying the conditions in Lemma 3.2. We will henceforth write gP instead of $g(P)$, where $P \in \mathcal{P}_n$ and $g \in G_n$. The group G_n acts on $\mathcal{P}_n$ by left multiplication, and the distinct orbits under this action yields a clopen partition of X into distinct *towers*, whose *floors* are made up of sets in $\mathcal{P}_n$. Specifically, the tower containing the set $P \in \mathcal{P}_n$ as a floor is $[P]_n = \{gP \mid g \in G_n\}$. Consider two consecutive partitions $\mathcal{P}_{n-1}$ and $\mathcal{P}_n$, and an element $P \in \mathcal{P}_n$, $n \geq 1$. Since $\mathcal{P}_{n-1} \prec \mathcal{P}_n$, there exists a unique $P' \in \mathcal{P}_{n-1}$ such that $P \subset P'$. We will refer to P' as a *superset* of P. Let $g \in G_{n-1} \subset G_n$. Then $gP \subset gP' \in [P']_{n-1}$, so the set $\{gP \mid g \in G_{n-1}\}$ is contained in the tower $[P']_{n-1}$. Also, for $g' \in G_n \setminus G_{n-1}$ we have $g'P \subset P''$ for some $P'' \in \mathcal{P}_{n-1}$, and so the set $\{g(g'P) \mid g \in G_{n-1}\}$ is contained in the tower $[P'']_{n-1}$. We may describe this by saying that the towers at the n'th level are *traversing* the towers at the $(n-1)$'th level. The vertex set V_n, $n \geq 0$, at the n'th level of our Bratteli diagram (V, E) will be in one-to-one correspondence with the towers at level n. Specifically, we set $V_n = \{[P]_n \mid P \in \mathcal{P}_n\}$. The edge set $E_n \subset E$ is defined to be $E_n = \{\{gP \mid g \in G_{n-1}\} \mid P \in \mathcal{P}_n\}$. The range $t(e)$ of $e = \{gP \mid g \in G_{n-1}\} \in E_n$ is defined by inclusion, i.e. $t(e) = [P]_n \in V_n$ since $e \subset [P]_n$. The source $i(e)$ of e is defined to be the element $[P']_{n-1} = \{gP' | g \in G_{n-1}\} \in V_{n-1}$, where P' is the unique superset of P lying in $\mathcal{P}_{n-1}$. Loosely speaking, we put an edge between a vertex $v \in V_n$ and a vertex $w \in V_{n-1}$ whenever the tower represented by v is traversing the tower represented by w in the way we have described above. Since $\mathcal{P}_0 = \{X\}$, we get a standard Bratteli diagram.

Observe that there is a bijective correspondence Π_n between $\mathcal{P}_n$ and the set of finite paths (or the associated cylinder sets) from $v_0 \in V_0$ to level n in the Bratteli diagram. In fact, given $P \in \mathcal{P}_n$ and $1 \leq k \leq n$, then the k'th edge of $\Pi_n(P)$ is the edge $\{gP' \mid g \in G_{k-1}\}$ between level $k-1$ and k which is uniquely determined by the requirement that $P' \in \mathcal{P}_k$ is a superset of P. In particular, this implies that the number of paths from v_0 to $v = [P]_n \in V_n$ is the same as the height of the tower $[P]_n$. Since the sequence $\{\mathcal{P}_n\}_{n=0}^\infty$ generates the topology of X, it is easy to see that this determines a homeomorphism $\Pi : X \to X_{(V,E)}$, defined by $\Pi(x) = \lim_{n\to\infty} \Pi_n(P_n)$, where $x \in P_n \in \mathcal{P}_n$ for all $n \geq 0$ (and so the descending sequence $\{P_n\}_{n=0}^\infty$ of clopen sets shrinks to a unique path in $X_{(V,e)}$).

Proof of Theorem 3.1. We have above described the construction of the Bratteli diagram (V, E) and the homeomorphism $\Pi : X \to X_{(V,E)}$. It remains to show that Π implements a conjugacy between (X, G) and $(X_{(V,E)}, H)$, where H is an AF-group acting on $X_{(V,E)}$.

Let $g \in G$. Define $h = \Pi \circ g \circ \Pi^{-1}$. We need to show that h acts on $X_{(V,E)}$ by permuting only the initial segment of paths that are cofinal from some specified vertex level, leaving the tail unchanged, and thus h will be an element of the full AF-group $\widetilde{H}(V, E) = \cup_{n=0}^\infty \widetilde{H}_n$. Now $g \in G_n$ for some n. Consider a path $x = (e_i)_{i=1}^\infty \in X_{(V,E)}$ passing through the vertex $v = [P]_n \in V_n$, where $P \in \mathcal{P}_n$.

As $[gP]_n = [P]_n$, we get that x is mapped by h to another path going through the same vertex. For $m > n$, the edge $e_m = \{g'P' \,|\, g' \in G_{m-1}\} \in E_m$, where $P' \in \mathcal{P}_m$, is fixed under the action of $g \in G_n \subset G_{m-1}$. Therefore $(hx)_m = e_m$, and so h acts solely by permuting finite paths from the top vertex to a given vertex v in V_n. We conclude that $H = \{\Pi \circ g \circ \Pi^{-1} \,|\, g \in G\}$ is a subgroup of $\widetilde{H}(V, E)$, and so is an AF-group. In fact, we get that $\Pi G_n \Pi^{-1} = H \cap \widetilde{H}_n = H_n$, $H = \cup_{n=0}^{\infty} H_n$.

We may identify the action of G on X and the conjugate action of H on the path space $X_{(V,E)}$. Clearly the G-orbit of an infinite path is a subset of its cofinality class. We show that they coincide. Assume $x = (e_i)_{i=1}^{\infty}$ and $y = (f_i)_{i=1}^{\infty}$ are two cofinal paths in $X_{(V,E)}$. We want to find $g \in G$ such that $x = gy$. Let n be such that $e_i = f_i$ for $i > n$ and let $P_1, P_2 \in \mathcal{P}_n$ be such that $(e_i)_{i=1}^{n} = \Pi_n(P_1)$ and $(f_i)_{i=1}^{n} = \Pi_n(P_2)$. Then P_1 and P_2 are floors in the same tower at level n, and so $t(e_n) = t(f_n) = v \in V_n$, and there is an element $g \in G_n$ which maps P_1 to P_2. Clearly g maps $\Pi_n(P_1)$ to $\Pi_n(P_2)$, and by the same argument as we gave above g fixes all edges below n. Hence $x = gy$. Observe that we have actually proved that g maps the cylinder set associated to $\Pi_n(P_1)$ onto the cylinder set associated to $\Pi_n(P_2)$, and hence G acts transitively (cf. Definition 2.2). So if G acts amply, then by Remark 2.5, G is the full AF-group $\widetilde{H}(V, E)$.

Finally, the last assertion follows form the well-known fact that a Bratteli diagram (V, E) is simple if and only if for each $x \in X_{(V,E)}$, the set of paths that are cofinal with x is dense in $X_{(V,E)}$. □

Corollary 3.3. *Let (X, G) be as in Theorem 3.1. Then the étale equivalence relation $(R_G, \mathcal{T}_G)$ introduced in Section 2 is an AF equivalence relation. In fact, R_G is isomorphic to $AF(V, E)$, where (V, E) is the Bratteli diagram in Theorem 3.1.*

Proof. By Theorem 3.1 we may assume that $X = X_{(V,E)}$ for some Bratteli diagram (V, E), and that G is an AF-group acting transitively on $X_{(V,E)}$. It is now easy to see that $(R_G, \mathcal{T}_G)$ is isomorphic to $AF(V, E)$. In fact, the graph of an element $g \in G$ is obviously related to the maps of cylinder sets that are relevant for the topology on $AF(V, E)$. We omit the details. □

The following theorem is essentially due to Krieger [9, Theorem 3.5 and Corollary 3.6]. However, we state it in terms of equivalence relations, and our proof is an easy consequence of Theorem 3.1.

Theorem 3.4. *For $i = 1, 2$, let (X_i, G_i) be a dynamical system, where X_i is a compact, metrizable, zero dimensional space, and G_i is a locally finite group acting amply on X_i, such that the fix-point sets $fix(g) = \{x \in X_i \,|\, gx = x\}$ are clopen for all $g \in G_i$. Then (X_1, G_1) is conjugate to (X_2, G_2) if and only if $(R_{G_1}, \mathcal{T}_{G_1})$ is isomorphic to $(R_{G_2}, \mathcal{T}_{G_2})$.*

Proof. One implication is immediate, namely that (X_1, G_1) conjugate to (X_2, G_2) implies that $(R_{G_1}, \mathcal{T}_{G_1})$ is isomorphic to $(R_{G_2}, \mathcal{T}_{G_2})$. For the converse implication, we may by Theorem 3.1 at start assume that $X_i = X_{(V^{(i)}, E^{(i)})}$, where $(V^{(i)}, E^{(i)})$ is a Bratteli diagram, and G_i is the full AF-group $\widetilde{H}(V^{(i)}, E^{(i)})$ associated to $(V^{(i)}, E^{(i)})$, $i = 1, 2$. As in the proof of Corollary 3.3 we have that $(R_{G_i}, \mathcal{T}_{G_i}) \simeq AF(V^{(i)}, E^{(i)})$, $i = 1, 2$. By Lemma 4.13 of [5] there exists an "aggregate" Bratteli diagram (V, E) such that telescoping (V, E) to odd levels yields a telescope of $(V^{(1)}, E^{(1)})$, and telescoping to even levels yields a telescope of $(V^{(2)}, E^{(2)})$. By

Remark 2.5 we have that

$$(X_{(V^{(1)},E^{(1)})}, \widetilde{H}(V^{(1)},E^{(1)})) \simeq (X_{(V,E)}, \widetilde{H}(V,E)) \simeq (X_{(V^{(2)},E^{(2)})}, \widetilde{H}(V^{(2)},E^{(2)})),$$

and this completes the proof. □

4. Free action of a locally finite group

In this section we consider the extreme opposite case of ample action, namely that the action of the locally finite group $G = \cup_{n=0}^{\infty} G_n$ on the Cantor set X is free, i.e. if $gx = x$ for some $x \in X$, $g \in G$, then $g = id$. For free actions condition (ii) in Lemma 3.2 becomes: If $g \in G_n$, $P \in \mathcal{P}_n$ and $gP \cap P \neq \emptyset$, then $g = id$. This will imply that the Bratteli diagram constructed in Section 3 will have the *equal path number property*, cf. Definition 4.1. Conversely, we will show that to any Bratteli diagram having the equal path number property, one can associate a (locally finite) AF-group acting freely on the path space.

Definition 4.1. A Bratteli diagram (V, E) is said to have the *equal path number property* (abbreviated e.p.n-property) if for any two vertices v and v' at the same level, the number of (finite) paths from the top vertex $v_0 \in V_0$ terminating at v is the same as the number terminating at v'. It is easy to see that this is equivalent to say that $|t^{-1}(v)| = |t^{-1}(v')|$ for all $v, v' \in V_n$, $n = 1, 2, \ldots$. In other words, the incidence matrix A_n between levels $(n-1)$ and n of (V, E) has constant row sums for $n = 1, 2, \ldots$. Obviously the e.p.n-property is preserved under telescoping.

Bratteli diagrams having the e.p.n-property arise in the study of Toeplitz flows. In fact, in the Bratteli-Vershik model $(V, E, \geq)$ (cf. [7, Theorem 4.6]) for Toeplitz flows the underlying Bratteli diagram (V, E) has the e.p.n-property [6]. Conversely, if (V, E) has the e.p.n-property, there exists a Toeplitz flow (X, T) such that the dimension group $K^0(X, T)$ associated to (X, T) is order-isomorphic to $K_0(V, E)$ [11].

For free actions we have the following version of Theorem 3.1, Corollary 3.3 and Theorem 3.4:

Theorem 4.2. *If $G = \cup_{n=0}^{\infty} G_n$ is a locally finite group acting freely and minimally on the Cantor set X, then there exists a simple Bratteli diagram (V, E) having the e.p.n-property such that R_G is isomorphic to AF(V, E). Also, (X, G) is conjugate to $(X_{(V,E)}, H)$, where $H = \cup_{n=0}^{\infty} H_n$ is an AF-group acting transitively and freely on $X_{(V,E)}$, such that the number of paths from $v_0 \in V_0$ to any vertex $v \in V_n$ is $|G_n| = |H_n|$, $n = 1, 2, \ldots$.*

Proof. We only need to show that the Bratteli diagram has the e.p.n-property, as the rest are immediate consequences of Theorem 3.1 and Corollary 3.3. Looking at the way we construct the Bratteli diagram (V, E) in the proof of Theorem 3.1, wee see that the height of a tower $[P]_n = v \in V_n$ at level n, where $P \in \mathcal{P}_n$, is equal to the order $|G_n|$ of G_n. This is an immediate consequence of the freeness of the action. So (V, E) has the e.p.n-property. □

We will now look at the converse problem, and this is treated in the following theorem.

Theorem 4.3. *Let (V, E) be a simple Bratteli diagram having the equal path number property. There exists a locally finite group $G = \cup_{n=0}^{\infty} G_n$ acting freely on the path space $X_{(V,E)}$ such that G is a transitive AF-group (with respect to (V, E)), and we* *have that R_G is isomorphic to $AF(V,E)$. In particular, if b_n is the number of paths from $v_0 \in V_0$ to a vertex at level n, we can choose G_n to be the cyclic group on b_n symbols.*

Proof. We will inductively construct a sequence $G_0 = \{id\} \subset G_1 \subset G_2 \subset \cdots \subset G_{n-1} \subset G_n \subset \dots$ of finite cyclic groups, where the action of G_n can be described by how it permutes the finite paths of length n, or rather the corresponding cylinder sets. It will be clear from our construction that G will be a transitive AF-group, and so we get by Corollary 3.3 that $R_G \simeq AF(V,E)$. It is clearly sufficient to describe how a generator g_n of G_n, $n = 0,1,2,\dots$, acts on $X_{(V,E)}$. First, let b_1 be the number of edges from v_0 to a vertex at level 1. This number does not depend upon the choice of vertex, because (V,E) has the e.p.n-property. Let G_1 be the cyclic group of order b_1, and let g_1 be a generator. Then g_1 shall act by cyclically permuting the first edge of any $x \in X_{(V,E)}$, keeping the tail unchanged. Specifically, if $x = (e_1, e_2, e_3, \dots) \in X_{(V,E)}$, we let $gx = (e, e_2, e_3, \dots) \in X_{(V,E)}$, (and so in particular $t(e) = t(e_1) = v \in V_1$), where e is the successor of $e_1 \in E_1$ in a linear ordering we give $t^{-1}(v)$, say $f_1, f_2, \cdots, f_{b_1}$, with the proviso that we define f_1 to be the successor of f_{b_1}. Let $v' \in V_1$ be another vertex. In the sequel we will use the order preserving bijection between $t^{-1}(v)$ and $t^{-1}(v')$ to identify these two edge subsets of E_1 – the relevance of this will become clear in what follows. We will describe how the cyclic group G_2 acts on $X_{(V,E)}$, keeping in mind that the action of G_2 must be compatible with the action of the subgroup G_1. Since the same type of argument works in general, i.e. going from G_{n-1} to G_n – the modifications that have to be made being obvious – we will content ourselves with showing this special case. Now $|G_2|$ equals b_2, the number of paths from $v_0 \in V_0$ to any vertex in V_2. Let g_2 be a generator for G_2. We may clearly assume that $g_2^r = g_1$, where $r = \frac{b_2}{b_1}$ = row sum of the incidence matrix $A_2 = (a_{ij})_{\substack{i=1,\dots,q\\ j=1,\dots,p}}$ between levels 1 and 2. We order the b_2 paths from $v_0 \in V_0$ to $v_i^{(2)} \in V_2 = \{v_1^{(2)}, v_2^{(2)}, \dots v_q^{(2)}\}$: Let $V_1 = \{v_1^{(1)}, v_2^{(1)}, \dots v_p^{(1)}\}$, which we give a linear ordering, say from left to right. A path from $v_0 \in V_0$ to $v_i^{(2)}$ passing through $v_j^{(1)} \in V_1$ gets the label (s,j,t), where $s \in \{1, \dots b_1\}$ tells which edge in E_1, in the ordering we introduced above, that the path follows. Furthermore $t \in \{1, \dots, a_{ij}\}$ tells which edge between $v_j^{(1)}$ and $v_i^{(2)}$ the path follows, where we give these edges an (arbitrary) linear ordering. Note that there are $b_1 \cdot \sum_{j=1}^{p} a_{ij} = b_1 r = b_2$ different labels. We order the paths from v_0 to $v_i^{(2)}$ lexicographically, i.e. $(s,j,t) > (s',j',t')$ if (i) $s > s'$ or (ii) $s = s'$ and $j > j'$, or (iii) $s = s'$, $j = j'$ and $t > t'$. Now let the generator $g_2 \in G_2$ act by cyclically permuting these edges (or, rather, the corresponding cylinder sets), keeping tails fixed, analogous to what we did at the first level. It is now straightforward to verify that this action is compatible with the action of G_1. In fact, one verifies easily that $g_1(x) = g_2^r(x)$ for all $x \in X_{(V,E)}$.

Finally, the action of G that we have defined is free. In fact, if $g \in G_n$ and $gx = x$ for some $x = (e_1, e_2, \dots, e_n, e_{n+1}, \dots) \in X_{(V,E)}$, then by definition of the action, g will fix (pointwise) the cylinder set associated to the (finite) path $(e_1, e_2, \dots, e_n)$, and hence g will fix every cylinder set of length n since the generator g_n of G_n cyclically permutes the cylinder sets (keeping tails fixed), analogous to what we explained for $n = 1$ and 2. So $g = id$. □

A natural question to ask is if the locally finite group we can associate to a Bratteli diagram having the e.p.n-property, acting as stated in Theorem 4.3, is in some sense unique? The answer to that question is answered by Proposition 4.4, Corollary 4.7 and Proposition 4.9.

Proposition 4.4. *Let (V, E) be a simple Bratteli diagram having the equal path number property. Assume a_n is the number of edges terminating at a vertex at level n, i.e. a_n is the row sum of the incidence matrix A_n between levels $(n-1)$ and n of (V, E). Then for any locally finite group $G = \cup_{n=0}^{\infty} G_n$, with $\frac{|G_n|}{|G_{n-1}|} = a_n$ (where we assume $G_0 = \{id\}$, $a_0 = 1$), one can define a free and transitive AF-action of G on $X_{(V,E)}$.*

Proof. We will describe how G_1 and G_2 acts on the cylinder sets (keeping tails fixed) at level 1 and level 2, respectively. From this it will be clear how one proceeds to define the action of G_n on cylinder sets at level n, $n > 2$. Let $G_1 = \{h_1, \dots, h_{a_1}\}$, and let G_1 act by left multiplication; i.e. label the edges meeting at any vertex v in V_1 by $h_1, h_2, \dots h_{a_1}$, and let the action of an element $h \in G_1$ on the cylinder set associated to the edge h_j, with $t(h_j) = v$, be defined by left multiplication, keeping tails fixed. Specifically, if $x = (h_j, e_2, e_3, \dots) \in X_{(V,E)}$, then $hx = (hh_j, e_2, e_3, \dots)$. Clearly, this is a free action, since $hx = x$ implies $hh_j = h_j$, and so $h = id$. Next we label the paths from $v_0 \in V_0$ to any $w \in V_2$ by elements of G_2, and then letting G_2 act on the associated cylinder sets by left multiplication, similarly to what we did above for G_1. Also, we have to make this action compatible with the action of $G_1 (\subset G_2)$. We proceed as follows: For $g \in G_2$, form the right coset $G_1 g = \{hg \,|h \in G_1\}$. This gives a partition of G_2 into $\frac{|G_2|}{|G_1|} = a_2$ disjoint cosets, each having $|G_1|$ elements. Pick a representative for each coset to get the set $\{g_1, g_2, \dots, g_{a_2}\}$. We label the edges meeting at any $w \in V_2$ by $g_1, g_2, \dots, g_{a_2}$. We label the path (h_i, g_j) from v_0 to $w \in V_2$, i.e. first following the edge labeled h_i, and then the edge labeled g_j (where $t(h_i) = i(g_j) = v \in V_1$), by $h_i g_j \in G_2$. We claim that if we let G_2 act by left multiplication on the cylinder sets associated to these paths, then we get what we want. To prove this is routine, and we omit the details. □

Remark 4.5. If we telescope the diagram (V, E) in Proposition 4.4 to levels $0 = n_0 < n_1 < n_2 < \dots$ to get the new diagram (V', E'), and we write $G = \cup_{k=0}^{\infty} G_{n_k}$, then it is easy to see that G_{n_k} acts on the cylinder sets at level k of (V', E') the same way as G_{n_k} acts on the cylinder sets at level n_k of (V, E), by making the obvious identification between paths of length k in (V', E') with paths of length n_k in (V, E).

Definition 4.6. Let $G = \cup_{n=0}^{\infty} G_n$ be a locally finite group, where $|G_0| = 1$. The *superorder* of G is the *generalized natural number* $N(G) = \Pi_{n=1}^{\infty} \frac{|G_n|}{|G_{n-1}|} = p_1^{k_1} p_2^{k_2} p_3^{k_3} \cdots$, where $p_1 < p_2 < \dots$ are the prime numbers, and $k_i \in \{0, 1, 2, \dots, \infty\}$. (We factor $\frac{|G_n|}{|G_{n-1}|} = a_n = p_1^{l_1^{(n)}} p_2^{l_2^{(n)}} \cdots$, and then $k_i = l_i^{(1)} + l_i^{(2)} + \cdots$.)

Corollary 4.7. *Let (V, E) be a simple Bratteli diagram having the e.p.n-property, and assume that $G = \cup_{n=0}^{\infty} G_n$ is a locally finite group acting freely and transitively (hence minimally) as an AF-group on $X_{(V,E)}$. Then for any locally finite group $H = \cup_{n=0}^{\infty} H_n$ such that $N(H) = N(G)$, there is a free and transitive AF-action of*

H on the path space of a Bratteli diagram (V', E') such that $(V, E) \sim (V', E')$, or, equivalently, $(R_G, \mathcal{T}_G) \simeq (R_H, \mathcal{T}_H)$ (cf. Theorem 1.7).

Proof. The last assertion, i.e. $(V, E) \sim (V'E')$ if and only if $(R_G, \mathcal{T}_G) \simeq (R_H, \mathcal{T}_H)$, is an immediate consequence of Theorem 4.3 and Theorem 1.7.

By Proposition 4.4 and Remark 4.5 it is sufficient to construct an ("aggregate") simple Bratteli diagram (W, F) having the e.p.n-property such that the following hold:

(i) Telescoping (W, F) to odd levels yields a telescope of (V, E).
(ii) Telescoping (W, F) to even levels yields a diagram (V', E') (with the e.p.n-property) such that the number of paths from $v'_0 \in V'_0$ to any vertex $v' \in V'_k$ equals $|H_{t_k}|$, $k = 1, 2, \dots$, where $0 < t_1 < t_2 < \dots$.

We describe the construction of the first three levels of (W, F), and it will then be clear how one proceeds to construct the whole diagram. Let $|G_1| = a_1$. Because $N(H) = N(G)$, we can find n_2 such that $|H_{n_2}| = a_1 \cdot a_2$, with $a_2 \in \mathbb{N}$. Next find $n_3 > 1 = n_1$ such that $|G_{n_3}| = a_1 \cdot a_2 \cdot a_3$, with $a_3 \in \mathbb{N}$.

Let $W_0 = V_0 = \{v_0\}$, $W_1 = V_1$ and $W_3 = V_{n_3}$. We let $F_1 = E_1$, retaining the range and source maps from (V, E), and so the number of edges terminating at any vertex in W_1 is a_1. We let W_2 consist of $|V_{n_3}| \cdot a_3$ vertices, and we connect each vertex in W_2 to exactly one of the vertices in W_3 with a single edge. Furthermore, each vertex in W_3 is connected to a_3 of the vertices in W_2 (by a single edge). This defines F_3, and so there are a_3 edges terminating at each vertex in W_3. We now define F_2 by the requirements that there are a_2 edges terminating at each vertex in W_2, and such that if we telescope between $W_1 (= V_1)$ and $W_3 (= V_{n_3})$, by deleting W_2, we get exactly the same as if we telescope between V_1 and V_{n_3} in (V, E). This is possible to achieve since the number of (finite) paths of (V, E) terminating at (any) vertex in V_{n_3}, and with source in V_1, is $a_2 \cdot a_3$. We omit the details, which are easy to establish. The rest of (W, F) is constructed likewise. Clearly telescoping (W, F) to odd levels yields a telescope of (V, E) to the levels $n_0 = 0 < n_1 = 1 < n_3 < n_5 < \dots$. Also, the diagram (V', E') obtained by telescoping (W, F) to even levels has the property stated in (ii), with $t_k = n_{2k}$. This finishes the proof. $\square$

Remark 4.8. It follows from Corollary 4.7 that there exist an abundance of examples of non-isomorphic locally finite groups acting freely and minimally on the Cantor set such that the associated (simple) C*-algebra crossed products are isomorphic. In fact, if $R_G \simeq R_H$ for two such groups G and H, then the associated C*-crossed products are isomorphic, cf. [10].

Given a dimension group D with order unit u, define $\mathbb{Q}(D) = \{d \in D \,|\, nd = mu, \text{ some } n, m \in \mathbb{Z}\} \subset D$. Then $\mathbb{Q}(D)$ is order isomorphic to a subgroup of the rational numbers $\mathbb{Q}$ (with the standard ordering), via the map $d \in \mathbb{Q}(D) \to \frac{m}{n} \in \mathbb{Q}$, where $nd = mu$. In particular, the order unit u of D is in $\mathbb{Q}(D)$ and is sent to $1 \in \mathbb{Q}$ by this map. $\mathbb{Q}(D)$ is called the *rational sub-dimension group* of D [6, Section 4.1]. Such groups are (order-)isomorphic to either $\mathbb{Z}$, or to subgroups of $\mathbb{Q}$ of the form $\{\frac{m}{a_1 a_2 \cdots a_k} | m \in \mathbb{Z}, k = 1, 2, \dots\}$, for some $\{a_n \,|\, a_n \geq 2, n = 1, 2, \dots\}$, cf. [2, Chapter 4]. If (V, E) is a simple Bratteli diagram with the e.p.n-property, and a_k is the number of edges terminating at (any) $v \in V_k$, $k = 1, 2, 3, \dots$, then it is easy to see that $\mathbb{Q}(K_0(V, E)) = \{\frac{m}{a_1 a_2 \cdots a_k} \,|\, m \in \mathbb{Z}, k = 1, 2, \dots\}$ (cf. also [6, Section 4.1]).

Let $N = p_1^{k_1} p_2^{k_2} p_3^{k_3} \cdots$ be a generalized natural number, and let n be a natural number with unique factorization $n = p_1^{l_1} p_2^{l_2} p_3^{l_3} \cdots$ (only a finite number of the l_i's are > 0). We say that n is a *divisor in* N if $l_i \leq k_i$ for $i = 1, 2, \ldots$, and we use the notation $n|N$. We let $\mathbb{Q}(N)$ denote the subgroup of $\mathbb{Q}$ defined by $\mathbb{Q}(N) = \{\frac{m}{n} \,|\, m \in \mathbb{Z}, n|N\}$.

We can now state the following proposition whose proof is immediate from the above.

Proposition 4.9. *Let $G = \cup_{n=0}^{\infty} G_n$ be a locally finite group with superorder $N = N(G)$. Let G act freely and minimally on the Cantor set X, and let R_G be the associated AF equivalence relation. Then $\mathbb{Q}(K_0(R_G)) \simeq \mathbb{Q}(N)$, where $K_0(R_G)$ is the simple dimension group associated to R_G (cf. Remark 1.8).*

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II. INDUCED SUBSYSTEMS ASSOCIATED TO A CANTOR MINIMAL SYSTEM

Joint work with Mats Molberg. Submitted to Colloquium Mathematicum.

INDUCED SUBSYSTEMS ASSOCIATED TO A CANTOR MINIMAL SYSTEM

HEIDI DAHL AND MATS MOLBERG

Abstract. Let (X,T) be a Cantor minimal system and let $(R,\mathcal{T})$ be the associated étale equivalence relation (the orbit equivalence relation). We show that for an arbitrary Cantor minimal system (Y,S) there exists a closed subset Z of X such that (Y,S) is conjugate to the subsystem $(Z,\widetilde{T})$, where $\widetilde{T}$ is the induced map on Z from T. We explore when we may choose Z to be a T-regular and/or a T-thin set, and we relate T-regularity of a set to R-étaleness. The latter concept plays an important role in the study of the orbit structure of minimal $\mathbb{Z}^d$-actions on the Cantor set, cf. [3], [4].

1. Main results

We state the two main theorems of this paper, postponing the proofs to later. In the next two sections we will give definitions of pertinent concepts, and state some properties and results that will be relevant for the proofs.

Theorem 1.1. *Let (X,T) be a Cantor minimal system, and let (Y,S) be an arbitrarily given Cantor minimal system. There exists a closed subset $Z \subset X$ such that all points in Z have finite return times under the action of T, and if $\widetilde{T} : Z \to Z$ is the induced map (i.e. $\widetilde{T}z = T^m z$, where $m = inf\{k \in \mathbb{N} \,|\, T^k z \in Z\}$), then $(Y,S) \simeq (Z,\widetilde{T})$. We can choose $Z \subset X$ to be a T-regular set if we allow one point x in Z to have infinite return time, appropriately defining $\widetilde{T}x$. Moreover, we can always choose Z to be a T-thin set in X, i.e. $\mu(Z) = 0$ for all T-invariant probability measures μ.*

Theorem 1.2. *Let (X,T) be a Cantor minimal system and let $(R,\mathcal{T})$ be the associated étale equivalence relation. Let Z be a non-empty closed subset of X. The following are equivalent:*

(i) *Z is T-regular, i.e. the (forward and backward) return time maps (with respect to Z) are continuous.*
(ii) *Z is R-étale, i.e. $R \cap (Z \times Z)$ is an étale equivalence relation.*
(iii) *Z is $R_{\{x\}}$-étale for all $x \in X$, where $R_{\{x\}}$ is obtained from R by splitting the T-orbit of x in the forward and backward T-orbits.*

2. Basic concepts

In this and the next section we will recall some basic definitions and results that we will need concerning Cantor minimal systems and étale equivalence relations. For details we refer to [1], [2], [5] and the survey article [6].

Let X be a locally compact and second countable (hence metrizable) Hausdorff space. An *étale equivalence relation* $R(\subset X \times X)$ on X is a countable equivalence relation (i.e. every equivalence class is at most countable), which has a topology $\mathcal{T}$

making it a locally compact topological groupoid, and with the added property that the range map, $r : R \longrightarrow X$, defined by $r((x,y)) = x$, is a local homeomorphism. Recall that r is a local homeomorphism if for all $(x,y) \in R$ there exists an open neighborhood $U_{(x,y)} \subset R$ of (x,y) such that

(i) $r(U_{(x,y)})$ is open in X;
(ii) $r : U_{(x,y)} \longrightarrow r(U_{(x,y)})$ is a homeomorphism.

Recall that the product of *composable pairs* $(x,y),(y,z) \in R$ is $(x,y)\cdot(y,z) = (x,z)$, and the *inverse* $(x,y)^{-1}$ of $(x,y) \in R$ is (y,x). We will denote an étale equivalence relation by $(R, \mathcal{T})$, or simply by R. We say that R is *minimal* if $[x]_R$ is dense in X for every $x \in X$, where $[x]_R = \{y \in X \mid (x,y) \in R\}$ is the equivalence class of x. The diagonal $\Delta = \{(x,x) \mid x \in X\}$ is a clopen subset of R, and is homeomorphic to X. It should be remarked that only rarely does the topology $\mathcal{T}$ on R coincide with the relative topology from $X \times X$. In general, $\mathcal{T}$ is finer than the relative topology. We will refer to $U_{(x,y)}$ as an *étale neighborhood*, and the local homeomorphism condition as the *étale condition*. It is easily seen that if $S(\subset X \times X)$ is an open subequivalence relation of R, then S is étale in the relative topology.

Let $(R_i, \mathcal{T}_i)$ be étale equivalence relations on X_i, $i = 1, 2$. We say that $(R_1, \mathcal{T}_1)$ is *isomorphic* to $(R_2, \mathcal{T}_2)$, and write $(R_1, \mathcal{T}_1) \cong (R_2, \mathcal{T}_2)$, if there exists a homeomorphism $F : X_1 \longrightarrow X_2$ such that

(i) $(x,y) \in R_1 \Longleftrightarrow (F(x), F(y)) \in R_2$;
(ii) $F \times F : (R_1, \mathcal{T}_1) \longrightarrow (R_2, \mathcal{T}_2)$, defined by $F \times F((x,y)) = (F(x), F(y))$ for $(x,y) \in R_1$, is a homeomorphism.

If condition (i) is satisfied, we say that $(R_1, \mathcal{T}_1)$ and $(R_2, \mathcal{T}_2)$ are *orbit equivalent*. (Note that condition (i) is equivalent to $F([x]_{R_1}) = [F(x)]_{R_2}$ for each $x \in X$.)

By an *action* of a countable (discrete) group G on a locally compact, second countable space X we mean a homomorphism $\alpha : G \rightarrow Homeo(X)$ such that $\alpha_g \circ \alpha_h = \alpha_{gh}$ for all $g, h \in G$. When the action is *free*, i.e. if $\alpha_g(x) = x$ for some $x \in X$, some $g \in G$, then g is the identity element of G, this gives rise to an étale equivalence relation R_G on X. That is, we let R_G be the *orbit equivalence relation* induced by α, where the equivalence class of $x \in X$ is the orbit $[x]_G = \{\alpha_g(x) | g \in G\}$. We give R_G the topology $\mathcal{T}_G$, which is obtained by transferring the (product) topology from the product space $X \times G$ using the map $(x,g) \longrightarrow (x, \alpha_g(x))$. (This map is bijective since the action α is free.) The resulting space $(R_G, \mathcal{T}_G)$ will be an étale equivalence relation on X.

We will be concerned with the following, which falls under the general scheme described above: Let (X,T) be a *Cantor minimal system*, i.e. X is the Cantor set and $T : X \rightarrow X$ is a minimal homeomorphism, where minimality means that the orbit $[x]_T = \{T^n x \mid n \in \mathbb{Z}\}$ is dense in X for all $x \in X$. By viewing (X,T) as a (free) $\mathbb{Z}$-action on X, where $1 \in \mathbb{Z}$ corresponds to T, we get an étale equivalence relation on X as described above.

Two Cantor minimal systems (X,T) and (Y,S) are *conjugate*, written $(X,T) \simeq (Y,S)$, if there exists a homeomorphism $h : X \rightarrow Y$ such that $h \circ T = S \circ h$. Conjugate Cantor minimal systems gives isomorphic orbit equivalence relations.

Let (X,T) be a Cantor minimal system. For a closed, non-empty subset Z of X, define $\lambda^+, \lambda^- : Z \longrightarrow \mathbb{N} \cup \{\infty\}$, where $\mathbb{N} \cup \{\infty\}$ is given the one point compactification topology, by

$$\lambda^+(z) = inf\{k \geq 1 \mid T^k z \in Z\},$$

$$\lambda^-(z) = inf\{k \geq 1, \mid T^{-k}z \in Z\}.$$

(We use the convention that inf of the empty set is ∞.) These maps are called the *forward* and *backward return time maps* with respect to Z. We say that $Z \subset X$ is *regular* with respect to T (or T-regular) if both maps λ^+ and λ^- are continuous.

Remark 2.1. The maps λ^+ and λ^- are lower semi-continuous. To see this, just observe that if $\lambda^+(z) = k$, then $T^i z$ is not in Z for $i = 1, \dots, k-1$, and since $X \setminus Z$ is open, there are open neighborhoods $U_{T^i z} \subset X$ around each of these points, each $U_{T^i z}$ disjoint from Z. Hence $V = \cap_{i=1}^{k-1} T^{-i}(U_{T^i z}) \cap Z$ is an open neighborhood of z in Z, and for all $z' \in V$ we have $\lambda^+(z') \geq k$. With a slight modification the argument goes through also for $\lambda^+(z) = \infty$, by considering $V_N = \cap_{i=1}^{N} T^{-i}(U_{T^i z}) \cap Z$ for each $N \in \mathbb{N}$. A similar proof can be given for λ^-.

Given an étale equivalence relation $(R, \mathcal{T})$ on a locally compact second countable space X and a (non-empty) closed subset Z of X we define $R|_Z = R \cap (Z \times Z)$ as an equivalence relation on Z. We say that Z is *R-étale* if $R|_Z$, given relative topology $\mathcal{T}|_Z$ from $\mathcal{T}$, is étale.

3. Bratteli diagrams as models for Cantor minimal systems and for AF-equivalence relations

The concept of a *Bratteli diagram* will be important for us, because it serves as a model, and as such as a crucial tool, for both AF-equivalence relations and for Cantor minimal systems (when an ordering is introduced). A Bratteli diagram (V, E) is a special directed infinite graph, consisting of a vertex set V, an edge set E and two maps $i, t : E \to V$ such that

(i) V is an infinite union of disjoint, non-empty finite sets; $V = \cup_{n=0}^{\infty} V_n$, and V_0 is a one-point set; $V_0 = \{v_0\}$.
(ii) E is an infinite union of disjoint, non-empty finite sets; $E = \cup_{n=1}^{\infty} E_n$.
(iii) The *source* (or *initial*) map i satisfies $i(E_n) \subset V_{n-1}$ for all $n \geq 1$, and $i^{-1}(v) \neq \emptyset$ for all $v \in V$.
(iii) The *range* (or *terminal*) map t satisfies $t(E_n) \subset V_n$ for all $n \geq 1$, and $t^{-1}(v) \neq \emptyset$ for all $v \in V \setminus V_0$.

For a Bratteli diagram (V, E) we denote by $X_{(V,E)}$ the set of all infinite paths in (V, E), where a path $x = (e_n)_{n=1}^{\infty}$ is a sequence of edges $e_1, e_2, \cdots$, such that $e_n \in E_n$ and $t(e_n) = i(e_{n+1})$ for all n. We can also talk about (finite) paths between a vertex $v \in V_n$ and a vertex $w \in V_m$, $m > n$, and it is obvious what we mean by that. If there exists an edge $e \in E_n$ with source $v \in V_{n-1}$ and range $u \in V_n$, we say that v is *connected* to u. We say that two paths $x = (e_n)_{n=1}^{\infty}$, $y = (f_n)_{n=1}^{\infty}$ in $X_{(V,E)}$ are *cofinal* if there exists an $N \in \mathbb{N}$ such that $e_n = f_n$ for all $n > N$. Henceforth we will only consider non-trivial Bratteli diagrams (V, E), i.e. $X_{(V,E)}$ is an infinite set.

We describe two operations that we can perform on a Bratteli diagram, turning it into new Bratteli diagrams that retain the basic properties of the original. These are *telescoping* and its converse, *symbol splitting*. Let (V, E) be a Bratteli diagram. Let $0 = t_0 < t_1 < t_2 < \dots$ be a sequence of natural numbers. Define a new Bratteli diagram (V', E') by setting $V'_n = V_{t_n}$ and $E'_n = \{$the set of all finite paths between $V_{t_{n-1}}$ and $V_{t_n}\}$. The range and source maps are the obvious ones. We say that (V', E') is a *telescope* of (V, E).

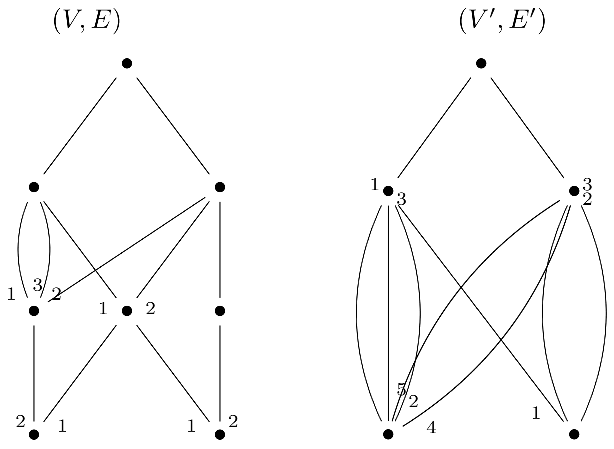

Figure 1. An illustration of telescoping

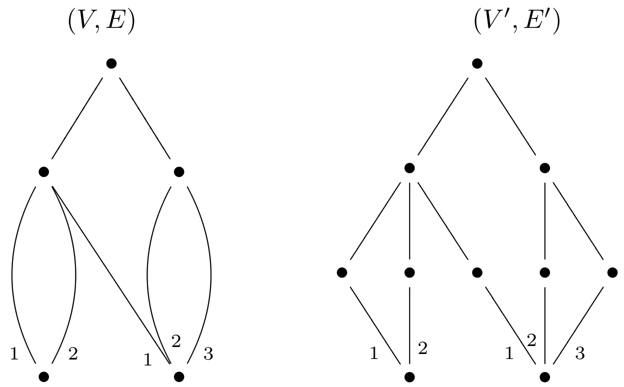

Figure 2. An illustration of symbol splitting

By the operation of symbol splitting we create a new diagram (V', E') from (V, E) by inserting new vertex levels. Let $V'_{2k} = V_k$ for $k \geq 0$, and let $|V'_{2k-1}| = |E_k|$ for $k \geq 1$. There is an obvious way of defining E'_{2k-1} and E'_{2k} such that by telescoping between levels $2k-1$ and $2k$ we get E_k. In other words, each edge in E_k is split in two by introducing a vertex in V'_{2k-1}. (See Figure 1 and Figure 2 for examples of telescoping and symbol splitting, respectively. (Disregard the ordering of the edges for the time being.)

A diagram is *simple* if it can be telescoped into a diagram (V', E') where each $v \in V'_{n-1}$ is connected to each $u \in V'_n$, all $n > 0$. For a simple Bratteli diagram (V, E), the path space $X_{(V,E)}$ becomes a Cantor set, where the *cylinder sets* $\{C_n(x) \,|\, x = (e_1, e_2, \cdots) \in X_{(V,E)},\ n \in \mathbb{N}\}$ form a clopen basis for the topology. Here $C_n(x) = \{y = (f_1, f_2, \cdots) \in X_{(V,E)} \,|\, f_1 = e_1, f_2 = e_2, \cdots, f_n = e_n\}$. We remark here that if we drop the condition that (V, E) is simple, then $X_{(V,E)}$ is still a zero-dimensional space, i.e. $X_{(V,E)}$ has a countable basis of clopen sets (consisting of the cylinder sets $C_n(x)$).

We can give a partial order to the edge set by giving a linear order to the set of edges $t^{-1}(v)$ for each vertex $v \in V \setminus V_0$. Let $(V, E, \geq)$ denote a Bratteli diagram equipped with a partial order $\geq$ on E, called an *ordered Bratteli diagram.* This induces a (partial) lexicographic order on the path space $X_{(V,E)}$. Specifically, $(e_i)_{i=1}^{\infty} > (f_i)_{i=1}^{\infty}$ if there exists $n \in \mathbb{N}$ such that $r(e_n) = r(f_n)$, $e_k = f_k$ for all $k > n$ and $e_n > f_n$. A path $x = (e_1, e_2, \cdots) \in X_{(V,E)}$ is a *maximal* (resp. *minimal*) *path* if all edges e_n are maximal (resp. minimal) in the linearly ordered set $r^{-1}(r(e_n))$.

Given an ordered Bratteli diagram $(V, E, \geq)$, the *Vershik map* $T_{(V,E)} : X_{(V,E)} \to X_{(V,E)}$ is defined such that a non-maximal path is mapped to its successor in the lexicographic order, while a maximal path is mapped to a minimal path. Let $(V, E, \geq)$ be an ordered Bratteli diagram, where (V, E) is simple. We say that $(V, E, \geq)$ is *properly ordered* if there exist exactly one maximal path and one minimal path. Then $(X_{(V,E)}, T_{(V,E)})$ is a Cantor minimal system, and we call such a system a *Bratteli-Vershik system.*

We state a basic theorem that we shall need, which we may call the model theorem for Cantor minimal systems, and we will refer to the properly ordered Bratteli diagram $(V, E \geq)$ occurring in the theorem as a *Bratteli-Vershik model* (for the given Cantor minimal system (X, T)).

Theorem 3.1. *(*[5, Thm.4.7], [6, Thm.4]*) Let (X, T, x) be a (pointed) Cantor minimal system, where $x \in X$. There exists a properly ordered Bratteli diagram $(V, E, \geq)$ such that (X, T, x) is (pointedly) conjugate to $(X_{(V,E)}, T_{(V,E)}, x_{min})$, where x_{min} is the unique minimal path in $X_{(V,E)}$. This means that the conjugating map $h : X \to X_{(V,E)}$ maps x to x_{min}.*

Remark 3.2. There is a natural way to introduce an ordering on a Bratteli diagram which is obtained from an ordered Bratteli diagram by either telescoping or symbol splitting, cf. [1, Section 3]. Both telescoping and symbol splitting yield natural homeomorphisms, preserving cofinality, between the original path space and the new path space, such that $(X_{(V,E)}, T_{(V,E)})$ is conjugate to $(X_{(V',E')}, T_{(V',E')})$, where $(V', E', \geq)$ is the ordered Bratteli diagram obtained from $(V, E, \geq)$ by a finite number of telescopings and/or symbol splittings.

The Bratteli diagram (V, E) induces an equivalence relation on $X_{(V,E)}$, denoted by $AF(V, E)$, namely, two paths are equivalent if and only if they are cofinal. Topologized appropriately (cf. [2, Example 2.7 (ii)]), $AF(V, E)$ becomes a so-called *AF-equivalence relation*, according to the following definition.

Definition 3.3. *An AF-equivalence relation R on a zero-dimensional space X is an étale equivalence relation $(R, \mathcal{T})$ such that $R = \cup_{n=1}^{\infty} R_n$, where $R_1 \subset R_2 \subset \cdots$ is an increasing sequence of subequivalence relations of R such that $R_n = (R_n, \mathcal{T}_n)$ is a compact étale equivalence relation (CEER) for all n. R is given the inductive limit topology, i.e. $U \subset \mathcal{T}$ iff $U \cap R_n \in \mathcal{T}_n$ for all n. We write $(R, \mathcal{T}) = \varinjlim (R_n, \mathcal{T}_n)$.*

In a similar way as ordered Bratteli diagrams serve as models for Cantor minimal systems, (unordered) Bratteli diagrams serve as models for AF-equivalence relations as stated in the following theorem.

Theorem 3.4. *(*[2, Thm. 3.9]*) Let $(R, T) = \varinjlim (R_n, \mathcal{T}_n)$ be an AF-equivalence relation on the zero-dimensional space X. There exists a Bratteli diagram (V, E) such that $(R, \mathcal{T})$ is isomorphic to the AF-equivalence relation $AF(V, E)$ associated to (V, E). Furthermore, (V, E) is simple if and only if $(R, \mathcal{T})$ is minimal.*

Combining Theorem 3.1 and Theorem 3.4 we get the following corollary.

Corollary 3.5. *(*[2, Theorem 2.4]*) Let (X,T) be a Cantor minimal system and let $(R,\mathcal{T})$ be the associated étale equivalence relation as described in Section 2. Let x be an arbitrary point in X. The subequivalence relation $R_{\{x\}}$ of R whose equivalence classes are the full T-orbits, except that the T-orbit of x is split into two at x — the forward orbit $\{T^n x \mid n \geq 1\}$ and the backward orbit $\{T^n x \mid n \leq 0\}$ — is open in R. Furthermore, $(R_{\{x\}},\mathcal{T}_{\{x\}})$ is an AF-equivalence relation on X, where $\mathcal{T}_{\{x\}}$ is the relative topology.*

Remark 3.6. It is noteworthy that if x_1 and x_2 are any two points in X, then $(R_{\{x_1\}},\mathcal{T}_{\{x_1\}}) \simeq (R_{\{x_2\}},\mathcal{T}_{\{x_2\}})$. This follows from [2, Lemma 4.13] and [5, Theorem 5.3].

Let (V,E) be a Bratteli diagram. By a *subdiagram* of (V,E) we mean a Bratteli diagram (W,F), where $W \subset V$, $F \subset E$ and $t(F) \cup \{v_0\} = i(F)$. The range and source maps of (W,F) are the restrictions of the range and source maps of (V,E). Note that a subdiagram (W,F) of a Bratteli diagram (V,E) is being telescoped or symbol splitted in an obvious way simultaneously as these operations are applied to (V,E). If $(V,E,\geq)$ is an ordered Bratteli diagram, a sub-diagram (W,F) of (V,E) will inherit the order in an obvious way. Note that if (W,F) is a sub-diagram of (V,E), then the topology of $AF(W,F)$ coincides with the relative topology from $AF(V,E)$, and so $AF(V,E)|_{X_{(W,F)}}$ is AF, and hence étale. With the terminology we have introduced we can say that $X_{(W,F)}$ is $AF(V,E)$-étale.

4. Proof of Theorem 1.1

The key to the proof is the following lemma, which illustrates what a useful tool Bratteli diagram models can be. In fact, by elementary and easy manipulations on a given Bratteli diagram (ordered or unordered) one can set the stage for proving non-trivial results that seems to be inaccessible otherwise.

Lemma 4.1. *Let (X,T) be a Cantor minimal system, and let $\{(l_k,n_k)\}_{k=1}^{\infty}$ be as sequence of pairs of natural numbers, where $l_k \geq 2$. There exists a Bratteli-Vershik model $(V,E,\geq)$ for (X,T) such that*

(i) $|V_k| \geq l_k,\ \forall k > 0$.
(ii) *x_{min} and x_{max} do not pass through the same vertex at any level $k \geq 1$ of (V,E), where x_{min} and x_{max} denote the unique minimal and maximal paths, respectively, in $X_{(V,E)}$.*
(iii) *If $V_{k-1} = \{v_1, v_2, \ldots, v_{m_{k-1}}\}$, $k \geq 1$, then for all vertices $w \in V_k$ and for all $v_i \in V_{k-1}$ we can choose n_k edges $\{e_{(i,j)}\}_{j=1}^{n_k}$ in E_k connecting w to v_i. Furthermore, the ordering of these edges are as follows: $e_{(1,1)} < e_{(2,1)} < \cdots < e_{(m_{k-1},1)} < e_{(1,2)} < e_{(2,2)} < \cdots < e_{(m_{k-1},2)} < \cdots < e_{(1,n_k)} < \cdots < e_{(m_{k-1},n_k)}$.*

Proof. Let $(V,E,\geq)$ be a Bratteli-Vershik model for (X,T). It is easy to see that by a succession of telescoping, symbol splitting and telescoping, in that order, one may satisfy condition (i) and (ii). Indeed, more can be achieved by the same token. If $(V',E',\geq)$ denotes the new ordered Bratteli diagram obtained, which by Remark 3.2 is again a Bratteli-Vershik model for (X,T), then we can assume that $|V'_{k-1}| \leq |V'_k|$ for all $k \geq 1$. Furthermore, we may assume that (V',E') is totally connected, that is, between any $v \in V'_{k-1}$ and $w \in V'_k$, there exists an edge $e \in E'_k$

connecting the two, i.e. $i(e) = v$ and $t(e) = w$. We observe that all the properties of $(V'E', \geq)$ listed above are preserved under telescopings of $(V', E', \geq)$.

So we may at start assume that $(V, E, \geq)$ has all the above-mentioned properties. We want to show that condition (iii) can be obtained by telescoping $(V, E, \geq)$, and this will finish the proof by the above remarks. Clearly we can choose a level $k_1 \geq 1$ such that if we telescope between level $k_0 = 0$ and k_1 of $(V, E, \geq)$, then (iii) is satisfied for $k = 1$. Assume we have telescoped $(V, E, \geq)$ between the levels $0 = k_0 < k_1 < \cdots < k_l$, such that (iii) is satisfied for $k = 1, 2, \ldots, l$. Now choose an arbitrary $u \in V_{k_l+1}$. By our assumption on total connection, there exists an edge e_i ranging at u and sourcing at v_i for every $i \in \{1, 2, \ldots, s\}$, where $V_{k_l} = \{v_1, v_2, \ldots, v_s\}$. By rearranging we may assume that $e_1 < e_2 < \cdots < e_s$. There exists a level $k_{l+1} > k_l + 1$ such that the number of paths from u to any vertex in $V_{k_{l+1}}$ is at least n_{k+1}. We telescope between level k_l and k_{l+1} of our diagram, and for each $v_i \in V_{k_l}$ and $w \in V_{k_{l+1}}$ we choose n_{k+1} paths arbitrarily between v_i and w, except that we require the first edge of each path to be e_i. Now it is easy to see, using the lexicographic way we order paths, that we may arrange these n_{k+1} paths, which becomes edges after telescoping, in such a way that they satisfy condition (iii) for $k = k_{l+1}$. Telescoping $(V, E, \geq)$ to levels $0 = k_0 < k_1 < k_2 < \cdots$, we get a diagram satisfying the three conditions of the lemma. □

Proof of Theorem 1.1. The idea is to use Lemma 4.1 to construct a Bratteli-Vershik model $(V, E, \geq)$ for (X, T), in which we can imbed a Bratteli- Vershik model $(W, F, \geq)$ for (Y, S), such that the ordering on $(W, F \geq)$ coincides with the one induced from $(V, E, \geq)$. This will obviously give a conjugacy $h : (Y, S) \to (Z, \widetilde{T})$, where $Z = X_{(W,F)}$, and $\widetilde{T} : Z \to Z$ is the induced map as described in the theorem.

Let $(W, F, \geq)$ be a Bratteli-Vershik model for (Y, S), where the maximal and the minimal paths pass through different vertices at each level. Let $l_k = |W_k|$ and let n_k be the maximal number of edges ranging at a vertex at level W_k, i.e. $n_k = max\{|t^{-1}(w)| \,|\, w \in W_k\}$. Let $(V, E, \geq)$ be a Bratteli-Vershik model for (X, T) satisfying the conditions of Lemma 4.1 with respect to the sequence $\{(l_k, n_k)\}_{k=1}^{\infty}$. We describe how to define a copy of (W, F) as a sub-diagram of (V, E), such that the order of $(W, F, \geq)$ coincides with the induced order from $(V, E, \geq)$.

For $k > 0$, choose $l_k = |W_k|$ vertices $\{v_1^{(k)}, v_2^{(k)}, \ldots, v_{l_k}^{(k)}\} \subset V_k$, including the ones that the unique maximal and minimal paths pass through, denoting these by $v_{max}^{(k)}$ and $v_{min}^{(k)}$, respectively. We denote the corresponding vertices in W_k by $w_{max}^{(k)}$ and $w_{min}^{(k)}$, respectively. Let g_k be a bijection between W_k and $\{v_i^{(k)}\}_{i=1}^{l_k}$, such that $g_k(w_{max}^{(k)}) = v_{max}^{(k)}$ and $g_k(w_{min}^{(k)}) = v_{min}^{(k)}$. We define $g_0(w_0) = v_0$, where $W_0 = \{w_0\}$ and $V_0 = \{v_0\}$.

Next we define an injective map h_k from F_k into E_k. For a vertex $w^{(k)} \in W_k$, let $\{f_s\}_{s=1}^{m}$ be the linearly ordered edges in F_k with range $w^{(k)}$, i.e. $f_1 < f_2 < \cdots < f_m$. Let $v^{(k)} = g_k(w^{(k)})$, and let $\{e_t\}_{t=1}^{n}$ be the linearly ordered edges in E_k with range $v^{(k)}$. Define $h_k(f_1) = e_{t_1}$, where e_{t_1} is the minimal edge in $\{e_t\}_{t=1}^{n}$ such that $g_{k-1}(i(f_1)) = i(e_{t_1})$. Note that if f_1 is an edge of the unique minimal path in $X_{(W,F)}$, then e_{t_1} is an edge of the unique minimal path in $X_{(V,E)}$. After having defined $h_k(f_l) = e_{t_l}$, where $l < m$ and $l + 1 < m$, we define $h_k(f_{l+1}) = e_{t_{l+1}}$, where $e_{t_{l+1}}$ is the minimal edge in $\{e_t\}_{t=1}^{n}$ greater than e_{t_l}, such that $g_{k-1}(i(f_{l+1})) = i(e_{t_{l+1}})$. If $l + 1 = m$, we define $h_k(f_m)$ to be the maximal edge e in $\{e_t\}_{t=1}^{n}$ such that $g_{k-1}(i(f_m)) = i(e)$. Note that if f_m is an edge of the unique maximal path in

$X_{(W,F)}$ then $h_k(f_m)$ will be an edge of the unique maximal path in $X_{(V,E)}$. The properties satisfied by the ordered Bratteli diagram $(V, E, \geq)$ entail that the map h_k is well-defined for $k = 1, 2, \ldots$.

Let $W' = \{g_k(w) \,|\, w \in W_k,\ k = 0, 1, 2 \cdots\}$, $F' = \{h_k(f) \,|\, f \in F_k,\ k = 1, 2, \cdots\}$. It is easy to see that (W', F') is a subdiagram of (V, F), and that (W', F') is a an obvious way isomorphic to (W, F). Transferring the order from $(W, F \geq)$ to (W', F') by this isomorphism we get a copy, $(W', F', \geq)$, of $(W, F, \geq)$. Hence the two associated Bratteli-Vershik systems are conjugate, with conjugating map $h : X_{(W,F)} \to X_{(W',F')}$ being defined by $h(x) = (h_1(f_1), h_2(f_2), \cdots) \in X_{(W',F')}$, where $x = (f_1, f_2, \cdots) \in X_{(W,F)}$. Furthermore, it follows by our definition of the pair of maps (g_{k-1}, h_k), for $k = 1, 2, \cdots$, that the ordering on $(W', F', \geq)$ coincides with the induced ordering from $(V, E, \geq)$, i.e. if $f_1, f_2 \in F'$ and $t(f_1) = t(f_2)$, then $f_1 < f_2$ in $(W', F', \geq)$ if and only if $f_1 < f_2$ in $(V, E, \geq)$. We conclude from all this that the proof of the first statement of the theorem is complete if we show that the return times to $Z = X_{(W',F')}$ is finite. But this follows from the fact that by our set-up the unique maximal and minimal paths in $X_{(W',F')}$ coincide with the unique maximal and minimal paths, respectively, in $X_{(V,E)}$. We omit the easy details.

The set $Z = X_{(W',F')}$ may not be regular since the forward return time map at x_{max}, and the backward return time map at x_{min}, may not be continuous. (Here x_{max} and x_{min} denote the (coinciding) unique maximal and minimal paths, respectively, in $X_{(W',F')}$ and $X_{(V,E)}$.) At all other paths it is easy to see that the two return time maps are continuous. By a slight modification of the construction of $(W', F', \geq)$ we can achieve that $Z = X_{(W',F')}$ is regular, but we pay the price that at the unique maximal and minimal paths in $X_{(W',F')}$ (which no longer coincide with the corresponding ones in $X_{(V,E)}$) the return time is no longer finite. Referring to the notation used above, we let $l_k = |W_k| + 2$. This time we avoid $v^{(k)}_{max}$ and $v^{(k)}_{min}$ for all k in the construction of the subdiagram (W', F'). It is easy to see that $Z = X_{(W',F')}$ is T-regular, and that $(Y, S) \simeq (Z, \widetilde{T})$, where we define $\widetilde{T}(y_{max}) = y_{min}$. (Here y_{max} and y_{min} denote the unique maximal and minimal paths, respectively, in $X_{(W',F')}$.) We have that in this case the forward return time of y_{max}, and the backward return time of y_{min}, are both infinite. However, the return time maps at both these points are continuous. We omit the details.

The last assertion of the theorem is easy to obtain. In fact, we can telescope (V, E), before we define the subdiagram (W', F'), such that the ratio of the number of paths from the top vertex in (V, E) to any vertex at level k to the corresponding number for (W, F), tend to zero as k goes to infinite. Then $Z = X_{(W',F')}$ is going to be a thin subset of $X_{(V,E)}$, a fact that is easily seen. This completes the proof. $\square$

We give a simple example to illustrate the construction done both in the proof of Lemma 4.1 and Theorem 1.1. We keep the above notation.

Example 4.2. *Let (Y, S) be the Sturmian flow with rotation number equal to the golden mean. The simplest Bratteli-Vershik model $(W, F, \geq)$ for (Y, S) is shown in Figure 3, cf.* [6, 3.3]. *In this case the parameters are $l_k = |W_k| = 2$ for all $k \geq 1$, and $n_k = 2$ for $k \geq 2$. Let (X, T) be the 2-odometer. The simplest Bratteli-Vershik model for (X, T) is shown to the left in Figure 4. Also in Figure 4 we indicate the manipulations done in order to get a Bratteli-Vershik model $(V, E, \geq)$ for (X, T) that is adapted for the construction of a copy $(W', F', \geq)$ of $(W, F, \geq)$ as a subdiagram. (Note that in this case we only need the operations of symbol*

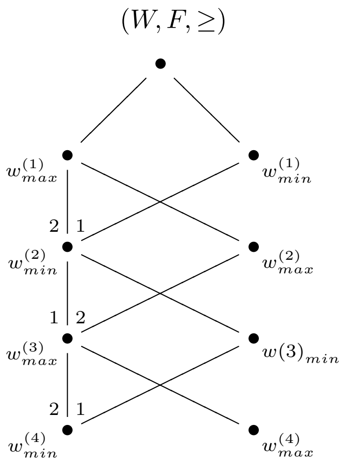

FIGURE 3. A Bratteli-Vershik model for (Y, S).

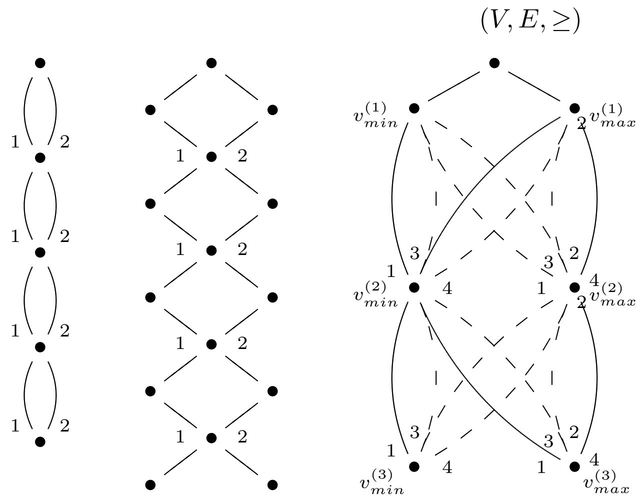

FIGURE 4. Starting with the two-odometer, we first do symbol splitting between every level, then we telescope to the sequence $0 < 1 < 5 < 9 < 13 < \dots$.

splitting and telescoping, in that order.) The edges belonging to $(W', F', \geq)$ *are solidly drawn.*

5. PROOF OF THEOREM 1.2

Proof of Theorem 1.2. We start by showing (i) $\Rightarrow$ (ii). For $k \in \mathbb{N}$, define the maps $\lambda_k^+, \lambda_k^- : Z \longrightarrow \mathbb{N} \cup \{\infty\}$, by

$$\lambda_k^+(z) = inf\{l \geq 1, \infty \mid \exists 0 < l_1 < ... < l_{k-1} < l_k = l \text{ s.t. } T^{l_i} z \in Z, i = 1, ..., k\},$$

$$\lambda_k^-(z) = inf\{l \geq 1, \infty \mid \exists 0 < l_1 < ... < l_{k-1} < l_k = l \text{ s.t. } T^{-l_i}z \in Z, i = 1, ..., k\}.$$

We call these maps the positive and negative k-th return time maps with respect to Z. We claim that λ_k^+ and λ_k^- is continuous for every $k \in \mathbb{N}$ if and only if λ^+ (respectively λ^-) is continuous. This follows by induction on k, and the observation that for all $z \in Z$ we have $\lambda_{k+1}^+(z) = \lambda_k^+(z) + \lambda^+(T^{\lambda_k^+(z)}(z))$. The claim concerning λ_k^- is completely analogues. (The modifications needed if $\lambda_k^{\pm}(y) = \infty$ are obvious.)

Suppose Z is regular. Let $z_0 \in Z$, and assume that the equivalence class of z_0 in Z is $[z_0]_{R|Z} = \{..., z_{-1}, z_0, z_1, ...\}$, where $z_i = T^{m_i}z_0$, and we have arranged the points such that $... < m_{-1} < m_0 = 0 < m_1 < ....$ We want to find an étale neighborhood $U_{(z_0,z)} \subset R|_Z$ for all possibilities of $z \in [z_0]_{R|Z}$. If $z = z_0$, then obviously $\Delta_X \cap (Z \times Z)$ is an étale neighborhood containing (z_0, z_0), where $\Delta_X = \{(x, x) \mid x \in X\}$ is the diagonal of R. Next assume z is in the positive orbit of z_0. Note that if $\lambda^+(z_0) = \infty$ then there are no points to check. Suppose $z = z_k$, i.e. $z = T^{m_k}z_0$. This means that $z_0 \in (\lambda_k^+)^{-1}(m_k)$, which is open in Z by continuity of λ_k^+. Let $V_{(z_0,z)}$ be an étale neighborhood of (z_0, z) in R, and let $U_{z_0} = (\lambda_k^+)^{-1}(m_k) \cap r(V_{(z_0,z)}) \subset Z$. Define $U_{(z_0,z)} = \{(x, T^{m_k}x) \mid x \in U_{z_0}\}$. This is an étale neighborhood of (z_0, z) in $R|_z$, a fact that is easily verified. If z is in the negative orbit the argument is analogous, using the continuity of λ_k^-.

To prove (ii) $\Rightarrow$ (iii), let $x \in X$. By Corollary 3.5 $R_{\{x\}}$ is open in R, and hence $R_{\{x\}}|_Z$ is open in $R|_Z$. As Z is R-étale, it follows that Z is $R_{\{x\}}$-étale, since every open sub-equivalence relation of an étale equivalence relation is étale.

To show (iii) $\Rightarrow$ (i) suppose that Z is not regular. We want to show that this implies that there exists $x \in X$ such that $R_x|_Z$ is not étale. In fact, we will show that $x = z \in Z$, where z is a point of discontinuity of either λ^+ or λ^-. So let $z \in Z$, and suppose λ^+ is not continuous at z. (A similar argument applies with respect to λ^-.) As λ^+ is always lower semi-continuous, it is not upper semi-continuous at z. So there exist a sequence $\{z_n\}_{n=1}^{\infty} \subset Z$ such that $z_n \longrightarrow z$ and $\lim_n \lambda^+(z_n) > \lambda^+(z)$. [Note: $\lim_n \lambda^+(z_n)$ may be infinite, but if so, the ensuring argument is unchanged.] Assuming $R|_Z$ is étale there exist an étale neighborhood $U_{(z,T^{\lambda^+(z)}z)}$ of $(z, T^{\lambda^+(z)}z)$ in $R|_Z$. Choose an open neighborhood V_z of z in X. Put $V_{(z,T^{\lambda^+(z)}z)} = \{(x, T^{\lambda^+(z)}x) | x \in V_z\}$. $V_{(z,T^{\lambda^+(z)}z)}$ is an étale neighborhood of $(z, T^{\lambda^+(z)}z)$ with respect to R. This implies that $U_{(z,T^{\lambda^+(z)}z)} \cap V_{(z,T^{\lambda^+(z)}z)}$ is another étale neighborhood of $(z, T^{\lambda^+(z)}z)$ with respect to $R|_Z$. This is clear since $R|_Z$ has the relative topology from R. However, $\{z_n\}_{n=1}^{\infty} \cap r(U_{(z,T^{\lambda^+(z)}z)} \cap V_{(z,T^{\lambda^+(z)}z)}) = \emptyset$, and so $r(U_{(z,T^{\lambda^+(z)}z)} \cap V_{(z,T^{\lambda^+(z)}z)})$ can not be open in Z, and hence $U_{(z,T^{\lambda^+(z)}z)} \cap V_{(z,T^{\lambda^+(z)}z)}$ is not an étale neighborhood of $(z, T^{\lambda^+(z)}z)$. This contradiction finishes the proof of (iii) $\Rightarrow$ (i), and so the proof of Theorem 1.2. □

Corollary 5.1. *Let (X, T) be a Cantor minimal system, and let $(R, \mathcal{T})$ be the associated étale equivalence relation. Let Z be a R-étale subset of X. For any $x \in X$ there exists a simple Bratteli diagram (V, E) and a homeomorphism $h : X_{(V,E)} \to X$ implementing an isomorphism $h \times h : AF(V, E) \to R_{\{x\}}$ such that $h^{-1}(Z) = X_{(W,F)}$ for some subdiagram (W, F) of (V, E).*

Proof. By Corollary 3.5, $R_{\{x\}}$ is an AF-equivalence relation for $x \in X$. By Theorem 1.2 we have that Z is $R_{\{x\}}$-étale. By [2, Theorem 3.11] we get the result. □

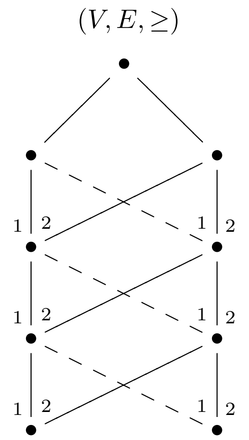

FIGURE 5. The subdiagram (W, F) of the Bratteli diagram (V, E) is obtained by deleting the dotted edges.

The example we will now exhibit is somewhat related to the above corollary, even though it illustrates a different aspect of the theory. Figure 5 shows that not every sub-diagram of a properly ordered, simple Bratteli-diagram is regular with respect to the Vershik map, and hence not all subdiagrams give rise to étale sub-equivalence relations by Theorem 1.2. Let $(X_{(V,E)}, T_{(V,E)})$ be the Bratteli-Vershik system associated to $(V, E, \geq)$ in Figure 5, and let x_{min} and x_{max} be the unique minimal and maximal paths, respectively, in $X_{(V,E)}$. Let $Z = X_{(W,F)}$. Now $\lambda^+(x_{\max}) = 1$, but there exists a sequence $\{z_n\}$ in Z converging to $x_{\max}$, such that $\lambda^+(z_n) \longrightarrow \infty$, which shows that λ^+ is not continuous at x_{max}.

Note also, referring again to Figure 5, that Z is $R_{\{x_{max}\}}$-étale, but not $R_{\{x_{min}\}}$-étale. This underscores the requirement (iii) of Theorem 1.2 , namely that Z should be $R_{\{x\}}$-étale for *all* $x \in X$.

We end this paper by giving the following result which extends Theorem 1.2 when the subset $Z \subset X$ satisfies a certain condition.

Corollary 5.2. *Let (X, T) be a Cantor minimal system and let $(R, \mathcal{T})$ be the associated étale equivalence relation. Let Z be a non-empty closed subset of X such that $\exists z_0 \in X$ with $[z_0]_T \cap Z$ contained in either $\{T^n z_0 | n \geq 1\}$ or $\{T^{-n} z_0 | n \geq 0\}$. (Recall that $[z_0]_T$ denotes the T-orbit $\{T^n z_0 \,|\, n \in \mathbb{Z}\}$ of z_0.) The following are equivalent.*

(i) *Z is regular;*
(ii) *Z is R-étale;*
(iii) *Z is $R_{\{x\}}$-étale for all $x \in X$;*
(iv) *There exists a simple Bratteli diagram (V, E), containing a sub-diagram (W, F), and a map $h : X_{(V,E)} \longrightarrow X$ such that $h \times h : AF(V, E) \longrightarrow R_{\{z_0\}}$ is an isomorphism and $h(X_{(W,F)}) = Z$.*

Proof. By Corollary 5.1 we have that $(ii) \Rightarrow (iv)$. We will prove $(iv) \Rightarrow (ii)$. The rest follows by Theorem 1.2.

Let (V, E) and (W, F) be as in (iv). Now $AF(W, F)$ is an AF-equivalence relation on $X_{(W,F)}$, and the topology coincides with the relative topology from $AF(V, E)$.

So $(h \times h)(AF(W,F) = R_{z_0}|_Z$ is an AF-equivalence relation, and hence étale, on $h(X_{(W,F)}) = Z$. This means that $R|_Z$ is étale, since $R|_Z = R_{\{z_0\}}|_Z$, and both $R|_Z$ and $R_{\{z_0\}}|_Z$ have relative topology from R. □

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III. DYNAMICAL CHOQUET SIMPLICES AND CANTOR MINIMAL SYSTEMS

DYNAMICAL CHOQUET SIMPLICES AND CANTOR MINIMAL SYSTEMS.

HEIDI DAHL

Abstract. We prove that if K is a dynamical simplex of probability measures on the Cantor set X, for which the extreme points, $\partial_e K$, of K is a finite set, then there exists a minimal homeomorphism $T : X \to X$ such that $M(X, T) = K$, where $M(X, T)$ denotes the T-invariant probability measures on X.

Introduction

For a Cantor minimal system (X, T), the set of T-invariant probability measures, with the w^*-topology, is known to be a Choquet simplex consisting of non-atomic (probability) measures of full support, and where the extreme points are ergodic measures which are mutually singular with respect to each other (cf. [14, Ch 6.2] and [11, Ch 10]). Downarowicz [3], and later Gjerde and Johansen [7] – using a Bratteli-Vershik approach – showed that in fact any metrizable Choquet simplex is affinely homeomorphic to the set of invariant measures of some $0-1$ Toeplitz flow.

In [1], Akin showed that if μ is a so-called *good measure* on the Cantor set X, there exists a uniquely ergodic transformation on X for which μ is the unique invariant probability measure. In this paper we extend Akin's definition of a good measure to Choquet simplices of probability measures, and show that for any Cantor minimal system (X, T) the set of T-invariant probability measures is what we will call a *dynamical simplex*. Also, given a dynamical simplex K of probability measures on the Cantor set X, where the extreme boundary $\partial_e K$ is a finite set, there exists a minimal homeomorphism on X having precisely K as its set of invariant measures. Our proofs are very different in spirit from Akin's in that we base our arguments on the approach initiated in [10], and further developed in [6] and [9]. In fact, by this method one can even sharpen Akin's result in the uniquely ergodic case. However, our goal is to prove a result for a general dynamical simplex, but we have only succeeded so far to prove it for finite-dimensional simplices. The relevant definitions and terminology will be presented below.

1. Dynamical simplices associated to Cantor minimal systems

Throughout this paper, let X denote the Cantor set and let $M(X)$ denote the set of probability measures on X. For a Cantor minimal system (X, T), we denote by $M(X, T)$ the set of all T-invariant probability measures on X, i.e. $M(X, T) = \{\mu \in M(X) \,|\, \mu(T^{-1}(A)) = \mu(A) \text{ for all Borel sets } A \subset X\}$.

Recall that a convex set $K \subset M(X)$ is a *Choquet simplex* if it is compact in the w^*-topology, and each $\mu \in K$ is represented by a unique probability measure supported on the extreme boundary $\partial_e K$ of K, cf [11]. If $\partial_e K$ is a finite set, K is affinely homeomorphic to a finite-dimensional simplex in a Euclidean space.

Definition 1.1. Let $K \subset M(X)$ be a Choquet simplex consisting of non-atomic probability measures with full support. We say that K is a *dynamical simplex* (abbreviated *D-simplex*) if it satisfies the following two conditions:

(i) For clopen subsets A and B of X with $\mu(A) < \mu(B)$ for all $\mu \in K$, there exists a clopen subset $B_1 \subset B$ such that $\mu(A) = \mu(B_1)$ for all $\mu \in K$ (*the subset condition*).

(ii) If μ, $\nu \in \partial_e K$, $\mu \neq \nu$, then μ and ν are mutually singular, i.e. there exists a measurable set $A \subset X$ such that $\mu(A) = 1$ and $\nu(A) = 0$.

Remark 1.2. *Part* (*i*) *is an obvious extension of the definition of a* good measure *given by Akin in* [1], *applied to a set of probability measures .*

As we pointed out above, it is well known that for any Cantor minimal system (X, T), the set $M(X, T)$ is a Choquet simplex of non-atomic probability measures with full support, where the extreme points are mutually singular. In fact, the following proposition, whose proof is due to Glasner and Weiss [9], shows that $M(X, T)$ is a dynamical simplex.

Proposition 1.3. [9, Lemma 2.5] *Let* (X, T) *be a Cantor minimal system. The set* $M(X, T)$ *of* T*-invariant probability measures on* X *is a D-simplex.*

Proof. We only need to show the subset condition. Assume A and B are clopen subsets of X such that $\mu(A) < \mu(B)$ for all $\mu \in M(X, T)$. Since $\mu(A) = \mu(A \cap B) + \mu(A \setminus B)$ and $\mu(B) = \mu(A \cap B) + \mu(B \setminus A)$, we can assume A and B to be disjoint. Define a function f by $f = \chi_B - \chi_A$, where χ_E denotes the characteristic function of a clopen set $E \subset X$, i.e. $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ otherwise. Choose c such that $0 < c < \inf\{\int_X f d\mu \mid \mu \in M(X, T)\}$. This is possible since $M(X, T)$ is a compact set. We claim that there exists $N_0 \in \mathbb{N}$ such that for all $x \in X$ and all $N \geq N_0$ we have $\frac{1}{N}\sum_{j=0}^{N-1} f(T^j x) \geq c$. If not, there exist a sequence $\{x_k\}_{k=1}^{\infty}$ in X, and an increasing sequence $\{N_k\}_{k=1}^{\infty}$ in $\mathbb{N}$, such that $\frac{1}{N_k}\sum_{j=0}^{N_k-1} f(T^j x_k) < c$ for all k. Then the sequence $\frac{1}{N_k}\sum_{j=0}^{N_k-1} \delta_{T^j x_k}$ converges in the w^*-topology to a T-invariant measure $\nu \in M(X, T)$ as $k \to \infty$, and $\int_X f d\nu \leq c$, giving a contradiction. This proves the claim.

Next we find a non-empty clopen set $D \subset X$ such that $D, T(D), \cdots, T^{N_0-1}(D)$ are mutually disjoint clopen subsets of X, and we construct the *Kakutani-Rohlin partition* of X with basis set D. That is, first define the return time map $\lambda : D \to \mathbb{N}$ by $\lambda(x) = \inf\{k > 0 \,|\, T^k x \in D\}$. Since D is compact, there are only finitely many different return times, say $\{h_1, h_2, \ldots, h_l\}$, and hence we can write D as the disjoint union $D = \cup_{j=1}^{l} D_j$, where D_j is the set of points in D having return time h_j. Note that $h_j \geq N_0$ for $j = 1, 2, \ldots, l$. Now X can be written as the disjoint union $X = \cup_{j=1}^{l} \cup_{i=0}^{h_j-1} T^i(D_j)$, and this is called the Kakutani-Rohlin partition of X (associated to T) with basis set D. The set $C_j = \cup_{i=0}^{h_j-1} T^i(D_j)$ is called the j'th *tower* of the partition (of hight h_j), while $T^i(D_j)$ is called a *floor* of the j'th tower. For a T-invariant probability measure μ, each floor in the same tower has the same μ-measure. We may assume by subdividing the towers that the partition is compatible with A and B, i.e. if $A \cap T^i(D_j) \neq \emptyset$ for some $1 \leq j \leq l$, some $0 \leq i < h_j$, then $T^i(D_j) \subset A$, and likewise for B. For a tower C_j in the partition, choose $x \in D_j$ and consider the sum $\frac{1}{h_j}\sum_{i=0}^{h_j-1} f(T^i x)$. Since $h_j \geq N_0$, this is a positive number $\geq c$. But as $f(T^i x) = 1$ if $T^i x \in B$ (hence $T^i(D_j) \subset B$), and $f(T^i x) = -1$ if $T^i x \in A$ (hence $T^i(D_j) \subset A$), and $f(T^i x) = 0$ otherwise, the number of floors of C_j contained in B is (strictly) greater than the number of floors of C_j contained in A. Hence we can find $B_1 \subset B$ with $\mu(A) = \mu(B_1)$ for all $\mu \in M(X,T)$, simply by picking out from each tower of the partition as many floors contained in B as there are floors contained in A in this tower. $\square$

2. From D-simplices to Cantor minimal systems

Assume that we start with an arbitrary D-simplex K of probability measures on the Cantor set X. The goal is to construct a minimal homeomorphism $T : X \to X$ such that $M(X,T) = K$. The main ingredients of our construction will be Bratteli diagrams and dimension groups. We give a brief overview of the two concepts below, for a more thorough discussion cf. [4], [6] and [10]. For a survey article, we refer to [13].

Definition 2.1. A *Bratteli diagram* (V,E) consists of a vertex set V, an edge set E and two maps; $r, s : E \to V$ (the *range* and *source* map, respectively), such that

(i) The vertex set is a disjoint union of non-empty finite sets; $V = \bigcup_{n=0}^{\infty} V_n$, and $V_0 = \{v_0\}$ is a one-point set.
(ii) The edge set is a disjoint union of non-empty finite sets; $E = \bigcup_{n=1}^{\infty} E_n$.
(iii) $r(E_n) \subset V_n$ and $r^{-1}(v) \neq \emptyset$ for all $v \in V \setminus V_0$.
(iv) $s(E_n) \subset V_{n-1}$ and $s(v) \neq \emptyset$ for all $v \in V$.

If we give a linear order to each of the V_n's, the Bratteli diagram can be coded in a sequence of *incidence matrices* $\{A_n\}_{n=1}^{\infty} = \{(a_{ij}^{(n)})\}_{n=1}^{\infty}$, where the entry $a_{ij}^{(n)}$ is the number of edges in E_n having range $v_i \in V_n = \{v_1, v_2, \cdots, v_{l_n}\}$ and source $u_j \in V_{n-1} = \{u_1, u_2, \cdots, u_{l_{n-1}}\}$.

For a Bratteli diagram (V,E), we define the *path space* $X_{(V,E)}$ to be the set of all infinite paths of the diagram, where a path $x = (e_n)_{n=1}^{\infty}$ is a sequence of edges $e_1, e_2, \cdots$ such that $e_n \in E_n$ and $r(e_n) = s(e_{n+1})$ for all $n > 0$. We will only consider non-trivial Bratteli diagrams, i.e. diagrams (V,E) for which $X_{(V,E)}$ is an infinite set. The path space is a totally disconnected metric space. When (V,E) is a *simple* Bratteli diagram, $X_{(V,E)}$ is a Cantor set. (Here simple means that for any n and any vertex $v \in V_n$, there exists $m > n$ such that v is connected to all vertices in V_m by a finite path.) For an infinite path $x = (e_i)_{i=1}^{\infty}$, we denote by $C_k(x)$ the k'th *cylinder set* associated to x, i.e. $C_k(x) = \{(f_n)_{n=1}^{\infty} \in X_{(V,E)} \mid f_n = e_n \text{ for } n = 1, 2, \ldots, k\}$. Define the range of $C_k(x)$ by $r(C_k(x)) = r(e_k) \in V_k$. The cylinder sets form a clopen basis for the topology of $X_{(V,E)}$. Let $P_{(V,E)}^{(k)}$ denote the set of finite paths starting at $v_0 \in V_0$ and ranging at a vertex in V_k. The set $P_{(V,E)}^{(k)}$ can be identified in an obvious way with the set of all cylinder sets ranging at level k (and hence with a clopen partition of X). We will make this identification whenever it is convenient.

Given a Bratteli diagram (V,E), one can define a partial order $\geq$ on the path space by, for each vertex $v \in V \setminus V_0$, choose a linear order on the set of edges with range v, and then use the lexicographic order on $X_{(V,E)}$. That is, $x = (e_n)_{n=1}^{\infty} > y = (f_n)_{n=1}^{\infty}$ if there exists $k \in \mathbb{N}$ such that $e_n = f_n$ for all $n > k$ and $e_k > f_k$. If the diagram is simple and the order is such that there are exactly one path in $X_{(V,E)}$ for which all edges are maximal and exactly one path in $X_{(V,E)}$ for which all edges are minimal, we say that $(V,E,\geq)$ is a *properly ordered* Bratteli diagram. For a properly ordered Bratteli diagram $(V,E,\geq)$, the *Vershik* (or *lexicographic*) map

$T_{(V,E)} : X_{(V,E)} \to X_{(V,E)}$, defined by sending a non-maximal path to its successor and the maximal path to the minimal path, is a minimal homeomorphism, and hence $(X_{(V,E)}, T_{(V,E)})$ is a Cantor minimal system.

In [10], the following basic result is proved.

Theorem 2.2. [10, Thm. 4.7]. *Let (X, T) be a Cantor minimal system. There exists a properly ordered Bratteli diagram $(V, E, \geq)$ such that (X, T) is conjugate to $(X_{(V,E)}, T_{(V,E)})$, i.e. there exists a homeomorphism $F : X \to X_{(V,E)}$ such that $T = F^{-1} \circ T_{(V,E)} \circ F$.*

Consider an invariant probability measure μ for $(X_{(V,E)}, T_{(V,E)})$. For $k \in \mathbb{N}$ and a path $x = (e_n)_{n=1}^{\infty} \in X_{(V,E)}$ having a non-maximal edge e_i, for some $i \leq k$, the Vershik map will map the cylinder set $C_k(x)$ to a cylinder set $C_k(y)$ such that $r(C_k(x)) = r(C_k(y))$. Hence all cylinder sets in $P^{(k)}_{(V,E)}$ having the same range also have the same μ-measure. Furthermore, if $C_k(x) = \cup_{i=1}^{R} C_{k+1}(y_i)$ is the disjoint partition of a cylinder set of length k into cylinder sets of length $(k + 1)$, then $\mu(C_k(x)) = \sum_{i=1}^{R} \mu(C_{k+1}(y_i))$.

Definition 2.3. A *dimension group* (G, G^+, u), with an order unit $u \in G^+$, is an ordered abelian group G, where

(i) The *positive cone* $G^+ \subset G$ satisfies
$G^+ + G^+ \subset G^+$,
$G^+ \cap (-G^+) = \{0\}$ and
$G^+ - G^+ = G$.

(ii) For all $g \in G$ there exists $n \in \mathbb{N}$ such that $g \leq nu$.

(iii) The Riesz interpolarization property holds; i.e. given $g_i, f_j \in G, i, j = 1, 2$ such that $g_i \leq f_j$ for $i, j = 1, 2$, then there exists $h \in G$ with $g_i \leq h \leq f_j$, $i, j = 1, 2$.

A dimension group is said to be *simple* if every $g \in G^+ \setminus \{0\}$ is an order unit. Also, the *infinitesimal subgroup*, $\operatorname{Inf} G$, of G is defined by $\operatorname{Inf} G = \{g \in G \,|\, -\epsilon u \leq g \leq \epsilon u$ for every $0 < \epsilon = \frac{p}{q} \in \mathbb{Q}$ (where $\frac{p}{q} a \leq b$ means $pa \leq qb)\}$

The dimension group $K_0(V, E)$ associated to a Bratteli diagram (V, E), is the dimension group obtained by taking the inductive limit of the sequence

$$\mathbb{Z}^{|V_0|} \xrightarrow{A_1} \mathbb{Z}^{|V_1|} \xrightarrow{A_2} \mathbb{Z}^{|V_2|} \xrightarrow{A_3} \cdots,$$

where A_n is the incidence matrix between level $(n-1)$ and level n of the diagram, and we assume all $\mathbb{Z}^{|V_n|}$ to have standard order. The canonical order unit is the element corresponding to $1 \in \mathbb{Z}^{|V_0|} = \mathbb{Z}$. $K_0(V, E)$ is a simple dimension group iff (V, E) is a simple Bratteli diagram. One can show (cf. [5]) that any dimension group is order-isomorphic to one associated to a Bratteli diagram. In [10] it is shown that any simple dimension group with order unit u, (G, G^+, u), $G \neq \mathbb{Z}$, is order isomorphic to $(K^0(X, T), K^0(X, T)^+, \mathbf{1})$ by a map sending u to $\mathbf{1}$ ($\mathbf{1}$, the canonical order unit of $K^0(X, T)$, corresponds to the constant function 1 on X), where (X, T) is some Cantor minimal system. (Here $K^0(X, T)$ is the countable abelian group obtained from $C(X, \mathbb{Z})$ (the continuous functions on X taking values in $\mathbb{Z}$) by dividing out with the coboundary $\{f - f \circ T^{-1} \,|\, f \in C(X, \mathbb{Z})\}$. $K^0(X, T)^+$ is the image of $C(X, \mathbb{Z})^+$ under the quotient map.) In fact, if (V, E) is a simple Bratteli diagram, one can introduce a partial ordering $\geq$, making $(V, E, \geq)$ a properly ordered Bratteli diagram. The Cantor minimal system $(X_{(V,E)}, T_{(V,E)})$ associated to $(V, E, \geq)$, has the property that $K^0(X_{(V,E)}, T_{(V,E)})$ is order isomorphic to $K_0(V, E)$ by a map respecting canonical order units. We will use the notation $K_0(V, E) \simeq K^0(X_{(V,E)}, T_{(V,E)})$. The map implementing the order isomorphism is natural, sending a cylinder set C (or rather its equivalence class) in $C(X_{(V,E)}, \mathbb{Z})$ to the group element in $K_0(V, E)$ corresponding to $r(C)$, i.e. the range of C. The survey article [13] gives an overview of all this. With this in mind, we remark that any positive element $g \in K_0(V, E)^+$ can be viewed as a (positive) linear combination of vertices at some vertex level k, and so we can write $g = \sum_{i=1}^{l} a_i C_k(x_i)$, where the $C_k(x_i)$'s are mutually disjoint cylinder sets in $P^{(k)}_{(V,E)}$, and a_i are positive integers.

Definition 2.4. A *state* on a dimension group (G, G^+, u) is a positive normalized homomorphism $s : G \to \mathbb{R}$, i.e. $s(G^+) \subset \mathbb{R}^+$ and $s(u) = 1$.

We let $S(G)$ denote the set of all states on (G, G^+, u), which is a compact and convex set, cf. [4, Ch. 4]. For a simple dimension group (G, G^+, u) the order of G is determined by $S(G)$. In fact, $G^+ = \{g \in G \,|\, s(g) > 0\,, \forall s \in S(G)\} \cup \{0\}$, cf. [5, Cor. 1.5] and [4, Ch. 4].

For the dimension group $K_0(V, E)$ associated to a properly ordered Bratteli diagram $(V, E, \leq)$, there is an affine bijective correspondence $s \to \mu_s$ between the set $S(K_0(V, E))$ of states on $K_0(V, E)$ and the set $M(X_{(V,E)}, T_{(V,E)})$ of $T_{(V,E)}$-invariant probability measures on $X_{(V,E)}$ (cf. [10, Theorem 5.5]), and the correspondence is such that $\mu_s(C_k(x)) = s(C_k(x))$ for all $s \in S(K_0(V, E))$. Here $C_k(x)$ is viewed both as an element of $K_0(V, E)$ and as a clopen subset of $X_{(V,E)}$, in accordance with the identification we have described above. By Theorem 2.2 we get a similar correspondence between states on $K^0(X, T)$ and $M(X, T)$, where (X, T) is a Cantor minimal system.

Definition 2.5. For a simplex $K \subset M(X)$, let $\mathrm{Aff}(K)$ be the set of *affine* continuous functions from K to $\mathbb{R}$, i.e. $\mathrm{Aff}(K) = \{\phi : K \to \mathbb{R} \,|\, \phi \text{ continuous, and } \phi(t\mu_1 + (1-t)\mu_2) = t\phi(\mu_1) + (1-t)\phi(\mu_2), \forall\, \mu_1, \mu_2 \in K,\, t \in [0, 1]\}$

Note that any real-valued continuous function $f \in C(X)$ gives rise to a continuous affine function $\hat{f} : K \to \mathbb{R}$ by $\hat{f}(\mu) = \int_X f d\mu$. One can show that $\{\hat{f} \,|\, f \in C(X)\}$ is uniformly dense in $\mathrm{Aff}(K)$ [2, Cor. I.1.5]. Define the strict order on $\mathrm{Aff}(K)$ by $\hat{f} < \hat{g}$ if and only if $\hat{f}(\mu) < \hat{g}(\mu)$ for all $\mu \in K$, and set $\mathrm{Aff}(K)^+ = \{\hat{f} \in \mathrm{Aff}(K) | \hat{f} > \hat{0}\} \cup \{\hat{0}\}$. The function $\hat{1}$ is the canonical order unit.

The following result is due to Effros, Handelman and Shen:

Theorem 2.6. [4, Thm. 4.5] *Suppose K is a Choquet simplex and that H is a uniformly dense subgroup of* $\mathrm{Aff}(K)$. *Then with the strict order, H is a simple dimension group ($\neq \mathbb{Z}$) such that $\mathit{Inf}\, H = \{0\}$.*

Definition 2.7. For a simplex $K \subset M(X)$, define the set $G(K) = \{\hat{f} : K \to \mathbb{R} \,|\, f \in C(X, \mathbb{Z})\}$, where $\hat{f}(\mu) = \int_X f\, d\mu$ for $\mu \in K$, and give $G(K)$ the (inherited) strict order from $\mathrm{Aff}(K)$. We will denote the positive cone by $G(K)^+$. So $G(K)^+ = G(K) \cap \mathrm{Aff}(K)^+$. We note that $G(K)$ is a subgroup of $\mathrm{Aff}(K)$.

In the rest of this section we assume that K is a D-simplex of probability measures on the Cantor set X. Furthermore, we assume $G(K) = \{\hat{f} : K \to \mathbb{R} \,|\, f \in$

$C(X,\mathbb{Z})\}$ is dense in $\mathrm{Aff}(K)$, such that according to Theorem 2.6, $(G(K), G(K)^+, \hat{1})$ is a simple dimension group with the canonical order unit $\hat{1}$ (note that $\hat{1}(\mu) = 1$ for $\mu \in K$). We note that the state space $S(G(K))$ of $G(K)$ can be identified with K. In fact, the map sending $\mu \in K$ to the state $\hat{\mu} : G(K) \to \mathbb{R}$, where $\hat{\mu}(f) = \int_X f d\mu$, $f \in G(K)$, is an affine isomorphism. This follows from [11, Prop. 1.1.].

Our next goal is to prove Theorem 2.11, which says that under these conditions there exists a Cantor minimal system (X,T) such that $K = M(X,T)$.

Definition 2.8. Given two systems (X, K_1), (Y, K_2), where X, Y are Cantor sets and $K_1 (\subset M(X))$, $K_2 (\subset M(Y))$ are convex sets. Then K_1 is *affinely isomorphic* to K_2, written $K_1 \cong K_2$, if there exists a homeomorphism $h : X \to Y$ such that h_* is an *affine* bijection between K_1 and K_2. The map h_* is defined by $h_*(\mu)(B) = \mu(h^{-1}(B))$, $\mu \in K_1$, for all Borel sets $B \subset Y$. (Note that this is equivalent to $\widehat{f \circ h}(\mu) = \hat{f}(h_*(\mu))$ for all $f \in C(Y)$. If $f = \chi_B$, B clopen, we get $h_*(\mu)(B) = \mu(h^{-1}(B))$, and this in turn implies the former.) We will say that h implements the affine isomorphism between K_1 and K_2.

Lemma 2.9. *Given (X, K_1) and (Y, K_2), where X, Y are Cantor sets and where $K_1 \subset M(X)$, $K_2 \subset M(Y)$ are D-simplices. Then $K_1 \cong K_2$ if and only if there exists an affine bijection $\Theta : K_1 \to K_2$, such that for any clopen set $A \subset X$, there exists a clopen set $B \subset Y$ such that $\mu(A) = \Theta(\mu)(B)$ for all $\mu \in K_1$, and, conversely, for any clopen set $B \subset Y$, there exists a clopen set $A \subset X$ such that $\nu(B) = \Theta^{-1}(\nu)(A)$ for all $\nu \in K_2$.*

Proof. One direction is an immediate consequence of Definition 2.8, setting $\Theta = h_*$.

Conversely, assume we have an affine bijection $\Theta : K_1 \to K_2$ satisfying the condition of the lemma. We want to construct a homeomorphism between X and Y which will implement an affine isomorphism between K_1 and K_2. Let $\{\mathcal{P}_n\}_{n=1}^\infty$ and $\{\mathcal{Q}_n\}_{n=1}^\infty$ be sequences of clopen partitions generating the topology on X and Y, respectively. Set $\mathcal{A}_1 = \mathcal{P}_1 = \{A_{11}, \cdots, A_{1k_1}\}$. By assumption on Θ, we can find a clopen set $B_{11} \subset Y$ such that $\mu(A_{11}) = \Theta(\mu)(B_{11})$ for all $\mu \in K_1$. If $\mathcal{A}_1 = \{A_{11}, A_{12}\}$, i.e. $k_1 = 2$ and so $A_{12} = X \setminus A_{11}$, we chose $B_{12} = Y \setminus B_{11}$. If $k_1 > 2$, we have that $\mu(A_{12}) < \mu(X \setminus A_{11})$ for all $\mu \in K_1$ (since μ has full support). There exists a clopen set $B'_{12} \subset Y$ such that $\mu(A_{12}) = \Theta(\mu)(B'_{12})$ for all $\mu \in K_1$. Since $\mu(A_{12}) < \mu(X \setminus A_{11})$ for all $\mu \in K_1$, we have $\Theta(\mu)(B'_{12}) < \Theta(\mu)(Y \setminus B_{11})$, and so – using the fact that K_2 is a D-simplex – we can find a clopen set $B_{12} \subset Y$ such that $\mu(A_{12}) = \Theta(\mu)(B_{12})$, and B_{12} is disjoint from B_{11}. We find $B_{13}, B_{14} \cdots B_{1(k_1-1)}$ similarly, and by choosing $B_{1k_1} = Y \setminus \cup_{j=1}^{k_1-1} B_{1j}$, we get a clopen partition $\mathcal{B}_1 = \{B_{11}, B_{12}, \cdots B_{1k_1}\}$ of Y. Then $\mathcal{B}_1$ is measurably comparable to $\mathcal{A}_1$, i.e. $\mu(A_{1i}) = \Theta(\mu)(B_{1i})$ for all $i = 1, 2, \ldots, k_1$. Set $\mathcal{B}_2 = \mathcal{B}_1 \vee \mathcal{Q}_1 = \{B_{21}, \ldots B_{2k_2}\}$. Using the same technique as described above, we find a clopen partition $\{A'_{21}, \ldots, A'_{2k_2}\}$ of X such that $\mu(A'_{2i}) = \Theta(\mu)(B_{2i})$, $i = 1, \ldots, k_2$, $\mu \in K_1$. This might not refine $\mathcal{A}_1$, but $\mathcal{B}_2$ is a refinement of $\mathcal{B}_1$, so for each $i \in \{1, \ldots, k_2\}$, $B_{2i} \subset B_{1j}$ for some $j \in \{1, \ldots, k_1\}$. Hence for all $\mu \in K_1$, $\mu(A'_{2i}) = \Theta(\mu)(B_{2i}) \leq \Theta(\mu)(B_{1j}) = \mu(A_{1j})$, with strict inequality if $B_{2i} \subsetneq B_{1j}$. Again using the subset condition of a D-simplex, this implies that we can find a clopen set $A_{2i} \subset A_{1j}$, with $\mu(A'_{2i}) = \mu(A_{2i})$. As before we can choose the sets such that $\mathcal{A}_2 = \{A_{21}, \cdots, A_{2k_2}\}$ becomes a clopen partition of X. It is still measurably comparable to $\mathcal{B}_2$, and it refines $\mathcal{A}_1$. Now set $\mathcal{A}_3 = \mathcal{A}_2 \vee \mathcal{P}_2$, and continue to switch back and forth between X and Y to obtain nested sequences $\{\mathcal{A}_n\}_{n=1}^\infty$,

$\{\mathcal{B}_n\}_{n=1}^{\infty}$ of generating clopen partitions of X and Y, respectively, such that the μ-measure of the sets in $\mathcal{A}_n$ coincide with the $\Theta(\mu)$-measure of the sets in $\mathcal{B}_n$, for all $\mu \in K_1$ and all n. There is a standard way to construct a Bratteli diagram from a nested sequence of clopen partitions of a Cantor set, where the sequence generates the topology. In our case, the Bratteli diagram (V, E) constructed by using the sequence $\{\mathcal{A}_n\}_{n=1}^{\infty}$ will be identical to the Bratteli diagram (W, F) constructed using $\{\mathcal{B}_n\}_{n=1}^{\infty}$. Consequently we have a homeomorphism $h : X \to Y$, defined by sending $x \in X$ to $y \in Y$ such that the path representation of x in (V, E) is equal to the path representation of y in (W, F). It remains to show that h_* is in fact an affine bijection between K_1 and K_2. But this is clear, since for all n and all sets $B_{ni} \in \mathcal{B}_n$ we have $h_*(\mu)(B_{ni}) = \mu(h^{-1}(B_{ni})) = \mu(A_{ni}) = \Theta(\mu)(B_{ni})$. Hence Θ is an affine bijection, and the proof is complete. □

Lemma 2.10. *Let $K \subset M(X)$ be a D-simplex of probability measures on the Cantor set X. Let $(V, E, \geq)$ be a properly ordered Bratteli diagram, such that $K_0(V, E) \simeq G(K)(= \{\hat{f} : K \to \mathbb{R} \,|\, f \in C(X, \mathbb{Z})\})$ as ordered dimension groups with order units. Then $M(X_{(V,E)}, T_{(V,E)}) \cong K$.*

Proof. For each $\mu \in K$ the function $\hat{\mu} : G(K) \to \mathbb{R}$ defined by $\hat{\mu}(\hat{f}) = \int_X f d\mu$, $\hat{f} \in G(K)$, is a state on $(G(K), G(K)^+, \hat{1})$, and as we remarked above, all states on $G(K)$ are of this form. By $G(K) \simeq K_0(V, E)$ and the affine bijective correspondence between $S(K_0(V, E))$ and $M(X_{(V,E)}, T_{(V,E)})$, we get an affine bijection $\Theta : K \to M(X_{(V,E)}, V_{(V,E)})$.

Let A be a clopen set in X. Then $f = \chi_A \in C(X, \mathbb{Z})$, and so the corresponding element $\hat{f}$ defined by $\hat{f}(\mu) = \int_X f d\mu$, $\mu \in K$, is in $G(K)^+ \simeq K_0(V, E)^+$, and is less or equal to the canonical order unit. Hence $\hat{f}$ can be viewed, according to the identification we have explained above, as a positive linear combination of mutually disjoint cylinder sets at some specified level k in (V, E); $\hat{f} = \sum_{i=1}^{l} C_k(x_i)$. So with $B = \bigcup_{i=1}^{l} C_k(x_i) (\subset X_{(V,E)})$, we get $\Theta(\mu)(B) = \mu(A)$ for all $\mu \in K$. Similarly, if we start with a clopen $B \subset X_{(V,E)}$, we can find a clopen $A \subset X$ such that $\Theta(\mu)(B) = \mu(A)$ for all $\mu \in K$. By Lemma 2.9 we get that $M(X_{(V,E)}, T_{(V,E)}) \cong K$. □

Putting these preliminary results together, we get the main theorem of this section:

Theorem 2.11. *Given a Cantor set X and a D-simplex $K \subset M(X)$. Assume $G(K)$ is uniformly dense in $\mathrm{Aff}(K)$. Then there exists a homeomorphism $T : X \to X$ such that (X, T) is a Cantor minimal system with $M(X, T) = K$.*

Proof. If $G(K)$ is uniformly dense in $\mathrm{Aff}(K)$, then $G(K)$ is a dimension group. Let $(V, E, \leq)$ be a properly ordered Bratteli diagram such that $G(K) \simeq K_0(V, E)$. By Lemma 2.10, $K \cong M(X_{(V,E)}, T_{(V,E)})$, where $X_{(V,E)}$ is the path space of (V, E) and $T_{(V,E)}$ is the Vershik-map. Assume $h : X \to X_{(V,E)}$ is a homeomorphism implementing the affine isomorphism. Then $T = h^{-1} \circ T_{(V,E)} \circ h : X \to X$ is a homeomorphism on X, and $M(X, T) = K$. In fact, let $A \subset X$ be a Borel set, and let $\mu \in K$. Then

$$\begin{aligned} \mu(T^{-1}(A)) &= \mu((h^{-1} \circ T_{(V,E)}^{-1} \circ h)(A)) = h_*(\mu)((T_{(V,E)}^{-1} \circ h)(A)) \\ &= h_*(\mu)(h(A)) = \mu((h^{-1} \circ h)(A)) = \mu(A), \end{aligned},$$

so $K \subset M(X,T)$. For the other inclusion, assume we have a T-invariant measure ν on X, and define $\tilde{\nu}$ on $X_{(V,E)}$ by $\tilde{\nu}(B) = \nu(h^{-1}(B)) = h_*(\nu)(B)$, where $B \subset X_{(V,E)}$ is Borel. Then $\tilde{\nu}$ is a probability measure, and

$$\begin{aligned} \tilde{\nu}(T^{-1}_{(V,E)}(B)) &= \nu((h^{-1} \circ T^{-1}_{(V,E)})(B)) = \nu((h^{-1} \circ T^{-1}_{(V,E)} \circ h \circ h^{-1})(B)) \\ &= \nu((T^{-1} \circ h^{-1})(B)) = \nu(h^{-1}(B)) = \tilde{\nu}(B), \end{aligned},$$

so $\tilde{\nu}$ is $T_{(V,E)}$-invariant, and hence $\tilde{\nu} \in M(X_{(V,E)}, T_{(V,E)})$. This implies that $\nu \in K$ and we conclude that $M(X,T) = K$. □

3. When is $G(K)$ dense in $\mathrm{Aff}(K)$?

For K a D-simplex of probability measures on the Cantor set X, we defined $G(K) = \{\hat{f} \mid f \in C(X,\mathbb{Z})\}$, where $\hat{f} : K \to \mathbb{R}$ is the continuous affine map defined by $\hat{f}(\mu) = \int_X f d\mu$, $\mu \in K$. If $G(K)$ is (uniformly) dense in $\mathrm{Aff}(K)$, then $G(K)$ becomes a simple dimension group (in the inherited ordering from $\mathrm{Aff}(K)$). Conversely, it is proved in [4, Thm.4.4] that if $G(K)$ is a simple dimension group, then necessarily $G(K)$ is dense in $\mathrm{Aff}(K)$. We proved in Theorem 2.11 that if $G(K)$ is dense in $\mathrm{Aff}(K)$, or equivalently, $G(K)$ is a simple dimension group, then there exists a minimal homeomorphism $T : X \to X$, such that $M(X,T) = K$. So the question becomes: When is $G(K)$ dense in $\mathrm{Aff}(K)$? We are presently not able to answer this question in general, but in this section we show that if $\partial_e K$ is a finite set, then the answer is affirmative.

The following definition is also found in [1], and, furthermore, Lemma 3.2 is contained in [1, Proposition 1.4]. We present an alternative proof, using Bratteli diagrams.

Definition 3.1. For a probability measure μ on X, define the *clopen value set*

$$S(\mu) = \{\mu(U) | U \text{ clopen in X }\}.$$

Lemma 3.2. *Let μ be a non-atomic probability measure on a Cantor set X. Then $S(\mu)$ is a countable dense subset of the unit interval $I = [0,1]$.*

Proof. The set $S(\mu)$ is countable because X has a countable clopen basis, and every clopen subset of X is a finite union of sets in the basis. To show that $S(\mu)$ is dense in I, we represent X as the path space of a Bratteli diagram (V,E). Given $\epsilon > 0$, there exists N such that $\mu(C_n(x)) < \epsilon$ for all $C_n(x) \in P^{(n)}_{(V,E)}$, $n \geq N$. Otherwise, using the fact that $\mu(C_k(x)) \geq \mu(C_{k+1}(x))$, there would be a path $x = (e_i)_{i=1}^{\infty}$ having measure greater than (or equal to) ϵ. But this would contradict that μ is non-atomic. Let $r \in I$ and $\epsilon > 0$ be given. We can find a level N such that $\mu(C_N(x)) < \epsilon$ for all $C_N(x) \in P^{(N)}_{(V,E)}$. Since $\sum_{C_N(x) \in P^{(N)}_{(V,E)}} \mu(C_N(x)) = 1$, we can find a clopen set U consisting of a union of such cylinder sets such that $|\mu(U) - r| < \epsilon$. This shows that $S(\mu)$ is dense in I. □

For a finite set $\{\mu_i\}_{i=1}^n \subset M(X)$, let $\bar{\mu} = [\mu_1, \mu_2, \cdots, \mu_n]$. We extend Definition 3.1.

Definition 3.3. For a finite set $\{\mu_i\}_{i=1}^n \subset M(X)$, define the set

$$S(\bar{\mu}) = \{[\mu_1(U), \mu_2(U), \cdots, \mu_n(U)] \, | U \text{ clopen in } X\} \subset I^n.$$

Lemma 3.4. *Let $\{\mu_i\}_{i=1}^n$ be mutually singular, non-atomic probability measures on the Cantor set X. Then $S(\bar{\mu})$ is dense in I^n.*

Proof. This is an easy corollary of Lyapunov's theorem, which says the following: Suppose $\nu_1, \cdots, \nu_k$ are non-atomic measures (not necessarily probability measures) on a σ-algebra $\mathcal{M}$. Define $\nu(E) = (\nu_1(E), \cdots, \nu_k(E))$, $E \in \mathcal{M}$. Then the range of ν is a compact convex subset of $\mathbb{R}^k$. (For a proof, see [12, Thm. 5.5].) In our case the measures $\{\mu_i\}_{i=1}^n$ are mutually singular, and it is easy to see that this implies that there exists for each $i \in \{1, \cdots, n\}$, a Borel set $E_i \subset X$ such that $\mu_j(E_i) = \delta_{ij}$, $j = 1, \cdots, n$. Hence by Lyapunov's theorem, $\{(\mu_1(E), \cdots, \mu_n(E)) \,|\, E \text{ Borel set in } X\} = I^n$. Now by the regularity of the measures $\mu_1, \cdots, \mu_n$, we conclude that $S(\bar{\mu})$ is dense in I^n. □

We are now able to show the following result:

Lemma 3.5. *If K is a D-simplex with only finitely many extreme points, $G(K)$ is dense in* $\text{Aff}(K)$.

Proof. Let $\partial_e K = \{\mu_i\}_{i=1}^n$ be the extreme points of K. By the previous lemma we have that $S(\bar{\mu})$ is dense in $[0,1]^n$. This means that given any affine function $\hat{g} : K \to \mathbb{R}$, with $\hat{g}(\mu_k) \in [0,1]$, $k = 1, \ldots, n$, we can by Lemma 3.4 find a clopen set $A \subset X$ such that the function $\hat{\chi}_A \in G(K)$ approximates $\hat{g}$ on $\mu_1, \cdots, \mu_n$ arbitrarily well, and so by Krein-Milman approximates $\hat{g}$ uniformly on K. If $\hat{g}$ takes values outside $[0,1]$, we can by a simple scaling argument transfer this to the first situation. □

Corollary 3.6. *Let $K \subset M(X)$ be a D-simplex of probability measures on the Cantor set X. Assume $\partial_e K$ is a finite set. Then there exists a minimal homeomorphism $T : X \to X$ such that $M(X,T) = K$.*

Remark 3.7. *The only thing that was used to prove that $G(K)$ is dense in* $\text{Aff}(K)$ *in Lemma 3.5, was that the extreme points in K are mutually singular non-atomic measures. One may ask if this is true in general. That is, if K is a simplex consisting of non-atomic probability measures on the Cantor set X, such that if $\mu_1, \mu_2 \in \partial_e K$, $\mu_1 \neq \mu_2$, then μ_1 is singular with respect to μ_2, is it then true that $G(K)$ is dense in* $\text{Aff}(K)$*?*

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