Tilting and Relative Theories in Subcategories

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Preface

The present thesis is the result of my research work at the Department of Mathematical Sciences at the Norwegian University of Science and Technology (NTNU) where I was employed between November 2003 to January 2008. The work has been made possible by the financial support from NTNU to which I extend my deep appreciations. It was supervised by Prof. Idun Reiten and Prof. Øyvind Solberg.

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Introduction

This thesis is divided into two parts. The first part is represented in Chapter 1 while the second part is the work in Chapters 2, 3 and 4. In the first part we will investigate a certain aspect of standard tilting theory, while the second part deals with relative theory in subcategories.

Tilting is a well-known concept in modern algebra. There are several types of tilting for different categories, for example, tilting modules for module categories, tilting complexes for derived categories, tilting objects for abelian categories, and so on. Cotilting is the dual concept of tilting [**22**].

Let Λ be an artin algebra and let $\operatorname{mod}\Lambda$ denote the category of finitely generated left Λ-modules. Suppose T is a tilting Λ-module of finite projective dimension ($\operatorname{pd}_\Lambda T < \infty$) and let Γ denote the opposite of $\operatorname{End}_\Lambda(T)$ (the notation is fixed throughout the introduction). Then it is well-known that DT, the dual of T, is a cotilting Γ-module. Moreover, the classical tilting functor $\operatorname{Hom}_\Lambda(T, \ )$ from $\operatorname{mod}\Lambda$ to $\operatorname{mod}\Gamma$ induces an equivalence between $T^\perp$, the category of all Λ-modules Y such that $\operatorname{Ext}^i_\Lambda(T, Y) = 0$ for all $i > 0$, and its image $\operatorname{Hom}_\Lambda(T, T^\perp)$ in $\operatorname{mod}\Gamma$, where the category $\operatorname{Hom}_\Lambda(T, T^\perp)$ is identified with ${}^\perp DT$, the category of all Γ-modules X such that $\operatorname{Ext}^i_\Gamma(X, DT) = 0$ for all $i > 0$. This equivalence is the cornerstone for tilting theory. Let us refer to this equivalence as the "classical tilting equivalence". It is also well-known that the global dimensions of Λ and Γ are related by the formula $\operatorname{gl.dim}\Lambda - \operatorname{pd}_\Lambda T \leqslant \operatorname{gl.dim}\Gamma \leqslant \operatorname{gl.dim}\Lambda + \operatorname{pd}_\Lambda T$ [**9**][**14**][**22**].

Auslander and Reiten [**5**] introduced a different equivalence between $\widecheck{\operatorname{add} T}$, the category of all Λ-modules with finite $\operatorname{add} T$-coresolution, and $\widehat{\operatorname{add} DT}$, the category of all Γ-modules with finite $\operatorname{add} DT$-resolution, where T was a special tilting Λ-module known as strong tilting module. While studying tilting theory for standardly stratified algebras, Agoston *et al.* [**1**] discovered a similar equivalence (but in another setting) between $\mathcal{F}_\Lambda(\Delta)$, the category of all Λ-modules filtered by the standard Λ-modules and $\mathcal{F}_\Gamma(\overline{\nabla})$, the category of all Γ-modules filtered by the proper co-standard Γ-modules. This equivalence was given by a special tilting Λ-module T known as the characteristic tilting module, which is defined by the equation $\mathcal{F}_\Lambda(\Delta) \cap \mathcal{F}_\Lambda(\overline{\nabla}) = \operatorname{add} T$. It is shown that if T is a characteristic tilting Λ-module, then there are equalities $\mathcal{F}_\Lambda(\Delta) = \widecheck{\operatorname{add} T}$ and $\mathcal{F}_\Gamma(\overline{\nabla}) = \widehat{\operatorname{add} DT}$ of subcategories [**1**]. Note that Γ is also known as *Ringel dual* (a brief survey of tilting theory for stratified algebras will be given in Section 1.1).

One of the main results of Chapter 1 generalizes [**1**] and [**5**]. We show that for any tilting module T over an artin algebra Λ, there is an equivalence between the subcategories $\widecheck{\operatorname{add} T}$ and $\widehat{\operatorname{add} DT}$. The result was also independently established by Happel and Unger [**20**].

Dlab [**15**] introduced the properly stratified algebras, which are standardly stratified algebras where $\mathcal{F}(\Delta) \subseteq \mathcal{F}(\overline{\Delta})$. The concept was studied further by Frisk and Mazorchuck [**17**] where the authors showed that if Λ is a standardly stratified algebra where the Ringel dual Γ is properly stratified, then the inverse of the tilting functor takes the characteristic tilting Γ-module to a certain tilting Λ-module H which is strong in the sense of [**5**]. This implies that the category of Λ-modules of finite projective dimension is contravariantly finite in $\operatorname{mod} \Lambda$, which is a sufficient condition for the finitistic dimension of Λ to be finite [**5**].

Suppose T is a tilting module over an artin algebra Λ. Then we show in Chapter 1 that the classical tilting functor $\operatorname{Hom}_\Lambda(T, \ )$ (and its inverse) preserves all (co)tilting modules in the subcategories $T^\perp$ and ${}^\perp DT$. We then apply this to find a sufficient condition for the finitistic dimension of Λ to be finite. This generalizes the above-mentioned result from [**17**].

Auslander and Solberg [**10, 11**] called the well-known tilting theory discussed earlier the “standard tilting theory”. In their work, they studied the relative homological algebra in the representation theory of artin algebras Λ and developed the “relative tilting theory” in $\operatorname{mod} \Lambda$. Subfunctors of the bifunctor $\operatorname{Ext}^1_\Lambda(\ , \ )$ are the main ingredients of the relative theory of Auslander and Solberg. Consider a subfunctor F in $\operatorname{mod} \Lambda$. Then F-(co)tilting modules are analogs of (co)tilting Λ-modules. It is shown that if there is an F-tilting module in $\operatorname{mod} \Lambda$, then $\mathcal{I}(F)$, the category of F-injective modules in $\operatorname{mod} \Lambda$, is of finite type. Let T be an F-tilting module in $\operatorname{mod} \Lambda$ with $\operatorname{pd}_F T$ finite. Then there is a generalization of the classical tilting equivalence. Denote by T_0 the Γ-module associated to $\operatorname{Hom}_\Lambda(T, \mathcal{I}(F))$. Then the image of the tilting functor restricted to $T^\perp$, $\operatorname{Hom}_\Lambda(T, T^\perp)$, is identified with ${}^\perp T_0$. Moreover, the Γ-module T_0 is cotilting. However, unlike in the classical case, the Γ-module DT is not necessarily cotilting, but a direct summand of T_0. The relative global dimension of Λ and the global dimension of Γ are related by the formula $\operatorname{gl.dim}_F \Lambda - \operatorname{pd}_F T \leqslant \operatorname{gl.dim} \Gamma \leqslant \operatorname{gl.dim}_F \Lambda + \nu(\operatorname{pd}_F T)$, where ν is a function of $\operatorname{pd}_F T$. [**10, 11**].

Let $\mathcal{C}'$ be an additive category which is closed under kernels and cokernels and suppose $\mathcal{C}$ is a functorially finite subcategory of $\mathcal{C}'$. Iyama [**23**] introduced an invariant of $\mathcal{C}'$ given by $\mathcal{C}$, namely the right and left $\mathcal{C}$-resolution dimensions of $\mathcal{C}'$. These are analogs of the projective and injective dimensions, where right and left $\mathcal{C}$-approximation "resolutions" are considered instead of projective and injective resolutions. A special example of this invariant occurs when $\mathcal{C}'$ is $\operatorname{mod}\Lambda$. In this case we refer to the right and left $\mathcal{C}$-resolution dimensions as the right and left $\mathcal{C}$-approximation dimensions. Let us call the maximum of the two invariants (the right and left $\mathcal{C}$-approximation dimensions) the $\mathcal{C}$-approximation dimension of $\operatorname{mod}\Lambda$.

Suppose $\mathcal{C}$ is closed under extensions and assume that the $\mathcal{C}$-approximation dimension of $\operatorname{mod}\Lambda$ is zero. Then it will be shown that $\mathcal{C}$ is naturally equivalent to a module category over an artin algebra. This means that a relative theory in $\mathcal{C}$ can be developed in the sense of [**10, 11**]. Let us refer to this theory as the relative theory in dimension "0". In the second part (Chapters 2, 3 and 4) of this thesis we will develop the relative theory in dimension "n" for certain subfunctors F of the bifunctor $\operatorname{Ext}^1_\Lambda(\ ,\ )$, where n is the $\mathcal{C}$-approximation dimension of $\operatorname{mod}\Lambda$.

Let $\mathcal{C}$ be a functorially finite subcategory of $\operatorname{mod}\Lambda$ which is closed under extensions and let $\mathcal{X}$ be a contravariantly finite subcategory of $\mathcal{C}$ (note that the notations and assumptions are fixed throughout the introduction). In Chapter 2 we investigate the subfunctors $F = F_{\mathcal{X}}$ in $\mathcal{C}$ and their properties. Among the properties, we show that $\mathcal{C}$ is closed under kernels of F-epimorphisms. Moreover, if the category of F-injective modules in $\mathcal{C}$, $\mathcal{I}_{\mathcal{C}}(F)$, is covariantly finite, then F has enough projectives and injectives. We also show that the subcategories $\mathcal{C}$ of $\operatorname{mod}\Lambda$ with $\mathcal{C}$-approximation dimension zero are equivalent to categories $\operatorname{mod}\Lambda/I$, where I is a ideal of Λ.

In Chapter 3 we investigate relative (co)tilting modules in subcategories $\mathcal{C}$ of $\operatorname{mod}\Lambda$. Consider a subfunctor F in $\mathcal{C}$ with enough projectives and injectives in $\mathcal{C}$. Suppose T is a F-tilting module in $\mathcal{C}$ with $\operatorname{pd}_F T = r$. Then we generalize the classical tilting equivalence. Suppose that the $\mathcal{C}$-approximation dimension of $\operatorname{mod}\Lambda$ is a nonnegative integer n. Then if there is an F-tilting module in $\mathcal{C}$, then it will be shown that $\mathcal{I}_{\mathcal{C}}(F)$ is of finite type. So we assume from now on that $\mathcal{I}_{\mathcal{C}}(F)$ is of finite type. Denote the Γ-module associated to $\operatorname{Hom}_\Lambda(T, \mathcal{I}_{\mathcal{C}}(F))$ by $T^0_{\mathcal{C}}$. Then we show that the image of the classical tilting functor restricted to $T^\perp_{\mathcal{C}}$, $\operatorname{Hom}_\Lambda(T, T^\perp_{\mathcal{C}})$, is identified with ${}^\perp T^0_{\mathcal{C}}$, where $T^\perp_{\mathcal{C}}$ denotes the category $T^\perp \cap \mathcal{C}$. Moreover, the Γ-module $T^0_{\mathcal{C}}$ is cotilting. However, we show that the Γ-module DT is not necessarily cotilting and we give an example which shows that DT is not a direct summand of $T^0_{\mathcal{C}}$ either. Nevertheless, we show that DT has a finite $\operatorname{add} T^0_{\mathcal{C}}$-resolution. We also show that $\operatorname{gl.dim}_F \mathcal{C}$, the relative global dimension of $\mathcal{C}$, and the global dimension of Γ are related by the formula $\operatorname{gl.dim}_F \mathcal{C} - \operatorname{pd}_F T \leqslant \operatorname{gl.dim} \Gamma \leqslant \operatorname{gl.dim}_F \mathcal{C} + \nu(n, r)$, where ν is a function of n and r.

If the $\mathcal{C}$-approximation dimension of $\operatorname{mod}\Lambda$ is infinite, then we have many examples where the Γ-module $T^0_{\mathcal{C}}$ is not cotilting. However, it is not known that the $\mathcal{C}$-approximation dimension of $\operatorname{mod}\Lambda$ being finite is necessary for $T^0_{\mathcal{C}}$ to be cotilting.

Erdmann and Sáenz [**16**] introduced the concept of a stratifying system. The concept was studied further by Marcos *et al.* [**26**], where the authors introduced the notion of an Ext-projective stratifying system. Suppose Θ is a stratifying system and let $\mathcal{F}(\Theta)$ denote the category of Λ-modules filtered by Θ. It is shown in [**30**] that $\mathcal{F}(\Theta)$ is functorially finite in $\operatorname{mod}\Lambda$. Denote by B the opposite algebra of $\operatorname{End}_\Lambda(Q)$, where Q is a direct sum of non-isomorphic Ext-projective modules in $\mathcal{F}(\Theta)$.

Assume that $\mathcal{F}(\Theta)$ is closed under extensions in $\operatorname{mod}\Lambda$ and consider the subfunctor $F = F_{\operatorname{add} Q}$ in $\mathcal{F}(\Theta)$. Then Q is the trivial F-tilting module in $\mathcal{F}(\Theta)$. It is also F-cotilting in $\mathcal{F}(\Theta)$, since the F-global dimension of $\mathcal{F}(\Theta)$ is finite [**26**] [**27**]. One of the main results of [**26**] is that B is standardly stratified and there is an equivalence between subcategories $\mathcal{F}_\Lambda(\Theta)$ and $\operatorname{Hom}_\Lambda(Q, \mathcal{F}_\Lambda(\Theta))$. Moreover, the category $\operatorname{Hom}_\Lambda(Q, \mathcal{F}_\Lambda(\Theta))$ is identified with $\operatorname{add} {}_BT$, where ${}_BT$, which is equal to $\operatorname{Hom}_\Lambda(Q, Y)$, is the characteristic tilting B-module. Here Y denotes the direct sum of non-isomorphic Ext-injective modules in $\mathcal{F}(\Theta)$.

Consider the subfunctor $F = F_{\mathcal{X}}$ in $\mathcal{C}$. Suppose T is an F-tilting F-cotilting module in $\mathcal{C}$. In Chapter 4 we generalize the above-mentioned result from [**26**]. We show that the Γ-module $T^0_{\mathcal{C}}$ is tilting and that the tilting functor induces an equivalence between subcategories $\operatorname{add} T_{\mathcal{C}}$ of $\mathcal{C}$ and $\operatorname{add} T^0_{\mathcal{C}}$ of $\operatorname{mod}\Gamma$. This is the main result of Chapter 4.

At the end of Chapter 4 we give some examples which illustrate the main results of the second part of this thesis. One of the examples shows that the subcategories $\operatorname{Hom}_\Lambda(T, T^{\perp}_{\mathcal{C}})$ and ${}^{\perp}T^0_{\mathcal{C}}$ of $\operatorname{mod}\Gamma$, where T is an F-tilting module in $\mathcal{C}$, coincide even if the $\mathcal{C}$-approximation dimension of $\operatorname{mod}\Lambda$ is not finite. We also give an example where the $\mathcal{C}$-approximation dimension of $\operatorname{mod}\Lambda$ is finite, but the Γ-module $T^0_{\mathcal{C}}$ is not tilting for an F-tilting module T in $\mathcal{C}$.

Unless otherwise stated, throughout this thesis Λ is a basic artin algebra and $\operatorname{mod}\Lambda$ denotes the category of all finitely generated left Λ-modules. Given a subcategory $\mathcal{A}$ of $\operatorname{mod}\Lambda$, $\operatorname{add}\mathcal{A}$ is the full subcategory of $\operatorname{mod}\Lambda$ containing all Λ-modules which are direct summands of finite direct sums of modules in $\mathcal{A}$. Denote by D the duality between left and right modules as given in [**6**, II.3].

Chapter 1

Equivalence of Subcategories and Tilting Functor

This chapter comprises the first part of this thesis. We shall look at equivalence of subcategories and the tilting functor.

Since our main results in this chapter are motivated by stratified algebras, we give a brief survey of tilting theory for stratified algebras in Section 1.1.

Let T be a tilting Λ-module and denote $\operatorname{End}_{\Lambda}(T)^{\mathrm{op}}$ by Γ. In section 1.2 we prove the first of the main result of this chapter. The result shows that there is an equivalence between subcategories $\widehat{\operatorname{add} T}$ of $\operatorname{mod} \Lambda$ and $\widehat{\operatorname{add} DT}$ of $\operatorname{mod} \Gamma$.

In Section 1.3 we state some preliminary results. Then we prove the other main result of this chapter, which shows that the tilting functor preserves (co)tilting modules in the subcategories $T^{\perp}$ of $\operatorname{mod} \Lambda$ and ${}^{\perp}DT$ of $\operatorname{mod} \Gamma$.

In the last section we provide a sufficient condition for the category of Λ-modules of finite projective dimension to be contravariantly finite in $\operatorname{mod} \Lambda$, which in turn will tell us something about the finitistic dimension of Λ. This generalizes [**17**].

1.1. Preliminaries

In this section we give a brief survey of stratified algebras and their relationship with tilting theory.

We first recall some definitions, notations and results from [**1**] [**5**] [**17**] [**29**] and [**30**]. For unexplained terminologies we refer the reader to [**6**] and [**31**].

A subcategory ω of $\operatorname{mod}\Lambda$ is said to be *selforthogonal* if $\operatorname{Ext}^i_\Lambda(\omega,\omega)=0$ for all $i>0$. Let $\mathcal{X}$ be a subcategory of $\operatorname{mod}\Lambda$. We denote by $\mathcal{X}^\perp$ the full subcategory of $\operatorname{mod}\Lambda$ containing all modules M such that $\operatorname{Ext}^i_\Lambda(\mathcal{X},M)=0$ for all $i>0$. Dually, one defines

$$^\perp\mathcal{X}=\{N \text{ in } \operatorname{mod}\Lambda \mid \operatorname{Ext}^i_\Lambda(N,\mathcal{X})=0\}$$

The notion of *$\mathcal{X}$-resolution* (*$\mathcal{X}$-coresolution*) is an analog of projective resolution (injective coresolution). Denote by $\widehat{\mathcal{X}}$ the full subcategory of $\operatorname{mod}\Lambda$ containing all modules M with a finite $\mathcal{X}$-resolution, that is, there is an exact sequence

$$0\to X_n\to\cdots\to X_1\to X_0\to M\to 0$$

with all X_i in $\operatorname{add}T$. The dual construction is denoted by $\check{\mathcal{X}}$, which is the full subcategory of $\operatorname{mod}\Lambda$ containing all modules N with a finite $\mathcal{X}$-coresolution. If $\mathcal{X}$ is selforthogonal, then it is easy to see that $\widehat{\mathcal{X}}$ is contained in $\mathcal{X}^\perp$ (and $\check{\mathcal{X}}$ is contained in $^\perp\mathcal{X}$) [**5**]. Let us denote by $\mathcal{P}^{<\infty}(\Lambda)$ ($\mathcal{I}^{<\infty}(\Lambda)$) the full subcategory of $\operatorname{mod}\Lambda$ consisting of modules with finite projective (injective) dimension.

Denote by $(\Lambda,\preceq)$ the algebra Λ with a fixed pre-order on the complete set $e_1,...,e_n$ of primitive orthogonal idempotents of Λ. Note that even though $\preceq$ can be any pre-order, we restrict ourselves to $\leqslant$, the natural order. For $1\leqslant i\leqslant n$, we denote by S_i the simple Λ-module corresponding to e_i. As usual, P_i and I_i denote the projective cover and injective envelope of S_i respectively.

For $1\leqslant i\leqslant n$, we define the *standard module*, Δ_i as the maximal factor module of P_i with no composition factor S_j for $j>i$. The proper *standard module*, $\overline{\Delta}_i$ is defined to be the maximal factor module of Δ_i where S_i occurs only once as a composition factor. Dually, one defines the *co-standard module* ∇_i and the *proper co-standard module* $\overline{\nabla}_i$ [**30**].

For an arbitrary set $\mathcal{S}$ of modules in $\operatorname{mod}\Lambda$, denote by $\mathcal{F}(\mathcal{S})$ the subcategory of all Λ-modules M which can be filtered by the modules in $\mathcal{S}$, that is, there is a filtration

$$0\subseteqq M_0\subseteqq M_1\subseteqq\cdots\subseteqq M_n=M$$

such that $M_i/M_{i-1}\cong S$, where S is in $\operatorname{add}\mathcal{S}$ for $1\leqslant i\leqslant n$. Note that if $\mathcal{F}(\mathcal{S})$ is closed under summands, then it is closed under extensions [**30**].

We are mostly interested in the subcategories $\mathcal{F}(\Delta)$, $\mathcal{F}(\overline{\Delta})$, $\mathcal{F}(\nabla)$ and $\mathcal{F}(\overline{\nabla})$ where the modules are filtered by standard, proper standard, co-standard and proper co-standard modules respectively.

Definition: [**17**]

(a) A pair $(\Lambda, \leqslant)$ is said to be a *standardly stratified* algebra if the kernel of the canonical epimorphism $P_i \twoheadrightarrow \Delta_i$ has a filtration whose subfactors are Δ_j for $j > i$.
(b) A standardly stratified algebra $(\Lambda, \leqslant)$ is called *properly stratified* if for all i, the standard module Δ_i has a filtration with subfactors isomorphic to $\overline{\Delta}_i$, equivalently $\mathcal{F}(\Delta)$ is contained in $\mathcal{F}(\overline{\Delta})$.
(c) A standardly stratified algebra $(\Lambda, \leqslant)$ in which $\mathcal{F}(\Delta) = \mathcal{F}(\overline{\Delta})$ is called a *quasihereditary algebra.*

Note that it is easy to see that (a) is equivalent to Λ being in $\mathcal{F}(\Delta)$.

Let $(\Lambda, \leqslant)$ be a standardly stratified algebra. It has been shown that the subcategory $\omega = \mathcal{F}(\Delta) \cap \mathcal{F}(\overline{\nabla})$, whose objects are modules with filtration with factors in both Δ and $\overline{\nabla}$, is closed under taking direct summands. The indecomposable modules in this subcategory are indexed by $\{1, 2, \ldots, n\}$ in a natural way.

Let T_i, for $1 \leqslant i \leqslant n$, be the indecomposable module in ω such that there are unique exact sequences $0 \to \Delta_i \to T_i \xrightarrow{\alpha_i} Z_i \to 0$ and $0 \to W_i \xrightarrow{\gamma_i} T_i \to \overline{\nabla}_i \to 0$ with Z_i in $\mathcal{F}(\{\Delta_j \colon j < i\})$ and W_i in $\mathcal{F}(\{\overline{\nabla}_j \colon j < i\})$. Then the module

$$T = \bigoplus_{i=1}^{n} T_i$$

in ω is called the *characteristic tilting module* [**1**][**29**] [**30**]. Moreover, T is uniquely defined and has the following properties.

Proposition 1.1.1. [**1**][**29**] *Let $(\Lambda, \leqslant)$ be a standardly stratified algebra and T the characteristic tilting module. Then*

(a) $\mathcal{F}(\Delta) \cap \mathcal{F}(\overline{\nabla}) = \operatorname{add} T.$
(b) $\mathcal{F}(\Delta) \subseteq {}^{\perp}T$ *(= for quasihereditary).*
(c) $\mathcal{F}(\Delta) = \widecheck{\operatorname{add}} T.$
(d) $\mathcal{F}(\overline{\nabla}) = T^{\perp}.$

In the case where $(\Lambda, \leqslant)$ is properly stratified, the subcategory $\sigma = \mathcal{F}(\overline{\Delta}) \cap \mathcal{F}(\nabla)$ is closed under taking summands. As in ω, the indecomposable Λ-modules in σ are naturally indexed by $\{1, 2, \ldots, n\}$.

Let C_i, for $0 \leqslant i \leqslant n$, denote the indecomposable module in σ such that there are unique exact sequences $0 \to Z_i \to C_i \to \nabla_i \to 0$ and $0 \to \overline{\Delta}_i \to C_i \to W_i \to 0$ with Z_i in $\mathcal{F}(\{\nabla_j \colon j < i\}$ and W_i in $\mathcal{F}(\{\overline{\Delta}_j \colon j < i\}$. Then the module

$$C = \bigoplus_{i=1}^{n} C_i$$

is called the *characteristic cotilting module* over Λ [**17**]. This module is uniquely defined with the following dual properties.

Proposition 1.1.2. *Let* $(\Lambda, \leqslant)$ *be a properly stratified algebra and* C *the characteristic cotilting module. Then*

1. $\mathcal{F}(\overline{\Delta}) \cap \mathcal{F}(\nabla) = \operatorname{add} C$.
2. $\mathcal{F}(\nabla) \subseteq C^{\perp}$.
3. $\mathcal{F}(\nabla) = \widehat{\operatorname{add} C}$.
4. $\mathcal{F}(\overline{\Delta}) = {}^{\perp}C$.

Let $(\Lambda, \leqslant)$ be a standardly stratified algebra and T be the characteristic tilting Λ-module. Then $\Gamma = \operatorname{End}_{\Lambda}(T)^{\mathrm{op}}$ is called the *Ringel dual*. Consider the functors $e_T = \operatorname{Hom}_{\Lambda}(T, \ ) \colon \operatorname{mod} \Lambda \to \operatorname{mod} \Gamma$ (the Ringel duality) and $f_T = D \operatorname{Hom}_{\Lambda}(\ , T) \colon \operatorname{mod} \Lambda \to \operatorname{mod} \Gamma$. Denote by $(\Gamma, \leqslant^{\mathrm{op}})$ the algebra Γ equipped with the order opposite to that of $(\Lambda, \leqslant)$. Then we have the following result.

Proposition 1.1.3. [**1**, Theorem 2.6] *Let* $(\Lambda, \leqslant)$ *be a standardly stratified algebra and* T *the characteristic tilting* Λ*-module. Let* $\Gamma = \operatorname{End}_{\Lambda}(T)^{\mathrm{op}}$*. Then we have the following for* $(\Gamma, \leqslant^{\mathrm{op}})$,

- (a) *For* $1 \leqslant i \leqslant n$
    - (i) $e_T({}_{\Lambda}\overline{\nabla}_i) = {}_{\Gamma}\overline{\Delta}_{n-i+1}$
    - (ii) $f_T({}_{\Lambda}\Delta_i) = {}_{\Gamma}\nabla_{n-i+1}$.
- (b) *The functor* e_T *induces an equivalence between* $\mathcal{F}({}_{\Lambda}\overline{\nabla})$ *and* $\mathcal{F}({}_{\Gamma}\overline{\Delta})$.
- (c) *The functor* f_T *induces an equivalence between* $\mathcal{F}({}_{\Lambda}\Delta)$ *and* $\mathcal{F}({}_{\Gamma}\nabla)$.
- (d) ${}_{\Gamma}\Gamma$ *is in* $\mathcal{F}({}_{\Gamma}\overline{\Delta})$*. In particular* $(\Gamma, \leqslant^{\mathrm{op}})$ *is a standardly stratified algebra.*
- (e) *The* Γ*-module* $e_T(D(\Lambda)) = DT$ *is the characteristic cotilting module.*
- (f) $\Lambda \cong \operatorname{End}_{\Gamma}(DT)^{\mathrm{op}}$*, and the ordering given by* DT *gives back the original ordering of* $(\Lambda, \leqslant)$.

1.2. Equivalence of Subcategories

Let T be a tilting Λ-module and denote $\operatorname{End}_{\Lambda}(T)^{\mathrm{op}}$ by Γ. Consider the functor $f_T = D \operatorname{Hom}_{\Lambda}(\ , T) \colon \operatorname{mod} \Lambda \to \operatorname{mod} \Gamma$. In this section we show that f_T induces an equivalence between the subcategories $\widecheck{\operatorname{add} T}$ of $\operatorname{mod} \Lambda$ and $\widehat{\operatorname{add} DT}$ of $\operatorname{mod} \Gamma$. This was also independently established in [**20**, Theorem 3.1].

In Proposition 1.1.3, the equivalence in (b) is well-known in tilting theory, since $\mathcal{F}({}_{\Lambda}\overline{\nabla}) = T^{\perp}$ and $\mathcal{F}({}_{\Gamma}\overline{\Delta}) = {}^{\perp}DT$. But the equivalence between the subcategories $\widecheck{\operatorname{add} T}$ and $\widehat{\operatorname{add} DT}$ in (c) is not well-known. In the case where T is strong tilting (i.e. $\widecheck{\operatorname{add} T} = \mathcal{P}^{<\infty}(\Lambda)$) the equivalence is also proved in [**5**, Proposition 6.6].

In the following result we show that this is true in general for any tilting Λ-module.

Theorem 1.2.1. *Let T be a tilting Λ-module, $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$ and DT be the corresponding cotilting Γ-module. Then the functor f_T induces an equivalence between the subcategories $\widecheck{\operatorname{add} T}$ and $\widehat{\operatorname{add} DT}$ of $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$ respectively.*

Proof. Define another functor $f'_T = \operatorname{Hom}_\Gamma(DT, \ ) \colon \operatorname{mod}\Gamma \to \operatorname{mod}\Lambda$. We want to show that the induced functor $f'_T \colon \widehat{\operatorname{add} DT} \to \widecheck{\operatorname{add} T}$ is an inverse equivalence of the induced functor $f_T \colon \widecheck{\operatorname{add} T} \to \widehat{\operatorname{add} DT}$.

First we show that $\operatorname{Im} f_T \subseteq \widehat{\operatorname{add} DT}$ and $\operatorname{Im} f'_T \subseteq \widecheck{\operatorname{add} T}$. Let X be in $\widecheck{\operatorname{add} T}$. Then by [**28**, Lemma 2.1] we have that $\operatorname{pd}_\Lambda X < \infty$, since $\operatorname{pd}_\Lambda T < \infty$. Let

$$0 \to P_m \to \cdots \to P_1 \to P_0 \to X \to 0 \tag{1}$$

be a projective resolution of X. Then since $\operatorname{Ext}^i_\Lambda(X, T) = 0$, we have that the functor f_T is exact on (1). Hence, $f_T(X)$ is in $\widehat{\operatorname{add} DT}$, since $f_T(\Lambda) = DT$.

On the other hand, let Y be in $\widehat{\operatorname{add} DT}$. Then by [**28**, Lemma 2.1] we have that $\operatorname{id}_\Gamma Y < \infty$ since $\operatorname{id}_\Gamma DT < \infty$. Let

$$0 \to Y \to I_0 \to I_1 \to \cdots \to I_n \to 0 \tag{2}$$

be an injective resolution of Y. Since $\operatorname{Ext}^i_\Gamma(DT, Y) = 0$, the functor f'_T is exact on (2). Hence $f'_T(Y)$ is in $\widecheck{\operatorname{add} T}$ since $f'_T(D\Gamma) = T$.

Now it remains to show that $f'_T f_T(X) \cong X$ for all X in $\widecheck{\operatorname{add} T}$ and $f_T f'_T(Y) \cong Y$ for all Y in $\widehat{\operatorname{add} DT}$. For the former, define a Λ-homomorphism

$$\Phi \colon X \to \operatorname{Hom}_\Gamma(DT, D\operatorname{Hom}_\Lambda(X, T))$$

by $\Phi(x) = \phi_x$, where $\phi_x \colon DT \to D\operatorname{Hom}_\Lambda(X, T)$. Moreover, for $f \colon X \to T$ and $g \in DT$ we have that $\phi_x(g)(f) = g(f(x))$. It can be shown that Φ is functorial. For $X = T$, we have

$$\begin{aligned} f'_T f_T(T) = f'_T(D\operatorname{Hom}_\Lambda(T, T)) &\cong f'_T(D\Gamma) \\ &= \operatorname{Hom}_\Gamma(DT, D\Gamma_\Gamma) \\ &\cong T \end{aligned}$$

Since $f'_T f_T$ commutes with finite direct sums, it follows that Φ is an isomorphism for all X in $\operatorname{add} T$. Now, let X be in $\widecheck{\operatorname{add} T}$, then we have an exact sequence $0 \to X \to T_0 \to T_1$ with T_0 and T_1 in $\operatorname{add} T$. By applying f_T to the above sequence, we get an exact sequence

$$0 \to f_T(X) \to {}_\Gamma I_0 \to {}_\Gamma I_1 \tag{3}$$

When we apply f'_T to (3) we get the following diagram

$$\begin{tikzcd}
0 \arrow[r] & f'_T f_T(X) \arrow[r] & f'_T f_T(T_0) \arrow[r] & f'_T f_T(T_1) \\
0 \arrow[r] & X \arrow[r] \arrow[u, "\Phi"] & T_0 \arrow[r] \arrow[u, "\wr"] & T_1 \arrow[u, "\wr"]
\end{tikzcd}$$

One can show that the above diagram is commutative and hence by the natural isomorphisms we have that Φ is an isomorphism for all X in $\operatorname{add} T$. To show that $f_T f'_T(Y) \cong Y$ for all Y in $\operatorname{add} DT$ is dual to what we have shown above. ☐

1.3. The Tilting Functor Preserves Tilting

Let T be a tilting Λ-module and DT the corresponding cotilting Γ-module, where $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$. In this section we show that, for the well known equivalence $e_T = \operatorname{Hom}_\Lambda(T, \ )\colon T^\perp \to {}^\perp DT$, the functors e_T and e_T^{-1} preserve (co)tilting modules. This means that e_T takes a (co)tilting Λ-module T' in $T^\perp$ to a (co)tilting Γ-module $e_T(T')$ in ${}^\perp DT$, while e_T^{-1} takes a (co)tilting Γ-module in ${}^\perp DT$ to a (co)tilting Λ-module in $T^\perp$.

Let T be a selforthogonal Λ-module. Consider the subcategory $T^\perp$ of $\operatorname{mod}\Lambda$. It is easy to see that $T^\perp$ is coresolving (that is, closed under extensions, cokernels of monomorphisms and contains the injective modules). Denote by $\mathcal{Y}_T$ the full subcategory of $T^\perp$ containing all Λ-modules C such that there is an exact sequence

$$\cdots \to T_i \xrightarrow{g_i} T_{i-1} \to \cdots \to T_1 \xrightarrow{g_1} T_0 \to C \to 0$$

with T_i in $\operatorname{add} T$ and $\operatorname{Im} g_i$ in $T^\perp$. The subcategory $\mathcal{Y}_T$ has the following properties.

Proposition 1.3.1. [**5**, Proposition 5.1] *Let T be a selforthogonal Λ-module. Then the subcategory $\mathcal{Y}_T$ is closed under extensions, cokernels of monomorphisms and direct summands.*

We have the following result from [**5**, Proposition 5.2(b);Theorem 5.4(b)]

Proposition 1.3.2. *Let T be a tilting Λ-module. Then*

(a) $\mathcal{Y}_T = T^\perp$.
(b) $\operatorname{add} T = {}^\perp(T^\perp)$.

Let T be a tilting Λ-module, $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$ and DT the corresponding cotilting Γ-module. Consider the functor

$$e_T = \operatorname{Hom}_\Lambda(T,)\colon \operatorname{mod}\Lambda \to \operatorname{mod}\Gamma.$$

Let T' be a (co)tilting Λ-module which is in $T^{\perp}$. Then one wonders if $e_T(T')$, which is in ${}^{\perp}DT$, is a (co)tilting Γ-module. For the case of the trivial (co)tilting Λ-modules, then this is true, since $e_T(T) = \Gamma$ and $e_T(D\Lambda) = DT$.

On the other hand, suppose Λ is standardly stratified and the Ringel dual Γ is properly stratified. Then there are both characteristic tilting and cotilting Γ-modules. By Proposition 1.1.3, the (characteristic) cotilting module is identified with $D(\Lambda)$.

In [**17**], it is pointed out that the (characteristic) tilting Γ-module is identified with a Λ-module H, that is $e_T^{-1}({}_{\Gamma}T) = H$. It is shown that the module H has very nice properties. Among the properties, H is a tilting Λ-module and $\widecheck{\operatorname{add} H} = \mathcal{P}^{<\infty}(\Lambda)$. The latter property is equivalent to H being strong in the sense of [**5**]. So if we apply e_T to H we get a tilting Γ-module, namely ${}_{\Gamma}T$.

The following result generalizes the above discussion. This is the main result of this section.

THEOREM 1.3.3. *Let T be a tilting Λ-module with $\operatorname{pd}_{\Lambda} T = r$. Let $\Gamma = \operatorname{End}_{\Lambda}(T)^{\mathrm{op}}$ and let DT be the corresponding cotilting Γ-module. Then*

(a) *T' in $T^{\perp}$ is a tilting Λ-module if and only if $e_T(T')$ in ${}^{\perp}DT$ is a tilting Γ-module.*
(b) *C in $T^{\perp}$ is a cotilting Λ-module if and only if $e_T(C)$ in ${}^{\perp}DT$ is cotilting Γ-module.*

To prove this result we need some lemmas. The following lemma generalizes [**29**, Proposition 1.5]. The lemma was originally given in [**21**].

LEMMA 1.3.4. *Let T be a tilting Λ-module. Then $T^{\perp} \cap \mathcal{P}^{<\infty}(\Lambda) = \widehat{\operatorname{add} T}$.*

PROOF. If X is in $\widehat{\operatorname{add} T}$, then X is in $\mathcal{P}^{<\infty}(\Lambda)$, since T is in $\mathcal{P}^{<\infty}(\Lambda)$. So the inclusion $\widehat{\operatorname{add} T} \subseteqq T^{\perp} \cap \mathcal{P}^{<\infty}(\Lambda)$ follows, since $\widehat{\operatorname{add} T}$ is contained in $T^{\perp}$. For the other inclusion, let X be in $T^{\perp} \cap \mathcal{P}^{<\infty}(\Lambda)$, then we have the following exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & T_d \arrow[r, "f_r"] & T_{d-1} \arrow[r] & \cdots \arrow[r] & T_1 \arrow[r, "f_1"] & T_0 \arrow[r] & X \arrow[r] & 0 \end{tikzcd}$$

with T_i in $\operatorname{add} T$, since T is tilting ([**5**, Dual of Theorem 5.4]). Denote $\operatorname{Im} f_i$ by X_i. Let $\operatorname{pd}_{\Lambda} X = d$, then by dimension shift, we have

$$\operatorname{Ext}^i_{\Lambda}(X_d, T^{\perp}) \cong \operatorname{Ext}^{d+i}_{\Lambda}(X, T^{\perp}) = (0) \qquad \text{for all } i > 0,$$

which means that X_d is in ${}^{\perp}(T^{\perp})$. Then by Proposition 1.3.2 we have that ${}^{\perp}(T^{\perp}) = \widecheck{\operatorname{add} T}$, so X_d is in $T^{\perp} \cap \widecheck{\operatorname{add} T} = \operatorname{add} T$. Therefore X is in $\widehat{\operatorname{add} T}$. □

We state without proof the dual of Lemma 1.3.4.

Lemma 1.3.5. *Let C be a cotilting Λ-module. Then* ${}^{\perp}C \cap \mathcal{I}^{<\infty}(\Lambda) = \widetilde{\operatorname{add} C}$.

The following lemma will also be needed.

Lemma 1.3.6. *Let T and T' be tilting Λ-modules with T' in $T^{\perp}$. Then T is in* $\widetilde{\operatorname{add} T'}$.

Proof. We know that ${}^{\perp}(T'^{\perp}) = \widetilde{\operatorname{add} T'}$ by Proposition 1.3.2, so we show that T is in ${}^{\perp}(T'^{\perp})$. For, let Y be in $T'^{\perp}$. Since T' is tilting we have an $\operatorname{add} T'$-resolution

$$\begin{tikzcd}\cdots \arrow[r] & T'_r \arrow[r, "f_r"] & T'_{r-1} \arrow[r] & \cdots \arrow[r] & T'_1 \arrow[r, "f_1"] & T'_0 \arrow[r] & Y \arrow[r] & 0\end{tikzcd}$$

with $Y_i = \operatorname{Im} f_i$ in $T'^{\perp}$. Let $\operatorname{pd}_{\Lambda} T = r$, which is finite since T is a tilting module. Since $\operatorname{Ext}^i_{\Lambda}(T, T'_j) = 0$ for all $i > 0$ and $j \geqslant 0$, we have

$$\operatorname{Ext}^i_{\Lambda}(T, Y) \cong \operatorname{Ext}^{i+r}_{\Lambda}(T, Y_r) = 0 \qquad \text{for all } i > 0.$$

Hence $\operatorname{Ext}^i_{\Lambda}(T, T'^{\perp}) = 0$ for all $i > 0$. So T is in ${}^{\perp}(T'^{\perp})$. ☐

Proof of *Theorem 1.3.3*. First we prove the sufficient condition of (a). Let T' be a tilting Λ-module in $T^{\perp}$. Then by Lemma 1.3.4 we have that T' is in $\widehat{\operatorname{add} T}$, and we have a finite $\operatorname{add} T$-resolution

$$0 \to T_s \to \cdots \to T_0 \to T' \to 0. \tag{4}$$

Applying e_T to (4), we get that $\operatorname{pd}_{\Gamma} e_T(T')$ is finite. Moreover, by Lemma 1.3.6 we have that $T \in \widetilde{\operatorname{add} T'}$. So, applying e_T to the $\operatorname{add} T'$-coresolution of T, we get that Γ is in $\operatorname{add} e_T(T')$.

By [**28**, Proposition 1.20], we have

$$\operatorname{Ext}^i_{\Gamma}(e_T(T'), e_T(T')) \xleftarrow{\sim} \operatorname{Ext}^i_{\Lambda}(T', T') = 0$$

Hence $e_T(T')$ is tilting Γ-module.

Next we prove the sufficient condition of (b). This means, we want to show that a cotilting Λ-module C in $T^{\perp}$ goes to a cotilting Γ-module $e_T(C)$ in ${}^{\perp}DT$. For, let C be a cotilting Λ-module with $\operatorname{id}_{\Lambda} C = t$, then C has a finite injective resolution, say

$$0 \to C \to I_0 \to I_1 \to \cdots \to I_t \to 0 \tag{5}$$

Applying e_T to (5), we get an exact sequence

$$0 \to e_T(C) \to DT_0 \to DT_1 \to \cdots \to DT_t \to 0$$

with DT_i in $\operatorname{add} DT$. Then, by induction and [**28**, Lemma 2.1] we have that $e_T(C)$ has finite injective dimension since DT has. In particular, $\operatorname{id}_{\Gamma} e_T(C) \leqslant r + t$.

By [**28**, Proposition 1.20], we have that

$$\operatorname{Ext}^i_{\Gamma}(e_T(C), e_T(C)) = 0 \text{ for } i > 0.$$

So it remains to show that $D\Gamma$ is in $\widehat{\operatorname{add} e_T(C)}$. For, let C be as above, then we have that $D\Lambda$ is in $\widehat{\operatorname{add} C}$. By applying e_T to the $\operatorname{add} C$-resolution of $D\Lambda$ we get that DT is in $\widehat{\operatorname{add} e_T(C)}$. But also, we have an exact sequence

$$0 \to DT_d \xrightarrow{f_d} DT_{d-1} \to \cdots \to DT_1 \xrightarrow{f_1} DT_0 \to D\Gamma \to 0$$

since DT is cotilting. Denote $\operatorname{Im} f_i$ by X_i. We show that $D\Gamma$ is in $\widehat{\operatorname{add} e_T(C)}$ by reverse induction on the length of DT-resolution of $D\Gamma$.

For $n = 1$, consider the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & V_1 \arrow[r, equal] \arrow[d] & V_1 \arrow[d] & & \\
0 \arrow[r] & Z_1 \arrow[r] \arrow[d] & e_T(C_0) \arrow[r] \arrow[d] & D\Gamma \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & DT_1 \arrow[r] \arrow[d] & DT_0 \arrow[r] \arrow[d] & D\Gamma \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

We have that DT_1 and V_1 are in $\widehat{\operatorname{add} e_T(C)}$. Hence Z_1 is in $\widehat{\operatorname{add} e_T(C)}$, since $\widehat{\operatorname{add} e_T(C)}$ is closed under extensions by Proposition 1.3.1.

For $n \geqslant 2$, consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & V_1 \arrow[r, equal] \arrow[d] & V_1 \arrow[d] & & \\
0 \arrow[r] & Z_1 \arrow[r] \arrow[d] & e_T(C_0) \arrow[r] \arrow[d] & D\Gamma \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X_1 \arrow[r] \arrow[d] & DT_0 \arrow[r] \arrow[d] & D\Gamma \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Then by reverse induction we have that X_1 is in $\widehat{\operatorname{add} e_T(C)}$. Since V_1 is in $\widehat{\operatorname{add} e_T(C)}$, and $\widehat{\operatorname{add} e_T(C)}$ is closed under extensions by Proposition 1.3.1, it follows that Z_1 is in $\widehat{\operatorname{add} e_T(C)}$. So, we have shown that if C is a cotilting Λ-module in $T^{\perp}$, then $e_T(C)$ cotilting Γ-module in ${}^{\perp}DT$.

Now we want to show that if V is a tilting Γ-module in ${}^{\perp}DT$, then $e_T^{-1}(V)$ is a tilting Λ-module in $T^{\perp}$, that is, the necessary condition of (a).

We know that $DV \in {T_{\Gamma}}^{\perp} \subseteq \operatorname{mod}\Gamma^{\mathrm{op}}$ is cotilting module. One can easily see that the functor

$$H = \operatorname{Hom}_{\Gamma^{\mathrm{op}}}(T_{\Gamma}, -)\colon {T_{\Gamma}}^{\perp} \to {}^{\perp}(DT)_{\Lambda}$$

is an equivalence of subcategories. By the sufficient condition of (b), we have that $H(DV)$ is a cotilting Λ^{op}-module which is in ${}^{\perp}(DT)_{\Lambda}$. Hence $DH(DV)$ is a tilting Λ-module in $T^{\perp}$.

Now if we can show that $DH(DV)$ coincides with $F^{-1}(V)$, we are done. For,

$$\begin{aligned} DH(DV) &= D\operatorname{Hom}_{\Gamma^{\mathrm{op}}}({}_{\Lambda}T_{\Gamma}, , DV) \\ &\cong D^2(V \otimes_{\Gamma^{\mathrm{op}}} T_{\Gamma}) \\ &\cong V \otimes_{\Gamma^{\mathrm{op}}} T_{\Gamma} \\ &\cong T_{\Gamma} \otimes_{\Gamma} V \\ &= e_T^{-1}(V) \end{aligned}$$

Therefore, $e_T^{-1}(V)$ is a tilting Λ-module.

Finally, we prove the necessary condition of (b), that is, if C is a cotilting Γ-module in ${}^{\perp}DT$, then $e_T^{-1}(V)$ is a cotilting Λ-module in $T^{\perp}$. This can be done using dual of the sufficient condition of (a). This completes the proof. □

1.4. Tilting theory and finitistic dimension

In this section we provide a sufficient condition for $\mathcal{P}^{<\infty}(\Lambda)$,the category of Λ-modules of finite projective dimension, to be contravariantly finite in $\operatorname{mod}\Lambda$. The subcategory $\mathcal{P}^{<\infty}(\Lambda)$ being contravariantly finite in $\operatorname{mod}\Lambda$ is a sufficient condition for the finitistic dimension of $\operatorname{mod}\Lambda$ to be finite [**5**].

Let M be a finitely generated Λ-module and denote by e_M the functor $\operatorname{Hom}_{\Lambda}(M, \ )$ from $\operatorname{mod}\Lambda$ to $\operatorname{mod}\Sigma$, where $\Sigma = \operatorname{End}_{\Lambda}(M)^{\mathrm{op}}$. Consider the following two cases:

Case 1. Let T be a strong tilting Λ-module (that is $\mathcal{P}^{<\infty}(\Lambda) = \widehat{\operatorname{add}T}$), denote $\operatorname{End}_{\Lambda}(T)^{\mathrm{op}}$ by Γ and let DT be the cotilting Γ-module corresponding to T. Then we know that $e_T\colon T^{\perp} \to {}^{\perp}DT$ is an equivalence. Consider a Γ-module X in ${}^{\perp}DT$ with $\operatorname{pd}_{\Gamma} X < \infty$. Then $e_T^{-1}(X)$ is in $\widehat{\operatorname{add}T} = T^{\perp} \cap \mathcal{P}^{<\infty}(\Lambda)$ (Lemma 1.3.4). Since $\mathcal{P}^{<\infty}(\Lambda) = \widehat{\operatorname{add}T}$, it follows that $e_T^{-1}(X)$ is in $\widehat{\operatorname{add}T}$. Hence $e_T^{-1}(X)$ is in $\widehat{\operatorname{add}T} \cap T^{\perp} = \operatorname{add}T$, which implies that X is in $\mathcal{P}(\Gamma)$. This means ${}^{\perp}DT \cap \mathcal{P}^{<\infty}(\Gamma) = \mathcal{P}(\Gamma)$.

Case 2. Let Λ be a standardly stratified algebra such that the Ringel dual $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$ is properly stratified (that is $\mathcal{F}({}_\Gamma\Delta) \subseteq \mathcal{F}({}_\Gamma\overline{\Delta})$). Denote by ${}_\Lambda T$ the characteristic tilting Λ-module and by ${}_\Gamma T$ the characteristic tilting Γ-module. Then $e_T \colon \mathcal{F}({}_\Lambda\overline{\nabla}) = {}_\Lambda T^\perp \to \mathcal{F}({}_\Gamma\overline{\Delta}) = {}^\perp DT$ is an equivalence, where DT is the cotilting Γ-module corresponding to ${}_\Lambda T$. By [**17**] we have that $e_T^{-1}({}_\Gamma T) = H$ is a tilting Λ-module with the property that $\mathcal{P}^{<\infty}(\Lambda) = \widecheck{\operatorname{add} H}$.

Now, let Y be a Γ-module in $\mathcal{F}({}_\Gamma\overline{\Delta}) \cap \mathcal{P}^{<\infty}(\Gamma)$. Then $e_T^{-1}(Y)$ is in $\widehat{\operatorname{add}{}_\Lambda T}$ which implies that $\operatorname{pd}_\Gamma e_T^{-1}(Y) < \infty$, since ${}_\Lambda T$ is tilting. Hence $e_T^{-1}(Y)$ is in $\widecheck{\operatorname{add} H}$, which means there is an exact sequence

$$0 \to e_T^{-1}(Y) \to H^0 \to H^1 \to \cdots \to H^s \to 0$$

with the H^i in $\operatorname{add} H$. Applying e_T to the above sequence we get that Y is in $\widecheck{\operatorname{add}{}_\Gamma T} = \mathcal{F}({}_\Gamma\Delta)$. Since $\mathcal{F}({}_\Gamma\Delta) \subseteq \mathcal{F}({}_\Gamma\overline{\Delta})$, we have that $\mathcal{F}({}_\Gamma\overline{\Delta}) \cap \mathcal{P}^{<\infty}(\Gamma) = \mathcal{F}({}_\Gamma\Delta)$.

In the two cases we considered above, we see that ${}^\perp DT \cap \mathcal{P}^{<\infty}(\Gamma)$ is equal to a subcategory of $\operatorname{mod}\Gamma$ associated with tilting Γ-modules, namely $\mathcal{P}(\Gamma)$ and $\widecheck{\operatorname{add}{}_\Gamma T}$. Moreover, these tilting Γ-modules are special (trivial in Case 1 and characteristic in Case 2). We also have that, in Case 1 the subcategory $\mathcal{P}^{<\infty}(\Lambda)$ is contravariantly finite in $\operatorname{mod}\Lambda$ by the definition of strong tilting module [**5**, Section 5 and 6] while in Case 2 the category $\mathcal{P}^{<\infty}(\Lambda)$ is contravariantly finite by [**17**, Theorem 4].

It is known that the category $\mathcal{P}^{<\infty}(\Lambda)$ being contravariantly finite in $\operatorname{mod}\Lambda$ is a sufficient condition for the finitistic dimension of Λ to be finite [**5**, Corollary 30].

Let T be a tilting Λ-module and denote $\operatorname{End}_\Lambda(T)^{\mathrm{op}}$ by Γ. We want to show that the condition ${}^\perp DT \cap \mathcal{P}^{<\infty}(\Gamma) = \widecheck{\operatorname{add} U}$, where U is a tilting Γ-module in ${}^\perp DT$ is sufficient for the subcategory $\mathcal{P}^{<\infty}(\Lambda)$ to be contravariantly finite in $\operatorname{mod}\Lambda$. Then the finitistic dimension of Λ would be finite [**5**]. This will generalize the two cases we considered above.

But first we state the following result, which will be very useful in proving the main result of this section.

Proposition 1.4.1. *Let T be a tilting Λ-module and consider a Λ-module X. Suppose X has finite $\widecheck{\operatorname{add} T}$-resolution. Then the following holds*

(i) *There is a short exact sequence $0 \to X \to E \to Y \to 0$ with E in $\widehat{\operatorname{add} T}$ and Y in $\widecheck{\operatorname{add} T}$.*
(ii) *There is a long exact sequence*

$$0 \to X \to E^0 \to E^1 \to \cdots$$

with the E^i in $\widehat{\operatorname{add} T}$.

Proof. i) We use reverse induction on the $\widecheck{\operatorname{add} T}$-resolution dimension of X.

If $\widecheck{\operatorname{add} T}\text{-}\operatorname{resdim}_\Lambda(X) = 0$, then X is in $\widecheck{\operatorname{add} T}$. Since T is tilting the claim follows.

Suppose that $\widecheck{\operatorname{add} T}\text{-}\operatorname{resdim}_\Lambda(X) = 1$ and consider a minimal $\widecheck{\operatorname{add} T}$-resolution $0 \to Y_1 \to Y_0 \to X \to 0$ of X. Then we have the following commutative exact diagram

(1)

$$\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & \\
\alpha_1\colon & 0 \arrow[r] & Y_1 \arrow[r] \arrow[d, equal] & Y_0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & Y_1 \arrow[r] & T_0^0 \arrow[r] \arrow[d] & E^0 \arrow[r] \arrow[d] & 0 \\
 & & & Y_0^1 \arrow[r, equal] \arrow[d] & Y_0^1 \arrow[d] & \\
 & & & 0 & 0 &
\end{tikzcd}$$

with Y_0^1 in $\widecheck{\operatorname{add} T}$ and T_0^0 in $\operatorname{add} T$. Then we use the middle row of (1) to construct the following commutative diagram

(2)

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
\theta_1\colon & 0 \arrow[r] & Y_1 \arrow[r] \arrow[d] & T_0^0 \arrow[r] \arrow[d] & E^0 \arrow[r] \arrow[d, equal] & 0 \\
 & 0 \arrow[r] & T_1^0 \arrow[r] \arrow[d] & K^0 \arrow[r] \arrow[d] & E^0 \arrow[r] & 0 \\
 & & Y_1^1 \arrow[r, equal] \arrow[d] & Y_1^1 \arrow[d] & & \\
 & & 0 & 0 & &
\end{tikzcd}$$

where Y_1^1 is in $\widecheck{\operatorname{add} T}$ and T_1^0 is in $\operatorname{add} T$. Since Y_1^1 is in ${}^\perp T$, we have that $K^0 \simeq T_0^0 \oplus Y_1^1$. Then we have an exact sequence $0 \to T_0^0 \oplus Y_1^1 \to T_0^0 \oplus T_1^{1,0} \to Y_1^{1,1} \to 0$ with $Y_1^{1,1}$ in $\widecheck{\operatorname{add} T}$ and $T_1^{1,0}$ in $\operatorname{add} T$. Denote $Y_1^{1,1}$ by ${}'Y$. Since ${}'Y$ is in ${}^\perp T$, we have the following commutative exact diagram

(3)
$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & & \\
\eta_1\colon & 0 \arrow[r] & T_0^1 \arrow[r] \arrow[d] & T_0^0 \oplus Y_1^1 \arrow[r] \arrow[d] & E^0 \arrow[r] \arrow[d, equal] & 0 \\
\theta_2\colon & 0 \arrow[r] & T_0^1 \oplus {}'Y \arrow[r] \arrow[d] & T_0^0 \oplus T_1^{1,0} \arrow[r] \arrow[d] & E^0 \arrow[r] & 0 \\
 & & {}'Y \arrow[r, equal] \arrow[d] & {}'Y \arrow[d] & & \\
 & & 0 & 0 & &
\end{tikzcd}$$

Replacing θ_1 in (2) by θ_2 we get an exact sequence $\eta_2\colon 0 \to T_1 \to T_0 \oplus Y_1^1 \to E^0 \to 0$, where T_0 and T_1 are in $\operatorname{add} T$. We then replace η_1 in (3) by η_2 and we get an exact sequence $\theta_3\colon 0 \to T_1 \oplus {}'Y \to T_0' \to E^0 \to 0$ with T_0' in $\operatorname{add} T$, where ${}'Y = Y_1^{1,1}$. Continuing with the process we get an exact sequence $0 \to T_1 \to T_0 \to E^0 \to 0$ with T_0 and T_1 in $\operatorname{add} T$ since the $\operatorname{add} T$-coresolution of the Y_i are finite. Hence E^0 is in $\widehat{\operatorname{add} T}$. From the first diagram we have the sequence $0 \to X \to E^0 \to Y_0^1 \to 0$ and the claim follows.

Now suppose that $\widecheck{\operatorname{add} T}\text{-}\operatorname{resdim}_\Lambda(X) = k > 1$. Then we have exact sequences $\alpha_2\colon 0 \to X_1 \to Y_0 \xrightarrow{f_0} X \to 0$ and $0 \to Y_k \to \cdots \to Y_1 \to X_1 \to 0$, where $X_1 = \operatorname{Ker} f_0$ and the Y_i are in $\widecheck{\operatorname{add} T}$. By the reverse induction on $\widecheck{\operatorname{add} T}\text{-}\operatorname{resdim}_\Lambda(X)$, we get an exact sequence $0 \to X_1 \to E^1 \to {}'Y \to 0$ with ${}'Y$ in $\widecheck{\operatorname{add} T}$ and E^1 in $\widehat{\operatorname{add} T}$. Replacing α_1 in (1) by α_2, we get an exact sequence $0 \to X_1 \to T_0^0 \to E^0 \to 0$ with T_0^0 in $\operatorname{add} T$, which we use to construct the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & X_1 \arrow[r] \arrow[d] & T_0^0 \arrow[r] \arrow[d] & E \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E^1 \arrow[r] \arrow[d] & K^0 \arrow[r] \arrow[d] & E \arrow[r] & 0 \\
 & {}'Y \arrow[r, equal] \arrow[d] & {}'Y \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

But since ${}'Y$ is in ${}^\perp T$, it follows that $K^0 \simeq T_0^0 \oplus {}'Y$. Note that for any Y in $\widecheck{\operatorname{add} T}$ we have that $\operatorname{Ext}_\Lambda^i(Y, E^1) = 0$ for all $i > 0$, since E^1 is in $\widehat{\operatorname{add} T}$. Then by using the middle row of the above diagram, we get the following commutative exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & E^1 \arrow[r] \arrow[d] & T_0^0 \oplus {}'Y \arrow[r] \arrow[d] & E^0 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & E^1 \oplus {}'Y^1 \arrow[r] \arrow[d] & T_0^0 \oplus T_1^0 \arrow[r] \arrow[d] & E^0 \arrow[r] & 0 \\
 & {}'Y^1 \arrow[r, equal] \arrow[d] & {}'Y^1 \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

since ${}'Y^1$ is in $\widecheck{\operatorname{add} T}$. Repeating the process as in the case $k = 1$, we get an exact sequence $0 \to E^1 \oplus T_1 \to T_0 \to E^0 \to 0$ with T_0 and T_1 in $\operatorname{add} T$ since the $\operatorname{add} T$-coresolution of the Y_i are finite. Since E^1 is in $\widehat{\operatorname{add} T}$, so is E^0. Our desired exact sequence is $0 \to X \to E^0 \to Y_0^1 \to 0$.

ii) Suppose X has a finite $\widecheck{\operatorname{add} T}$-resolution. Then by (i) we have an exact sequence $0 \to X \to E \to Y \to 0$ with E in $\widehat{\operatorname{add} T}$ and Y in $\widecheck{\operatorname{add} T}$. Then the claim follows by induction, since Y has a finite $\widecheck{\operatorname{add} T}$-resolution. ☐

As an immediate consequence we have the following result, which will be used to prove the main result of this section.

COROLLARY 1.4.2. *Let T be a tilting Λ-module and consider a Λ-module X with $\operatorname{pd}_\Lambda X < \infty$. Then there is a long exact sequence*

$$0 \to X \to E^0 \to E^1 \to \cdots$$

with the E^i in $\widehat{\operatorname{add} T}$.

PROOF. Assume that $\operatorname{pd}_\Lambda X < \infty$. Then by Proposition 1.4.1 we have an exact sequence $0 \to X \to E \to Y \to 0$ with E in $\widehat{\operatorname{add} T}$, since Λ is in $\widecheck{\operatorname{add} T}$. But since $\operatorname{pd}_\Lambda T < \infty$, it follows that $\operatorname{pd}_\Lambda E < \infty$. Hence $\operatorname{pd}_\Lambda Y < \infty$ by [**28**, Lemma 2.1]. Then the result follows by induction. ☐

Let T be a tilting Λ-module, denote $\operatorname{End}_\Lambda(T)^{\mathrm{op}}$ by Γ and let DT be the cotilting Γ-module corresponding to T. Then we have that the functor e_T induces an equivalence between subcategories $T^\perp$ of $\operatorname{mod}\Gamma$ and ${}^\perp DT$ of $\operatorname{mod}\Gamma$.

Denote the subcategory ${}^\perp DT \cap \mathcal{P}^{<\infty}(\Gamma)$ by $\mathcal{P}^{<\infty}({}^\perp DT)$. Then we have the following result which generalizes [**5**] and [**17**] for tilting modules T. This is the main result of this section.

Theorem 1.4.3. *Let T be a tilting Λ-module and let U be a tilting Γ-module in ${}^{\perp}DT$ such that $\mathcal{P}^{<\infty}({}^{\perp}DT) = \widecheck{\operatorname{add} U}$. Denote $e_T^{-1}(U)$ by H. Then we have the following*

(i) *The Λ-module H is tilting.*
(ii) *$\mathcal{P}^{<\infty}(\Lambda) = \widecheck{\operatorname{add} H}$. In particular, $\mathcal{P}^{<\infty}(\Lambda)$ is contravariantly finite.*
(iii) *The little finitistic dimension of Λ is finite.*

Proof. i) This follows by Theorem 1.3.3, since U is a tilting Γ-module in ${}^{\perp}DT$.

ii) Since H is tilting, the inclusion $\mathcal{P}^{<\infty}(\Lambda) \supseteq \widecheck{\operatorname{add} H}$ follows.

Let X be in $\mathcal{P}^{<\infty}(\Lambda)$. Then by Corollary 1.4.2 we have an exact sequence

$$0 \to X \to E^0 \to E^1 \to \cdots \tag{4}$$

with the E^i in $\widehat{\operatorname{add} H}$ since H is tilting. Then for each $j \geqslant 0$ we have an exact sequence

$$0 \to H_t \to \cdots \to H_1 \to H_0 \to E^j \to 0$$

with the H_i in $\operatorname{add} H$. Applying the functor e_T to the above sequence we get the following exact sequence

$$0 \to U_t \to \cdots \to U_1 \to U_0 \to e_T(E^j) \to 0$$

with the U_i in $\operatorname{add} U$. Since U is a tilting Γ-module, it follows that $\operatorname{pd}_\Gamma e_T(E^j) < \infty$. Hence $e_T(E^j)$ is in $\widecheck{\operatorname{add} U}$. This means that $e_T(E^j)$ is in $\widecheck{\operatorname{add} U} \cap U^{\perp} = \operatorname{add} U$, hence E^j is in $\operatorname{add} H$ for all $j \geqslant 0$. Then from (4) we get an exact sequence

$$0 \to X \xrightarrow{f^0} H^0 \xrightarrow{f^1} H^1 \to \cdots \to H^{r-1} \xrightarrow{f^r} H^r \to \cdots$$

with the H^i in $\operatorname{add} H$. Denote $\operatorname{Im} f^i$ by X^i and let $\operatorname{pd}_\Lambda H = r$. Then applying the functor e_H to the above sequence and using dimension shift we get that

$$\operatorname{Ext}^i_\Lambda(H, X^r) \simeq \operatorname{Ext}^{i+r}_\Lambda(H, X) = 0 \text{ for all } i > 0.$$

This implies that X^r is in $H^{\perp}$. Since $\operatorname{pd}_\Lambda X^r < \infty$, it follows that X^r is in $\widehat{\operatorname{add} H}$. Applying the functor e_T to the $\operatorname{add} H$-resolution of X^r, we infer that $\operatorname{pd}_\Gamma e_T(X^r) < \infty$.

On the other hand, since H is in $T^{\perp}$ and X^r is in $\widehat{\operatorname{add} H}$, we have that X^r is in $T^{\perp}$. Hence $e_T(X^r)$ is in ${}^{\perp}DT$. But since $\operatorname{pd}_\Gamma e_T(X^r) < \infty$, it follows that $e_T(X^r)$ is in $\widecheck{\operatorname{add} U}$. We also have that $e_T(X^r)$ is in $\widehat{\operatorname{add} U} \subseteq U^{\perp}$. Then it follows that $e_T(X^r)$ is in $U^{\perp} \cap \widecheck{\operatorname{add} U} = \operatorname{add} U$. Hence X^r is in $\operatorname{add} H$. Therefore, X is in $\widecheck{\operatorname{add} H}$ and we have that $\mathcal{P}^{<\infty}(\Lambda) = \widecheck{\operatorname{add} H}$.

Since H is tilting, it follows that $\mathcal{P}^{<\infty}(\Lambda)$ is contravariantly finite by (ii) and [**5**, Propostion 3.3 and 5.2].

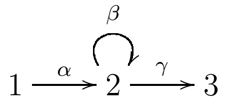
iii) Follows from (ii) and [**5**, Corollary 3.10]. □

1.4.1. Examples

In this subsection we consider examples to illustrate Theorem 1.4.3.

The following example shows that Case 1 is not contained in Case 2. Moreover, Theorem 1.4.3 is trivial on this example.

Example 1.4.4. Let Λ be given by the quiver



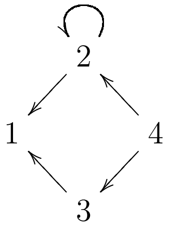
with radical square-zero relations. Denote by P_i and S_i the indecomposable projective and simple corresponding to the vertex i. Then the pair $(\Lambda, \leqslant)$ is not standardly stratified. The only non-trivial tilting Λ-module is $T = P_1 \oplus P_2 \oplus P_2/S_3$, which is also strong. So $\mathcal{P}^{<\infty}(\Lambda) = \widehat{\operatorname{add} T}$ is contravariantly finite.

The following example is covered by both Case 1 and 2.

Example 1.4.5. Let Λ be given by the quiver in Example 1.4.4 with relations $\gamma\alpha = 0 = \beta$ and $\gamma\beta\alpha = 0$. The pair $(\Lambda, \leqslant)$ is properly stratified. The characteristic tilting Λ-module is $T = D\Lambda$, which is strong. The Ringel dual $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$ is properly stratified with respect to $\leqslant^{\mathrm{op}}$.

The following example is neither covered by Case 1 nor by Case 2.

Example 1.4.6. Let Λ be given by the quiver



with radical square-zero relations. Denote by P_1 and S_i the indecomposable projective and simple corresponding to the vertex i.

Then the pair $(\Lambda, \leqslant)$ is standardly stratified, but not properly stratified. The characteristic tilting Λ-module is Λ itself, so the Ringel dual is not properly stratified with respect to $\leqslant^{\mathrm{op}}$, since it is not standardly stratified.

The Λ-module $T = P_3 \oplus X \oplus P_2 \oplus P_4$, where X denotes the module ${}^{3}{}_{1}{}^{2}{}_{2}$, is tilting, but not strong. Let $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$ and denote by Q_i, J_i and S_i the indecomposable projective, injective and simple Γ-module corresponding to the vertex i. The radical filtration of Q_i and J_i, for $j = 1, \ldots, 4$, look like:

$$Q_1\colon\ 1 \qquad Q_2\colon\ \begin{smallmatrix} & 2 & \\ 1 & & 3 \\ & & 2 \\ & & 3 \end{smallmatrix} \qquad Q_3\colon\ \begin{smallmatrix} 3 \\ 2 \\ 3 \end{smallmatrix} \qquad Q_4\colon\ \begin{smallmatrix} & 4 & \\ 2 & & 2 \\ 1 & & 3 \end{smallmatrix} \qquad J_1\colon\ \begin{smallmatrix} 4 \\ 2 \\ 1 \end{smallmatrix} \qquad J_2\colon\ \begin{smallmatrix} 2 & & \\ 3 & 4 & 4 \\ & 2 & \end{smallmatrix}$$

$$J_3\colon\ \begin{smallmatrix} 2 & & \\ 3 & & 4 \\ & 2 & \\ & 3 & \end{smallmatrix} \qquad J_4\colon\ 4$$

The Γ-module $DT = \begin{smallmatrix} & 2 & \\ 1 & & 3 \end{smallmatrix} \oplus J_3 \oplus J_1 \oplus S_4$ and ${}^{\perp}DT$ consists of the following Γ-modules.

$$\Gamma \qquad \begin{smallmatrix} & 2 & \\ 1 & & 3 \end{smallmatrix} \qquad S_4 \qquad \begin{smallmatrix} 2 \\ 3 \\ 2 \\ 3 \end{smallmatrix} \qquad \begin{smallmatrix} 4 \\ 2 \\ 3 \end{smallmatrix} \qquad M\colon\ \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \qquad S_3 \qquad \begin{smallmatrix} 2 \\ 3 \end{smallmatrix} \qquad \begin{smallmatrix} 3 & & 4 & \\ & 2 & & 2 \\ & 3 & & 1 \end{smallmatrix} \qquad \begin{smallmatrix} 3 & & 4 \\ & 2 & \\ & 3 & \end{smallmatrix}$$

$$\begin{smallmatrix} & 2 & & & & \\ 1 & & 3 & & 4 & \\ & & & 2 & & 2 \\ & & & 3 & & 1 \end{smallmatrix} \qquad \begin{smallmatrix} 2 & & & \\ 3 & & 4 & \\ & 2 & & 2 \\ & & & 1 \end{smallmatrix} \qquad \begin{smallmatrix} & 2 & & & \\ 1 & & 3 & & 4 \\ & & & 2 & \\ & & & 3 & \end{smallmatrix} \qquad J_1 \qquad J_3$$

It is not difficult to see that

$$\mathcal{P}^{<\infty}({}^{\perp}DT) = \operatorname{add}\{\Gamma, Q_2/S_1, M, Q_4/M\} = \overset{\smile}{\operatorname{add} U},$$

where $U = Q_2/S_1 \oplus Q_1 \oplus Q_4 \oplus Q_4/M$ is a tilting Γ-module. Then by Theorem 1.4.3 we have that $e_T(U) = H = \begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \oplus X \oplus P_4 \oplus \begin{smallmatrix} 4 \\ 2 \end{smallmatrix}$ is a tilting Λ-module with the property that $\mathcal{P}^{<\infty}(\Lambda) = \overset{\smile}{\operatorname{add} H}$.

Remark. In Example 1.4.1 our algebra Λ is of finite type while Γ is of infinite type. Hence it is easier to show that $\mathcal{P}^{<\infty}(\Lambda) = \overset{\smile}{\operatorname{add} H}$ than to show $\mathcal{P}^{<\infty}({}^{\perp}DT) = \overset{\smile}{\operatorname{add} U}$. However, there are cases where one tilts from infinite type to finite type. Those cases are the ones where the theorem is more interesting.

Chapter 2

Relative Theory in Subcategories

Let Λ be an artin algebra and let $\operatorname{mod}\Lambda$ be the category of finitely generated left Λ-modules. Auslander and Solberg [**10**] investigated the theory of relative homological algebra for $\operatorname{mod}\Lambda$, where the main ingredients were subfunctors. Let us refer to this theory as "the relative theory in" $\operatorname{mod}\Lambda$. In this chapter we develop "the relative theory in $\mathcal{C}$", where the $\mathcal{C}$ are functorially finite subcategories of $\operatorname{mod}\Lambda$ which are closed under extensions.

As mentioned in the introduction, functorially finite subcategories $\mathcal{C}$ of $\operatorname{mod}\Lambda$ give a nice invariant of Λ, namely the $\mathcal{C}$-approximation dimension (which in [**23**] was defined in general and was called $\mathcal{C}$-resolution dimension). Suppose that the $\mathcal{C}$-approximation dimension of $\operatorname{mod}\Lambda$ is zero. Then it will be shown that $\mathcal{C}$ is equivalent to $\operatorname{mod}\Lambda/I$, where I is an ideal in Λ.

Let $\mathcal{C}$ be a functorially finite subcategory of $\operatorname{mod}\Lambda$ which is closed under extensions and assume that the $\mathcal{C}$-approximation dimension is zero. Then by the above discussion, relative theory in $\mathcal{C}$ makes sense, since $\mathcal{C}$ is canonically equivalent to a module category over an artin algebra. This motivates us to develop a relative theory in functorially finite subcategories $\mathcal{C}$ of $\operatorname{mod}\Lambda$ with $\mathcal{C}$-approximation dimension greater than zero.

Now we explain the content of this chapter. In Section 2.1 we recall some well-known definitions and give some general properties of $\mathcal{C}$. In Section 2.2 we give a brief survey of relative theory in $\operatorname{mod}\Lambda$. In the survey, we recall the definition of a subfunctor F in $\operatorname{mod}\Lambda$ and look at some of its properties in $\operatorname{mod}\Lambda$. Then we define a subfunctor F in $\mathcal{C}$ and give some properties of F in $\mathcal{C}$. In Section 2.3 we study relative (co)resolving subcategories of the subcategory $\mathcal{C}$, which will be used in Chapter 3.

In the last section we introduce the notion of the right (left) $\mathcal{C}$-approximation dimension of $\operatorname{mod}\Lambda$. Then we characterize subcategories $\mathcal{C}$ of $\operatorname{mod}\Lambda$

with $\mathcal{C}$-approximation dimension equal to zero. Suppose that the $\mathcal{C}$-approximation dimension of $\operatorname{mod}\Lambda$ is finite. Then we show that any sequence of short F-exact sequences in $\operatorname{mod}\Lambda$ with all the middle terms in $\mathcal{C}$, will eventually be in $\mathcal{C}$. This will be very useful in Chapter 3.

2.1. Preliminaries

In this section we recall some definitions from [**7**] and give some preliminary results. Among the results, we show that functorially finite subcategories $\mathcal{C}$ of $\operatorname{mod}\Lambda$ which are closed under extensions in $\operatorname{mod}\Lambda$ have enough Ext-projectives and Ext-injectives. Moreover the subcategories of Ext-projectives and Ext-injectives are respectively contravariantly and covariantly finite in $\mathcal{C}$.

Let $\mathcal{C}$ be a subcategory of $\operatorname{mod}\Lambda$. An exact sequence in $\mathcal{C}$ is an exact sequence in $\operatorname{mod}\Lambda$ with all the terms in $\mathcal{C}$. A module Y in $\mathcal{C}$ is said to be Ext-*injective* if $\operatorname{Ext}^1_\Lambda(X,Y)=0$ for all X in $\mathcal{C}$. We denote the subcategory of Ext-injective modules in $\mathcal{C}$ by $\mathcal{I}(\mathcal{C})$.

A subcategory $\mathcal{C}$ is said to have *enough* Ext-*injectives* if for all C is $\mathcal{C}$ there is an exact sequence $0\to C\xrightarrow{f} I\to C^1\to 0$ with I Ext-injective and C^1 in $\mathcal{C}$. Note that if $\mathcal{C}$ has enough Ext-injectives and is closed under extensions in $\mathcal{C}$, then any map $g\colon C\to I'$ with I' in $\mathcal{I}(\mathcal{C})$ factors through f (i.e. there exists a map $h\colon I\to I'$ such that $g=hf$).

The notions of Ext-*projective* module and *enough* Ext-*projectives* are defined dually. The subcategory of Ext-projective modules in $\mathcal{C}$ is denoted by $\mathcal{P}(\mathcal{C})$.

When $\mathcal{C}$ has enough Ext-projectives, then for all C in $\mathcal{C}$ there is an exact sequence $0\to C_1\to P\xrightarrow{g} C\to 0$ with P in $\mathcal{P}(\mathcal{C})$ and C_1 in $\mathcal{C}$. So if $\mathcal{C}$ has enough Ext-projectives and is closed under extensions in $\mathcal{C}$, then any map $f\colon P'\to C$ with P' in $\mathcal{P}(\mathcal{C})$ factors through g.

Let $\mathcal{D}$ be a subcategory of $\operatorname{mod}\Lambda$ containing a subcategory $\mathcal{C}$. Given a module M in $\mathcal{D}$, a sequence $0\to Y\to C\xrightarrow{g} M$ with C in $\mathcal{C}$ is said to be a *right* $\mathcal{C}$-*approximation* of M if the sequence $0\to(C',Y)\to(C',C)\xrightarrow{(C',g)}(C',M)\to 0$ is exact for all C' in $\mathcal{C}$. A right $\mathcal{C}$-approximation is called a *minimal* right $\mathcal{C}$-approximation if g is right minimal, that is, if every endomorphism $s\colon C\to C$ satisfying $g=gs$ is an isomorphism.

A minimal right $\mathcal{C}$-approximation is unique up to isomorphism. A module has a right $\mathcal{C}$-approximation if and only if it has a minimal right $\mathcal{C}$-approximation. We denote the minimal right $\mathcal{C}$-approximation of M by $0\to Y_M\to C_M\xrightarrow{g_M} M$. A subcategory $\mathcal{C}$ is said to be *contravariantly finite* in $\mathcal{D}$ if every Λ-module in $\mathcal{D}$ has a right $\mathcal{C}$-approximation.

Dually, one defines the notions of *left (minimal)* $\mathcal{C}$*-approximation* and *covariantly finite* subcategory of $\mathcal{D}$. A subcategory $\mathcal{C}$ is said to be *functorially finite* in $\mathcal{C}$ if it is both contravariantly and covariantly finite in $\mathcal{D}$.

Let $\mathcal{C}$ be a contravariantly finite subcategory of $\operatorname{mod}\Lambda$. Then by [**7**, Lemma 3.11] we have that $\mathcal{C}$ has a finite cocover, that is, there is some Y in $\operatorname{add}\mathcal{C}$ such that $\mathcal{C}$ is contained in $\operatorname{Sub}Y$, where $\operatorname{Sub}Y$ denotes the subcategory of $\operatorname{mod}\Lambda$ consisting of objects which are submodules of finite direct sums of copies of Y. Suppose $\mathcal{C}$ is closed under extensions in $\operatorname{mod}\Lambda$. Then we have the following result which is an analog of [**7**, Lemma 3.11].

Proposition 2.1.1. *Let* $\mathcal{C}$ *be a contravariantly finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions. Then every* X *in* $\mathcal{C}$ *has an* $\mathcal{I}(\mathcal{C})$*-coresolution.*

To prove the result we need to show that the full subcategory $\mathcal{E}$ of $\operatorname{mod}\Lambda$ consisting of all Y such that $\operatorname{Ext}^1_\Lambda(X,Y) = 0$ for all X in $\mathcal{C}$ is covariantly finite in $\operatorname{mod}\Lambda$. To do this, we use the following proposition which is the dual of [**5**, Proposition 1.8].

Proposition 2.1.2. *Suppose* $\mathcal{J}$ *is a subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions such that* $\operatorname{Ext}^1_\Lambda(\ ,A)\mid_{\mathcal{J}}$ *is finitely generated for all* A *in* $\operatorname{mod}\Lambda$. *Then the subcategory*

$$\mathcal{K} = \{Y \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^1_\Lambda(\mathcal{J},Y) = 0\}$$

is covariantly finite in $\operatorname{mod}\Lambda$.

When $\mathcal{C}$ is contravariantly finite in $\operatorname{mod}\Lambda$, then $\operatorname{Ext}^1_\Lambda(\ ,A)\mid_{\mathcal{C}}$ is finitely generated for all A in $\operatorname{mod}\Lambda$. For, the exact sequence $0 \to A \to I(A) \to \Omega^{-1}(A) \to 0$, where $I(A)$ is the injective envelope of A, gives rise to an exact sequence of functors

$$(*)\ 0 \to (\ ,A) \to (\ ,I(A)) \to (\ ,\Omega^{-1}(A)) \to \operatorname{Ext}^1_\Lambda(-,A) \to 0.$$

Let $X \to \Omega^{-1}(A)$ be a right $\mathcal{C}$-approximation of $\Omega^{-1}(A)$. Then we have an exact sequence of functors $(\ ,X)\mid_{\mathcal{C}} \twoheadrightarrow (\ ,\Omega^{-1}(A))\mid_{\mathcal{C}}$. Restricting $(*)$ to $\mathcal{C}$, we get

$$(\ ,X)\mid_{\mathcal{C}} \twoheadrightarrow \operatorname{Ext}^1_\Lambda(\ ,A)\mid_{\mathcal{C}}$$

This is equivalent to saying that $\operatorname{Ext}^1_\Lambda(\ ,A)\mid_{\mathcal{C}}$ is finitely generated.

Our subcategory $\mathcal{C}$ in Proposition 2.1.1 satisfies the conditions of Proposition 2.1.2. Hence the subcategory $\mathcal{E}$ is covariantly finite and contains the injective Λ-modules.

Proof of *Proposition 2.1.1.* Let X be in $\mathcal{C}$. Then we have a minimal left $\mathcal{E}$-approximation $0 \to X \to E^X \to Z^X \to 0$ of X, which is a monomorphism, since $D\Lambda$ is in $\mathcal{E}$. Then by [**5**, Corollary 1.7] we have that Z^X is in $\mathcal{C}$. Since $\mathcal{C}$ is closed under extensions, it implies that E^X is in $\mathcal{C} \cap \mathcal{E} = \mathcal{I}(\mathcal{C})$. Then the result follows by induction. □

We state without proof the dual of Proposition 2.1.1.

Proposition 2.1.3. *Let $\mathcal{C}$ be covariantly finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions. Then every Y in $\mathcal{C}$ has a $\mathcal{P}(\mathcal{C})$-resolution.*

Let $\mathcal{P}^{\leqslant n}(\Lambda)$ denote the category of Λ-modules of projective dimension at most n, where n is a nonnegative integer. Denote by J the direct sum of all non-isomorphism indecomposable Ext-injective Λ-modules in $\mathcal{P}^{\leqslant n}(\Lambda)$. Then we have the following equivalent statements, where the first two parts are [**21**, Theorems 2.1 and 2.2].

Theorem 2.1.4. *The following are equivalent:*

- (i) *$\mathcal{P}^{\leqslant n}(\Lambda)$ is contravariantly finite.*
- (ii) *J is a tilting Λ-module.*
- (iii) *$\mathcal{P}^{\leqslant n}(\Lambda) = \overbrace{\operatorname{add} J}$.*

Proof. (i)⇔(ii) Follows by [**21**, Theorems 2.1 and 2.2].

(ii)⇒(iii) Assume J tilting. Then the inclusion $\overbrace{\operatorname{add} J} \subseteq \mathcal{P}^{\leqslant n}(\Lambda)$ follows. Let C be in $\mathcal{P}^{\leqslant n}(\Lambda)$. Then by Proposition 2.1.1, we have an exact sequence

$$0 \to C \xrightarrow{f_0} J_0 \xrightarrow{f_1} J_1 \to \cdots \qquad (*)$$

with J_i in $\operatorname{add} J$ and $\operatorname{Im} f_i$ in $\mathcal{P}^{\leqslant n}(\Lambda)$ for all i. Denote $\operatorname{Im} f_i$ by C_i. Consider the full subcategory

$$(\mathcal{P}^{\leqslant n}(\Lambda))^{\perp_1} = \{Y \in \operatorname{mod}\Lambda \mid \operatorname{Ext}^1_\Lambda(\mathcal{P}^{\leqslant n}(\Lambda), Y) = 0\}$$

of $\operatorname{mod}\Lambda$. Then by [**5**, Lemma 3.2] we have that $(\mathcal{P}^{\leqslant n}(\Lambda))^{\perp_1} = (\mathcal{P}^{\leqslant n}(\Lambda))^{\perp}$, since $\mathcal{P}^{\leqslant n}(\Lambda)$ is resolving.

Now let M be in $\mathcal{P}^{\leqslant n}(\Lambda)$. Applying $\operatorname{Hom}_\Lambda(M, \ )$ to (*), and using the fact that $(\mathcal{P}^{\leqslant n}(\Lambda))^{\perp_1} = (\mathcal{P}^{\leqslant n}(\Lambda))^{\perp}$ we get that

$$\operatorname{Ext}^i_\Lambda(M, C_r) \simeq \operatorname{Ext}^{i+r}_\Lambda(M, C) = 0 \text{ for all } i > 0$$

Hence C_r is in $(\mathcal{P}^{\leqslant n}(\Lambda))^{\perp_1} \cap \mathcal{P}^{\leqslant n}(\Lambda) = \operatorname{add} J$. Therefore, $\mathcal{P}^{\leqslant n}(\Lambda) = \overbrace{\operatorname{add} J}$.

(iii)⇒(i) Follows by [**5**, Propositions 3.3 and 5.2], since J is tilting. □

The following is consequence of Propositions 2.1.1 and 2.1.3.

Corollary 2.1.5. *Let $\mathcal{C}$ be functorially finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions. Then*

- (a) *$\mathcal{C}$ has enough* Ext*-projectives and* Ext*-injectives.*
- (b) *The subcategory $\mathcal{P}(\mathcal{C})$ is contravariantly finite in $\mathcal{C}$.*
- (c) *The subcategory $\mathcal{I}(\mathcal{C})$ is covariantly finite in $\mathcal{C}$.*

We recall the following definition from [**10**]. A subcategory $\mathcal{X}$ of $\mathcal{C}$ is said to be a *generator* for $\mathcal{C}$ if it contains $\mathcal{P}(\mathcal{C})$. Dually one defines *cogenerator* subcategory for $\mathcal{C}$. In the next result we show that contravariantly finite subcategories which are generator are very nice.

Lemma 2.1.6. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions. Let $\mathcal{X}$ be a contravariantly finite subcategory of $\mathcal{C}$ which is a generator for $\mathcal{C}$. Consider a right $\mathcal{X}$-approximation $0 \to Y \to X \xrightarrow{g} C \to 0$ of C in $\mathcal{C}$. Then Y is in $\mathcal{C}$.*

Proof. Since $\mathcal{C}$ has enough Ext-projectives by Corollary 2.1.5, there is, for any C in $\mathcal{C}$, an exact sequence $0 \to C_1 \to P \xrightarrow{p} C \to 0$ with P in $\mathcal{P}(\mathcal{C})$ and C_1 in $\mathcal{C}$. Then, we have the following exact commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & C_1 \arrow[r, equal] \arrow[d] & C_1 \arrow[d] & \\
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & Y \oplus P \arrow[r] \arrow[d] & P \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & Y \arrow[r] & X \arrow[r, "g"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

since g is a right $\mathcal{X}$-approximation of C. But since $\mathcal{C}$ is closed under extensions and summands, it follows that Y is in $\mathcal{C}$. ☐

We have the following dual of Lemma 2.1.6

Lemma 2.1.7. *Let $\mathcal{C}$ a functorially finite subcategory of* $\operatorname{mod}\Lambda$. *Let $\mathcal{Y}$ be a covariantly finite cogenerator subcategory of $\mathcal{C}$. Consider a left $\mathcal{Y}$-approximation $0 \to C \xrightarrow{g} Y^C \to Z \to 0$ of C in $\mathcal{C}$. Then Z is in $\mathcal{C}$.*

In the next sections we deal with relative theory, so pullbacks and pushouts will play an important role. We have the following useful result.

Lemma 2.1.8. *Consider the following commutative diagram in* $\operatorname{mod}\Lambda$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & Y \arrow[r, equal] \arrow[d] & Y \arrow[d] & & \\
0 \arrow[r] & A \arrow[r, "\gamma"] \arrow[d, "\delta"] & B \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r, "g"'] \arrow[d] & M \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
 & X \arrow[r, equal] \arrow[d] & X \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

with exact rows and columns. Then A is a pullback of f and g.

Proof. Consider a pullback E of f and g. Then we have a unique map $d\colon A \to E$ (one can show that d is a monomorphism). Let $\beta\colon E \to L$, and denote $\operatorname{Coker}\beta$ by X'. Then we have that the map $i\colon \operatorname{Im}\delta \to \operatorname{Im}\beta$ is a monomorphism (by Snake Lemma), so that the map $h\colon X \to X'$ is an epimorphism. Consider the map $h'\colon X' \to X$ such that the diagram commutes. Since $h' \circ h = 1_X$, we have that h is an monomorphism. Therefore d is an isomorphism. □

2.2. Subfunctors in Subcategories and their Properties

Let $\mathcal{C}$ be a functorially finite subcategory of $\operatorname{mod}\Lambda$ which is closed under extensions. In this section we study subfunctors in $\mathcal{C}$. We first recall some background on subfunctors in $\operatorname{mod}\Lambda$ from [**10**]. Then we study a special subfunctor $F = F_{\mathcal{X}}$ in $\mathcal{C}$, where $\mathcal{X}$ is a contravariantly finite subcategory of $\mathcal{C}$.

2.2.1. Background on Subfunctors

Let F be an (additive) sub-bifunctor of the bifunctor

$$\operatorname{Ext}^1_\Lambda(\ ,\ )\colon (\operatorname{mod}\Lambda)^{\mathrm{op}} \times \operatorname{mod}\Lambda \to \mathrm{Ab},$$

where $(\operatorname{mod}\Lambda)^{\mathrm{op}}$ denotes the opposite category of $\operatorname{mod}\Lambda$. Then F is said to be a (additive) subfunctor of $\operatorname{Ext}^1_\Lambda(\ ,\ )$ in $\operatorname{mod}\Lambda$. A short exact sequence $\eta\colon 0 \to A \to B \to C \to 0$ is called an *F-exact* sequence if η is in $F(C, A)$. Any pullback, pushout and Baer sum of an F-exact sequence is again F-exact [**10**].

In particular, a subfunctor F determines a collection of short exact sequences which is closed under pushouts, pullbacks and Baer sums. Conversely, given a collection of short exact sequences which is closed under pushouts, pullbacks and Baer sums, it gives rise to a subfunctor of $\operatorname{Ext}^1_\Lambda(\ ,\ )$ in the obvious way [**10**].

Let $\mathcal{P}(F)$ be a subcategory of $\operatorname{mod}\Lambda$ consisting of all Λ-modules P such that if $0 \to A \to B \to C \to 0$ is F-exact, then the sequence $0 \to (P, A) \to (P, B) \to (P, C) \to 0$ is exact. The objects in $\mathcal{P}(F)$ are called *projective modules* of the subfunctor F or *F-projectives.* If $\mathcal{P}(\Lambda)$ denotes the category of projective Λ-modules, then $\mathcal{P}(\Lambda)$ is contained in $\mathcal{P}(F)$.

An additive subfunctor F is said to have *enough projectives* if for every A in $\operatorname{mod}\Lambda$ there exists an F-exact sequence $0 \to A' \to P \to A \to 0$ with P in $\mathcal{P}(F)$. The definitions of *F-injectives* and *enough injectives* are dual.

Let $\mathcal{Z}$ be a subcategory of $\operatorname{mod}\Lambda$. Define

$$F_{\mathcal{Z}}(C,A)=\{0\to A\to B\to C\to 0 \mid (\mathcal{Z},B)\to(\mathcal{Z},C)\to 0 \text{ is exact}\}$$

for each pair of modules A and C in $\operatorname{mod}\Lambda$. Dually one defines

$$F^{\mathcal{Z}}(C,A)=\{0\to A\to B\to C\to 0 \mid (B,\mathcal{Z})\to(A,\mathcal{Z})\to 0 \text{ is exact}\}$$

for each pair of modules A and C in $\operatorname{mod}\Lambda$. It is shown in [**10**, Proposition 1.7] that these constructions give (additive) subfuntors of $\operatorname{Ext}^1_\Lambda(\ ,\ )$. Moreover, we have the following from [**10**].

Proposition 2.2.1. [**10**, Proposition 1.8 and 1.10] *Let $\mathcal{Z}$ be an additive subcategory of* $\operatorname{mod}\Lambda$. *Then*

(a) $F_{\mathcal{Z}}=F^{D\operatorname{Tr}\mathcal{Z}}$.
(b) $\mathcal{P}(F_{\mathcal{Z}})=\mathcal{Z}\cup\mathcal{P}(\Lambda)$

2.2.2. Subfunctors F in the Subcategory $\mathcal{C}$

Let $\mathcal{C}$ be a functorially finite subcategory of $\operatorname{mod}\Lambda$ which is closed under extensions. Consider a subfunctor F in $\operatorname{mod}\Lambda$. We say F *is a subfunctor in* $\mathcal{C}$ if we consider only the F-projectives and F-injectives which are in $\mathcal{C}$.

Our aim is to study a subfunctor F in $\mathcal{C}$. First we want to find the subcategories of F-projectives and F-injectives in $\mathcal{C}$. We denote these subcategories by $\mathcal{P}_{\mathcal{C}}(F)$ and $\mathcal{I}_{\mathcal{C}}(F)$ respectively. We fix the following notation.

Notation. Unless specified otherwise F denotes a subfunctor $F_{\mathcal{X}}$, where $\mathcal{X}$ is a contravariantly finite generator subcategory of $\mathcal{C}$.

Let F be a subfunctor in $\mathcal{C}$. By definition, P is in $\mathcal{P}_{\mathcal{C}}(F)$ if and only if the sequence $0\to(P,A)\to(P,B)\to(P,C)\to 0$ is exact whenever $0\to A\to B\to C\to 0$ is F-exact in $\mathcal{C}$. By Proposition 2.2.1, we have that $\mathcal{X}$ is contained in $\mathcal{P}_{\mathcal{C}}(F)$.

Let P be an F-projective in $\mathcal{C}$, then by Lemma 2.1.6 we have an F-exact sequence $0\to C'\to X\xrightarrow{f}P\to 0$ in $\mathcal{C}$ with X in $\mathcal{X}$. Since P is in $\mathcal{P}_{\mathcal{C}}(F)$, the identity map 1_P factors through f. So P is a direct summand of X, hence it is in $\mathcal{X}$.

The following summarizes the above discussion.

Proposition 2.2.2. *Let $\mathcal{C}$, F and $\mathcal{X}$ be as before. Then $\mathcal{P}_{\mathcal{C}}(F)=\mathcal{X}$.*

Note that since $\mathcal{X}$ contains $\mathcal{P}(\mathcal{C})$, then $\mathcal{P}_{\mathcal{C}}(F)$ is the restriction of $\mathcal{P}(F)$ to $\mathcal{C}$.

Now we want to find the F-injective modules in $\mathcal{C}$. The category $\mathcal{I}_{\mathcal{C}}(F)$ is not necessarily equal to the restriction of $\mathcal{I}(F)$ to $\mathcal{C}$. By definition I is in $\mathcal{I}_{\mathcal{C}}(F)$ if and only if the sequence $0\to\operatorname{Hom}(C,I)\to\operatorname{Hom}(B,I)\to$

$\mathrm{Hom}(A, I) \to 0$ is exact whenever $0 \to A \to B \to C \to 0$ is F-exact in $\mathcal{C}$. First we need several results from [**24**] (see also [**2**][**8**][**25**]).

The following lemma is the part (b) of [**24**, Lemma 2.1]. The proof is just the dual of the part (a), so it was not given in [**24**]. We give it here since it is not long. The result is a generalization of Wakamatsu's lemma [**33**].

Lemma 2.2.3. [**24**, Lemma 2.1(b)] *Let $\mathcal{C}$ be a contravariantly finite subcategory of* $\mathrm{mod}\,\Lambda$ *which is closed under extensions and let Z be a Λ-module. Then the natural transformation*

$$\mathrm{Ext}^1_\Lambda(\ ,g_Z)\colon\ \mathrm{Ext}^1_\Lambda(\ ,C_Z)|_{\mathcal{C}} \to \mathrm{Ext}^1_\Lambda(\ ,Z)|_{\mathcal{C}}$$

restricted to $\mathcal{C}$ is a monomorphism of contravariant functors, where g_Z denotes the minimal right $\mathcal{C}$-approximation of Z.

Proof. Consider the exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C_Z \arrow[r, "f"] \arrow[d, "g_Z"] & W \arrow[r] \arrow[d] & N \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Z \arrow[r] & V \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

with N in $\mathcal{C}$. Suppose the bottom row splits. Then $g_Z = sf$, for some morphism $s\colon W \to Z$. Since $\mathcal{C}$ is closed under extensions, W is in $\mathcal{C}$, so there is a morphism $h\colon W \to C_Z$ such that $s = g_Z h$. But then, we get that $g_Z = g_Z hf$. Since g_Z is right minimal, it follows that hf is an isomorphism, so that the top row splits. Hence the homomorphism

$$\mathrm{Ext}^1_\Lambda(N, g_Z)\colon\ \mathrm{Ext}^1_\Lambda(N, C_Z) \to \mathrm{Ext}^1_\Lambda(N, Z)$$

is a monomorphism. ☐

The following result is consequence of [**24**, Theorem 3.4].

Corollary 2.2.4. *Let $\mathcal{C}$ be a contravariantly finite subcategory of* $\mathrm{mod}\,\Lambda$ *which is closed under extensions. Let Y be in* $\mathrm{mod}\,\Lambda$*, and consider a succession of minimal right $\mathcal{C}$-approximations $Y_1 \hookrightarrow C_0 \to Y$, $Y_2 \hookrightarrow C_1 \to Y_1$, ... Then for $i > 0$, C_i is* Ext*-injective in $\mathcal{C}$.*

Proof. We know that $\mathrm{Ext}^1_\Lambda(\mathcal{C}, Y_i) = 0$ for all $i > 0$ by Wakamatsu's lemma. By Lemma 2.2.3 the map $\mathrm{Ext}^1_\Lambda(\mathcal{C}, f_i)\colon\ \mathrm{Ext}^1_\Lambda(\mathcal{C}, C_i) \to \mathrm{Ext}^1_\Lambda(\mathcal{C}, Y_i)$ is a monomorphism for all $i \geqslant 0$. Therefore $\mathrm{Ext}^1_\Lambda(\mathcal{C}, C_i) = 0$ for all $i > 0$, that is, C_i is Ext-injective for all $i > 0$. ☐

Note that if $Y = I$ is an injective Λ-module, then C_0 in Corollary 2.2.4 is Ext-injective in $\mathcal{C}$ [**7**, Lemma 3.5].

We recall the notions of covariant and contravariant defect of a short exact sequence [**6**]. Given a short exact sequence $\delta\colon 0 \to L \to M \to N \to 0$ in $\mathrm{mod}\,\Lambda$, the *covariant defect* δ_* and the *contravariant defect* δ^* of δ are the subfunctors of $\operatorname{Ext}^1_\Lambda(N,\ )$ and $\operatorname{Ext}^1_\Lambda(\ ,L)$ respectively, defined by the exact sequences

$$0 \to \operatorname{Hom}_\Lambda(N,\ ) \to \operatorname{Hom}_\Lambda(M,\ ) \to \operatorname{Hom}_\Lambda(L,\ ) \to \delta_* \to 0$$

and

$$0 \to \operatorname{Hom}_\Lambda(\ ,L) \to \operatorname{Hom}_\Lambda(\ ,M) \to \operatorname{Hom}_\Lambda(\ ,N) \to \delta^* \to 0$$

Now we state the following result from [**24**] and we give a different proof.

Proposition 2.2.5. [**24**, Proposition 2.5(b)] *Let $\mathcal{C}$ be a a contravariantly finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions. Let $\delta\colon 0 \to L \xrightarrow{f} M \to N \to 0$ be an exact sequence in $\mathcal{C}$. For all Z in* $\operatorname{mod}\Lambda$*, the morphism*

$$\operatorname{Hom}_\Lambda(L,g_Z)\colon \operatorname{Hom}_\Lambda(L,Z_\mathcal{C}) \to \operatorname{Hom}_\Lambda(L,Z)$$

induces an isomorphism $\delta_*(C_Z) \xrightarrow{\sim} \delta_*(Z)$.

Proof. Let $0 \to L \to M \to N \to 0$ be exact in $\mathcal{C}$ and let $0 \to Y_Z \to C_Z \xrightarrow{g_Z} Z$ be the minimal right $\mathcal{C}$-approximation of Z. Consider the following commutative diagram

$$\begin{tikzcd}
{}_\Lambda(M,C_Z) \arrow[r, "{(f,C_Z)}"] \arrow[d, "{(M,g_Z)}"] & {}_\Lambda(L,C_Z) \arrow[r, "\alpha"] \arrow[d, "{(L,g_Z)}"] & \delta_*(C_Z) \arrow[r] \arrow[d, "\delta_*(g_Z)"] & 0 \\
{}_\Lambda(M,Z) \arrow[r, "{(f,Z)}"] \arrow[d] & {}_\Lambda(L,Z) \arrow[r, "\beta"] \arrow[d] & \delta_*(Z) \arrow[r] & 0 \\
0 & 0 & &
\end{tikzcd}$$

Since (L,g_Z) and β are epimorphisms, it follows that $\delta_*(g_Z)$ is an epimorphism. On the other hand we have the following commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] \\
0 \arrow[r] & \delta_*(C_Z) \arrow[r, "i"] \arrow[d, "\delta_*(g_Z)"] & \operatorname{Ext}^1_\Lambda(N,C_Z) \arrow[d, "{\operatorname{Ext}^1_\Lambda(N,g_Z)}"] \\
0 \arrow[r] & \delta_*(Z) \arrow[r, "j"] \arrow[d] & \operatorname{Ext}^1_\Lambda(N,Z) \\
 & 0 &
\end{tikzcd}$$

Since i and $\operatorname{Ext}^1_\Lambda(N,g_Z)$ (by Lemma 2.2.3) are monomorphisms, it follows that $\delta_*(g_Z)$ is a monomorphism, hence it is an isomorphism. ☐

We have the following consequence of Proposition 2.2.5 which will be very useful for finding the relative F-injectives.

Corollary 2.2.6. *Let* $0 \to A \to B \to C \to 0$ *be exact in* $\mathcal{C}$*, and let* X *be in* $\operatorname{mod}\Lambda$*. The the following are equivalent.*

(i) $\operatorname{Hom}_\Lambda(X, B) \to \operatorname{Hom}_\Lambda(X, C)$ *is an epimorphism.*
(ii) $\operatorname{Hom}_\Lambda(B, C_{(\operatorname{DTr} X)}) \to \operatorname{Hom}_\Lambda(A, C_{(\operatorname{DTr} X)})$ *is an epimorphism.*

Proof. (i)⇔(ii) $\delta^*(X) = 0$ if and only if $\delta_*(\operatorname{DTr} X) = 0$ by [**6**, Theorem 4.1], but $\delta_*(\operatorname{DTr} X) \simeq \delta_*(C_{(\operatorname{DTr} X)})$ by Proposition 2.2.5. □

Now, the F-injectives in $\mathcal{C}$ are given by the following result. This is an analog of [**10**, Corollary 1.6].

Proposition 2.2.7. *Let* $\mathcal{C}$ *be a functorially finite subcategory which is closed under extensions. Then*

(a) $\mathcal{I}_{\mathcal{C}}(F) = C_{(\operatorname{DTr} \mathcal{P}_{\mathcal{C}}(F))} \cup \mathcal{I}(\mathcal{C})$.
(b) $\mathcal{P}_{\mathcal{C}}(F) = C^{\operatorname{TrD} \mathcal{I}_{\mathcal{C}}(F)} \cup \mathcal{P}(\mathcal{C})$.

Proof. (a) See Corollary 2.2.6.

(b) Dual of Corollary 2.2.6. □

Remark. Nothing can be said about the size of the subcategories $\mathcal{P}_{\mathcal{C}}(F)$ and $\mathcal{I}_{\mathcal{C}}(F)$ at the moment. But later we will see that if there exists an F-tilting module in $\mathcal{C}$, then $\mathcal{P}_{\mathcal{C}}(F)$ and $\mathcal{I}_{\mathcal{C}}(F)$ are of finite type.

Let $\mathcal{C}$ be a functorially finite subcategory of $\operatorname{mod}\Lambda$ which is closed under extensions. We study some properties of the subfunctor F in $\mathcal{C}$. A subfunctor F' in $\mathcal{C}$ is said to have *enough projectives* if for each C in $\mathcal{C}$ there exists an F'-exact sequence $0 \to C_1 \to P \to C \to 0$ with P in $\mathcal{P}_{\mathcal{C}}(F')$ and C_1 in $\mathcal{C}$. The notion of *enough injectives* is defined dually.

By Lemma 2.1.6, we have that F has enough projectives in $\mathcal{C}$. Moreover, the following lemma shows that $\mathcal{C}$ is closed under kernels of F-*epimorphisms.*

Proposition 2.2.8. *Let* $\mathcal{C}$ *be a functorially finite subcategory which is closed under extensions. Let* $0 \to C_1 \to C_2 \to C_3 \to 0$ *be an* F*-exact sequence with* C_2, C_3 *in* $\mathcal{C}$*, then* C_1 *is in* $\mathcal{C}$*.*

Proof. Let $0 \to C_1 \to C_2 \to C_3 \to 0$ be an F-exact sequence with C_2, C_3 in $\mathcal{C}$. By Lemma 2.1.6, we have a right F-$\mathcal{X}$-approximation $0 \to Y \to X \to C_3 \to 0$ of C_3 with Y in $\mathcal{C}$, where $\mathcal{X}$ is as before. From the following commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & Y \arrow[r, equal] \arrow[d] & Y \arrow[d] & \\
0 \arrow[r] & C_1 \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_1 \arrow[r] & C_2 \arrow[r] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

we have that E is in $\mathcal{C}$, since $\mathcal{C}$ is closed under extensions. The exact sequence $\eta_1\colon 0 \to C_1 \to E \to X \to 0$ is F-exact, since it is a pullback of an F-exact sequence. Then we have that the sequence η_1 splits, since X in $\mathcal{P}(F)$. Hence $E \simeq C_1 \oplus X$. Since $\mathcal{C}$ is closed under extensions and direct summands, it follows that C_1 is contained in $\mathcal{C}$. $\square$

Now, let us consider the subfunctor $F^{\mathcal{I}_\mathcal{C}(F)}$ given by $\mathcal{I}_\mathcal{C}(F)$. Let M be a Λ-module with a surjective $\mathcal{C}$-approximation. Then we have the F-exact sequence $\eta\colon 0 \to Y_M \xrightarrow{g} C_M \to M \to 0$. If Y_M is in $\mathcal{C}$, then it is in $\mathcal{I}_\mathcal{C}(F)$ since $\mathcal{I}(\mathcal{C})$ is contained in $\mathcal{I}_\mathcal{C}(F)$. Assume Y_M is nonzero, then the identity map 1_{Y_M} does not factor through g. Therefore η is not $F^{\mathcal{I}_\mathcal{C}(F)}$-exact.

Dually, given N in $\operatorname{mod}\Lambda$, the exact sequence $0 \to N \to C^N \to Z^N \to 0$ is not F-exact whenever X^N is a nonzero Λ-module in $\mathcal{C}$. So outside $\mathcal{C}$ we may not have $F = F^{\mathcal{I}_\mathcal{C}(F)}$. But inside $\mathcal{C}$ we have the following result.

Corollary 2.2.9. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions. Then $F|_\mathcal{C} = F^{\mathcal{I}_\mathcal{C}(F)}|_\mathcal{C}$.*

We have that the subfunctor F has enough projectives by Lemma 2.1.6. The following result shows that F has enough injectives under certain conditions.

Corollary 2.2.10. *If $\mathcal{I}_\mathcal{C}(F)$ is covariantly finite in $\mathcal{C}$, then F has enough injectives.*

Proof. Suppose $\mathcal{I}_\mathcal{C}(F)$ is covariantly finite in $\mathcal{C}$. Since $\mathcal{I}_\mathcal{C}(F)$ is a cogenerator for $\mathcal{C}$, for each C in $\mathcal{C}$ there is, by Lemma 2.1.7, an exact sequence $\eta\colon 0 \to C \to I \to C^1 \to 0$ with I in $\mathcal{I}_\mathcal{C}(F)$ and C^1 in $\mathcal{C}$ such that $0 \to (C^1, \mathcal{I}_\mathcal{C}(F)) \to (I, \mathcal{I}_\mathcal{C}(F)) \to (C, \mathcal{I}_\mathcal{C}(F)) \to 0$ is exact. Hence the sequence η is $F^{\mathcal{I}_\mathcal{C}(F)}$-exact. Then by Corollary 2.2.9 it follows that η is F-exact, since it is in $\mathcal{C}$. Thus F has enough injectives. $\square$

Suppose $\mathcal{I}_\mathcal{C}(F)$ is covariantly finite in $\mathcal{C}$. Then the following lemma, which is a 'dual' of Lemma 2.2.8, shows that $\mathcal{C}$ is closed under cokernels of *$F^{\mathcal{I}_\mathcal{C}(F)}$-monomorphisms.*

Proposition 2.2.11. *Let* $0 \to C_1 \to C_2 \to C_3 \to 0$ *be an* $F^{\mathcal{I}_{\mathcal{C}}(F)}$*-exact with* C_1, C_2 *in* $\mathcal{C}$. *Assume* $\mathcal{I}_{\mathcal{C}}(F)$ *is covariantly finite in* $\mathcal{C}$. *Then* C_3 *is in* $\mathcal{C}$.

2.3. Relative (co)resolving in Subcategories

Let $\mathcal{C}$ be a functorially finite subcategory of $\operatorname{mod}\Lambda$ which is closed under extensions. Let F be a subfunctor in $\mathcal{C}$. In this section we study F-(co)resolving subcategories in $\mathcal{C}$. We also give some of the preliminary results needed for studying F-(co)tilting in $\mathcal{C}$.

This section is mostly devoted to showing that the results about F-(co)resolving subcategories in $\operatorname{mod}\Lambda$ hold for F-(co)resolving subcategories in $\mathcal{C}$. First we give some definitions.

A subcategory $\mathcal{J}$ of $\mathcal{C}$ is said to be closed under F*-extensions* in $\mathcal{C}$ if for each F-exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$ with A and C in $\mathcal{J}$, we have that B is in $\mathcal{J}$.

Let $\mathcal{Z}$ be a subcategory of $\mathcal{C}$. A (minimal) right F*-*$\mathcal{Z}$*-approximation* of a module C in $\mathcal{C}$ is an F-exact sequence $\eta\colon 0 \to Y \to Z \to C \to 0$ where η is a (minimal) right $\mathcal{Z}$-approximation of C.

A subcategory $\mathcal{Z}$ is said to be F*-contravariantly* finite if every C in $\mathcal{C}$ has a right F-$\mathcal{Z}$-approximation. Dually, one defines (minimal) left F*-*$\mathcal{Z}$*-approximation* and an F*-covariantly* finite subcategory.

We state the following important result which is an analog of [**5**, Proposition 1.4].

Lemma 2.3.1. *Suppose* $\mathcal{Y}$ *is a subcategory of* $\mathcal{C}$ *which is closed under* F*-extensions and let* $\mathcal{Z} = \{Z \in \mathcal{C} \mid \operatorname{Ext}^1_F(Z, \mathcal{Y}) = 0$. *Then the following are equivalent for* C *in* $\mathcal{C}$.

(a) *The functor* $\operatorname{Ext}^1_F(C, \ )\,|_{\mathcal{Y}}\colon \mathcal{Y} \to \mathrm{Ab}$ *is finitely generated.*
(b) *There exists a minimal right* F*-*$\mathcal{Z}$*-approximation* $0 \to Y \to Z \to C \to 0$ *with* Y *in* $\mathcal{Y}$.

Proof. The proof is similar to the case of $\operatorname{mod}\Lambda$. The only difference is that we require that $\mathcal{C}$ is closed under F-exact sequences. Since $\mathcal{C}$ is closed under extensions, it is indeed closed under F-extensions. The result will then follow. □

Here is another result which is an analog of [**5**, Proposition 1.8]. The proof follows easily from Lemma 2.3.1.

Proposition 2.3.2. *Let* $\mathcal{J}$ *be a subcategory of* $\mathcal{C}$ *which is closed under* F*-extensions and such that* $\operatorname{Ext}^1_F(C, \ )\,|_{\mathcal{J}}$ *is finitely generated for all* C *in* $\mathcal{C}$. *Then we have the following:*

(a) *The subcategory* $\mathcal{Z} = \{Z \in \mathcal{C} \mid \mathrm{Ext}^1_F(Z, \mathcal{J}) = 0\}$ *is* F*-contravariantly in* $\mathcal{C}$*, closed under* F*-extensions and contains* $\mathcal{P}_{\mathcal{C}}(F)$*.*

(b) *The subcategory* $\mathcal{Y} = \{Y \in \mathcal{C} \mid \mathrm{Ext}^1_F(\mathcal{Z}, Y) = 0\}$ *is* F*-covariantly finite in* $\mathcal{C}$*, closed under* F*-extensions and contains* $\mathcal{I}_{\mathcal{C}}(F)$*.*

The following is another result, also an analog of [**5**, Dual of Proposition 1.10].

Proposition 2.3.3. *Suppose* $\mathcal{Y}$ *is a covariantly finite subcategory of* $\mathcal{C}$ *which is closed under* F*-extensions and contains* $\mathcal{I}_{\mathcal{C}}(F)$*. Let* $\mathcal{Z} = \{Z \in \mathcal{C} \mid \mathrm{Ext}^1_F(Z, \mathcal{Y}) = 0\}$*. Then* $\mathcal{Y} = \{C \in \mathcal{C} \mid \mathrm{Ext}^1_F(\mathcal{Z}, C) = 0\}$*.*

Proof. By definition $\mathcal{Y}$ is contained in $\{C \in \mathcal{C} \mid \mathrm{Ext}^1_F(\mathcal{Z}, C) = 0\}$. Suppose $\mathrm{Ext}^1_F(\mathcal{Z}, C) = 0$. Since $\mathcal{Y}$ is covariantly finite in $\mathcal{C}$ and contains $\mathcal{I}_{\mathcal{C}}(F)$, there is a minimal left F-$\mathcal{Z}$-approximation $0 \to C \to Y^C \to Z^C \to 0$ of C. Since $\mathcal{Y}$ is also closed under F-extensions, by Wakamatsu's lemma [**33**] we have that Z^C is in $\mathcal{Z}$. Hence the sequence $0 \to C \to Y^C \to Z^C \to 0$ splits, so that C is a summand of Y^C and the result follows since $\mathcal{Y}$ is closed under summands. □

A subcategory $\mathcal{Z}$ of $\mathcal{C}$ is said to be F*-resolving* in $\mathcal{C}$ if it satisfies the conditions (a) it is closed under F-extensions, (b) if $0 \to A \to B \to C \to 0$ is F-exact and B and C are in $\mathcal{Z}$, then A is in $\mathcal{Z}$ and (c) it contains $\mathcal{P}_{\mathcal{C}}(F)$. Let $\mathcal{A}$ be a subcategory of $\mathcal{C}$, then ${}^{\perp}\mathcal{A}$ is the full subcategory of $\mathcal{C}$ defined as follows:

$$ {}^{\perp}\mathcal{A} = \{Z \in \mathcal{C} \mid \mathrm{Ext}^i_F(Z, \mathcal{A}) = 0 \text{ for all } i > 0\}. $$

It is easy to check that ${}^{\perp}\mathcal{A}$ is F-resolving in $\mathcal{C}$ for any subcategory $\mathcal{A}$ of $\mathcal{C}$. Dually, one defines F*-coresolving* in $\mathcal{C}$ and $\mathcal{A}^{\perp}$, which is F-coresolving for a subcategory $\mathcal{A}$ of $\mathcal{C}$.

We state a couple of results which are analogs of [**5**, Proposition 3.1] and [**5**, Dual of Proposition 3.3].

Proposition 2.3.4. *Suppose* $\mathcal{Y}$ *is an* F*-coresolving subcategory of* $\mathcal{C}$*. Let* $\mathcal{Z} = \{Z \in \mathcal{C} \mid \mathrm{Ext}^1_F(Z, \mathcal{Y}) = 0\}$*. Then*

(a) $\mathcal{Z} = {}^{\perp}\mathcal{Y}$*.*

(b) $\mathcal{Z}$ *is an* F*-resolving subcategory of* $\mathcal{C}$*.*

Proof. Same as [**11**, Lemma 2.1] □

Proposition 2.3.5. *Let* $\mathcal{Y}$ *be an* F*-coresolving* F*-covariantly finite subcategory of* $\mathcal{C}$*. Then*

(a) $\mathcal{Z} = {}^{\perp}\mathcal{Y}$ *is an* F*-resolving* F*-contravariantly finite subcategory of* $\mathcal{C}$*.*

(b) $\mathcal{Y} = \mathcal{Z}^{\perp} = ({}^{\perp}\mathcal{Y})^{\perp}$*.*

(c) *Every* C *in* $\mathcal{C}$ *has a minimal left* F*-*$\mathcal{Y}$*-approximation* $0 \to C \to Y^C \to Z^C \to 0$ *with* Z^C *in* $\mathcal{Z}$*.*

(d) *Every C in $\mathcal{C}$ has minimal right F-$\mathcal{Z}$-approximation* $0 \to Y_C \to Z_C \to C \to 0$ *with* Y_C *in* $\mathcal{Y}$.

Proof. (a) Since $\mathcal{Y}$ is F-covariantly finite in $\mathcal{C}$, we have that $\mathcal{Z} = \{Z \in \mathcal{C} \mid \operatorname{Ext}^1_F(Z, \mathcal{Y}) = 0\}$ is F-contravariantly finite in $\mathcal{C}$ (Proposition 2.3.2). By Proposition 2.3.4 we have that $\mathcal{Z} = {}^{\perp}\mathcal{Y}$.

(b) Since $\mathcal{Y}$ is F-covariantly finite, closed under F-extensions and contains $\mathcal{I}_{\mathcal{C}}(F)$, we have by Proposition 2.3.3 that $\mathcal{Y} = \{Y \in \mathcal{C} \mid \operatorname{Ext}^1_F(\mathcal{Z}, Y) = 0\}$. By the dual of Proposition 2.3.4 we have $\mathcal{Y} = \mathcal{Z}^{\perp}$, since $\mathcal{Z}$ is F-resolving.

(c) Follows from Lemma 2.3.1.

(d) Dual of (c). □

Now, we want to generalize the notion of generator defined earlier to subcategories of $\mathcal{C}$. Let $\mathcal{Z}$ be a subcategory of $\mathcal{C}$. Then a subcategory ω in $\mathcal{Z}$ is called *F-generator* for $\mathcal{Z}$ if for each Z in $\mathcal{Z}$ there is an F-exact sequence $0 \to Z_1 \to W \to Z \to 0$ with W in ω and Z_1 in $\mathcal{Z}$. Dually, one defines *F-cogenerator* for $\mathcal{Z}$.

Let $\mathcal{Y}$ be F-covariantly finite F-coresolving in $\mathcal{C}$. Then the *F-coresolution dimension* of a Λ-module C with respect to $\mathcal{Y}$ is defined to be the minimum of all n including infinity such that there exists an F-exact sequence

$$0 \to C \to Y^0 \to Y^1 \to \cdots \to Y^{n-1} \to Y^n \to 0$$

where the Y^i are in $\mathcal{Y}$. We denote this dimension by $\mathcal{Y}\text{-coresdim}_F M$. If $\mathcal{W}$ is a subcategory of $\operatorname{mod}\Lambda$, then $\mathcal{Y}\text{-}\operatorname{coresdim}_F(\mathcal{W})$ is defined to be $\sup\{\mathcal{Y}\text{-}\operatorname{coresdim}_F Z \mid Z \in \mathcal{W}\}$.

We state the following result which we will use to prove the next proposition. This is an analog of [**3**, Dual of Theorem 1.1].

Proposition 2.3.6. *Let $\mathcal{U}$ be an F-extension closed subcategory of $\mathcal{C}$ and $\omega \subseteq \mathcal{U}$ an F-generator for $\mathcal{U}$. Then for each C in $\check{\mathcal{U}}$ there is an F-exact sequence* $0 \to C \to U^C \to Z^C \to 0$ *with U^C in $\mathcal{U}$ and Z^C in $\check{\omega}$.*

Proof. We prove this using induction on $\mathcal{U}\text{-coresdim}_F C = n$. Let $0 \to C \to U^0 \xrightarrow{d^1} \cdots \xrightarrow{d^n} U^n \to 0$ an F-exact sequence with each U^i in $\mathcal{U}$. If $n = 0$, our desired sequence is $0 \to C \xrightarrow{1_C} C \to 0 \to 0$.

For $n = 1$, we have the following commutative F-exact diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & U_1 \arrow[r, equal] \arrow[d] & U_1 \arrow[d] & \\
0 \arrow[r] & C \arrow[r] \arrow[d, equal] & E_1 \arrow[r] \arrow[d] & W_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C \arrow[r] & U^0 \arrow[r] \arrow[d] & U^1 \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

with W_1 in ω and U_1 in $\mathcal{U}$. Since $\mathcal{U}$ is closed under F-extensions, E_1 in $\mathcal{U}$. Then our desired F-exact sequence is $0 \to C \to E_1 \to W_1 \to 0$. Moreover, we get the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & U_0 \arrow[r, equal] \arrow[d] & U_0 \arrow[d] & & \\
0 \arrow[r] & E_0 \arrow[r] \arrow[d] & W_0 \arrow[r] \arrow[d] & W_1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d] & W_1 \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

with W_0 in ω and U_0 in $\mathcal{U}$. The sequence $0 \to E_0 \to W_0 \to W_1 \to 0$ is F-exact by [**11**, Theorem 1.4]. Hence we have an F-exact sequence $0 \to U_0 \to E_0 \to C \to 0$ with E_0 in $\breve{\omega}$ and U_0 in $\mathcal{U}$.

Now suppose $n > 0$ and set $\operatorname{Im} d^1 = K$, then we have the F-exact sequences $0 \to C \to U^0 \to K \to 0$ and $0 \to K \to U^1 \to \cdots \to U^n \to 0$. By induction there is an F-exact sequence $0 \to U^K \to W^K \to K \to 0$ with W^K in $\breve{\omega}$ and U^K in $\mathcal{U}$. Then we have the following commutative F-exact diagram.

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & U^K \arrow[r, equal] \arrow[d] & U^K \arrow[d] & \\
0 \arrow[r] & C \arrow[r] \arrow[d, equal] & U^0_K \arrow[r] \arrow[d] & W^K \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C \arrow[r] & U^0 \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since $\mathcal{U}$ is closed under F-extensions, U^0_K is in $\mathcal{U}$. Then we can choose $0 \to C \to U^0_K \to W^K \to 0$ as our desired F-exact sequence. □

Let $\mathcal{Y}$ be a subcategory of $\mathcal{C}$. A subcategory $\mathcal{W}$ is an Ext_F*-projective generator* of $\mathcal{Y}$ if (i) $\mathcal{W}$ is contained in $\mathcal{Y}$, (ii) $\mathcal{W}$ is contained in ${}^{\perp}\mathcal{Y}$ and (iii) for every Y in $\mathcal{Y}$ there exists an F-exact sequence $0 \to Y' \to W \to Y \to 0$ in $\mathcal{Y}$ with W in $\mathcal{W}$.

The following result is an analog of [**11**, Theorem 2.4].

PROPOSITION 2.3.7. *Let* $\mathcal{Y}$ *be an* F*-coresolving subcategory of* $\mathcal{C}$ *with* Ext_F*-projective generator* ω*. If* $\check{\mathcal{Y}} = \mathcal{C}$*, we have the following.*

(a) *The subcategory* $\mathcal{Y}$ *is* F*-covariantly finite in* $\mathcal{C}$*.*
(b) ${}^{\perp}\mathcal{Y} \cap \mathcal{C} = \check{\omega} \cap \mathcal{C}$*.*

PROOF. (a) Since $\check{\mathcal{Y}} = \mathcal{C}$, we have an F-exact sequence $0 \to C \xrightarrow{g} Y^C \to Z^C \to 0$ for all C in $\mathcal{C}$ with Y^C in $\mathcal{C}$ and Z^C in $\check{\omega}$ by Proposition 2.3.6. Since $\check{\omega}$ is in ${}^{\perp}\mathcal{Y}$, we have that $\operatorname{Ext}^i_F(Z^C, \mathcal{Y}) = 0$ for all $i > 0$. Hence g is a left $\mathcal{Y}$-approximation, by [**3**, Theorem 2.3].

(b) By the dual of [**3**, Proposition 3.6], we have that ${}^{\perp}\mathcal{Y} = \check{\omega}$. Then the result follows. ☐

2.4. Approximation Dimension

Let $\mathcal{C}$ be a subcategory of $\operatorname{mod}\Lambda$. In this section we define $\mathcal{C}$-approximation dimension. Then we characterize subcategories $\mathcal{C}$ with $\mathcal{C}$-approximation equal to zero. Suppose the $\mathcal{C}$-approximation dimension of $\operatorname{mod}\Lambda$ is finite. Then we show that any long relative exact sequence in $\operatorname{mod}\Lambda$ with all the middle terms in $\mathcal{C}$ is eventually in $\mathcal{C}$. This will be used to prove the main results in the next chapter.

Let $\mathcal{C}$ be a contravariantly finite subcategory of $\operatorname{mod}\Lambda$. For any M in $\operatorname{mod}\Lambda$, consider a succession $0 \to Y_1 \to C_0 \xrightarrow{g_0} M$, $0 \to Y_2 \to C_0 \xrightarrow{g_1} Y_1$, ... of minimal right $\mathcal{C}$-approximations. Then, the complex

$$(*) \qquad \cdots \to C_t \xrightarrow{g_t} C_{t-1} \to \cdots \to C_1 \xrightarrow{g_1} C_0 \xrightarrow{g_0} M$$

is called a right $\mathcal{C}$*-approximation resolution* of M. In [**23**], this was defined in general for a contravariantly finite subcategory $\mathcal{C}$ in a additive category $\mathcal{C}'$ with kernels and cokernels. There the right $\mathcal{C}$-approximation resolution was called right $\mathcal{C}$-resolution.

Let us denote the $\operatorname{Ker} g_i$ in $(*)$ by Y_{i+1}. We write $r\mathcal{C}\text{-}\operatorname{app.dim}(M) = n$ if there exists a smallest nonnegative integer n in the right $\mathcal{C}$-approximation resolution of M, such that $Y_{n+1} = 0$. If no such integer exists, we write $r\mathcal{C}\text{-}\operatorname{app.dim}(M) = \infty$. We call $r\mathcal{C}\text{-}\operatorname{app.dim}(M)$ the right $\mathcal{C}$*-approximation dimension* of M. Then for $\operatorname{mod}\Lambda$ we have

$$r\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda) = \sup\{r\mathcal{C}\text{-}\operatorname{app.dim}(M) \mid M \in \operatorname{mod}\Lambda\}.$$

Example 2.4.1. If $\mathcal{C}$ is closed under factor modules, then it is known that every right $\mathcal{C}$-approximation is a monomorphism [**7**, Proposition 4.8]. Hence

$$r\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda) = 0.$$

Dually, one can define left *$\mathcal{C}$-approximation resolution* of M, left *$\mathcal{C}$-approximation dimension* of $\operatorname{mod}\Lambda$, denoted by $l\mathcal{C}$-app. dim(mod Λ), for a covariantly finite subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$. We have the following proposition relating the two approximation dimensions when $\mathcal{C}$ is of finite type.

Proposition 2.4.2. *Let $\mathcal{C}$ be a subcategory of finite type in* $\operatorname{mod}\Lambda$. *Then* $r\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda)$ *is finite if and only if* $l\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda)$ *is finite. Moreover, in this case they differ by at most* 2.

Proof. Assume $r\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda)$ is finite. Let $C = \oplus_{i=1}^{t} C_i$ for all isomorphism classes of indecomposable Λ-modules C_i in $\mathcal{C}$ and let $\Sigma = \operatorname{End}_\Lambda(C)^{\mathrm{op}}$. Since $r\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda)$ is finite, we have that

$$\operatorname{gl.dim}(\Sigma) \leqslant r\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda) + 2.$$

Now, Let $r\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda) = n$ and consider a left $\mathcal{C}$-approximation resolution

$$M \to C^0 \to C^1 \to \cdots \to C^n \to C^{n+1} \to \cdots$$

of M. Applying $\operatorname{Hom}_\Lambda(\ ,C)$ to the above sequence, we get a projective resolution of $\operatorname{Hom}_\Lambda(M,C)$ over Σ^{op}. But since $\operatorname{gl.dim}\Sigma^{\mathrm{op}} \leqslant n+2$, we have that $\operatorname{Hom}_\Lambda(M^j, C) = 0$ for $j > n+2$, so that any left $\mathcal{C}$-approximation of M^j is 0 for $j > n+2$. Hence $l\mathcal{C}\text{-}\operatorname{app.dim}(M)$ is at most $n+2$, which is finite.

The other implication is dual. □

Remark: Proposition 2.4.2 holds if $\mathcal{C}$ is a functorially finite subcategory of $\operatorname{mod}\Lambda$ [**23**, Section 1].

Let $\mathcal{C}$ be a functorially finite subcategory of $\operatorname{mod}\Lambda$. The $\mathcal{C}$-approximation dimension of $\operatorname{mod}\Lambda$, $\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda)$, is defined to be

$$\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda) = \max\{l\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda), r\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda)\}.$$

The following is a nice corollary of Proposition 2.4.2.

Corollary 2.4.3. *Let $\mathcal{C}$ be a subcategory of* $\operatorname{mod}\Lambda$ *which is closed under factor modules. Then* $\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda) \leqslant 2$.

Proof. Follows from Example 2.4.1 and Proposition 2.4.2. □

Let $\mathcal{C}$ be equal to $\operatorname{mod}\Lambda$. Then $\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda) = 0$. However, $\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda)$ being zero does not necessarily mean that $\mathcal{C} = \operatorname{mod}\Lambda$, as shown below.

Corollary 2.4.4. *Let $\mathcal{C}$ be a subcategory of* $\operatorname{mod}\Lambda$ *which is closed under submodules and factor modules. Then*

$$\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda) = 0.$$

In general, $\mathcal{A}$-app. dim$(\mathcal{B})$ can be defined, where $\mathcal{A}$ is functorially finite subcategory of a category $\mathcal{C}'$ with kernels and cokernels [**23**].

2.4.1. Approximation Dimension Zero

In this section we want to characterize functorially finite subcategories $\mathcal{C}$ with $\mathcal{C}$-approximation dimension zero.

The following result shows that functorially finite subcategories with finite approximation dimension zero are the same as those which are closed under factor modules and submodules.

Proposition 2.4.5. *Let $\mathcal{C}$ be an additive functorially finite subcategory of* $\operatorname{mod}\Lambda$*. Then* $\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda) = 0$ *if and only if $\mathcal{C}$ is closed under factor modules and submodules.*

Proof. Let $\mathcal{C}$ functorially finite subcategory of $\operatorname{mod}\Lambda$ and assume that $\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda) = 0$. We show that $\mathcal{C}$ is closed under factor modules. Let C be in $\mathcal{C}$ and M be a factor module of C. Since $\mathcal{C}$ is functorially finite, we have a right $\mathcal{C}$-approximation $C_M \xrightarrow{g_M} M$ of M. Since $\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda) = 0$, we have that g_M is a monomorphism. Then we have the following commutative diagram

$$\begin{tikzcd}
& C \arrow[dl, dashed] \arrow[d, "l"] & \\
C_M \arrow[r, hook, "g_M"] & M \arrow[r, dashed] \arrow[d] & 0 \\
& 0 &
\end{tikzcd}$$

But since l is an epimorphism we have that g_M is an epimorphism, hence isomorphism. Therefore M is in $\mathcal{C}$. To show that $\mathcal{C}$ is closed under submodules is dual.

The converse follows by Corollary 2.4.4. □

Now we want to characterize subcategories of $\operatorname{mod}\Lambda$ closed under factor modules and submodules. But first we recall a well-known concept.

Let $\mathcal{C}$ be a subcategory of $\operatorname{mod}\Lambda$. Recall that the *annihilator of* $\mathcal{C}$, $\operatorname{ann}_\Lambda \mathcal{C}$ is defined as:

$$\operatorname{ann}_\Lambda \mathcal{C} = \bigcap_{C\in\mathcal{C}} \operatorname{ann}_\Lambda(C)$$

where the annihilator of a module C is defined as:

$$\operatorname{ann}_\Lambda(C) = \{\lambda \in \Lambda \mid \lambda \cdot C = 0\}.$$

It is well-known that $\operatorname{ann}_\Lambda \mathcal{C}$ is an ideal of Λ. Then we have the following useful lemma.

LEMMA 2.4.6. *Let* $\mathcal{C}$ *be an additive subcategory of* $\operatorname{mod}\Lambda$*. Then* $\operatorname{ann}_\Lambda \mathcal{C} = \operatorname{ann}_\Lambda(C)$ *for some* C *in* $\mathcal{C}$*. Moreover, there is an embedding* $\Lambda/I \hookrightarrow C$*, where* $I = \operatorname{ann}_\Lambda(\mathcal{C})$*.*

PROOF. Consider Λ-modules C_1 and C_2 in $\mathcal{C}$ and denote $\operatorname{ann}_\Lambda(C_1)$ by I_1. Then we have that $\operatorname{ann}_\Lambda(C_1 \oplus C_2) = I_2$ is contained I_1. Continuing with this process we get a descending chain $\cdots \subseteq I_{n+1} \subseteq I_n \subseteq \cdots \subseteq I_2 \subseteq I_1$ of ideals in Λ with $I_n = \operatorname{ann}_\Lambda(C_1 \oplus \cdots \oplus C_n)$. But since Λ is artin, there exist a nonnegative integer t such that $I_t = I_{t+1} = I_{t+2} = \cdots$ regardless of what one adds to $C_1 \oplus \cdots \oplus C_t$. Then we have that $\operatorname{ann}_\Lambda \mathcal{C} = \operatorname{ann}_\Lambda(C) = I_t$, where $C = C_1 \oplus \cdots \oplus C_t$.

Now, choose elements $c_1, \ldots, c_t$ in C such that

$$\bigcap_{i=1}^{t} \operatorname{ann}_\Lambda(c_i) = \operatorname{ann}_\Lambda(C)$$

and define a Λ-morphism $f\colon \Lambda \to C$ by $f(1) = (c_1, \ldots, c_t)$. Then it is easy to see that $\operatorname{Ker} f = I$. Hence the result follows. $\square$

The following result shows that the subcategories of $\operatorname{mod}\Lambda$ which are closed under submodules and factor modules are abelian.

PROPOSITION 2.4.7. *Let* $\mathcal{C}$ *be an additive subcategory of* $\operatorname{mod}\Lambda$ *which is closed under factor modules and submodules. Then* $\mathcal{C}$ *is equivalent to* $\operatorname{mod}\Lambda/I$*, where* $I = \operatorname{ann}_\Lambda \mathcal{C}$*.*

PROOF. If C is in $\mathcal{C}$, then $IC = 0$, so C is in $\operatorname{mod}\Lambda/I$.

Now, let M be in $\operatorname{mod}\Lambda/I$. By Corollary 2.4.6 we have an embedding $\Lambda/I \hookrightarrow C$ for some C in $\mathcal{C}$. Hence Λ/I in in $\mathcal{C}$ since $\mathcal{C}$ is closed under submodules. But since $\mathcal{C}$ is closed under factor modules, we have that M is in $\mathcal{C}$. $\square$

Let $\mathcal{C}$ and I be as before and consider the algebra morphism $\varphi\colon \Lambda \to \Lambda/I$. Then φ induces an exact functor $G_\varphi\colon \operatorname{mod}(\Lambda/I) \to \operatorname{mod}\Lambda$, which is an embedding. We have that $\operatorname{Im} G_\varphi = \mathcal{C}$. It is easy to see that G_φ and its inverse preserve exact sequences and exact diagrams. Hence they preserve pushouts, pullbacks and Baer sums. Since these (pushouts pullbacks and Baer sums) determine subfunctors, it follows that G_φ and its inverse preserve subfunctors too. Hence $\mathcal{C}$ and $\operatorname{mod}(\Lambda/I)$ have the same relative theory.

Note that the factor category $\operatorname{mod}\Lambda/I$, in Proposition 2.4.7, is not necessarily closed under extensions in $\operatorname{mod}\Lambda$ [**4**]. However, if $\mathcal{C}$ is closed under extensions, then $\operatorname{mod}\Lambda/I$ is also closed under extensions in $\operatorname{mod}\Lambda$ (by using the functor G_φ above).

Now, we combine Proposition 2.4.5 and 2.4.7 to get the following crucial result for subcategories $\mathcal{C}$ with $\mathcal{C}$- app. dim(mod Λ) $= 0$.

Corollary 2.4.8. *Let $\mathcal{C}$ be an additive functorially finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions and assume* $\mathcal{C}\text{- app.}\dim(\operatorname{mod}\Lambda) = 0$. *Then $\mathcal{C}$ is canonically equivalent to* $\operatorname{mod}\Sigma$, *where Σ is a quotient algebra of Λ. Moreover,* $\operatorname{mod}\Sigma$ *inherits the relative theory in $\mathcal{C}$ and vice versa.*

2.4.2. Approximation Dimension $n > 0$

Let $\mathcal{C}$ be a functorially finite subcategory of $\operatorname{mod}\Lambda$ which is closed under extensions. Let $\mathcal{X}$ be a contravariantly finite subcategory of $\mathcal{C}$ which is a generator for $\mathcal{C}$. Consider the subfunctor $F = F_{\mathcal{X}}$ in $\mathcal{C}$. In this subsection we look at some relationship between $\mathcal{C}$ and $\operatorname{mod}\Lambda$ which will be useful later. We show that any long F-exact sequence in $\operatorname{mod}\Lambda$ with the middle terms in $\mathcal{C}$ is eventually in $\mathcal{C}$.

Let $\mathcal{T}$ be a subcategory of $\mathcal{C}$ and let M be in $\operatorname{mod}\Lambda$. Suppose M has an F-$\mathcal{T}$-resolution. The following result will help us to find right $\mathcal{C}$-approximations of all the $\mathcal{T}$-syzygies of the $\mathcal{T}$-resolution of M. Moreover, the result establishes a nice relationship between kernels of the right $\mathcal{C}$-approximations of all the $\mathcal{T}$-syzygies of the $\mathcal{T}$-resolution of M and the 'syzygies' of the right $\mathcal{C}$-approximation resolution of M. This relationship will be used to prove the next result.

Lemma 2.4.9. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions. Consider a minimal right $\mathcal{C}$-approximation resolution*

$$\cdots \to C_{i+s+1} \xrightarrow{g_{i+s+1}} C_{i+s} \to \cdots \to C_{i+1} \xrightarrow{g_{i+1}} C_i \xrightarrow{g_i} M_i$$

of M_i for some $i \geqslant 0$. Denote $\operatorname{Ker} g_{i+j}$ *by Y_{i+j+1} for $j \geqslant 0$ and let $M_i = Y_i$. Let $0 \to M_{i+j+1} \to T_{i+j} \to M_{i+j} \to 0$ be an F-exact sequence with T_{i+j} in $\mathcal{C}$ for $j \geqslant 0$. Then there is a right $\mathcal{C}$-approximation $0 \to Y'_{i+j+1} \to C'_{i+j} \to M_{i+j}$ with $Y_{i+j+1} = Y'_{i+j+1}$ for $j \geqslant 0$.*

Proof. We prove this by induction on j. For $j = 0$, we have $M_i = Y_i$, so $Y_{i+1} = Y'_{i+1}$.

For $j = 1$, consider the following commutative F-exact diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & & M_{i+1} \arrow[r, equal] \arrow[d] & M_{i+1} \arrow[d] & \\
 & 0 \arrow[r] & Y_{i+1} \arrow[r] \arrow[d, equal] & Y_{i+1} \oplus T_i \arrow[r] \arrow[d, "\alpha"] & T_i \arrow[r] \arrow[d] & 0 \\
\theta_1\colon & 0 \arrow[r] & Y_{i+1} \arrow[r] & C_i \arrow[r] \arrow[d] & M_i \arrow[r] \arrow[d] & 0 \\
 & & & 0 & 0 &
\end{tikzcd}$$

(1)

and let $X \xrightarrow{p} C_i$ be an epimorphism with X in $\mathcal{X}$. Since $0 \to M_{i+1} \to Y_{i+1} \oplus T_i \xrightarrow{\alpha} C_i \to 0$ is F-exact, we have that p factors through α. Moreover, since $\eta\colon 0 \to Y_{i+2} \to C_{i+1} \oplus T_i \xrightarrow{\begin{pmatrix} g_{i+1} & 1_{T_i} \end{pmatrix}} Y_{i+1} \oplus T_i$ is a right $\mathcal{C}$-approximation of $Y_{i+1} \oplus T_i$, we have that p factors through $f = \alpha \circ \begin{pmatrix} g_i & 1_{T_i} \end{pmatrix}$. Hence f is onto, since p is onto. Then we use the F-exact sequence $0 \to M_{i+1} \to Y_{i+1} \oplus T_i \xrightarrow{\alpha} C_i \to 0$ to construct the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & Y_{i+2} \arrow[r, equal] \arrow[d] & Y_{i+2} \arrow[d] & & \\
0 \arrow[r] & C'_{i+1} \arrow[r] \arrow[d, "g'_{i+1}"] & C_{i+1} \oplus T_i \arrow[r, "f"] \arrow[d, "\begin{pmatrix} g_i & 1_{T_i} \end{pmatrix}"] & C_i \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M_{i+1} \arrow[r, "\delta"] \arrow[d] & Y_{i+1} \oplus T_i \arrow[r, "\alpha"] \arrow[d] & C_i \arrow[r] & 0 \\
 & N \arrow[r, equal] \arrow[d] & N \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

By the earlier discussion, we have that the exact sequence $0 \to C'_{i+1} \to C_{i+1} \oplus T_i \xrightarrow{f} C_i \to 0$ is F-exact. Then by Proposition 2.2.8, C'_{i+1} is in $\mathcal{C}$.

Our aim is to show that $\theta_2\colon 0 \to Y_{i+2} \to C'_{i+1} \xrightarrow{g'_{i+1}} M_{i+1}$ is a right $\mathcal{C}$-approximation of M_{i+1}. If C'_{i+1} were a pullback of δ and $\begin{pmatrix} g_i & 1_{T_i} \end{pmatrix}$, then by the universal property of pullbacks, θ_2 would be a right $\mathcal{C}$-approximation, since η is a right $\mathcal{C}$-approximation of $Y_{i+1} \oplus T_i$. But by Lemma 2.1.8, C'_{i+1} is indeed a pullback of δ and $\begin{pmatrix} g_i & 1_{T_i} \end{pmatrix}$. Hence the sequence θ_2 is a right $\mathcal{C}$-approximation, and we have $Y'_{i+2} = Y_{i+2}$.

For $j > 1$ we replace the sequence θ_1 in (1) by θ_j and continue from there. Then the result will follow by induction. $\square$

The following result, which is a consequence of Lemma 2.4.9, shows that any long F-exact sequence in $\operatorname{mod}\Lambda$ with the middle terms in $\mathcal{C}$ is eventually in $\mathcal{C}$.

Corollary 2.4.10. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions. Assume $\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda) = n < \infty$. Fix an integer $t \geqslant 0$, and let $0 \to M_{i+1} \to T_i \to M_i \to 0$ be F-exact in* $\operatorname{mod}\Lambda$ *with T_i in $\mathcal{C}$ for $i \geqslant t$. Then M_{t+n} is in $\mathcal{C}$. In general, M_i is in $\mathcal{C}$ for $i \geqslant t+n$.*

Proof. By Lemma 2.4.9 we have the following commutative exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & C'_{t+n} \arrow[r] \arrow[d, "g'_{t+n}"] & C_{i+n} \oplus T_{t+n-1} \arrow[r] \arrow[d] & C'_{t+n-1} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M_{t+n} \arrow[r] & Y_{t+n} \oplus T_{t+n-1} \arrow[r] & C'_{t+n-1} \arrow[r] & 0
\end{tikzcd}$$

where g'_{t+n} is a right $\mathcal{C}$-approximation of M_{t+n}. Since T_{t+n} maps onto M_{t+n}, we have that g'_{t+n} is an epimorphism, and hence an isomorphism. Therefore M_{t+n} is in $\mathcal{C}$. Then by Lemma 2.2.8 M_i is in $\mathcal{C}$ for all $i \geqslant t+n$. □

Chapter 3

Relative Tilting, Approximation and Global Dimensions

Let Λ be an artin algebra and let $\operatorname{mod}\Lambda$ denote the category of finitely generated left Λ-modules. In [**11**] a general theory of relative (co)tilting modules in $\operatorname{mod}\Lambda$ was developed. In this chapter we shall develop a relative tilting theory in subcategories of $\operatorname{mod}\Lambda$.

Let $\mathcal{C}$ be a functorially finite subcategory of $\operatorname{mod}\Lambda$ which is closed under extensions. Suppose the $\mathcal{C}$-approximation dimension of $\operatorname{mod}\Lambda$ is zero. Then it has been shown (in Section 2.4) that $\mathcal{C}$ is equivalent to a module category over an artin algebra. This means that a "relative (co)tilting theory in $\mathcal{C}$" can be developed. Refer to this theory as the relative (co)tilting theory of dimension "0" in $\mathcal{C}$. In this chapter we develop a relative (co)tilting theory of dimension "n" in $\mathcal{C}$, where n denotes the $\mathcal{C}$-approximation dimension of $\operatorname{mod}\Lambda$.

Let $\mathcal{X}$ be a contravariantly finite subcategory of $\mathcal{C}$ and consider the subfunctor $F = F_{\mathcal{X}}$ in $\mathcal{C}$. In Section 3.1 we define the notion of relative tilting in $\mathcal{C}$. Suppose T is an F-tilting module in $\mathcal{C}$ and denote $\operatorname{End}_\Lambda(T)^{\text{op}}$ by Γ. Then we state some preliminary results, which are generalizations of some results in [**11**]. Then we prove one of the main results of this chapter. The result shows that the tilting functor $\operatorname{Hom}_\Lambda(T,\ )$ induces an equivalence between subcategories $T_{\mathcal{C}}^{\perp}$ of $\mathcal{C}$ and $\operatorname{Hom}_\Lambda(T, T_{\mathcal{C}}^{\perp})$ of $\operatorname{mod}\Gamma$, where $T_{\mathcal{C}}^{\perp}$ denotes the subcategory $T^{\perp} \cap \mathcal{C}$. We also show that if there exists a relative tilting module in $\mathcal{C}$, then the category of F-projective modules in $\mathcal{C}$ is of finite type.

In Section 3.2 we state more preliminary results, which also generalize some results in [**11**]. We then prove the other main result of this chapter. The result states that if the $\mathcal{C}$-approximation dimension of $\operatorname{mod}\Lambda$ is finite, then the image of the tilting functor restricted to $T_{\mathcal{C}}^{\perp}$ is identified with a subcategory ${}^{\perp}T_{\mathcal{C}}^{0}$ of $\operatorname{mod}\Gamma$, where $T_{\mathcal{C}}^{0}$ denotes the Γ-module associated to $\operatorname{Hom}_{\Lambda}(T, \mathcal{I}_{\mathcal{C}}(F))$. Moreover, we show that the $\operatorname{Hom}_{\Lambda}(T, T_{\mathcal{C}}^{\perp})$-resolution dimension of $\operatorname{mod}\Gamma$ is finite. This will help us to show that the Γ-module $T_{\mathcal{C}}^{0}$ is cotilting.

In the last section we look at the relationship between relative global dimension of $\mathcal{C}$ and the global dimension of Γ.

3.1. Relative Tilting in Subcategories

Let $\mathcal{C}$ be a functorially finite subcategory of $\operatorname{mod}\Lambda$ which is closed under extensions. Let $\mathcal{X}$ be a contravariantly finite subcategory of $\mathcal{C}$ which is a generator for $\mathcal{C}$. Consider the subfunctor $F = F_{\mathcal{X}}$ in $\mathcal{C}$. Suppose T is an F-tilting module in $\mathcal{C}$ and let $\Gamma = \operatorname{End}_{\Lambda}(T)^{\mathrm{op}}$. In this section we will show that the tilting functor $\operatorname{Hom}_{\Lambda}(T, \ ) \colon \operatorname{mod}\Lambda \to \operatorname{mod}\Gamma$ induces an equivalence between the subcategories $T_{\mathcal{C}}^{\perp}$ and $(T, T_{\mathcal{C}}^{\perp})$ of $\operatorname{mod}\Lambda$ and $\operatorname{mod}\Gamma$ respectively. Then we show that $(\mathcal{P}_{\mathcal{C}}(F), T)$ is a tilting Γ^{op}-module and we then use this to show that $\mathcal{P}_{\mathcal{C}}(F)$ is of finite type.

We need that our subfunctor F has enough projectives and injectives. We know that F has enough projectives in $\mathcal{C}$ (since $\mathcal{P}_{\mathcal{C}}(F) = \mathcal{X}$). Suppose $\mathcal{I}_{\mathcal{C}}(F)$ is covariantly finite in $\mathcal{C}$. Then by Corollary 2.2.10 we have that F has enough injectives in $\mathcal{C}$. From now on we assume that $\mathcal{I}_{\mathcal{C}}(F)$ is covariantly finite in $\mathcal{C}$.

First we define the concept of F-tilting in $\mathcal{C}$.

Definition. A Λ-module T is called *F-tilting* in $\mathcal{C}$ if

(i) T is in $\mathcal{C}$.
(ii) $\operatorname{Ext}_F^i(T, T) = 0$ for all $i > 0$.
(iii) $\operatorname{pd}_F T < \infty$.
(iv) For all P in $\mathcal{P}_{\mathcal{C}}(F)$ there is an F-exact sequence $0 \to P \to T_0 \to T_1 \to \cdots \to T_s \to 0$ with T_i in $\operatorname{add} T$.

An *F-cotilting* module in $\mathcal{C}$ is defined dually.

Let ω be a subcategory of $\operatorname{mod}\Lambda$, then ω is said to be *F-selforthogonal* if $\operatorname{Ext}_F^i(\omega, \omega) = 0$ for all $i > 0$.

Let T be an F-selforthogonal Λ-module in $\mathcal{C}$. Define $T^{\perp}$ to be the full subcategory of $\operatorname{mod}\Lambda$ consisting of all modules Y with $\operatorname{Ext}_F^i(T, Y) = 0$ for all $i > 0$. It has been shown in [**11**] that $T^{\perp}$ is F-coresolving in $\operatorname{mod}\Lambda$. Denote $T^{\perp} \cap \mathcal{C}$ by $T_{\mathcal{C}}^{\perp}$. We then denote by $\mathcal{Y}_T^{\mathcal{C}}$ the full subcategory of all Λ-modules A in $T_{\mathcal{C}}^{\perp}$ such that there is an F-exact sequence

$$\cdots \to T_s \xrightarrow{f_s} T_{s-1} \to \cdots \to T_1 \xrightarrow{f_1} T_0 \to A \to 0$$

with T_i in $\operatorname{add} T$ and $\operatorname{Im} f_i$ in $T_{\mathcal{C}}^{\perp}$. Then we have the following result which is a generalization of [**5**, Dual of Proposition 5.1].

PROPOSITION 3.1.1. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod} \Lambda$ *which is closed under extensions. For an F-selforthogonal Λ-module T in $\mathcal{C}$ the subcategory $\mathcal{Y}_T^{\mathcal{C}}$ is closed under*

(a) *F-extensions.*
(b) *cokernels of F-monomorphisms.*
(c) *direct summands.*

PROOF. (a) Let $0 \to A \to B \to C \to 0$ be an F-exact sequence in $\mathcal{C}$ with A and C in $\mathcal{Y}_T^{\mathcal{C}}$. We want to show that B is in $\mathcal{Y}_T^{\mathcal{C}}$. We have F-exact sequences $0 \to K \to T_0 \to A \to 0$ and $0 \to L \to T_0' \to C \to 0$ in $\mathcal{C}$ with T_0 and T_0' in $\operatorname{add} T$. Consider the commutative F-exact pullback diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & L \arrow[r, equal] \arrow[d] & L \arrow[d] & & \\
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & W \arrow[r] \arrow[d] & T_0' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

Since A is in $\mathcal{Y}_T^{\mathcal{C}}$ which is contained in $T_{\mathcal{C}}^{\perp}$, we have that $\operatorname{Ext}_F^1(T_0', A) = 0$. Hence the sequence $0 \to A \to W \to T_0' \to 0$ is split exact, and we have $W \simeq A \oplus T_0'$. Then we have the following commutative exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & K \arrow[r, equal] \arrow[d] & K \arrow[d] & & \\
0 \arrow[r] & M \arrow[r] \arrow[d] & T_0 \oplus T_0' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r] \arrow[d] & A \oplus T_0' \arrow[r] \arrow[d] & B \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

with M in $\mathcal{C}$, since $\mathcal{C}$ is closed under extensions. We show that $\eta\colon 0 \to M \to T_0 \oplus T_0' \to B \to 0$ is F-exact. For, the vertical middle sequence in the diagram is F-exact, since it is a direct sum of two F-exact sequences [**11**, Lemma 1.1]. The sequence $0 \to K \to M \to L \to 0$ is F-exact, since it is a pullback of an F-exact sequence. Since $F = F_{\mathcal{X}}$ it follows that η is F-exact [**11**, Theorem 1.4].

Now K and L are in $T_{\mathcal{C}}^{\perp}$ which is closed under F-extensions, and this implies that M is in $T_{\mathcal{C}}^{\perp}$. Using the fact that $0 \to K \to M \to L \to 0$ is F-exact, and L and K are in $\mathcal{Y}_T^{\mathcal{C}}$, we have by induction that B is in $\mathcal{Y}_T^{\mathcal{C}}$.

(b & c) Let $0 \to A \to B \to C \to 0$ be an F-exact sequence in $\mathcal{C}$ with B in $\mathcal{Y}_T^{\mathcal{C}}$. Then we have an F-exact sequence $0 \to B_1 \to T_0 \to B \to 0$ in $\mathcal{C}$ with T_0 in $\operatorname{add} T$ and B_1 in $\mathcal{Y}_T^{\mathcal{C}}$. Consider the following commutative exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & B_1 \arrow[r, equal] \arrow[d] & B_1 \arrow[d] & & \\
0 \arrow[r] & C_1 \arrow[r] \arrow[d, "s"] & T_0 \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

The sequence $\gamma\colon 0 \to B_1 \to C_1 \to A \to 0$ is F-exact, since it is pullback of an F-exact sequence. Again, since $F = F_{\mathcal{X}}$, it follows that the sequence $\beta\colon 0 \to C_1 \to T_0 \to C \to 0$ is F-exact.

(b) Assume that A is in $\mathcal{Y}_T^{\mathcal{C}}$. We want to show that C is in $\mathcal{Y}_T^{\mathcal{C}}$. Since $\mathcal{Y}_T^{\mathcal{C}}$ is closed under F-extensions, it follows that C_1 is in $\mathcal{Y}_T^{\mathcal{C}}$. The sequence γ is F-exact as shown above, hence the module C_1 is in $\mathcal{Y}_T^{\mathcal{C}}$. But then $T^{\perp}$ is closed under cokernels of F-monomorphisms, so C is in $T^{\perp}$. Hence $\mathcal{Y}_T^{\mathcal{C}}$ is closed under cokernels of monomorphisms of F-exact sequences.

(c) Assume that $B \simeq A \oplus C$. We want to show that C is in $\mathcal{Y}_T^{\mathcal{C}}$. The sequence $0 \to B_1 \to C_1 \oplus C \xrightarrow{(s,1_C)} A \oplus C \to 0$ is F-exact by [**11**, Lemma 1.1] (since it is a direct sum of two F-exact sequences). Since B and B_1 are in $\mathcal{Y}_T^{\mathcal{C}}$, we have that $C_1 \oplus C$ is in $\mathcal{Y}_T^{\mathcal{C}}$, so that C_1 is in $T_{\mathcal{C}}^{\perp}$. Then we have the following commutative F-exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & B_2 \arrow[r, equal] \arrow[d] & B_2 \arrow[d] & & \\
0 \arrow[r] & C_2 \arrow[r] \arrow[d] & T_1 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C \arrow[r] \arrow[d] & C \oplus C_1 \arrow[r] \arrow[d] & C_1 \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

By using the above argument we get that $C_2 \oplus C_1$ is in $\mathcal{Y}_T^{\mathcal{C}}$, so that C_2 is in $T_{\mathcal{C}}^{\perp}$. Then by repeating the process with $0 \to C_i \to C_i \oplus C_{i+1} \to C_{i+1} \to 0$,

$i > 0$, we get by induction that C is in $\mathcal{Y}_T^{\mathcal{C}}$. It is easy to see that all C_i are in $\mathcal{C}$. Therefore $\mathcal{Y}_T^{\mathcal{C}}$ is closed under direct summands. $\square$

When our F-selforthogonal module T is F-tilting in $\mathcal{C}$ we have the following result, which is a generalization of [**11**, Dual of Theorem 3.2]. Denote $\widetilde{\operatorname{add} T} \cap \mathcal{C}$ by $\widetilde{\operatorname{add} T_{\mathcal{C}}}$.

Proposition 3.1.2. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions. Let T be an F-tilting module in $\mathcal{C}$. Then we have the following.*

(a) *The subcategory $\mathcal{Y}_T^{\mathcal{C}} = T_{\mathcal{C}}^{\perp}$ is F-coresolving F-covariantly finite in $\mathcal{C}$ with $\mathcal{Y}_T^{\mathcal{C}}$-$\operatorname{coresdim}_F \mathcal{C}$ finite.*

(b) *The subcategory $\widetilde{\operatorname{add} T_{\mathcal{C}}} = {}^{\perp}(\mathcal{Y}_T^{\mathcal{C}}) \cap \mathcal{C}$ is F-resolving F-contravariantly finite in $\mathcal{C}$ with $\operatorname{pd}_F \widetilde{\operatorname{add} T_{\mathcal{C}}}$ finite.*

Proof. (a) Let T be an F-tilting module in $\mathcal{C}$ with $\operatorname{pd}_F T = r$. We want to show that $\mathcal{Y}_T^{\mathcal{C}} = T_{\mathcal{C}}^{\perp}$. Let A be in $T_{\mathcal{C}}^{\perp}$ and consider an F-exact sequence $0 \to A_0 \to P \to A \to 0$ with P in $\mathcal{P}_{\mathcal{C}}(F)$. Then A_0 is in $\mathcal{C}$, by Proposition 2.2.8. Since $\mathcal{P}_{\mathcal{C}}(F)$ is contained in $\widetilde{\operatorname{add} T_{\mathcal{C}}}$, we have an F-exact sequence $0 \to P \to T_0 \to T_1 \to \cdots \to T_r \to 0$ with T_i in $\operatorname{add} T$. Consider the short F-exact sequence $0 \to P \to T_0 \to K \to 0$. Then we have the following F-exact commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & A_0 \arrow[r, equal] \arrow[d] & A_0 \arrow[d] & & \\
0 \arrow[r] & P \arrow[r] \arrow[d] & T_0 \arrow[r] \arrow[d] & K \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & K \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

by [**11**, Theorem 1.4]. By dimension shift we get that

$$\operatorname{Ext}_F^1(K, T^{\perp}) = \operatorname{Ext}_F^n(T_n, T^{\perp}) = 0.$$

So the F-exact sequence $0 \to A \to E \to K \to 0$ splits. Hence we get an exact sequence $0 \to L \to T_0 \to A \to 0$. By the following commutative exact diagram

$$\begin{tikzcd}
0 \arrow[r] & A_0 \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & L \arrow[r] & T_0 \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

the sequence $0 \to L \to T_0 \to A \to 0$ is F-exact. Then by Proposition 2.2.8 we have that L is in $\mathcal{C}$.

Since $\operatorname{add} T$ is contravariantly finite in $\mathcal{C}$, there is a map $T_0^{'} \to A$ such that any map $T \to A$ factors through $T_0 \oplus T_0^{'} \to A$, where $T_0^{'}$ is in $\operatorname{add} T$. We need to show that $\eta\colon 0 \to M \to T_0 \oplus T_0^{'} \to A \to 0$ is F-exact. We have the following commutative exact pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] \arrow[d] & T_0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] & T_0 \oplus T_0^{'} \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

and therefore η is F-exact, since it is a pushout of an F-exact sequence. Moreover, M is in $\mathcal{C}$ by Proposition 2.2.8.

Now, applying $\operatorname{Hom}_\Lambda(T, -)$ to η, we get that $\operatorname{Ext}_F^i(T, M) = 0$ for all $i > 1$, and since $\operatorname{add} T$ is contravariantly finite in $\mathcal{C}$ we get that $\operatorname{Ext}_F^1(T, M) = 0$. Hence M is in $T_{\mathcal{C}}^{\perp}$. Continuing this process with M we get, by induction, that A is in $\mathcal{Y}_T^{\mathcal{C}}$. This shows that $T_{\mathcal{C}}^{\perp}$ is contained in $\mathcal{Y}_T^{\mathcal{C}}$. The other inclusion follows by definition of $\mathcal{Y}_T^{\mathcal{C}}$, hence we have $T_{\mathcal{C}}^{\perp} = \mathcal{Y}_T^{\mathcal{C}}$.

Next we prove that $\mathcal{Y}_T^{\mathcal{C}}$-$\operatorname{coresdim}_F \mathcal{C}$ is finite. Since $\operatorname{pd}_F T = r$, we have that $\operatorname{Ext}_F^i(T, A) = 0$ for $i > r$ and for all A in $\mathcal{C}$. Consider an F-injective coresolution of A. By dimension shift we have that $\operatorname{Ext}_F^i(T, \Omega_F^{-r}(A)) \simeq \operatorname{Ext}_F^{i+r}(T, A) = 0$ for all $i > 0$ and for all A in $\mathcal{C}$. Hence $\Omega_F^{-r}(A)$ is in $T_{\mathcal{C}}^{\perp} = \mathcal{Y}_T^{\mathcal{C}}$, so that $\mathcal{Y}_T^{\mathcal{C}}$-$\operatorname{coresdim}_F \mathcal{C} \leqslant r$, since $\mathcal{I}_{\mathcal{C}}(F)$ is contained in $\mathcal{Y}_T^{\mathcal{C}}$.

Since $\mathcal{Y}_T^{\mathcal{C}} = T_{\mathcal{C}}^{\perp}$, the subcategory $\operatorname{add} T$ is an Ext_F-projective generator for $\mathcal{Y}_T^{\mathcal{C}}$. We also have that $\mathcal{Y}_T^{\mathcal{C}}$- $\operatorname{coresdim}_F \mathcal{C}$ is finite, so by Proposition 2.3.7 the subcategory $\mathcal{Y}_T^{\mathcal{C}}$ is covariantly finite in $\mathcal{C}$. This completes the proof of (a).

(b) By Proposition 2.3.7 we have that $\widehat{\operatorname{add} T_{\mathcal{C}}} = {}^{\perp}(\mathcal{Y}_T^{\mathcal{C}}) \cap \mathcal{C}$. Since $\mathcal{Y}_T^{\mathcal{C}}$ is F-coresolving covariantly finite in $\mathcal{C}$ by (a), the subcategory $\widehat{\operatorname{add} T_{\mathcal{C}}}$ is F-resolving contravariantly finite in $\mathcal{C}$ by Proposition 2.3.5. By (a) $\mathcal{Y}_T^{\mathcal{C}}$-$\operatorname{coresdim}_F \mathcal{C} \leqslant r$, therefore the subcategory $\widehat{\operatorname{add} T_{\mathcal{C}}}$ is contained in $\mathcal{P}^{\leqslant r}(F)$ (the category of Λ-modules with F-projective dimension at most r) by the dual of [**11**, Theorem 2.5] $\square$

We restate Lemma 1.3.4 for the relative theory in subcategories. The proof is similar, so it will not be given. We denote $\widehat{\operatorname{add} T} \cap \mathcal{C}$ by $\widehat{\operatorname{add} T_{\mathcal{C}}}$.

Lemma 3.1.3. *Let T be an F-tilting module in $\mathcal{C}$. Then $T_{\mathcal{C}}^{\perp} \cap \mathcal{P}_{\mathcal{C}}^{<\infty}(F) = \widehat{\operatorname{add} T_{\mathcal{C}}}$.*

Next we show that the tilting functor is fully faithful on the category $\mathcal{Y}_T^{\mathcal{C}}$.

Let T be in $\mathcal{C}$ and $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$. Consider the tilting functor

$$\operatorname{Hom}_\Lambda(T, \ )\colon \operatorname{mod} \Lambda \to \operatorname{mod} \Gamma.$$

Then we have the following lemma which is an analog of [**11**, Dual of Lemma 3.3].

Lemma 3.1.4. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions. If T is an F-tilting Λ-module in $\mathcal{C}$, then the functor*

$$\operatorname{Hom}_\Lambda(T,\ )\colon \mathcal{Y}_T^{\mathcal{C}} \to \operatorname{mod}\Gamma$$

is an F-exact fully faithful covariant functor.

Proof. Define a map

$$\Phi\colon\ \operatorname{Hom}_\Lambda(Y,A) \to \operatorname{Hom}_\Gamma((T,Y),(T,A))$$

by $\Phi(f) = \psi\colon (T,Y) \to (T,A)$. Then for $g\colon T \to Y$, the map $\psi(g)\colon T \to A$ is given by $\psi(g)(t) = fg(t)$. It is easy to see that Φ is functorial in both variables.

Now, it is not difficult to see that

$$\operatorname{Hom}_\Lambda(T,\ )\colon\ \operatorname{Hom}_\Lambda(Z,A) \to \operatorname{Hom}_\Gamma((T,Z),(T,A))$$

is an isomorphism for any A in $\operatorname{mod}\Lambda$ and Z in $\operatorname{add}T$. Let Y be in $\mathcal{Y}_T^{\mathcal{C}} = T_{\mathcal{C}}^{\perp}$. Then there is an F-exact sequence $T_1 \xrightarrow{g} T_0 \to Y \to 0$, where $\operatorname{Im} g$ and $\operatorname{Ker} g$ are in $T_{\mathcal{C}}^{\perp}$. The sequence

$$\operatorname{Hom}_\Lambda(T,T_1) \to \operatorname{Hom}_\Lambda(T,T_0) \to \operatorname{Hom}_\Lambda(T,Y) \to 0 \qquad (*)$$

is exact. Applying $\operatorname{Hom}_\Gamma(-,\operatorname{Hom}_\Lambda(T,A))$, with A in $\operatorname{mod}\Lambda$, to (*) we get the following commutative exact diagram

$$\begin{tikzcd}
0 \arrow[r] & {}_{\Gamma}((T,Y),(T,A)) \arrow[r] & {}_{\Gamma}((T,T_0),(T,A)) \arrow[r] & {}_{\Gamma}((T,T_1),(T,A)) \\
0 \arrow[r] & \operatorname{Hom}_\Lambda(Y,A) \arrow[u] \arrow[r] & \operatorname{Hom}_\Lambda(T_0,A) \arrow[u, "\wr"'] \arrow[r] & \operatorname{Hom}_\Lambda(T_1,A). \arrow[u, "\wr"']
\end{tikzcd}$$

Hence by functorial isomorphism we have $\operatorname{Hom}_\Lambda(Y,A) \simeq \operatorname{Hom}_\Gamma((T,Y),(T,A))$ for all A in $\operatorname{mod}\Lambda$ and all Y in $\mathcal{Y}_T^{\mathcal{C}} = T_{\mathcal{C}}^{\perp}$.

Next, if $0 \to Y \to B \to C \to 0$ is an F-exact sequence with Y in $\mathcal{Y}_T^{\mathcal{C}}$, then the sequence $0 \to (T,Y) \to (T,B) \to (T,C) \to 0$ is exact, since Y is in $T^{\perp}$. Hence $\operatorname{Hom}_\Lambda(T,\ )$ is F-exact. □

The following is a consequence of Lemma 3.1.4.

Corollary 3.1.5. *Let T be an F-tilting module in $\mathcal{C}$ and* $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$. *Then*

$$\operatorname{Hom}_\Lambda(T,\ )\colon\ \operatorname{Ext}_F^i(Y,Y') \to \operatorname{Ext}_\Gamma^i((T,Y),(T,Y'))$$

is an isomorphism for all Y and Y' in $\mathcal{Y}_T^{\mathcal{C}}$ functorial in both variables.

Proof. Similar to the dual of [**11**, Proposition 3.7]. □

Let T be a tilting module in $\operatorname{mod}\Lambda$, $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$ and DT the corresponding cotilting Γ-module. It is well known that the tilting functor $(T,\ )\colon \operatorname{mod}\Lambda \to \operatorname{mod}\Gamma$ induces an equivalence between the categories $T^{\perp}(= \mathcal{Y}_T$ by the dual of [**5**, Theorem 5.4]) of $\operatorname{mod}\Lambda$ and $(T, T^{\perp})$ of $\operatorname{mod}\Gamma$, where the image $(T, T^{\perp})$ is identified with the subcategory ${}^{\perp}DT$. This was also established for relative tilting modules in $\operatorname{mod}\Lambda$ [**11**].

Let F be a subfunctor in $\operatorname{mod}\Lambda$. Let T be an F-tilting module in $\operatorname{mod}\Lambda$ and denote $\operatorname{End}_{\Lambda}(T)^{\mathrm{op}}$ by Γ. Then it can be shown (by using duality in [**11**]) that the tilting functor induces the same equivalence as in the standard case. But this time the image $(T, T^{\perp})$ is identified with the category ${}^{\perp}(T, \mathcal{I}(F))$, where $(T, \mathcal{I}(F))$ is a cotilting Γ-module.

Our aim is to show that this (in the above discussion) also holds for relative tilting modules T in subcategories. In the present section we prove the existence of an equivalence between the subcategory $\mathcal{Y}_T^{\mathcal{C}}$ of $\mathcal{C}$ and its image $(T, \mathcal{Y}_T^{\mathcal{C}})$ in $\operatorname{mod}\Gamma$. The identification of the subcategory which corresponds to the image $(T, \mathcal{Y}_T^{\mathcal{C}})$ of $(T,\ )$ will be done in the next section.

Let T be an F-tilting Λ-module in $\mathcal{C}$ and $\Gamma = \operatorname{End}_{\Lambda}(T)^{\mathrm{op}}$. We have seen that $\mathcal{Y}_T^{\mathcal{C}} = T_{\mathcal{C}}^{\perp}$. Since $\operatorname{Hom}_{\Lambda}(T,\ )\colon \mathcal{Y}_T^{\mathcal{C}} \to \operatorname{mod}\Gamma$ is a fully faithful functor by Lemma 3.1.4, we have that

$$DY = \operatorname{Hom}_{\Lambda}(Y, D\Lambda) \simeq \operatorname{Hom}_{\Gamma}((T, Y), (T, D\Lambda)) \simeq \operatorname{Hom}_{\Gamma}((T, Y), DT)$$

for all Y in $\mathcal{Y}_T^{\mathcal{C}}$. Applying the duality D to the above isomorphism we get the isomorphism

$$Y \simeq D\operatorname{Hom}_{\Gamma}((T, Y), DT) \simeq T \otimes_{\Gamma} \operatorname{Hom}_{\Lambda}(T, Y).$$

Hence $\mathcal{Y}_T^{\mathcal{C}} \simeq T \otimes_{\Gamma} (T, \mathcal{Y}_T^{\mathcal{C}})$. Therefore $\mathcal{Y}_T^{\mathcal{C}}$ is equivalent to $(T, \mathcal{Y}_T^{\mathcal{C}})$ in $\operatorname{mod}\Gamma$. The following result, which summarizes the above discussion, shows that there is an equivalence between subcategories $\mathcal{Y}_T^{\mathcal{C}}$ of $\mathcal{C}$ and $(T, \mathcal{Y}_T^{\mathcal{C}})$ of $\operatorname{mod}\Gamma$. This is a generalization of the dual of [**11**, Corollary 3.6].

THEOREM 3.1.6. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions. Let T be an F-tilting module in $\mathcal{C}$ and $\Gamma = \operatorname{End}_{\Lambda}(T)^{\mathrm{op}}$.*

(a) *The functor* $\operatorname{Hom}_{\Lambda}(T,\ )\colon \mathcal{C} \to \operatorname{mod}\Gamma$ *induces an equivalence between $\mathcal{Y}_T^{\mathcal{C}}$ and $(T, \mathcal{Y}_T^{\mathcal{C}})$.*
(b) *The functor* $\operatorname{Hom}_{\Lambda}(T,\ )\colon \mathcal{C} \to \operatorname{mod}\Gamma$ *induces an equivalence between $\mathcal{I}_{\mathcal{C}}(F)$ and $(T, \mathcal{I}_{\mathcal{C}}(F))$.*

If T is a standard tilting Λ-module, then we have that the Γ-modules $(T, D\Lambda_{\Lambda})$ and $D(\Lambda, T)$ coincide. But for relative tilting modules this is not always the case.

We want to show that the Γ^{op}-module $(\mathcal{P}_{\mathcal{C}}(F), T)$ is a tilting Γ^{op}-module. This will imply that the module $D(\mathcal{P}_{\mathcal{C}}(F), T)$ is a cotilting Γ-module by duality. But first we need the following results.

LEMMA 3.1.7. *For all W in $\widetilde{\operatorname{add} T_{\mathcal{C}}}$ and all C in* $\operatorname{mod}\Lambda$ *the homomorphism*

$$\operatorname{Hom}_{\Lambda}(\ , T)\colon (C, W) \to {}_{\Gamma^{\mathrm{op}}}((W, T), (C, T))$$

is an isomorphism functorial in both variables.

Proof. The proof is similar to that of [**11**, Lemma 3.3]. ☐

The following is a consequence of the above result, where the proof is similar to that of [**11**, Proposition 3.7].

Corollary 3.1.8. *For W in $\widetilde{\operatorname{add} T_{\mathcal{C}}}$ and C in ${}^{\perp}T_{\mathcal{C}}$ the homomorphism*

$$\operatorname{Hom}_{\Lambda}(\ ,T)\colon\ \operatorname{Ext}^i_F(C,W) \to \operatorname{Ext}^i_{\Gamma^{\mathrm{op}}}((W,T),(C,T)) \quad \textit{for all } i>0$$

is an isomorphism functorial in both variables.

Now we show that $(\mathcal{P}_{\mathcal{C}}(F),T)$ is a tilting Γ^{op}-module.

Proposition 3.1.9. *Let $\mathcal{C}$ be a subcategory of $\operatorname{mod}\Lambda$ which is closed under extensions. Let T be an F-tilting Λ-module in $\mathcal{C}$ with $\operatorname{pd}_F T = r$. Denote $\operatorname{End}_{\Lambda}(T)^{\mathrm{op}}$ by Γ. Then $(\mathcal{P}_{\mathcal{C}}(F),T)$ is a tilting Γ^{op}-module. Moreover, $(\mathcal{P}_{\mathcal{C}}(F),T)$ is of finite type.*

Proof. Since $\mathcal{P}_{\mathcal{C}}(F)$ is in $\widetilde{\operatorname{add} T_{\mathcal{C}}}$ it is easy to see that $\mathcal{P}_{\mathcal{C}}(F)$ is in ${}^{\perp}T_{\mathcal{C}}$. We then have

$$0 = \operatorname{Ext}^i_F(\mathcal{P}_{\mathcal{C}}(F),\mathcal{P}_{\mathcal{C}}(F)) \simeq \operatorname{Ext}^i_{\Gamma^{\mathrm{op}}}((\mathcal{P}_{\mathcal{C}}(F),T),(\mathcal{P}_{\mathcal{C}}(F),T)) \text{ for all } i>0,$$

so that

$$\operatorname{Ext}^i_{\Gamma^{\mathrm{op}}}((\mathcal{P}_{\mathcal{C}}(F),T),(\mathcal{P}_{\mathcal{C}}(F),T)) = 0 \quad \text{for all} \quad i>0.$$

Since T is F-tilting we have, for all P in $\mathcal{P}_{\mathcal{C}}(F)$, an F-exact sequence $0 \to P \to T_0 \to \cdots \to T_s \to 0$ with T_i in $\operatorname{add} T$. Applying the functor $(\ ,T)$ to the sequence we get that $\operatorname{pd}_{\Gamma^{\mathrm{op}}}(\mathcal{P}_{\mathcal{C}}(F),T)$ is finite. Since $\operatorname{pd}_F T$ is finite, there is a minimal F-projective resolution $0 \to P_r \to \cdots \to P_1 \to P_0 \to T \to 0$. Applying $(\ ,T)$ to the F-projective resolution of T we get that Γ^{op} is in $\widetilde{\operatorname{add}}(\mathcal{P}_{\mathcal{C}}(F),T)$. Therefore $(\mathcal{P}_{\mathcal{C}}(F),T)$ is a tilting Γ^{op}-module.

By the corollary to [**28**, Proposition 1.18] we have that, for all P in $\mathcal{P}_{\mathcal{C}}(F)$, the module (P,T) is a direct summand of

$$\operatorname{add} \bigoplus_{i=0}^{r} (P_i,T),$$

where the P_i are in $\mathcal{P}_{\mathcal{C}}(F)$. Hence $(\mathcal{P}_{\mathcal{C}}(F),T)$ is of finite type. ☐

Now we want to show that $\mathcal{P}_{\mathcal{C}}(F)$ is of finite type whenever there is an F-tilting module in $\mathcal{C}$. But we need the following result.

Lemma 3.1.10. *Let F be a subfunctor in $\mathcal{C}$ and consider the functor $\operatorname{Hom}_{\Lambda}(\ ,T)\colon\ \operatorname{mod}\Lambda \to \operatorname{mod}\Gamma$. Then*

(a) $\operatorname{Hom}_{\Lambda}(\ ,T)$ *induces a duality between $\widetilde{\operatorname{add} T_{\mathcal{C}}}$ and $(\widetilde{\operatorname{add} T_{\mathcal{C}}},T)$.*
(b) $\operatorname{Hom}_{\Lambda}(\ ,T)$ *induces a duality between $\mathcal{P}_{\mathcal{C}}(F)$ and $(\mathcal{P}_{\mathcal{C}}(F),T)$.*

Proof. (a) Let W be in $\widetilde{\operatorname{add} T_{\mathcal{C}}}$. Then by Lemma 3.1.7 we have that

$$\begin{aligned} W \simeq (\Lambda, W) &\simeq {}_{\Gamma}((W,T),(\Lambda,T)) \\ &\simeq {}_{\Gamma}((W,T),T_{\Gamma}). \end{aligned}$$

Hence $\widetilde{\operatorname{add} T_{\mathcal{C}}} \simeq (\widetilde{\operatorname{add} T_{\mathcal{C}}}, T)$.

(b) Follows from (a) since $\mathcal{P}_{\mathcal{C}}(F)$ is contained in $\widetilde{\operatorname{add} T_{\mathcal{C}}}$. ☐

The following result is a consequence of Proposition 3.1.9.

Corollary 3.1.11. *The subcategory* $\mathcal{P}_{\mathcal{C}}(F)$ *is of finite type.*

Proof. Follows from Proposition 3.1.9 and Lemma 3.1.10. ☐

3.2. Relative Tilting and Finite Approximation Dimension

Let $\mathcal{C}$ be a functorially finite subcategory of $\operatorname{mod} \Lambda$ which is closed under extensions. Let $\mathcal{X}$ be a contravariantly finite subcategory of $\mathcal{C}$ which is a generator for $\mathcal{C}$. Consider the subfunctor $F = F_{\mathcal{X}}$ in $\mathcal{C}$. Suppose T is an F-tilting module in $\mathcal{C}$ and let $\Gamma = \operatorname{End}_{\Lambda}(T)^{\mathrm{op}}$. In this section we show that the image of the equivalence given in the previous section, namely $(T, \mathcal{Y}_T^{\mathcal{C}})$ is identified with the subcategory ${}^{\perp}(T, \mathcal{I}_{\mathcal{C}}(F))$. Moreover, we show that the Γ-module $(T, \mathcal{I}_{\mathcal{C}}(F))$ is cotilting.

Let $\mathcal{C}$ be a functorially finite subcategory of $\operatorname{mod} \Lambda$ which is closed under extensions and assume the $\mathcal{C}$-approximation dimension of $\operatorname{mod} \Lambda$ is zero. Then, by Corollary 2.4.8, we have that $\mathcal{C}$ is canonically equivalent to $\operatorname{mod} \Sigma$, where Σ is a quotient algebra of Λ. Moreover, we have that $\mathcal{C}$ and $\operatorname{mod} \Sigma$ have the same relative theory. Let T be an F-tilting module in $\mathcal{C}$ and denote $\operatorname{End}_{\Lambda}(T)^{\mathrm{op}}$ by Γ. Then by the duals of [**11**, Proposition 3.8] and [**11**, Theorem 3.13] we have that $(T, \mathcal{Y}_T^{\mathcal{C}}) = {}^{\perp}(T, \mathcal{I}_{\mathcal{C}}(F))$ and $(T, \mathcal{I}_{\mathcal{C}}(F))$ is a cotilting Γ-module.

For $\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod} \Lambda) = \infty$, we give examples in Section 4.2 which show that $(T, \mathcal{I}_{\mathcal{C}}(F))$ is not always a cotilting Γ-module.

Now assume that the $\mathcal{C}$-approximation of $\operatorname{mod} \Lambda$ is greater than zero, but finite. Let T be an F-tilting module in $\mathcal{C}$ and denote $\operatorname{End}_{\Lambda}(T)^{\mathrm{op}}$ by Γ. We want to show that the subcategory $(T, \mathcal{Y}_T^{\mathcal{C}}) = {}^{\perp}(T, \mathcal{I}_{\mathcal{C}}(F))$ and $(T, \mathcal{I}_{\mathcal{C}}(F))$ is a cotilting Γ-module.

But first we need several preliminary results. The following result is an analog of [**11**, Dual of Lemma 2.9].

Lemma 3.2.1. *Let* $\mathcal{C}$ *be a functorially finite subcategory of* $\operatorname{mod} \Lambda$ *which is closed under extensions. Let* T *be an* F*-tilting module in* $\mathcal{C}$ *and let* $\Gamma =$

$\operatorname{End}_\Lambda(T)^{\mathrm{op}}$. *Then, the map*

$$\Psi\colon \operatorname{Hom}_\Lambda(W,T)\otimes_\Gamma \operatorname{Hom}_\Lambda(T,Y) \to \operatorname{Hom}_\Lambda(W,Y)$$

given by $\psi(f\otimes g) = g\circ f$ *is an isomorphism for all* W *in* $\widecheck{\operatorname{add} T_{\mathcal{C}}}$ *and* Y *in* $\mathcal{Y}_T^{\mathcal{C}}$ *and is functorial in both variables.*

Proof. For W in $\operatorname{add} T$ we have that

$$_\Lambda(W,T)\otimes_\Gamma {}_\Lambda(T,Y) \simeq \operatorname{Hom}_\Lambda(W,Y).$$

Let W be in $\widecheck{\operatorname{add} T_{\mathcal{C}}}$. We have an F-exact sequence $0\to W\to T_0\to T_1$. Since $\widecheck{\operatorname{add} T_{\mathcal{C}}}$ is in ${}^{\perp}T_{\mathcal{C}}$, we have that the sequence

$$_\Lambda(T_1,T)\to {}_\Lambda(T_0,T)\to {}_\Lambda(W,T)\to 0$$

is exact. Applying $-\otimes_\Gamma {}_\Lambda(T,Y)$, with Y in $\mathcal{Y}_T^{\mathcal{C}}$, to the above sequence we get the following commutative diagram

$$\begin{tikzcd}
(T_1,T)\otimes_\Gamma (T,Y) \arrow[r] \arrow[d, "\wr"'] & (T_0,T)\otimes_\Gamma (T,Y) \arrow[r] \arrow[d, "\wr"'] & (W,T)\otimes_\Gamma (T,Y) \arrow[r] \arrow[d] & 0 \\
(T_1,Y) \arrow[r] & (T_0,Y) \arrow[r] & (W,Y) \arrow[r] & 0
\end{tikzcd}$$

Since ${}^{\perp}\mathcal{Y}_T^{\mathcal{C}} = \widecheck{\operatorname{add} T_{\mathcal{C}}}$ by Proposition 3.1.2, the lower sequence is exact. Hence $\Psi\colon \operatorname{Hom}_\Lambda(W,T)\otimes_\Gamma \operatorname{Hom}_\Lambda(T,Y)\to \operatorname{Hom}_\Lambda(W,Y)$ is an isomorphism. It is easy to see that the isomorphism is functorial in both variables. □

The following result is an analog of [**11**, Dual of Lemma 3.10].

Lemma 3.2.2. *Let* $\mathcal{C}$ *be a functorially finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions. If* T *is* F*-tilting in* $\mathcal{C}$*, then* $\operatorname{id}_\Gamma D(\widecheck{\operatorname{add} T_{\mathcal{C}}},T)\leqslant \operatorname{pd}_F T$*, where* $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$*. In particular,* $\operatorname{id}_\Gamma D(\mathcal{P}(\mathcal{C}),T)\leqslant \operatorname{pd}_F T$.

Proof. Let $\operatorname{pd}_F T = r$ and let W in $\widecheck{\operatorname{add} T_{\mathcal{C}}}$. Consider a minimal F-exact $\operatorname{add} T$-coresolution $0\to W\to T_0\xrightarrow{f_1} T_1\to\cdots\xrightarrow{f_s} T_s\to 0$ of W. Denote $\operatorname{Ker} f_i$ by W_{i-1} for $0<i\leqslant s$. It is easy to see that all W_i are in ${}^{\perp}T_{\mathcal{C}}$. Then by using dimension shift we get that W_r is in $T_{\mathcal{C}}^{\perp}$. By [**28**, Lemma 2.1] we have that $\operatorname{pd}_\Lambda W_r<\infty$. Hence W_r is in $\widehat{\operatorname{add} T_{\mathcal{C}}}$, by Lemma 1.3.4. So W_r is in ${}^{\perp}T_{\mathcal{C}}\cap \widehat{\operatorname{add} T_{\mathcal{C}}} = \operatorname{add} T$. Hence $s\leqslant r$. When one applies $(\ ,T)$ to the $\operatorname{add} T$-coresolution of W, one gets that $\operatorname{pd}_{\Gamma^{\mathrm{op}}}(W,T)\leqslant r$, which is the same as saying that $\operatorname{id}_\Gamma D(W,T)\leqslant r$. Since $\mathcal{P}_{\mathcal{C}}(\mathcal{C})$ is contained in $\widecheck{\operatorname{add} T_{\mathcal{C}}}$, it follows that $\operatorname{id}_\Gamma D(\mathcal{P}_{\mathcal{C}}(\mathcal{C}),T)\leqslant r$. □

We have the following nice corollary.

Corollary 3.2.3. *Let* $\mathcal{C}$ *be a functorially finite subcategory of* $\operatorname{mod}\Lambda$ *and assume that* $\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda) = n<\infty$*. Let* T *be an* F*-tilting module in* $\mathcal{C}$ *with* $\operatorname{pd}_F T = r$ *and let* $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$*. Then* $\operatorname{id}_\Gamma DT\leqslant r+n$.

Proof. We prove this by induction on n. For $n = 0$, see Corollary 2.4.8 and the dual of [**11**, Lemma 3.10]. For $n = 1$, we have a left $\mathcal{C}$-approximation resolution (presentation) $\Lambda \xrightarrow{f^0} C^0 \xrightarrow{f^1} C^1 \to 0$ of Λ. By the dual of Corollary 2.2.4 we have that C^0 and C^1 are in $\mathcal{P}(\mathcal{C})$. Applying $D(\ ,T)$ to the sequence we get the exact sequence

$$0 \to D(\Lambda, T) \to D(C^0, T) \to D(C^1, T) \to 0.$$

By Lemma 3.2.2 we have that $\operatorname{id}_\Gamma D(C^i, T) \leqslant r$ for $i = 0, 1$. Hence, by [**28**, Lemma 2.1] (see also [**31**]) we have that $\operatorname{id}_\Gamma DT \leqslant r + 1$.

Now suppose that $n > 1$. Then we have a left $\mathcal{C}$-approximation resolution $\Lambda \xrightarrow{f^0} C^0 \xrightarrow{f^1} C^1 \to \cdots \to C^n \to 0$ of Λ. Applying $D(\ ,T)$ to the sequence we get the exact sequence

$$0 \to DT \to D(C^0, T) \to D(C^1, T) \to \cdots \to D(C^n, T) \to 0.$$

Denote $\operatorname{Ker} D(f^i, T)$ by L^i. Then by induction we have that $\operatorname{id}_\Gamma L^1 \leqslant r + n - 1$. Again by [**28**, Lemma 2.1] it follows that $\operatorname{id}_\Gamma DT \leqslant r + n$. □

The following lemma will be very useful.

Lemma 3.2.4. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions and assume $\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda) = n < \infty$. Let T be an F-tilting module in $\mathcal{C}$ with* $\operatorname{pd}_F T = r$. *Let M be a Λ-module and consider a succession $M_1 \hookrightarrow T_0 \to M$, $M_2 \hookrightarrow T_1 \to M_1$, ... of minimal right* $\operatorname{add} T$*-approximations. Then $0 \to M_{i+1} \to T_i \to M_i \to 0$ is F-exact for $i \geqslant r + n + 1$.*

Proof. Let us denote $\operatorname{End}_\Lambda(T)^{\mathrm{op}}$ by Γ. From the complex

$$\cdots \to T_2 \to T_1 \to T_0 \to M$$

we get a minimal projective resolution

$$\cdots \to (T, T_1) \to (T, T_0) \to (T, M) \to 0$$

of (T, M) over Γ. By Lemma 3.2.2, we have that

$$\operatorname{Ext}^j_\Gamma((T, M_i), D(\widetilde{\operatorname{add} T_{\mathcal{C}}}, T)) = 0 \text{ for all } j > 0 \text{ and } i > r.$$

So if one applies the functor $\operatorname{Hom}_\Gamma(\ , D(W, T))$, for W in $\widetilde{\operatorname{add} T_{\mathcal{C}}}$, to the sequence

$$\cdots \to (T, T_{r+1}) \to \cdots \to (T, T_r) \to (T, M_r) \to 0$$

it remains exact. Let W be in $\widetilde{\operatorname{add} T_{\mathcal{C}}}$. Then we have the following commutative diagram by the adjoint isomorphism and Lemma 3.2.1

$$\begin{tikzcd}
((T,M_r),D(W,T)) \arrow[r] \arrow[d, "\wr"] & ((T,T_r),D(W,T)) \arrow[r] \arrow[d, "\wr"] & ((T,T_{r+1}),D(W,T)) \arrow[r] \arrow[d, "\wr"] & \cdots \\
D((W,T)\otimes_\Gamma (T,M_r)) \arrow[r] & D((W,T)\otimes_\Gamma (T,T_r)) \arrow[r] & D((W,T)\otimes_\Gamma (T,T_{r+1})) \arrow[r] & \cdots \\
D((W,M_r)) \arrow[r] \arrow[u] & D((W,T_r)) \arrow[r] \arrow[u, "\wr"] & D((W,T_{r+1})) \arrow[r] \arrow[u, "\wr"] & \cdots
\end{tikzcd}$$

Since the middle row in the above diagram is exact, we have that the sequence

$$0 \to (W, M_{i+1}) \to (W, T_i) \to (W, M_i) \to 0 \tag{1}$$

is exact for $i \geqslant r+1$. In particular, (1) is exact for Q in $\mathcal{P}_{\mathcal{C}}(F)$, since $\mathcal{P}_{\mathcal{C}}(F)$ is contained in $\widetilde{\operatorname{add} T_{\mathcal{C}}}$.

Now, since $\mathcal{C}$-app. dim(mod Λ) $= n$, we have for any $P \in \mathcal{P}(\Lambda)$ a minimal left $\mathcal{C}$-approximation resolution

$$P \xrightarrow{f^0} C^0 \xrightarrow{f^1} C^1 \to \cdots \to C^{l-1} \xrightarrow{f^l} C^l \to 0 \tag{2}$$

with $l \leqslant n$. Denote $\operatorname{Coker} f^{i-1}$ by Z^i for $0 < i < l$. Note that by the dual of Corollary 2.2.4 the C^i are in $\mathcal{P}_{\mathcal{C}}(F)$ for $0 \leqslant i \leqslant n$. We want to show that the sequence $0 \to (P, M_{i+1}) \to (P, T_i) \to (P, M_i) \to 0$ is exact for all $i \geqslant r+n+1$ by using induction on n. For $n = 0$, it follows from Corollary 2.4.8 and the dual of [**11**, Propostion].

For $n = 1$, we combine (1) and (2) to get the following exact sequence of complexes

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & (C^1,T_{r+2}) \arrow[r] \arrow[d] & (C^0,T_{r+2}) \arrow[r] \arrow[d] & (P,T_{r+2}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (C^1,T_{r+1}) \arrow[r] \arrow[d] & (C^0,T_{r+1}) \arrow[r] \arrow[d] & (P,T_{r+1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (C^1,M_{r+1}) \arrow[r] \arrow[d] & (C^0,M_{r+1}) \arrow[r] \arrow[d] & (P,M_{r+1}) & \\
 & 0 & 0 & &
\end{tikzcd}$$

By the long exact sequence (of complexes) [**31**], we have that the sequence $0 \to (P, M_{i+1}) \to (P, T_i) \to (P, M_i) \to 0$ is exact for all $i \geqslant r+2$. Therefore the sequence $0 \to M_{i+1} \to T_i \to M_i \to 0$ is exact for $i \geqslant r+2$. Then by (1), the sequence is F-exact.

For $n > 1$, we have the sequence (2). By induction and using (1) and (2), we get that the sequence

$$0 \to (Z^{n-k}, M_{i+1}) \to (Z^{n-k}, T_i) \to (Z^{n-k}, M_i) \to 0$$

is exact for $i \geqslant r+1+k$ and $0 < k \leqslant n$. In particular, for $k = n$, we get that the sequence $0 \to M_{i+1} \to T_i \to M_i \to 0$ is exact for $i \geqslant r+n+1$. Then by (1) it is F-exact. $\square$

Remark. Let B be in $\operatorname{mod}\Gamma$ and consider a projective resolution of B. Then the Γ-module $\Omega^j_\Gamma(B)$ has a preimage in $\operatorname{mod}\Lambda$ for $j \geqslant 2$. However $\Omega^1_\Gamma(B)$ does not necessarily has a preimage in $\operatorname{mod}\Lambda$.

Now we show that $(T, \mathcal{Y}^{\mathcal{C}}_T) = {}^\perp(T, \mathcal{I}_{\mathcal{C}}(F))$ for a functorially finite subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ which is closed under extensions and $\mathcal{C}$- app. dim(mod Λ) is finite. This result is an generalization of [**11**, Dual of Proposition 3.8].

Proposition 3.2.5. *Let $\mathcal{C}$ be functorially finite a subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions and assume* $\mathcal{C}$- app. dim(mod Λ) $= n < \infty$. *Let T be an F-tilting module in $\mathcal{C}$ with* $\operatorname{pd}_F T = r$ *and let* $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$. *Then,* $\operatorname{Ext}^i_\Gamma(B, (T, \mathcal{I}_{\mathcal{C}}(F))) = 0$ *for all $i > 0$ if and only if B is in* $\operatorname{Hom}_\Lambda(T, \mathcal{Y}^{\mathcal{C}}_T)$.

Proof. Let Y be in $\mathcal{Y}^{\mathcal{C}}_T$. Then, by Corollary 3.1.5 we have that $0 = \operatorname{Ext}^i_F(Y, \mathcal{I}_{\mathcal{C}}(F)) \simeq \operatorname{Ext}^i_\Gamma((T,Y), (T, \mathcal{I}_{\mathcal{C}}(F)))$. So $(T,Y) = B$ is in ${}^\perp(T, \mathcal{I}_{\mathcal{C}}(F))$.

Conversely, let B be a Γ-module such that $\operatorname{Ext}^i_\Gamma(B, (T, \mathcal{I}_{\mathcal{C}}(F))) = 0$ for $i > 0$. Let

$$\operatorname{Hom}_\Lambda(T, T_1) \xrightarrow{(T,f_1)} \operatorname{Hom}_\Lambda(T, T_0) \to B \to 0$$

be a minimal projective presentation of B. By Lemma 3.1.4 the above sequence is induced by $T_1 \xrightarrow{f_1} T_0$. Denote $\operatorname{Ker} f_1$ by M_2. Let $0 \to M_3 \to T_2 \to M_2$, $0 \to M_4 \to T_3 \to M_3$,... be a succession of minimal left $\operatorname{add} T$-approximations. Then we get a complex (a minimal right $\operatorname{add} T$-approximation resolution)

$$\cdots \to T_4 \xrightarrow{f_4} T_3 \xrightarrow{f_3} T_2 \to M_2$$

and the exact sequence

$$(3) \qquad \cdots \to (T, T_s) \to (T, T_{s-1}) \to \cdots \to (T, T_1) \to (T, T_0) \to B \to 0$$

is a minimal projective resolution of B over Γ. Denote $\Omega^1_\Gamma(B)$ by B_1. Applying $\operatorname{Hom}_\Gamma(-, (T, I))$, with I in $\mathcal{I}_{\mathcal{C}}(F)$, to the resolution of B, we get the following exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & {}_\Gamma(B, (T,I)) \arrow[r] & {}_\Gamma((T,T_0), (T,I)) \arrow[r] & {}_\Gamma((T,T_1), (T,I)) \arrow[r] & \cdots \\
0 \arrow[r] & \operatorname{Hom}_\Lambda(T \otimes_\Gamma B, I) \arrow[r] \arrow[u, "\wr"'] & \operatorname{Hom}_\Lambda(T_0, I) \arrow[r] \arrow[u, "\wr"'] & \operatorname{Hom}_\Lambda(T_1, I) \arrow[r] \arrow[u, "\wr"'] & \cdots
\end{tikzcd}$$

by Lemma 3.1.4 and the adjoint isomorphism. The cohomology of the upper row is $\operatorname{Ext}^i_\Gamma(B, (T, \mathcal{I}_{\mathcal{C}}(F)) = 0$ for $i > 0$. So the sequence

$$(4) \qquad 0 \to (T \otimes_\Gamma B, I) \to (T_0, I) \to \cdots \to (T_r, I) \to (T_{r+1}, I) \to \cdots$$

is exact.

On the other hand, since $\mathcal{C}$-app. $\dim(\mathcal{I}(\Lambda)) = n$, we have, for all I in $\mathcal{I}(\Lambda)$, a minimal right $\mathcal{C}$-approximation resolution

$$0 \to C_l \xrightarrow{g_l} \cdots \to C_1 \xrightarrow{g_1} C_0 \xrightarrow{g_0} I \tag{5}$$

with $l \leqslant n$. Denote $\operatorname{Ker} g_i$ by Y_{i+1} for $0 \leqslant i < n$. By Corollary 2.2.4 the modules C_i are in $\mathcal{I}(F)$ for $0 \leqslant i \leqslant n$. Then by the adjoint isomorphism, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (T \otimes_\Gamma B, C_l) \arrow[r] \arrow[d, "\wr"] & \cdots \arrow[r] & (T \otimes_\Gamma B, C_0) \arrow[r] \arrow[d, "\wr"] & (T \otimes_\Gamma B, I) \arrow[d, "\wr"] & \\
0 \arrow[r] & (B, (T, C_l)) \arrow[r] & \cdots \arrow[r] & (B, (T, C_0)) \arrow[r] & (B, (T, I)) \arrow[r] & \operatorname{Ext}^1_\Gamma(B, (T, Y_1))
\end{tikzcd}$$

with $l \leqslant n$. By dimension shift we have

$$\operatorname{Ext}^1_\Gamma(B, (T, Y_1)) \simeq \operatorname{Ext}^n_\Gamma(B, (T, C_n)) = 0$$

since C_n is in $\mathcal{I}_{\mathcal{C}}(F)$. So the top row in the above diagram is exact.

Now, combining (4) and (5) we get the following exact sequence of complexes

$$\begin{tikzcd}
& 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & (T \otimes_\Gamma B, C_l) \arrow[r] \arrow[d] & \cdots \arrow[r] & (T \otimes_\Gamma B, C_0) \arrow[r] \arrow[d] & (T \otimes_\Gamma B, I) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (T_0, C_l) \arrow[r] \arrow[d] & \cdots \arrow[r] & (T_0, C_0) \arrow[r] \arrow[d] & (T_0, I) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (T_1, C_l) \arrow[r] \arrow[d] & \cdots \arrow[r] & (T_1, C_0) \arrow[r] \arrow[d] & (T_1, I) \arrow[r] \arrow[d] & 0 \\
& \vdots & & \vdots & \vdots &
\end{tikzcd}$$

with $l \leqslant n$. By the long exact sequence (of complexes) [**31**], we have that the sequence

$$0 \to (T \otimes_\Gamma B, I) \to (T_0, I) \to \cdots \to (T_r, I) \to (T_{r+1}, I) \to \cdots$$

is exact for all I in $\mathcal{I}(\Lambda)$. Hence

$$0 \to M_{r+2n} \to T_{r+2n-1} \to \cdots \to T_0 \to T \otimes_\Gamma B \to 0 \tag{6}$$

is exact.

By Lemma 3.2.4 we have that $0 \to M_{i+1} \to T_i \to M_i \to 0$ is F-exact for all $i \geqslant r + n + 1$. Then using Corollary 2.4.10 we get that M_i is in $\mathcal{C}$ for $i \geqslant r + 2n + 1$. But, then by (4) we have that (6) is $F^{\mathcal{I}_{\mathcal{C}}(F)}$-exact. Hence by Proposition 2.2.11, M_i for $2 \leqslant i \leqslant r + 2n + 1$, $T \otimes_\Gamma B_1$ and $T \otimes_\Gamma B$ are in $\mathcal{C}$. But since $F_{\mathcal{X}}|_{\mathcal{C}} = F^{\mathcal{I}_{\mathcal{C}}(F)}|_{\mathcal{C}}$ by Corollary 2.2.9, we have that (6) is F-exact.

From (3) and (6) we have that

$$\operatorname{Ext}^1_F(T, M_i) = 0 \text{ for } 2 < i \leqslant r + 2n + 1.$$

The F-exact sequence $0 \to M_{i+1} \to T_i \to M_i \to 0$ gives

$$\operatorname{Ext}^{j+1}_F(T, M_{i+1}) \simeq \operatorname{Ext}^j_F(T, M_i) \text{ for } j > 0 \text{ and } 2 < i \leqslant r + 2n + 1.$$

By dimension shift, we have that

$$\operatorname{Ext}^j_F(T, M_{r+2n+1}) = 0 \text{ for } 0 < j < r + 1.$$

Since $\operatorname{pd}_F T = r$, it follows that M_{r+2n+1} is in $\mathcal{Y}^{\mathcal{C}}_T = T^{\perp}_{\mathcal{C}}$.

By Proposition 3.1.2, the subcategory $\mathcal{Y}^{\mathcal{C}}_T$ is F-coresolving, hence, by using the fact that (6) is F-exact we have that $T \otimes_\Gamma B$, $T \otimes_\Gamma B_1$ and M_i, for $i = 2, \ldots, r + 2n + 1$, are in $\mathcal{Y}^{\mathcal{C}}_T$. Let $V = \operatorname{Ext}^1_F(T, T \otimes_\Gamma B_1)$. Then, from the commutative exact diagram

$$\begin{tikzcd}
\cdots \arrow[r] & (T,T_2) \arrow[r] & (T,T_1) \arrow[r] & (T,T_0) \arrow[r] & (T, T \otimes_\Gamma B) \arrow[r] & V \arrow[r] & 0 \\
\cdots \arrow[r] & (T,T_2) \arrow[r] \arrow[u, equal] & (T,T_1) \arrow[r] \arrow[u, equal] & (T,T_0) \arrow[r] \arrow[u, equal] & B \arrow[r] \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

we have $(T, T \otimes_\Gamma B) \simeq B$, since $V = 0$. Therefore B is in $(T, \mathcal{Y}^{\mathcal{C}}_T)$ and the result follows. ☐

Remark. Note that $\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda)$ being finite is sufficient but not necessary for the equality $(T, \mathcal{Y}^{\mathcal{C}}_T) = {}^{\perp}(T, \mathcal{I}_{\mathcal{C}}(F))$. We illustrate this in Example 4.2.1.

Next we want to show that $(T, \mathcal{I}_{\mathcal{C}}(F))$ is a standard cotilting Γ-module. The following result will help us to achieve our goal. The result also shows that the $(T, \mathcal{Y}^{\mathcal{C}}_T)\text{-}\operatorname{coresdim}(\operatorname{mod}\Gamma)$ is finite when $\mathcal{C}$ is a functorially finite subcategory of $\operatorname{mod}\Lambda$ which is closed under extensions and $\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda)$ is finite. This result is a generalization of [**11**, Proposition 3.11].

Proposition 3.2.6. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions and assume $\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda) = n < \infty$. Let T be an F-tilting module in $\mathcal{C}$ with $\operatorname{pd}_F T = r$ and let $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$. Then $\widehat{(T, \mathcal{Y}^{\mathcal{C}}_T)} = \operatorname{mod}\Gamma$ and*

$$(T, \mathcal{Y}^{\mathcal{C}}_T)\text{-}\operatorname{resdim}(\operatorname{mod}\Gamma) \leqslant \nu(n,r) = \begin{cases} 2 + n & r = 0 \\ 3 + 2n & r = 1 \\ r + 2n + 1 & r \geqslant 2 \end{cases}$$

Proof. Let $(T, T_{-1}) \to (T, T_{-2}) \to B \to 0$ be a minimal projective presentation of B in $\operatorname{mod}\Gamma$. By Lemma 3.1.4 the presentation is induced by $T_{-1} \xrightarrow{f} T_{-2}$. Denote $\operatorname{Ker} f$ by M_0, then we have that $\Omega^2_\Gamma(B) = (T, M_0)$.

For $r = 0$, we have that $T = \mathcal{P}_{\mathcal{C}}(F)$, so that $\mathcal{Y}_T^{\mathcal{C}} = \mathcal{C}$. From the right $\mathcal{C}$-approximation resolution of M_0, we have the sequence

$$\begin{tikzcd}
0 \arrow[r] & C_l \arrow[r] & \cdots \arrow[r] & C_1 \arrow[rr] \arrow[dr, "f_1"] & & C_0 \arrow[rr] \arrow[dr, "f_0"] & & T_{-1} \arrow[r, "f"] & T_{-2} \\
& & & & Y_1 \arrow[ur, hook] & & M_0 \arrow[ur, hook] & &
\end{tikzcd}$$

with $l \leqslant n$, since $\mathcal{C}$-app. dim(mod $\Lambda) = n$. We then have the exact sequence

$$0 \to (T, C_l) \to \cdots \to (T, C_0) \to (T, T_{-1}) \to (T, T_{-2}) \to B \to 0.$$

But since $\mathcal{Y}_T^{\mathcal{C}} = \mathcal{C}$, it follows that $\widehat{(T, \mathcal{Y}_T^{\mathcal{C}})} = \operatorname{mod}\Gamma$ and

$$(T, \mathcal{Y}_T^{\mathcal{C}})\text{- resdim}(\operatorname{mod}\Gamma) \leqslant 2 + n.$$

For $r > 0$, let $0 \to M_1 \to T_0 \to M_0$, $0 \to M_2 \to T_1 \to M_1$, ... be a succession of minimal right $\operatorname{add} T$-approximations. Then we get a complex

$$\cdots \to T_2 \to T_1 \to T_0 \to M_0$$

and the exact sequence

$$\cdots \to (T, T_1) \to (T, T_0) \to (T, T_{-1}) \to (T, T_{-2}) \to B \to 0$$

is a minimal projective resolution of B in $\operatorname{mod}\Gamma$.

Assume that $r \geqslant 2$. Since $\mathcal{C}$- app. dim(mod $\Lambda) = n$, it follows, by Lemma 3.2.4 that the sequence $0 \to M_{i+1} \to T_i \to M_i \to 0$ is F-exact for all $i \geqslant r + n - 1$.

Now, by Corollary 2.4.10, we have that M_i is in $\mathcal{C}$ for $i \geqslant r + 2n - 1$. Moreover, by (1) in the proof of Lemma 3.2.4, we have that

$$\operatorname{Ext}_F^1(\widecheck{\operatorname{add} T_{\mathcal{C}}}, M_i) = 0 \text{ for } i > r + 2n - 1.$$

Using the fact that $0 \to M_{i+1} \to T_i \to M_i \to 0$ is F-exact for $i \geqslant r + 2n - 1$ and $\widecheck{\operatorname{add} T_{\mathcal{C}}}$ is in ${}^{\perp}T$, we have that

$$\operatorname{Ext}_F^j(\widecheck{\operatorname{add} T_{\mathcal{C}}}, M_i) \simeq \operatorname{Ext}_F^{j+1}(\widecheck{\operatorname{add} T_{\mathcal{C}}}, M_{i+1}) \text{ for } j > 0 \text{ and } i \geqslant r + 2n - 1.$$

By dimension shift we have

$$\operatorname{Ext}_F^i(\widecheck{\operatorname{add} T_{\mathcal{C}}}, M_{2r+2n-1}) = 0 \text{ for } 0 < i < r + 1.$$

Since $\widecheck{\operatorname{add} T_{\mathcal{C}}}$ is contained in $\mathcal{P}^r(F)$ we have that $M_{2r+2n-1}$ is in $(\widecheck{\operatorname{add} T_{\mathcal{C}}})^{\perp} \simeq \mathcal{Y}_T^{\mathcal{C}}$.

But since $\mathcal{Y}_T^{\mathcal{C}}$ is F-coresolving and $0 \to M_{i+1} \to T_i \to M_i \to 0$ is F-exact for $i \geqslant r + 2n$, we have that M_i is in $\mathcal{Y}_T^{\mathcal{C}}$ for $r + 2n - 1 \leqslant i \leqslant 2r + 2n - 1$. Hence $(T, M_{r+2n-1}) = \Omega_{\Gamma}^{r+2n+1}(B)$ is in $(T, \mathcal{Y}_T^{\mathcal{C}})$. Therefore

$$(T, \mathcal{Y}_T^{\mathcal{C}})\text{- resdim}(\operatorname{mod}\Gamma) \leqslant r + 2n + 1.$$

If $r = 1$, the proof of the case $r \geqslant 2$ plus the remark after Lemma 3.2.4 can be used to show that M_{2n+1} is in $\mathcal{Y}_T^{\mathcal{C}}$. Hence $(T, M_{2n+1}) = \Omega_\Gamma^{3+2n}(B)$ is in $(T, \mathcal{Y}_T^{\mathcal{C}})$ and we have that

$$(T, \mathcal{Y}_T^{\mathcal{C}})\text{-}\operatorname{resdim}(\operatorname{mod}\Gamma) \leqslant 3 + 2n.$$

$\square$

Remark. $\mathcal{C}$-app. dim(mod Λ) being finite is sufficient for $\widehat{(T, \mathcal{Y}_T^{\mathcal{C}})} = \operatorname{mod}\Gamma$, but it is not known if the assumption is necessary.

We are now in position to show that $\operatorname{Hom}_\Lambda(T, \mathcal{I}_{\mathcal{C}}(F))$ is a cotilting module in $\operatorname{mod}\Gamma$ when $\mathcal{C}$ is a functorially finite subcategory of $\operatorname{mod}\Lambda$ which is closed under extensions and $\mathcal{C}$-app. dim(mod Λ) is finite. This result is a generalization of [**11**, Dual of Theorem 3.13].

THEOREM 3.2.7. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions and assume $\mathcal{C}$-*app. dim(mod Λ) $= n < \infty$. *Let T be an F-tilting module in $\mathcal{C}$ with $\operatorname{pd}_F T = r$ and let $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$. Then we have the following:*

(a) *The subcategory $(T, \mathcal{Y}_T^{\mathcal{C}}) = {}^\perp(T, \mathcal{I}_{\mathcal{C}}(F))$ is resolving contravariantly finite in* $\operatorname{mod}\Gamma$ *with* $(T, \mathcal{Y}_T^{\mathcal{C}})$-resdim(mod Γ) $\leqslant \nu(n, r)$.

(b) *The subcategory $(T, \mathcal{Y}_T^{\mathcal{C}})^\perp = \widehat{(T, \mathcal{I}_{\mathcal{C}}(F))}$ is a coresolving covariantly finite subcategory of* $\operatorname{mod}\Gamma$ *with* $\operatorname{id}_\Gamma(T, \mathcal{I}_{\mathcal{C}}(F)) \leqslant \nu(n, r)$.

(c) $(T, \mathcal{Y}_T^{\mathcal{C}}) \cap (T, \mathcal{Y}_T^{\mathcal{C}})^\perp = (T, \mathcal{I}_{\mathcal{C}}(F))$.

(d) *The subcategory $(T, \mathcal{I}_{\mathcal{C}}(F)) = \operatorname{add} T_{\mathcal{C}}^0$ for a cotilting Γ-module $T_{\mathcal{C}}^0$ with $\operatorname{id}_\Gamma T_{\mathcal{C}}^0 \leqslant \nu(n, r)$. In particular, $(T, \mathcal{Y}_T^{\mathcal{C}}) = \mathcal{Y}_{T_{\mathcal{C}}^0} = {}^\perp T_{\mathcal{C}}^0$.*

PROOF. (a) The subcategory $(T, \mathcal{Y}_T^{\mathcal{C}}) = {}^\perp(T, \mathcal{I}_{\mathcal{C}}(F))$ is resolving with

$$(T, \mathcal{Y}_T^{\mathcal{C}})\text{-}\operatorname{resdim}(\operatorname{mod}\Gamma) \leqslant \nu(n, r)$$

by Proposition 3.2.5 and Proposition 3.2.6. Since $(T, \mathcal{I}_{\mathcal{C}}(F))$ is an Ext-injective cogenerator of $(T, \mathcal{Y}_T^{\mathcal{C}})$ in $\operatorname{mod}\Gamma$ and $\widehat{(T, \mathcal{Y}_T^{\mathcal{C}})} = \operatorname{mod}\Gamma$ (by Proposition 3.2.5), the subcategory $(T, \mathcal{Y}_T^{\mathcal{C}})$ is contravariantly finite [**5**, Proposition 5.2].

(b) By [**3**, Theorem 2.3] we have that $(T, \mathcal{Y}_T^{\mathcal{C}})^\perp = \widehat{(T, \mathcal{I}_{\mathcal{C}}(F))}$. Then by [**5**, Proposition 3.3] it follows that $(T, \mathcal{Y}_T^{\mathcal{C}})^\perp$ is coresolving covariantly finite in $\operatorname{mod}\Gamma$, since $(T, \mathcal{Y}_T^{\mathcal{C}})$ is contravariantly finite resolving.

By (a) we have that $(T, \mathcal{Y}_T^{\mathcal{C}})$-resdim(mod Γ) $\leqslant \nu(n, r)$, so by [**5**, Proposition 5.3] it follows that $\widehat{(T, \mathcal{I}_{\mathcal{C}}(F))} \subseteq \mathcal{I}^{\nu(n,r)}(\Gamma)$. Therefore

$$\operatorname{id}_\Gamma(T, \mathcal{I}_{\mathcal{C}}(F)) \leqslant \nu(n, r).$$

(c) We have that $(T, \mathcal{Y}_T^{\mathcal{C}}) \cap (T, \mathcal{Y}_T^{\mathcal{C}})^\perp = (T, \mathcal{Y}_T^{\mathcal{C}}) \cap \widehat{(T, \mathcal{I}_{\mathcal{C}}(F))}$. So $(T, \mathcal{I}_{\mathcal{C}}(F))$ is contained in $(T, \mathcal{Y}_T^{\mathcal{C}}) \cap (T, \mathcal{Y}_T^{\mathcal{C}})^\perp$. Let (T, Y) be in $(T, \mathcal{Y}_T^{\mathcal{C}}) \cap (T, \mathcal{Y}_T^{\mathcal{C}})^\perp$. Then there is an exact sequence

$$(1) \qquad 0 \to (T,I_s) \to \cdots \xrightarrow{(T,f_2)} (T,I_1) \xrightarrow{(T,f_1)} (T,I_0) \xrightarrow{(T,f_0)} (T,Y) \to 0$$

with I_j in $\mathcal{I}_{\mathcal{C}}(F)$ for all $j \leqslant s$. Since $(T,\mathcal{Y}_T^{\mathcal{C}})$ is resolving, we have that $\operatorname{Coker}(T,f_i) = (T,Y_{i-1})$ with Y_{i-1} in $\mathcal{Y}_T^{\mathcal{C}}$ for all $i > 0$. Since (T,Y) is in ${}^{\perp}(T,\mathcal{I}_{\mathcal{C}}(F))$ we have that the functor $(\ ,(T,\mathcal{I}_{\mathcal{C}}(F)))$ is exact on (1). Applying $(\ ,(T,J))$, for J in $\mathcal{I}_{\mathcal{C}}(F)$, on (1) we get the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & ((T,Y),(T,J)) \arrow[r] & ((T,I_0),(T,J)) \arrow[r] & \cdots \arrow[r] & ((T,I_s),(T,J)) \arrow[r] & 0 \\
0 \arrow[r] & (Y,J) \arrow[r] \arrow[u] & (I_0,J) \arrow[r] \arrow[u] & \cdots \arrow[r] & (I_s,J) \arrow[u] &
\end{tikzcd}$$

By Lemma 3.1.4 the sequence

$$(2) \qquad 0 \to (Y,J) \to (I_0,J) \to \cdots \to (I_s,J) \to 0$$

is exact.

Now, since $\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda) = n < \infty$, we have a right $\mathcal{C}$-approximation resolution

$$(3) \qquad 0 \to C_l \to \cdots \to C_1 \to C_0 \to D\Lambda$$

of $D\Lambda$ with $l \leqslant n$. Combining (2) and (3) we get the following commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d, dashed] & \\
0 \arrow[r] & (Y,C_l) \arrow[r] \arrow[d] & \cdots \arrow[r] & (Y,C_0) \arrow[r] \arrow[d] & (Y,D\Lambda) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (I_0,C_l) \arrow[r] \arrow[d] & \cdots \arrow[r] & (I_0,C_0) \arrow[r] \arrow[d] & (I_0,D\Lambda) \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & (I_s,C_l) \arrow[r] \arrow[d] & \cdots \arrow[r] & (I_s,C_0) \arrow[r] \arrow[d] & (I_s,D\Lambda) \arrow[r] \arrow[d, dashed] & 0 \\
 & 0 & & 0 & 0 &
\end{tikzcd}$$

which is exact by the snake lemma. Hence the sequence

$$(4) \qquad 0 \to I_s \to \cdots \to I_1 \to I_0 \to Y \to 0$$

is exact. We have that (4) is indeed F-exact by using (2) and Corollary 2.2.9. Since I_s is in $\mathcal{I}_{\mathcal{C}}(F)$, the sequence $0 \to I_s \to I_{s-1} \to Y_{s-1} \to 0$ splits and hence Y_{s-1} is in $\mathcal{I}_{\mathcal{C}}(F)$. By induction we have that Y is in $\mathcal{I}_{\mathcal{C}}(F)$. Therefore $(T,\mathcal{Y}_T^{\mathcal{C}}) \cap (T,\mathcal{Y}_T^{\mathcal{C}})^{\perp} = (T,\mathcal{I}_{\mathcal{C}}(F))$.

(d) Since $(T, \mathcal{Y}_T^{\mathcal{C}})$ is a resolving contravariantly finite subcategory of $\operatorname{mod}\Gamma$ with $\widehat{(T, \mathcal{Y}_T^{\mathcal{C}})} = \operatorname{mod}\Gamma$, we have, by [**5**, Theorem 5.5], that

$$(T, \mathcal{Y}_T^{\mathcal{C}}) \cap (T, \mathcal{Y}_T^{\mathcal{C}})^{\perp} = \operatorname{add} T_{\mathcal{C}}^{0}$$

for a cotilting Γ-module $T_{\mathcal{C}}^{0}$. By (c) we have $(T, \mathcal{Y}_T^{\mathcal{C}}) \cap (T, \mathcal{Y}_T^{\mathcal{C}})^{\perp} = (T, \mathcal{I}_{\mathcal{C}}(F))$. This completes the proof. ☐

The following is an immediate consequence of the above theorem. The result is an analog of the dual of [**11**, Corollary 3.14].

Corollary 3.2.8. *The subcategory $\mathcal{I}_{\mathcal{C}}(F)$ is of finite type.*

Proof. Since $\mathcal{I}_{\mathcal{C}}(F)$ is equivalent to $(T, \mathcal{I}_{\mathcal{C}}(F))$ by Proposition 3.1.6(b) and $(T, \mathcal{I}_{\mathcal{C}}(F))$ is of finite type by Theorem 3.2.7(d), the subcategory $\mathcal{I}_{\mathcal{C}}(F)$ is of finite type. ☐

By the above result we have that if $\mathcal{I}_{\mathcal{C}}(F)$ is of infinite type, then there is no F-tilting Λ-module in $\mathcal{C}$.

It can be shown that (by the dual of [**11**, Proposition 3.15]) if T is an F-tilting Λ-module in $\operatorname{mod}\Lambda$ and $\Gamma = \operatorname{End}_{\Lambda}(T)^{\mathrm{op}}$, then DT is direct summand of a cotilting Γ-module T_0, where $\operatorname{add} T_0 = (T, \mathcal{I}(F))$. This is not the case for an F-tilting Λ-module T in a functorially finite subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ with $\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda) = n$, where $0 \leqslant n < \infty$. We illustrate this by the following example.

Example 3.2.9. Let Λ be given by the quiver

$$\begin{tikzcd}
 & 2 \arrow[dl, "\alpha"'] & \\
1 & & 4 \arrow[ul, "\gamma"'] \arrow[dl, "\delta"] \\
 & 3 \arrow[ul, "\beta"] &
\end{tikzcd}$$

with relations $\alpha\gamma = 0$. Denote by P_i, I_i and S_i the indecomposable projective, injective and simple Λ-module corresponding to the vertex i respectively. Let $\mathcal{C} = \operatorname{add}\{P_1, P_2, S_2, P_4, C_1, C_2, I_1, I_2, I_4\}$, where the radical filtrations of C_1 and C_2 look like:

$$\begin{array}{ccccc} & & & 4 & \\ 2 & & 3 & & 2 \\ & 1 & & & \end{array} \qquad\qquad \begin{array}{c} 4 \\ 3 \\ 1 \end{array}$$

respectively. It can be (easily) shown that $\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda) = 1$. The Ext-projectives in $\mathcal{C}$ are $\mathcal{P}(\mathcal{C}) = \operatorname{add}\{P_1, P_2, P_4, I_2\}$, while the Ext-injectives are $\mathcal{I}(\mathcal{C}) = \operatorname{add}\{I_1, I_2, P_2, I_4\}$. Since $\operatorname{mod}\Lambda$ is of finite type, every subcategory of $\operatorname{mod}\Lambda$ is functorially finite ([**5**, Proposition 1.2]).

Let $F = F_{\mathcal{X}}$ where $\mathcal{X} = \mathcal{P}(\mathcal{C}) \cup \operatorname{add} S_4$. Then the F-projectives are given by $\mathcal{P}_{\mathcal{C}}(F) = \mathcal{X}$ by Proposition 2.2.2, while the F-injectives are given by $\mathcal{I}_{\mathcal{C}}(F) = \operatorname{add} Y \cup \mathcal{I}(\mathcal{C})$, where $Y = C_{D\operatorname{Tr} S_4}$, by using Proposition 2.2.7. Denote the direct sum of all indecomposable F-projective Λ-modules in $\mathcal{C}$ by P. Then P is the trivial F-tilting module in $\mathcal{C}$.

Let $\Gamma = \operatorname{End}_\Lambda(P)^{\mathrm{op}}$ and denote by Q_i, J_i and S_i the indecomposable projective, injective and simple Λ-module corresponding to the vertex i respectively. The radical filtrations for Q_i and J_i, for $i = 1, \dots, 5$, look like:

$$Q_1\colon\ {\scriptstyle 1} \qquad Q_2\colon\ \begin{smallmatrix}2\\1\end{smallmatrix} \qquad Q_3\colon\ \begin{smallmatrix}&3&\\1&&2\end{smallmatrix} \qquad Q_4\colon\ \begin{smallmatrix}4\\3\\2\end{smallmatrix} \qquad Q_5\colon\ \begin{smallmatrix}5\\4\\3\end{smallmatrix} \qquad J_1\colon\ \begin{smallmatrix}2&&3\\&1&\end{smallmatrix} \qquad J_2\colon\ \begin{smallmatrix}4\\3\\2\end{smallmatrix}$$

$$J_3\colon\ \begin{smallmatrix}5\\4\\3\end{smallmatrix} \qquad J_4\colon\ \begin{smallmatrix}5\\4\end{smallmatrix} \qquad J_5\colon\ {\scriptstyle 5}\,.$$

By Theorem 3.2.7(d) the module $T^0_{\mathcal{C}} = J_1 \oplus Q_4 \oplus Q_2 \oplus Q_5 \oplus \begin{smallmatrix}2&&3&\\&1&&2\end{smallmatrix}$ is a cotilting Γ-module. The module (T, I_3) is a direct summand of DT, but it is not a direct summand of $T^0_{\mathcal{C}}$. So DT is not a direct summand of $T^0_{\mathcal{C}}$.

But observe that in Example 3.2.9 we have that DT is in $\widehat{\operatorname{add} T^0_{\mathcal{C}}}$. This is true in general as shown by the following result.

Proposition 3.2.10. *Let T be an F-tilting module in a functorially finite subcategory $\mathcal{C}$ of* $\operatorname{mod}\Lambda$ *with $\mathcal{C}$-$\operatorname{app.dim}(\operatorname{mod}\Lambda) = n$, where $0 \leqslant n < \infty$. Then DT is in $(\widehat{T, \mathcal{I}_{\mathcal{C}}(F)})$.*

Proof. Consider the right $\mathcal{C}$-approximation resolution

$$0 \to C_l \to \cdots \to C_1 \to C_0 \to D\Lambda$$

of $D\Lambda$, where $l \leqslant n$. When applying the functor $(T, \ )$ to the above resolution, we get the exact sequence

$$0 \to (T, C_l) \to \cdots \to (T, C_1) \to (T, C_0) \to (T, D\Lambda) \to 0.$$

By Lemma 2.2.4 we have that C_i is in $\mathcal{I}_{\mathcal{C}}(F)$ for $0 \leqslant i \leqslant n$. Hence (T, C_i) in $\operatorname{add} T^0_{\mathcal{C}}$ for $0 \leqslant i \leqslant n$. Therefore DT is in $(\widehat{T, \mathcal{I}_{\mathcal{C}}(F)})$. □

3.3. Relative Tilting and Global Dimension

In this section we show some relationship between the F-global dimension of $\mathcal{C}$ and the global dimension of Γ, which generalizes [**11**].

Let $\mathcal{C}$ be a functorially finite subcategory of $\operatorname{mod}\Lambda$ which is closed under extensions and assume $\mathcal{C}$-$\operatorname{app.dim}(\operatorname{mod}\Lambda) = n < \infty$. Let $\mathcal{X}$ be a contravariantly finite subcategory of $\mathcal{C}$ which is a generator for $\mathcal{C}$. Consider the subfunctor $F = F_{\mathcal{X}}$ in $\mathcal{C}$. Throughout this section we assume that $\mathcal{I}_{\mathcal{C}}(F)$ is covariantly finite in $\mathcal{C}$. We fix an F-tilting module T in $\mathcal{C}$ with $\operatorname{pd}_F T = r$ and denote $\operatorname{End}_\Lambda(T)^{\mathrm{op}}$ by Γ.

If T is F-tilting in $\operatorname{mod}\Lambda$, then it can be shown that (using duality [**11**, Section 4]) the relative (or F-) global dimension of Λ, $\operatorname{gl.dim}_F \Lambda$, and the global dimension of Γ, $\operatorname{gl.dim}\Gamma$, are related by the formula $\operatorname{gl.dim}_F \Lambda - \operatorname{pd}_F T \leqslant \operatorname{gl.dim}\Gamma \leqslant \nu(\operatorname{pd}_F T) + \operatorname{gl.dim}_F \Lambda$.

Denote by $\operatorname{gl.dim}_F \mathcal{C}$ the relative (or F-) global dimension of $\mathcal{C}$. Then we show that $\operatorname{gl.dim}_F \mathcal{C}$ and $\operatorname{gl.dim}\Gamma$ satisfy a similar formula, namely $\operatorname{gl.dim}_F \mathcal{C} - \operatorname{pd}_F T \leqslant \operatorname{gl.dim}\Gamma \leqslant \nu(n,r) + \operatorname{gl.dim}_F \mathcal{C}$, where $\nu(n,r)$ is the upper bound of $(\mathcal{Y}_T^{\mathcal{C}}\text{-}\operatorname{resdim}(\operatorname{mod}\Gamma)$ (see Proposition 3.2.6).

The main result in this section is given below. The result is a generalization of [**11**, Dual of Proposition 4.1].

Proposition 3.3.1. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions and assume* $\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda) = n < \infty$. *Let T be an F-tilting module in $\mathcal{C}$ with* $\operatorname{pd}_F T = r$ *and let* $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$. *Then*

$$\operatorname{gl.dim}_F \mathcal{C} - \operatorname{pd}_F T \leqslant \operatorname{gl.dim}\Gamma \leqslant \nu(n,r) + \operatorname{gl.dim}_F \mathcal{C}.$$

Proof. First we want to prove that

$$\operatorname{gl.dim}\Gamma \leqslant \nu(n,r) + \operatorname{gl.dim}_F \mathcal{C}.$$

If $\operatorname{gl.dim}_F \mathcal{C}$ is infinite, then there is nothing to prove, so we assume that it is finite. For all Y in $\mathcal{Y}_T^{\mathcal{C}}$ there is an F-exact sequence

$$0 \to Y \to I_0 \to I_1 \to \cdots \to I_s \to 0$$

with I_i in $\mathcal{I}_{\mathcal{C}}(F)$ and $s \leqslant \operatorname{gl.dim}_F \mathcal{C}$. When we apply $\operatorname{Hom}_\Lambda(T, \ )$ to the above sequence we get the following exact sequence

$$0 \to (T, Y) \to (T, I_0) \to \cdots \to (T, I_s) \to 0.$$

By Theorem 3.2.7(b) we have that $\operatorname{id}_\Gamma(T, \mathcal{I}_{\mathcal{C}}(F)) \leqslant \nu(n,r)$, hence it follows that $\operatorname{id}_\Gamma(T, \mathcal{Y}_T^{\mathcal{C}}) \leqslant \nu(n,r) + \operatorname{gl.dim}_F \mathcal{C}$. By Proposition 3.2.6 we have that $\Omega_F^{\nu(n,r)}(B)$ is in $(T, \mathcal{Y}_T^{\mathcal{C}})$ for all B in $\operatorname{mod}\Gamma$. Hence

$$\operatorname{id}_\Gamma B \leqslant \operatorname{id}_\Gamma(T, Y) \leqslant \nu(n,r) + \operatorname{gl.dim}_F \mathcal{C}$$

for all Y in $\mathcal{Y}_T^{\mathcal{C}}$, since Γ is in $(T, \mathcal{Y}_T^{\mathcal{C}})$. Therefore we have shown that $\operatorname{gl.dim}\Gamma \leqslant \nu(n,r) + \operatorname{gl.dim}_F \mathcal{C}$.

Now we want to show that $\operatorname{gl.dim}_F \mathcal{C} \leqslant \operatorname{pd}_F T + \operatorname{gl.dim}\Gamma$. If $\operatorname{gl.dim}\Gamma$ is infinite, there is nothing to prove, so we assume that it is finite. By the dual of [**11**, Proposition 3.7] we have $\operatorname{Ext}_F^i(C, A) \simeq \operatorname{Ext}_\Gamma^i((T, C), (T, A))$ for all A, and C in $\mathcal{Y}_F^{\mathcal{C}}$. So $\operatorname{Ext}_F^i(C, A) = 0$ for $i > \operatorname{gl.dim}\Gamma$.

We claim that if $\operatorname{Ext}_F^i(\mathcal{Y}_F^{\mathcal{C}}, B) = 0$ for all $i > j$ then $\operatorname{Ext}_F^i(\ , B) = 0$ for all $i > j$, equivalently $\Omega_F^{-j}(B)$ is in $\mathcal{I}_{\mathcal{C}}(F)$. For, let N be in $\mathcal{C}$. By Proposition 3.1.2, we have that $\mathcal{Y}_T^{\mathcal{C}}\text{-}\operatorname{coresdim}_F \mathcal{C} = r$ is finite, so we have an F-exact sequence

$$0 \to N \to Y_0 \to \cdots \to Y_r \to 0$$

with Y_i in $\mathcal{Y}_T^{\mathcal{C}}$. Applying $(\ , B)$ to the above sequence and using dimension shift, we get the following

$$\operatorname{Ext}_F^i(N, B) \simeq \operatorname{Ext}_F^{i+r}(Y_r, B) = 0 \quad \text{for all } i > j.$$

So $\operatorname{Ext}_F^i(N, B) = 0$ for all $i > j$ and for all N in $\mathcal{C}$, which is equivalent to saying that $\Omega_F^{-j}(B)$ is in $\mathcal{I}_{\mathcal{C}}(F)$. Hence the claim follows.

Now since $\operatorname{Ext}_F^i(C, A) = 0$ for $i > \operatorname{gl.dim}\Gamma$ for all C and A in $\mathcal{Y}_T^{\mathcal{C}}$, we have, by the above claim, that $\Omega_F^{-\operatorname{gl.dim}\Gamma}(A)$ is in $\mathcal{I}_{\mathcal{C}}(F)$. By Proposition 3.1.2 we have $\mathcal{Y}_T^{\mathcal{C}}\text{-}\operatorname{coresdim}_F \mathcal{C} \leqslant r$.

Since $\mathcal{I}_{\mathcal{C}}(F)$ is contained in $\mathcal{Y}_T^{\mathcal{C}}$, we have an F-exact sequence $0 \to N \to I_0 \to \cdots \to I_{r-1} \to \Omega_F^{-r}(N) \to 0$ with $\Omega_F^{-r}(N)$ in $\mathcal{Y}_T^{\mathcal{C}}$ for all N in $\mathcal{C}$. So $\operatorname{id}_F N \leqslant r + \operatorname{gl.dim}\Gamma$ for all N in $\mathcal{C}$. Therefore, we have that

$$\operatorname{gl.dim}_F \mathcal{C} \leqslant \operatorname{pd}_F T + \operatorname{gl.dim}\Lambda$$

and the result follows. □

Chapter 4

Relative Theory and Stratifying Systems

Let Λ be an artin algebra and let $\operatorname{mod}\Lambda$ denote the category of finitely generated left Λ modules. In this chapter we shall look at the relationship between relative theory in subcategories and stratifying systems. Throughout this chapter $\mathcal{C}$ denotes a functorially finite subcategory of $\operatorname{mod}\Lambda$ which is closed under extensions. We fix a subfunctor $F = F_{\mathcal{X}}$ in $\mathcal{C}$, where $\mathcal{X}$ is a contravariantly finite generator subcategory of $\mathcal{C}$.

Erdmann and Sáenz [**16**] introduced the concept of a stratifying system. The concept was studied further by Marcos *et al.* [**26**], where the authors introduced the notion of an Ext-projective stratifying system. Suppose Θ is a stratifying system and let $\mathcal{F}(\Theta)$ denote the category of Λ-modules filtered by Θ. Let Q denote the direct sum of all non-isomorphic indecomposable Ext-projective modules in $\mathcal{F}(\Theta)$. One of the main results of [**26**] states that the algebra $B = \operatorname{End}_\Lambda(Q)^{\mathrm{op}}$ is standardly stratified and the functor $\operatorname{Hom}_\Lambda(Q,\ )$ induces an equivalence between the subcategories $\mathcal{F}_\Lambda(\Theta)$ and $\mathcal{F}_B(\Delta)$. Moreover, $\mathcal{F}_\Gamma(\Delta) = \widecheck{\operatorname{add}{}_BT}$, where ${}_BT$ is the characteristic tilting B-module.

Let T be an F-tilting F-cotilting module in $\mathcal{C}$ and denote $\operatorname{End}_{\mathcal{C}}(T)^{\mathrm{op}}$ by Γ. In Section 4.1 we prove the main result of this chapter, which shows that the Γ-module $\operatorname{Hom}_\Lambda(T, \mathcal{I}_{\mathcal{C}}(F))$ is tilting. Moreover, there is an equivalence between the subcategories $\widehat{\operatorname{add} T_{\mathcal{C}}}$ of $\mathcal{C}$ and $\widecheck{(T, \mathcal{I}_{\mathcal{C}}(F))}$ of $\operatorname{mod}\Gamma$. We then show that the above-mentioned result from [**26**] is a corollary to the main result of this chapter. In Section 4.1.2 we first show that if $\mathcal{C}$-approximation dimension of $\operatorname{mod}\Lambda$ is finite, then Γ is an artin Gorenstein algebra, which generalizes [**12**, Proposition 3.1]. We then construct quasihereditary algebras using relative theory in subcategories.

In the third section we consider some examples which illustrate the theory we have developed.

All subcategories of $\operatorname{mod}\Lambda$ will be full additive subcategories which are closed under isomorphisms and summands.

4.1. Relative Tilting Cotilting Modules in Subcategories

Consider the subfunctor $F = F_{\mathcal{X}}$ in $\mathcal{C}$. Let T be an F-tilting F-cotilting module in $\mathcal{C}$ and denote $\operatorname{End}_\Lambda(T)^{\mathrm{op}}$ by Γ. In this section we show that the module $(T, \mathcal{I}_\mathcal{C}(F))$ is a standard tilting Γ-module and the tilting functor induces an equivalence between $\widehat{\operatorname{add} T_\mathcal{C}}$ and $(T, \widehat{\operatorname{add} T_\mathcal{C}})$. Moreover we show that the image $(T, \widehat{\operatorname{add} T_\mathcal{C}})$ of the functor is identified with the category $(T, \widehat{\mathcal{I}_\mathcal{C}(F)})$.

Let T be an F-tilting F-cotilting module in $\mathcal{C}$ and denote $\operatorname{End}_\Lambda(T)^{\mathrm{op}}$ by Γ. In the next result we show that the Γ-module $(T, \mathcal{I}_\mathcal{C}(F))$ is tilting and the tilting functor induces an equivalence between $\widehat{\operatorname{add} T_\mathcal{C}}$ and $(T, \widehat{\operatorname{add} T_\mathcal{C}})$. This is the main result of this section.

Theorem 4.1.1. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions. Let T be an F-tilting F-cotilting module in $\mathcal{C}$ and let* $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$. *Then*

(a) *The Γ-module $(T, \mathcal{I}_\mathcal{C}(F))$ is tilting with projective dimension at most* $\operatorname{id}_F T$. *Moreover, $(T, \mathcal{I}_\mathcal{C}(F))$ is of finite type.*
(b) *The functor* $\operatorname{Hom}_\Lambda(T, \ ) \colon \operatorname{mod}\Lambda \to \operatorname{mod}\Gamma$ *induces and equivalence between* $\widehat{\operatorname{add} T_\mathcal{C}}$ *and* $(T, \widehat{\operatorname{add} T_\mathcal{C}})$.

Proof. (a) By Corollary 3.1.5, we have that

$$\operatorname{Ext}^i_\Gamma((T, \mathcal{I}_\mathcal{C}(F)), (T, \mathcal{I}_\mathcal{C}(F))) \simeq \operatorname{Ext}^i_F(\mathcal{I}_\mathcal{C}(F), \mathcal{I}_\mathcal{C}(F)) = 0$$

since $\mathcal{I}_\mathcal{C}(F)$ is contained in $\mathcal{Y}^\mathcal{C}_T$. Since T is F-cotilting module in $\mathcal{C}$, we have an F-exact sequence.

$$0 \to T_m \to \cdots \to T_1 \to T_0 \to \mathcal{I}_\mathcal{C}(F) \to 0$$

with T_i in $\operatorname{add} T$ and $m \leqslant \operatorname{id}_F T$. Applying the functor $(T, \ )$ to the above sequence we get that $\operatorname{pd}_\Gamma(T, \mathcal{I}_\mathcal{C}(F))$ is finite. In particular, $\operatorname{pd}_\Gamma(T, \mathcal{I}_\mathcal{C}(F)) \leqslant \operatorname{id}_F T$. We also have an F-exact sequence

$$0 \to T \to I_0 \to I_1 \to \cdots \to I_s \to 0 \tag{1}$$

with the I_i in $\mathcal{I}_\mathcal{C}(F)$, since T is F-cotilting. Applying $\operatorname{Hom}_\Lambda(T, \ )$ to (1) we get that Γ is in $(T, \widehat{\mathcal{I}_\mathcal{C}(F)})$. Therefore $(T, \mathcal{I}_\mathcal{C}(F))$ is a standard tilting Γ-module.

By the corollary to [**28**, Proposition 1.8] we have that (T, I), for all I in $\mathcal{I}_{\mathcal{C}}(F)$, is a direct summand of

$$\operatorname{add} \bigoplus_{i=0}^{s} (T, I_i)$$

with the I_i in $\mathcal{I}_{\mathcal{C}}(F)$. Hence $(T, \mathcal{I}_{\mathcal{C}}(F))$ is of finite type.

(b) This follows from Theorem 3.1.6, since $\widehat{\operatorname{add} T_{\mathcal{C}}}$ is contained in $T_{\mathcal{C}}^{\perp}$. $\square$

The following result show that $\operatorname{gl.dim}_F \mathcal{C}$ being finite is sufficient for Theorem 4.1.1.

Corollary 4.1.2. *Let T be an F-tilting module in $\mathcal{C}$ and assume that* $\operatorname{gl.dim}_F \mathcal{C}$ *is finite. Then T is F-cotilting module in $\mathcal{C}$.*

Proof. That T is F-selforthogonal and has finite F-injective dimension follows, since T is F-tilting and $\operatorname{gl.dim}_F \mathcal{C}$ is finite. Since $\operatorname{gl.dim}_F \mathcal{C}$ is finite and T is an F-tilting module in $\mathcal{C}$, we have that $T_{\mathcal{C}}^{\perp} = \widehat{\operatorname{add} T}$ by Lemma 3.1.3. Therefore $\mathcal{I}_{\mathcal{C}}(F)$ has finite F-$\operatorname{add} T$-resolution. $\square$

The following is also a consequence of the Theorem 4.1.1.

Corollary 4.1.3. *The subcategory $\mathcal{I}_{\mathcal{C}}(F)$ is of finite type.*

Proof. By Theorem 4.1.1 (a) we have that $(T, \mathcal{I}_{\mathcal{C}}(F))$ is of finite type. By Theorem 4.1.1 (b) there is an equivalence between $\mathcal{I}_{\mathcal{C}}(F)$ and $(T, \mathcal{I}_{\mathcal{C}}(F))$. Then the claim follows. $\square$

Now we want to show that the subcategories $(T, \widehat{\operatorname{add} T_{\mathcal{C}}})$ and $(T, \widecheck{\mathcal{I}_{\mathcal{C}}(F)})$ coincide. We need the following results.

Lemma 4.1.4. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod} \Lambda$ *which is closed under extensions. Let T be an F-tilting module in $\mathcal{C}$ and let* $\Gamma = \operatorname{End}_{\Lambda}(T)^{\mathrm{op}}$. *Assume* $\operatorname{pd}_{\Gamma}(T, \mathcal{I}_{\mathcal{C}}(F))$ *is finite. Then DT is in $(T, \mathcal{I}_{\mathcal{C}}(F))^{\perp}$.*

Proof. Since $\mathcal{C}$ is functorially finite in $\operatorname{mod} \Lambda$, we have a right $\mathcal{C}$-approximation resolution

$$\cdots \to C_1 \xrightarrow{g_1} C_0 \xrightarrow{g_0} D\Lambda$$

of $D\Lambda$. Denote $\operatorname{Ker} g_i$ by Y_{i+1} for $i \geqslant 0$. Applying $(T, \ )$ to the above sequence we get an exact sequence

$$\cdots \to (T, C_1) \to (T, C_0) \to (T, D\Lambda) \to 0. \tag{2}$$

since T is in $\mathcal{C}$. Consider the short exact sequence $0 \to (T, Y_{j+1}) \to (T, C_j) \to (T, Y_j) \to 0$. Applying $((T, \mathcal{I}_{\mathcal{C}}(F)), \ )$ to the sequence we get the following commutative diagram by Lemma 3.1.4

$$\begin{tikzcd}
0 \arrow[r] & ((T,I),(T,Y_{j+1})) \arrow[r] & ((T,I),(T,C_j)) \arrow[r] & ((T,I),(T,Y_j)) & \\
0 \arrow[r] & (I,Y_{j+1}) \arrow[r] \arrow[u, "\wr"] & (I,C_j) \arrow[r] \arrow[u, "\wr"] & (I,Y_j) \arrow[r] \arrow[u, "\wr"] & 0
\end{tikzcd}$$

(3)

Since I is in $\mathcal{C}$ we have that the bottom row of (3) is exact. Hence the top row of (3) is exact. Thus the functor $((T,I),\ )$, for I in $\mathcal{I}_{\mathcal{C}}(F)$, is exact on (2). Then we have

$$\operatorname{Ext}^1_\Gamma((T,I),(T,Y_j)) = 0 \text{ for all } j > 0$$

Let s be a nonnegative integer, then by dimension shift we have that

$$\operatorname{Ext}^i_\Gamma((T,I),(T,Y_s)) = 0 \text{ for all } i > 0 \text{ and for all } s \geqslant \operatorname{pd}_\Gamma(T,I).$$

But by the assumption we have that $\operatorname{pd}_\Gamma(T,\mathcal{I}_{\mathcal{C}}(F))$ is finite. Hence (T,Y_s) is in $(T,\mathcal{I}_{\mathcal{C}}(F))^\perp$ for $s > \operatorname{pd}_\Gamma(T,I)$. Then by using the fact that $(T,\mathcal{I}_{\mathcal{C}}(F))^\perp$ is closed under cokernels of monomorphisms and (2), it follows by induction that DT is in $(T,\mathcal{I}_{\mathcal{C}}(F))^\perp$. □

As an immediate consequence of the above result we have the following.

Corollary 4.1.5. *The functor*

$$T\otimes_\Gamma \ \simeq D(\ ,DT)\colon \operatorname{mod}\Gamma \to \operatorname{mod}\Lambda$$

is exact on $(\widetilde{T,\mathcal{I}_{\mathcal{C}}(F)})$.

Proof. Let Y be in $(\widetilde{T,\mathcal{I}_{\mathcal{C}}(F)})$. Then we have an exact sequence

$$0 \to Y \to (T,I_0) \to (T,I_1) \to \cdots \to (T,I_q) \to 0$$

with the I_j in $\mathcal{I}_{\mathcal{C}}(F)$. Applying $(\ ,DT)$ to the above sequence, and then using dimension shift and Lemma 4.1.4 we get that

$$\operatorname{Ext}^i_\Gamma(Y,DT) \simeq \operatorname{Ext}^{i+q}_\Gamma((T,I_q),DT) = 0 \text{ for all } i > 0.$$

Then the claim follows. □

We now show that the subcategory $(T,\widehat{\operatorname{add} T_{\mathcal{C}}})$ is identified with the subcategory $(\widetilde{T,\mathcal{I}_{\mathcal{C}}(F)})$.

Proposition 4.1.6. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions. Let T be an F-tilting F-cotilting module in $\mathcal{C}$ and let* $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$. *Then* $(T,\widehat{\operatorname{add} T_{\mathcal{C}}}) \simeq (\widetilde{T,\mathcal{I}_{\mathcal{C}}(F)})$.

Proof. By Theorem 4.1.1(b) Z is in $\widehat{\operatorname{add} T_{\mathcal{C}}}$ if and only if (T,Z) is in $(T,\widehat{\operatorname{add} T_{\mathcal{C}}})$. Let Z be in $\widehat{\operatorname{add} T_{\mathcal{C}}}$. Then we have an F-exact sequence

$$0 \to Z \to T_0 \to T_1 \to \cdots \to T_m \to 0$$

with the T_i in $\operatorname{add} T$. Since $\operatorname{id}_F T$ is finite, we have that $\operatorname{id}_F Z$ is finite by [**28**, Lemma 2.1(1)]. Let $0 \to Z \to I_0 \to \cdots \to I_s \to 0$ be an F-injective resolution of Z. Applying $(T,\ )$ to the resolution of Z we get an exact sequence

$$0 \to (T,Z) \to (T,I_0) \to (T,I_1) \to \cdots \to (T,I_s) \to 0,$$

thus, (T,Z) is in $\widecheck{(T,\mathcal{I}_{\mathcal{C}}(F))}$. Hence $(T,\widehat{\operatorname{add} T_{\mathcal{C}}})$ is contained in $\widecheck{(T,\mathcal{I}_{\mathcal{C}}(F))}$.

Now let Y be in $\widecheck{(T,\mathcal{I}_{\mathcal{C}}(F))}$. Then we have an exact sequence

$$0 \to Y \to (T,I_0) \to (T,I_1) \to \cdots \to (T,I_s) \to 0$$

with the I_i in $\mathcal{I}_{\mathcal{C}}(F)$. By Theorem 4.1.1(a) we have that $\operatorname{pd}_\Gamma(T,I_j) < \infty$, hence $\operatorname{pd}_\Gamma Y < \infty$ (by [**28**, Lemma 2.1(4)]). Consider a projective resolution

$$0 \to P_t \to \cdots \to P_1 \to P_0 \to Y \to 0$$

of Y over Γ. Denote $\Omega^i_\Gamma(Y)$ by Y_i. Note that all Y_i are in $\widecheck{(T,\mathcal{I}_{\mathcal{C}}(F))}$, since $(T,\mathcal{I}_{\mathcal{C}}(F))$ is tilting. Applying $T\otimes_\Gamma$ to the above sequence we get the following exact sequence

$$0 \to T\otimes_\Gamma P_t \to \cdots \to T\otimes_\Gamma P_1 \to T\otimes_\Gamma P_0 \to T\otimes_\Gamma Y \to 0 \tag{4}$$

by Corollary 4.1.5. But since $T\otimes_\Gamma \Gamma \simeq T$ we get that (4) is isomorphic to the following exact sequence

$$0 \to T_t \to \cdots \to T_1 \to T_0 \to T\otimes_\Gamma Y \to 0. \tag{5}$$

We need to show that (5) is F-exact. But by using the adjoint isomorphism and the fact that the Y_j are in ${}^\perp(T,\mathcal{I}_{\mathcal{C}}(F))$, we have that the functor $\operatorname{Hom}_\Lambda(\ ,J)$, for J in $\mathcal{I}_{\mathcal{C}}(F)$, is exact on (4). Hence (5) is $F^{\mathcal{I}_{\mathcal{C}}(F)}$-exact. But then by Proposition 2.2.11 we have that (5) is in $\mathcal{C}$. So (5) is F-exact by using Corollary 2.2.9. Therefore $T\otimes_\Gamma Y$ is in $\widehat{\operatorname{add} T_{\mathcal{C}}}$. Then using Theorem 4.1.1(b) we get that $(T,T\otimes_\Gamma Y)$ is in $(T,\widehat{\operatorname{add} T_{\mathcal{C}}})$. But by [**28**, Lemma 1.9], we have that $Y \simeq (T,T\otimes_\Gamma Y)$. Therefore Y is in $(T,\widehat{\operatorname{add} T_{\mathcal{C}}})$. This completes the proof. ☐

4.1.1. Stratifying Systems

In this subsection we look at the relationship between relative theory and stratifying systems. We show how a relative theory can be defined in a subcategory associated with a stratifying system. Then we show that the main result of this chapter generalizes one of the main results of [**26**].

But first we recall the definition of a stratifying system.

Definition.[**16**, Definition 1.1] Let R be a finite dimensional algebra. A *stratifying system* $\Theta = (\Theta, \preceq)$ of size t consists of a set $\Theta = \{\theta(i)\}_{i=1}^t$ of indecomposable R-modules and a total order $\preceq$ on the set $\{1,2,\ldots,t\}$ satisfying the following conditions:

(i) $\operatorname{Hom}_R(\theta(j),\theta(i)) = 0$ for $j \succ i$,

(ii) $\mathrm{Ext}^1_R(\theta(j), \theta(i)) = 0$ for $j \geq i$.

As before, $\mathcal{F}(\Theta)$ denotes the subcategory of $\mathrm{mod}\, R$ consisting of all modules having filtrations with quotients isomorphic to the $\theta(i)$'s. The subcategory $\mathcal{F}(\Theta)$ is functorially finite in $\mathrm{mod}\, R$ [**30**]. If $\mathcal{F}(\Theta)$ is closed under direct summands, then it is closed under extensions [**30**].

Let Θ be a stratifying system and let $\mathcal{C} = \mathcal{F}(\Theta)$. Then $\mathcal{P}(\mathcal{C}) = \mathrm{add}\, Q$, where $Q = \bigoplus_{i=1}^t Q(i)$. The module $Q(i)$, for $i = 1, \dots, t$, is given by the exact sequence $0 \to K(i) \to Q(i) \to \theta(i) \to 0$ such that $K(i)$ is in $\mathcal{F}(\{\theta(j) \colon j \succ i\})$. Dually, $\mathcal{I}(\mathcal{C}) = \mathrm{add}\, Y$, where $Y = \bigoplus_{i=1}^t Y(i)$. The module $Y(i)$, for $i = 1, \dots, t$, is given by the exact sequence $0 \to \theta(i) \to Y(i) \to L(i) \to 0$ such that $L(i)$ is in $\mathcal{F}(\{\theta(j) \colon j \prec i\})$ [**26**] [**27**].

Now, since $\mathcal{C}$ is functorially finite in $\mathrm{mod}\, \Lambda$ and is closed under extensions, we have that $\mathcal{C}$ has enough Ext-projectives and Ext-injectives by Corollary 2.1.5 in Chapter 1. Then by [**26**, Corollary 2.11] and [**16**, Lemma 1.5] we have that $\mathrm{gl.dim}\, \mathcal{C}$ is finite. It is easy to see that $\mathcal{P}(\mathcal{C})$ and $\mathcal{I}(\mathcal{C})$ are contravariantly and covariantly finite subcategories of $\mathcal{C}$, respectively.

Consider the subfunctor $F = F_{\mathcal{X}}$, where $\mathcal{X} = \mathcal{P}(\mathcal{C})$. Then F is the trivial subfunctor in $\mathcal{C}$ with $\mathrm{gl.dim}_F\, \mathcal{C}$ finite. We have that $\mathcal{P}_{\mathcal{C}}(F) = \mathrm{add}\, Q$ and $\mathcal{I}_{\mathcal{C}}(F) = \mathrm{add}\, Y$. Let T be the trivial F-tilting module Q in $\mathcal{C}$ and let $\Gamma = \mathrm{End}_\Lambda(T)^{\mathrm{op}}$. Then we have the following result which is a consequence of Theorem 4.1.1 and Proposition 4.1.6.

Theorem 4.1.7. [**26**, Theorem 3.1, 3.2] *Let Θ be a stratifying system and consider the category $\mathcal{F}(\Theta)$. Then*

(a) $\mathrm{Hom}_\Lambda(T, Y)$ *is a tilting* Γ*-module.*
(b) *The functor* $\mathrm{Hom}_\Lambda(T, \ ) \colon \mathrm{mod}\, \Lambda \to \mathrm{mod}\, \Gamma$ *induces an equivalence between* $\mathcal{F}(\Theta)$ *and* $\mathrm{Hom}_\Lambda(T, \mathcal{F}(\Theta))$.
(c) $(T, \mathcal{F}(\Theta)) = \widecheck{(T, Y)}$.

Proof. (a) and (b) follow from Theorem 4.1.1, while (c) follows from Proposition 4.1.6. □

4.1.2. Construction of Gorenstein and Quasihereditary Algebras

In this section we construct Gorenstein algebras as endomorphism algebras of relative tilting relative cotilting modules. We then construct quasihereditary algebras from stratifying systems.

Recall that an algebra Λ is said to be *Gorenstein* if $\mathrm{id}_\Lambda \Lambda$ and $\mathrm{id}_{\Lambda^{\mathrm{op}}} \Lambda^{\mathrm{op}}$ are both finite. If Λ is also artin (or an algebra which admits duality), then we have that $\operatorname{id}_{\Lambda^{\mathrm{op}}} \Lambda^{\mathrm{op}}$ is finite if and only if $\operatorname{pd}_\Lambda D(\Lambda^{\mathrm{op}})$ is finite [**12**]. We have the following result which is a generalization of [**12**, Proposition 3.1].

Proposition 4.1.8. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions and assume* $\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda) = n < \infty$*. Let T be an F-tilting F-cotilting module in $\mathcal{C}$ and $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$. Then Γ is an artin Gorenstein algebra with both* $\operatorname{id}_\Gamma \Gamma$ *and* $\operatorname{pd}_\Gamma D(\Gamma^{\mathrm{op}})$ *at most* $\operatorname{id}_F T + \nu(n, r)$*.*

Proof. By Theorem 4.1.1 we have that $(T, \mathcal{I}_\mathcal{C}(F))$ is a tilting Γ-module with $\operatorname{pd}_\Gamma(T, \mathcal{I}_\mathcal{C}(F)) \leqslant \operatorname{id}_F T$. So we have an exact sequence

$$0 \to \Gamma \to (T, I_0) \to (T, I_1) \to \cdots \to (T, I_s) \to 0$$

with the (T, I_j) in $(T, \mathcal{I}_\mathcal{C}(F))$ and $s \leqslant \operatorname{id}_F T$. Then by Theorem 3.2.7 we have that $\operatorname{id}_\Gamma \Gamma \leqslant \operatorname{id}_F T + \nu(n, r)$.

On the other hand, we have, by Theorem 3.2.7, an exact sequence

$$0 \to (T, I_t) \to \cdots \to (T, I_1) \to (T, I_0) \to D(\Gamma^{\mathrm{op}}) \to 0$$

with the (T, I_j) in $(T, \mathcal{I}_\mathcal{C}(F))$ and $t \leqslant \nu(n, r)$, since $(T, \mathcal{I}_\mathcal{C}(F))$ is a cotilting Γ-module. Hence

$$\operatorname{pd}_\Gamma D(\Gamma^{\mathrm{op}}) \leqslant \operatorname{id}_F T + \nu(n, r).$$

Therefore Γ is artin Gorenstein. □

The following result gives us an important subclass of Gorenstein algebras, namely a class of algebras of finite global dimensions.

Proposition 4.1.9. *Let $\mathcal{C}$ be a functorially finite subcategory of* $\operatorname{mod}\Lambda$ *which is closed under extensions. Let T be an F-tilting module in $\mathcal{C}$. Assume* $\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda)$ *and* $\operatorname{gl.dim}_F \mathcal{C}$ *are finite. Then* $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$ *has finite global dimension.*

Proof. Follows easily from Proposition 3.3.1. □

The following result, which is a consequence of Proposition 4.1.9, gives a sufficient condition for obtaining a quasihereditary algebra for a given stratifying system Θ. Let Q denote the direct sum of non-isomorphism indecomposable Ext-projective modules in $\mathcal{F}(\Theta)$.

Corollary 4.1.10. *Let Θ be a stratifying system and Q be as above. Assume* $\mathcal{F}(\Theta)\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda)$ *is finite. Then* $\operatorname{End}_\Lambda(Q)^{\mathrm{op}}$ *is quasihereditary.*

Proof. Define a subfunctor $F = F_\mathcal{X}$, where $\mathcal{X} = \operatorname{add} Q$. Then we have that $\operatorname{gl.dim}_F \mathcal{F}(\Theta)$ is finite. By [**26**, Theorem 0.1] we have that $\operatorname{End}_\Lambda(Q)^{\mathrm{op}}$ is a standardly stratified algebra. But then by Proposition 4.1.9 we have that $\operatorname{End}_\Lambda(Q)^{\mathrm{op}}$ has finite global dimension. Hence $\operatorname{End}_\Lambda(Q)^{\mathrm{op}}$ is quasihereditary by using [**1**, Theorem 2.4]. □

4.2. Examples

In this section we consider some examples. Most examples will illustrate the theory we have developed. But we also give examples where the theory does not work. We show that if $\mathcal{C}$- app. dim(mod Λ) $= \infty$, then $(T, \mathcal{I}_{\mathcal{C}}(F))$ is not a (co)tilting Γ-module, where T is an F-tilting module in $\mathcal{C}$ and $\Gamma = \operatorname{End}_{\Lambda}(T)^{\mathrm{op}}$. We also give an example where $\mathcal{C}$- app. dim(mod Λ) $< \infty$, but the Γ-module $(T, \mathcal{I}_{\mathcal{C}}(F))$ is not tilting.

The following example illustrates the remark after Proposition 3.2.5. The example shows that $\mathcal{C}$- app. dim(mod Λ) being finite is not necessary for the equality ${}^{\perp}(T, \mathcal{I}_{\mathcal{C}}(F)) = (T, \mathcal{Y}_T^{\mathcal{C}})$.

Example 4.2.1. Let Λ be an algebra given by the quiver

$$\begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, bend left, "\beta_1"] & 2 \arrow[l, bend left, "\beta_2"] \end{tikzcd}$$

with radical square-zero relations. Denote by P_i, I_i and S_i the indecomposable projective, injective and simple Λ-modules corresponding to the vertex i. Let $\mathcal{C} = \mathcal{F}(\Theta)$ where $\Theta = \{P_1/S_2, P_2\}$. Note that $\mathcal{C}$ is closed under summands, so it is closed under extensions by [**30**]. $\mathcal{C}$ is functorially finite since it is of finite type. A right $\mathcal{C}$-approximation resolution of S_1 is

$$\cdots \to P_1/S_2 \to P_1/S_2 \to S_1 \to 0,$$

then by Proposition 2.4.2 we have $\mathcal{C}$- app. dim(mod Λ) $= \infty$. We have $\mathcal{P}(\mathcal{C}) = \mathcal{I}(\mathcal{C}) = \mathcal{C}$. Let $F = F_{\mathcal{X}}$ where $\mathcal{X} = \mathcal{P}(\mathcal{C})$. Then the only F-tilting module up to isomorphism is $T = P_1/S_2 \oplus P_2$. We have $\mathcal{I}_{\mathcal{C}}(F) = \mathcal{P}_{\mathcal{C}}(F) = \mathcal{C}$.

Let $\Gamma = \operatorname{End}_{\Lambda}(T)^{\mathrm{op}}$ and denote by Q_i the projective Γ-module corresponding to the vertex i. It can be shown that $(T, \mathcal{Y}_T^{\mathcal{C}}) = (T, \mathcal{C}) = {}^{\perp}(T, \mathcal{I}_{\mathcal{C}}(F))$.

The following example illustrates the main result of Section 4.1 which says that the Γ-module $(T, \mathcal{I}_{\mathcal{C}}(F))$ is tilting. It also shows that the Γ-module $(T, \mathcal{I}_{\mathcal{C}}(F))$ is not cotilting.

Example 4.2.2. Let Λ be an algebra given by the quiver in Example 4.2.1 with relations $\alpha^2 = 0$, $\beta_1\beta_2 = 0$ and $\beta_1\alpha = \alpha\beta_2 = 0$. Let $\theta_1 = P_1/P_2$ and $\theta_2 = P_2$. Then $\mathcal{C} = \mathcal{F}(\Theta) = \operatorname{add}\{\theta_1, P_1, P_2\}$ is closed under direct summands, hence $\mathcal{C}$ is closed under extensions. A right $\mathcal{C}$-approximation resolution of S_2 is

$$\cdots \to P_1/P_2 \to P_1/P_2 \to P_2 \to S_2 \to 0.$$

Then by Proposition 2.4.2 we have that $\mathcal{C}$- app. dim(mod Λ) $= \infty$. Let $\mathcal{X} = \mathcal{C}$ and consider the subfunctor $F = F_{\mathcal{X}}$. Then we have that $\mathcal{P}_{\mathcal{C}}(F) = \mathcal{I}_{\mathcal{C}}(F) = \mathcal{C}$ and gl. $\dim_F \mathcal{C} = 0$. There is only one F-tilting module in $\mathcal{C}$ up to isomorphism, namely the trivial F-tilting module $T = P_1 \oplus \theta_1 \oplus P_2$. Let

$\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$ and denote by Q_i and I_i the projective and injective Γ-module corresponding to the vertex i. Then the radical filtrations of Q_i and I_i, for $i = 1, 2, 3$, look as follows:

$$Q_1\colon \begin{array}{cccccc} & & 1 & & & \\ & 2 & & 3 & & \\ 1 & & & & 2 & \\ & & & & & 1 \end{array} \qquad Q_2\colon \begin{array}{c} 2 \\ 1 \\ 2 \\ 1 \end{array} \qquad Q_3\colon \begin{array}{c} 3 \\ 2 \\ 1 \end{array} \qquad I_1\colon \begin{array}{ccccc} 2 & & & & 1 \\ & 1 & & 3 & \\ & & 2 & & \\ & & 1 & & \end{array} \qquad I_2\colon \begin{array}{ccccc} 2 & & & & 1 \\ & 1 & & 3 & \\ & & 2 & & \end{array} \qquad I_3\colon \begin{array}{c} 1 \\ 3 \end{array}$$

The module $(T, \mathcal{I}_{\mathcal{C}}(F))$ is Γ itself, so it is a tilting Γ-module. It can be easily seen that $\operatorname{id}_\Gamma Q_3 = \infty$. Hence Γ is not a cotilting module over itself.

Question 1. Let $\mathcal{C}$ be a functorially finite subcategory of $\operatorname{mod}\Lambda$ which is closed under extensions and assume $\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda) = \infty$. Let $\mathcal{X}$ be a contravariantly finite generator subcategory of $\mathcal{C}$ and consider the subfunctor $F = F_{\mathcal{X}}$ in $\mathcal{C}$. Is $(T, \mathcal{I}_{\mathcal{C}}(F))$ a tilting Γ-module, where T is an arbitrary F-tilting module in $\mathcal{C}$?

If T is an F-tilting F-cotilting module in $\mathcal{C}$, then the answer is given in Section 4.1. But if T is F-tilting but not F-cotilting, then we have the following example.

Example 4.2.3. Let Λ be an algebra given by the quiver

$$\begin{tikzcd} 1 \arrow[loop left] \arrow[r] & 2 \arrow[loop right] \end{tikzcd}$$

with radical square-zero relations. Denote by P_i, I_i and S_i the indecomposable projective, injective and simple Λ-module corresponding to the vertex i respectively. Let $\mathcal{C} = \operatorname{add}\{S_1, P_2, M, I_1, I_2\}$, where M is given by the following radical filtration:

$$M\colon \begin{array}{cccc} & 1 & & 2 \\ 1 & & 2 & \end{array}$$

The subcategory $\mathcal{C}$ is closed under extensions. The right $\mathcal{C}$-approximation resolution of S_2 is

$$\cdots \to I_2 \to I_2 \to S_2 \to 0.$$

Then by Proposition 2.4.2 we have that $\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda) = \infty$. Since Λ is of finite type, all subcategories of $\operatorname{mod}\Lambda$ are functorially finite as in the previous example. We have $\mathcal{P}(\mathcal{C}) = \operatorname{add}\{P_2, M\}$ and $\mathcal{I}(\mathcal{C}) = \operatorname{add}\{I_1, I_2\}$. Let $F = F_{\mathcal{X}}$ be the trivial subfunctor in $\mathcal{C}$, that is $\mathcal{X} = \mathcal{P}(\mathcal{C})$. Then we have $\mathcal{P}_{\mathcal{C}}(F) = \mathcal{P}(\mathcal{C})$ and $\mathcal{I}_{\mathcal{C}}(F) = \mathcal{I}(\mathcal{C})$. We consider the trivial F-tilting module $T = P_2 \oplus M$ in $\mathcal{C}$. It can be (easily) shown that $\operatorname{id}_F T = \infty$. Hence T is not an F-cotilting Γ-module.

The algebra $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$ is given by the quiver

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\beta"] \arrow[loop right, "\gamma"] \end{tikzcd}$$

with relations $\alpha\beta\alpha = 0$, $\gamma\alpha = 0$ and $\beta\gamma = 0$. Denote by Q_i and J_i the projective and injective Γ-module corresponding to the vertex i. Then the radical filtrations of Q_i and J_i, for $i = 1, 2, 3$, look as follows:

$$Q_1\colon \begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix} \qquad Q_2\colon \begin{smallmatrix} & 2 & & & \\ 2 & & 1 & & \\ & & & 2 & \\ & & & & 1 \end{smallmatrix} \qquad J_1\colon \begin{smallmatrix} 2 \\ 1 \\ 2 \\ 1 \end{smallmatrix} \qquad J_2\colon \begin{smallmatrix} 2 & & & \\ & 1 & & 2 \\ & & 2 & \end{smallmatrix}$$

Denote by U the the direct sum of all indecomposable modules in $\mathcal{I}_{\mathcal{C}}(F)$. Then the Γ-module $(T, U) = Q_2/Q_1 \oplus J_1$. It can be easily seen that $\operatorname{pd}_\Gamma J_1 = \infty$. Hence (T, U) is not a tilting Γ-module. It can also be seen that $\operatorname{id}_\Gamma Q_2/Q_1 = \infty$, hence (T, U) is not a cotilting module.

Now we consider examples where $\mathcal{C}$ has $\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda) < \infty$. In the following example we illustrate Theorem 3.2.7. The shows that the Γ-module $(T, \mathcal{I}_{\mathcal{C}}(F))$ is cotilting.

Example 4.2.4. Let Λ be an algebra given by the quiver

$$\begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[loop above, "\beta"] \arrow[r, "\gamma"] & 3 \end{tikzcd}$$

with relations $\gamma\alpha = 0 = \beta^2$ and $\gamma\beta\alpha = 0$. Denote by P_i, I_i and S_i the indecomposable projective, injective and simple Λ-module corresponding to the vertex i respectively. Let $\mathcal{C} = \operatorname{add}\{S_2, P_2, I_2, L, M, N\}$, where L, M and N are given by the following radical filtrations:

$$L\colon \begin{smallmatrix} 2 \\ 2 \end{smallmatrix} \qquad M\colon \begin{smallmatrix} & 2 & \\ 3 & & 2 \end{smallmatrix} \qquad N\colon \begin{smallmatrix} 2 \\ 3 \end{smallmatrix}.$$

Then $\mathcal{C}$ is closed under extensions. Again, $\mathcal{C}$ is functorially finite, since Λ is of finite type. It can be shown that $\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda) \leqslant 1$. The subcategories $\mathcal{P}(\mathcal{C}) = \operatorname{add}\{P_2, I_3\}$ and $\mathcal{I}(\mathcal{C}) = \operatorname{add}\{I_3, L\}$. It is easy to see that $\mathcal{C}$ has enough Ext-projectives and Ext-injectives, hence $\mathcal{P}(\mathcal{C})$ and $\mathcal{I}(\mathcal{C})$ are respectively contravariantly and covariantly finite in $\mathcal{C}$ by Corollary 2.1.5 (or one could use [**5**, Theorem 1.6]). Let $F = F_{\mathcal{X}}$ be the trivial subfunctor in $\mathcal{C}$. Let $T = P_2 \oplus I_3$, which is the trivial F-tilting module in $\mathcal{C}$. We have an F-exact sequence $0 \to P_2 \to I_3 \oplus I_3 \to L \to 0$, so that $\operatorname{id}_F T < \infty$ and $\mathcal{I}_{\mathcal{C}}(F)$ is in $\widehat{\operatorname{add} T_{\mathcal{C}}}$. Hence T is also an F-cotilting module in $\mathcal{C}$.

The algebra $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$ is given by the quiver

$$\begin{tikzcd} 1 \arrow[loop left, "x"] & 2 \arrow[l] \end{tikzcd}$$

with relations $x^2 = 0$. Denote by Q_i and J_i the projective and injective Γ-module corresponding to vertex i. Then the radical filtrations of Q_i and J_i, for $i = 1, 2$, look as follows:

$$Q_1\colon \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \qquad Q_2\colon \begin{smallmatrix} 2 \\ 1 \\ 1 \end{smallmatrix} \qquad J_1\colon \begin{smallmatrix} & & & 2 \\ 2 & & 1 & \\ & 1 & & \end{smallmatrix} \qquad J_2\colon 2\,.$$

The Γ-module $V = Q_2 \oplus J_1$, where $\operatorname{add} V = (T, \mathcal{I}_{\mathcal{C}}(F))$, is cotilting with $\operatorname{id}_\Gamma V = 1$ by Theorem 3.2.7. Since T is an F-tilting F-cotilting module in $\mathcal{C}$, we have that (T, V) is a tilting Γ-module by Theorem 4.1.1.

Remark. In the case where $\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda) < \infty$, the problem in Question 1 also arises.

Let $\mathcal{C}$ be a subcategory of $\operatorname{mod}\Lambda$ which is closed under extensions and assume $\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda) < \infty$. If T is an F-tilting F-cotilting module in $\mathcal{C}$, then the answer is given in Section 4.1. Otherwise, we have the following example.

Example 4.2.5. Consider Λ and $\mathcal{C}$ as in Example 4.2.4. Let $F = F_{\mathcal{X}}$, where $\mathcal{X} = \mathcal{P}(\mathcal{C}) \cup \operatorname{add} M$, then we have that $\mathcal{I}_{\mathcal{C}}(F) = \mathcal{I}(\mathcal{C}) \cup \operatorname{add} N$. The Λ-module $T = I_3 \oplus L \oplus M$ is an F-tilting module in $\mathcal{C}$ with $\operatorname{pd}_F T = 1$. It can be shown that $\operatorname{id}_F T = \infty$, hence T is not F-cotilting in $\mathcal{C}$.

Let $\Gamma = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$ and denote by Q_i, J_i and S_i the projective, injective and simple Γ-module corresponding to vertex i. Then the radical filtrations of Q_i and J_i, for $i = 1, 2, 3$, look as follows:

$$Q_1\colon \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \qquad Q_2\colon \begin{smallmatrix} & 2 & & \\ & & 3 & \\ 1 & & & \\ & & 2 & \\ & 3 & & 1 \end{smallmatrix} \qquad Q_3\colon \begin{smallmatrix} & 3 & \\ & 2 & \\ 1 & & 3 \end{smallmatrix} \qquad J_1\colon \begin{smallmatrix} 2 \\ 3 \\ 2 \\ 1 \end{smallmatrix} \qquad J_2\colon \begin{smallmatrix} 2 \\ 3 \\ 2 \end{smallmatrix} \qquad J_3\colon \begin{smallmatrix} & 2 & \\ & & 3 \\ 1 & & \\ & & 2 \\ & 3 & \end{smallmatrix}.$$

It is easy to see that the Γ-module $V = P_1 \oplus P_2 \oplus S_3$, where $\operatorname{add} V = (T, \mathcal{I}_{\mathcal{C}}(F))$, is cotilting with $\operatorname{id}_\Gamma V = 2$ (or one could use Theorem 3.2.7). But we can easily see that $\operatorname{pd}_\Gamma S_3 = \infty$, hence (T, V) is not a tilting Γ-module.

The following example illustrates Corollary 4.1.10.

Example 4.2.6. Let Λ be given by the quiver

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[d] \\ 4 \arrow[u] & 3 \arrow[l] \end{tikzcd}$$

with radical cube-zero relations. As usual P_i, I_i and S_i denote the indecomposable projective, injective and simple module corresponding to the vertex i. Denote by L_i the module $P_i/\operatorname{soc} P_i$ corresponding to the vertex i.

Let $\Theta = \{S_1, S_2, S_3, P_4\}$. Then $\mathcal{C} = \mathcal{F}(\Theta)$ is closed under summands, hence it is closed under extensions by [**30**]. Since Λ is of finite type, $\mathcal{C}$ is functorially finite. It can be shown that $\mathcal{C}$ is closed under submodules. So by Proposition 2.4.2 we have that $\mathcal{C}\text{-}\operatorname{app.dim}(\operatorname{mod}\Lambda) \leqslant 2$. We have that $\mathcal{P}(\mathcal{C}) = \operatorname{add}\{S_3, L_2, P_1, P_4\}$ and $\mathcal{I}(\mathcal{C}) = \operatorname{add}\{S_1, L_1, I_2, I_3\}$. Let $F = F_{\mathcal{X}}$, where $\mathcal{X} = \mathcal{P}(\mathcal{C})$. Then $\operatorname{gl.dim}_F \mathcal{C} \leqslant 1$.

Consider the trivial F-tilting module Q and let $\Gamma = \operatorname{End}_\Lambda(Q)^{\mathrm{op}}$. Then Γ is given by $3 \leftarrow 2 \leftarrow 1 \leftarrow 4$ with radical cube-zero relations. It is easy to see that Γ is quasihereditary with respect to the natural order.

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