Algebraic thinking

Jeremy Hodgen, Reinhard Oldenburg & Heidi Strømskag

1. Introduction

Algebra is one of the most extensively researched areas in mathematics education. Over the past forty years, many researchers have addressed the problems associated with the learning and teaching of algebra in school and beyond. Hence, as Radford noted in his plenary lecture at CERME 6, this raises the question of "whether or not there is really something new to say about algebraic thinking" (p. XXXIV). Having reviewed the work on algebraic thinking at CERME, our answer to Radford's question is most emphatically "yes". Whilst this body of work at CERME has extended our understandings of algebraic thinking, it also demonstrates how we have yet to reach a consensus on some of the fundamental questions associated with the teaching and learning of algebra. Like the discipline as a whole, the Algebraic Thinking working group has a long history. The group has featured at all CERME conferences except CERME 2. A total of 146 papers have been presented with authors representing 29 countries across the world. In this chapter, we cannot refer to all these papers individually. Rather, we highlight the main themes that have been discussed, highlighting what are in our view the key papers that contribute to these themes.

We begin with a clarification of what algebraic thinking is. On this basis, various topics of algebra are described before the particular issues of their teaching and learning is discussed. We conclude with an evaluation and critique of CERME algebraic thinking research as a whole. Finally, we consider potential future avenues of work.

2. The nature of algebraic thinking

2.1. Definitions of algebraic thinking

Drawing on Kaput (2008), we try to provide concise definitions of algebra and algebraic thinking: Whereas algebra is a cultural artefact – a body of knowledge embedded in educational systems across the world, algebraic thinking is a human activity – an activity from which algebra emerges. Since CERME 3, the title of the group is Algebraic Thinking. This title reflects that the research reported in the group is into students' ways of doing, thinking, and talking about algebra, and further, into teachers' ways of dealing with algebra in terms of instructional design and implementation. According to Kaput (2008), school algebra has two core aspects: algebra as generalisation and expression of generalisations (see section 2.4) in increasingly systematic, conventional symbol systems; and, algebra as syntactically guided action on symbols within conventional symbol systems. He claims, further, that these aspects are embodied in three strands of school algebra: algebra as the study of structures and systems abstracted from computations and relations; algebra as the study of functions, relations, and joint variation; and, algebra as the application of a cluster of modelling languages (both inside and outside of mathematics).

Another model of school algebra is proposed by Kieran (2004), where she describes three interrelated principal activities of algebra: generational activity; transformational activity; and, global/meta-level activity. The generational activities involve the creation of algebraic expressions and equations such as (i) equations that represent quantitative problem situations; (ii) expressions of generality arising from shape patterns or numerical sequences; and (iii) expressions of the rules that determine numerical relationships. The transformational activities involve syntactically-guided manipulation of formalisms including: collecting like terms; factoring; expanding brackets; simplifying expressions; exponentiation with polynomials; and, solving equations. The global/meta-level activities involve activities for which algebra is used as a tool, and include: problem solving; modelling and predicting; studying structure and change; analysing relationships; and, generalising and proving.

In comparison, generational activity in Kieran’s model parallels (but is not equivalent to) Kaput’s first core aspect; transformational activity parallels Kaput’s second core aspect; and, global/meta-level activity contains Kaput’s second and third strands of algebra.

2.2 Theoretical papers and research frameworks

Debates about theory have been a constant within the working group. However, there are relatively few purely theoretical papers. In this section, we discuss three such papers (Bergsten, 1999; Dörfler, 2007; Godino et al., 2015). A more common approach is to make a theoretical contribution that is rooted in an analysis of empirical data (Schwartz, Herschkowitz, & Dreyfus, 2002; Rinvold & Lorange, 2010).

On semiotics, Bergsten (1999) discusses figurative aspects of algebraic symbolism in light of Lakoff and Johnson’s theory of image schemata in order to better understand the development of Arcavi’s symbol sense. Bergsten’s hypothesis is that there is a dynamical interplay between form and content, facilitated by the use of image schemata (e.g., the notion of equation can be seen as formalisation of properties of the balance schema). Further, on semiotics, Dörfler (2007), gives a narrative account of a hypothetical learning process concerned with matrices. Drawing on Peirce and his own publications, Dörfler makes the hypothesis that the matrix – in the form of a diagram (iconic sign) – is not just a means, but rather the very object of mathematical activity (referred to as diagrammatic reasoning). A third contribution on semiotics is that of Radford (2010). Based on recent conceptions of thinking as it is conceptualised by anthropology, semiotics and neurosciences, he suggests that thinking is a complex form of reflection mediated by the senses, the body, signs and artefacts. Exemplified by the context of pattern generalisation, Radford suggests a classification of three forms of algebraic thinking: factual; contextual; and symbolic. The classroom data he presents provides a glimpse of the ontogenetic journey of students on their route to algebraic thinking. It stresses some of the challenges they have to overcome when passing from factual to contextual to symbolic thinking, in particular the changes that have to be accomplished in modes of signification.

Godino et al. (2015) discuss their developing model of algebraic thinking. They extend their previous three-level model of proto-algebraic reasoning in primary education by including three more advanced levels of algebraic reasoning in secondary education. The six-level model of algebraic reasoning is based on an onto-semiotic approach to mathematical knowledge and instruction, where the advanced levels involve use of parameters to represent families of functions, and the study of algebraic structures themselves. By describing and exemplifying (theoretically) the six algebraisation levels, Godino and colleagues point at potential impact of the model on teacher education.

Abstraction is naturally a recurrent theme of algebraic thinking. Whereas many researchers take a cognitive stance to abstraction (rooted in Piaget), Schwartz, Herschkowitz, and Dreyfus (2002) take a context-dependent stance to abstraction. They analyse an interview with a pair of Grade 7 students carrying out an algebraic proof, using their previously proposed model for the genesis of abstraction. The model is operational in the sense that its components are three observable epistemic actions: recognising; building with; and constructing. Another contribution on abstraction is made by Rinvold and Lorange (2010), where they propose a cognitive allegory theory in analogy with metaphor theory. Their hypothesis is that narrative text problems that can represent or create something else that is more abstract, have the potential to become prototypical text problems (allegories) for mathematical models. Rinvold and Lorange's argument is based on empirical data from teacher education, where a narrative text problem (corresponding to a linear congruence equation) is used by three student teachers as a prototype for a subsequent task that corresponds to the same mathematical topic.

Of course, theory has a crucial role in framing research and the examination of how theoretical and research frameworks can be employed to investigate algebraic thinking has featured at every CERME conference. It is striking how the number and range of theories drawn on in papers has increased over time and, at CERME 9 in 2015, a total of 25 different theoretical frameworks were used. This presents some challenges in terms of communicating between and across these frameworks (and this relates to chapter 18 of this book: "Theoretical approaches in mathematics education research"). The theoretical frameworks used in research on algebraic thinking can be categorised in three groups that have different scales. (1) Conceptual frameworks are skeletal structures of justification, rather than structures of explanation based on a formal theory. Some of these are: models for conceptualising algebra and algebraic thinking; frameworks of variables and equation solving; frameworks of teaching of linear algebra; frameworks of functions and functional thinking. (2) General theories of teaching and learning are frameworks where algebra is the focal topic "imported" into the framework by the researcher using it. Some of these are: semiotic theory; genetic epistemology; theory of sense and reference; theory of mediating tools; cognitive theory of instrument use. (3) Holistic theories are frameworks that encompass a methodology for instructional design. These include: the theory of didactical situation in mathematics; the anthropological theory of the didactic; and, variation theory. This plethora of theories raises two important questions. First, what research problems in the teaching and learning of algebra are related to which theoretical frameworks? It seems to us that many theoretical approaches could benefit from better articulation in terms of its description, explanation, prediction and scope – that is, what is it a theory of and for? Second, to what extent are these different theoretical approaches complementary or contradictory? Clearly, there is potential here to draw on the research outlined in chapter 18 of this book.

2.3 Insights from the historical studies of algebra and mathematics

From an epistemological point of view, algebra is a complex subject. It is dominated by abstract concepts that relate to more concrete entities in a subtle way that has evolved in the history of mathematics over many centuries. Thus, it is natural that several contributions clarified foundational issues by incorporating insights from history and philosophy, as noted by Drouhard, Panizza, Puig, and Radford (2006).

The genetic development of algebra as seen in the history of algebra gives valuable insights that are interesting not just for their own sake, but also because they can influence the development of modern teaching approaches. This has been exemplified by Chiappini (2011) who used Peacock's (1940) distinction between symbolic and arithmetical algebra to guide activities in a modern microworld and explain students learning needs in coping with negative numbers. In Peacock's (as referred by Chiappini) arithmetical algebra the meaning of expressions involving letters is completely determined by the laws of arithmetic. An expression like $a - b$ over the domain of natural numbers is thus only sensible if $a > b$, while in symbolic algebra it has sense by transferring the characteristics of the operation to new objects. The ALNUSET microworld (essentially an interactive number line) allows students to discover the meaning of expressions like $a - b$ by extending the domain while keeping operational characteristics – and thus passing from arithmetical to symbolic algebra.

Bagni (2006) has studied the development of equations and in-equations and inequality. He concludes that in order to avoid breaks between sense and denotation of algebraic expressions, an integrated introduction of equations and inequalities from a functional point of view is adequate and should focus on the concept of boundary points as they are solutions of the corresponding equation.

Both papers (as well as others not mentioned) support the didactical version of Haeckel's law, namely that the historical and the individual genesis of meaning have parallels.

2.4 Generalisation

Generalisation is a topic that is deeply integrated into the nature of algebra as has been shown in section 2.1. Virtually all contributions touch on it to some extent. However, some papers have dealt with generalisation explicitly. Chua and Hoyles (2011) investigate the generalisations used by different groups of Singaporean students working with number patterns. They found no differences for linear relationships, where both normal programme (average attaining) and express programme (higher attaining) students tended to use a numerical approach. However, for patterns based on quadratic relationships, a greater proportion of express programme students favoured a constructive diagrammatic strategy, whilst the normal programme students tended to use a numerical method. Similarly, Cañadas, Castro and Castro (2011) examined how the presentation of generalisation tasks affects the approach used by Spanish Grade 9 and 10 students. They discuss three tasks presented in different ways, diagrammatically, verbally and numerically. They found that students had a very strong tendency to use a numerical approach and that students were more likely to use a formal algebraic approach where a problem was presented numerically. Bolea, Bosch, and Gascón (2004) also discuss the dominance of numerical approaches, which they suggest is strongly related to teachers' understandings of school algebra. They argue that for the introduction of an algebraic modelling approach to counter this.

2.5 Early Algebra

The issue of early algebra and the relationship (or transition) between arithmetic and algebra has been a recurrent – and hotly debated – theme, which touches on the nature of algebraic thinking. Some systems, e.g., Portugal, have introduced an explicit strand of early algebra within their curriculum and a number of papers have examined the implementation of this in primary classrooms. Mestre and Oliviera (2013) describe and analyse a teaching experiment in which Grade 4 students are introduced to the use of informal symbols as quasi-variables. They find that such an approach has benefits for the development of algebraic thinking, particularly in moving from equations involving specific unknowns to equations expressing generalisations about arithmetic. In contrast, however, Gerhard (2013) argues that a generalised number approach may hinder the development of algebraic thinking. She highlights in particular an over-reliance on repeated addition as a model of multiplication and suggests that Davydov's more geometrically-based approach has the potential to overcome this gap.

Others have examined the role of discourse in early algebraic thinking. Caspi and Sfard (2011) investigate the discourse of Israeli Grade 7 students as they move from informal meta-arithmetic toward formal algebra. By examining a historical example, they show how students‘ discourse, whilst informal and ambiguous, contains some algebra-like features, not normally found in everyday discourse. Dooley (2011) examines a group of primary students in Ireland aged nine to eleven years. She uses the epistemic actions of recognising, building-with and constructing to analyse and describe the development of algebraic reasoning amongst the students. She argues that in some case the use of "vague" language facilitated this development.

Pittalis, Pitta-Pantazi, and Christou (2015) take a different approach by examining the development of number sense amongst Grade 1 students. Based on their analysis of test data over the course of 12 months, they suggest that algebraic arithmetic has a positive effect on the development of the other number sense components, particularly conventional arithmetic.

The promise of early algebra can only be fulfilled if it improves algebraic competence in the long run. Isler et al. (2017) show that students who had taken part in an early algebra program in Grade 3 show significantly better abilities in representing functional relationships when they are in Grade 6. This applies for the ability to express the relationship verbally as well as symbolically. Interestingly, both age groups were more successful with symbolic representations than with verbal representations.

3. Topics within algebra

Within the wide range of algebraic topics or more general topics that can be handled by algebraic methods, research has focused mainly on some central issues that are taught in many curricula such as functions and linear equations.

An early example is the work of Bazzini (1999) where the difficulty in symbolizing relations between variables is addressed by using questionnaires and interviews. It turned out that the translation from a situation into the symbolic language is a key issue, i.e. often the change from one semiotic register to another (natural language and symbolic language) does not occur appropriately.

Quadratic equations have been investigated by Didiş, Baş, & Erbaş (2011). They found that students tend to solve quadratic equations as quickly as possible without paying much attention to their structures and thus ignore special cases. Similarly, in a study probing the features of quadratic equations perceived by Grade 9 and 10 students in Germany, Block (2015) used the novice-expert-paradigm to identify flexible algebraic action. He found that students tend to focus on just one feature and thus their ability to act flexibly is limited. He, therefore, proposes that students should spend time categorising quadratic equations in different ways.

Basic models of logarithmic functions are discussed in Weber (2017). He identifies four such models, namely multiplicative measure (how often can you divide by the base), digit counting, decrease of hierarchy level and inverse to exponentiation. While most textbooks give preference to the last model, Weber suggests that the first gives a better operational start.

For general polynomial functions, Douady (1999) worked with qualitative reasoning (topological arguments, e.g. does the sign change around a zero?) and found that this fosters students' understanding. She argues that the study of zeros and their multiplicity should be augmented by (at least implicit) arguments about continuity.

In recent years, several papers address issues relating to equivalence and students' understanding of the equals sign, dealing with the well-known distinction between operational and relational meanings of the equal sign. Alexandrou-Leonidou and Philippou (2011) report on a teaching experiment with Cypriot primary students that enabled students to develop this dual meaning of the equal sign and showed that this in turn had a significant effect on students' ability to solve equations. Zwetzschler and Prediger (2013) argue that, whilst previous research has examined the learning of equivalence in *transformational* situations, little attention has been devoted to the equivalence of expressions in *generational* activities, where algebraic expressions are understood as "pattern generalizers of arithmetical or geometrical pattern" (p. 559). They highlight student understanding of the connection between geometric shapes and algebraic expressions as a conceptual barrier. Jones (2010) reports on a computer-based task designed to enable primary students to develop a relational understanding of the equal sign. This work suggests that students experience difficulties in coordinating two aspects of the relational meaning, sameness and exchanging.

4. Teaching and learning algebraic thinking

4.1 Students' difficulties, misconceptions and partial understandings

As is evident from the preceding discussion, the issue of student difficulties has been a central theme. Several authors have analysed difficulties in new areas such as Postelnicu's (2013) work on linear functions. Bazzini (1999) highlights the persistence of such errors even when students receive what is considered good teaching.

Building on critiques of the cognitivist literature on misconceptions, discussions at early CERME conferences have shifted attention away from categorising errors and misconceptions towards analysing students' algebraic activity from non-cognitive perspectives. Drawing particularly on socio-cultural/historical, anthropological and semiotic theories (e.g., Radford, 2010, see discussion above), discussion has focused on how context can enable students to understand algebraic symbolism (Drouhard et al., 2006, p. 638).

In recent conferences, there has been a resurgence of more cognitive perspectives, but with a focus on how these difficulties can be overcome. Several papers, for example, have replicated aspects of Küchemann's (1981) work relating to generalised number and the meaning of letters. Broadly these papers indicate that these findings still hold, subject to some minor variation due to curricular or cultural factors (e.g., Hadjidemetriou, Pampaka, Petridou, Williams, & Wo, 2007). Hodgen, Küchemann, Brown, and Coe (2010) suggest that the earlier teaching of algebra in England does not appear to have produced better understanding.

Others have examined the inter-relationship of syntactical and semantic understanding in order to better understand how to enable the learning of algebraic symbolisation. Malara and Iaderosa (1999) argue that there is a conflict between additive and multiplicative notation that creates difficulties as students move from arithmetic to algebra and suggest that this difficulty may be overcome by promoting semantic and metacognitive activity. However, Oldenburg, Hodgen and Küchemann (2013) show that the distinction between syntactic and semantic aspects of algebra is not straightforward to make empirically.

4.2 Teaching experiments and design research studies

Design based research is where a *tool* (a product or process) is designed, developed and refined through cycles of enactment, observation, analysis and redesign, with systematic feedback from those involved. The goal is transformative; new teaching and learning opportunities are created and studied in terms of their impact on teachers, students and other actors. Examples of *principled* design-based research are curriculum development and didactical engineering (e.g., Brousseau, 1997). The limited length of CERME papers is a challenge of design studies, so small-scale teaching experiments are more frequently reported.

The focus of Douady (1999) is the elements of a didactical engineering related to teaching of polynomial functions in secondary school, where the didactical hypothesis is that students need to conceive polynomial functions both from an algebraic and a topological perspective. This involves the premise that the study of zeroes and their multiplicity must be performed in relation to properties of continuity and differentiability, at least implicitly. An important principle of the design, which is rooted in the theory of didactical situations (TDS), is illustrated by the way the class is organised and how the knowledge unfolds: Conjectures proposed by the teacher are examined by the students, where the decision about the validity of the conjectures is instrumental in students' development of the target knowledge. Douady describes how the engineering has been inserted in a work decided at national level. Strømskag (2015) uses TDS' methodological principle – that the knowledge at stake is integrated in a *situation* as the optimal solution to a problem – in a teaching experiment within a teacher education programme. The experiment is concerned with a general numerical statement about odd numbers and square numbers, and she describes how the design of the milieu is related to the nature of the knowledge aimed at (an equivalence statement).

Principles of task design is the topic of Ainley, Bills, & Wilson's (2004) paper. They propose *purpose* and *utility* as design principles of a sequence of tasks for the teaching and learning of algebra in the first years of secondary school, based on the use of spreadsheets. Kieran's (2004) triadic model of activities of school algebra is used as a framework for the task design, where Ainley and colleagues have attempted to achieve a balance between generational-, transformational-, and global/meta-level activities.

Drawing on a design science approach, Gerhard (2011) uses task-centred interviews with secondary students to exemplify the use of an analytic tool – an interdependence model that examines how new algebraic knowledge interacts with prior arithmetic knowledge. In the interview tasks, arithmetic rather than algebraic word problems were chosen, where a modification was done in terms of substituting numbers by letters. Gerhard argues that it is important to distinguish between the transition from arithmetical to algebraic thinking, and the transition from numbers to variables. Framed within Bussi and Mariotti's (2008) theory of semiotic mediation, Maffei and Mariotti (2011) use Aplusix CAS to examine the interplay between different representations of algebraic expressions. They emphasise how the semiotic potential of the different artefacts does not emerge spontaneously; hence, a specific didactic organisation in terms of task design and teacher's actions is necessary.

Design-based research is theory driven. It is important to describe the principles of the underlying theory and explain how these are related to the design itself. For this to happen, it is necessary to be explicit about the principles of the design. In the studies referred to above, this is done to varying degrees. The relationship between theory and design principles (including tasks) is important to include at future CERME conferences.

4.3 Technology

The last two decades have seen a significant increase in the availability and sophistication of digital technology and discussions of the potential impact of these new tools on the teaching of algebra have been going on over the whole period. It is remarkable that even early CERME papers show a great awareness of the challenge to turn computers into useful instruments that support the learning process.

Spreadsheets have been a class of tools that have been investigated by many researchers mainly to support functional approaches to algebra and to bridge the gap between arithmetic and algebra (e.g., Ainley et al., 2004). In contrast, Hewitt (2011) reports on the use of a bespoke package, Grid Algebra, based on an underlying multiplication grid, which has been designed to help students create and discuss expressions with numbers and symbols. He shows how this can be used to enable primary students to engage with relatively complex manipulation and thus begin to “express generality”.

Computer algebra systems (CAS) have been a big issue for some time, although interest in this technology seems to be decreasing. Nevertheless, there have been a lot of deep investigations in this area and especially important research has been conducted on developing these tools further into purposeful artefacts. Maffei and Mariotti (2011) have investigated how the structure of algebraic expressions can be made explicit by use of the Aplusix CAS (see discussion above). They find that the semiotic potential of the artefact does not emerge spontaneously, but that it needs a specific didactic organisation and that natural language plays a central role in this. The three representation systems that are available to denote mathematical expressions (natural language, standard representation and tree representation) serve different purposes. The role of natural language – beyond communication with the teacher – is to focus on specific features of both the standard representation and the tree representation. Thus, it is the language of a meta-discourse.

Lagrange (2013) illustrates how theories about the organisation of cognitive processes have influenced the design of the Casyopee system and how this helps students to bridge the gap between a real-world situation and its symbolic description by use of dynamic geometry within a computer algebra environment. Similar to Maffei and Mariotti (2011), this work points at the importance of natural language in structuring situations.

Viewing the whole landscape of the use of technology for the learning and teaching algebra it is apparent that the orchestration is of crucial importance, but many questions remain unsettled, e.g. the question of how specific or general good teaching software should be. The use of computers parallels in a sense the use of symbolic expressions in algebra as in both cases it is difficult to enable students to use their full power.

4.4 Teachers and teacher education

Teachers and teacher education pervade many of the papers. Yet, surprisingly few tackle the issue directly. This may be due to the very active strands of work at CERME that are represented by two chapters in this book in these areas. Nevertheless, several papers do address these issues directly. Ayalon and Even (2010) and Kilhamn (2013) each highlight how teaching materials are enacted differently by different teachers. Novotna and Sarrazy (2006) show how the degree of variation in the problems set by teachers is related to successful student problem-solving. Mason (2007) discusses the design of a course for teachers, Developing Thinking in Algebra, in which pedagogy, didactics and mathematics are “interwoven” – an approach strongly informed by his own research, the Discipline of Noticing.

5. Evaluation and critique of research on algebraic thinking

The corpus of work on algebraic thinking at CERME is both extensive and impressive. Whilst we intend to celebrate this, our evaluation and critique will identify a number of problematic issues. It is perhaps unsurprising that the WG has been dominated by small-scale studies that are difficult to generalise. One of the strengths of the algebraic thinking group has been its focus on work in progress. This has enabled discussions that have influenced ongoing studies theoretically, methodologically and analytically. However, whilst the results of many of these studies have been reported elsewhere, this inevitably means that collection of papers presents a somewhat partial picture of algebraic thinking work over this period.

Most papers present single-country studies and there have been few bi- or cross-national studies in CERME or beyond. The CERME work as a whole highlights many interesting similarities and differences between different systems. But without specific studies that address this, generalising research findings from one country to another is inherently difficult.

Overall, and again perhaps surprisingly, there is more empirical than theoretical or conceptual research and the balance appears to be stable over time. Within empirical research, qualitative methods dominate slightly and most quantitative studies are either intervention or cross section studies. There is a lack of research on the long-term development of algebraic thinking, particularly longitudinal research within cohorts of students. In our view, the quality of quantitative studies has improved over the years. Whilst inferential statistics dominates, there are a few papers that have used more recent modelling methods, such as Rasch analysis or structural equation modelling. (See, for example, Izsák, Remillard & Templin, 2016, for a discussion of how such statistical modeling approaches can address critical questions in mathematics education.) We suggest that research on algebraic thinking might draw on methodological (and theoretical) approaches from other fields such as cognitive science. For example, to investigate embodied cognition, Henz, Oldenburg, and Schöllhorn (2015) examined the electroencephalographic brain activity of university students, whilst they performed algebraic, geometric, and numerical reasoning tasks. Initial pilot results suggest that bodily movement has a positive effect on the cognitive processing of demanding mathematical tasks.

It is a considerable strength that the papers are focused on algebraic thinking. This allows a depth of discussion and consideration that is valuable. But it is also a potential weakness and there are times when thinking about algebraic thinking could benefit from insights from other strands within CERME, in particular the work on teachers, technology, theory and number. We also consider that there are opportunities to learn from didactical strategies in other areas. Proulx (2013) raises this issue in relation to mental mathematics and argues that too little attention has been devoted to understanding mental mathematics activities with objects other than numbers.

6. Looking forward

The opportunity to review the corpus of CERME work on algebraic thinking is also an opportunity to look forward and it is clear that there are still important issues to be resolved in algebraic thinking.

The papers presented at CERME attest to the fact that students around the world still encounter what Radford (2010) refers to as *legendary difficulties* with algebra and teachers still struggle to overcome these difficulties. This raises important, and we believe urgent, questions for research on algebraic thinking.

An area that has received little attention in research on algebraic thinking is mathematical tasks – we see the need for a more systematic analysis of mathematical tasks used as instruments in classrooms, and in research on teaching and learning of algebra.

We have been somewhat surprised that no substantive literature reviews of algebraic thinking have been presented at CERME. Of course, there are literature reviews elsewhere, but there is a real need to reach a consensus on what is already known and what research questions remain open. We think that literature reviews have considerable potential to help address the question of what algebra and algebraic thinking are and to help resolve some of the theoretical difficulties that we referred to above. To reiterate this point, research an algebraic thinking draws on a wide and diverse range of theories and there is a real need to examine how these theories interact and align (or not). Without doing this, it is difficult to see how we can build a strong and coherent program of research.

As we have already noted, CERME is dominated by small-scale studies. For example, many interesting and potentially promising approaches to teaching algebraic thinking have been presented. Indeed, some of these produce very impressive results. But these are almost all conducted in a very small number of settings and often taught by the researchers themselves. Hence, important questions relate to the scaling up of research and to the communication of research beyond the research community – to teachers, policy-makers and others. A similar argument can be made in favour of more longitudinal studies. A feature of algebra is its complexity and this may render teaching interventions that are successful in the short run but unfortunate in the long run because they might lead to misconceptions (e.g., variables as real-world objects). However, according to our experience, the structured discussion process of CERME is optimal for spotting critical issues and inspiring further research that will, hopefully, improve algebra education so that more students can use it as a valuable tool in their everyday and academic lives.

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