Petter Andreas Bergh

Hochschild cohomology, complexity and support varieties

PhD-thesis

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ACKNOWLEDGEMENTS

First of all I thank my supervisor Øyvind Solberg for his guidance and constant support during my time as a PhD-student. Already from the start he urged me to go wherever my interests would take me, and the result is a thesis whose topics seemingly are quite distant from each other. Nevertheless, he has always shown genuine interest for my work. I thank him for everything from his mathematical insight to his cooking skills.

The algebra group at the Department of mathematical sciences, NTNU, is an ideal research group for a PhD-student. Thanks to all its members, and thanks to all my friends at the department for many reasons, both mathematical and non-mathematical. It has been a pleasure.

I thank the Department of Mathematics at the University of Nebraska, Lincoln, for inviting me and for their hospitality during my visit there in the spring of 2004. In particular I thank my host Luchezar Avramov and also Srikanth Iyengar for fruitful discussions and conversations, especially on the important examples in the paper "Complexity and periodicity". I learned some of the central homological techniques in that paper from Lucho on a paper napkin during a tea-and-cake break. Thanks also to the Research Council of Norway for financial support during the visit.

I thank my brother Lars Magnus for making me realize in 1998 that mathematics was the right thing for me. Had it not been for him I would not have chosen an academic career.

Finally, I thank my future wife Stine for being who she is, and for showing great patience and understanding even when I read and think mathematics at times when I should not.

Trondheim, November 2006
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INTRODUCTION

This PhD-thesis consists of the five papers

- *On the Hochschild (co)homology of quantum exterior algebras*, to appear in Comm. Algebra,
- *Complexity and periodicity*, Coll. Math. 104 (2006), no. 2, 169-191,
- *Twisted support varieties*,
- *Modules with reducible complexity*, to appear in J. Algebra,
- *On support varieties for modules over complete intersections*, to appear in Proc. Amer. Math. Soc.

These papers are roughly divided into two groups; the first three study modules over Artin algebras using techniques from Hochschild cohomology, whereas the last two papers study modules over commutative Noetherian local rings, in particular modules over complete intersections. In what follows we give a brief introduction to the central topics in this thesis.

Hochschild cohomology. Motivated by his study of derivations of associative algebras and Lie algebras in [Ho1], and by the Eilenberg-Mac Lane approach to the cohomology theory of groups, Hochschild introduced in his 1945 paper [Ho2] the cohomology theory now known as Hochschild cohomology.

Let k be a commutative Artin ring and Λ an Artin k-algebra, that is, as a k-module Λ is finitely generated. For a Λ-Λ-bimodule B, the *Hochschild complex* $C^*(\Lambda, B)$ is the cochain complex

$$\cdots \to C^{n-1}(\Lambda, B) \xrightarrow{d^{n-1}} C^n(\Lambda, B) \xrightarrow{d^n} C^{n+1}(\Lambda, B) \to \cdots$$

of k-modules and maps in which the k-modules are defined by

$$C^n(\Lambda, B) = \begin{cases} 0 & \text{for } n < 0, \\ B & \text{for } n = 0, \\ \operatorname{Hom}_k(\Lambda^{\otimes n}, B) & \text{for } n > 0, \end{cases}$$

(tensor products over k) and whose differential is defined by $d^0(b)(\lambda) = \lambda b - b\lambda$ for $b \in B, \lambda \in \Lambda$ and

$$\begin{aligned} d^n(f)(\lambda_1 \otimes \cdots \otimes \lambda_{n+1}) &= \lambda_1 f(\lambda_2 \otimes \cdots \otimes \lambda_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i f(\lambda_1 \otimes \cdots \otimes \lambda_i \lambda_{i+1} \otimes \cdots \otimes \lambda_{n+1}) \\ &+ (-1)^{n+1} f(\lambda_1 \otimes \cdots \otimes \lambda_n)\lambda_{n+1} \end{aligned}$$

for $n > 0, f \in C^n(\Lambda, B)$ and $\lambda_1 \otimes \cdots \otimes \lambda_{n+1} \in \Lambda^{\otimes(n+1)}$. The nth *Hochschild cohomology group of* Λ *with coefficients in* B, denoted $\operatorname{HH}^n(\Lambda, B)$, is the cohomology group $\operatorname{Ker} d^n / \operatorname{Im} d^{n-1}$ of $C^*(\Lambda, B)$. When $B = \Lambda$ the Hochschild cohomology group $\operatorname{HH}^n(\Lambda, \Lambda)$ is denoted by $\operatorname{HH}^n(\Lambda)$ and simply called the nth *Hochschild cohomology group of* Λ.

The low dimensional Hochschild cohomology groups may be interpreted explicitly. It follows immediately from the definition that $\operatorname{HH}^0(\Lambda, B)$ is the set of all $b \in B$ such that $\lambda b = b\lambda$ for every $\lambda \in \Lambda$, in particular we see that $\operatorname{HH}^0(\Lambda)$ coincides with the center of Λ. As for $\operatorname{HH}^1(\Lambda, B)$, let $\operatorname{Der}_k(\Lambda, B)$ be the k-module of all k-derivations (or crossed homomorphisms) of Λ on B, i.e.

$$\operatorname{Der}_k(\Lambda, B) = \{d \in \operatorname{Hom}_k(\Lambda, B) \mid d(\lambda_1\lambda_2) = \lambda_1 d\lambda_2 + (d\lambda_1)\lambda_2\}.$$

Let $\operatorname{Der}_k^*(\Lambda, B)$ be the submodule of $\operatorname{Der}_k(\Lambda, B)$ consisting of all inner derivations (or principal crossed homomorphisms), i.e. the set of all $d \in \operatorname{Der}_k(\Lambda, B)$ such that there exists an element $b \in B$ with $d\lambda = \lambda b - b\lambda$ for all $\lambda \in \Lambda$. Then $\mathrm{H}^1(\Lambda, B) = \operatorname{Der}_k(\Lambda, B)/\operatorname{Der}_k^*(\Lambda, B)$. Note that $\operatorname{Der}_k^*(\Lambda, \Lambda) = 0$ if and only if Λ is commutative.

Now suppose Λ is projective as a k-module, and denote the enveloping algebra $\Lambda \otimes \Lambda^{\mathrm{op}}$ of Λ by Λ^{e}. Following [CaE, IX.6] and [Hap], for each $n \in \mathbb{N} \cup \{0\}$ let Q_n denote the n-fold tensor product of Λ (with $Q_0 = k$), and define $P_n = Q_{n+2} = \Lambda^{\otimes(n+2)}$. We give P_n a Λ^{e}-module structure (that is, a Λ-Λ-bimodule structure) by defining

$$(\lambda \otimes \lambda')(\lambda_0 \otimes \cdots \otimes \lambda_{n+1}) = \lambda\lambda_0 \otimes \cdots \otimes \lambda_{n+1}\lambda',$$

a scalar action under which P_n actually becomes a projective bimodule. Now for each $n \geq 0$, define $d_n \colon P_n \to P_{n-1}$ by

$$\lambda_0 \otimes \cdots \otimes \lambda_{n+1} \mapsto \sum_{i=0}^{n} (-1)^i \lambda_0 \otimes \cdots \otimes \lambda_i\lambda_{i+1} \otimes \cdots \otimes \lambda_{n+1},$$

where $P_{-1} = \Lambda$. The sequence

$$\mathbb{S} : \cdots \to P_3 \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \Lambda \to 0$$

is exact, and is therefore a Λ^{e}-projective resolution of Λ. It is called the *standard resolution* (or *Bar-resolution*) of Λ. Now for any bimodule B the Hochschild complex $C^*(\Lambda, B)$ is isomorphic to the complex $\operatorname{Hom}_{\Lambda^{\mathrm{e}}}(\mathbb{S}_\Lambda, B)$, where $\mathbb{S}_\Lambda$ denotes the truncated standard resolution of Λ. Therefore the Hochschild cohomology group $\mathrm{HH}^n(\Lambda, B)$ is isomorphic to $\operatorname{Ext}_{\Lambda^{\mathrm{e}}}^n(\Lambda, B)$.

Support varieties over Artin algebras. In [Car] Carlson introduced the notion of cohomological support varieties for modules over a group algebra of a finite group. These varieties are defined in terms of the maximal ideal spectrum of the group cohomology ring, a ring Evens showed in [Eve] is finitely generated. More precisely, the variety of a module is the set of all maximal ideals in the cohomology ring containing the annihilator ideal defined by the Ext-algebra of the module.

In [SnS] Snashall and Solberg developed a theory of support varieties for arbitrary Artin algebras, using Hochschild cohomology instead of group cohomology. Let k be a commutative Artin ring and Λ an Artin k-algebra which is projective as a k-module. As $\mathrm{HH}^n(\Lambda)$ is isomorphic to $\operatorname{Ext}_{\Lambda^{\mathrm{e}}}^n(\Lambda, \Lambda)$, the graded k-module $\mathrm{HH}^*(\Lambda) = \oplus_{n=0}^{\infty} \mathrm{HH}^n(\Lambda)$ is a ring under the Yoneda product, called the *Hochschild cohomology ring* of Λ. This ring is graded commutative, and for every left Λ-module M the tensor map

$$\begin{array}{rcl} - \otimes_\Lambda M \colon \operatorname{Ext}_{\Lambda^{\mathrm{e}}}^n(\Lambda, \Lambda) & \to & \operatorname{Ext}_\Lambda^n(M, M) \\ \eta & \mapsto & \eta \otimes_\Lambda M \end{array}$$

induces a homomorphism $\mathrm{HH}^*(\Lambda) \to \operatorname{Ext}_\Lambda^*(M, M)$ of graded rings. The *support variety* of M is the set of all maximal ideals in the commutative ring $\mathrm{HH}^{2*}(\Lambda)$ containing the annihilator ideal of $\operatorname{Ext}_\Lambda^*(M, M)$.

Given certain finite generation hypotheses introduced in [EHSST], the support varieties defined in terms of Hochschild cohomology to a large extent behave precisely as the group cohomological support varieties. For example, the varieties detect modules having finite projective dimension, i.e. the support variety of a module is trivial if and only if the module has finite projective dimension. Moreover, every closed homogeneous variety in the maximal ideal spectrum is realizable as the variety of a module, and if a module decomposes then its variety decomposes accordingly. These properties are desirable in any variety theory.

Strictly speaking, the support variety of a Λ-module is defined relative to *some* commutative subalgebra H of $\mathrm{HH}^*(\Lambda)$, not necessarily the "even" subalgebra $\mathrm{HH}^{2*}(\Lambda)$. However, in [Sol] Solberg gave a completely elementary argument showing that the finite generation hypotheses mentioned hold for some $H \subseteq \mathrm{HH}^*(\Lambda)$ if and only if they hold for $\mathrm{HH}^{2*}(\Lambda)$. Thus the underlying geometric object may be taken to be the commutative algebra $\mathrm{HH}^{2*}(\Lambda)$, the advantage being that the varieties are defined relative to an object known *a priori*.

Complexity. The notion of complexity was introduced by Alperin in [Alp] as a means of studying minimal projective resolutions of modules over group algebras. However, the definition applies to all rings for which it makes sense to speak of minimal projective resolutions of modules, in particular Artin algebras and commutative Noetherian local rings.

Let k be a commutative Artin ring and Λ an Artin k-algebra. Then every finitely generated left Λ-module M has a projective cover $P_0 \twoheadrightarrow M$ which is unique up to isomorphism, and hence also a unique minimal projective resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0.$$

For each $n \geq 0$ the well defined integer $\ell_k(P_n)$ is called the nth *Betti number* of M, and denoted by $\beta_n(M)$. The *complexity* of M, denoted $\operatorname{cx} M$, is defined by

$$\operatorname{cx} M \overset{\text{def}}{=} \inf\{c \in \mathbb{N} \cup \{0\} \mid \exists a \in \mathbb{R} \text{ such that } \beta_n(M) \leq an^{c-1} \text{ for } n \gg 0\}.$$

Thus the complexity of M measures how the sequence $\beta_0(M), \beta_1(M), \beta_2(M), \ldots$ of Betti numbers behaves with respect to polynomial growth. From the definition we see that M has complexity 0 if and only if it has finite projective dimension, and that it has complexity less than or equal to 1 if and only if its sequence of Betti numbers is bounded. Over commutative Noetherian local rings every finitely generated module has a minimal *free* resolution, hence in this case the complexity of a module is defined in terms of the ranks of the modules in its minimal free resolution.

The importance of the notion of complexity is emphasized by the result of Carlson in [Car] saying that the complexity of a module over a group algebra is equal to the dimension of its cohomological support variety. This is also the case for the support varieties defined in terms of Hochschild cohomology, given the finite generation hypotheses introduced in [EHSST]. Moreover, the same holds for the cohomological support varieties defined for modules over complete intersections.

Support varieties over complete intersections. Let $(A, \mathfrak{m}, k)$ be a commutative Noetherian local ring. If the completion $\widehat{A}$ of A with respect to the $\mathfrak{m}$-adic topology is the residue ring of a regular local ring modulo an ideal generated by a regular sequence, then A is called a *complete intersection*. The terminology originates from algebraic geometry; the coordinate ring of an affine variety V over an algebraically closed field is a complete intersection if the defining ideal is generated by the least possible number of elements, namely $\operatorname{codim} V$, the codimension of the variety. The ideal is then generated by a regular sequence, and the variety is the intersection of $\operatorname{codim} V$ hypersurfaces, each corresponding to one of the elements in the regular sequence.

Let c be the codimension of A, i.e. $c = \dim_k(\mathfrak{m} / \mathfrak{m}^2) - \dim A$. As shown in [Avr], there exists a polynomial ring $\widehat{A}[\chi_1, \ldots, \chi_c]$ in c commuting cohomological operators of degree 2 (the *Eisenbud operators*, see [Eis]), satisfying the following:

for every finitely generated $\widehat{A}$-modules X and Y there is a homomorphism

$$\widehat{A}[\chi_1,\dots,\chi_c] \to \operatorname{Ext}^*_{\widehat{A}}(X,X)$$

of graded rings under which $\operatorname{Ext}^*_{\widehat{A}}(X,Y)$ is a finitely generated $\widehat{A}[\chi_1,\dots,\chi_c]$-module. Consequently $\operatorname{Ext}^*_{\widehat{A}}(X,Y)\otimes_{\widehat{A}} k$ is a finitely generated graded module over the polynomial ring $k[\chi_1,\dots,\chi_c]$ via the canonical isomorphism $\widehat{A}[\chi_1,\dots,\chi_c]\otimes_{\widehat{A}} k \simeq k[\chi_1,\dots,\chi_c]$. Denote this polynomial ring by H. The *support variety* of a finitely generated A-module M is the algebraic set

$$\{\alpha\in\tilde{k}^c \mid f(\alpha)=0 \text{ for all } f\in\operatorname{Ann}_H\Big(\operatorname{Ext}^*_{\widehat{A}}(\widehat{M},\widehat{M})\otimes_{\widehat{A}} k\Big)\},$$

where $\tilde{k}$ is the algebraic closure of k and $\widehat{M}$ is the $\mathfrak{m}$-adic completion of M.

Support varieties for modules over complete intersections were introduced by Avramov in [Avr], and in their paper [AvB] Avramov and Buchweitz showed that these varieties share many of the properties of varieties over group algebras. For instance, the dimension of the variety of a module equals the complexity of the module, hence the varieties detect modules of finite projective dimension.

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I.

ON THE HOCHSCHILD (CO)HOMOLOGY OF QUANTUM EXTERIOR ALGEBRAS

ABSTRACT

We compute the Hochschild cohomology and homology of the algebra $\Lambda = k\langle x, y\rangle/(x^2, xy + qyx, y^2)$ with coefficients in ${}_1\Lambda_\psi$ for every degree preserving k-algebra automorphism $\psi\colon \Lambda \to \Lambda$. As a result we obtain several interesting examples of the homological behavior of Λ as a bimodule.

This paper is to appear in Comm. Algebra.

ON THE HOCHSCHILD (CO)HOMOLOGY OF QUANTUM EXTERIOR ALGEBRAS

PETTER ANDREAS BERGH

Introduction

Throughout this paper, let k be a field and $q \in k$ a nonzero element which is *not* a root of unity. Denote by Λ the k-algebra

$$\Lambda = k\langle x, y\rangle/(x^2, xy + qyx, y^2),$$

and by Λ^{e} its enveloping algebra $\Lambda^{\mathrm{op}} \otimes_k \Lambda$. All modules considered are assumed to be right modules.

During the last years, this 4-dimensional graded Koszul algebra, whose module category was classified in [Sch], has provided several examples (or rather counterexamples) giving negative answers to homological conjectures and questions. Among these are the conjecture of Auslander on local Ext-limitations (see [Aus, page 815], [JoS] and [Sma]) and the question of Happel on the relation between the global dimension and the vanishing of the Hochschild cohomology (see [Hap] and [BGMS]).

We shall study the Hochschild cohomology and homology of Λ. More precisely, for every degree preserving k-algebra automorphism $\psi\colon \Lambda \to \Lambda$ we compute $\mathrm{HH}^*(\Lambda, {}_1\Lambda_\psi) = \mathrm{Ext}^*_{\Lambda^{\mathrm{e}}}(\Lambda, {}_1\Lambda_\psi)$ and $\mathrm{HH}_*(\Lambda, {}_1\Lambda_\psi) = \mathrm{Tor}^{\Lambda^{\mathrm{e}}}_*(\Lambda, {}_1\Lambda_\psi)$, that is, the Hochschild cohomology and homology of Λ with coefficients in the twisted bimodule ${}_1\Lambda_\psi$ (the action of Λ^{e} on ${}_1\Lambda_\psi$ is defined as $\lambda(\lambda_1 \otimes \lambda_2) = \lambda_1\lambda\psi(\lambda_2)$). As a result we obtain several interesting examples, both in cohomology and homology, of the homological behavior of Λ as a bimodule.

1. The Hochschild Homology

Denote by D the usual k-dual $\mathrm{Hom}_k(-, k)$, and consider the map $\phi\colon {}_\Lambda\Lambda \to D(\Lambda_\Lambda)$ of left Λ-modules defined by

$$\phi(1)(\alpha + \beta x + \gamma y + \delta yx) \overset{\mathrm{def}}{=} \delta.$$

It is easy to show that this is an injective map and hence also an isomorphism since $\dim_k \Lambda = \dim_k D(\Lambda)$, and therefore Λ is a Frobenius algebra by definition. Now take any element $\lambda \in \Lambda$, and consider the element $\phi(1) \cdot \lambda \in D(\Lambda)$ (we consider $D(\Lambda)$ as a Λ-Λ-bimodule). As ϕ is surjective, there is an element $\lambda' \in \Lambda$ such that $\lambda' \cdot \phi(1) = \phi(\lambda') = \phi(1) \cdot \lambda$, and the map $\lambda \mapsto \lambda'$ defines a k-algebra automorphism $\nu^{-1}\colon \Lambda \to \Lambda$ whose inverse ν is called the *Nakayama automorphism* of Λ (with respect to the map ϕ). Straightforward calculations show that $x \cdot \phi(1) = \phi(1) \cdot (-q^{-1}x)$ and $y \cdot \phi(1) = \phi(1) \cdot (-qy)$, hence since x and y generate Λ over k we see that ν is the degree preserving map defined by

$$x \mapsto -q^{-1}x, \quad y \mapsto -qy.$$

The map ϕ induces a bimodule isomorphism ${}_1\Lambda_{\nu^{-1}} \simeq D(\Lambda)$, which in turn gives an isomorphism ${}_{(\nu\psi)^{-1}}\Lambda_{\nu^{-1}} \simeq {}_{(\nu\psi)^{-1}} D(\Lambda)_1$ for any automorphism ψ of Λ. Furthermore, since ${}_{(\nu\psi)^{-1}}\Lambda_{\nu^{-1}}$ is isomorphic to ${}_1\Lambda_\psi$ and ${}_{(\nu\psi)^{-1}}D(\Lambda)_1 = D({}_1\Lambda_{(\nu\psi)^{-1}})$, we get an isomorphism ${}_1\Lambda_\psi \simeq D({}_1\Lambda_{(\nu\psi)^{-1}})$ of bimodules. Now from [CaE, Proposition VI.5.1] we get

$$\begin{aligned}\operatorname{HH}^n(\Lambda, {}_1\Lambda_\psi) &= \operatorname{Ext}^n_{\Lambda^e}(\Lambda, {}_1\Lambda_\psi) \\ &\simeq \operatorname{Ext}^n_{\Lambda^e}\left(\Lambda, D({}_1\Lambda_{(\nu\psi)^{-1}})\right) \\ &\simeq D\left(\operatorname{Tor}_n^{\Lambda^e}(\Lambda, {}_1\Lambda_{(\nu\psi)^{-1}})\right) \\ &= D\left(\operatorname{HH}_n(\Lambda, {}_1\Lambda_{(\nu\psi)^{-1}})\right),\end{aligned}$$

thus when computing the (dimension of the) Hochschild cohomology group $\operatorname{HH}^n(\Lambda, {}_1\Lambda_\psi)$ we are also computing the Hochschild homology group $\operatorname{HH}_n(\Lambda, {}_1\Lambda_{(\nu\psi)^{-1}})$. Moreover, as ψ ranges over all degree preserving k-algebra automorphisms of Λ, so does $(\nu\psi)^{-1}$.

2. The Cohomology Complex

We start by recalling the construction of the minimal bimodule projective resolution of Λ from [BGMS]. Define the elements

$$f_0^0 = 1, \quad f_0^1 = x, \quad f_1^1 = y,$$
$$f_{-1}^n = 0 = f_{n+1}^n \text{ for each } n \geq 0,$$

and for each $n \geq 2$ define elements $\{f_i^n\}_{i=0}^n \subseteq \underbrace{\Lambda \otimes_k \cdots \otimes_k \Lambda}_{n \text{ copies}}$ inductively by

$$f_i^n = f_{i-1}^{n-1} \otimes y + q^i f_i^{n-1} \otimes x.$$

Denote by P^n the Λ^e-projective module $\bigoplus_{i=0}^n \Lambda \otimes_k f_i^n \otimes_k \Lambda$, and by $\tilde{f}_i^n$ the element $1 \otimes f_i^n \otimes 1 \in P^n$ (and $\tilde{f}_0^0 = 1 \otimes 1$). The set $\{\tilde{f}_i^n\}_{i=0}^n$ generates P^n as a Λ^e-module. Now define a map $\delta_n \colon P^n \to P^{n-1}$ by

$$\tilde{f}_i^n \mapsto \left[x\tilde{f}_i^{n-1} + (-1)^n q^i \tilde{f}_i^{n-1} x\right] + \left[q^{n-i} y \tilde{f}_{i-1}^{n-1} + (-1)^n \tilde{f}_{i-1}^{n-1} y\right].$$

It is shown in [BGMS] that

$$(\mathbb{P}, \delta) \colon \cdots \to P^{n+1} \xrightarrow{\delta_{n+1}} P^n \xrightarrow{\delta_n} P^{n-1} \to \cdots$$

is a minimal Λ^e-projective resolution of Λ. Denote the direct sum of n copies of ${}_1\Lambda_\psi$ by ${}_1\Lambda_\psi^n$, and consider its standard k-basis $\{e_i^{n-1}, xe_i^{n-1}, ye_i^{n-1}, yxe_i^{n-1}\}_{i=0}^{n-1}$. Define a map $d_n \colon {}_1\Lambda_\psi^n \to {}_1\Lambda_\psi^{n+1}$ by

$$\lambda e_i^{n-1} \mapsto \left[x\lambda + (-1)^n q^i \lambda\psi(x)\right] e_i^n + \left[q^{n-i-1} y\lambda + (-1)^n \lambda\psi(y)\right] e_{i+1}^n.$$

Applying $\operatorname{Hom}_{\Lambda^e}(-, {}_1\Lambda_\psi)$ to the resolution $(\mathbb{P}, \delta)$, keeping in mind that $\operatorname{Hom}_{\Lambda^e}(P^n, {}_1\Lambda_\psi)$ and ${}_1\Lambda_\psi^{n+1}$ are isomorphic as k-vector spaces, we get the commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & \operatorname{Hom}_{\Lambda^e}(P^{n-1}, {}_1\Lambda_\psi) \arrow[r, "\delta_n^*"] \arrow[d, "\wr"] & \operatorname{Hom}_{\Lambda^e}(P^n, {}_1\Lambda_\psi) \arrow[r, "\delta_{n+1}^*"] \arrow[d, "\wr"] & \cdots \\ \cdots \arrow[r] & {}_1\Lambda_\psi^n \arrow[r, "d_n"] & {}_1\Lambda_\psi^{n+1} \arrow[r, "d_{n+1}"] & \cdots \end{tikzcd}$$

of k-vector spaces.

In order to compute $\operatorname{HH}^n(\Lambda, {}_1\Lambda_\psi) = \operatorname{Ext}^n_{\Lambda^e}(\Lambda, {}_1\Lambda_\psi)$ for $n > 0$ we compute the cohomology $\operatorname{Ker} d_{n+1} / \operatorname{Im} d_n$ of the bottom complex in the above commutative diagram. We do this by finding $\dim_k \operatorname{Im} d_n$; once we know $\dim_k \operatorname{Im} d_n$, we obtain $\dim_k \operatorname{Ker} d_n$ (and therefore also $\dim_k \operatorname{Ker} d_{n+1}$) from the equation

$$\dim_k \operatorname{Ker} d_n + \dim_k \operatorname{Im} d_n = \dim_k {}_1\Lambda_\psi^n = 4n.$$

We then have

$$\dim_k \mathrm{HH}^n(\Lambda, {}_1\Lambda_\psi) = \dim_k \operatorname{Ker} d_{n+1} - \dim_k \operatorname{Im} d_n.$$

Now let ψ be a degree preserving k-algebra automorphism of Λ. Then there are elements $\alpha_1, \alpha_2, \beta_1, \beta_2 \in k$ such that $\psi(x) = \alpha_1 x + \alpha_2 y$ and $\psi(y) = \beta_1 x + \beta_2 y$. Since $\psi(x^2) = \psi(y^2) = \psi(xy + qyx) = 0$, we have the relations $\alpha_1\alpha_2 = \beta_1\beta_2 = \alpha_2\beta_1 = 0$. If $\alpha_2 \neq 0$, then $\alpha_1 = \beta_1 = 0$, implying $x \notin \operatorname{Im}\psi$. Similarly, if $\beta_1 \neq 0$, then $\alpha_2 = \beta_2 = 0$, implying $y \notin \operatorname{Im}\psi$. Therefore $\alpha_2 = \beta_1 = 0$, and this forces α_1 and β_2 to be nonzero. Thus the degree preserving k-algebra automorphisms of Λ are precisely those defined by

$$x \mapsto \alpha x, \quad y \mapsto \beta y$$

for two arbitrary nonzero elements $\alpha, \beta \in k$. For such an automorphism, the result of applying d_n to the basis vectors $\{e_i^{n-1}, xe_i^{n-1}, ye_i^{n-1}, yxe_i^{n-1}\}_{i=0}^{n-1}$ of ${}_1\Lambda_\psi^n$ is

$$\begin{array}{rcl}
e_i^{n-1} & \mapsto & \left[1 + (-1)^n q^i \alpha\right] xe_i^n + \left[q^{n-i-1} + (-1)^n \beta\right] ye_{i+1}^n \\
xe_i^{n-1} & \mapsto & \left[q^{n-i-1} + (-1)^{n+1} q\beta\right] yxe_{i+1}^n \\
ye_i^{n-1} & \mapsto & \left[-q + (-1)^n q^i \alpha\right] yxe_i^n \\
yxe_i^{n-1} & \mapsto & 0
\end{array}$$

for $0 \leq i \leq n-1$. Note that the inequality $\dim_k \operatorname{Im} d_n \leq 2n+1$ always holds.

3. The Hochschild Cohomology

We start by computing $\mathrm{HH}^0(\Lambda, {}_1\Lambda_\psi)$. Rather than computing this vector space directly using the identifications

$$\begin{array}{rcl}
\mathrm{HH}^0(\Lambda, {}_1\Lambda_\psi) & = & \{z \in {}_1\Lambda_\psi \mid \lambda \cdot z = z \cdot \lambda \text{ for all } \lambda \in \Lambda\} \\
& = & \{z \in {}_1\Lambda_\psi \mid \lambda z = z\psi(\lambda) \text{ for all } \lambda \in \Lambda\} \\
& = & \{z \in {}_1\Lambda_\psi \mid xz = z\psi(x), yz = z\psi(y), yxz = z\psi(yx)\} \\
& = & \{z \in {}_1\Lambda_\psi \mid xz = \alpha zx, yz = \beta zy, yxz = \alpha\beta zyx\},
\end{array}$$

we use our cohomology complex and the isomorphism $\mathrm{HH}^0(\Lambda, {}_1\Lambda_\psi) \simeq \operatorname{Ker} d_1$. From the above we see that the map d_1 is defined by

$$\begin{array}{rcl}
e_0^0 & \mapsto & [1-\alpha]xe_0^1 + [1-\beta]ye_1^1 \\
xe_0^0 & \mapsto & [1+q\beta]yxe_1^1 \\
ye_0^0 & \mapsto & -[q+\alpha]yxe_0^1 \\
yxe_0^0 & \mapsto & 0,
\end{array}$$

and so calculation gives

$$\dim_k \mathrm{HH}^0(\Lambda, {}_1\Lambda_\psi) = \begin{cases} 3 & \text{when } \alpha = -q, \beta = -q^{-1} \\ 2 & \text{when } \alpha = 1, \beta = 1 \\ 2 & \text{when } \alpha = -q, \beta \neq -q^{-1} \\ 2 & \text{when } \alpha \neq -q, \beta = -q^{-1} \\ 1 & \text{otherwise} \end{cases}$$

when the characteristic of k is not 2. In the characteristic 2 case we replace $-q, -q^{-1}$ and 1 in the above formula by $\pm q, \pm q^{-1}$ and ± 1, respectively.

Now we turn to the cohomology groups $\mathrm{HH}^n(\Lambda, {}_1\Lambda_\psi)$ for $n > 0$. To compute the dimension of $\operatorname{Im} d_n$, we distinguish between four possible cases depending on whether or not α and β belong to the set

$$\Sigma = \{\pm q^i\}_{i \in \mathbb{Z}}.$$

3.1. **The case $\alpha, \beta \notin \Sigma$:**
This is the easiest case; $d_n(e_i^{n-1}), d_n(xe_i^{n-1})$ and $d_n(ye_i^{n-1})$ are all nonzero for $0 \leq i \leq n-1$, hence $\dim_k \operatorname{Im} d_n = 2n+1$ for all n. Then $\dim_k \operatorname{Ker} d_n = 2n-1$, implying $\dim_k \operatorname{Ker} d_{n+1} = 2n+1$ and therefore that $\operatorname{HH}^n(\Lambda, {}_1\Lambda_\psi) = 0$ for $n > 0$.

Remark. We can relate the vanishing of cohomology to the conjecture of Tachikawa stating that over a selfinjective ring the only finitely generated modules having no self extensions are the projective ones. Namely, let M be a finitely generated Λ-module such that Λ has no bimodule extensions by $\operatorname{Hom}_k(M, M)$. Then from [CaE, Corollary IX.4.4] we get

$$\operatorname{Ext}_\Lambda^n(M, M) \simeq \operatorname{HH}^n(\Lambda, \operatorname{Hom}_k(M, M)) = 0$$

for $n > 0$. Since Tachikawa's conjecture holds for Λ (see [Sch, Proposition 4.2]), the module M must be projective and therefore (Λ is local) isomorphic to Λ^t for some $t \in \mathbb{N}$. This gives

$$\operatorname{Hom}_k(M, M) \simeq \left(\Lambda \otimes_k D(\Lambda)\right)^{t^2} \simeq (\Lambda^{\mathrm{e}})^{t^2}.$$

In particular, if $\operatorname{HH}^n(\Lambda, {}_1\Lambda_\psi) = 0$ for $n > 0$ then there cannot exist a Λ-module M such that $\operatorname{Hom}_k(M, M)$ is isomorphic to ${}_1\Lambda_\psi$, since this would imply the contradiction ${}_1\Lambda_\psi \simeq (\Lambda^{\mathrm{e}})^{t^2}$.

3.2. **The case $\alpha \in \Sigma, \beta \notin \Sigma$:**
Since $\left[q^{n-i-1} + (-1)^n \beta\right]$ and $\left[q^{n-i-1} + (-1)^{n+1} q\beta\right]$ are nonzero, we have $d_n(e_i^{n-1}) \neq 0$ and $d_n(xe_i^{n-1}) \neq 0$ for $0 \leq i \leq n-1$. Therefore $\dim_k \operatorname{Im} d_n \geq 2n$, and the problem is now whether or not the basis vector yxe_0^n belongs to $\operatorname{Im} d_n$. This is the case if and only if $d_n(ye_0^{n-1}) \neq 0$, that is, if and only if

$$-q + (-1)^n \alpha \neq 0 \tag{C1}$$

holds. We now break down this case into three cases.

(i) The case $\alpha \neq \pm q$:
Since (C1) holds we have $\dim_k \operatorname{Im} d_n = 2n+1$, and so $\operatorname{HH}^n(\Lambda, {}_1\Lambda_\psi) = 0$ for $n > 0$.

(ii) The case $\alpha = q$:
When the characteristic of k is not 2, the condition (C1) holds if and only if n is odd. Therefore

$$\dim_k \operatorname{Im} d_n = \begin{cases} 2n & \text{for } n \text{ even} \\ 2n+1 & \text{for } n \text{ odd,} \end{cases}$$

and so

$$\dim_k \operatorname{Ker} d_{n+1} = \begin{cases} 2n+1 & \text{for } n \text{ even} \\ 2n+2 & \text{for } n \text{ odd.} \end{cases}$$

This gives $\dim_k \operatorname{HH}^n(\Lambda, {}_1\Lambda_\psi) = 1$ for $n > 0$.

When k is of characteristic 2 we see that (C1) never holds, hence $\dim_k \operatorname{Im} d_n = 2n$. Then $\dim_k \operatorname{Ker} d_{n+1} = 2n+2$, giving $\dim_k \operatorname{HH}^n(\Lambda, {}_1\Lambda_\psi) = 2$ for $n > 0$.

(iii) The case $\alpha = -q$:
As in the previous case, we get $\dim_k \operatorname{HH}^n(\Lambda, {}_1\Lambda_\psi) = 1$ for $n > 0$ when k is not of characteristic 2, and $\dim_k \operatorname{HH}^n(\Lambda, {}_1\Lambda_\psi) = 2$ for $n > 0$ in the characteristic 2 case.

Remark. From this case we obtain an example showing that symmetry in the vanishing of Ext over Λ^{e} does not hold. Namely, define ψ by

$$x \mapsto q^{-1}x, \quad y \mapsto \beta y$$

for some element β not contained in Σ. For $n > 0$, case (i) above gives $\operatorname{Ext}_{\Lambda^{\mathrm{e}}}^n(\Lambda, {}_1\Lambda_\psi) = 0$, whereas from case (ii) we see that $\dim_k \operatorname{Ext}_{\Lambda^{\mathrm{e}}}^n(\Lambda, {}_1\Lambda_{\psi^{-1}})$ is either 1 or 2, depending on the characteristic of k. Now it is easy to see that $\operatorname{Ext}^n_{\Lambda^e}(\Lambda, {}_1\Lambda_{\psi^{-1}})$ is isomorphic to $\operatorname{Ext}^n_{\Lambda^e}({}_1\Lambda_\psi, \Lambda)$.

3.3. **The case $\alpha \notin \Sigma, \beta \in \Sigma$:**
The algebra Λ is isomorphic to the algebra $k\langle u,v\rangle/(u^2, uv + q^{-1}vu, v^2)$ via the map

$$x \mapsto v, \quad y \mapsto u,$$

hence this case is symmetric to the case $\alpha \in \Sigma, \beta \notin \Sigma$ treated above. Namely, when $\beta \neq \pm q^{-1}$ the result is as in (i), whereas when $\beta = \pm q^{-1}$ the result is as in (ii) and (iii).

3.4. **The case $\alpha \in \Sigma, \beta \in \Sigma$:**
The basis vectors yxe^n_0 and yxe^n_n can only be the image of ye^{n-1}_0 and xe^{n-1}_{n-1}, respectively, whereas for $1 \leq i \leq n-1$ the basis vector yxe^n_i can be the image of both ye^{n-1}_i and xe^{n-1}_{i-1}. Therefore we break this case down into four cases, each depending on whether or not $\alpha = \pm q$ and $\beta = \pm q^{-1}$.

(i) The case $\alpha = \pm q, \beta = \pm q^{-1}$:
We have

$$d_n(e^{n-1}_i) = \left[1 \pm (-1)^n q^{i+1}\right] xe^n_i + \left[q^{n-i-1} \pm (-1)^n q^{-1}\right] ye^n_{i+1},$$

and since $i+1 \geq 1$ the term $\left[1 \pm (-1)^n q^{i+1}\right]$ must be nonzero. Therefore $d_n(e^{n-1}_i) \neq 0$. Applying d_n to xe^{n-1}_i and ye^{n-1}_i gives $\left[q^{n-i-1} \pm (-1)^{n+1}\right] yxe^n_{i+1}$ and $\left[-q \pm (-1)^n q^{i+1}\right] yxe^n_i$, respectively, hence when the characteristic of k is not 2 we get

$$\begin{aligned}
d_n(xe^{n-1}_i) &= \begin{cases} 0 & \text{for } i = n-1, \beta = q^{-1}, n \text{ even} \\ 0 & \text{for } i = n-1, \beta = -q^{-1}, n \text{ odd} \\ \neq 0 & \text{otherwise,} \end{cases} \\
d_n(ye^{n-1}_i) &= \begin{cases} 0 & \text{for } i = 0, \alpha = q, n \text{ even} \\ 0 & \text{for } i = 0, \alpha = -q, n \text{ odd} \\ \neq 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

There are four possible pairs (α, β) to consider. If $\alpha = q$ and $\beta = q^{-1}$, then the above gives

$$\dim_k \operatorname{Im} d_n = \begin{cases} 2n-1 & \text{for } n \text{ even} \\ 2n+1 & \text{for } n \text{ odd,} \end{cases}$$

and therefore

$$\dim_k \operatorname{Ker} d_{n+1} = \begin{cases} 2n+1 & \text{for } n \text{ even} \\ 2n+3 & \text{for } n \text{ odd.} \end{cases}$$

This implies $\dim_k \operatorname{HH}^n(\Lambda, {}_1\Lambda_\psi) = 2$ for $n > 0$. Similar computation gives $\dim_k \operatorname{HH}^n(\Lambda, {}_1\Lambda_\psi) = 2$ for $n > 0$ also for the other three possible pairs.
When k is of characteristic 2 then

$$\begin{aligned}
d_n(xe^{n-1}_i) &= \begin{cases} 0 & \text{for } i = n-1 \\ \neq 0 & \text{otherwise,} \end{cases} \\
d_n(ye^{n-1}_i) &= \begin{cases} 0 & \text{for } i = 0 \\ \neq 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

and so $\dim_k \operatorname{Im} d_n = 2n-1$ for all $n > 0$. Consequently $\dim_k \operatorname{HH}^n(\Lambda, {}_1\Lambda_\psi) = 4$ for $n > 0$.

Remark. Note that, in this particular case, we have computed the (dimension of the) Hochschild homology $\operatorname{HH}_*(\Lambda) = \operatorname{Tor}^{\Lambda^e}_*(\Lambda, \Lambda)$ of Λ. Namely, it follows from

Section 1 that for each $n > 0$ the k-vector spaces $\mathrm{HH}^n(\Lambda,_1\Lambda_{\nu^{-1}})$ and $D(\mathrm{HH}_n(\Lambda))$ are isomorphic, where ν is the Nakayama automorphism

$$x \mapsto -q^{-1}x, \quad y \mapsto -qy$$

of Λ. Then ν^{-1} is defined by

$$x \mapsto -qx, \quad y \mapsto -q^{-1}y,$$

hence ${}_1\Lambda_{\nu^{-1}}$ is precisely the sort of bimodule we have just considered in terms of Hochschild cohomology. Consequently $\dim_k \mathrm{HH}_n(\Lambda) = 2$ for $n > 0$ when the characteristic of k is not 2, whereas $\dim_k \mathrm{HH}_n(\Lambda) = 4$ for $n > 0$ in the characteristic 2 case.

(ii) The case $\alpha = \pm q, \beta \neq \pm q^{-1}$:

As in (i) the element $d_n(e_i^{n-1})$ is nonzero for $0 \leq i \leq n-1$. Moreover, since $\beta \neq \pm q^{-1}$ the basis element yxe_n^n always lies in $\operatorname{Im} d_n$, as does the basis elements yxe_i^n for $1 \leq i \leq n-1$. Therefore $\dim \operatorname{Im} d_n \geq 2n$, and the question is whether or not yxe_0^n belongs to $\operatorname{Im} d_n$, i.e. whether or not $d_n(ye_0^{n-1})$ is nonzero.

When the characteristic of k is not 2, then from (i) we see that

$$d_n(ye_0^{n-1}) = \begin{cases} 0 & \text{for } \alpha = q, n \text{ even} \\ 0 & \text{for } \alpha = -q, n \text{ odd} \\ \neq 0 & \text{otherwise,} \end{cases}$$

and computation gives $\dim_k \mathrm{HH}^n(\Lambda, {}_1\Lambda_\psi) = 1$ for $n > 0$. However, when the characteristic of k is 2 then $d_n(ye_0^{n-1}) = 0$, giving $\dim_k \operatorname{Im} d_n = 2n$ and consequently $\dim_k \mathrm{HH}^n(\Lambda, {}_1\Lambda_\psi) = 2$ for $n > 0$.

(iii) The case $\alpha \neq \pm q, \beta = \pm q^{-1}$:

Using the isomorphism $\Lambda \simeq k\langle u, v\rangle/(u^2, uv + q^{-1}vu, v^2)$ from the case $\alpha \notin \Sigma, \beta \in \Sigma$, we see that the present case is symmetric to the case (ii) above. Thus when the characteristic of k is not 2 then $\dim_k \mathrm{HH}^n(\Lambda, {}_1\Lambda_\psi) = 1$ for $n > 0$, whereas $\dim_k \mathrm{HH}^n(\Lambda, {}_1\Lambda_\psi) = 2$ for $n > 0$ in the characteristic 2 case.

(iv) The case $\alpha \neq \pm q, \beta \neq \pm q^{-1}$:

We now have $\alpha = \pm q^s$ and $\beta = \pm q^t$ where $s \in \mathbb{Z} \setminus \{1\}$ and $t \in \mathbb{Z} \setminus \{-1\}$, and therefore the basis elements yxe_0^n and yxe_n^n both lie in the image of d_n. To compute the dimension of $\operatorname{Im} d_n$, we must find out when $d_n(e_i^{n-1}) = 0$ for some $0 \leq i \leq n-1$ and when $yxe_i^n \notin \operatorname{Im} d_n$ for some $1 \leq i \leq n-1$. We have

$$(*) \qquad \begin{gathered} \left\{ d_n(e_i^{n-1}) = 0 \text{ for some } 0 \leq i \leq n-1 \right\} \\ \Updownarrow \\ \left\{ 1 + (-1)^n q^i \alpha = 0 \text{ and } q^{n-i-1} + (-1)^n \beta = 0 \right\} \end{gathered}$$

and

$$(**) \qquad \begin{gathered} \{ yxe_i^n \notin \operatorname{Im} d_n \text{ for some } 1 \leq i \leq n-1 \} \\ \Updownarrow \\ \left\{ -q + (-1)^n q^i \alpha = 0 \text{ and } q^{n-i} + (-1)^{n+1} q\beta = 0 \right\}, \end{gathered}$$

and when the characteristic of k is not 2 this happens precisely when we have the following:

$$\text{(C2)} \qquad s \leq 0, \quad t \geq 0, \quad \alpha = (-1)^{t-s} q^s, \quad \beta = (-1)^{t-s} q^t.$$

In the characteristic 2 case we may relax this condition; in this case $(*)$ and $(**)$ occur precisely when we have the following:

$$\text{(C3)} \qquad s \leq 0, \quad t \geq 0.$$

However, both when the characteristic of k is not 2 and (C2) holds, and in the characteristic 2 case when (C3) holds, we see that $(*)$ occurs for $n = t - s + 1$, whereas $(**)$ occurs for $n = t - s + 2$.

Therefore, when the characteristic of k is not 2 the dimension of $\operatorname{Im} d_n$ is given by

$$\dim_k \operatorname{Im} d_n = \begin{cases} 2n & \text{when (C2) holds and } n = t - s + 1 \\ 2n & \text{when (C2) holds and } n = t - s + 2 \\ 2n + 1 & \text{otherwise,} \end{cases}$$

implying the dimension of $\operatorname{Ker} d_{n+1}$ is given by

$$\dim_k \operatorname{Ker} d_{n+1} = \begin{cases} 2n + 2 & \text{when (C2) holds and } n = t - s \\ 2n + 2 & \text{when (C2) holds and } n = t - s + 1 \\ 2n + 1 & \text{otherwise.} \end{cases}$$

Consequently , the dimension of $\mathrm{HH}^n(\Lambda, {}_1\Lambda_\psi)$ for $n > 0$ is given by

$$\dim_k \mathrm{HH}^n(\Lambda, {}_1\Lambda_\psi) = \begin{cases} 1 & \text{when (C2) holds and } n = t - s \\ 2 & \text{when (C2) holds and } n = t - s + 1 \\ 1 & \text{when (C2) holds and } n = t - s + 2 \\ 0 & \text{otherwise.} \end{cases}$$

When the characteristic of k is 2, we obtain the exact same formulas, but with (C2) replaced by (C3).

Remark. (i) In the case considered above, we see that the cohomology is zero except possibly in three degrees, depending on what conditions s, t, α and β satisfy. As a consequence, we construct a counterexample to the following conjecture by Auslander (see [Aus, page 815]): if M is a finitely generated module over an Artin algebra Γ, then there exists a number n_M such that for any finitely generated module N we have

$$\operatorname{Ext}^i_\Gamma(M, N) = 0 \text{ for } i \gg 0 \Rightarrow \operatorname{Ext}^i_\Gamma(M, N) = 0 \text{ for } i \geq n_M.$$

The first counterexample to this conjecture appeared in [JoS], where the algebra considered was a finite dimensional commutative Noetherian local Gorenstein algebra. A counterexample over our algebra $\Lambda = k\langle x, y\rangle/(x^2, xy + qyx, y^2)$ was given in [Sma].

As for a counterexample using Hochschild cohomology, define for each natural number t an automorphism $\psi\colon \Lambda \to \Lambda$ by $x \mapsto q^{-t}x$ and $y \mapsto q^t y$, and denote the bimodule ${}_1\Lambda_\psi$ by M_t. Then condition (C2)/(C3) is satisfied (with $s = -t, \alpha = q^{-t}$ and $\beta = q^t$), and so

$$\operatorname{Ext}^n_{\Lambda^{\mathrm{e}}}(\Lambda, M_t) = \begin{cases} \neq 0 & \text{for } n = 2t + 2 \\ 0 & \text{for } n > 2t + 2. \end{cases}$$

(ii) Even though the above conjecture of Auslander fails to hold in general, Auslander himself proved (unpublished, see [Aus, page 815]) that if the conjecture holds for the enveloping algebra Γ^{e} of a finite dimensional algebra Γ over a field, then the finitistic dimension

$$\sup\{\operatorname{pd}_\Gamma X \mid X \text{ finitely generated } \Gamma\text{-module with } \operatorname{pd}_\Gamma X < \infty\}$$

of Γ is finite. In view of the above remark, we see that the converse to this result does not hold; our algebra Λ, being selfinjective, trivially has finite finitistic dimension, whereas the conjecture of Auslander does not hold for Λ^{e}.

(iii) The computation of the Hochschild cohomology $\mathrm{HH}^*(\Lambda) = \operatorname{Ext}^*_{\Lambda^{\mathrm{e}}}(\Lambda, \Lambda)$ of Λ is covered by the last of the above cases. When ψ is the identity automorphism we have $s = t = 0$, and the condition (C2)/(C3) is satisfied. This gives

$$\dim_k \mathrm{HH}^n(\Lambda) = \begin{cases} 2 & \text{for } n = 1 \\ 1 & \text{for } n = 2 \\ 0 & \text{for } n \geq 3, \end{cases}$$

and so our algebra Λ is a counterexample to the following question raised by Happel in [Hap]: if the Hochschild cohomology groups of a finite dimensional algebra vanish in high degrees, does the algebra have finite global dimension? The counterexample above first appeared in [BGMS], where it was shown that the generating function $\sum_{n=0}^{\infty} \mathrm{HH}^n(\Lambda)t^n$ of $\mathrm{HH}^n(\Lambda)$ is $2 + 2t + t^2$.

The converse to the question of Happel is always true when the algebra modulo its Jacobson radical is separable over the ground field. More specifically, if Γ is a finite dimensional algebra over a field K with Jacobson radical $\mathfrak{r}$, and the semisimple algebra $\Gamma / \mathfrak{r}$ is separable over K, then by [EIN, §3] the implication

$$\mathrm{gl.\,dim}\, \Gamma < \infty \Rightarrow \mathrm{pd}_{\Gamma^e} \Gamma < \infty$$

holds. In particular the Hochschild cohomology groups $\mathrm{HH}^n(\Gamma) = \mathrm{Ext}^n_{\Gamma^e}(\Gamma, \Gamma)$ and the homology groups $\mathrm{HH}_n(\Gamma) = \mathrm{Tor}_n^{\Gamma^e}(\Gamma, \Gamma)$ of Γ vanish for $n \gg 0$ when the global dimension of Γ is finite. It is not known whether the vanishing of the Hochschild homology groups in high degrees for a finite dimensional algebra implies the global dimension of the algebra is finite.

(iv) Because of the equality $\dim_k \mathrm{HH}^n(\Lambda, {}_1\Lambda_\psi) = \dim_k \mathrm{HH}_n(\Lambda, {}_1\Lambda_{(\nu\psi)^{-1}})$ which follows from Section 1, the somewhat strange behavior in Hochschild cohomology revealed in the last case considered above can also be transferred to Hochschild homology. When the automorphism ψ is given by $\psi(x) = \pm q^s x$ and $\psi(y) = \pm q^t y$, then the automorphism $\theta \stackrel{\text{def}}{=} (\nu\psi)^{-1}$, where ν is the Nakayama automorphism, is given by

$$x \mapsto \mp q^{1-s} x, \quad y \mapsto \mp q^{-(t+1)} y.$$

Thus for such an automorphism θ, when $s \in \mathbb{Z} \setminus \{1\}$ and $t \in \mathbb{Z} \setminus \{-1\}$ we get

$$\dim_k \mathrm{HH}_n(\Lambda, {}_1\Lambda_\theta) = \begin{cases} 1 & \text{when (C2)/(C3) holds and } n = t - s \\ 2 & \text{when (C2)/(C3) holds and } n = t - s + 1 \\ 1 & \text{when (C2)/(C3) holds and } n = t - s + 2 \\ 0 & \text{otherwise.} \end{cases}$$

when $n > 0$. In these formulas condition (C2) applies when the characteristic of k is not 2, and condition (C3) applies in the characteristic 2 case.

Acknowledgement

I would like to thank my supervisor Øyvind Solberg for valuable suggestions and comments.

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II.

COMPLEXITY AND PERIODICITY

ABSTRACT

Let M be a finitely generated module over an Artin algebra. By considering the lengths of each module in the minimal projective resolution of M, we obtain its Betti sequence. This sequence must be bounded if M is eventually periodic, but the converse fails to hold in general. We give conditions under which this holds, using techniques from Hochschild cohomology. We also provide a result which under certain conditions guarantees the existence of periodic modules. Finally, we study the case when an element in the Hochschild cohomology ring "generates" the periodicity of a module.

This paper has been published in Coll. Math. 104 (2006), no. 2, 169-191.

COMPLEXITY AND PERIODICITY

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1. Introduction

This paper is devoted to investigating connections between periodicity and complexity for modules over Artin algebras, as was done in [Eis] for modules over both group rings of finite groups and commutative Noetherian local rings. More specifically, let M be a finitely generated module over an Artin algebra. By considering the lengths of each module in the minimal projective resolution of M, we obtain its Betti sequence. If M is eventually periodic, i.e. if its minimal projective resolution becomes periodic from some step on, then the Betti sequence of M must be bounded. The converse, however, fails to hold in general, that is, it need not be true that M is eventually periodic even though its Betti sequence is bounded. We give conditions under which this holds, using techniques from Hochschild cohomology. In addition we provide a result which under certain conditions guarantees the existence of periodic modules.

One reason for restricting our attention to Artin algebras is that we need to make sure that every finitely generated module has a unique minimal projective resolution. Therefore, throughout this paper, we let k be a commutative Artin ring and Λ an Artin k-algebra with Jacobson radical $\mathfrak{r}$. We fix a finitely generated Λ-module M with a minimal projective resolution

$$(\mathbb{P}, d)\colon\ \cdots \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0,$$

i.e. $\operatorname{Ker} d_i \subseteq \mathfrak{r} P_i$. The integers $\beta_n(M) = \ell_k(P_n)$ are called the *Betti numbers* of M, and they are all finite since a module is finitely generated over k whenever it is finitely generated over Λ. Moreover, these integers are well defined since any minimal projective resolution is unique up to isomorphism. Thus, we may associate the infinite sequence

$$\beta_0(M), \beta_1(M), \beta_2(M), \dots$$

to M, and this sequence is called the *Betti sequence* of M. Over a commutative Noetherian local ring it is customary to define the Betti numbers of a finitely generated module as the ranks of the modules in its minimal free resolution. However, this would not make sense in our setting since projective modules need not be free.

We say that M is *periodic* if there is an integer $p \geq 1$ such that M is isomorphic to $\Omega_\Lambda^p(M)$ (the p'th syzygy in the minimal projective resolution $\mathbb{P}$), and the least such integer p is the *period* of M. Furthermore, M is *eventually periodic* if one of its syzygies (in the minimal projective resolution) is periodic. Clearly, if M has this last property, then its Betti sequence is bounded. The converse is not true in general. A counterexample was given by R. Schulz in [Sch, Proposition 4.1], where he considered finite dimensional algebras of the form $k\langle x, y\rangle/(x^2, xy + qyx, y^2)$, for k a field and $q \in k$ a nonzero element.

In [Eis] D. Eisenbud proved that the converse does hold over group rings of finite groups, and that it also holds in the commutative Noetherian local setting when the rings considered are complete intersections. In fact, it was shown that over a hypersurface (that is, a complete intersection of codimension one) *any* minimal free resolution eventually becomes periodic. In the same paper it was therefore conjectured that over a commutative Noetherian local ring a module having bounded Betti numbers must be periodic.

However, as in the case of Artin algebras, the conjecture fails to hold in general. An example of this was given in [GaP]. Here V. Gasharov and I. Peeva considered the commutative local finite dimensional k-algebra $(R, \mathfrak{m}, k) = k[x_1, x_2, x_3, x_4, x_5]/\, \mathfrak{a}$, where $\mathfrak{a}$ is the ideal generated by the quadratic forms

$$x_1^2, \quad x_2^2, \quad x_5^2, \quad x_3x_4, \quad x_3x_5, \quad x_4x_5, \quad x_1x_4 + x_2x_4,$$
$$x_4^2 - x_2x_5 + x_1x_5, \quad \alpha x_1x_3 + x_2x_3, \quad x_3^2 - x_2x_5 + \alpha x_1x_5$$

for a nonzero element $\alpha \in k$ having infinite order in the multiplicative group $k \setminus \{0\}$. They constructed the free resolution

$$\cdots \xrightarrow{d_3} R^2 \xrightarrow{d_2} R^2 \xrightarrow{d_1} R^2 \xrightarrow{d_0} M \to 0,$$

where the maps are given by the matrices

$$d_n = \begin{pmatrix} x_1 & \alpha^n x_3 + x_4 \\ 0 & x_2 \end{pmatrix}$$

for $n \geq 0$, and $M = \operatorname{Im} d_0$. The module M has a constant Betti sequence, but is *not* eventually periodic.

We now briefly describe the contents of the three main sections of this paper. The first section is devoted to constructing a certain chain endomorphism on the minimal projective resolution of a module, given a finite generation hypothesis similar to the one used in [EHSST]. This chain map eventually becomes surjective, and we use this to prove the main result; whenever the finite generation hypothesis holds, a module has bounded Betti numbers precisely when it is eventually periodic. In some cases (Theorem 2.5) we may also determine what the period is.

In the second section we develop a method for "reducing" the complexity of a module. More precisely, given a module M satisfying certain conditions (and having finite nonzero complexity), we construct a new module closely related to M and having complexity exactly one less than that of M. Iterating this procedure, we end up with a module having complexity one (i.e. having bounded Betti numbers), and this module must be periodic in view of the main result in the first section.

Finally, in the third section we study the case when the period of an eventually periodic module is "generated" by an element in the Hochschild cohomology ring. As with many other concepts, the inspiration comes from the group ring case, where one uses the group cohomology ring instead of the Hochschild cohomology ring.

2. Preliminary Results

The existence of eventually periodic modules of infinite projective dimension - and therefore of nonzero periodic modules - is far from obvious in general. In the next section we prove a result which under certain circumstances guarantees the existence of such modules. The proof is based on the main result of this section, which for a module gives a sufficient condition under which having a bounded Betti sequence is equivalent to being periodic.

The following proposition is the key to the main results. It guarantees regular elements for graded modules, provided we go "far enough" out in the grading.

Proposition 2.1. *Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a commutative Noetherian graded k-algebra of finite type over k (that is, each A_i is a finitely generated k-module), generated as an A_0-algebra by homogeneous elements $a_1, \dots, a_r$ of positive degrees. If $N =$*

$\bigoplus_{i=0}^{\infty} N_i$ is a finitely generated graded A-module, then there exists a homogeneous element $\eta \in A$ of positive degree, such that the multiplication map

$$N_i \xrightarrow{\eta} N_{i+|\eta|}$$

is a k-monomorphism for $i \gg 0$. Moreover, we can pick this η such that for some j, the degree of a_j divides $|\eta|$.

Proof. Consider the graded ideal $A^+ = \bigoplus_{i=1}^{\infty} A_i$ of A and the graded submodule

$$(0 :_N A^+) = \{x \in N \mid A^+ x = 0\}$$

of N. Since N is Noetherian, this submodule is finitely generated. Moreover, since it is annihilated by A^+ it is a finitely generated A_0-module ($A_0 = A/A^+$). Now A_0, being finitely generated over k, is Artinian, hence $(0 :_N A^+)$ has the descending chain condition on submodules. Therefore there exists an integer w such that $(0 :_N A^+)_i = 0$ for $i \geq w$.

Consider the graded A-submodule $N_{\geq w} = \bigoplus_{i=w}^{\infty} N_i$ of N. Since this is a finitely generated module, its set of associated prime ideals is finite and consists of graded ideals each of which is the annihilator of a homogeneous element (see, for example, [BrH, Lemma 1.5.6]), and whose union is the set of zero-divisors on $N_{\geq w}$. If A^+ is contained in any of these primes, then A^+ annihilates a nonzero homogeneous element of $N_{\geq w}$, a contradiction. Therefore, by the graded version of the "prime avoidance" lemma (see, for example [BrH, Lemma 1.5.10]), there exists a homogeneous $N_{\geq w}$-regular element η in A^+, obviously of positive degree. Since $a_1, \dots, a_r$ generate A^+, a slight modification of the proof of [BrH, Lemma 1.5.10] shows that η can be chosen so that the degree of a_j divides that of η for some j. □

From now on, we assume that Λ is projective (or, equivalently, flat) as a k-module. We denote by Λ^{e} its enveloping algebra $\Lambda \otimes_k \Lambda^{\mathrm{op}}$, and by $\mathrm{HH}^*(\Lambda)$ its Hochschild cohomology ring. Since Λ is projective as a k-module, we have

$$\mathrm{HH}^*(\Lambda) = \bigoplus_{i=0}^{\infty} \mathrm{Ext}^i_{\Lambda^{\mathrm{e}}}(\Lambda, \Lambda)$$

with Yoneda product as multiplication. For two Λ-modules X and Y we denote the graded k-module $\bigoplus_{i=0}^{\infty} \mathrm{Ext}^i_{\Lambda}(X, Y)$ by $\mathrm{Ext}^*_{\Lambda}(X, Y)$, and this is a left and right $\mathrm{HH}^*(\Lambda)$-module via the ring homomorphisms

$$- \otimes_{\Lambda} Y \colon \mathrm{HH}^*(\Lambda) \to \mathrm{Ext}^*_{\Lambda}(Y, Y)$$
$$- \otimes_{\Lambda} X \colon \mathrm{HH}^*(\Lambda) \to \mathrm{Ext}^*_{\Lambda}(X, X)$$

followed by Yoneda composition. The left and right scalar multiplications on this module are closely related as follows (see [SnS, Corollary 1.3]): for homogeneous elements $\eta \in \mathrm{HH}^*(\Lambda)$ and $\theta \in \mathrm{Ext}^*_{\Lambda}(X, Y)$ we have $\eta\theta = (-1)^{|\eta||\theta|}\theta\eta$, where $|\eta|$ and $|\theta|$ denote the degrees of these elements. In particular, we see that $\mathrm{Ext}^*_{\Lambda}(X, Y)$ is finitely generated as a left $\mathrm{HH}^*(\Lambda)$-module if and only if it is finitely generated as a right $\mathrm{HH}^*(\Lambda)$-module.

In view of the counterexamples provided by Schulz, Gasharov and Peeva, we need to impose some restrictions in order to be able to prove that M is eventually periodic whenever its Betti numbers are bounded. The assumption we introduce is a "local variant" of those used in [EHSST] to develop the theory of support varieties for Artin algebras, and it enables us to use well known techniques from commutative algebra to obtain our results.

Assumption (Fg). Given the finitely generated Λ-module M, there exists a commutative Noetherian graded subalgebra $H = \bigoplus_{i=0}^{\infty} H^i$ of the Hochschild cohomology ring $\mathrm{HH}^*(\Lambda)$, with the property that $H^0 = \mathrm{HH}^0(\Lambda)$ (the center of Λ) and that the module $\mathrm{Ext}^*_{\Lambda}(M, \Lambda/\mathfrak{r})$ is finitely generated over H.

Now we apply Proposition 2.1 to H and the graded H-module $\operatorname{Ext}^*_\Lambda(M, \Lambda/\mathfrak{r})$. This we can do because H, being Noetherian, is generated as an algebra over H^0 by a finite set of homogeneous elements of positive degrees. We obtain from the homogeneous element granted by Proposition 2.1 a negative degree chain endomorphism $\{\xi_0, \xi_1, \dots\}\colon \mathbb{P} \to \mathbb{P}$ (where $\mathbb{P}$ is the minimal projective resolution of M) which eventually becomes *surjective*, that is, the map ξ_i is an epimorphism for $i \gg 0$.

Proposition 2.2. *Assume* **Fg** *holds. Then there exists an integer $n \geq 1$ with a map $\xi\colon \Omega^n_\Lambda(M) \to M$ and a chain map $\{\xi_0, \xi_1, \dots\}\colon \mathbb{P} \to \mathbb{P}$ over ξ of degree $-n$, such that ξ_i is surjective for $i \gg 0$.*

Proof. From Proposition 2.1 we see that there exists an integer w and a homogeneous element $\eta \in H^{|\eta|}$, with $|\eta| \geq 1$, such that the multiplication map

$$\operatorname{Ext}^i_\Lambda(M, \Lambda/\mathfrak{r}) \xrightarrow{\eta} \operatorname{Ext}^{i+|\eta|}_\Lambda(M, \Lambda/\mathfrak{r})$$

is injective for $i \geq w$. Moreover, since $(\mathbb{P}, d)$ is a minimal projective resolution we have $\operatorname{Im} d_i \subseteq \mathfrak{r} P_{i-1}$, implying that the differential in the complex $\operatorname{Hom}_\Lambda(\mathbb{P}, \Lambda/\mathfrak{r})$ is zero. Therefore $\operatorname{Ext}^i_\Lambda(M, \Lambda/\mathfrak{r}) = \operatorname{Hom}_\Lambda(P_i, \Lambda/\mathfrak{r})$. Now consider the element η and the maps it induces. The action on $\operatorname{Ext}^i_\Lambda(M, \Lambda/\mathfrak{r})$ is via the map $-\otimes_\Lambda M\colon \operatorname{HH}^*(\Lambda) \to \operatorname{Ext}^*_\Lambda(M, M)$, followed by Yoneda composition. Let ξ denote the image of η in $\operatorname{Ext}^*_\Lambda(M, M)$. It can be interpreted as a Λ-linear map $\Omega^{|\eta|}_\Lambda(M) \xrightarrow{\xi} M$, and so by the Comparison Theorem there exist Λ-linear maps $\{\xi_0, \xi_1, \xi_2, \dots\}$ making the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_{|\eta|+i+1} \arrow[r, "d_{|\eta|+i+1}"] \arrow[d, "\xi_{i+1}"] & P_{|\eta|+i} \arrow[r, "d_{|\eta|+i}"] \arrow[d, "\xi_i"] & \cdots \arrow[r] & P_{|\eta|} \arrow[r] \arrow[d, "\xi_0"] & \Omega^{|\eta|}_\Lambda(M) \arrow[r] \arrow[d, "\xi"] & 0 \\
\cdots \arrow[r] & P_{i+1} \arrow[r, "d_{i+1}"] & P_i \arrow[r, "d_i"] & \cdots \arrow[r] & P_0 \arrow[r, "d_0"] & M \arrow[r] & 0
\end{tikzcd}$$

commute. We show ξ_i is surjective for $i \geq w$.

An element $\theta \in \operatorname{Ext}^i_\Lambda(M, \Lambda/\mathfrak{r})$ can be interpreted as a Λ-linear map $P_i \xrightarrow{\theta} \Lambda/\mathfrak{r}$ (having the property $\theta \circ d_{i+1} = 0$), and then scalar multiplication by η is given by $\theta\eta = \theta \circ \xi_i\colon P_{i+|\eta|} \to \Lambda/\mathfrak{r}$. Thus the map $\eta \cdot (-)\colon \operatorname{Hom}_\Lambda(P_i, \Lambda/\mathfrak{r}) \to \operatorname{Hom}_\Lambda(P_{i+|\eta|}, \Lambda/\mathfrak{r})$ is simply given by

$$\xi_i^*\colon f \mapsto f \circ \xi_i,$$

and we know it is injective for $i \geq w$. Applying $\operatorname{Hom}_\Lambda(-, \Lambda/\mathfrak{r})$ to the exact sequence

$$P_{|\eta|+i} \xrightarrow{\xi_i} P_i \to \operatorname{Coker} \xi_i \to 0$$

therefore shows that $\operatorname{Hom}_\Lambda(\operatorname{Coker} \xi_i, \Lambda/\mathfrak{r}) = 0$ for $i \geq w$. Now if X is any finitely generated nonzero Λ-module, then $X/\mathfrak{r}X$ is nonzero by Nakayama's Lemma. This factor module is semisimple, and since every simple Λ-module occurs as a direct summand of $\Lambda/\mathfrak{r}$, there exists a nonzero map $X \to \Lambda/\mathfrak{r}$. This shows $\operatorname{Coker} \xi_i = 0$ for $i \geq w$. By taking $n = |\eta|$, we are done. $\square$

We now prove the main result of this section, which gives sufficient (and necessary) conditions for a module to be eventually periodic. Having Proposition 2.2 at hand, the proof is only a formality, as it reduces to simply comparing lengths in the minimal projective resolution of a module.

Theorem 2.3. *If* **Fg** *holds, then M has bounded Betti numbers if and only if it is an eventually periodic module.*

Proof. From the previous proposition, there exist integers $w \geq 0$ and $n \geq 1$ such that we have n sequences of surjective maps

$$\cdots \xrightarrow{\xi_{w+i+2n}} P_{w+i+2n} \xrightarrow{\xi_{w+i+n}} P_{w+i+n} \xrightarrow{\xi_{w+i}} P_{w+i} \qquad 0 \leq i < n.$$

Considering the lengths of these modules over k, we see that we have non-decreasing sequences $\beta_{w+i}(M) \leq \beta_{w+i+n}(M) \leq \cdots$ for $0 \leq i < n$. Now if the Betti numbers of M are bounded, then these sequences must all eventually stabilize. Thus there is an integer t such that ξ_i is bijective for $i \geq t$, and diagram chasing in the commutative diagram

$$\begin{tikzcd} P_{n+t+1} \arrow[r, "d_{n+t+1}"] \arrow[d, "\xi_{t+1}"] & P_{n+t} \arrow[r] \arrow[d, "\xi_t"] & \Omega_\Lambda^{n+t}(M) \arrow[r] & 0 \\ P_{t+1} \arrow[r, "d_{t+1}"] & P_t \arrow[r] & \Omega_\Lambda^t(M) \arrow[r] & 0 \end{tikzcd}$$

with exact rows provides an isomorphism $\psi\colon \Omega_\Lambda^{n+t}(M) \to \Omega_\Lambda^t(M)$. □

From this result we obtain some insight into the structure of the sequence of Betti numbers of M. Determining how these sequences grow is a problem which has been studied for a long time in the commutative Noetherian local setting. In [Avr], L. Avramov asked whether the Betti sequence $(b_0, b_1, b_2, \dots)$ of a finitely generated module over such a ring is eventually non-decreasing, whereas a somewhat weaker question was asked by M. Ramras in [Ra2]; is it true that either $(b_0, b_1, b_2, \dots)$ is eventually constant, or $\lim_{i\to\infty} b_i = \infty$? With these questions in mind, we include the following corollary to Proposition 2.2 and Theorem 2.3. The result shows that we can split the "tail" of the sequence of Betti numbers of M into a finite number of sequences, each of which is either eventually constant or strictly increasing, depending on whether M has bounded Betti numbers or not.

Corollary 2.4. *Let the setting be as in the theorem. There exist integers $w \geq 0$ and $n \geq 1$ such that the n sequences*

$$(\beta_{w+i+jn}(M))_{j=0}^{\infty} \qquad 0 \leq i < n$$

are non-decreasing. In fact, these sequences are all eventually constant if the Betti numbers of M are bounded, and strictly increasing if not. In particular, if the Betti numbers of M are unbounded, then $\lim_{i\to\infty} \beta_i(M) = \infty$.

Proof. The n non-decreasing sequences were given in the proof of the theorem. It is clear that if the Betti numbers of M are bounded, then the sequences are all eventually constant. In the case when there is no bound on the Betti numbers, the module M is not eventually periodic, and then ξ_i cannot be bijective for any $i \geq w$. For if ξ_t was bijective for some $t \geq w$, then since ξ_{t+1} is surjective we would (as in the proof of theorem) have an isomorphism $\Omega_\Lambda^{n+t}(M) \simeq \Omega_\Lambda^t(M)$. □

Although Theorem 2.3 provides a tool for determining whether or not a module M is eventually periodic, it does not indicate *when* the minimal projective resolution of M becomes periodic, nor what the period actually *is*, contrary to the commutative local case. In [Eis, Theorem 4.1] it is shown that over a commutative local complete intersection A any minimal free resolution whose Betti sequence is bounded becomes periodic of period 2 after at most $\dim A + 1$ steps.

Question. Given the assumptions of Theorem 2.3, does there exist a computable integer s such that the minimal projective resolution of M becomes periodic after at most s steps?

As to determining the period, the degree n of the element η, which we obtain from Proposition 2.1, is of course a candidate, but all that is certain is that the (eventual) period must divide n. However, imposing moderate restrictions on the rings k and H, we get a much stronger result.

Theorem 2.5. *Suppose k contains an infinite field, and let H be generated as an algebra over H^0 by homogeneous elements $a_1, \dots, a_r$ of positive degrees. Then if* **Fg** *holds and M has bounded Betti numbers, the eventual period of M divides $\operatorname{lcm}(|a_1|, \dots, |a_r|)$.*

In particular, if $|a_i| = 1$ for all i then the eventual period of M is 1, and if $|a_i| \leq 2$ for all i then the Betti sequence of M is eventually constant.

Proof. From Proposition 2.1 we know there exists an integer w and a homogeneous element $\eta \in H$, of positive degree, such that η is regular on the H-module $E_{\geq w} = \bigoplus_{i=w}^{\infty} \operatorname{Ext}_{\Lambda}^{i}(M, \Lambda / \mathfrak{r})$. Now let $\{\mathfrak{p}_i\}_{i=1}^{s}$ be the (finite) set of associated primes of $E_{\geq w}$, and denote $\operatorname{lcm}(|a_1|, \dots, |a_r|)$ by u. Then a suitable power of each a_i belongs to H^u, and so if $H^u \subseteq \mathfrak{p}_j$ for some j, then each a_i belongs to $\mathfrak{p}_j$. This implies that the ideal $H^+ = \bigoplus_{i=1}^{\infty} H^i$ is contained in $\mathfrak{p}_j$, contradicting the fact that η, which is an element of H^+, is regular on $E_{\geq w}$. Therefore $H^u \not\subseteq \mathfrak{p}_i$ for $1 \leq i \leq s$, implying $H^u \cap \mathfrak{p}_i \subset H^u$ (strict inclusion). In particular, if k' is an infinite field contained in k, then $H^u \cap \mathfrak{p}_i$ is a proper k'-subspace of H^u. Since over an infinite field no vector space can be written as a finite union of proper subspaces, we get the (strict) inclusion

$$(H^u \cap \mathfrak{p}_1) \cup \dots \cup (H^u \cap \mathfrak{p}_s) \subset H^u,$$

and so H^u must contain an element η' which is not contained in $\mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_s$. This union is the set of all zero-divisors on $E_{\geq w}$, hence η' is regular on this module. In the proofs of Proposition 2.2 and Theorem 2.3 we may replace η by η', thus proving the first statement.

Suppose $|a_i| \leq 2$ for all i. Clearly, if each a_i is of degree 1 then the eventual period of M is 1, and the sequence of Betti numbers of M is eventually constant. If one of the generators is of degree 2, then the eventual period is either 1 or 2. If it is 2, then for some integer $N \geq 0$ we have $\Omega_{\Lambda}^{i}(M) \simeq \Omega_{\Lambda}^{i+2}(M)$ for $i \geq N$. By taking the alternate sum of the k-dimensions in the exact sequence

$$0 \to \Omega_{\Lambda}^{i+2}(M) \to P_{i+1} \to P_i \to \Omega_{\Lambda}^{i}(M) \to 0,$$

and recalling that this sum has to be zero, we get $\beta_i(M) = \beta_{i+1}(M)$ for $i \geq N$. Therefore the sequence of Betti numbers of M is eventually constant also in this case. $\square$

The results we have proved in this section (and also the main result in the next section) depend on the assumption that $\operatorname{Ext}_{\Lambda}^{*}(M, \Lambda / \mathfrak{r})$ is finitely generated as a module over H, and therefore also as a module over $\operatorname{HH}^*(\Lambda)$. As the action of the Hochschild cohomology ring on this module factors through the rings $\operatorname{Ext}_{\Lambda}^{*}(M, M)$ and $\operatorname{Ext}_{\Lambda}^{*}(\Lambda / \mathfrak{r}, \Lambda / \mathfrak{r})$ via ring homomorphisms, the assumption forces $\operatorname{Ext}_{\Lambda}^{*}(M, \Lambda / \mathfrak{r})$ to be finitely generated as a module over both these latter rings. In which situations this happens has been studied in the commutative case. Let $(A, \mathfrak{m}, k')$ be a commutative Noetherian local ring, and N a finitely generated A-module whose so-called *complete intersection dimension* (this was first defined in [AGP2]) over A is finite (as happens for example when A is a complete intersection). Then L. Avramov and L.-C. Sun proved in [AvS] that the graded module $\operatorname{Ext}_{A}^{*}(N, k')$ is finitely generated over $\operatorname{Ext}_{A}^{*}(k', k')$, whereas L. Avramov, V. Gasharov and I. Peeva proved in [AGP2] that the module is also finitely generated over $\operatorname{Ext}_{A}^{*}(N, N)$.

Question. When is $\operatorname{Ext}^*_\Lambda(M, \Lambda/\mathfrak{r})$ finitely generated over (one of) $\operatorname{Ext}^*_\Lambda(M, M)$ and $\operatorname{Ext}^*_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$?

It is not difficult to see that $\operatorname{Ext}^*_\Lambda(M, \Lambda/\mathfrak{r})$ is finitely generated over $\operatorname{Ext}^*_\Lambda(M, M)$ whenever M is periodic; suppose $\Omega^p_\Lambda(M) = M$ and let μ denote the extension

$$0 \to M \to P_{p-1} \to \cdots \to P_0 \to M \to 0.$$

If $n \geq p$, say $n = qp + i$ where $0 \leq i < p$, then an element of $\operatorname{Ext}^n_\Lambda(M, \Lambda/\mathfrak{r})$ can be written as $f\mu^q$ where $f \in \operatorname{Ext}^i_\Lambda(M, \Lambda/\mathfrak{r})$. Since $\operatorname{Ext}^i_\Lambda(M, \Lambda/\mathfrak{r})$ is finitely generated over k for $0 \leq i < p$, the result follows.

As to the finiteness of $\operatorname{Ext}^*_\Lambda(M, \Lambda/\mathfrak{r})$ as an H-module, a criterion for when this always happens was given in [EHSST, Proposition 1.4]. This result is actually much stronger, as it states that $\operatorname{Ext}^*_\Lambda(X, Y)$ is finite over H for all finite Λ-modules X and Y if and only if $\operatorname{Ext}^*_\Lambda(\Lambda/\mathfrak{r}, \Lambda/\mathfrak{r})$ is finite over H.

3. Reducing Complexity

We can use Theorem 2.3 to construct eventually periodic modules - and therefore also periodic modules - of infinite projective dimension (the modules having finite projective dimension are not very interesting in the context of eventual periodicity). This is done by considering an (almost) arbitrary module and from it obtain a new module whose minimal projective resolution behaves "nicer".

Let $X = \bigoplus_{n=0}^{\infty} X_n$ be a graded k-module of finite type. The *rate of growth* of X, denoted $\gamma(X)$, is defined as

$$\gamma(X) = \inf\{t \in \mathbb{N}_0 \mid \exists a \in \mathbb{R} \text{ such that } \ell_k(X_n) \leq an^{t-1} \text{ for } n \gg 0\},$$

and it may be finite or infinite (here $\mathbb{N}_0$ denotes $\mathbb{N} \cup \{0\}$). Now consider our module M with the minimal projective resolution (P_i, d_i). The *complexity* of M, denoted $\operatorname{cx}_\Lambda M$, is defined as the rate of growth of the graded k-module $\bigoplus_{n=0}^{\infty} P_n$, that is

$$\operatorname{cx}_\Lambda M = \inf\{t \in \mathbb{N}_0 \mid \exists a \in \mathbb{R} \text{ such that } \beta_n(M) \leq an^{t-1} \text{ for } n \gg 0\}.$$

Thus the complexity of M indicates how the sequence of Betti numbers behaves with respect to polynomial growth. From the definition we see that M has complexity 0 if and only if it has finite projective dimension, and that it has complexity less than or equal to 1 if and only if its sequence of Betti numbers is bounded. The main result of this section gives the existence of a new module whose complexity is exactly one less than that of M. The proof uses the identity

$$\operatorname{cx}_\Lambda M = \gamma(\operatorname{Ext}^*_\Lambda(M, \Lambda/\mathfrak{r})),$$

which provides a method for computing the complexity of a module. This identity follows from the identities (see the paragraphs following [Ben, Definition 5.3.3])

$$\begin{aligned} \gamma(\operatorname{Ext}^*_\Lambda(M, \Lambda/\mathfrak{r})) &= \max\{\gamma(\operatorname{Ext}^*_\Lambda(M, S)) \mid S \text{ simple } \Lambda\text{-module}\} \\ \beta_n(M) &= \sum_{S \text{ simple}} \frac{\ell_k(P_S)}{\ell_k(\operatorname{Hom}_\Lambda(S, S))} \cdot \ell_k(\operatorname{Ext}^n_\Lambda(M, S)), \end{aligned}$$

where P_S denotes the projective cover of the simple module S.

Given a homogeneous element η in $\operatorname{HH}^*(\Lambda)$ of positive degree, we can interpret it as a Λ^e-linear map $\eta\colon \Omega^{|\eta|}_{\Lambda^e}(\Lambda) \to \Lambda$, where $\Omega^i_{\Lambda^e}(\Lambda)$ denotes the i'th syzygy in the minimal projective Λ^e-resolution of Λ. Let Q_i denote the i'th module in this resolution. By taking pushout we obtain the exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega_{\Lambda^e}^{|\eta|}(\Lambda) \arrow[r] \arrow[d, "\eta"] & Q_{|\eta|-1} \arrow[r] \arrow[d] & \Omega_{\Lambda^e}^{|\eta|-1}(\Lambda) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Lambda \arrow[r] & K_\eta \arrow[r] & \Omega_{\Lambda^e}^{|\eta|-1}(\Lambda) \arrow[r] & 0
\end{tikzcd}$$

of Λ^e-modules, whose bottom row we denote by ζ_η. Since $\Omega_{\Lambda^e}^{|\eta|-1}(\Lambda)$ is projective as a right (and left) Λ-module, the exact sequence ζ_η splits when considered as a sequence of right (and left) Λ-modules. Applying $-\otimes_\Lambda M$ therefore gives the exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega_{\Lambda^e}^{|\eta|}(\Lambda)\otimes_\Lambda M \arrow[r] \arrow[d, "\eta\otimes_\Lambda M"] & Q_{|\eta|-1}\otimes_\Lambda M \arrow[r] \arrow[d] & \Omega_{\Lambda^e}^{|\eta|-1}(\Lambda)\otimes_\Lambda M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] & K_\eta\otimes_\Lambda M \arrow[r] & \Omega_{\Lambda^e}^{|\eta|-1}(\Lambda)\otimes_\Lambda M \arrow[r] & 0
\end{tikzcd}$$

of left Λ-modules, whose bottom row we denote by $\zeta_\eta \otimes_\Lambda M$. Even though ζ_η splits when considered as a sequence of left Λ-modules, this is not necessarily the case for the new sequence. In fact, from [EHSST, Proposition 2.2] we see that the sequence splits if and only if η annihilates $\operatorname{Ext}_\Lambda^*(M,M)$.

The module $K_\eta \otimes_\Lambda M$ is going to be the one having complexity one less than that of M. However, as the above shows, the element η cannot be chosen arbitrarily, for if $\zeta_\eta \otimes_\Lambda M$ splits then the complexity of $K_\eta \otimes_\Lambda M$ equals that of M. To see this, note that in a split short exact sequence the complexity of the middle term equals the maximum of the complexities of the end terms, and that the complexities of the end term modules in $\zeta_\eta \otimes_\Lambda M$ are equal since $\Omega_{\Lambda^e}^{|\eta|-1}(\Lambda)\otimes_\Lambda M$ is a syzygy of M (it does not matter that $\Omega_{\Lambda^e}^{|\eta|-1}(\Lambda)\otimes_\Lambda M$ in general is not a syzygy in the *minimal* projective resolution of M, since projectively equivalent modules are of equal complexity). Thus we must pick an η not annihilating $\operatorname{Ext}_\Lambda^*(M,M)$.

Let N be any Λ-module. Applying the functor $\operatorname{Hom}_\Lambda(-,N)$ to $\zeta_\eta \otimes_\Lambda M$ gives the long exact sequence

$$\operatorname{Hom}_\Lambda(M,N) \xrightarrow{\partial_\eta} \operatorname{Ext}_\Lambda^{|\eta|}(M,N) \to \operatorname{Ext}_\Lambda^1(K_\eta\otimes_\Lambda M,N) \to \operatorname{Ext}_\Lambda^1(M,N) \xrightarrow{\partial_\eta}$$

$$\vdots$$

$$\xrightarrow{\partial_\eta} \operatorname{Ext}_\Lambda^{i+|\eta|-1}(M,N) \to \operatorname{Ext}_\Lambda^i(K_\eta\otimes_\Lambda M,N) \to \operatorname{Ext}_\Lambda^i(M,N) \xrightarrow{\partial_\eta}$$

$$\vdots$$

for Ext (we shall refer to this sequence as $\mathrm{ES}(M,N,\eta)$), where we have replaced $\operatorname{Ext}_\Lambda^i(\Omega_{\Lambda^e}^{|\eta|-1}(\Lambda)\otimes_\Lambda M,N)$ by $\operatorname{Ext}_\Lambda^{i+|\eta|-1}(M,N)$ for $i \geq 1$ using dimension shift. By making these replacements, *the new connecting homomorphism* $\operatorname{Ext}_\Lambda^i(M,N) \xrightarrow{\partial_\eta} \operatorname{Ext}_\Lambda^{i+|\eta|}(M,N)$ *is just multiplication by* $(-1)^i\eta$, a fact which is vital for the proof of the main theorem. To see this, note that applying $\operatorname{Hom}_\Lambda(-,N)$ to the above commutative diagram gives rise to a commutative diagram of long exact sequences in Ext. Tracing the connecting homomorphism ∂_η then gives the desired result. Whenever we refer to the exact sequence $\mathrm{ES}(M,N,\eta)$, we shall drop the sign $(-1)^i$ in front of the multiplication map induced by η, as it is of no relevance.

Note that if **Fg** holds, then $\operatorname{Ext}_\Lambda^*(K_\eta \otimes_\Lambda M, \Lambda / \mathfrak{r})$ is a finitely generated H module, regardless of the choice of η. To see this, consider the exact sequence

$$\bigoplus_{i=1}^{\infty} \operatorname{Ext}_\Lambda^{i+|\eta|-1}(M, \Lambda / \mathfrak{r}) \to \bigoplus_{i=1}^{\infty} \operatorname{Ext}_\Lambda^{i}(K_\eta \otimes_\Lambda M, \Lambda / \mathfrak{r}) \to \bigoplus_{i=1}^{\infty} \operatorname{Ext}_\Lambda^{i}(M, \Lambda / \mathfrak{r})$$

induced by $\operatorname{ES}(M, \Lambda / \mathfrak{r}, \eta)$. Both the end terms are finitely generated over H, hence so is the middle term because H is Noetherian. Since in addition $\operatorname{Hom}_\Lambda(K_\eta \otimes_\Lambda M, \Lambda / \mathfrak{r})$ is finitely generated over k (which sits inside H^0), the claim follows.

In addition to giving us an important tool for computing the complexity of a module, the equality $\operatorname{cx}_\Lambda M = \gamma(\operatorname{Ext}_\Lambda^*(M, \Lambda / \mathfrak{r}))$ implies that the modules we work with have finite complexity. For if **Fg** holds, then the rate of growth of $\operatorname{Ext}_\Lambda^*(M, \Lambda / \mathfrak{r})$ is not more than that of H, since it is a quotient of a finitely generated free H-module. Now as in the proof of Proposition 2.2, there is a finite set $\{a_1, \dots, a_r\}$ in H of homogeneous elements of positive degrees, generating H as an algebra over H^0. By the Hilbert-Serre Theorem (see [Ben, Proposition 5.3.1]) and [Ben, Proposition 5.3.2], we have that $\gamma(H)$ equals the order of the pole at $t = 1$ of a certain rational function $g(t)/\prod_{i=1}^{r}(1 - t^{|a_i|})$, where $g \in \mathbb{Z}[t]$. Hence the rate of growth of H is not more than r, implying $\operatorname{cx}_\Lambda M \leq r$.

The aim is now to pick an η such that the rate of growth of $\operatorname{Ext}_\Lambda^*(K_\eta \otimes_\Lambda M, \Lambda / \mathfrak{r})$ is one less than the rate of growth of $\operatorname{Ext}_\Lambda^*(M, \Lambda / \mathfrak{r})$. That such an element exists is a consequence of the following result.

Proposition 3.1. *Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a commutative Noetherian graded k-algebra of finite type over k, generated as an A_0-algebra by homogeneous elements $a_1, \dots, a_r$ with $|a_i| = n_i > 0$. Let $N = \bigoplus_{i=0}^{\infty} N_i$ be a finitely generated graded A-module, and pick a homogeneous element $\eta \in A$ as in* Proposition 2.1. *Let $w \in \mathbb{N}$ be an integer such that $\eta \colon N_i \to N_{i+|\eta|}$ is injective for $i \geq w$, define V_{i-w} to be the cokernel of this map, and denote by V the graded k-vector space $\bigoplus_{i=0}^{\infty} V_i$. If $\gamma(N) > 0$, then $\gamma(V) = \gamma(N) - 1$.*

Proof. Consider the Poincaré series $P(N, t) = \sum_{i=0}^{\infty} \ell_k(N_i) t^i$ of N. By the Hilbert-Serre Theorem we have

$$P(N, t) = \frac{f(t)}{\prod_{i=1}^{r}(1 - t^{n_i})},$$

for some $f(t) \in \mathbb{Z}[t]$, and from [Ben, Proposition 5.3.2] we see that $\gamma(N)$ equals the order of the pole of $P(N, t)$ at $t = 1$. By assumption this integer is strictly greater than zero.

Now consider the exact sequences

$$0 \to N_i \xrightarrow{\eta} N_{i+|\eta|} \to V_{i-w} \to 0$$

for $i \geq w$. Taking k-module lengths we get $\ell_k(V_i) = \ell_k(N_{i+w+|\eta|}) - \ell_k(N_{i+w})$ for $i \geq 0$, giving

$$P(V, t) = \sum_{i=0}^{\infty} \ell_k(V_i) t^i = \sum_{i=0}^{\infty} \ell_k(N_{i+w+|\eta|}) t^i - \sum_{i=0}^{\infty} \ell_k(N_{i+w}) t^i.$$

Multiplying this equation by $t^{w+|\eta|}$ gives

$$\begin{aligned} t^{w+|\eta|} P(V, t) &= \sum_{i=0}^{\infty} \ell_k(N_{i+w+|\eta|}) t^{i+w+|\eta|} - t^{|\eta|} \sum_{i=0}^{\infty} \ell_k(N_{i+w}) t^{i+w} \\ &= (1 - t^{|\eta|}) P(N, t) + g(t), \end{aligned}$$

where $g(t)$ is some polynomial in $\mathbb{Z}[t]$, and therefore

$$\begin{aligned} P(V,t) &= \frac{(1-t^{|\eta|})P(N,t)}{t^{w+|\eta|}} + \frac{g(t)}{t^{w+|\eta|}} \\ &= \frac{(1-t^{|\eta|})f(t)}{t^{w+|\eta|}\prod_{i=1}^{r}(1-t^{n_i})} + \frac{g(t)}{t^{w+|\eta|}}. \end{aligned}$$

Thus the order of the pole of $P(V,t)$ at $t=1$ is one less than that of $P(N,t)$, showing $\gamma(V) = \gamma(N) - 1$. □

We now return to the setting given at the beginning of this section. With Proposition 3.1 at hand, the main theorem is merely a corollary.

Theorem 3.2 (Reducing Complexity)**.** *Assume* **Fg** *holds and that M does not have finite projective dimension. Then there exists a homogeneous element $\eta \in H$ of positive degree such that* $\operatorname{cx}_\Lambda(K_\eta \otimes_\Lambda M) = \operatorname{cx}_\Lambda M - 1$.

Proof. We use the previous proposition with $A = H$ and $N = \operatorname{Ext}^*_\Lambda(M, \Lambda/\mathfrak{r})$. There is an integer $w \in \mathbb{N}$ and a homogeneous element $\eta \in H$ of positive degree such that the multiplication map $\operatorname{Ext}^i_\Lambda(M,\Lambda/\mathfrak{r}) \xrightarrow{\eta} \operatorname{Ext}^{i+|\eta|}_\Lambda(M,\Lambda/\mathfrak{r})$ is injective for $i \geq w$. From the long exact sequence $\operatorname{ES}(M, \Lambda/\mathfrak{r}, \eta)$ we then get short exact sequences

$$0 \to \operatorname{Ext}^i_\Lambda(M,\Lambda/\mathfrak{r}) \xrightarrow{\eta} \operatorname{Ext}^{i+|\eta|}_\Lambda(M,\Lambda/\mathfrak{r}) \to \operatorname{Ext}^{i+1}_\Lambda(K_\eta \otimes_\Lambda M, \Lambda/\mathfrak{r}) \to 0$$

for $i \geq w$. Using Proposition 3.1 we now get

$$\begin{aligned} \operatorname{cx}_\Lambda(K_\eta \otimes_\Lambda M) &= \gamma(\operatorname{Ext}^*_\Lambda(K_\eta \otimes_\Lambda M, \Lambda/\mathfrak{r})) \\ &= \gamma(\bigoplus_{i=w+1}^{\infty} \operatorname{Ext}^i_\Lambda(K_\eta \otimes_\Lambda M, \Lambda/\mathfrak{r})) \\ &= \gamma(\operatorname{Ext}^*_\Lambda(M,\Lambda/\mathfrak{r})) - 1 \\ &= \operatorname{cx}_\Lambda M - 1. \end{aligned}$$

□

As a corollary we get a result which under certain conditions guarantees the existence of nonzero periodic modules. As mentioned in the beginning of the previous section, the existence of such modules is not obvious. For example, in [Ra1], M. Ramras introduced a nonempty class of commutative Noetherian local rings called BNSI rings (short for "Betti numbers strictly increase" rings), which are rings for which every non-free module has a strictly increasing sequence of Betti numbers. There exist a lot of finite dimensional algebras which are BNSI rings, for example regular local rings of dimension at least two modulo any positive power of the maximal ideal. Clearly, such rings cannot have nonzero periodic modules.

Corollary 3.3. *Suppose* **Fg** *holds and that M has infinite projective dimension. Then Λ has a nonzero periodic module.*

Proof. Let d denote the complexity of M (by assumption $d > 0$). If $d = 1$ then M is eventually periodic by Theorem 2.3, whereas if $d > 1$ the previous theorem provides homogeneous elements $\eta_1, \dots, \eta_{d-1} \in H$ having the property

$$\operatorname{cx}_\Lambda(K_{\eta_i} \otimes_\Lambda \cdots \otimes_\Lambda K_{\eta_1} \otimes_\Lambda M) = d - i$$

for $1 \leq i \leq d-1$. Denote the module $K_{\eta_{d-1}} \otimes_\Lambda \cdots \otimes_\Lambda K_{\eta_1} \otimes_\Lambda M$ by X. This module has complexity one, and from the discussion prior to Proposition 3.1 we see that $\operatorname{Ext}^*_\Lambda(X, \Lambda/\mathfrak{r})$ is a finitely generated H-module. Using Theorem 2.3 once more, we get that X is eventually periodic.

Now if we take an eventually periodic Λ-module of infinite projective dimension, one of its syzygies is a nonzero periodic module. $\square$

Remark. (i) The existence of a periodic Λ-module implies the existence of a periodic module having period 1; if M is isomorphic to $\Omega_\Lambda^p(M)$, where $p \geq 1$, then the module $\bigoplus_{i=0}^{p-1} \Omega_\Lambda^i(M)$ is periodic of period 1 (here $\Omega_\Lambda^0(M) = M$).

(ii) Suppose Λ is a BNSI ring. Then there cannot exist a non-free Λ-module M for which **Fg** holds, i.e. $\operatorname{Ext}_\Lambda^*(M, \Lambda / \mathfrak{r})$ is not finitely generated over any commutative Noetherian graded subalgebra of $\operatorname{HH}^*(\Lambda)$; if such a module did exist, then the corollary would imply the existence of a nonzero periodic Λ-module.

4. Generating Periodicity

In this section we consider the case when the period of an eventually periodic module is "detected" by a homogeneous element in the Hochschild cohomology ring. We start by recalling the group ring case.

Assume k is an algebraically closed field and let G be a finite group. A nonzero homogeneous element $\theta \in \operatorname{H}^{|\theta|}(G,k) = \operatorname{Ext}_{kG}^{|\theta|}(k,k)$ can be interpreted as a surjective kG-homomorphism $\theta \colon \Omega_{kG}^{|\theta|}(k) \to k$, the kernel of which it is customary to denote by L_θ. The cohomological variety of L_θ is easily computed; from [Car, Lemma 2.3] we have that $\operatorname{V}_G(L_\theta)$ equals $\operatorname{V}_G(\theta)$, i.e. the set of maximal ideals in the group cohomology ring $\operatorname{H}^{\cdot}(G,k)$ containing θ. Now let N be a finitely generated kG-module. If $\operatorname{V}_G(\theta) \cap \operatorname{V}_G(N) = \{0\}$, then from the above and the equality $\operatorname{V}_G(X \otimes_k Y) = \operatorname{V}_G(X) \cap \operatorname{V}_G(Y)$, which holds for all finitely generated kG-modules X and Y, we get $\operatorname{V}_G(L_\theta \otimes_k N) = \{0\}$. This is equivalent to $L_\theta \otimes_k N$ being projective, and it follows from the proof of [Ben, Theorem 5.10.4] that N is isomorphic to $\Omega_{kG}^{|\theta|}(N) \oplus P$, where P is a projective module. This gives

$$\Omega_{kG}^{|\theta|}(N) \simeq \Omega_{kG}^{|\theta|}(\Omega_{kG}^{|\theta|}(N) \oplus P) \simeq \Omega_{kG}^{2|\theta|}(N),$$

showing $\Omega_{kG}^{|\theta|}(N)$ is periodic and therefore that N is isomorphic to a direct sum of a periodic module and a projective module. If N contains no nonzero projective summand we must have $N \simeq \Omega_{kG}^{|\theta|}(N)$, with the period of N dividing $|\theta|$, and in this case the element θ is said to generate the periodicity of N.

Returning to the setting given in the previous sections, with k a commutative Artin ring, Λ an Artin k-algebra (assumed to be projective as a k-module) and M a finitely generated Λ-module, let η be a nonzero element in $\operatorname{HH}^*(\Lambda)$ of positive degree. Instead of considering the kernel of the corresponding Λ^{e}-linear map $\eta \colon \Omega_{\Lambda^{\mathrm{e}}}^{|\eta|}(\Lambda) \to \Lambda$ (which is not necessarily surjective), we look at the pushout K_η and the tensor module $K_\eta \otimes_\Lambda M$, as we did in the last section.

Proposition 4.1. *If the Λ-module $K_\eta \otimes_\Lambda M$ has finite projective dimension, then M is eventually periodic with period dividing $|\eta|$.*

Proof. Denote the projective dimension of $K_\eta \otimes_\Lambda M$ by d. From the long exact sequence $\operatorname{ES}(M, \Lambda / \mathfrak{r}, \eta)$ we see that scalar multiplication by η induces k-isomorphisms

$$\operatorname{Ext}_\Lambda^i(M, \Lambda / \mathfrak{r}) \xrightarrow{\eta} \operatorname{Ext}_\Lambda^{i+|\eta|}(M, \Lambda / \mathfrak{r})$$

for $i > d$. Now let $\cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$ be the minimal projective resolution of M, and recall the proof of Proposition 2.2. If we denote by ξ the image of η in $\operatorname{Ext}_\Lambda^*(M,M)$, represented by a map $\Omega_\Lambda^{|\eta|}(M) \xrightarrow{\xi} M$, then in the commutative diagram (obtained from the Comparison Theorem)

$$\begin{tikzcd}
\cdots \arrow[r] & P_{|\eta|+i+1} \arrow[r, "d_{|\eta|+i+1}"] \arrow[d, "\xi_{i+1}"] & P_{|\eta|+i} \arrow[r, "d_{|\eta|+i}"] \arrow[d, "\xi_i"] & \cdots \arrow[r] & P_{|\eta|} \arrow[r] \arrow[d, "\xi_0"] & \Omega_\Lambda^{|\eta|}(M) \arrow[r] \arrow[d, "\xi"] & 0 \\
\cdots \arrow[r] & P_{i+1} \arrow[r, "d_{i+1}"] & P_i \arrow[r, "d_i"] & \cdots \arrow[r] & P_0 \arrow[r, "d_0"] & M \arrow[r] & 0
\end{tikzcd}$$

we have that ξ_i is surjective for $i > d$. Applying $\operatorname{Hom}_\Lambda(-, \Lambda / \mathfrak{r})$ to the exact sequences

$$0 \to \operatorname{Ker} \xi_i \to P_{|\eta|+i} \xrightarrow{\xi_i} P_i \to 0$$

(for $i > d$) then gives $\operatorname{Hom}_\Lambda(\operatorname{Ker} \xi_i, \Lambda / \mathfrak{r}) = 0$, hence $\operatorname{Ker} \xi_i = 0$. This shows ξ_i is bijective for $i > d$, and as in the proof of Theorem 2.3 we see that $\Omega_\Lambda^i(M) \simeq \Omega_\Lambda^{i+|\eta|}(M)$ for $i > d$. □

The proposition motivates the following definition of an element generating the periodicity of a module:

Definition 4.2. Let M be a Λ-module of infinite projective dimension. A nonzero homogeneous element $\eta \in \operatorname{HH}^*(\Lambda)$ of positive degree *generates the periodicity* of M if the Λ-module $K_\eta \otimes_\Lambda M$ has finite projective dimension.

Note that in the special case when $K_\eta \otimes_\Lambda M$ is projective, all but the end terms in the exact sequence

$$0 \to M \to K_\eta \otimes_\Lambda M \to Q_{|\eta|-2} \otimes_\Lambda M \to \cdots \to Q_0 \otimes_\Lambda M \to M \to 0$$

are projective, hence M is isomorphic to $\Omega_\Lambda^{|\eta|}(M) \oplus P$, where P is a projective Λ-module. As in the group ring case above, this implies that $\Omega_\Lambda^{|\eta|}(M) \simeq \Omega_\Lambda^{2|\eta|}(M)$, and therefore M is isomorphic to the direct sum of a periodic module and a projective module. In particular, if M contains no nonzero projective summand then it is periodic.

Also note that, whenever there exists an element η generating the periodicity of M, for every Λ-module N we see from the long exact sequence $\operatorname{ES}(M, N, \eta)$ that multiplication

$$\operatorname{Ext}_\Lambda^i(M, N) \xrightarrow{\eta} \operatorname{Ext}_\Lambda^{i+|\eta|}(M, N)$$

is an isomorphism for $i \gg 0$. From the proof of the previous proposition we see that the converse is also true, hence the following result.

Proposition 4.3. *A nonzero homogeneous element $\eta \in \operatorname{HH}^*(\Lambda)$ generates the periodicity of M if and only if scalar multiplication by η induces k-isomorphisms*

$$\operatorname{Ext}_\Lambda^i(M, N) \xrightarrow{\eta} \operatorname{Ext}_\Lambda^{i+|\eta|}(M, N)$$

for every Λ-module N and $i \gg 0$.

As an immediate corollary we obtain the following finite generation result.

Corollary 4.4. *If there exists an element η generating the periodicity of M, then $\operatorname{Ext}_\Lambda^*(M, N)$ is a finitely generated $\operatorname{HH}^*(\Lambda)$-module for every Λ-module N. In particular $\operatorname{Ext}_\Lambda^*(M, M)$ and $\operatorname{Ext}_\Lambda^*(M, \Lambda / \mathfrak{r})$ are finitely generated.*

Proof. Denoting $\operatorname{pd}_\Lambda(K_\eta \otimes_\Lambda M)$ by d we see that the k-bases of $\operatorname{Hom}_\Lambda(M, N), \operatorname{Ext}_\Lambda^1(M, N), \dots, \operatorname{Ext}_\Lambda^{|\eta|+d}(M, N)$ together form a generating set. □

Remark. We actually obtain a stronger result, namely that for every Λ-module N the k-module $\operatorname{Ext}_\Lambda^*(M,N)$ is a finitely generated module over the commutative Noetherian graded subalgebra of $\operatorname{HH}^*(\Lambda)$ generated by $\operatorname{HH}^0(\Lambda)$ and η (in particular **Fg** holds). As the element η is not nilpotent, this subalgebra is actually the polynomial ring in one variable over $\operatorname{HH}^0(\Lambda)$ (the center of Λ).

It is natural to ask whether an element generating the periodicity of M always exists when M is a periodic module. The following two examples show that this is not the case. In the first example the algebra is selfinjective, whereas in the second it is commutative local.

Example 4.5. Let k be a field, and consider the 4-dimensional algebra

$$\Lambda = k\langle x,y\rangle/(x^2, xy+qyx, y^2)$$

where $0 \neq q \in k$ is not a root of unity. Let M be any 2-dimensional k-vector space having a basis $\{u,v\}$, say. Straightforward computation shows that defining

$$xu = 0, xv = 0$$
$$yu = v, yv = 0$$

gives a Λ-module structure on M (this module was also studied in [Sch]). Define a Λ-linear map $p\colon \Lambda \to M$ by $1 \mapsto u$. This is a surjective map, and as a vector space $\operatorname{Ker} p$ has $\{x, xy\}$ as a basis. Therefore $\operatorname{Ker} p$ is contained in the radical of Λ, showing p is the projective cover of M. Define a k-linear map $f\colon M \to \operatorname{Ker} p$ by $u \mapsto x$ and $v \mapsto xy$. This is an isomorphism, and direct computation shows it is Λ-linear. Hence $M \simeq \Omega_\Lambda^1(M)$, and so M is periodic of period 1.

Suppose $0 \neq \eta \in \operatorname{HH}^{|\eta|}(\Lambda)$ is an element generating the periodicity of M. Then from Proposition 4.3 scalar multiplication by η induces isomorphisms $\operatorname{Ext}_\Lambda^i(M,M) \simeq \operatorname{Ext}_\Lambda^{i+|\eta|}(M,M)$ for $i \gg 0$. Therefore multiplication by any power of η also induces isomorphisms. However, from [BGMS] we have that $\operatorname{HH}^n(\Lambda) = 0$ for $n \geq 3$, in particular η is nilpotent. Hence $\operatorname{Ext}_\Lambda^i(M,M) = 0$ for $i \gg 0$, and since for each $i > 1$ we have

$$\operatorname{Ext}_\Lambda^i(M,M) \simeq \operatorname{Ext}_\Lambda^1(\Omega_\Lambda^{i-1}(M),M) \simeq \operatorname{Ext}_\Lambda^1(M,M),$$

we get $\operatorname{Ext}_\Lambda^1(M,M) = 0$. Therefore M is projective, since it is periodic of period 1, and this is a contradiction.

Example 4.6. Let k be an algebraically closed field of characteristic different from 2, and let $k[x_1,x_2,x_3,x_4]$ be the polynomial ring in four variables over k. Denote by R the finite dimensional local k-algebra $k[x_1,x_2,x_3,x_4]/\mathfrak{a}$, where $\mathfrak{a}$ is the ideal generated by the quadratic forms

$$\begin{gathered} x_1^2, \quad x_1x_2 - x_3^2, \quad x_1x_3 - x_2x_4, \\ x_1x_4, \quad x_2^2 + x_3x_4, \quad x_2x_3, \quad x_4^2. \end{gathered}$$

Define two R-endomorphisms $\phi, \psi\colon R^2 \to R^2$ by the matrices

$$\phi = \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix}, \qquad \psi = \begin{pmatrix} x_4 & -x_3 \\ -x_2 & x_1 \end{pmatrix},$$

and let $M = \operatorname{Im}\psi$. In [AGP1] it is shown that $\operatorname{Im}\psi = \operatorname{Ker}\phi$ and $\operatorname{Im}\phi = \operatorname{Ker}\psi$, hence M is periodic of period 2 and has the minimal free resolution

$$\cdots \to R^2 \xrightarrow{\phi} R^2 \xrightarrow{\psi} R^2 \xrightarrow{\phi} R^2 \xrightarrow{\psi} M \to 0.$$

As a k-vector space M is 8-dimensional, and the elements

$$v_1 = \left(\begin{smallmatrix} x_4 \\ -x_2 \end{smallmatrix}\right), v_2 = \left(\begin{smallmatrix} -x_3 \\ x_1 \end{smallmatrix}\right),$$
$$v_3 = \left(\begin{smallmatrix} 0 \\ x_1x_2 \end{smallmatrix}\right), v_4 = \left(\begin{smallmatrix} x_1x_3 \\ 0 \end{smallmatrix}\right), v_5 = \left(\begin{smallmatrix} x_1x_3 \\ x_3x_4 \end{smallmatrix}\right), v_6 = \left(\begin{smallmatrix} x_3x_4 \\ 0 \end{smallmatrix}\right), v_7 = \left(\begin{smallmatrix} -x_1x_2 \\ x_1x_3 \end{smallmatrix}\right), v_8 = \left(\begin{smallmatrix} 0 \\ x_1x_3 \end{smallmatrix}\right)$$

form a basis. Furthermore, as a graded R-module M consists of two nonzero graded components M_0 and M_1, with $\{v_1, v_2\}$ as a basis for M_0 and $\{v_3, \dots, v_8\}$ as a basis for M_1. The elements v_1, v_2 generate M as an R-module.

Denote by μ the extension $0 \to M \to R^2 \xrightarrow{\phi} R^2 \xrightarrow{\psi} M \to 0$ in $\operatorname{Ext}^2_R(M, M)$. Then for all $n \geq 0$ every element of $\operatorname{Ext}^{2n}_R(M, M)$ is of the form $f\mu^n$ for some $f \in \operatorname{Hom}_R(M, M)$. Now suppose $0 \neq \eta \in \operatorname{HH}^{|\eta|}(R)$ is an element generating the periodicity of M. Then $|\eta|$ is an even number (since M is of period 2, i.e. $\Omega^1_R(M) \not\simeq M$), and so the image of η in $\operatorname{Ext}^*_R(M, M)$ under the ring homomorphism $-\otimes_R M$ can be written as $f\mu^n$, where f is an R-endomorphism of M and $n = |\eta|/2$. Since $M = M_0 \oplus M_1$ as a graded module, we can write f as $h+g$, where h and g are homogeneous endomorphisms of degree 0 and 1, respectively (see the paragraphs following [BrH, Theorem 1.5.8]). Therefore the element $f\mu^n$ is the sum of two elements $h\mu^n$ and $g\mu^n$ of different internal degrees in $\operatorname{Ext}^{|\eta|}_R(M, M)$. Since the map $- \otimes_R M \colon \operatorname{HH}^{|\eta|}(R) \to \operatorname{Ext}^{|\eta|}_R(M, M)$ preserves internal grading we must have that $h\mu^n$ and $g\mu^n$ both lie in the image of $- \otimes_R M$.

Suppose h is nonzero. Since the degree of this map is zero, we must have that $h(v_1) = c_1v_1 + c_2v_2$ and $h(v_2) = c_3v_1 + c_4v_2$ for elements $c_i \in k$. As $v_6 = x_3v_1 = -x_4v_2$ and h is R-linear, direct computation of $h(v_6)$ gives

$$c_1v_6 + c_2v_7 = c_3v_8 + c_4v_6.$$

Thus $c_2 = c_3 = 0$ and $c_1 = c_4$, and we see that h is nothing but a scalar $c \in k$ times the identity on M. Therefore the element $c\mu^n$ belongs to the image of $- \otimes_R M$, and multiplying with the inverse of c we get that μ^n also lies in this image. From [SnS, Corollary 1.3] we see that every element of $\operatorname{Im}(-\otimes_R M)$ belongs to the graded center of $\operatorname{Ext}^*_R(M, M)$, and so μ^n, which is of even degree, is a central element. We show that this is not the case.

An element of $\operatorname{Ext}^1_R(M, M)$ can be considered as an R-linear map $\theta \colon R^2 \to M$ such that $\theta \circ \psi = 0$. Define three such elements $\theta_1, \theta_2, \theta_3$ by

$$\begin{aligned}
\theta_1 \quad &: \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \mapsto v_6, \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \mapsto 0,\\
\theta_2 \quad &: \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \mapsto 0, \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \mapsto v_4,\\
\theta_3 \quad &: \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \mapsto 0, \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) \mapsto v_5.
\end{aligned}$$

It is easy to check that these elements are linearly independent, and when they are multiplied on the right by any power of μ they stay linearly independent. Direct computation (lifting maps along the minimal free resolution of M) gives

$$\begin{aligned}
\mu\theta_1 &= (\theta_1/2 - \theta_2 + \theta_3/2)\mu\\
\mu\theta_2 &= \theta_2\mu\\
\mu\theta_3 &= (-\theta_1/2 - \theta_2 + 3\theta_3/2)\mu,
\end{aligned}$$

and so by induction we get $\mu^n\theta_1 = ([1 - n/2]\theta_1 - n\theta_2 + [n/2]\theta_3)\mu^n$ for all $n \geq 1$. This shows that μ^n is not a central element, implying h must be zero.

As a consequence, the image of η in $\operatorname{Ext}^*_R(M, M)$ is $g\mu^n$, where g is a homogeneous degree one R-endomorphism of M. In particular, since M only lives in two degrees, we must have $g^2 = 0$. From Proposition 4.3, the element η induces isomorphisms

$$\operatorname{Ext}^i_\Lambda(M, M) \xrightarrow{\eta} \operatorname{Ext}^{i+|\eta|}_\Lambda(M, M)$$

for $i \gg 0$, and in particular there is an $i > 0$ such that $\eta \cdot g\mu^i \neq 0$ (because $g\mu^n \neq 0$ implies $g\mu^i \neq 0$, since the element μ^j is not a zerodivisor for any j). But $g\mu^n$, being the image of η, is a central element, giving $\eta \cdot g\mu^i = (g\mu^n)g\mu^i = g(g\mu^n)\mu^i = 0$, a contradiction. Therefore there does not exist a nonzero homogeneous element of positive degree in $\operatorname{HH}^*(R)$ generating the periodicity of M.

These examples show that an element generating the periodicity of an eventually periodic module does not exist in general. However, when dealing with "suitably nice" algebras, such an element always exists. Suppose M is an eventually periodic Λ-module satisfying **Fg**. Then Theorem 3.2 implies the existence of a homogeneous element $\eta \in \operatorname{HH}^*(\Lambda)$ of positive degree with the property that $K_\eta \otimes_\Lambda M$ has finite projective dimension over Λ, and so this η generates the periodicity of M.

The following example shows that it may be difficult to decide whether or not there exists an element generating the periodicity of a module, even when the algebra is commutative local and selfinjective.

Example 4.7. Let k be a field of characteristic different from 2, and let $k[x_1, x_2, x_3, x_4, x_5]$ be the polynomial ring in five variables over k. Denote by R the finite dimensional local k-algebra $k[x_1, x_2, x_3, x_4, x_5]/\mathfrak{a}$, where $\mathfrak{a}$ is the ideal generated by the quadratic forms

$$x_1^2, \quad x_2^2, \quad x_5^2, \quad x_3x_4, \quad x_3x_5, \quad x_4x_5, \quad x_3^2 - x_4^2,$$
$$x_1x_3 + x_2x_3, \quad x_1x_4 + x_2x_4, \quad x_3^2 - x_2x_5 + x_1x_5.$$

It is shown in [GaP] that R is graded selfinjective with Hilbert series $1+5t+5t^2+t^3$, and that the R-endomorphism $d\colon R^2 \to R^2$ defined by

$$d = \begin{pmatrix} x_1 & x_3 + x_4 \\ 0 & x_2 \end{pmatrix}$$

satisfies $\operatorname{Im} d = \operatorname{Ker} d$. Letting $M = \operatorname{Im} d$, we have that M is periodic of period 1, with minimal free resolution

$$\cdots \to R^2 \xrightarrow{d} R^2 \xrightarrow{d} R^2 \xrightarrow{d} M \to 0.$$

Denote by μ the extension $0 \to M \to R^2 \xrightarrow{d} M \to 0$ in $\operatorname{Ext}_R^1(M, M)$. Then for all $n \geq 0$ every element of $\operatorname{Ext}_R^n(M, M)$ is of the form $f\mu^n$ for some endomorphism f of M.

Now suppose $0 \neq \eta \in \operatorname{HH}^{|\eta|}(R)$ is an element generating the periodicity of M. Then the image of η in $\operatorname{Ext}_R^*(M, M)$ can be written as $f\mu^{|\eta|}$, and as in the previous example this implies that $\mu^{|\eta|}$ belongs to the image of $- \otimes_R M\colon \operatorname{HH}^{|\eta|}(R) \to \operatorname{Ext}_R^{|\eta|}(M, M)$ (in this case M is graded and "lives" in three degrees, i.e. $M = M_0 \oplus M_1 \oplus M_2$). In particular $\mu^{|\eta|}$ belongs to the graded center of $\operatorname{Ext}_R^*(M, M)$.

An element of $\operatorname{Ext}_R^i(M, M)$ can be considered as an R-linear map $\theta\colon R^2 \to M$ with the property $\theta \circ d = 0$. In our case we have that $\operatorname{Ext}_R^i(M, M)$ is 12-dimensional, and the maps

$$\theta_1 : e_1 \mapsto \left(\begin{smallmatrix} x_1 \\ 0 \end{smallmatrix}\right), e_2 \mapsto \left(\begin{smallmatrix} x_3+x_4 \\ x_2 \end{smallmatrix}\right) \qquad \theta_7 : e_1 \mapsto 0, e_2 \mapsto \left(\begin{smallmatrix} 0 \\ x_2x_5 \end{smallmatrix}\right)$$

$$\theta_2 : e_1 \mapsto \left(\begin{smallmatrix} x_1x_2 \\ 0 \end{smallmatrix}\right), e_2 \mapsto 0 \qquad \theta_8 : e_1 \mapsto 0, e_2 \mapsto \left(\begin{smallmatrix} x_2x_5 \\ -x_1x_3 \end{smallmatrix}\right)$$

$$\theta_3 : e_1 \mapsto \left(\begin{smallmatrix} x_1x_5 \\ 0 \end{smallmatrix}\right), e_2 \mapsto 0 \qquad \theta_9 : e_1 \mapsto 0, e_2 \mapsto \left(\begin{smallmatrix} x_2x_5 \\ -x_1x_4 \end{smallmatrix}\right)$$

$$\theta_4 : e_1 \mapsto \left(\begin{smallmatrix} 0 \\ x_1x_2 \end{smallmatrix}\right), e_2 \mapsto 0 \qquad \theta_{10} : e_1 \mapsto \left(\begin{smallmatrix} x_1x_3-x_1x_4 \\ 0 \end{smallmatrix}\right), e_2 \mapsto 0$$

$$\theta_5 : e_1 \mapsto 0, e_2 \mapsto \left(\begin{smallmatrix} 0 \\ x_1x_2x_5 \end{smallmatrix}\right) \qquad \theta_{11} : e_1 \mapsto 0, e_2 \mapsto \left(\begin{smallmatrix} x_1x_3 \\ 0 \end{smallmatrix}\right)$$

$$\theta_6 : e_1 \mapsto 0, e_2 \mapsto \left(\begin{smallmatrix} 0 \\ x_1x_2 \end{smallmatrix}\right) \qquad \theta_{12} : e_1 \mapsto \left(\begin{smallmatrix} 0 \\ x_1x_4-x_1x_3 \end{smallmatrix}\right), e_2 \mapsto 0$$

represent a basis (here $e_1 = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and $e_2 = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$). Direct computation shows that $\mu\theta_j = \theta_j\mu$ for $j = 1, \dots 10$, while $\mu\theta_j = -\theta_j\mu$ for $j = 11, 12$, and therefore μ^n cannot belong to the graded center of $\operatorname{Ext}_R^*(M, M)$ when n is odd. However, the element μ^2 does belong to the graded center, hence it is possible that a homogeneous element of even degree in $\mathrm{HH}^{|\eta|}(R)$ generates the periodicity of M.

We end with a result providing a positive answer to the following natural question: if the product or the sum of two homogeneous elements in the Hochschild cohomology ring generates the periodicity of a module, do the elements themselves (or one of them) generate the periodicity? As a corollary, we obtain a result analogous to [Ben, Corollary 5.10.6], which states that if the group cohomology ring of a finite group G is finitely generated over a subring generated by homogeneous elements $x_1, \dots, x_t$, and N is a periodc kG-module (where k is an algebraically closed field), then one of the x_i generates the periodicity of N.

Proposition 4.8. *Let $\eta_1 \in \mathrm{HH}^{|\eta_1|}(\Lambda)$ and $\eta_2 \in \mathrm{HH}^{|\eta_2|}(\Lambda)$ be nonzero homogeneous elements of positive degrees.*

(i) *If $0 \neq \eta_1\eta_2$ generates the periodicity of M, then η_1 and η_2 both generate the periodicity.*
(ii) *Suppose Λ is Gorenstein and k is an algebraically closed field. If $|\eta_1| = |\eta_2|$, and the sum $\eta_1 + \eta_2$ generates the periodicity of M, then either η_1 or η_2 generates the periodicity.*

Proof. (i) From Proposition 4.3 we have that multiplication
$$\mathrm{Ext}^i_\Lambda(M, \Lambda / \mathfrak{r}) \xrightarrow{\eta_1\eta_2} \mathrm{Ext}^{i+|\eta_1|+|\eta_2|}_\Lambda(M, \Lambda / \mathfrak{r})$$
is a k-isomorphism for $i \gg 0$, and since $\mathrm{HH}^*(\Lambda)$ is graded commutative the element $\eta_2\eta_1$ also induces isomorphisms. Therefore the map
$$\mathrm{Ext}^i_\Lambda(M, \Lambda / \mathfrak{r}) \xrightarrow{\eta_2} \mathrm{Ext}^{i+|\eta_2|}_\Lambda(M, \Lambda / \mathfrak{r})$$
is injective for $i \gg 0$, whereas the map
$$\mathrm{Ext}^{i+|\eta_1|}_\Lambda(M, \Lambda / \mathfrak{r}) \xrightarrow{\eta_2} \mathrm{Ext}^{i+|\eta_1|+|\eta_2|}_\Lambda(M, \Lambda / \mathfrak{r})$$
is surjective for $i \gg 0$. It follows from the long exact sequence $\mathrm{ES}(M, \Lambda / \mathfrak{r}, \eta)$ that $\mathrm{Ext}^{i+|\eta_1|+1}_\Lambda(K_{\eta_2} \otimes_\Lambda M, \Lambda / \mathfrak{r}) = 0$ for $i \gg 0$, hence $K_{\eta_2} \otimes_\Lambda M$ has finite projective dimension. Similarly $K_{\eta_1} \otimes_\Lambda M$ has finite projective dimension.

(ii) Denote $\eta_1 + \eta_2$ by η, and let H be the commutative Noetherian graded subalgebra of $\mathrm{HH}^*(\Lambda)$ generated by $\mathrm{HH}^0(\Lambda)$ and η. As mentioned in the remark following Corollary 4.4, the graded k-vector space $\mathrm{Ext}^*_\Lambda(M, \Lambda / \mathfrak{r})$ is a finitely generated H-module, and consequently **Fg** holds. We may therefore apply the theory of support varieties from [EHSST].

Recall that if X is a finitely generated Λ-module, then the support variety $\mathrm{V}_H(X)$ is defined to be the variety in $\operatorname{MaxSpec} H$ defined by the annihilator of $\mathrm{Ext}^*_\Lambda(X, \Lambda / \mathfrak{r})$ in H. Denote the annihilator ideal associated to M by $\mathfrak{a}$. Since $\mathrm{cx}_\Lambda M = 1$ and $\mathrm{cx}_\Lambda(K_\eta \otimes_\Lambda M) = 0$, it follows from [EHSST, Proposition 1.1] that the variety $\mathrm{V}_H(M) = \mathrm{V}_H(\mathfrak{a})$ is one dimensional, whereas $\mathrm{V}_H(K_\eta \otimes_\Lambda M)$ is zero dimensional. By [EHSST, Proposition 3.3] the latter variety equals $\mathrm{V}_H(\eta) \cap V_H(\mathfrak{a}) = \mathrm{V}_H(\langle \eta \rangle + \mathfrak{a})$ (note that in order to apply [EHSST, Proposition 3.3] both **Fg** and the assumption that Λ is Gorenstein is needed, as a substitute for the stronger finite generation hypothesis used in that paper).

Suppose neither of the inclusions $\sqrt{\mathfrak{a}} \subseteq \sqrt{\langle \eta_i \rangle + \mathfrak{a}}$ is strict for $i = 1, 2$. Then
$$\sqrt{\mathfrak{a}} = \sqrt{\langle \eta_1 \rangle + \mathfrak{a}} = \sqrt{\langle \eta_1 \rangle + \sqrt{\mathfrak{a}}} = \sqrt{\langle \eta_1 \rangle + \sqrt{\langle \eta_2 \rangle + \mathfrak{a}}} = \sqrt{\langle \eta_1 \rangle + \langle \eta_2 \rangle + \mathfrak{a}},$$
and since
$$\sqrt{\mathfrak{a}} \subseteq \sqrt{\langle \eta \rangle + \mathfrak{a}} \subseteq \sqrt{\langle \eta_1 \rangle + \langle \eta_2 \rangle + \mathfrak{a}}$$
we get the equality $\sqrt{\mathfrak{a}} = \sqrt{\langle \eta \rangle + \mathfrak{a}}$. Since the variety defined by any ideal equals that defined by the radical of the ideal, we get $\mathrm{V}_H(M) = \mathrm{V}_H(K_\eta \otimes_\Lambda M)$, which is

impossible. Therefore $\sqrt{\mathfrak{a}} \subset \sqrt{\langle \eta_i \rangle + \mathfrak{a}}$ (strict inclusion) at least for one $j \in \{1, 2\}$, and as k is algebraically closed and H is a finitely generated k-algebra, Hilbert's Nullstellensatz gives the strict inclusion $\mathrm{V}_H(K_{\eta_j} \otimes_\Lambda M) \subset \mathrm{V}_H(M)$. The variety $\mathrm{V}_H(K_{\eta_j} \otimes_\Lambda M)$ is then zero dimensional, and consequently $K_{\eta_j} \otimes_\Lambda M$ has finite projective dimension. □

Corollary 4.9. *Suppose* Λ *is Gorenstein and* k *is an algebraically closed field, and that* $\mathrm{HH}^*(\Lambda)$ *is generated over a subalgebra by homogeneous elements* $x_1, \dots, x_t$*. If* M *is an eventually periodic module and there exists an element in* $\mathrm{HH}^*(\Lambda)$ *generating the periodicity, then one of the* x_i *generates the periodicity. In particular, the period of* M *divides the degree of one of the* x_i*.*

Remark. Proposition 4.8 enables us to strengthen Theorem 2.5 in the case when Λ is Gorenstein and k is an algebraically closed field; when **Fg** holds and H is generated as an algebra over H^0 by homogeneous elements $a_1, \dots, a_r$ of positive degrees, then if M has bounded Betti numbers it is eventually periodic with period dividing one of $|a_i|$.

Acknowledgements

I would like to thank The Department of Mathematics at the University of Nebraksa, Lincoln, for their hospitality during my visit there in the spring of 2004. In particular I thank my host Luchezar Avramov and also Srikanth Iyengar for fruitful discussions and comments. Thanks also to my supervisor Øyvind Solberg and to the Norwegian Research Council for financial support.

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III.

TWISTED SUPPORT VARIETIES

ABSTRACT

We define and study twisted support varieties for modules over an Artin algebra, where the twist is induced by an automorphism of the algebra. Under a certain finite generation hypothesis we show that the twisted variety of a module satisfies Dade's Lemma and is one dimensional precisely when the module is periodic with respect to the twisting automorphism. As a special case we obtain results on $D\,\mathrm{Tr}$-periodic modules over Frobenius algebras.

This paper is work in progress.

TWISTED SUPPORT VARIETIES

PETTER ANDREAS BERGH

1. Introduction

In this paper we define and study a new type of cohomological support varieties, called *twisted support varieties*, for modules over Artin algebras, varieties sharing many of the properties of those defined for Artin algebras in [EHSST] and [SnS]. In those papers the underlying geometric object used to define support varieties was the Hochschild cohomology ring of an algebra, whereas our geometric object is a "twisted" Hochschild cohomology ring, the twist being induced by an automorphism of the algebra.

By introducing a finite generation hypothesis similar to the one used in [Be1] we are able to relate the dimension of the twisted variety of a module to the polynomial growth of the ranks of the modules in its minimal projective resolution. In particular Dade's Lemma holds, that is, the twisted variety of a module is trivial if and only if the module has finite projective dimension. This property is highly desirable in any cohomological variety theory. Moreover, the modules whose twisted varieties are one dimensional are precisely those which are periodic with respect to the twisting automorphism, a concept defined in Section 4. As a special case we obtain results on $D\operatorname{Tr}$-periodic modules over Frobenius algebras.

Both when studying the twisted Hochschild cohomology ring and twisted support varieties we illustrate the theory with examples taken from the well known representation theory of the four dimensional Frobenius algebra

$$k\langle x, y\rangle/(x^2, xy + qyx, y^2),$$

where k is a field and $q \in k$ is not a root of unity. Indeed, detecting the $D\operatorname{Tr}$-periodic modules over this algebra by means of support varieties was the motivation for this paper in the first place.

2. Preliminaries

Throughout we let k be a commutative Artin ring and Λ an indecomposable Artin k-projective k-algebra with Jacobson radical $\mathfrak{r}$. Unless otherwise specified all modules considered are finitely generated left modules. We denote by $\operatorname{mod}\Lambda$ the category of all finitely generated left Λ-modules, and we fix a nonzero module $M \in \operatorname{mod}\Lambda$ with minimal projective resolution

$$\mathbb{P}\colon \cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0.$$

Let $F\colon \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda$ be a k-functor. It follows from a theorem of Watts (see [Rot, Corollary 3.34]) that there exists a bimodule Q having the property that F is naturally equivalent to the functor $Q \otimes_\Lambda -$, and that Q may be chosen to be $F(\Lambda)$. Since Q is in $\operatorname{mod}\Lambda$ it has finite k-length and is therefore finitely generated also as a right module, and this implies that Q is projective as a right module (it is flat). Suppose Q is projective also as a left module (but not necessarily as a bimodule). Then if P is a projective Λ-module the functor $\operatorname{Hom}_\Lambda(F(P), -)$, being naturally isomorphic to $\operatorname{Hom}_\Lambda(P, \operatorname{Hom}_\Lambda(Q, -))$, is exact, hence $F(P)$ is projective.

For such a functor F and a positive integer $t \in \mathbb{N}$, define a homogeneous product in the graded k-module

$$\operatorname{Ext}_{\Lambda}^{t*}(F^*(M), M) = \bigoplus_{n=0}^{\infty} \operatorname{Ext}_{\Lambda}^{tn}(F^n(M), M)$$

as follows; if η and θ are two homogeneous elements in $\operatorname{Ext}_{\Lambda}^{t*}(F^*(M), M)$, then

$$\eta\theta \stackrel{\text{def}}{=} \eta \circ F^{|\eta|/t}(\theta),$$

where "$\circ$" denotes the Yoneda product. In other words, if η and θ are given as exact sequences

$$\begin{aligned} \eta \colon 0 \to M \to A_{tn-1} \to \cdots \to A_0 \to F^n(M) \to 0 \\ \theta \colon 0 \to M \to B_{tm-1} \to \cdots \to B_0 \to F^m(M) \to 0, \end{aligned}$$

then $\eta\theta$ is given as the exact sequence obtained from the Yoneda product of the sequence η with the sequence

$$F^n(\theta) \colon 0 \to F^n(M) \to F^n(B_{tm-1}) \to \cdots \to F^n(B_0) \to F^{m+n}(M) \to 0.$$

Furthermore, for a Λ-module N, define a homogeneous right scalar action on $\operatorname{Ext}_{\Lambda}^{t*}(F^*(M), N)$ from $\operatorname{Ext}_{\Lambda}^{t*}(F^*(M), M)$ as follows; if $\mu \in \operatorname{Ext}_{\Lambda}^{t*}(F^*(M), N)$ and $\eta \in \operatorname{Ext}_{\Lambda}^{t*}(F^*(M), M)$ are homogeneous elements, then $\mu\eta \stackrel{\text{def}}{=} \mu \circ F^{|\mu|/t}(\eta)$.

Lemma 2.1. *Extending the homogeneous product and scalar product defined above to all graded elements makes* $\operatorname{Ext}_{\Lambda}^{t*}(F^*(M), M)$ *into a graded k-algebra and* $\operatorname{Ext}_{\Lambda}^{t*}(F^*(M), N)$ *into a graded right* $\operatorname{Ext}_{\Lambda}^{t*}(F^*(M), M)$*-module.*

Proof. The product in $\operatorname{Ext}_{\Lambda}^{t*}(F^*(M), M)$ is associative, and the right distributive law holds; if θ_1, θ_2 and η are homogeneous elements such that θ_1 and θ_2 are of the same degree tn, then

$$(\theta_1 + \theta_2)\eta = (\theta_1 + \theta_2) \circ F^n(\eta) = \theta_1 \circ F^n(\eta) + \theta_2 \circ F^n(\eta) = \theta_1\eta + \theta_2\eta.$$

Now suppose the degree of η is tm. We can represent θ_i as a map

$$\Omega_{\Lambda}^{tn}(F^n(M)) \xrightarrow{f_{\theta_i}} M,$$

and since the functor F^m preserves direct limits (in particular pushouts) and projective resolutions (though it may not preserve *minimal* resolutions), we see that the map

$$F^m(\Omega_{\Lambda}^{tn}(F^n(M))) \xrightarrow{F^m(f_{\theta_i})} F^m(M)$$

represents $F^m(\theta_i)$. As F^m is additive we have $F^m(f_{\theta_1}+f_{\theta_2}) = F^m(f_{\theta_1})+F^m(f_{\theta_2})$ as elements of $\operatorname{Hom}_{\Lambda}(F^m(\Omega_{\Lambda}^{tn}(F^n(M))), F^m(M))$, and this implies that $F^m(\theta_1+\theta_2) = F^m(\theta_1)+F^m(\theta_2)$. Therefore the left distributive law also holds for the homogeneous product;

$$\begin{aligned} \eta(\theta_1 + \theta_2) &= \eta \circ (F^m(\theta_1 + \theta_2)) = \eta \circ (F^m(\theta_1) + F^m(\theta_2)) \\ &= \eta \circ F^m(\theta_1) + \eta \circ F^m(\theta_2) = \eta\theta_1 + \eta\theta_2. \end{aligned}$$

Extending this homogeneous product in the natural way we see that $\operatorname{Ext}_{\Lambda}^{t*}(F^*(M), M)$ becomes a graded k-algebra, and similar arguments show that $\operatorname{Ext}_{\Lambda}^{t*}(F^*(M), N)$ is a graded right $\operatorname{Ext}_{\Lambda}^{t*}(F^*(M), M)$-module. $\square$

Example. Suppose k is a field and Λ is selfinjective, and consider a bimodule B in the stable category $\underline{\operatorname{mod}}\,\Lambda^{\mathrm{e}}$ of finitely generated Λ^{e}-modules modulo projectives.

The Auslander-Reiten translation $\tau_{\Lambda^e} = D\operatorname{Tr}$ is a self-equivalence on $\underline{\operatorname{mod}}\,\Lambda^e$, and we can endow the graded k-vector space

$$\mathbb{A}(B,\tau_{\Lambda^e}) = \operatorname{Hom}_{\Lambda^e}(B,B) \oplus \bigoplus_{i=1}^{\infty} \underline{\operatorname{Hom}}_{\Lambda^e}(\tau_{\Lambda^e}^{i}(B),B)$$

with a product as follows; for two homogeneous elements $f \in \mathbb{A}(B,\tau_{\Lambda^e})_m$ and $g \in \mathbb{A}(B,\tau_{\Lambda^e})_n$ we define fg to be the composition $f \circ \tau_{\Lambda^e}^{m}(g) \in \mathbb{A}(B,\tau_{\Lambda^e})_{m+n}$. In this way $\mathbb{A}(B,\tau_{\Lambda^e})$ becomes a graded k-algebra. Now for any selfinjective Artin algebra Γ the functors τ_{Γ}^{i} and $\Omega_{\Gamma}^{2i}\mathcal{N}^i$ are isomorphic by [ARS, Proposition IV.3.7], where $\mathcal{N}$ is the Nakayama functor $D\operatorname{Hom}_{\Gamma}(-,\Gamma)$. Therefore the orbit algebra $\mathbb{A}(B,\tau_{\Lambda^e})$ of the bimodule B has the form

$$\operatorname{Hom}_{\Lambda^e}(B,B) \oplus \bigoplus_{i=1}^{\infty} \underline{\operatorname{Hom}}_{\Lambda^e}(\Omega_{\Lambda^e}^{2i}(\mathcal{N}^i B),B),$$

giving an isomorphism

$$\mathbb{A}(B,\tau_{\Lambda^e}) \simeq \bigoplus_{i=0}^{\infty} \operatorname{Ext}_{\Lambda^e}^{2i}(\mathcal{N}^i B,B)$$

of graded k-algebras.

The algebra $\mathbb{A}(\Lambda,\tau_{\Lambda^e})$ is called the *Auslander-Reiten orbit algebra* of Λ, and these algebras have been extensively studied by Z. Pogorzały. In [Po1] it was shown that they are invariant under stable equivalences of Morita type between symmetric algebras; if Γ and Δ are finite dimensional symmetric K-algebras (where K is a field) stably equivalent of Morita type, then $\mathbb{A}(\Gamma,\tau_{\Gamma^e})$ and $\mathbb{A}(\Delta,\tau_{\Delta^e})$ are isomorphic K-algebras. This was generalized in [Po2] to arbitrary finite dimensional selfinjective algebras. In [Po4] the Auslander-Reiten orbit algebras of a class of finite dimensional basic connected selfinjective Nakayama K-algebras were computed. Namely, it was shown that if Γ is such an algebra of τ_{Γ^e}-period 1, then $\mathbb{A}(\Gamma,\tau_{\Gamma^e}) \simeq K[x]$ if Γ is a radical square zero algebra, and if not then there exists a natural number t such that $\mathbb{A}(\Gamma,\tau_{\Gamma^e}) \simeq K[x,y]/(y^t)$. In [Po3] τ-periodicity was investigated using similar techniques as was used in [Sch] to study syzygy-periodicity.

3. Twisted Hochschild cohomology

The underlying geometric object used in [EHSST] and [SnS] to define support varieties was the Hochschild cohomology ring $\operatorname{HH}^*(\Lambda,\Lambda) = \operatorname{Ext}_{\Lambda^e}^{*}(\Lambda,\Lambda) = \bigoplus_{n=0}^{\infty}\operatorname{Ext}_{\Lambda^e}^{n}(\Lambda,\Lambda)$ of Λ (where Λ^e is the enveloping algebra $\Lambda \otimes_k \Lambda^{\mathrm{op}}$ of Λ). For an Artin k-algebra Γ and any pair of Λ-Γ-bimodules X and Y the tensor map

$$-\otimes_{\Lambda} X\colon\ \operatorname{Ext}_{\Lambda^e}^{*}(\Lambda,\Lambda) \to \operatorname{Ext}_{\Lambda\otimes_k\Gamma^{\mathrm{op}}}^{*}(X,X)$$

is a homomorphism of graded k-algebras, making $\operatorname{Ext}_{\Lambda\otimes_k\Gamma^{\mathrm{op}}}^{*}(X,Y)$ a left and right $\operatorname{Ext}_{\Lambda^e}^{*}(\Lambda,\Lambda)$-module via the maps $-\otimes_{\Lambda} Y$ and $-\otimes_{\Lambda} X$, respectively (followed by Yoneda product). Now since Λ is projective as a k-module, it follows from [Yon, Proposition 3] that for any two Λ-Λ bimodules B and B' which are both projective as right Λ-modules, and for any homogeneous elements $\eta \in \operatorname{Ext}_{\Lambda^e}^{*}(B,B')$ and $\theta \in \operatorname{Ext}_{\Lambda\otimes_k\Gamma^{\mathrm{op}}}^{*}(X,Y)$, the Yoneda relation

$$(\dagger)\qquad (\eta\otimes_{\Lambda} Y)\circ(B\otimes_{\Lambda}\theta) = (-1)^{|\eta||\theta|}(B'\otimes_{\Lambda}\theta)\circ(\eta\otimes_{\Lambda} X)$$

holds (see also [SnS, Theorem 1.1]). By specializing to the case $\Gamma = \Lambda$ and $B = B' = X = Y = \Lambda$ we see that the Hochshild cohomology ring $\operatorname{HH}^*(\Lambda,\Lambda)$ of Λ is graded commutative, whereas the case $\Gamma = k$ and $B = B' = \Lambda$ shows that for any pair of Λ-modules X and Y the left and right scalar actions from $\operatorname{HH}^*(\Lambda,\Lambda)$ on $\operatorname{Ext}_{\Lambda}^{*}(X,Y)$ are related graded commutatively.

Denote the commutative even subalgebra $\bigoplus_{n=0}^{\infty} \operatorname{Ext}_{\Lambda^e}^{2n}(\Lambda, \Lambda)$ of $\operatorname{HH}^*(\Lambda, \Lambda)$ by H^{ev}. The *support variety* of the Λ-module M is the subset

$$\mathrm{V}(M) \stackrel{\text{def}}{=} \{\mathfrak{m} \in \operatorname{MaxSpec} H^{\mathrm{ev}} \mid \operatorname{Ann}_{H^{\mathrm{ev}}} \operatorname{Ext}_{\Lambda}^*(M, M) \subseteq \mathfrak{m}\}$$

of the maximal ideal spectrum of H^{ev}. As shown in [EHSST] and [SnS], the theory of support varieties is rich and in many ways similar to the theory of cohomological varieties for groups, especially under the hypothesis that H^{ev} is Noetherian and that $\operatorname{Ext}_{\Lambda}^*(X, Y)$ is a finitely generated H^{ev}-module for all Λ-modules X and Y.

In order to obtain a partly similar theory of twisted support varieties, the underlying geometric object we use is a "twisted" version of the Hochschild cohomology ring. Let Γ be any ring and let $\rho, \phi \colon \Gamma \to \Gamma$ be two ring automorphisms. If X is a left Γ-module and B is a Γ-Γ-bimodule, denote by ${}_{\phi}X$ and ${}_{\phi}B_{\rho}$ the left module and bimodule whose scalar actions are defined by $\gamma \cdot x = \phi(\gamma)x$ and $\gamma_1 \cdot b \cdot \gamma_2 = \phi(\gamma_1) b \rho(\gamma_2)$ for $\gamma, \gamma_1, \gamma_2 \in \Gamma$, $x \in X$, $b \in B$. The functor assigning to each Γ-module X the twisted module ${}_{\rho}X$ is exact and isomorphic to the functor ${}_{\rho}\Gamma_1 \otimes_{\Gamma} -$, and it preserves projective modules and minimal resolutions when the latter makes sense.

Fix a k-algebra automorphism

$$\psi \colon \Lambda \to \Lambda$$

and a positive integer $t \in \mathbb{N}$, and consider the graded k-module

$$\operatorname{HH}^{t*}({}_{\psi^*}\Lambda_1, \Lambda) = \operatorname{Ext}_{\Lambda^e}^{t*}({}_{\psi^*}\Lambda_1, \Lambda) = \bigoplus_{n=0}^{\infty} \operatorname{Ext}_{\Lambda^e}^{tn}({}_{\psi^n}\Lambda_1, \Lambda).$$

By the above and Lemma 2.1 the pairing

$$\begin{array}{rcl} \operatorname{Ext}_{\Lambda^e}^{tm}({}_{\psi^m}\Lambda_1, \Lambda) \times \operatorname{Ext}_{\Lambda^e}^{tn}({}_{\psi^n}\Lambda_1, \Lambda) & \to & \operatorname{Ext}_{\Lambda^e}^{t(m+n)}({}_{\psi^{m+n}}\Lambda_1, \Lambda) \\ (\eta, \theta) & \mapsto & \eta \circ_{\psi^m} \theta \end{array}$$

defines a multiplication under which $\operatorname{HH}^{t*}({}_{\psi^*}\Lambda_1, \Lambda)$ becomes a graded k-algebra, i.e. if η and θ are homogeneous elements given as exact sequences

$$\begin{aligned} \eta \colon 0 \to \Lambda \to E_{tm-1} \to \cdots \to E_0 \to {}_{\psi^m}\Lambda_1 \to 0 \\ \theta \colon 0 \to \Lambda \to T_{tn-1} \to \cdots \to T_0 \to {}_{\psi^n}\Lambda_1 \to 0, \end{aligned}$$

then their product $\eta\theta$ is given as the Yoneda product of the sequence η with the sequence

$${}_{\psi^m}\theta \colon 0 \to {}_{\psi^m}\Lambda_1 \to {}_{\psi^m}(T_{tn-1})_1 \to \cdots \to {}_{\psi^m}(T_0)_1 \to {}_{\psi^{m+n}}\Lambda_1 \to 0.$$

Before giving an example, recall that a finite dimensional algebra Γ over a field K is *Frobenius* if ${}_{\Gamma}\Gamma$ and $D(\Gamma_{\Gamma})$ are isomorphic as left Γ-modules, where D denotes the usual K-dual $\operatorname{Hom}_K(-, K)$. For such an algebra Γ, let $\varphi_l \colon {}_{\Gamma}\Gamma \to D(\Gamma_{\Gamma})$ be an isomorphism. Let $y \in \Gamma$ be any element, and consider the linear functional $\varphi_l(1) \cdot y \in D(\Gamma)$, i.e. the K-linear map $\Gamma \to K$ defined by $\gamma \mapsto \varphi_l(1)(y\gamma)$. Since φ_l is surjective there is an element $x \in \Gamma$ having the property that $\varphi_l(x) = \varphi_l(1) \cdot y$, giving $x \cdot \varphi_l(1) = \varphi_l(1) \cdot y$ since φ_l is a map of left Γ-modules. It is not difficult to show that the map $y \mapsto x$ defines a K-algebra automorphism on Γ, and its inverse ν_{Γ} is called the *Nakayama automorphism* of Γ (with respect to φ_l). Thus ν_{Γ} is defined by $\varphi_l(1)(\gamma x) = \varphi_l(1)(\nu_{\Gamma}(x)\gamma)$ for all $\gamma \in \Gamma$. This automorphism is unique up to an inner automorphism; if $\varphi_l' \colon {}_{\Gamma}\Gamma \to D(\Gamma_{\Gamma})$ is another isomorphism of left modules yielding a Nakayama automorphism ν_{Γ}', then there exists an invertible element $z \in \Gamma$ such that $\nu_{\Gamma} = z\nu_{\Gamma}'z^{-1}$. Note that φ_l is an isomorphism between the bimodules ${}_1\Gamma_{\nu_{\Gamma}^{-1}}$ and $D(\Gamma)$, and since $\nu_{\Gamma}^{-1} \colon {}_{\nu_{\Gamma}}\Gamma_1 \to {}_1\Gamma_{\nu_{\Gamma}^{-1}}$ is an isomorphism of bimodules we see that ${}_{\nu_{\Gamma}}\Gamma_1 \simeq D(\Gamma)$.

As $D(\Gamma_{\Gamma})$ is an injective left Γ-module, we see that a Frobenuis algebra is always left selfinjective, but in fact the definition is left-right symmetric. For if $\varphi_l \colon {}_\Gamma\Gamma \to D(\Gamma_\Gamma)$ is an isomorphism of left Γ-modules, we can dualize and get an isomorphism $D(\varphi_l) \colon D^2(\Gamma_\Gamma) \to D({}_\Gamma\Gamma)$ of right modules, and composing with the natural isomorphism $\Gamma_\Gamma \simeq D^2(\Gamma_\Gamma)$ we get an isomorphism $\varphi_r \colon \Gamma_\Gamma \to D({}_\Gamma\Gamma)$ of right Γ-modules. Moreover, we can view this last map as an isomorphism $\varphi_r \colon {}_{\Gamma^{\mathrm{op}}}\Gamma^{\mathrm{op}} \to D(\Gamma^{\mathrm{op}}_{\Gamma^{\mathrm{op}}})$ of left Γ^{op}-modules, thereby giving an isomorphism $\varphi_l \otimes \varphi_r \colon \Gamma \otimes_K \Gamma^{\mathrm{op}} \to D(\Gamma_\Gamma) \otimes_K D(\Gamma^{\mathrm{op}}_{\Gamma^{\mathrm{op}}})$ of Γ-Γ-bimodules. As $D(\Gamma_\Gamma) \otimes_K D(\Gamma^{\mathrm{op}}_{\Gamma^{\mathrm{op}}})$ is isomorphic to $D(\Gamma \otimes_K \Gamma^{\mathrm{op}})$ as a Γ-Γ-bimodule, we see that the enveloping algebra $\Gamma^{\mathrm{e}} = \Gamma \otimes_K \Gamma^{\mathrm{op}}$ of Γ is also Frobenius.

Example. Suppose k is a field and Λ is Frobenius. Recall from the example in Section 2 that for a bimodule B in the stable category $\underline{\mathrm{mod}}\,\Lambda^{\mathrm{e}}$ the orbit algebra

$$\mathbb{A}(B, \tau_{\Lambda^{\mathrm{e}}}) = \mathrm{Hom}_{\Lambda^{\mathrm{e}}}(B, B) \oplus \bigoplus_{i=1}^{\infty} \underline{\mathrm{Hom}}_{\Lambda^{\mathrm{e}}}(\tau^i_{\Lambda^{\mathrm{e}}}(B), B)$$

is isomorphic to the algebra

$$\bigoplus_{i=0}^{\infty} \mathrm{Ext}^{2i}_{\Lambda^{\mathrm{e}}}(\mathcal{N}^i B, B),$$

where $\mathcal{N}$ is the Nakayama functor $D\,\mathrm{Hom}_{\Lambda^{\mathrm{e}}}(-, \Lambda^{\mathrm{e}})$. This functor maps the projective cover of a simple Λ^{e}-module to its injective envelope, and is therefore isomorphic as a functor to $D(\Lambda^{\mathrm{e}}) \otimes_{\Lambda^{\mathrm{e}}} -$. As Λ is a Frobenius algebra, so is Λ^{e}, and we know that $D(\Lambda^{\mathrm{e}})$ is isomorphic to ${}_{\nu_{\Lambda^{\mathrm{e}}}}\Lambda^{\mathrm{e}}{}_1$ as a Λ^{e}-Λ^{e}-bimodule, giving an isomorphism

$$\mathbb{A}(B, \tau_{\Lambda^{\mathrm{e}}}) \simeq \bigoplus_{i=0}^{\infty} \mathrm{Ext}^{2i}_{\Lambda^{\mathrm{e}}}({}_{\nu^i_{\Lambda^{\mathrm{e}}}}B, B)$$

of graded k-algebras.

Next we address the question of graded commutativity in $\mathrm{HH}^{t*}({}_{\psi^*}\Lambda_1, \Lambda)$. For two homogeneous elements

$$\eta \colon 0 \to \Lambda \to E_{tm-1} \to \cdots \to E_0 \to {}_{\psi^m}\Lambda_1 \to 0$$
$$\theta \colon 0 \to \Lambda \to T_{tn-1} \to \cdots \to T_0 \to {}_{\psi^n}\Lambda_1 \to 0$$

the Yoneda relation (†) gives the equality

$$\eta \circ ({}_{\psi^m}\Lambda_1 \otimes_\Lambda \theta) = (-1)^{|\eta||\theta|}\theta \circ (\eta \otimes_\Lambda {}_{\psi^n}\Lambda_1),$$

in which the left hand side is the product $\eta\theta$ (we can identify ${}_{\psi^m}\Lambda_1 \otimes_\Lambda \theta$ with ${}_{\psi^m}\theta$). However, the extension on the right hand side is in general *not* the product $\theta\eta$; it is the Yoneda product of θ with $\eta_{\psi^{-n}}$, and although we have bimodule isomorphisms ${}_1\Lambda_{\psi^{-n}} \simeq {}_{\psi^n}\Lambda_1$ and ${}_{\psi^m}\Lambda_{\psi^{-n}} \simeq {}_{\psi^{m+n}}\Lambda_1$ we cannot in general identify $\eta_{\psi^{-n}}$ with ${}_{\psi^n}\eta$ unless $n = 0$. For reasons to be explained in the next section, we make the following definition.

Definition 3.1. A commutative graded subalgebra $H \subseteq \mathrm{HH}^{t*}({}_{\psi^*}\Lambda_1, \Lambda)$ is *strongly commutative* if $\eta_{\psi^{-1}} = {}_\psi\eta$ for all homogeneous elements $\eta \in H$.

In light of the Yoneda relation all we can say about graded commutativity in $\mathrm{HH}^{t*}({}_{\psi^*}\Lambda_1, \Lambda)$ is that a homogeneous degree zero element commutes with everything, which is not unexpected since $\mathrm{HH}^0({}_{\psi^0}\Lambda_1, \Lambda) = \mathrm{Ext}^0_{\Lambda^{\mathrm{e}}}(\Lambda, \Lambda) = \mathrm{Z}(\Lambda)$ (the center of Λ). The next result gives a criterion under which commutativity relations hold, but first recall the following from [CaE, page 174-175]: for each $n \geq 0$ let Q_n denote the n-fold tensor product of Λ over k (with $Q_0 = k$), and define

$$B^n = Q_{n+2} = \underbrace{\Lambda \otimes_k \cdots \otimes_k \Lambda}_{n+2 \text{ copies of } \Lambda}.$$

We give B^n a Λ^{e}-module structure by defining

$$\lambda(\lambda_0 \otimes \cdots \otimes \lambda_{n+1})\lambda' = \lambda\lambda_0 \otimes \cdots \otimes \lambda_{n+1}\lambda',$$

and as A^{e}-modules we then have an isomorphism $B^n \simeq \Lambda^{\mathrm{e}} \otimes_k Q_n$. Note that B^n is Λ^{e}-projective since the functor $\operatorname{Hom}_{\Lambda^{\mathrm{e}}}(\Lambda^{\mathrm{e}} \otimes_k Q_n, -)$ is naturally isomorphic to $\operatorname{Hom}_k(Q_n, -)$ by adjointness. Now for each $n \geq 1$, define $d \colon B^n \to B^{n-1}$ by

$$\lambda_0 \otimes \cdots \otimes \lambda_{n+1} \mapsto \sum_{i=0}^{n} (-1)^i \lambda_0 \otimes \cdots \otimes \lambda_i \lambda_{i+1} \otimes \cdots \otimes \lambda_{n+1}.$$

The sequence

$$\mathbb{B} : \cdots \to B^3 \xrightarrow{d} B^2 \xrightarrow{d} B^1 \xrightarrow{d} B^0 \xrightarrow{\mu} \Lambda \to 0,$$

where μ is the "multiplication map", is an exact Λ^{e}-projective resolution called the *standard resolution* (or *Bar-resolution*) of Λ.

Proposition 3.2. *Let $\eta \in \operatorname{HH}^{t*}({}_{\psi^*}\Lambda_1, \Lambda)$ be a homogeneous element of degree tm represented by the map $f_\eta \colon {}_{\psi^m}B_1^{tm} \to \Lambda$. If for some $n \geq 1$*

$$f_\eta(\psi^{-n}(\lambda_0) \otimes \cdots \otimes \psi^{-n}(\lambda_{tm}) \otimes 1) = \psi^{-n} f_\eta(\lambda_0 \otimes \cdots \otimes \lambda_{tm} \otimes 1)$$

for all $\lambda_0 \otimes \cdots \otimes \lambda_{tm} \otimes 1$ in ${}_{\psi^m}B_1^{tm}$, then $\eta_{\psi^{-n}} = {}_{\psi^n}\eta$, and consequently $\eta\theta = (-1)^{|\eta||\theta|}\theta\eta$ for every homogeneous element $\theta \in \operatorname{HH}^{t*}({}_{\psi^*}\Lambda_1, \Lambda)$ such that tn divides $|\theta|$. In particular, if

$$f_\eta(\psi^{-1}(\lambda_0) \otimes \cdots \otimes \psi^{-1}(\lambda_{tm}) \otimes 1) = \psi^{-1} f_\eta(\lambda_0 \otimes \cdots \otimes \lambda_{tm} \otimes 1)$$

for all $\lambda_0 \otimes \cdots \otimes \lambda_{tm} \otimes 1$ in ${}_{\psi^m}B_1^{tm}$, then η belongs to the graded center of $\operatorname{HH}^{t*}({}_{\psi^*}\Lambda_1, \Lambda)$.

Proof. Suppose the given condition on f_η holds for n. When viewing $\eta_{\psi^{-n}}$ as a tm-fold extension of ${}_{\psi^n}\Lambda_1$ by ${}_{\psi^{m+n}}\Lambda_1$, we use the bimodule isomorphisms $\psi^{-n} \colon {}_{\psi^{m+n}}\Lambda_1 \to {}_{\psi^m}\Lambda_{\psi^{-n}}$ and $\psi^n \colon {}_1\Lambda_{\psi^{-n}} \to {}_{\psi^n}\Lambda_1$. A lifting of the former along ${}_{\psi^{m+n}}\mathbb{B}_1$ is given by the maps

$$\begin{array}{rcl} (\psi^{-n})^{\otimes i+2} \colon {}_{\psi^{m+n}}B_1^i & \to & {}_{\psi^m}B^i_{\psi^{-n}} \\ \lambda_0 \otimes \cdots \otimes \lambda_{i+1} & \mapsto & \psi^{-n}(\lambda_0) \otimes \cdots \otimes \psi^{-n}(\lambda_{i+1}) \end{array}$$

for $i \geq 0$, giving the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & {}_{\psi^{m+n}}B_1^{tm} \arrow[r, "d"] \arrow[d, "(\psi^{-n})^{\otimes tm+2}"] & {}_{\psi^{m+n}}B_1^{tm-1} \arrow[r, "d"] \arrow[d, "(\psi^{-n})^{\otimes tm+1}"] & \cdots \arrow[r] & {}_{\psi^{m+n}}\Lambda_1 \arrow[r] \arrow[d, "\psi^{-n}"] & 0 \\
\cdots \arrow[r] & {}_{\psi^m}B^{tm}_{\psi^{-n}} \arrow[r, "d"] \arrow[d, "f_\eta"] & {}_{\psi^m}B^{tm-1}_{\psi^{-n}} \arrow[r, "d"] & \cdots \arrow[r] & {}_{\psi^m}\Lambda_{\psi^{-n}} \arrow[r] & 0 \\
 & {}_1\Lambda_{\psi^{-n}} \arrow[d, "\psi^n"] & & & & \\
 & {}_{\psi^n}\Lambda_1 & & & &
\end{tikzcd}$$

with exact rows. Therefore the extension $\eta_{\psi^{-n}}$ is represented by the map composite map

$$\psi^n \circ f_\eta \circ (\psi^{-n})^{\otimes tm+2} \colon {}_{\psi^{m+n}}B_1^{tm} \to {}_{\psi^n}\Lambda_1,$$

under which the image of an element $\lambda_0 \otimes \cdots \otimes \lambda_{tm+1} \in {}_{\psi^{m+n}}B_1^{tm}$ is easily seen to be $f_\eta(\lambda_0 \otimes \cdots \otimes \lambda_{tm+1})$ because of the assumption on f_η.

This shows that we may identify the extensions $\eta_{\psi^{-n}}$ and ${}_{\psi^n}\eta$, and by induction we see that $\eta_{\psi^{-jn}} = {}_{\psi^{jn}}\eta$ for all $j \geq 1$. Consequently, if $\theta \in \operatorname{HH}^{t*}({}_{\psi^*}\Lambda_1, \Lambda)$ is a homogeneous element such that tn divides $|\theta|$, say $|\theta| = tnj$, then the Yoneda relation gives

$$\eta\theta = (-1)^{|\eta||\theta|}\theta \circ \eta_{\psi^{-jn}} = (-1)^{|\eta||\theta|}\theta \circ {}_{\psi^{jn}}\eta = (-1)^{|\eta||\theta|}\theta\eta.$$

□

We end this section with an example of a Frobenius algebra for which the twisted Hochschild cohomology ring with respect to the Nakayama automorphism and $t = 2$ is strongly commutative.

Example. Let k be a field of odd characteristic and $q \in k$ a nonzero element which is *not* a root of unity, and denote by Λ the k-algebra

$$\Lambda = k\langle x, y\rangle/(x^2, xy + qyx, y^2).$$

Denote by D the usual k-dual $\operatorname{Hom}_k(-, k)$, and consider the map $\varphi\colon {}_\Lambda\Lambda \to D(\Lambda_\Lambda)$ of left Λ-modules defined by

$$\varphi(1)(\alpha + \beta x + \gamma y + \delta yx) \stackrel{\text{def}}{=} \delta.$$

It is easy to show that this is an injective map and hence also an isomorphism since $\dim_k \Lambda = \dim_k D(\Lambda)$, and therefore Λ is a Frobenius algebra by definition. Straightforward calculations show that $x \cdot \varphi(1) = \varphi(1) \cdot (-q^{-1}x)$ and $y \cdot \varphi(1) = \varphi(1) \cdot (-qy)$, hence since x and y generate Λ over k we see that the Nakayama automorphism ν (with respect to φ) is the degree preserving map defined by

$$x \mapsto -q^{-1}x, \quad y \mapsto -qy.$$

In [Be2] the Hochschild cohomology of this 4-dimensional graded Koszul algebra was studied, and for every degree preserving k-algebra automorphism $\psi\colon \Lambda \to \Lambda$ the cohomology groups $\operatorname{HH}^*(\Lambda, {}_1\Lambda_\psi) = \operatorname{Ext}^*_{\Lambda^e}(\Lambda, {}_1\Lambda_\psi)$ were computed. In particular, from [Be2, 3.4(iv)] we get

$$\dim_k \operatorname{Ext}^{2n}_{\Lambda^e}(\Lambda, {}_1\Lambda_{\nu^n}) = \begin{cases} 1 & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

when $n > 0$, whereas $\operatorname{Ext}^0_{\Lambda^e}(\Lambda, {}_1\Lambda_{\nu^0}) = \operatorname{Z}(\Lambda)$ is two dimensional (the elements 1 and yx form a basis for the center of Λ). Moreover, it is not difficult to see that $\operatorname{Ext}^{2n}_{\Lambda^e}(\Lambda, {}_1\Lambda_{\nu^n})$ is isomorphic to $\operatorname{Ext}^{2n}_{\Lambda^e}({}_{\nu^n}\Lambda_1, \Lambda)$. Therefore the algebra $\operatorname{HH}^{4*}({}_{\nu^{2*}}\Lambda_1, \Lambda) = \bigoplus_{n=0}^\infty \operatorname{Ext}^{4n}_{\Lambda^e}({}_{\nu^{2n}}\Lambda_1, \Lambda)$ is two dimensional in degree zero and one dimensional in the positive degrees.

We now recall the construction of the minimal bimodule projective resolution of Λ from [BGMS]. Define the elements

$$f_0^0 = 1, \quad f_0^1 = x, \quad f_1^1 = y,$$

$$f_{-1}^n = 0 = f_{n+1}^n \text{ for each } n \geq 0,$$

and for each $n \geq 2$ define elements $\{f_i^n\}_{i=0}^n \subseteq \underbrace{\Lambda \otimes_k \cdots \otimes_k \Lambda}_{n \text{ copies}}$ inductively by

$$f_i^n = f_{i-1}^{n-1} \otimes y + q^i f_i^{n-1} \otimes x.$$

Denote by F^n the Λ^e-projective module $\bigoplus_{i=0}^n \Lambda \otimes_k f_i^n \otimes_k \Lambda$, and by $\tilde{f}_i^n$ the element $1 \otimes f_i^n \otimes 1 \in F^n$ (and $\tilde{f}_0^0 = 1 \otimes 1$). The set $\{\tilde{f}_i^n\}_{i=0}^n$ generates F^n as a Λ^e-module. Now define a map $\delta\colon F^n \to F^{n-1}$ by

$$\tilde{f}_i^n \mapsto \left[x\tilde{f}_i^{n-1} + (-1)^n q^i \tilde{f}_i^{n-1} x\right] + \left[q^{n-i} y \tilde{f}_{i-1}^{n-1} + (-1)^n \tilde{f}_{i-1}^{n-1} y\right].$$

It is shown in [BGMS] that

$$(\mathbb{F}, \delta)\colon \cdots \to F^{n+1} \xrightarrow{\delta} F^n \xrightarrow{\delta} F^{n-1} \to \cdots$$

is a minimal Λ^{e}-projective resolution of Λ.

For each $m \geq 1$ consider the Λ^{e}-linear map

$$\begin{aligned} g_{4m} \colon {}_{\nu^{2m}}F_1^{4m} & \rightarrow \Lambda \\ \tilde{f}_i^{4m} & \mapsto \begin{cases} 1 & \text{for } i = 2m \\ 0 & \text{for } i \neq 2m. \end{cases} \end{aligned}$$

The only generators in ${}_{\nu^{2m}}F_1^{4m+1}$ which can possibly map to $\tilde{f}_{2m}^{4m}$ under the map ${}_{\nu^{2m}}F_1^{4m+1} \xrightarrow{\delta} {}_{\nu^{2m}} F_1^{4m}$ are $\tilde{f}_{2m}^{4m+1}$ and $\tilde{f}_{2m+1}^{4m+1}$, and so it follows from the equalities

$$\begin{aligned} g_{4m} \circ \delta(\tilde{f}_{2m}^{4m+1}) &= g_{4m}(x\tilde{f}_{2m}^{4m} - q^{2m}\tilde{f}_{2m}^{4m}x) \\ &= g_{4m}\left([q^{2m}x] \cdot \tilde{f}_{2m}^{4m} - \tilde{f}_{2m}^{4m} \cdot [q^{2m}x]\right) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} g_{4m} \circ \delta(\tilde{f}_{2m+1}^{4m+1}) &= g_{4m}(q^{2m}y\tilde{f}_{2m}^{4m} - \tilde{f}_{2m}^{4m}y) \\ &= g_{4m}\left(y \cdot \tilde{f}_{2m}^{4m} - \tilde{f}_{2m}^{4m} \cdot y\right) \\ &= 0 \end{aligned}$$

that g_{4m} belongs to the kernel of the map $\delta^* \colon \operatorname{Hom}_{\Lambda^{\mathrm{e}}}({}_{\nu^{2m}}F_1^{4m}, \Lambda) \rightarrow \operatorname{Hom}_{\Lambda^{\mathrm{e}}}({}_{\nu^{2m}}F_1^{4m+1}, \Lambda)$. Moreover, there cannot exist a Λ^{e}-linear map ${}_{\nu^{2m}}F_1^{4m-1} \rightarrow \Lambda$ making the diagram

$$\begin{tikzcd} {}_{\nu^{2m}}F_1^{4m} \arrow[r, "\delta"] \arrow[d, "g_{4m}"] & {}_{\nu^{2m}}F_1^{4m-1} \arrow[dl] \\ \Lambda & \end{tikzcd}$$

commute, since δ is a map of graded degree 1, whereas the degree of g_{4m} is 0. This shows that g_{4m} represents a generator for the one dimensional space $\operatorname{Ext}_{\Lambda^{\mathrm{e}}}^{4m}({}_{\nu^{2m}}\Lambda_1, \Lambda)$. Note that if we multiply this map with yx, then there *does* exist a map making the above diagram commute, hence whenever we multiply any homogeneous element of positive degree in $\operatorname{HH}^{4*}({}_{\nu^{2*}}\Lambda_1, \Lambda)$ with the element $yx \in \operatorname{HH}^0({}_{\nu^0}\Lambda_1, \Lambda)$ we get zero.

For each $0 \leq i \leq 4$ define Λ^{e}-linear maps $\bar{g}_{4m+i} \colon {}_{\nu^{2(m+1)}}F_1^{4m+i} \rightarrow_{\nu^2} F_1^i$ by

$$\bar{g}_{4m} \colon \tilde{f}_i^{4m} \mapsto \begin{cases} \tilde{f}_0^0 & \text{for } i = 2m \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{g}_{4m+1} \colon \tilde{f}_i^{4m+1} \mapsto \begin{cases} q^{2m}\tilde{f}_0^1 & \text{for } i = 2m \\ \tilde{f}_1^1 & \text{for } i = 2m+1 \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{g}_{4m+2} \colon \tilde{f}_i^{4m+2} \mapsto \begin{cases} q^{4m}\tilde{f}_0^2 & \text{for } i = 2m \\ q^{2m}\tilde{f}_1^2 & \text{for } i = 2m+1 \\ \tilde{f}_2^2 & \text{for } i = 2m+2 \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{g}_{4m+3} \colon \tilde{f}_i^{4m+3} \mapsto \begin{cases} q^{4m}\tilde{f}_1^3 & \text{for } i = 2m+1 \\ q^{2m}\tilde{f}_2^3 & \text{for } i = 2m+2 \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{g}_{4m+4} \colon \tilde{f}_i^{4m+4} \mapsto \begin{cases} q^{4m}\tilde{f}_2^4 & \text{for } i = 2m \\ 0 & \text{otherwise.} \end{cases}$$

Then for each $1 \leq i \leq 4$ the diagram

$$\begin{tikzcd} {}_{\nu^{2(m+1)}}F_1^{4m+i} \arrow[r, "\delta"] \arrow[d, "\bar{g}_{4m+i}"] & {}_{\nu^{2(m+1)}}F_1^{4m+i-1} \arrow[d, "\bar{g}_{4m+i-1}"] \\ {}_{\nu^2}F_1^{i} \arrow[r, "\delta"] & {}_{\nu^2}F_1^{i-1} \end{tikzcd}$$

commutes, hence the maps $\bar{g}_{4m+i}$ for $0 \leq i \leq 4$ provide a lifting of ${}_{\nu^{2(m+1)}}F_1^{4m} \xrightarrow{\nu^2 g_{4m}} {}_{\nu^2}\Lambda_1$ along the projective bimodule resolution ${}_{\nu^{2(m+1)}}\mathbb{F}_1$ of ${}_{\nu^{2(m+1)}}\Lambda_1$. Moreover, the composition $g_4 \circ \bar{g}_{4m+4}$ equals the map $q^{4m}g_{4m+4}$. Therefore, if we denote by θ the extension in $\operatorname{Ext}^4_{\Lambda^e}({}_{\nu^2}\Lambda_1, \Lambda)$ corresponding to the map g_4, we see that for each $m \geq 2$ the element θ^m is the (nonzero) extension in $\operatorname{Ext}^{4m}_{\Lambda^e}({}_{\nu^{2m}}\Lambda_1, \Lambda)$ corresponding to the map $q^{4(1+2+\cdots+m-1)}g_{4m}$.

As a result, we see that we have an isomorphism

$$\operatorname{HH}^{4*}({}_{\nu^{2*}}\Lambda_1, \Lambda) \simeq k[x] \times k$$

of graded k-algebras, with x of degree 4, and in particular $\operatorname{HH}^{4*}({}_{\nu^{2*}}\Lambda_1, \Lambda)$ is commutative. Moreover, it is easily shown that we may identify the extensions $\theta_{\nu^{-2}}$ and ${}_{\nu^2}\theta$, and so $\operatorname{HH}^{4*}({}_{\nu^{2*}}\Lambda_1, \Lambda)$ is actually strongly commutative.

4. Twisted support varieties

As mentioned, in order to obtain a theory of twisted support varieties the underlying geometric object we shall use is the twisted Hochschild cohomology ring $\operatorname{HH}^{t*}({}_{\psi^*}\Lambda_1, \Lambda) = \operatorname{Ext}^{t*}_{\Lambda^e}({}_{\psi^*}\Lambda_1, \Lambda)$ studied in the previous section (where $\Lambda \xrightarrow{\psi} \Lambda$ is an automorphism and $t \in \mathbb{N}$). Similarly as for $\operatorname{HH}^{t*}({}_{\psi^*}\Lambda_1, \Lambda)$, the graded k-module $\operatorname{Ext}^{t*}_{\Lambda}({}_{\psi^*}M, M) = \bigoplus_{n=0}^{\infty} \operatorname{Ext}^{tn}_{\Lambda}({}_{\psi^n}M, M)$ is a k-algebra, and for each Λ-module N the graded k-module $\operatorname{Ext}^{t*}_{\Lambda}({}_{\psi^*}M, N) = \bigoplus_{n=0}^{\infty} \operatorname{Ext}^{tn}_{\Lambda}({}_{\psi^n}M, N)$ becomes a graded right $\operatorname{Ext}^{t*}_{\Lambda}({}_{\psi^*}M, M)$-module by defining

$$\zeta\mu \overset{\text{def}}{=} \zeta \circ {}_{\psi^m}\mu$$

for $\zeta \in \operatorname{Ext}^{tm}_{\Lambda}({}_{\psi^m}M, N)$ and $\mu \in \operatorname{Ext}^{tn}_{\Lambda}({}_{\psi^n}M, M)$. Moreover, the tensor map

$$\begin{array}{rccc} - \otimes_\Lambda M \colon & \operatorname{HH}^{t*}({}_{\psi^*}\Lambda_1, \Lambda) & \to & \operatorname{Ext}^{t*}_{\Lambda}({}_{\psi^*}M, M) \\ & \eta & \mapsto & \eta \otimes_\Lambda M \end{array}$$

is a homomorphism of graded k-algebras, thus making $\operatorname{Ext}^{t*}_{\Lambda}({}_{\psi^*}M, N)$ a graded right $\operatorname{HH}^{t*}({}_{\psi^*}\Lambda_1, \Lambda)$-module. Similarly $\operatorname{Ext}^{t*}_{\Lambda}({}_{\psi^*}M, N)$ becomes a graded left $\operatorname{HH}^{t*}({}_{\psi^*}\Lambda_1, \Lambda)$ via the homomorphism

$$\begin{array}{rccc} - \otimes_\Lambda N \colon & \operatorname{HH}^{t*}({}_{\psi^*}\Lambda_1, \Lambda) & \to & \operatorname{Ext}^{t*}_{\Lambda}({}_{\psi^*}N, N) \\ & \eta & \mapsto & \eta \otimes_\Lambda N \end{array}$$

of algebras.

We now make the following assumption:

Assumption. Given the automorphism $\Lambda \xrightarrow{\psi} \Lambda$ and the integer $t \in \mathbb{N}$, the graded subalgebra $H = \bigoplus_{n=0}^{\infty} H^{tn}$ of $\operatorname{HH}^{t*}({}_{\psi^*}\Lambda_1, \Lambda)$ is strongly commutative with $H^0 = \operatorname{HH}^0(\Lambda, \Lambda) = \operatorname{Z}(\Lambda)$.

Why do we restrict ourselves to strongly commutative algebras instead of "ordinary" commutative algebras? The answer is that we want the bifunction $\operatorname{Ext}^{t*}_{\Lambda}({}_{\psi^*}-, -)$, which maps the pair (M, N) of Λ-modules to the H-module $\operatorname{Ext}^{t*}_{\Lambda}({}_{\psi^*}M, N)$, to preserve maps. Consider two homogeneous elements $\zeta \in \operatorname{Ext}^{tm}_{\Lambda}({}_{\psi^m}M, N)$ and $\eta \in \operatorname{Ext}^{tn}_{\Lambda^e}({}_{\psi^n}\Lambda_1, \Lambda)$. By definition $\zeta \cdot \eta$ is the extension $\zeta \circ {}_{\psi^m}(\eta \otimes_\Lambda M) = \zeta \circ ({}_{\psi^m}\eta \otimes_\Lambda M)$, and if ${}_{\psi^m}\eta = \eta_{\psi^{-m}}$ we get ${}_{\psi^m}\eta \otimes_\Lambda M =$

$\eta_{\psi^{-m}} \otimes_\Lambda M = \eta \otimes_\Lambda {}_{\psi^m}M$. Hence in this case we may identify the right scalar product $\zeta \cdot \eta$ with $(\Lambda \otimes_\Lambda \zeta) \circ (\eta \otimes_\Lambda {}_{\psi^m}M)$, and then by the Yoneda relation (†) from the previous section we get

$$\zeta \cdot \eta = (\Lambda \otimes_\Lambda \zeta) \circ (\eta \otimes_\Lambda {}_{\psi^m}M) = (-1)^{|\zeta||\eta|}(\eta \otimes_\Lambda N) \circ ({}_{\psi^n}\Lambda_1 \otimes_\Lambda \zeta).$$

However, the extension $(\eta \otimes_\Lambda N) \circ ({}_{\psi^n}\Lambda_1 \otimes_\Lambda \zeta)$ is precisely the left scalar product $\eta \cdot \zeta$, and so if H is strongly commutative we see that

$$\zeta \cdot \eta = (-1)^{|\zeta||\eta|}\eta \cdot \zeta$$

for all homogeneous elements $\eta \in H$ and $\zeta \in \operatorname{Ext}_\Lambda^{t*}({}_{\psi^*}M, N)$.

The fact that the left and right scalar multiplications basically coincide is what makes $\operatorname{Ext}_\Lambda^{t*}({}_{\psi^*}-, -)$ preserve maps. To see this, let $f\colon M \to M'$ be a Λ-homomorphism. This map induces a homomorphism $\hat{f}\colon \operatorname{Ext}_\Lambda^{t*}({}_{\psi^*}M', N) \to \operatorname{Ext}_\Lambda^{t*}({}_{\psi^*}M, N)$ of graded groups, under which the image of a homogeneous element

$$\theta\colon 0 \to N \to X_{tn} \to \cdots \to X_1 \to {}_{\psi^n}M' \to 0$$

is the extension θf given by the commutative diagram

$$\begin{tikzcd}
\theta f\colon 0 \arrow[r] & N \arrow[r] \arrow[d, equal] & X_{tn} \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & X_2 \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d] & {}_{\psi^n}M \arrow[r] \arrow[d, "f"] & 0 \\
\theta\colon 0 \arrow[r] & N \arrow[r] & X_{tn} \arrow[r] & \cdots \arrow[r] & X_2 \arrow[r] & X_1 \arrow[r] & {}_{\psi^n}M' \arrow[r] & 0
\end{tikzcd}$$

in which the module Y is a pullback. For a homogeneous element $\eta \in H^{tm}$ we then get

$$\begin{aligned}
\hat{f}(\theta \cdot \eta) &= (-1)^{|\eta||\theta|}\hat{f}(\eta \cdot \theta) \\
&= (-1)^{|\eta||\theta|}\hat{f}\left((\eta \otimes_\Lambda N) \circ {}_{\psi^m}\theta\right) \\
&= (-1)^{|\eta||\theta|}(\eta \otimes_\Lambda N) \circ {}_{\psi^m}(\theta f) \\
&= (-1)^{|\eta||\theta|}\eta \cdot \hat{f}(\theta) \\
&= \hat{f}(\theta) \cdot \eta,
\end{aligned}$$

showing $\hat{f}$ is a homomorphism of H-modules. Similarly $\operatorname{Ext}_\Lambda^{t*}({}_{\psi^*}-, -)$ preserves maps in the second argument.

Remark. It should also be noted that in many cases considering only strongly commutative algebras is not a severe restriction. For example, let H be a commutative Noetherian graded subalgebra of $\operatorname{HH}^{t*}({}_{\psi^*}\Lambda_1, \Lambda)$, as will be the case in most of the results in this section. Then by [Be1, Proposition 2.1] there exists an integer w and a homogeneous element $\eta \in H$ of positive degree, say $|\eta| = tn$, such that the multiplication map

$$H^{ti} \xrightarrow{\eta\cdot} H^{t(i+n)}$$

is injective for $i \geq w$. Now for any homogeneous element $\theta \in H^{tm}$ the product $\theta \cdot \eta$ is by definition the extension $\theta \circ {}_{\psi^m}\eta = (\theta \otimes_\Lambda \Lambda) \circ ({}_{\psi^m}\Lambda_1 \otimes_\Lambda \eta)$, which by the Yoneda relation (†) from Section 3 equals the extension $(-1)^{|\eta||\theta|}(\Lambda \otimes_\Lambda \eta) \circ (\theta \otimes_\Lambda {}_{\psi^n}\Lambda_1) = (-1)^{|\eta||\theta|}\eta \circ \theta_{\psi^{-n}}$. However, the extension $\eta \circ \theta_{\psi^{-n}}$ corresponds to the product $\eta \cdot {}_{\psi^{-n}}\theta_{\psi^{-n}}$ in H, and so since H is commutative we get

$$\eta \cdot \theta = \theta \cdot \eta = (-1)^{|\eta||\theta|}\eta \cdot {}_{\psi^{-n}}\theta_{\psi^{-n}}.$$

Now if $|\theta| \geq tw$ then $\theta = (-1)^{|\eta||\theta|}{}_{\psi^{-n}}\theta_{\psi^{-n}}$ due to the injectivity of the multiplication map induced by η, and twisting both sides in this equality by ψ^n from the left we get ${}_{\psi^n}\theta = (-1)^{|\eta||\theta|}\theta_{\psi^{-n}}$.

The algebra H, being Noetherian, is generated over H^0 by a finite set of homogeneous elements, say $\{x_0, \dots, x_r\}$. Suppose now that k contains an infinite field. Then from the proof of [Be1, Theorem 2.5] we see that we may choose the element η above such that $|\eta| = \operatorname{lcm}\{|x_0|, \dots, |x_r|\}$ In particular, if all the generators $x_0, \dots, x_r$ are of the same degree, i.e. $|x_i| = t$, then the subalgebra $H^0 \oplus \bigoplus_{i=w}^{\infty} H^{ti}$ of H is strongly commutative.

Having made all the necessary assumptions, we are now ready to define twisted support varieties. For two Λ-modules X and Y, denote by $\mathrm{A}_H^{\psi}(X,Y)$ the ideal $\operatorname{Ann}_H \operatorname{Ext}_{\Lambda}^{t*}({}_{\psi^*}X, Y)$, i.e. the annihilator in H of $\operatorname{Ext}_{\Lambda}^{t*}({}_{\psi^*}X, Y)$. This ideal is graded since $\operatorname{Ext}_{\Lambda}^{t*}({}_{\psi^*}X, Y)$ is a graded H-module. As for ordinary support varieties, the defining ideal for the twisted support variety of M is $\mathrm{A}_H^{\psi}(M, \Lambda / \mathfrak{r})$.

Definition 4.1. The *twisted support variety* $\mathrm{V}_H^{\psi}(M)$ of M (with respect to the automorphism ψ, the integer t and the ring H) is the subset

$$\mathrm{V}_H^{\psi}(M) \stackrel{\text{def}}{=} \{\mathfrak{m} \in \operatorname{MaxSpec} H \mid \mathrm{A}_H^{\psi}(M, \Lambda / \mathfrak{r}) \subseteq \mathfrak{m}\}$$

of the maximal ideal spectrum of H.

Note that since M and therefore also $\operatorname{Ext}_{\Lambda}^{t*}({}_{\psi^*}M, \Lambda / \mathfrak{r})$ is nonzero, the degree zero part of $\mathrm{A}_H^{\psi}(M, \Lambda / \mathfrak{r})$ must be contained in the radical $\operatorname{rad} H^0$ of the local ring $H^0 = \mathrm{Z}(\Lambda)$ (if not then $\mathrm{A}_H^{\psi}(M, \Lambda / \mathfrak{r})$ contains the identity in H). Hence the twisted variety $\mathrm{V}_H^{\psi}(M)$ always contains the unique graded maximal ideal

$$\mathfrak{m}_H = \operatorname{rad} H^0 \oplus H^t \oplus H^{2t} \oplus \cdots$$

of H, and we shall say the variety is *trivial* if it does not contain any other element.

The following proposition lists some elementary and more or less expected properties of twisted varieties.

Proposition 4.2. (i) *The ideal $\mathrm{A}_H^{\psi}(M, M)$ is also a defining ideal for the twisted variety of M, i.e.*

$$\mathrm{V}_H^{\psi}(M) = \{\mathfrak{m} \in \operatorname{MaxSpec} H \mid \mathrm{A}_H^{\psi}(M, M) \subseteq \mathfrak{m}\}.$$

(ii) *If $\operatorname{Ext}_{\Lambda}^{tn}({}_{\psi^n}M, M) = 0$ for $n \gg 0$, then $\mathrm{V}_H^{\psi}(M)$ is trivial. In particular, the twisted variety of M is trivial whenever the projective or injective dimension of M is finite.*

(iii) *Given nonzero Λ-modules M' and M'' and an exact sequence*

$$0 \to M' \to M \to M'' \to 0,$$

the relation

$$\mathrm{V}_H^{\psi}(M) \subseteq \mathrm{V}_H^{\psi}(M') \cup \mathrm{V}_H^{\psi}(M'')$$

holds. Moreover, if $\Omega_{\Lambda}^{t-1}(M') \neq 0$, then the relation

$$\mathrm{V}_H^{\psi}(M'') \subseteq \mathrm{V}_H^{\psi}(M) \cup \mathrm{V}_H^{\psi}(\Omega_{\Lambda}^{t-1}({}_{\psi}M'))$$

holds, and if $M'' = \Omega_{\Lambda}^{t-1}(M''')$ for some module M''', then the relation

$$\mathrm{V}_H^{\psi}(M') \subseteq \mathrm{V}_H^{\psi}(M) \cup \mathrm{V}_H^{\psi}({}_{\psi^{-1}}M''')$$

holds.

(iv) *If M' and M'' are nonzero Λ-modules such that $M = M' \oplus M''$, then $\mathrm{V}_H^{\psi}(M) = \mathrm{V}_H^{\psi}(M') \cup \mathrm{V}_H^{\psi}(M'')$.*

Proof. For a Λ-module N, denote by $\mathrm{V}_H^{\psi}(M,N)$ the variety whose defining ideal is $\mathrm{A}_H^{\psi}(M,N)$. To prove (i), we must show that $\mathrm{V}_H^{\psi}(M,\Lambda/\mathfrak{r})$ and $\mathrm{V}_H^{\psi}(M,M)$ are equal, and we start by proving by induction on the length of N that $\mathrm{V}_H^{\psi}(M,N)$ is contained in $\mathrm{V}_H^{\psi}(M,\Lambda/\mathfrak{r})$. For any simple Λ-module S the ideal $\mathrm{A}_H^{\psi}(M,\Lambda/\mathfrak{r})$ is contained in $\mathrm{A}_H^{\psi}(M,S)$ since S is a summand of $\Lambda/\mathfrak{r}$, hence $\mathrm{V}_H^{\psi}(M,S)\subseteq\mathrm{V}_H^{\psi}(M,\Lambda/\mathfrak{r})$. If the length of N is greater than 2, take any simple submodule $S\subset N$ and consider the exact sequence

$$0\to S\to N\to N/S\to 0.$$

This sequence induces the exact sequence

$$\operatorname{Ext}_{\Lambda}^{t*}({}_{\psi^*}M,S)\to\operatorname{Ext}_{\Lambda}^{t*}({}_{\psi^*}M,N)\to\operatorname{Ext}_{\Lambda}^{t*}({}_{\psi^*}M,N/S)$$

of H-modules, from which the inclusion

$$\mathrm{A}_H^{\psi}(M,S)\cdot\mathrm{A}_H^{\psi}(M,N/S)\subseteq\mathrm{A}_H^{\psi}(M,N)$$

follows. This implies that $\mathrm{V}_H^{\psi}(M,N)\subseteq\mathrm{V}_H^{\psi}(M,S)\cup\mathrm{V}_H^{\psi}(M,N/S)$, and since both $\mathrm{V}_H^{\psi}(M,S)$ and $\mathrm{V}_H^{\psi}(M,N/S)$ are contained in $\mathrm{V}_H^{\psi}(M,\Lambda/\mathfrak{r})$ by induction, we get $\mathrm{V}_H^{\psi}(M,N)\subseteq\mathrm{V}_H^{\psi}(M,\Lambda/\mathfrak{r})$. In particular $\mathrm{V}_H^{\psi}(M,M)$ is contained in $\mathrm{V}_H^{\psi}(M,\Lambda/\mathfrak{r})$. The reverse inclusion $\mathrm{V}_H^{\psi}(M,\Lambda/\mathfrak{r})\subseteq\mathrm{V}_H^{\psi}(M,M)$ follows from the inclusion $\mathrm{A}_H^{\psi}(M,M)\subseteq\mathrm{A}_H^{\psi}(M,\Lambda/\mathfrak{r})$ of ideals in H, thus proving (i).

Now suppose $\operatorname{Ext}_{\Lambda}^{tn}({}_{\psi^n}M,M)=0$ for $n\gg 0$, let $\mathfrak{m}\subset H$ be a maximal ideal containing $\mathrm{A}_H^{\psi}(M,\Lambda/\mathfrak{r})$, and let $x\in H^+=\bigoplus_{n=1}^{\infty}H^{tn}$ be a homogeneous element. As the scalar action from H on $\operatorname{Ext}_{\Lambda}^{t*}({}_{\psi^*}M,\Lambda/\mathfrak{r})$ factors through $\operatorname{Ext}_{\Lambda}^{t*}({}_{\psi^*}M,M)$, some power of x must lie in $\mathrm{A}_H^{\psi}(M,\Lambda/\mathfrak{r})$. Therefore x must be an element of $\mathfrak{m}$, implying $\mathfrak{m}$ is a graded ideal. But $\mathfrak{m}_H$ is the only graded maximal ideal of H. This proves (ii).

Next suppose we are given an exact sequence as in (iii). The sequence induces the exact sequence

$$\operatorname{Ext}_{\Lambda}^{t*}({}_{\psi^*}M'',\Lambda/\mathfrak{r})\to\operatorname{Ext}_{\Lambda}^{t*}({}_{\psi^*}M,\Lambda/\mathfrak{r})\to\operatorname{Ext}_{\Lambda}^{t*}({}_{\psi^*}M',\Lambda/\mathfrak{r})$$

of H-modules, giving the inclusion

$$\mathrm{A}_H^{\psi}(M',\Lambda/\mathfrak{r})\cdot\mathrm{A}_H^{\psi}(M'',\Lambda/\mathfrak{r})\subseteq\mathrm{A}_H^{\psi}(M,\Lambda/\mathfrak{r})$$

of ideals of H. The relation $\mathrm{V}_H^{\psi}(M)\subseteq\mathrm{V}_H^{\psi}(M')\cup\mathrm{V}_H^{\psi}(M'')$ now follows.

The original exact sequence also induces the two exact sequences

$$0\to\operatorname{Hom}_{\Lambda}(M'',\Lambda/\mathfrak{r})\to\operatorname{Hom}_{\Lambda}(M,\Lambda/\mathfrak{r})$$
$$\operatorname{Ext}_{\Lambda}^{tn-1}({}_{\psi^n}M',\Lambda/\mathfrak{r})\to\operatorname{Ext}_{\Lambda}^{tn}({}_{\psi^n}M'',\Lambda/\mathfrak{r})\to\operatorname{Ext}_{\Lambda}^{tn}({}_{\psi^n}M,\Lambda/\mathfrak{r})$$

for $n\geq 1$, in which we may identify $\operatorname{Ext}_{\Lambda}^{tn-1}({}_{\psi^n}M',\Lambda/\mathfrak{r})$ with $\operatorname{Ext}_{\Lambda}^{t(n-1)}({}_{\psi^{n-1}}\Omega_{\Lambda}^{t-1}({}_{\psi}M'),\Lambda/\mathfrak{r})$. Consequently the inclusion

$$\mathrm{A}_H^{\psi}(M,\Lambda/\mathfrak{r})\cdot\mathrm{A}_H^{\psi}(\Omega_{\Lambda}^{t-1}({}_{\psi}M'),\Lambda/\mathfrak{r})\subseteq\mathrm{A}_H^{\psi}(M'',\Lambda/\mathfrak{r})$$

holds whenever $\Omega_{\Lambda}^{t-1}(M')\neq 0$, giving $\mathrm{V}_H^{\psi}(M'')\subseteq\mathrm{V}_H^{\psi}(M)\cup\mathrm{V}_H^{\psi}(\Omega_{\Lambda}^{t-1}({}_{\psi}M'))$.

Finally, the short exact sequence induces the exact sequence

$$\operatorname{Ext}_{\Lambda}^{tn}({}_{\psi^n}M,\Lambda/\mathfrak{r})\to\operatorname{Ext}_{\Lambda}^{tn}({}_{\psi^n}M',\Lambda/\mathfrak{r})\to\operatorname{Ext}_{\Lambda}^{tn+1}({}_{\psi^n}M'',\Lambda/\mathfrak{r})$$

for $n\geq 0$. If $M''=\Omega_{\Lambda}^{t-1}(M''')$ for some module M''', then we may identify $\operatorname{Ext}_{\Lambda}^{tn+1}({}_{\psi^n}M'',\Lambda/\mathfrak{r})$ with $\operatorname{Ext}_{\Lambda}^{t(n+1)}({}_{\psi^{n+1}}({}_{\psi^{-1}}M'''),\Lambda/\mathfrak{r})$, and the inclusion

$$\mathrm{A}_H^{\psi}(M,\Lambda/\mathfrak{r})\cdot\mathrm{A}_H^{\psi}({}_{\psi^{-1}}M''',\Lambda/\mathfrak{r})\subseteq\mathrm{A}_H^{\psi}(M',\Lambda/\mathfrak{r})$$

holds. This gives $\mathrm{V}_H^{\psi}(M')\subseteq\mathrm{V}_H^{\psi}(M)\cup\mathrm{V}_H^{\psi}({}_{\psi^{-1}}M''')$, and the proof of (iii) is complete.

To prove (iv), note that by (iii) the inclusion $\mathrm{V}_H^{\psi}(M) \subseteq \mathrm{V}_H^{\psi}(M') \cup \mathrm{V}_H^{\psi}(M'')$ holds, whereas the inclusions $\mathrm{A}_H^{\psi}(M, \Lambda / \mathfrak{r}) \subseteq \mathrm{A}_H^{\psi}(M', \Lambda / \mathfrak{r})$ and $\mathrm{A}_H^{\psi}(M, \Lambda / \mathfrak{r}) \subseteq \mathrm{A}_H^{\psi}(M'', \Lambda / \mathfrak{r})$ give $\mathrm{V}_H^{\psi}(M') \subseteq \mathrm{V}_H^{\psi}(M)$ and $\mathrm{V}_H^{\psi}(M'') \subseteq \mathrm{V}_H^{\psi}(M)$. □

As a corollary, we obtain a result whose analogue in the theory of support varieties says that varieties are invariant under the syzygy operator.

Corollary 4.3. *If $\Omega_\Lambda^t(M)$ is nonzero, then $\mathrm{V}_H^{\psi}(M) = \mathrm{V}_H^{\psi}(\Omega_\Lambda^t({}_{\psi}M))$.*

Proof. The exact sequence

$$0 \to \Omega_\Lambda^1(M) \to P_0 \to M \to 0$$

gives $\mathrm{V}_H^{\psi}(M) \subseteq \mathrm{V}_H^{\psi}(P_0) \cup \mathrm{V}_H^{\psi}(\Omega_\Lambda^{t-1}({}_{\psi}\Omega_\Lambda^1(M)))$, and since $\mathrm{V}_H^{\psi}(P_0)$ is trivial we get $\mathrm{V}_H^{\psi}(M) \subseteq \mathrm{V}_H^{\psi}(\Omega_\Lambda^t({}_{\psi}M))$. On the other hand, the exact sequence

$$0 \to \Omega_\Lambda^t({}_{\psi}M) \to {}_{\psi}P_{t-1} \to \Omega_\Lambda^{t-1}({}_{\psi}M) \to 0$$

gives $\mathrm{V}_H^{\psi}(\Omega_\Lambda^t({}_{\psi}M)) \subseteq \mathrm{V}_H^{\psi}({}_{\psi}P_{t-1}) \cup \mathrm{V}_H^{\psi}({}_{\psi^{-1}}({}_{\psi}M))$, and since $\mathrm{V}_H^{\psi}({}_{\psi}P_{t-1})$ is trivial we get $\mathrm{V}_H^{\psi}(\Omega_\Lambda^t({}_{\psi}M)) \subseteq \mathrm{V}_H^{\psi}(M)$. □

We illustrate this last result with an example.

Example. Suppose k is a field and Λ is a Frobenius algebra, let $\psi = \nu_\Lambda$ be the Nakayama automorphism of Λ, and take $t = 2$. We saw in the example following Lemma 2.1 that the Auslander-Reiten translation $\tau = D\,\mathrm{Tr}$ is isomorphic to $\Omega_\Lambda^2 \mathcal{N}$, where $\mathcal{N}$ is the Nakayama functor $D\,\mathrm{Hom}_\Lambda(-, \Lambda)$. Moreover, in the example prior to Proposition 3.2 we saw that the latter is isomorphic to ${}_{\nu}\Lambda_1 \otimes_\Lambda -$. Therefore, from the corollary above we get

$$\mathrm{V}_H^{\nu}(M) = \mathrm{V}_H^{\nu}(\Omega_\Lambda^2({}_{\nu}M)) = \mathrm{V}_H^{\nu}(\Omega_\Lambda^2({}_{\nu}\Lambda_1 \otimes_\Lambda M)) = \mathrm{V}_H^{\nu}(\tau(M))$$

whenever $\Omega_\Lambda^2(M)$ is nonzero.

A fundamental feature within the theory of support varieties for modules over both group algebras of finite groups and complete intersections is the finite generation of $\mathrm{Ext}^*(X, Y)$ (where X and Y are modules over the ring in question) as a module over the commutative Noetherian graded ring of cohomological operators (see [Ben] and [Car] for the group ring case and [Avr] and [AvB] for the complete intersection case). In order to obtain a similar theory for selfinjective algebras in [EHSST], it was necessary to *assume* that the same finite generation hypothesis held, since there exist selfinjective algebras for which it does not hold.

We now introduce a finite generation hypothesis similar to that used in [Be1], where instead of assuming finite generation of $\mathrm{Ext}^*(X, Y)$ for *all* modules X and Y, a local variant focusing on a single module was used.

Assumption ($\mathbf{Fg}(M, H, \psi, t)$). Given the automorphism $\Lambda \xrightarrow{\psi} \Lambda$ and the integer $t \in \mathbb{N}$, there exists a strongly commutative Noetherian graded subalgebra $H = \bigoplus_{n=0}^{\infty} H^{tn}$ of $\mathrm{HH}^{t*}({}_{\psi^*}\Lambda_1, \Lambda)$ such that $H^0 = \mathrm{HH}^0(\Lambda, \Lambda) = \mathrm{Z}(\Lambda)$, and with the property that $\mathrm{Ext}_\Lambda^{t*}({}_{\psi^*}M, \Lambda / \mathfrak{r})$ is a finitely generated H-module.

As mentioned in the remark prior to Definition 4.1, considering only strongly commutative algebras instead of "ordinary" commutative algebras is not a severe restriction. In fact, the following result shows that if $\mathbf{Fg}(M, H, \psi, t)$ holds for a commutative Noetherian graded algebra H which is not necessarily strongly commutative, then there exist a positive integer s and a strongly commutative Noetherian subalgebra $H' \subseteq H$ for which $\mathbf{Fg}(M, H', \psi^s, ts)$ holds.

Proposition 4.4. *Let $\Lambda \xrightarrow{\psi} \Lambda$ be an automorphism and $t \in \mathbb{N}$ an integer, and suppose there exists a commutative Noetherian graded subalgebra $H = \bigoplus_{n=0}^{\infty} H^{tn}$ of $\operatorname{HH}^{t*}({}_{\psi^*}\Lambda_1, \Lambda)$ such that $H^0 = \operatorname{HH}^0(\Lambda, \Lambda) = \operatorname{Z}(\Lambda)$, and with the property that $\operatorname{Ext}_\Lambda^{t*}({}_{\psi^*}M, \Lambda / \mathfrak{r})$ is a finitely generated H-module. Then there exist a positive integer s and a strongly commutative Noetherian graded subalgebra H' of H for which $\mathbf{Fg}(M, H', \psi^s, ts)$ holds*

Proof. From the remark prior to Definition 4.1 we see that there exist two positive integers n and w such that ${}_{\psi^n}\theta = \theta_{\psi^{-n}}$ for every homogeneous element $\theta \in H$ with $|\theta| \geq tw$. Let $\eta_1, \dots, \eta_r \in H$ be homogeneous elements of positive degrees generating H as an algebra over H^0, and denote the integer wn by s. Then the subalgebra $H' = H^0[\eta_1^s, \dots, \eta_r^s]$ of H is strongly commutative with respect to the automorphism ψ^s, and $\operatorname{Ext}_\Lambda^{(ts)*}({}_{(\psi^s)^*}M, \Lambda / \mathfrak{r})$ is a finitely generated H'-module. $\square$

We now show that introducing the above finite generation hypothesis enables us to compute the dimension of the twisted variety of a module. Recall first that if $X = \bigoplus_{n=0}^{\infty} X_n$ is a graded k-module of finite type (that is, the k-length of X_n is finite for all n), then the *rate of growth* of X, denoted $\gamma(X)$, is defined as

$$\gamma(X) \stackrel{\text{def}}{=} \inf\{c \in \mathbb{N} \cup \{0\} \mid \exists a \in \mathbb{R} \text{ such that } \ell_k(X_n) \leq an^{c-1} \text{ for } n \gg 0\}.$$

For a Λ-module Y with minimal projective resolution $\cdots \to Q_1 \to Q_0 \to Y \to 0$, we define the *$m$-complexity* of Y (where $m \geq 1$ is a number) as the rate of growth of $\bigoplus_{n=0}^{\infty} Q_{mn}$, and denote it by $\operatorname{cx}^m Y$. Note that the 1-complexity of a module coincides with the usual notion of complexity, and that the identity

$$\operatorname{cx}^m Y = \gamma\left(\operatorname{Ext}_\Lambda^{m*}(Y, \Lambda / \mathfrak{r})\right)$$

always holds (the latter can be seen by adopting the arguments given in [Be1, Section 3]).

The following result allows us to compute the dimension of $\operatorname{V}_H^\psi(M)$ in terms of the t-complexity of M provided $\mathbf{Fg}(M, H, \psi, t)$ is satisfied. In particular Dade's Lemma holds.

Proposition 4.5. *If $\mathbf{Fg}(M, H, \psi, t)$ holds, then $\dim \operatorname{V}_H^\psi(M) = \operatorname{cx}^t M$. In particular $\operatorname{V}_H^\psi(M)$ is trivial if and only if M has finite projective dimension.*

Proof. Adopting the arguments used to prove [Ben, Proposition 5.7.2] and [EHSST, Proposition 2.1] gives $\dim \operatorname{V}_H^\psi(M) = \gamma\left(\operatorname{Ext}_\Lambda^{t*}({}_{\psi^*}M, \Lambda / \mathfrak{r})\right)$. For any Λ-module Y and any k-automorphism $\Lambda \xrightarrow{\phi} \Lambda$ there is an isomorphism $Y \simeq {}_\phi Y$ of k-modules, and so if $\cdots \to Q_1 \to Q_0 \to Y \to 0$ is a minimal projective resolution we see that the m-complexity of Y equals the rate of growth of $\bigoplus_{n=0}^{\infty} {}_{\phi^n}(Q_{mn})$. In particular the equalities

$$\operatorname{cx}^t M = \gamma\left(\bigoplus_{n=0}^{\infty} {}_{\psi^n}(P_{tn})\right) = \gamma\left(\operatorname{Ext}_\Lambda^{t*}({}_{\psi^*}M, \Lambda / \mathfrak{r})\right)$$

hold, where

$$\mathbb{P}\colon \cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$$

is the minimal projective resolution of M. $\square$

Note that whenever $\mathbf{Fg}(M, H, \psi, t)$ is satisfied the dimension of $\operatorname{V}_H^\psi(M)$, and therefore also the t-complexity of M, must be finite; since H is commutative graded Noetherian it is generated as an algebra over H^0 by a finite set $\{x_0, \dots, x_r\}$ of homogeneous elements of positive degrees, giving $\gamma\left(\operatorname{Ext}_\Lambda^{t*}({}_{\psi^*}M, \Lambda / \mathfrak{r})\right) \leq r$ (see the discussion prior to [Be1, Proposition 3.1]). It then follows from the proof of the above proposition that $\operatorname{cx}^t M \leq r$.

The next result gives a sufficient and necessary condition for the variety to be one dimensional. Recall that a Λ-module Y is *periodic* if there exists a positive integer p such that $Y \simeq \Omega_\Lambda^p(Y)$, whereas it is *eventually periodic* if $\Omega_\Lambda^i(Y)$ is periodic for some $i \geq 0$. For a k-automorphism $\Lambda \xrightarrow{\phi} \Lambda$ we define Y to be *ϕ-periodic* if there exists a positive integer p such that $Y \simeq \Omega_\Lambda^p({}_\phi Y)$, and *eventually ϕ-periodic* if $\Omega_\Lambda^i(Y)$ is ϕ-periodic for some $i \geq 0$.

Proposition 4.6. *If* $\mathbf{Fg}(M, H, \psi, t)$ *holds, then* $\dim \mathrm{V}_H^\psi(M) = 1$ *if and only if* M *is eventually* ψ^i*-periodic for some* $i \geq 1$*. Moreover, when this occurs there is a positive integer* w *such that* $\Omega_\Lambda^{tj}(M) \simeq \Omega_\Lambda^{t(j+w)}({}_{\psi^w}M)$ *for some* $j \geq 0$*.*

Proof. If M is eventually ψ^i-periodic, then the sequence

$$\ell_k(P_0), \ell_k(P_1), \ell_k(P_2), \dots$$

must be bounded, that is, the 1-complexity of M is 1. But then the sequence

$$\ell_k(P_0), \ell_k(P_t), \ell_k(P_{2t}), \dots$$

is also bounded, that is, the t-complexity of M is also 1, and consequently $\dim \mathrm{V}_H^\psi(M) = 1$.

Conversely, suppose the latter of the above sequences is bounded. By [Be1, Proposition 2.1] there exists a homogeneous element $\eta \in H$ of positive degree, say $|\eta| = tw$, such that the multiplication map

$$\mathrm{Ext}_\Lambda^{ti}({}_{\psi^i}M, \Lambda/\mathfrak{r}) \xrightarrow{\cdot\eta} \mathrm{Ext}_\Lambda^{t(i+w)}({}_{\psi^{i+w}}M, \Lambda/\mathfrak{r})$$

is injective for $i \gg 0$. Represent the element $\eta \otimes_\Lambda M \in \mathrm{Ext}_\Lambda^{t*}({}_{\psi^*}M, M)$ by a map $f_\eta \colon \Omega_\Lambda^{tw}({}_{\psi^w}M) \to M$. Then for each $i \geq 0$ there is a map $f_i \colon {}_{\psi^w}(P_{tw+i}) \to P_i$ making the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & {}_{\psi^w}(P_{tw+2}) \arrow[r, "d_{tw+2}"] \arrow[d, "f_2"] & {}_{\psi^w}(P_{tw+1}) \arrow[r, "d_{tw+1}"] \arrow[d, "f_1"] & {}_{\psi^w}(P_{tw}) \arrow[r] \arrow[d, "f_0"] & \Omega_\Lambda^{tw}({}_{\psi^w}M) \arrow[r] \arrow[d, "f_\eta"] & 0 \\
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] & P_0 \arrow[r, "d_0"] & M \arrow[r] & 0
\end{tikzcd}$$

with exact rows commute. If $\theta \in \mathrm{Ext}_\Lambda^{t*}({}_{\psi^*}M, \Lambda/\mathfrak{r})$ is a homogeneous element, say $|\theta| = tn$, and represented by a map $f_\theta \colon {}_{\psi^n}(P_{tn}) \to \Lambda/\mathfrak{r}$, then $\theta\eta \in \mathrm{Ext}_\Lambda^{t(w+n)}({}_{\psi^{w+n}}M, \Lambda/\mathfrak{r})$ is represented by the composite map

$${}_{\psi^{w+n}}(P_{t(w+n)}) \xrightarrow{{}_{\psi^n}f_{tn}} {}_{\psi^n}(P_{tn}) \xrightarrow{f_\theta} \Lambda/\mathfrak{r}.$$

For any $i \geq 0$ the complex ${}_{\psi^i}\mathbb{P}$ is a minimal projective resolution of ${}_{\psi^i}M$, and therefore $\mathrm{Ext}_\Lambda^{ti}({}_{\psi^i}M, \Lambda/\mathfrak{r}) = \mathrm{Hom}_\Lambda({}_{\psi^i}(P_{ti}), \Lambda/\mathfrak{r})$. Moreover, the multiplication map

$$\mathrm{Ext}_\Lambda^{ti}({}_{\psi^i}M, \Lambda/\mathfrak{r}) \xrightarrow{\cdot\eta} \mathrm{Ext}_\Lambda^{t(i+w)}({}_{\psi^{i+w}}M, \Lambda/\mathfrak{r})$$

is just the map

$$\begin{array}{ccc}
\mathrm{Hom}_\Lambda({}_{\psi^i}(P_{ti}), \Lambda/\mathfrak{r}) & \xrightarrow{({}_{\psi^i}f_{ti})^*} & \mathrm{Hom}_\Lambda({}_{\psi^{i+w}}(P_{t(i+w)}), \Lambda/\mathfrak{r}) \\
g & \mapsto & g \circ {}_{\psi^i} f_{ti},
\end{array}$$

and since the exact sequence

$${}_{\psi^{i+w}}(P_{t(i+w)}) \xrightarrow{{}_{\psi^i}f_{ti}} {}_{\psi^i}(P_{ti}) \to {}_{\psi^i}(\mathrm{Coker}\, f_{ti}) \to 0$$

shows that the kernel of the multiplication map is isomorphic to $\operatorname{Hom}_\Lambda({}_{\psi^i}(\operatorname{Coker} f_{ti}), \Lambda / \mathfrak{r})$, we see that $\operatorname{Coker} f_{ti} = 0$ for $i \gg 0$. Consequently, for each $i \gg 0$ there exists a surjective map $\Omega_\Lambda^{t(w+i)}({}_{\psi^w}M) \twoheadrightarrow \Omega_\Lambda^{ti}(M)$ and therefore also a sequence

$$\cdots \twoheadrightarrow \Omega_\Lambda^{t(3w+i)}({}_{\psi^{3w}}M) \twoheadrightarrow \Omega_\Lambda^{t(2w+i)}({}_{\psi^{2w}}M) \twoheadrightarrow \Omega_\Lambda^{t(w+i)}({}_{\psi^w}M) \twoheadrightarrow \Omega_\Lambda^{ti}(M)$$

of surjections. However, by assumption the sequence

$$\ell_k(P_0), \ell_k(P_t), \ell_k(P_{2t}), \dots$$

is bounded, and therefore $\Omega_\Lambda^{t((q+1)w+i)}({}_{\psi^{(q+1)w}}M)$ must be isomorphic to $\Omega_\Lambda^{t(qw+i)}({}_{\psi^{qw}}M)$ for large q and i. By setting $j = qw + i$ and twisting with the automorphism ψ^{-qw}, we see that $\Omega_\Lambda^{tj}(M) \simeq \Omega_\Lambda^{t(j+w)}({}_{\psi^w}M)$ for $j \gg 0$. $\square$

As a particular case of the proposition we obtain the following result on $D\,\mathrm{Tr}$-periodicity over a Frobenius algebra.

Proposition 4.7. *Suppose k is a field and Λ is a Frobenius algebra, and let $\Lambda \xrightarrow{\nu} \Lambda$ be a Nakayama automorphism. If M does not have a nonzero projective summand and* $\mathbf{Fg}(M, H, \nu^n, 2n)$ *holds for some $n \geq 1$, then* $\dim \mathrm{V}_H^\nu(M) = 1$ *if and only if $M \simeq \tau^p(M)$ for some $p \geq 1$.*

Proof. By the previous proposition the variety $\mathrm{V}_H^\nu(M)$ is one dimensional if and only if there is a positive integer w such that $\Omega_\Lambda^{2nj}(M) \simeq \Omega_\Lambda^{2n(j+w)}({}_{\nu^{nw}}M)$ for some $j \geq 0$. Taking cosyzygies we see that the latter happens precisely when $M \simeq \Omega_\Lambda^{2nw}({}_{\nu^{nw}}M)$, that is, when $M \simeq \tau^{nw}(M)$. $\square$

We illustrate this last result with an example.

Example. Let k be an algebraically closed field of odd characteristic and $q \in k$ a nonzero element which is *not* a root of unity, and denote by Λ the k-algebra

$$\Lambda = k\langle x, y \rangle / (x^2, xy + qyx, y^2).$$

We saw in the example following Proposition 3.2 that the Nakayama automorphism ν of Λ is defined by

$$x \mapsto -q^{-1}x, \quad y \mapsto -qy,$$

and that $\mathrm{HH}^{4*}({}_{\nu^{2*}}\Lambda_1, \Lambda)$ is isomorphic to $k[\theta] \times k$ with θ of degree 4. In particular $\mathrm{HH}^{4*}({}_{\nu^{2*}}\Lambda_1, \Lambda)$ is strongly commutative, and we denote this ring by H.

For elements $\alpha, \beta \in k$, denote by $M_{(\alpha,\beta)}$ the Λ-module $\Lambda(\alpha x + \beta y)$ (see [Sma] for a counterexample, using the module $M_{(1,1)}$, to a question raised by Auslander, a question for which a counterexample was first given in [JoS]). Consider the module $M = M_{(1,\beta)}$ for $\beta \neq 0$, and for each $i \geq 0$ let $P_i = \Lambda$. The sequence

$$\mathbb{P}\colon \ \cdots \to P_3 \xrightarrow{\cdot(x+q^3\beta y)} P_2 \xrightarrow{\cdot(x+q^2\beta y)} P_1 \xrightarrow{\cdot(x+q\beta y)} P_0 \xrightarrow{\cdot(x+\beta y)} M \to 0$$

is a minimal projective resolution of M, hence since ${}_{\nu^n}\mathbb{P}$ is a minimal projective resolution of ${}_{\nu^n}M$ for any $n \geq 1$ we see that $\operatorname{Ext}_\Lambda^{2n}({}_{\nu^n}M, k) = \operatorname{Hom}_\Lambda({}_{\nu^n}\Lambda, k)$ is one dimensional. We shall prove that $\mathbf{Fg}(M, H, \nu, 2)$ holds.

Recall from the example following Proposition 3.2 the minimal bimodule projective resolution

$$(\mathbb{F}, \delta)\colon \ \cdots \to F^{n+1} \xrightarrow{\delta} F^n \xrightarrow{\delta} F^{n-1} \to \cdots$$

of Λ, where the set $\{\tilde{f}_i^n\}_{i=0}^n$ generates F^n as a Λ^{e}-module and the differential $\delta\colon F^n \to F^{n-1}$ is given by

$$\tilde{f}_i^n \mapsto \left[x\tilde{f}_i^{n-1} + (-1)^n q^i \tilde{f}_i^{n-1} x \right] + \left[q^{n-i} y \tilde{f}_{i-1}^{n-1} + (-1)^n \tilde{f}_{i-1}^{n-1} y \right].$$

The element $\theta \in H^4$ represented by the map

$$\begin{array}{rcl} g_4 \colon {}_{\nu^2}F_1^4 & \to & \Lambda \\ \tilde{f}_i^4 & \mapsto & \begin{cases} 1 & \text{for } i = 2 \\ 0 & \text{for } i \neq 2 \end{cases} \end{array}$$

generates H as an algebra over H^0. The resolution $\mathbb{F} \otimes_\Lambda M$ is also a projective resolution of M, and defining Λ-linear maps

$$\begin{array}{rcl} h_n \colon P_n & \to & F^n \otimes_\Lambda M \\ 1 & \mapsto & \left(\sum_{i=0}^{n} q^{\frac{i(i+1)}{2}} \beta^i \tilde{f}_i^n \right) \otimes (x + \beta y) \end{array}$$

gives a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & P_2 \arrow[r, "\cdot(x+q^2\beta y)"] \arrow[d, "h_2"] & P_1 \arrow[r, "\cdot(x+q\beta y)"] \arrow[d, "h_1"] & P_0 \arrow[r, "\cdot(x+\beta y)"] \arrow[d, "h_0"] & M \arrow[r] \arrow[d, equal] & 0 \\ \cdots \arrow[r] & F^2 \otimes_\Lambda M \arrow[r, "\delta\otimes 1"] & F^1 \otimes_\Lambda M \arrow[r, "\delta\otimes 1"] & F^0 \otimes_\Lambda M \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

with exact rows. Consequently, the element $\theta \otimes_\Lambda M \in \operatorname{Ext}_\Lambda^4({}_{\nu^2}M, M)$ is represented by the composite map

$$\begin{array}{rcl} \bar{f}_\theta \colon {}_{\nu^2}(P_4) & \xrightarrow{(g_4\otimes 1)\circ h_4} & \Lambda \otimes_\Lambda M \simeq M \\ 1 & \mapsto & q^3\beta^2(x + \beta y). \end{array}$$

Now for each $i \geq 0$ define $f_i \colon {}_{\nu^2}(P_{i+4}) \to_{\nu^2} (P_i)$ by $1 \mapsto q^{2i+3}\beta^2$. We then obtain a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & {}_{\nu^2}(P_7) \arrow[r, "\cdot(x+q^7\beta y)"] \arrow[d, "f_3"] & {}_{\nu^2}(P_6) \arrow[r, "\cdot(x+q^6\beta y)"] \arrow[d, "f_2"] & {}_{\nu^2}(P_5) \arrow[r, "\cdot(x+q^5\beta y)"] \arrow[d, "f_1"] & {}_{\nu^2}(P_4) \arrow[r] \arrow[d, "f_0"] \arrow[dr, "\bar{f}_\theta"] & \cdots \\ \cdots \arrow[r] & P_3 \arrow[r, "\cdot(x+q^3\beta y)"] & P_2 \arrow[r, "\cdot(x+q^2\beta y)"] & P_1 \arrow[r, "\cdot(x+q\beta y)"] & P_0 \arrow[r, "\cdot(x+\beta y)"] & M \arrow[r] & 0 \end{tikzcd}$$

with exact rows, hence if $\mu \in \operatorname{Ext}_\Lambda^{2n}({}_{\nu^n}M, k) = \operatorname{Hom}_\Lambda({}_{\nu^n}(P_{2n}), k)$ is an element represented by a map $\bar{f}_\mu \colon {}_{\nu^n}(P_{2n}) \to k$ we see that the element $\mu \cdot \theta \in \operatorname{Ext}_\Lambda^{2(n+2)}({}_{\nu^{(n+2)}}M, k)$ is represented by the composite map

$${}_{\nu^{(n+2)}}(P_{2(n+2)}) \xrightarrow{\nu^n(f_{2n})} {}_{\nu^n}(P_{2n}) \xrightarrow{\bar{f}_\mu} k.$$

Moreover, this composition is nonzero whenever μ is nonzero, since f_{2n} is just multiplication with $q^{4n+3}\beta^2$. Therefore, since $\operatorname{Ext}_\Lambda^{2n}({}_{\nu^n}M, k)$ is one dimensional for each $n \geq 1$, the H-module $\operatorname{Ext}_\Lambda^{2*}({}_{\nu^*}M, k)$ is finitely generated; it is generated as an H-module by any k-basis in $\operatorname{Ext}_\Lambda^0({}_{\nu^0}M, k) = \operatorname{Hom}_\Lambda(M, k)$ together with any nonzero elements $\mu_1 \in \operatorname{Ext}_\Lambda^2({}_\nu M, k)$ and $\mu_2 \in \operatorname{Ext}_\Lambda^4({}_{\nu^2}M, k)$.

The above shows that $\mathbf{Fg}(M_{(1,\beta)}, H, \nu, 2)$ holds for $\beta \neq 0$, and since the 2-complexity of $M_{(1,\beta)}$ is obviously 1 we see from Proposition 4.5 that the variety $\mathrm{V}_H^\nu(M_{(1,\beta)})$ is one dimensional. From Proposition 4.7 we conclude that $M_{(1,\beta)} \simeq \tau^w(M_{(1,\beta)})$ for some $w \geq 1$. Indeed, in [LiS] it is shown that $M_{(1,\beta)}$ is isomorphic to $\tau(M_{(1,\beta)})$.

Recall that each nonprojective indecomposable Λ-module is annihilated by yx, and is therefore a module over the algebra $\Lambda/(yx)$ (see [Sch, Section 4]). The latter is stably equivalent to the Kronecker algebra, an equivalence under which the representation

$$\begin{tikzcd} k \arrow[r, bend left, "\beta"] \arrow[r, bend right, "\alpha"'] & k \end{tikzcd}$$

corresponds to the module $M_{(\alpha,\beta)}$. Denoting $M_{(\alpha,\beta)}$ by $M^1_{(\alpha,\beta)}$, it follows from the well known representation theory of the Kronecker algebra (see, for example, [ARS] and [ASS]) that the indecomposable τ-periodic Λ-modules are divided into distinct countable classes $\{M^i_{(\alpha,\beta)}\}_{i=1}^{\infty}$, one for each pair $(\alpha,\beta) \in \{(0,1)\} \cup \{(1,\beta)\}_{\beta \in k}$, such that for each $i \geq 1$ there exists a short exact sequence

$$0 \to M^i_{(\alpha,\beta)} \to M^{i-1}_{(\alpha,\beta)} \oplus M^{i+1}_{(\alpha,\beta)} \to M^i_{(\alpha,\beta)} \to 0,$$

where $M^0_{(\alpha,\beta)} = 0$. Now each such exact sequence induces an exact sequence

$$\operatorname{Ext}^{2*}_{\Lambda}({}_{\nu^*}M^i_{\alpha}, k) \to \operatorname{Ext}^{2*}_{\Lambda}({}_{\nu^*}M^{i-1}_{\alpha}, k) \oplus \operatorname{Ext}^{2*}_{\Lambda}({}_{\nu^*}M^{i+1}_{\alpha}, k) \to \operatorname{Ext}^{2*}_{\Lambda}({}_{\nu^*}M^i_{\alpha}, k)$$

of H-modules, and so since H is Noetherian and $\operatorname{Ext}^{2*}_{\Lambda}({}_{\nu^*}M^1_{(1,\beta)}, k)$ is a finitely generated H-module whenever β is nonzero, an induction argument shows that $\operatorname{Ext}^{2*}_{\Lambda}({}_{\nu^*}M^i_{(1,\beta)}, k)$ is a finitely generated H-module for any $i \geq 1$ and $\beta \neq 0$. We conclude that $\mathbf{Fg}(M^i_{(1,\beta)}, H, \nu, 2)$ holds for *all* the modules $\{M^i_{(1,\beta)}\}_{i=1}^{\infty}$ when β is nonzero.

However, there are two more classes of indecomposable τ-periodic Λ-modules, namely $\{M^i_{(1,0)}\}_{i=0}^{\infty}$ and $\{M^i_{(0,1)}\}_{i=0}^{\infty}$. Do these modules satisfy the finite generation hypothesis? The answer is no, and to see this, consider the module $M_{(1,0)} = \Lambda x$. Letting $P_i = \Lambda$ for each $i \geq 0$ we see that the sequence

$$\cdots \to P_3 \xrightarrow{\cdot x} P_2 \xrightarrow{\cdot x} P_1 \xrightarrow{\cdot x} P_0 \xrightarrow{\cdot x} M_{(1,0)} \to 0$$

is a minimal projective resolution of $M_{(1,0)}$, and defining Λ-linear maps

$$\begin{array}{rcl} h_n \colon P_n & \to & F^n \otimes_{\Lambda} M_{(1,0)} \\ 1 & \mapsto & \tilde{f}^n_0 \otimes x \end{array}$$

gives a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & P_1 \arrow[r, "\cdot x"] \arrow[d, "h_1"] & P_0 \arrow[r, "\cdot x"] \arrow[d, "h_0"] & M_{(1,0)} \arrow[r] \arrow[d, equal] & 0 \\ \cdots \arrow[r] & F^1 \otimes_{\Lambda} M_{(1,0)} \arrow[r, "\delta\otimes 1"] & F^0 \otimes_{\Lambda} M_{(1,0)} \arrow[r] & M_{(1,0)} \arrow[r] & 0 \end{tikzcd}$$

with exact rows. Since the map h_4 does not "hit" the generator $\tilde{f}^4_2 \otimes x \in F^4 \otimes_{\Lambda} M_{(1,0)}$, we see that the element $\theta \otimes_{\Lambda} M \in \operatorname{Ext}^4_{\Lambda}({}_{\nu^2}M, M)$ is represented by the zero map. This shows that $\operatorname{Ext}^{2*}_{\Lambda}({}_{\nu^*}M^i_{(1,0)}, k)$ cannot be a finitely generated H-module, and a similar argument shows that the same is true for the module $M_{(0,1)}$.

Note also that the finite generation condition cannot hold for any nonzero indecomposable nonprojective Λ-module which is not τ-periodic; if X is such a module and $\operatorname{Ext}^{2*}_{\Lambda}({}_{\nu^*}X, k)$ is finitely generated over H, then the rate of growth of $\operatorname{Ext}^{2*}_{\Lambda}({}_{\nu^*}X, k)$ is not more than that of H. However, the latter equals the Krull dimension of H, thus $\gamma\left(\operatorname{Ext}^{2*}_{\Lambda}({}_{\nu^*}X, k)\right) \leq \gamma(H) = 1$. Since X is nonprojective we conclude that the rate of growth of $\operatorname{Ext}^{2*}_{\Lambda}({}_{\nu^*}X, k)$ is 1, and so by Proposition 4.5 the variety $\mathrm{V}^{\nu}_{H}(X)$ is one dimensional. But then Proposition 4.7 implies X is τ-periodic, a contradiction.

Returning to the general theory, we now impose the finite generation hypothesis on both M and $\Omega^1_{\Lambda}(M)$. The following result shows that, in this situation, if the variety of M is nontrivial (that is, when M does not have finite projective dimension) then there exists a homogeneous element in H "cutting down" the variety by one dimension. Recall first that if $\eta \in \mathrm{HH}^{t*}({}_{\psi^*}\Lambda_1, \Lambda)$ is a homogeneous element, say $\eta \in \mathrm{Ext}^{tm}_{\Lambda^e}({}_{\psi^m}\Lambda_1, \Lambda)$, then it can be represented by a Λ^e-linear map $f_\eta \colon \Omega^{tm}_{\Lambda^e}({}_{\psi^m}\Lambda_1) \to \Lambda$. This map yields a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^{tm}_{\Lambda^e}({}_{\psi^m}\Lambda_1) \arrow[r] \arrow[d, "f_\eta"] & {}_{\psi^m}(P^{tm-1}) \arrow[r] \arrow[d] & \Omega^{tm-1}_{\Lambda^e}({}_{\psi^m}\Lambda_1) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Lambda \arrow[r] & K_\eta \arrow[r] & \Omega^{tm-1}_{\Lambda^e}({}_{\psi^m}\Lambda_1) \arrow[r] & 0
\end{tikzcd}$$

with exact rows, in which we have denoted by P^i the ith module in the minimal projective bimodule resolution of Λ. Note that up to isomorphism the module K_η is independent of the map f_η chosen to represent η.

Proposition 4.8. *If both* $\mathbf{Fg}(M, H, \psi, t)$ *and* $\mathbf{Fg}(\Omega^1_\Lambda(M), H, \psi, t)$ *hold and* M *does not have finite projective dimension, then there exists a homogeneous element* $\eta \in H$ *of positive degree such that* $\dim \mathrm{V}^\psi_H(\Omega^1_{\Lambda^e}(K_\eta) \otimes_\Lambda M) = \dim \mathrm{V}^\psi_H(M) - 1$.

Proof. By assumption the H-modules $\mathrm{Ext}^{t*}_\Lambda({}_{\psi^*}M, \Lambda/\mathfrak{r})$ and $\mathrm{Ext}^{t*}_\Lambda({}_{\psi^*}\Omega^1_\Lambda(M), \Lambda/\mathfrak{r})$ are finitely generated, hence by slightly generalizing the proof of [Be1, Proposition 2.1] we see that there exists a homogeneous element $\eta \in H$ of positive degree, say $\eta \in H^{tm} \subseteq \mathrm{Ext}^{tm}_{\Lambda^e}({}_{\psi^m}\Lambda_1, \Lambda)$, such that the multiplication maps

$$\begin{array}{rcl}
\mathrm{Ext}^{ti}_\Lambda({}_{\psi^i}M, \Lambda/\mathfrak{r}) & \xrightarrow{\cdot\eta} & \mathrm{Ext}^{t(i+m)}_\Lambda({}_{\psi^{i+m}}M, \Lambda/\mathfrak{r}) \\
\mathrm{Ext}^{ti}_\Lambda({}_{\psi^i}\Omega^1_\Lambda(M), \Lambda/\mathfrak{r}) & \xrightarrow{\cdot\eta} & \mathrm{Ext}^{t(i+m)}_\Lambda({}_{\psi^{i+m}}\Omega^1_\Lambda(M), \Lambda/\mathfrak{r})
\end{array}$$

are both k-monomorphisms for $i \gg 0$. Consider the short exact sequence

$$0 \to \Lambda \to K_\eta \to \Omega^{tm-1}_{\Lambda^e}({}_{\psi^m}\Lambda_1) \to 0$$

obtained from η. As $\Omega^{tm-1}_{\Lambda^e}({}_{\psi^m}\Lambda_1)$ is right Λ-projective, the sequence splits when considered as a sequence of right Λ-modules, and consequently the sequence

$$0 \to M \to K_\eta \otimes_\Lambda M \to \Omega^{tm-1}_{\Lambda^e}({}_{\psi^m}\Lambda_1) \otimes_\Lambda M \to 0$$

is exact. For each $i \geq 0$ the latter sequence induces a long exact sequence

$$\begin{gathered}
\mathrm{Ext}^{ti}_\Lambda({}_{\psi^i}(K_\eta \otimes_\Lambda M), \Lambda/\mathfrak{r}) \to \mathrm{Ext}^{ti}_\Lambda({}_{\psi^i}M, \Lambda/\mathfrak{r}) \xrightarrow{\partial_{ti}} \\
\mathrm{Ext}^{t(i+m)}_\Lambda({}_{\psi^{i+m}}M, \Lambda/\mathfrak{r}) \to \mathrm{Ext}^{ti+1}_\Lambda({}_{\psi^i}(K_\eta \otimes_\Lambda M), \Lambda/\mathfrak{r}) \to \\
\mathrm{Ext}^{ti+1}_\Lambda({}_{\psi^i}M, \Lambda/\mathfrak{r}) \xrightarrow{\partial_{ti+1}} \mathrm{Ext}^{t(i+m)+1}_\Lambda({}_{\psi^{i+m}}M, \Lambda/\mathfrak{r})
\end{gathered}$$

in which we have replaced $\mathrm{Ext}^j_\Lambda({}_{\psi^i}(\Omega^{tm-1}_{\Lambda^e}({}_{\psi^m}\Lambda_1) \otimes_\Lambda M), \Lambda/\mathfrak{r})$ with $\mathrm{Ext}^{j+tm-1}_\Lambda({}_{\psi^{i+m}}M, \Lambda/\mathfrak{r})$, due to the fact that ${}_{\psi^i}(\Omega^{tm-1}_{\Lambda^e}({}_{\psi^m}\Lambda_1) \otimes_\Lambda M)$ is a $(tm-1)$th syzygy of ${}_{\psi^{i+m}}M$. By [Mac, Theorem III.9.1] the connecting homomorphism ∂_j is then the Yoneda product with the extension $(-1)^j {}_{\psi^i}(\eta \otimes_\Lambda M)$, in particular we see that ∂_{ti} is scalar multiplication with $(-1)^{ti}\eta$.

Now consider the connecting homomorphism ∂_{ti+1}. Applying the Yoneda relation (†) from Section 3 to η and the short exact sequence

$$\theta \colon 0 \to \Omega^1_\Lambda(M) \to P_0 \to M \to 0$$

gives the relation

$$(\eta \otimes_\Lambda \Omega^1_\Lambda(M)) \circ ({}_{\psi^m}\Lambda_1 \otimes_\Lambda \theta) = (-1)^{tm}(\Lambda \otimes_\Lambda \theta) \circ (\eta \otimes_\Lambda M),$$

which we may twist by ψ^i to obtain the relation

$${}_{\psi^i}(\eta \otimes_\Lambda \Omega^1_\Lambda(M)) \circ {}_{\psi^{i+m}}\theta = (-1)^{tm}{}_{\psi^i}\theta \circ {}_{\psi^i}(\eta \otimes_\Lambda M).$$

This gives a commutative diagram

$$\begin{tikzcd}
\operatorname{Ext}_{\Lambda}^{ti}({}_{\psi^i}\Omega_{\Lambda}^1(M), \Lambda/\mathfrak{r}) \arrow[r, "{}_{\psi^i}\theta"] \arrow[d, "\cdot(-1)^{t(i+m)+1}\eta"] & \operatorname{Ext}_{\Lambda}^{ti+1}({}_{\psi^i}M, \Lambda/\mathfrak{r}) \arrow[d, "\partial_{ti+1}"] \\
\operatorname{Ext}_{\Lambda}^{t(i+m)}({}_{\psi^{i+m}}\Omega_{\Lambda}^1(M), \Lambda/\mathfrak{r}) \arrow[r, "{}_{\psi^{i+m}}\theta"] & \operatorname{Ext}_{\Lambda}^{t(i+m)+1}({}_{\psi^{i+m}}M, \Lambda/\mathfrak{r})
\end{tikzcd}$$

in which the horizontal maps are isomorphisms, hence the connecting homomorphism ∂_{ti+1} is also basically just scalar multiplication with η, as was the case with ∂_{ti}. Consequently they are both injective for $i \gg 0$, giving a short exact sequence

$$(\dagger\dagger) \quad 0 \to {}^{ti}_{\Lambda}({}_{\psi^i}M, \Lambda/\mathfrak{r}) \xrightarrow{\cdot\eta} {}^{t(i+m)}_{\Lambda}({}_{\psi^{i+m}}M, \Lambda/\mathfrak{r}) \to {}^{ti+1}_{\Lambda}({}_{\psi^i}(K_\eta \otimes_\Lambda M), \Lambda/\mathfrak{r}) \to 0$$

for large i (in which we have used the short hand notion ${}^{j}_{\Lambda}(-,-)$ for $\operatorname{Ext}^j_\Lambda(-,-)$). Note that we may identify $\operatorname{Ext}_{\Lambda}^{ti+1}({}_{\psi^i}(K_\eta \otimes_\Lambda M), \Lambda/\mathfrak{r})$ with $\operatorname{Ext}_{\Lambda}^{ti}({}_{\psi^i}(\Omega^1_{\Lambda^e}(K_\eta) \otimes_\Lambda M), \Lambda/\mathfrak{r})$; since K_η is right Λ-projective the projective bimodule cover

$$0 \to \Omega^1_{\Lambda^e}(K_\eta) \to Q \to K_\eta \to 0$$

of K_η splits as a sequence of right Λ-modules, and therefore stays exact when tensored with M. In addition, the Λ-module $Q \otimes_\Lambda M$ is projective, hence

$$\operatorname{Ext}_{\Lambda}^{ti+1}({}_{\psi^i}(K_\eta \otimes_\Lambda M), \Lambda/\mathfrak{r}) \simeq \operatorname{Ext}_{\Lambda}^{ti}({}_{\psi^i}(\Omega^1_{\Lambda^e}(K_\eta) \otimes_\Lambda M), \Lambda/\mathfrak{r}).$$

Consider now the H-module $\operatorname{Ext}_{\Lambda}^{t*}({}_{\psi^*}(\Omega^1_{\Lambda^e}(K_\eta) \otimes_\Lambda M), \Lambda/\mathfrak{r})$, and let w be an integer such that the sequence $(\dagger\dagger)$ is exact for $i \geq w$. Then the submodule $\bigoplus_{i=w}^{\infty} \operatorname{Ext}_{\Lambda}^{ti}({}_{\psi^i}(\Omega^1_{\Lambda^e}(K_\eta) \otimes_\Lambda M), \Lambda/\mathfrak{r})$ is finitely generated over H, being a factor module of the submodule $\bigoplus_{i=w}^{\infty} \operatorname{Ext}_{\Lambda}^{t(i+m)}({}_{\psi^{i+m}}M, \Lambda/\mathfrak{r})$ of the finitely generated H-module $\operatorname{Ext}_{\Lambda}^{t*}({}_{\psi^*}M, \Lambda/\mathfrak{r})$. But then the H-module $\operatorname{Ext}_{\Lambda}^{t*}({}_{\psi^*}(\Omega^1_{\Lambda^e}(K_\eta) \otimes_\Lambda M), \Lambda/\mathfrak{r})$ must be finitely generated itself, since each graded part $\operatorname{Ext}_{\Lambda}^{tj}({}_{\psi^j}(\Omega^1_{\Lambda^e}(K_\eta) \otimes_\Lambda M), \Lambda/\mathfrak{r})$ is finitely generated over H^0. Also, from [Be1, Proposition 3.1] we get

$$\gamma\left(\bigoplus_{i=w}^{\infty} \operatorname{Ext}_{\Lambda}^{ti}({}_{\psi^i}(\Omega^1_{\Lambda^e}(K_\eta) \otimes_\Lambda M), \Lambda/\mathfrak{r})\right) = \gamma\left(\bigoplus_{i=w}^{\infty} \operatorname{Ext}_{\Lambda}^{t(i+m)}({}_{\psi^{i+m}}M, \Lambda/\mathfrak{r})\right) - 1,$$

and since for any Λ-module X the rate of growth of $\operatorname{Ext}_{\Lambda}^{t*}({}_{\psi^*}X, \Lambda/\mathfrak{r})$ equals that of $\bigoplus_{i=w}^{\infty} \operatorname{Ext}_{\Lambda}^{ti}({}_{\psi^i}X, \Lambda/\mathfrak{r})$ we get

$$\gamma\left(\operatorname{Ext}_{\Lambda}^{t*}({}_{\psi^*}(\Omega^1_{\Lambda^e}(K_\eta) \otimes_\Lambda M), \Lambda/\mathfrak{r})\right) = \gamma\left(\operatorname{Ext}_{\Lambda}^{t*}({}_{\psi^*}M, \Lambda/\mathfrak{r})\right) - 1.$$

Therefore the equality $\operatorname{cx}^t\left(\Omega^1_{\Lambda^e}(K_\eta) \otimes_\Lambda M\right) = \operatorname{cx}^t M - 1$ holds, and so from Proposition 4.5 we conclude that $\dim \operatorname{V}_H^{\psi}(\Omega^1_{\Lambda^e}(K_\eta) \otimes_\Lambda M) = \dim \operatorname{V}_H^{\psi}(M) - 1$. $\square$

Finally we turn to the setting in which $\mathbf{Fg}(X, H, \psi, t)$ holds for *all* Λ-modules X, and derive two corollaries from Proposition 4.8. Observe first that if $\mathbf{Fg}(\Lambda/\mathfrak{r}, H, \psi, t)$ holds, then $\mathbf{Fg}(S, H, \psi, t)$ holds for every simple Λ-module S, and so by induction on the length of a module we see that $\mathbf{Fg}(X, H, \psi, t)$ holds for every Λ-module X; namely, if $\ell(X) \geq 2$, choose a submodule $Y \subset X$ such that $\ell(Y) = \ell(X) - 1$. The exact sequence

$$0 \to Y \to X \to X/Y \to 0$$

induces the exact sequence

$$\operatorname{Ext}_{\Lambda}^{t*}({}_{\psi^*}(X/Y), \Lambda/\mathfrak{r}) \to \operatorname{Ext}_{\Lambda}^{t*}({}_{\psi^*}X, \Lambda/\mathfrak{r}) \to \operatorname{Ext}_{\Lambda}^{t*}({}_{\psi^*}Y, \Lambda/\mathfrak{r})$$

of H-modules, and since the end terms are finite over H, so is the middle term.

Corollary 4.9. *If* $\mathbf{Fg}(\Lambda / \mathfrak{r}, H, \psi, t)$ *holds and* $\dim \mathrm{V}_H^{\psi}(M) = d > 0$*, then there exist homogeneous elements* $\eta_1, \dots, \eta_{d-1} \in H$ *of positive degrees such that the module*

$$\Omega^1_{\Lambda^e}(K_{\eta_{d-1}}) \otimes_\Lambda \cdots \otimes_\Lambda \Omega^1_{\Lambda^e}(K_{\eta_1}) \otimes_\Lambda M$$

is eventually ψ^i*-periodic for some* $i \geq 1$.

Proof. This is a direct consequence of Proposition 4.6 and Proposition 4.8. □

Corollary 4.10. *Suppose* k *is a field and* Λ *is a Frobenius algebra, and let* $\Lambda \xrightarrow{\nu} \Lambda$ *be a Nakayama automorphism. If* $\mathbf{Fg}(\Lambda / \mathfrak{r}, H, \nu^n, 2n)$ *holds for some* $n \geq 1$ *and* $\dim \mathrm{V}_H^{\nu}(M) = d > 0$*, then there exist homogeneous elements* $\eta_1, \dots, \eta_{d-1} \in H$ *of positive degrees such that every nonzero nonprojective indecomposable summand of*

$$\Omega^1_{\Lambda^e}(K_{\eta_{d-1}}) \otimes_\Lambda \cdots \otimes_\Lambda \Omega^1_{\Lambda^e}(K_{\eta_1}) \otimes_\Lambda M$$

is τ*-periodic.*

Proof. This is a direct consequence of Proposition 4.7 and Proposition 4.8. □

Acknowledgement

I would like to thank my supervisor Øyvind Solberg for valuable suggestions and comments on this paper.

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IV.

MODULES WITH REDUCIBLE COMPLEXITY

ABSTRACT

For a commutative Noetherian local ring we define and study the class of modules having reducible complexity, a class containing all modules of finite complete intersection dimension. Various properties of this class of modules are given, together with results on the vanishing of homology and cohomology.

This paper is to appear in J. Algebra.

MODULES WITH REDUCIBLE COMPLEXITY

PETTER ANDREAS BERGH

1. Introduction

The complexity of a finitely generated module over a commutative Noetherian local ring is a measure on a polynomial scale of the growth of the ranks of the free modules in its minimal free resolution. Over a local complete intersection every finitely generated module has finite complexity. In fact, it follows from [Gul, Theorem 2.3] that this property characterizes a local complete intersection, and that it is equivalent to the complexity of the residue field being finite.

In [AGP] a certain finiteness condition was defined under which a module *behaves* homologically like a module over a complete intersection. Namely, a module M over a commutative Noetherian local ring A has finite *complete intersection dimension* if there exist local rings R and Q and a diagram $A \to R \twoheadleftarrow Q$ of local homomorphisms (called a *quasi-deformation* of A) such that $A \to R$ is faithfully flat, $R \twoheadleftarrow Q$ is surjective with kernel generated by a regular sequence, and $\operatorname{pd}_Q(R \otimes_A M)$ is finite. There is of course a reason behind the choice of terminology; it was shown in [AGP] that a local ring is a complete intersection if and only if all its finitely generated modules have finite complete intersection dimension, and that this is equivalent to the finiteness of the complete intersection dimension of the residue field. Moreover, it was shown that if the projective dimension of a module is finite, then it is equal to the complete intersection dimension of the module.

We shall study a class of modules whose complexity is "reducible" (defined in the next section), a class which contains all modules of finite complete intersection dimension. In particular, we investigate for this class of modules the vanishing of homology and cohomology, and generalize several of the results in [ArY], [ChI], [Jo1] and [Jo2].

2. reducible complexity

Throughout this paper we let $(A, \mathfrak{m}, k)$ be a commutative Noetherian local ring, and we suppose all modules are finitely generated. We fix a finitely generated nonzero A-module M with minimal free resolution

$$\mathbf{F}_M \colon \cdots \to F_2 \to F_1 \to F_0 \to M \to 0.$$

The rank of F_n, i.e. the integer $\dim_k \operatorname{Ext}_A^n(M, k)$, is the n'th *Betti number* of M, and we denote this by $\beta_n(M)$. The *complexity* of M, denoted $\operatorname{cx} M$, is defined as

$$\operatorname{cx} M = \inf\{t \in \mathbb{N}_0 \mid \exists a \in \mathbb{R} \text{ such that } \beta_n(M) \leq an^{t-1} \text{ for } n \gg 0\},$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In general the complexity of a module may be infinite, whereas it is zero if and only if the module has finite projective dimension.

We now give the main definition, which is inductive and defines the class of modules having reducible complexity as a subcategory of the category of (finitely generated) A-modules having finite complexity. However, before stating the definition, recall that the n'th *syzygy* of an A-module X with minimal free resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to X \to 0$$

is the cokernel of $P_{n+1} \to P_n$ and denoted by $\Omega_A^n(X)$ (note that $\Omega_A^0(X) = X$), and it is unique up to isomorphism. For an A-module Y and a homogeneous element $\eta \in \operatorname{Ext}_A^*(X,Y)$, choose a map $f_\eta \colon \Omega_A^{|\eta|}(X) \to Y$ representing η, and denote by K_η the pushout of this map and the inclusion $\Omega_A^{|\eta|}(X) \hookrightarrow P_{|\eta|-1}$. As parallel maps in a pushout diagram have isomorphic cokernels, we obtain an exact sequence

$$0 \to Y \to K_\eta \to \Omega_A^{|\eta|-1}(X) \to 0.$$

Note that the middle term K_η is independent (up to isomorphism) of the map f_η chosen to represent η.

Definition 2.1. *Let $\mathcal{C}_A$ denote the category of all A-modules having finite complexity. The subcategory $\mathcal{C}_A^r \subseteq \mathcal{C}_A$ of modules having* reducible complexity *is defined inductively as follows:*

(i) *Every module of finite projective dimension belongs to $\mathcal{C}_A^r$.*
(ii) *A module $X \in \mathcal{C}_A$ with $\operatorname{cx} X > 0$ belongs to $\mathcal{C}_A^r$ if there exists a homogeneous element $\eta \in \operatorname{Ext}_A^*(X,X)$ of positive degree such that $\operatorname{cx} K_\eta < \operatorname{cx} X, \operatorname{depth} K_\eta = \operatorname{depth} X$ and $K_\eta \in \mathcal{C}_A^r$. We say that the element η* reduces the complexity *of M.*

Remark. The condition $\operatorname{depth} K_\eta = \operatorname{depth} X$ is not very strong; suppose for example that $\operatorname{depth} \Omega_A^i(X) \leq \operatorname{depth} A$ for all i (this always happens when A is Cohen-Macaulay). Then we must have $\operatorname{depth} X \leq \operatorname{depth} \Omega_A^{|\eta|-1}(X)$, implying the equality $\operatorname{depth} K_\eta = \operatorname{depth} X$.

Note the trivial fact that if every A-module has reducible complexity, then A must be a complete intersection since then by definition every module has finite complexity. The following result shows that the converse is also true, in fact every module of finite complete intersection dimension has reducible complexity. Moreover, if A is Cohen-Macaulay and M has reducible complexity, then so does any syzygy of M.

Proposition 2.2. (i) *Every module of finite complete intersection dimension has reducible complexity.*
(ii) *If A is Cohen-Macaulay and M has reducible complexity, then so does the kernel of any surjective map $F \twoheadrightarrow M$ when F is free. In particular, any syzygy of M has reducible complexity.*
(iii) *if B is a local ring and $A \to B$ a faithfully flat local homomorphism, then if M has reducible complexity, so does the B-module $B \otimes_A M$.*

Proof. (i) If M has finite complete intersection dimension, then from [AGP, Proposition 5.2] it follows that the complexity of M is finite. We argue by induction on $\operatorname{cx} M$ that M has reducible complexity, the case $\operatorname{cx} M = 0$ following from the definition. Suppose therefore that the complexity of M is nonzero. By [AGP, Proposition 7.2] there exists an eventually surjective chain map of degree $-n$ (where $n > 0$) on the minimal free resolution of M. This chain map corresponds to an element η of $\operatorname{Ext}_A^n(M,M)$, giving the exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega_A^n(M) \arrow[r] \arrow[d, "f_\eta"] & F_{n-1} \arrow[r] \arrow[d] & \Omega_A^{n-1}(M) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] & K_\eta \arrow[r] & \Omega_A^{n-1}(M) \arrow[r] & 0
\end{tikzcd}$$

of A-modules.

Now consider the exact sequence involving K_η. Since M and $\Omega_A^{n-1}(M)$ both have finite complete intersection dimension, then so does K_η. The exact sequence gives rise to a long exact sequence

$$\cdots \to \operatorname{Ext}_A^i(K_\eta, k) \to \operatorname{Ext}_A^i(M,k) \xrightarrow{\partial_\eta} \operatorname{Ext}_A^{i+n}(M,k) \to \operatorname{Ext}_A^{i+1}(K_\eta,k) \to \cdots,$$

where $\operatorname{Ext}_A^i(\Omega_A^{n-1}(M),k)$ has been identified with $\operatorname{Ext}_A^{i+n-1}(M,k)$. It follows from [Mac, Theorem III.9.1] that the connecting homomorphism $\partial_\eta \colon \operatorname{Ext}_A^i(M,k) \to \operatorname{Ext}_A^{i+n}(M,k)$ is then scalar multiplication by $(-1)^i\eta$ (we think of $\operatorname{Ext}_A^*(M,k)$ as a graded right module over the graded ring $\operatorname{Ext}_A^*(M,M)$), and reversing the arguments in the proof of [Ber, Proposition 2.2] shows that this is an injective map for $i \gg 0$. Consequently the equality $\beta_{i+1}(K_\eta) = \beta_{i+n}(M) - \beta_i(M)$ holds for large values of i. By [AGP, Theorem 5.3] the complexities $\operatorname{cx} M$ and $\operatorname{cx} K_\eta$ equal the orders of the poles at $t = 1$ of the Poincaré series $\sum \beta_n(M)t^n$ and $\sum \beta_n(K_\eta)t^n$, respectively, thus $\operatorname{cx} K_\eta = \operatorname{cx} M - 1$.

It remains only to show that $\operatorname{depth} K_\eta = \operatorname{depth} M$, but this is easy; from [AGP, Theorem 1.4] we have $0 \le \text{CI-dim}\, \Omega_A^i(M) = \operatorname{depth} A - \operatorname{depth} \Omega_A^i(M)$ for each $i \ge 0$, and this implies the inequality $\operatorname{depth} \Omega_A^i(M) \le \operatorname{depth} \Omega_A^{i+1}(M)$. In particular we have $\operatorname{depth} M \le \operatorname{depth} \Omega_A^{n-1}(M)$, and therefore $\operatorname{depth} K_\eta$ must equal $\operatorname{depth} M$.

(ii) Let L denote the kernel of the surjective map $F \twoheadrightarrow M$. Again we argue by induction on $\operatorname{cx} M$. If the projective dimension of M is finite, then so is the projective dimension of L, and we are done. Suppose therefore $\operatorname{cx} M$ is nonzero, and let $\eta \in \operatorname{Ext}_A^*(M,M)$ be an element reducing the complexity of M. By the Horseshoe Lemma we have an exact commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L \arrow[r] \arrow[d] & K \oplus Q \arrow[r] \arrow[d] & \Omega_A^{|\eta|-1}(L) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F \arrow[r] \arrow[d] & F' \oplus Q \arrow[r] \arrow[d] & F'' \oplus Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[d] & K_\eta \arrow[r] \arrow[d] & \Omega_A^{|\eta|-1}(M) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where F' and F'' are free modules, and Q is a free module such that $\Omega_A^{|\eta|}(M) \oplus Q \simeq \Omega_A^{|\eta|-1}(L)$. If M and K_η are maximal Cohen-Macaulay, then so are L and $K \oplus Q$, and if not then $\operatorname{depth} L = \operatorname{depth} M + 1$ and $\operatorname{depth}(K \oplus Q) = \operatorname{depth} K_\eta + 1$. In either case we see that $\operatorname{depth} L$ equals $\operatorname{depth}(K \oplus Q)$. Moreover, we have $\operatorname{cx}(K \oplus Q) = \operatorname{cx} K_\eta < \operatorname{cx} M = \operatorname{cx} L$, and so by induction we are done.

Note that the upper horizontal short exact sequence in the above diagram corresponds to an element θ in $\operatorname{Ext}_A^{|\eta|}(L,L)$ reducing the complexity of L. A map $f_\theta \colon \Omega_A^{|\eta|}(L) \to L$ representing θ is obtained by lifting a map f_η representing η along the minimal free resolution of M, as in the diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega_A^{|\eta|+1}(M) \arrow[r] \arrow[d, dotted, "\Omega_A(f_\eta)"] & F_{|\eta|} \arrow[r] \arrow[d, dotted] & \Omega_A^{|\eta|}(M) \arrow[r] \arrow[d, "f_\eta"] & 0 \\
0 \arrow[r] & L \arrow[r] & F \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

In this way we obtain a map $\Omega_A(f_\eta)\colon \Omega_A^{|\eta|+1}(M) \to L$, and adding to $\Omega_A^{|\eta|+1}(M)$ a free module Q' such that $\Omega_A^{|\eta|+1}(M) \oplus Q' \simeq \Omega_A^{|\eta|}(L)$, we obtain a map $\Omega_A^{|\eta|}(L) \to L$ representing θ.

(iii) If X is any A-module with a minimal free resolution $\mathbf{F}_X$, then the complex $B \otimes_A \mathbf{F}_X$ is a minimal free B-resolution of $B \otimes_A X$, giving the equality $\operatorname{cx}_A X = \operatorname{cx}_B(B \otimes_A X)$. Moreover, if Y is another A-module then from [Mat, Theorem 23.3] we have $\operatorname{depth}_A X - \operatorname{depth}_A Y = \operatorname{depth}_B(B \otimes_A X) - \operatorname{depth}_B(B \otimes_A Y)$. Hence if $\eta \in \operatorname{Ext}_A^*(M,M)$ reduces the complexity of M, the element $B \otimes_A \eta \in \operatorname{Ext}_B^*(B \otimes_A M, B \otimes_A M)$ reduces the complexity of $B \otimes_A M$. $\square$

Thus the class of modules having finite complete intersection dimension is contained in the class of modules having reducible complexity. However, the following example shows that the inclusion is strict in general; there are a lot of modules having reducible complexity but whose complete intersection dimension is infinite.

Example. Suppose M is periodic of period $p \geq 3$ (i.e. $\Omega_A^p(M)$ is isomorphic to M), and that $\operatorname{depth} M \leq \operatorname{depth} \Omega_A^{p-1}(M)$. Then we have an exact sequence

$$0 \to M \to F_{p-1} \to \Omega_A^{p-1}(M) \to 0,$$

and we have $0 = \operatorname{cx} F_{p-1} = \operatorname{cx} M - 1$ and $\operatorname{depth} F_{p-1} = \operatorname{depth} M$. Therefore M has reducible complexity, and it cannot be of finite complete intersection dimension since then by [AGP, Theorem 7.3] the period would have been two.

An example of such a module was given in [GaP, Section3]; let $(A, \mathfrak{m}, k)$ be the commutative local finite dimensional k-algebra $k[x_1, x_2, x_3, x_4]/\,\mathfrak{a}$, where $\mathfrak{a}$ is the ideal generated by the quadratic forms

$$x_1^2, \quad x_2^2, \quad x_3^2, \quad x_4^2, \quad x_3x_4, \quad x_1x_4 + x_2x_4, \quad \alpha x_1x_3 + x_2x_3$$

for a nonzero element $\alpha \in k$. The complex

$$\cdots \to A^2 \xrightarrow{d_2} A^2 \xrightarrow{d_1} A^2 \xrightarrow{d_0} M \to 0,$$

where the maps are given by the matrices

$$d_n = \begin{pmatrix} x_1 & \alpha^n x_3 + x_4 \\ 0 & x_2 \end{pmatrix},$$

is a minimal free resolution of the module $M := \operatorname{Im} d_0$, and so if α has finite order p we see that M is periodic of period p.

It is worth mentioning that there do exist examples of modules over Gorenstein rings whose complete intersection dimension is not finite but which have reducible complexity (see [GaP, Proposition 3.1] for an example similar to that above).

Now let X and Y be arbitrary A-modules, and $\theta_1 \in \operatorname{Ext}_A^*(X,X)$ and $\theta_2 \in \operatorname{Ext}_A^*(X,Y)$ two homogeneous elements. The following lemma, motivated by [EHSST, Lemma 7.2], links K_{θ_1} and K_{θ_2} to $K_{\theta_2\theta_1}$, and will be a key ingredient in several of the forthcoming results.

Lemma 2.3. *If $\theta_1 \in \operatorname{Ext}_A^*(X,X)$ and $\theta_2 \in \operatorname{Ext}_A^*(X,Y)$ are two homogeneous elements, then there exists an exact sequence*

$$0 \to \Omega_A^{|\theta_2|}(K_{\theta_1}) \to K_{\theta_2\theta_1} \oplus F \to K_{\theta_2} \to 0$$

of A-modules, where F is free.

Proof. Denote $|\theta_i|$ by n_i for $i = 1, 2$. The element θ_1 gives rise to an exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega_A^{n_1}(X) \arrow[r, "i"] \arrow[d, "f_{\theta_1}"] & Q_{n_1-1} \arrow[r, "\pi"] \arrow[d, "h"] & \Omega_A^{n_1-1}(X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r, "j"] & K_{\theta_1} \arrow[r] & \Omega_A^{n_1-1}(X) \arrow[r] & 0
\end{tikzcd}$$

where Q_n denotes the n'th module in the minimal free resolution of X. Letting $Q \xrightarrow{g} X$ be a surjection, where Q is free, we can modify the diagram and obtain

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \operatorname{Ker}(h, jg) \arrow[r, equal] \arrow[d] & \operatorname{Ker}(h, jg) \arrow[d] & & \\
0 \arrow[r] & \Omega_A^{n_1}(X) \oplus Q \arrow[r, "\left(\begin{smallmatrix} i & 0 \\ 0 & 1 \end{smallmatrix}\right)"] \arrow[d, two heads, "(f_{\theta_1}, g)"] & Q_{n_1-1} \oplus Q \arrow[r, "(\pi, 0)"] \arrow[d, two heads, "(h, jg)"] & \Omega_A^{n_1-1}(X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r, "j"] \arrow[d] & K_{\theta_1} \arrow[r] \arrow[d] & \Omega_A^{n_1-1}(X) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Since $\operatorname{Ker}(h, jg)$ is isomorphic to $\Omega_A^1(K_{\theta_1}) \oplus Q'$ for some free module Q', the left vertical exact sequence yields an exact sequence

$$0 \to \Omega_A^1(K_{\theta_1}) \oplus Q' \to \Omega_A^{n_1}(X) \oplus Q \xrightarrow{(f_{\theta_1}, g)} X \to 0$$

on which we can apply the Horseshoe Lemma and obtain an exact sequence

$$\mu \colon 0 \to \Omega_A^{n_2}(K_{\theta_1}) \to \Omega_A^{n_1+n_2-1}(X) \oplus F \xrightarrow{(\Omega_A^{n_2-1}(f_{\theta_1}), s)} \Omega_A^{n_2-1}(X) \to 0,$$

where F is free and $F \xrightarrow{s} \Omega_A^{n_2-1}(X)$ is a map.

The definition of cohomological products and the pushout properties of $K_{\theta_2\theta_1}$ give a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & K_{\theta_2\theta_1} \arrow[r, "w"] \arrow[d, "t"] & \Omega_A^{n_1+n_2-1}(X) \arrow[r] \arrow[d, "\Omega_A^{n_2-1}(f_{\theta_1})"] & 0 \\
0 \arrow[r] & Y \arrow[r] & K_{\theta_2} \arrow[r, "v"] & \Omega_A^{n_2-1}(X) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Adding F to $K_{\theta_2\theta_1}$ and $\Omega_A^{n_1+n_2-1}(X)$ in the right-most square, we obtain the exact commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \Omega_A^{n_2}(K_{\theta_1}) \arrow[r, equal] \arrow[d] & \Omega_A^{n_2}(K_{\theta_1}) \arrow[d] & \\
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & K_{\theta_2\theta_1} \oplus F \arrow[r, "\left(\begin{smallmatrix} w & 0 \\ 0 & 1 \end{smallmatrix}\right)"] \arrow[d, "(t,r)"] & \Omega_A^{n_1+n_2-1}(X) \oplus F \arrow[r] \arrow[d, "(\Omega_A^{n_2-1}(f_{\theta_1}),s)"] & 0 \\
0 \arrow[r] & Y \arrow[r] & K_{\theta_2} \arrow[r, "v"] \arrow[d] & \Omega_A^{n_2-1}(X) \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

in which the right vertical exact sequence is μ and $F \xrightarrow{r} K_{\theta_2}$ is a map with the property that $s = vr$. The left vertical exact sequence is of the form we are seeking. $\square$

We end this section with two results on modules over complete intersections. Recall that a maximal Cohen-Macaulay (or "MCM" from now on) *approximation* of an A-module X is an exact sequence

$$0 \to Y_X \to C_X \to X \to 0$$

where C_X is MCM and Y_X has finite injective dimension, and that a *hull of finite injective dimension* of X is an exact sequence

$$0 \to X \to Y^X \to C^X \to 0$$

where C^X is MCM and Y^X has finite injective dimension. These notions were introduced in [AuB], where it was shown that every finitely generated module over a commutative Noetherian ring admitting a dualizing module has a MCM approximation and a hull of finite injective dimension. The following result provides a simple proof of the complete intersection case, using a technique similar to the proof of the main result in [Bak] and the fact that over a complete intersection every module has reducible complexity.

Proposition 2.4. *Suppose A is a complete intersection.*

(i) *If $\eta \in \operatorname{Ext}_A^{|\eta|}(M,M)$ reduces the complexity of M, then so does η^t for $t \geq 1$.*
(ii) *Every A-module has a MCM approximation and a hull of finite injective dimension.*

Proof. (i) Using Lemma 2.3 it is easily proved by induction on t that $\operatorname{cx} K_{\eta^t} \leq \operatorname{cx} K_\eta$.

(ii) Fix an exact sequence

$$0 \to Y \to C \to M \to 0$$

where C is MCM (the minimal free cover of M, for example). If the complexity of Y is nonzero, let $\eta \in \operatorname{Ext}_A^{|\eta|}(Y,Y)$ be an element reducing it, and choose an integer $t \geq 1$ with the property that $\Omega_A^{t|\eta|-1}(Y)$ is MCM. The element η^t is given by the exact sequence

$$0 \to Y \to K_{\eta^t} \to \Omega_A^{t|\eta|-1}(Y) \to 0,$$

and by (i) it also reduces the complexity of Y. From the pushout diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & Y \arrow[r] \arrow[d] & K_{\eta^t} \arrow[r] \arrow[d] & \Omega_A^{t|\eta|-1}(Y) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & \Omega_A^{t|\eta|-1}(Y) \arrow[r] & 0 \\
 & M \arrow[r, equal] \arrow[d] & M \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

we obtain the exact sequence

$$0 \to Y' \to C' \to M \to 0$$

(with $Y' = K_{\eta^t}$), where C' is MCM and $\operatorname{cx} Y' < \operatorname{cx} Y$. Repeating the process we eventually obtain a MCM approximation of M, since over a Gorenstein ring a module has finite injective dimension precisely when it has finite projective dimension.

As for a hull of finite injective dimension, fix an exact sequence

$$0 \to M \to Y \to C \to 0$$

where C is MCM (obtained for example from a suitable power of an element in $\operatorname{Ext}_A^*(M, M)$ reducing the complexity of M). If the complexity of Y is nonzero, choose as above an element $\eta \in \operatorname{Ext}_A^{|\eta|}(Y, Y)$ reducing it, and let $t \geq 1$ be an integer such that $\Omega_A^{t|\eta|-1}(Y)$ is MCM. From the pushout diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & M \arrow[r, equal] \arrow[d] & M \arrow[d] & & \\
0 \arrow[r] & Y \arrow[r] \arrow[d] & K_{\eta^t} \arrow[r] \arrow[d] & \Omega_A^{t|\eta|-1}(Y) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & \Omega_A^{t|\eta|-1}(Y) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

we obtain the exact sequence

$$0 \to M \to Y' \to C' \to 0$$

(with $Y' = K_{\eta^t}$), where C' is MCM and $\operatorname{cx} Y' < \operatorname{cx} Y$. Repeating the process we eventually obtain a hull of finite injective dimension of M. $\square$

3. Vanishing results

This section investigates the vanishing of cohomology and homology for a module having reducible complexity, and *the following is assumed throughout*:

Assumption. The module M has reducible complexity, and N is a nonzero A-module. If $\operatorname{cx} M > 0$, then there exist A-modules $K_1, \dots, K_c$ and a set of cohomological elements $\{\eta_i \in \operatorname{Ext}_A^{|\eta_i|}(K_{i-1}, K_{i-1})\}_{i=1}^c$ given by exact sequences

$$0 \to K_{i-1} \to K_i \to \Omega_A^{|\eta_i|-1}(K_{i-1}) \to 0$$

for $i = 1, \dots, c$ (where $K_0 = M$), satisfying $\operatorname{depth} K_i = \operatorname{depth} M, \operatorname{cx} K_i < \operatorname{cx} K_{i-1}$ and $\operatorname{cx} K_c = 0$ (such elements η_i must exist by Definition 2.1).

For an A-module N, we define $\mathrm{q}^A(M,N)$ and $\mathrm{p}^A(M,N)$ by

$$\begin{aligned} \mathrm{q}^A(M,N) &= \sup\{n \mid \operatorname{Tor}_n^A(M,N) \neq 0\}, \\ \mathrm{p}^A(M,N) &= \sup\{n \mid \operatorname{Ext}_A^n(M,N) \neq 0\}. \end{aligned}$$

The definition of modules having reducible complexity suggests that when proving results about $\mathrm{q}^A(M,N)$ and $\mathrm{p}^A(M,N)$, we use induction on the complexity of M.

The first result and its corollary (which considers a conjecture of Auslander and Reiten) consider the vanishing of cohomology, and generalize [ArY, Theorem 4.2 and Theorem 4.3].

Theorem 3.1. *The following are equivalent.*

(i) *There exists an integer* $t > \operatorname{depth} A - \operatorname{depth} M$ *such that* $\operatorname{Ext}_A^{t+i}(M,N) = 0$ *for* $0 \leq i \leq |\eta_1| + \cdots + |\eta_c| - c$.

(ii) $\mathrm{p}^A(M,N) < \infty$.

(iii) $\mathrm{p}^A(M,N) = \operatorname{depth} A - \operatorname{depth} M$.

Proof. We only need to show the implication (i) $\Rightarrow$ (iii), and we do this by induction on $\operatorname{cx} M$. If the projective dimension of M is finite, then by the Auslander-Buchsbaum formula it is equal to $\operatorname{depth} A - \operatorname{depth} M$. Since N is finitely generated, we have $N \neq \mathfrak{m} N$ by Nakayama's Lemma, hence there exists a nonzero element $x \in N \setminus \mathfrak{m} N$. The map $A \to N$ defined by $1 \mapsto x$ then gives rise to a nonzero element of $\operatorname{Ext}_A^{\operatorname{pd} M}(M,N)$, and therefore $\mathrm{p}^A(M,N) = \operatorname{depth} A - \operatorname{depth} M$.

Now suppose the complexity of M is nonzero, and consider the exact sequence

$$(\dagger) \qquad 0 \to M \to K_1 \to \Omega_A^{|\eta_1|-1}(M) \to 0.$$

The vanishing interval for $\operatorname{Ext}_A^i(M,N)$ implies that $\operatorname{Ext}_A^{t+i}(K_1,N) = 0$ for $0 \leq i \leq |\eta_2| + \cdots + |\eta_c| - (c-1)$, and so by induction $\mathrm{p}^A(K_1,N) = \operatorname{depth} A - \operatorname{depth} K_1$. Since we have equalities $\mathrm{p}^A(M,N) = \mathrm{p}^A(K_1,N)$ and $\operatorname{depth} M = \operatorname{depth} K_1$, we are done. $\square$

Corollary 3.2. $\mathrm{p}^A(M,M) = \operatorname{pd} M$.

Proof. Suppose $\mathrm{p}^A(M,M) < \infty$. If the projective dimension of M is not finite, i.e. if $\operatorname{cx} M > 0$, then consider the exact sequence $(\dagger)$ representing η_1 (from the proof of Theorem 3.1), where $K_1 = K_{\eta_1}$. Since η_1 is nilpotent there is an integer t such that $\eta_1^t = 0$, and therefore $\operatorname{cx} K_{\eta_1^t} = \operatorname{cx} M$. But using Lemma 2.3 we see that $\operatorname{cx} K_{\eta_1^i} \leq \operatorname{cx} K_{\eta_1}$ for all $i \geq 1$, and since $\operatorname{cx} K_{\eta_1} < \operatorname{cx} M$ we have reached a contradiction. Therefore the projective dimension of M is finite and equal to $\operatorname{depth} A - \operatorname{depth} M$ by the Auslander-Buchsbaum formula, and from Theorem 3.1 we see that $\mathrm{p}^A(M,M) = \operatorname{pd} M$. $\square$

The next result is a homology version of Theorem 3.1, and it is closely related to [Jo1, Theorem 2.1].

Theorem 3.3. *The following are equivalent.*

(i) *There exists an integer* $t > \operatorname{depth} A - \operatorname{depth} M$ *such that* $\operatorname{Tor}_{t+i}^A(M,N) = 0$ *for* $0 \leq i \leq |\eta_1| + \cdots + |\eta_c| - c$.
(ii) $\mathrm{q}^A(M,N) < \infty$.
(iii) $\operatorname{depth} A - \operatorname{depth} M - \operatorname{depth} N \leq \mathrm{q}^A(M,N) \leq \operatorname{depth} A - \operatorname{depth} M$.

Proof. We only need to show the implication (i) $\Rightarrow$ (iii), and we do this by induction on $\operatorname{cx} M$. The case $\operatorname{pd} M < \infty$ follows from the Auslander-Buchsbaum formula and [ChI, Remark 8], so suppose therefore $\operatorname{cx} M > 0$, and consider the exact sequence (†) from the proof of Theorem 3.1. Since $\operatorname{Tor}_{t+i}^A(K_1,N) = 0$ for $0 \leq i \leq |\eta_2| + \cdots + |\eta_c| - (c-1)$, we get by induction that the inequalities hold for K_1 and N. But as in the previous proof we have $\mathrm{q}^A(M,N) = \mathrm{q}^A(K_1,N)$ and $\operatorname{depth} M = \operatorname{depth} K_1$, hence the inequalities hold for M and N. $\square$

The following result contains half of [ChI, Theorem 3] (and a version of the first half of [ArY, Theorem 2.5]) and the main result from [Jo2] for Cohen-Macaulay rings, which says that the integer $\mathrm{q}^A(M,N)$ can be computed locally. It also establishes the *depth formula* provided N is maximal Cohen-Macaulay.

Theorem 3.4. (i) *If* $\mathrm{q}^A(M,N)$ *is finite and* $\operatorname{depth} \operatorname{Tor}_{\mathrm{q}^A(M,N)}^A(M,N) = 0$, *then*

$$\mathrm{q}^A(M,N) = \operatorname{depth} A - \operatorname{depth} M - \operatorname{depth} N.$$

(ii) *If A is Cohen-Macaulay and* $\mathrm{q}^A(M,N)$ *is finite, then the equality*

$$\mathrm{q}^A(M,N) = \sup\{\operatorname{ht} \mathfrak{p} - \operatorname{depth} M_{\mathfrak{p}} - \operatorname{depth} N_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} A\}$$

holds.

(iii) *If* $\mathrm{q}^A(M,N) = 0$ *and N is maximal Cohen-Macaulay, then the depth formula holds for M and N, i.e.*

$$\operatorname{depth} M + \operatorname{depth} N = \operatorname{depth} A + \operatorname{depth}(M \otimes N).$$

Proof. (i) We argue by induction on $\operatorname{cx} M$, the case $\operatorname{pd} M < \infty$ following from [Aus, Theorem 1.2] and the Auslander-Buchsbaum formula. Suppose therefore that the complexity of M is nonzero, and consider the exact sequence (†) from the proof of Theorem 3.1. Since $\operatorname{Tor}_{\mathrm{q}^A(M,N)}^A(M,N)$ is a submodule of $\operatorname{Tor}_{\mathrm{q}^A(M,N)}^A(K_1,N)$, the latter is also of depth zero, hence by induction and the equalities $\mathrm{q}^A(K_1,N) = \mathrm{q}^A(M,N), \operatorname{depth} K_1 = \operatorname{depth} M$ we are done.

(ii) Suppose $\mathrm{q}^A(M,N)$ is finite, and let $\mathfrak{p} \subseteq A$ be a prime ideal. If M has finite projective dimension, so has $M_{\mathfrak{p}}$, and from Theorem 3.3 we get $\mathrm{q}^{A_{\mathfrak{p}}}(M_{\mathfrak{p}},N_{\mathfrak{p}}) \geq \operatorname{ht} \mathfrak{p} - \operatorname{depth} M_{\mathfrak{p}} - \operatorname{depth} N_{\mathfrak{p}}$. If $\operatorname{cx} M > 0$, consider the exact sequences

$$0 \to K_{i-1} \to K_i \to \Omega_A^{|\eta_i|-1}(K_{i-1}) \to 0$$

for $i = 1, \dots, c$ (where $K_0 = M$), satisfying $\operatorname{depth} K_i = \operatorname{depth} M, \operatorname{cx} K_i < \operatorname{cx} K_{i-1}$ and $\operatorname{cx} K_c = 0$. Localizing at $\mathfrak{p}$, we see that $\operatorname{depth}(K_i)_{\mathfrak{p}} = \operatorname{depth} M_{\mathfrak{p}}$ (as in the remark following Definition 2.1), and that $\mathrm{q}^{A_{\mathfrak{p}}}((K_i)_{\mathfrak{p}},N_{\mathfrak{p}}) = \mathrm{q}^{A_{\mathfrak{p}}}((K_{i-1})_{\mathfrak{p}},N_{\mathfrak{p}})$. As K_c has finite projective dimension we get

$$\begin{aligned} \mathrm{q}^{A_{\mathfrak{p}}}(M_{\mathfrak{p}},N_{\mathfrak{p}}) &= \mathrm{q}^{A_{\mathfrak{p}}}((K_c)_{\mathfrak{p}},N_{\mathfrak{p}}) \\ &\geq \operatorname{ht} \mathfrak{p} - \operatorname{depth}(K_c)_{\mathfrak{p}} - \operatorname{depth} N_{\mathfrak{p}} \\ &= \operatorname{ht} \mathfrak{p} - \operatorname{depth} M_{\mathfrak{p}} - \operatorname{depth} N_{\mathfrak{p}}, \end{aligned}$$

hence since $\mathrm{q}^A(M,N) \geq \mathrm{q}^{A_{\mathfrak{p}}}(M_{\mathfrak{p}},N_{\mathfrak{p}})$ the inequality

$$\mathrm{q}^A(M,N) \geq \sup\{\operatorname{ht} \mathfrak{p} - \operatorname{depth} M_{\mathfrak{p}} - \operatorname{depth} N_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} A\}$$

holds.

For the reverse inequality, choose any associated prime $\mathfrak{p}$ of $\operatorname{Tor}^A_{\mathrm{q}^A(M,N)}(M,N)$. Then $\mathrm{q}^A(M,N) = \mathrm{q}^{A_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p})$ and $\operatorname{depth} \operatorname{Tor}^{A_\mathfrak{p}}_{\mathrm{q}^{A_\mathfrak{p}}(M_\mathfrak{p},N_\mathfrak{p})}(M_\mathfrak{p}, N_\mathfrak{p}) = 0$, and a small adjustment of the proof of (i) above gives

$$\mathrm{q}^{A_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) = \operatorname{ht} \mathfrak{p} - \operatorname{depth} M_\mathfrak{p} - \operatorname{depth} N_\mathfrak{p}.$$

(iii) Again we argue by induction on $\operatorname{cx} M$, where the case $\operatorname{pd} M < \infty$ follows from [Aus, Theorem 1.2]. Suppose $\operatorname{cx} M > 0$, and consider the exact sequence (†). We have $\mathrm{q}^A(K_1, N) = 0$, hence by induction the depth formula holds for K_1 and N. Since $\operatorname{depth} K_1 = \operatorname{depth} M$, we only have to prove that the equality $\operatorname{depth}(M \otimes N) = \operatorname{depth}(K_1 \otimes N)$ holds. For each $i \geq 0$ we have an exact sequence

$$0 \to \Omega^{i+1}_A(M) \otimes N \to F_i \otimes N \to \Omega^i_A(M) \otimes N \to 0,$$

and as N is maximal Cohen-Macaulay we must have that $\operatorname{depth}\big(\Omega^i_A(M) \otimes N\big)$ is at most $\operatorname{depth}\big(\Omega^{i+1}_A(M) \otimes N\big)$. In particular the inequality $\operatorname{depth}(M \otimes N) \leq \operatorname{depth}\Big(\Omega^{|\eta_1|-1}_A(M) \otimes N\Big)$ holds, and therefore when tensoring the sequence (†) with N we see that $\operatorname{depth}(M \otimes N) = \operatorname{depth}(K_1 \otimes N)$. $\square$

The next result deals with symmetry in the vanishing of Ext. It was shown in [AvB] that if X and Y are modules over a complete intersection A, then $\operatorname{Ext}^i_A(X, Y) = 0$ for $i \gg 0$ if and only if $\operatorname{Ext}^i_A(Y, X) = 0$ for $i \gg 0$. This was generalized in [HuJ] to a class of local Gorenstein rings named "AB rings", a class properly containing the class of complete intersections. Another generalization appeared in [Jør], where techniques from the theory of derived categories were used to show that symmetry in the vanishing of Ext holds for modules of finite complete intersection dimension over local Gorenstein rings.

Theorem 3.5. *If A is Gorenstein then the implication*

$$\mathrm{p}^A(N, M) < \infty \Rightarrow \mathrm{p}^A(M, N) < \infty$$

holds. In particular, symmetry in the vanishing of Ext *holds for modules with reducible complexity over a local Gorenstein ring.*

Proof. Define the integer w (depending on M) by

$$w = \operatorname{depth} A - \operatorname{depth} M + |\eta_1| + \cdots + |\eta_c| - c.$$

If for some integer $i \geq 1$ we have $\operatorname{Tor}^A_i(M, N) = \cdots = \operatorname{Tor}^A_{i+w}(M, N) = 0$, then $\operatorname{Tor}^A_i(M, N) = 0$ for all $i \geq 1$ by Theorem 3.3. The result now follows from [Jør, Theorem 1.7 and Proposition 2.2]. $\square$

The final result deals with the vanishing of homology for two modules when A is a complete intersection. Namely, in this situation the homology modules are given as the homology modules of two modules of finite projective dimension, due to the fact that every module over a complete intersection has reducible complexity.

Theorem 3.6. *Suppose A is a complete intersection, and let X and Y be A-modules such that $\operatorname{Tor}^A_i(X, Y) = 0$ for $i \gg 0$. Then there exist A-modules X' and Y', both of finite projective dimension, such that $\operatorname{depth} X = \operatorname{depth} X', \operatorname{depth} Y = \operatorname{depth} Y'$ and $\operatorname{Tor}^A_i(X, Y) \simeq \operatorname{Tor}^A_i(X', Y')$ for $i > 0$.*

Proof. If the complexity of one of X and Y, say X, is nonzero, choose a homogeneous element $\eta \in \operatorname{Ext}^*_A(X, X)$ reducing the complexity. By Proposition 2.4(i) any power of η also reduces the complexity of X, so choose an integer t such that $\operatorname{Tor}^A_i(\Omega^{t|\eta|-1}_A(X), Y) = 0$ for $i > 0$. The element η^t is given by the short exact sequence

$$0 \to X \to K_{\eta^t} \to \Omega^{t|\eta|-1}_A(X) \to 0,$$

therefore by the choice of t we see that $\operatorname{Tor}_i^A(X,Y)$ and $\operatorname{Tor}_i^A(K_{\eta^t},Y)$ are isomorphic for $i>0$. Since A is Cohen-Macaulay we automatically have $\operatorname{depth} X = \operatorname{depth} K_{\eta^t}$, and repeating this process we eventually obtain what we want. $\square$

This result has consequences for the study of the rigidity of Tor over Noetherian local rings. This study was initiated by M. Auslander in his 1961 paper [Aus], in which he proved his famous rigidity theorem; if X and Y are modules over an unramified regular local ring R, and $\operatorname{Tor}_n^R(X,Y)=0$ for some $n \geq 1$, then $\operatorname{Tor}_i^R(X,Y)=0$ for all $i \geq n$ (recall that a regular local ring $(S,\mathfrak{m}_S)$ is said to be *ramified* if it is of characteristic zero while its residue class field has characteristic $p>0$ and p is an element of $\mathfrak{m}_S^2$). In 1966 S. Lichtenbaum extended Auslander's rigidity theorem to all regular local rings (see [Lic]), and subsequently Peskine and Szpiro conjectured in [PeS] that the theorem holds for all Noetherian local rings provided one of the modules in question has finite projective dimension. A counterexample to the conjecture was provided by Heitmann in [Hei], where a Cohen-Macaulay ring R together with R-modules X and Y were given, for which $\operatorname{pd} X = 2$ and $\operatorname{Tor}_1^R(X,Y)=0$, while $\operatorname{Tor}_2^R(X,Y) \neq 0$.

However, whether the rigidity of Tor holds for Noetherian local rings provided *both* modules involved have finite projective dimension is unknown. If this holds over complete intersections, then Theorem 3.6 shows that the conjecture of Peskine and Szpiro also holds for such rings (i.e. rigidity of Tor holds provided one of the modules involved has finite projective dimension). In fact, the theorem shows that if rigidity holds over a complete intersection R provided both modules have finite projective dimension, then rigidity holds over R for all modules X and Y satisfying $\operatorname{Tor}_i^R(X,Y)=0$ for $i \gg 0$.

4. A generalization

In this final section we discuss a situation which slightly generalizes the concept of reducible complexity. Instead of letting M have reducible complexity as in Definition 2.1, we make the following assumption:

Assumption. The complexity of M is finite, and if it is nonzero then there exist local rings $\{R_i\}_{i=1}^c$ such that for each $i \in \{1,\dots,c\}$ there is a faithfully flat local homomorphism $R_{i-1} \to R_i$ (where $R_0 = A$), an R_i-module K_i, an integer n_i and an exact sequence

$$0 \to R_i \otimes_{R_{i-1}} K_{i-1} \to K_i \to \Omega_{R_i}^{n_i}(R_i \otimes_{R_{i-1}} K_{i-1}) \to 0$$

(where $K_0 = M$) satisfying $\operatorname{depth}_{R_i} K_i = \operatorname{depth}_{R_i}(R_i \otimes_{R_{i-1}} K_{i-1}), \operatorname{cx}_{R_i} K_i < \operatorname{cx}_{R_{i-1}} K_{i-1}$ and $\operatorname{pd}_{R_c} K_c < \infty$.

Of course, if M has reducible complexity then by choosing each R_i to be A we see that the assumption is satisfied. Now let $S \to T$ be any faithfully flat local homomorphism, and X and Y any (finitely generated) S-modules. If $\mathbf{F}_X$ is a minimal S-free resolution of X, then the complex $T \otimes_S \mathbf{F}_X$ is a minimal T-free resolution of $T \otimes_S X$, and by [EGA, Proposition (2.5.8)] we have natural isomorphisms

$$\begin{aligned} \operatorname{Hom}_T(T \otimes_S \mathbf{F}_X, T \otimes_S Y) &\simeq T \otimes_S \operatorname{Hom}_S(\mathbf{F}_X, Y), \\ (T \otimes_S \mathbf{F}_X) \otimes_T (T \otimes_S Y) &\simeq T \otimes_S (\mathbf{F}_X \otimes_S Y). \end{aligned}$$

Therefore we have isomorphisms

$$\begin{aligned} \operatorname{Ext}_T^i(T \otimes_S X, T \otimes_S Y) &\simeq T \otimes_S \operatorname{Ext}_S^i(X,Y), \\ \operatorname{Tor}_i^T(T \otimes_S X, T \otimes_S Y) &\simeq T \otimes_S \operatorname{Tor}_i^S(X,Y), \end{aligned}$$

and as T is faithfully S-flat we then get

$$\begin{array}{rcl}\operatorname{Ext}^i_T(T\otimes_S X, T\otimes_S Y)=0 & \Leftrightarrow & \operatorname{Ext}^i_S(X,Y)=0,\\ \operatorname{Tor}^T_i(T\otimes_S X, T\otimes_S Y)=0 & \Leftrightarrow & \operatorname{Tor}^S_i(X,Y)=0.\end{array}$$

We then get the equalities

$$\begin{array}{rcl}\operatorname{cx}_S X & = & \operatorname{cx}_T(T\otimes_S X)\\ \operatorname{p}^S(X,Y) & = & \operatorname{p}^T(T\otimes_S X, T\otimes_S Y)\\ \operatorname{q}^S(X,Y) & = & \operatorname{q}^T(T\otimes_S X, T\otimes_S Y)\\ \operatorname{depth}_S X-\operatorname{depth}_S Y & = & \operatorname{depth}_T(T\otimes_S X)-\operatorname{depth}_T(T\otimes_S Y),\end{array}$$

where the one involving depth follows from [Mat, Theorem 23.3].

Using the above facts it is easy to see that both Theorem 3.1 and Corollary 3.2 remain true in this new situation, as does Theorem 3.3 if we drop the left inequality in (iii).

Suppose now that M has finite complete intersection dimension. Then [AGP, Proposition 7.2] and an argument similar to the proof of Proposition 2.2(i) show that M satisfies this new assumption *and that* $n_i=1$ *for each* $1\leq i\leq c$. Consequently, the vanishing intervals in Theorem 3.1(i) and Theorem 3.3(i) are of length $\operatorname{cx}_A M+1$, as in [AvB, Theorem 4.7] and [Jo1, Theorem 2.1]. Moreover, we obtain [AvB, Theorem 4.2], which says that M is of finite projective dimension if and only if $\operatorname{Ext}^{2n}_A(M,M)=0$ for some $n\geq 1$. To see this, note that when $\operatorname{cx}_A M>0$ the extension

$$0\to R_1\otimes_A M\to K_1\to \Omega^1_{R_1}(R_1\otimes_A M)\to 0$$

corresponds to an element $\theta\in \operatorname{Ext}^2_{R_1}(R_1\otimes_A M, R_1\otimes_A M)$. If $\operatorname{Ext}^{2n}_A(M,M)=0$ for some $n\geq 1$, then $\operatorname{Ext}^{2n}_{R_1}(R_1\otimes_A M, R_1\otimes_A M)$ also vanishes, hence $\theta^{2n}=0$. As in the proof of Corollary 3.2 we obtain the contradiction

$$\begin{array}{rcl}\operatorname{cx}_A M & = & \operatorname{cx}_{R_1}(R_1\otimes_A M)\\ & = & \operatorname{cx}_{R_1} K_{\theta^{2n}}\\ & \leq & \operatorname{cx}_{R_1} K_1\\ & < & \operatorname{cx}_A M,\end{array}$$

showing that we cannot have $\operatorname{Ext}^{2n}_A(M,M)=0$ for some $n\geq 1$ when M is of positive complexity.

Acknowledgements

I would like to thank Dave Jorgensen and my supervisor Øyvind Solberg for valuable suggestions and comments on this paper.

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V.

ON SUPPORT VARIETIES FOR MODULES OVER COMPLETE INTERSECTIONS

ABSTRACT

Let $(A, \mathfrak{m}, k)$ be a complete intersection of codimension c, and $\tilde{k}$ the algebraic closure of k. We show that every homogeneous algebraic subset of $\tilde{k}^c$ is the cohomological support variety of an $\widehat{A}$-module, and that the projective variety of a complete indecomposable maximal Cohen-Macaulay A-module is connected.

This paper is to appear in Proc. Amer. Math. Soc.

ON SUPPORT VARIETIES FOR MODULES OVER COMPLETE INTERSECTIONS

PETTER ANDREAS BERGH

1. Introduction

Support varieties for modules over complete intersections were defined by Avramov in [Avr], and Avramov and Buchweitz showed in [AvB] that these varieties to a large extent behave precisely like the cohomological varieties of modules over group algebras of finite groups. Further illustrating this are the two main results in this paper, the first of which says that every homogeneous variety is realized as the variety of some module (over the completed ring). The second is a version of Carlson's result [Car, Theorem 1'] on varieties for modules over group algebras of finite groups. Namely, we prove that if the variety of a module decomposes as the union of two closed subvarieties having trivial intersection, then the (completion of the) minimal maximal Cohen-Macaulay approximation of the module decomposes accordingly.

Throughout this paper we let $(A, \mathfrak{m}, k)$ be a commutative Noetherian local complete intersection, i.e. the completion $\widehat{A}$ of A with respect to the $\mathfrak{m}$-adic topology is the residue ring of a regular local ring modulo an ideal generated by a regular sequence. We denote by c the codimension of A, that is, the integer $\mu(\mathfrak{m}) - \dim A$, where $\mu(\mathfrak{m})$ is the minimal number of generators for $\mathfrak{m}$. All modules are assumed to be finitely generated.

We now recall the definition of support varieties for modules over complete intersections; details can be found in [Avr, Section 1] and [AvB, Section 2]. Let M be an A-module and $\widehat{M} = \widehat{A} \otimes_A M$ its $\mathfrak{m}$-adic completion, and let $\widehat{A}[\chi_1, \dots, \chi_c]$ be the polynomial ring in the c commuting Eisenbud operators of cohomological degree 2 (where the integer c is the codimension of A). There is a homomorphism $\widehat{A}[\chi_1, \dots, \chi_c] \to \operatorname{Ext}^*_{\widehat{A}}(\widehat{M}, \widehat{M})$ of graded rings under which $\operatorname{Ext}^*_{\widehat{A}}(\widehat{M}, N)$ is a finitely generated graded $\widehat{A}[\chi_1, \dots, \chi_c]$-module for any $\widehat{A}$-module N, making $\operatorname{Ext}^*_{\widehat{A}}(\widehat{M}, N) \otimes_{\widehat{A}} k$ a finitely generated graded module over the polynomial ring $k[\chi_1, \dots, \chi_c]$ via the canonical isomorphism $\widehat{A}[\chi_1, \dots, \chi_c] \otimes_{\widehat{A}} k \simeq k[\chi_1, \dots, \chi_c]$. We denote $k[\chi_1, \dots, \chi_c]$ by H and $\operatorname{Ext}^*_{\widehat{A}}(\widehat{M}, N) \otimes_{\widehat{A}} k$ by $E(\widehat{M}, N)$. The *support variety* $\mathrm{V}(M)$ of M is the algebraic set

$$\mathrm{V}(M) = \{\alpha = (\alpha_1, \dots, \alpha_c) \in \tilde{k}^c \mid f(\alpha) = 0 \text{ for all } f \in \operatorname{Ann}_H E(\widehat{M}, \widehat{M})\},$$

where $\tilde{k}$ is the algebraic closure of k. This is equal to the algebraic set defined by the annihilator in H of $E(\widehat{M}, k)$.

For an ideal $\mathfrak{a}$ of H we denote by $\mathrm{V}_H(\mathfrak{a})$ the algebraic set in $\tilde{k}^c$ defined by $\mathfrak{a}$, i.e.

$$\mathrm{V}_H(\mathfrak{a}) = \{\alpha = (\alpha_1, \dots, \alpha_c) \in \tilde{k}^c \mid f(\alpha) = 0 \text{ for all } f \in \mathfrak{a}\}.$$

Note that the variety $\mathrm{V}(M)$ of M is the set $\mathrm{V}_H\left(\operatorname{Ann}_H E(\widehat{M}, \widehat{M})\right)$, and if f is an element of H then $\mathrm{V}_H(f)$ is the set of all elements in $\tilde{k}^c$ on which f vanishes.

2. Realizing support varieties

Before proving the main results we need some notation. Let R be a commutative Noetherian local ring and X an R-module with minimal free resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to X \to 0,$$

and denote by $\Omega_R^n(X)$ the n'th syzygy of X. For an R-module Y, a homogeneous element $\eta \in \operatorname{Ext}_R^*(X,Y)$ can be represented by a map $f_\eta \colon \Omega_R^{|\eta|}(X) \to Y$, giving the pushout diagram

$$\begin{tikzcd} 0 \arrow[r] & \Omega_R^{|\eta|}(X) \arrow[r] \arrow[d, "f_\eta"] & P_{|\eta|-1} \arrow[r] \arrow[d] & \Omega_R^{|\eta|-1}(X) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r] & K_\eta \arrow[r] & \Omega_R^{|\eta|-1}(X) \arrow[r] & 0 \end{tikzcd}$$

with exact rows. The module K_η is independent, up to isomorphism, of the map f_η chosen as a representative for η. If $\theta \in \operatorname{Ext}_R^*(X,X)$ is another homogeneous element, then their Yoneda product $\eta\theta \in \operatorname{Ext}_R^*(X,Y)$ is a homogeneous element of degree $|\eta| + |\theta|$. The following lemma links K_η and K_θ to $K_{\eta\theta}$ via a short exact sequence, and will be a key ingredient in the proof of the decomposition theorem in the next section.

Lemma 2.1 ([Ber, Lemma 2.3])**.** *If $\theta \in \operatorname{Ext}_R^*(X,X)$ and $\eta \in \operatorname{Ext}_R^*(X,Y)$ are two homogeneous elements, then there exists an exact sequence*

$$0 \to \Omega_A^{|\eta|}(K_\theta) \to K_{\eta\theta} \oplus F \to K_\eta \to 0$$

of R-modules, where F is free.

Now suppose R is Gorenstein and X is a maximal Cohen-Macaulay (or "MCM" from now on) module. Then there exists a *complete resolution*

$$\mathbb{P}\colon \cdots \to P_2 \to P_1 \to P_0 \xrightarrow{d} P_{-1} \to P_{-2} \to \cdots$$

of M, i.e. a doubly infinite exact sequence of free modules in which $\operatorname{Im} d$ is isomorphic to X. For an integer $n \in \mathbb{Z}$ the *stable cohomology module* $\widehat{\operatorname{Ext}}_R^n(X,Y)$ is defined as the n'th homology of the complex $\operatorname{Hom}_R(\mathbb{P}, Y)$. If X and Y are $\widehat{A}$-modules and X is MCM, then $\widehat{\operatorname{Ext}}_{\widehat{A}}^*(X,Y) = \bigoplus_{i=-\infty}^{\infty} \widehat{\operatorname{Ext}}_{\widehat{A}}^i(X,Y)$ is a module over the ring $\widehat{A}[\chi_1, \dots, \chi_c]$ of cohomology operators, and the exact same proof as the one used to prove [EHSST, Lemma 4.2] shows that for any prime ideal $\mathfrak{q} \neq (\chi_1, \dots, \chi_c)$ of $\widehat{A}[\chi_1, \dots, \chi_c]$ the $\widehat{A}[\chi_1, \dots, \chi_c]$-modules $\widehat{\operatorname{Ext}}_{\widehat{A}}^*(X,Y)_{\mathfrak{q}}$ and $\operatorname{Ext}_{\widehat{A}}^*(X,Y)_{\mathfrak{q}}$ are isomorphic.

We are now ready to prove the first result, whose corollary shows that every homogeneous algebraic set is the variety of some $\widehat{A}$-module

Theorem 2.2. *Let $\eta \in H^+ = (\chi_1, \dots, \chi_c)$ be a homogeneous element, and let $\overline{\eta} \in \widehat{A}[\chi_1, \dots, \chi_c]$ be a homogeneous element such that $\overline{\eta} \otimes 1$ corresponds to η when viewing the latter as an element of $\widehat{A}[\chi_1, \dots, \chi_c] \otimes_{\widehat{A}} k$. Furthermore, let Y be an $\widehat{A}$-module, and denote the image of $\overline{\eta}$ in $\operatorname{Ext}_{\widehat{A}}^*(Y,Y)$ by η^Y. Then we have an inclusion*

$$\operatorname{V}(K_{\eta^Y}) \subseteq \operatorname{V}(Y) \cap \operatorname{V}_H(\eta),$$

and equality holds whenever Y is MCM.

Proof. Consider the exact sequence

$$0 \to Y \to K_{\eta^Y} \to \Omega_{\widehat{A}}^{|\eta|-1}(Y) \to 0$$

representing η^Y. Since varieties are invariant under syzygies we have $\mathrm{V}(K_{\eta^Y}) \subseteq \mathrm{V}(Y)$. Moreover, a proof similar to the proof of [EHSST, Proposition 4.1(b)] shows that $(\eta^Y)^2 \operatorname{Ext}^*_{\widehat{A}}(K_{\eta^Y}, k) = 0$, and therefore the element $\eta^2 \in H$ is contained in $\operatorname{Ann}_H E(K_{\eta^Y}, k)$. This gives the inclusion $\mathrm{V}(K_{\eta^Y}) \subseteq \mathrm{V}_H(\eta^2) = \mathrm{V}_H(\eta)$, proving the first half of the lemma.

Now suppose that Y is MCM, and let $\mathfrak{p} \neq H^+$ be a prime ideal of H containing η and $\operatorname{Ann}_H E(Y,k)$. This prime ideal corresponds to a prime ideal $\overline{\mathfrak{p}} \neq (\chi_1, \dots, \chi_c)$ of $\widehat{A}[\chi_1, \dots, \chi_c]$ containing $\overline{\eta}$ and the annihilator of $\operatorname{Ext}^*_{\widehat{A}}(Y,k)$. Suppose $\overline{\mathfrak{p}}$ does not contain the annihilator of $\operatorname{Ext}^*_{\widehat{A}}(K_{\eta^Y}, k)$. The exact sequence from the beginning of the proof induces a long exact sequence

$$\cdots \to \widehat{\operatorname{Ext}}^n_{\widehat{A}}(K_{\eta^Y}, k) \to \widehat{\operatorname{Ext}}^n_{\widehat{A}}(Y,k) \xrightarrow{\eta^Y} \widehat{\operatorname{Ext}}^{n+|\eta|}_{\widehat{A}}(Y,k) \to \widehat{\operatorname{Ext}}^{n+1}_{\widehat{A}}(K_{\eta^Y}, k) \to \cdots$$

in stable cohomology, which in turn gives the exact sequence

$$0 \to \widehat{\operatorname{Ext}}^{*+|\eta|-1}_{\widehat{A}}(Y,k)/\eta^Y \widehat{\operatorname{Ext}}^{*-1}_{\widehat{A}}(Y,k) \to \widehat{\operatorname{Ext}}^{*}_{\widehat{A}}(K_{\eta^Y}, k)$$

of $\widehat{A}[\chi_1, \dots, \chi_c]$-modules. Now recall from the discussion prior to this theorem that $\widehat{\operatorname{Ext}}^*_{\widehat{A}}(W,Z)_{\overline{\mathfrak{p}}} \simeq \operatorname{Ext}^*_{\widehat{A}}(W,Z)_{\overline{\mathfrak{p}}}$ for any $\widehat{A}$-modules W and Z with W MCM. Since $\overline{\mathfrak{p}}$ does not contain the annihilator of $\operatorname{Ext}^*_{\widehat{A}}(K_{\eta^Y}, k)$, we see by localizing the above exact sequence at $\overline{\mathfrak{p}}$ that $\widehat{\operatorname{Ext}}^*_{\widehat{A}}(Y,k)_{\overline{\mathfrak{p}}} = \eta^Y \widehat{\operatorname{Ext}}^*_{\widehat{A}}(Y,k)_{\overline{\mathfrak{p}}}$. But $\widehat{\operatorname{Ext}}^*_{\widehat{A}}(Y,k)_{\overline{\mathfrak{p}}}$, being isomorphic to $\operatorname{Ext}^*_{\widehat{A}}(Y,k)_{\overline{\mathfrak{p}}}$, is finitely generated over $\widehat{A}[\chi_1, \dots, \chi_c]_{\overline{\mathfrak{p}}}$, hence Nakayama's Lemma implies $\operatorname{Ext}^*_{\widehat{A}}(Y,k)_{\overline{\mathfrak{p}}} = 0$. This contradicts the assumption that $\overline{\mathfrak{p}}$ contains the annihilator of $\operatorname{Ext}^*_{\widehat{A}}(Y,k)$, and therefore $\overline{\mathfrak{p}}$ must contain the annihilator of $\operatorname{Ext}^*_{\widehat{A}}(K_{\eta^Y}, k)$. But then $\operatorname{Ann}_H E(K_{\eta^Y}, k) \subseteq \mathfrak{p}$, giving the inclusion

$$\sqrt{\operatorname{Ann}_H E(K_{\eta^Y}, k)} \subseteq \sqrt{(\eta, \operatorname{Ann}_H E(Y,k))}$$

of ideals in H, and consequently we get $\mathrm{V}(Y) \cap \mathrm{V}_H(\eta) \subseteq \mathrm{V}(K_{\eta^Y})$. $\square$

Corollary 2.3. *Every closed homogeneous variety in $\tilde{k}^c$ is the variety of some MCM $\widehat{A}$-module.*

Proof. Let $\eta_1, \dots, \eta_t$ be homogeneous elements in H^+, and let Y be an MCM $\widehat{A}$-syzygy of k. Then $\mathrm{V}(Y) = \mathrm{V}(k) = \tilde{k}^c$, hence by the lemma we have $\mathrm{V}(K_{\eta_1^Y}) = \mathrm{V}(Y) \cap \mathrm{V}_H(\eta_1) = \mathrm{V}_H(\eta_1)$. Repeating the process with $\eta_2, \dots, \eta_t$ we end up with an MCM module K such that

$$\mathrm{V}(K) = \mathrm{V}_H(\eta_1) \cap \cdots \cap \mathrm{V}_H(\eta_t) = \mathrm{V}_H(\eta_1, \dots, \eta_t).$$

$\square$

3. Decomposition

Before proving the next result, recall that an *MCM-approximation* of an A-module X is an exact sequence

$$0 \to Y_X \to C_X \xrightarrow{f} X \to 0$$

where C_X is MCM and Y_X has finite injective dimension. The approximation is *minimal* if the map f is right minimal, that is, if every map $C_X \xrightarrow{g} C_X$ satisfying $f = fg$ is an isomorphism. This notion was introduced in [AuB], where it was shown that every finitely generated module over a commutative Noetherian ring admitting a dualizing module has an MCM-approximation. Moreover, it follows from the remark following [Mar, Theorem 18] that every finitely generated module over a commutative local Gorenstein ring has a minimal MCM-approximation, which is unique up to isomorphism. In particular this applies to our setting, where A is a local complete intersection. Furthermore, since $A \to \widehat{A}$ is a faithfully flat local homomorphism, an A-module Z has finite projective dimension if and only if the $\widehat{A}$-module $\widehat{Z}$ has finite projective dimension, and it follows from [Mat, Theorem 23.3] that Z is MCM if and only if $\widehat{Z}$ is MCM. Therefore, by [Mar, Proposition 19] and the fact that over a Gorenstein ring the modules having finite injective dimension are precisely those having finite projective dimension, we see that

$$0 \to Y_X \to C_X \xrightarrow{f} X \to 0$$

is a minimal MCM-approximation if and only if

$$0 \to \widehat{Y}_X \to \widehat{C}_X \xrightarrow{\widehat{f}} \widehat{X} \to 0$$

is a minimal MCM-approximation.

We are now ready to prove the second main result. It is the commutative complete intersection version of Carlson's famous theorem (see [Car]) from modular representation theory; if the variety V of a kG-module L (where k is an algebraically closed field and G is a finite group) decomposes as $V = V_1 \cup V_2$, where V_1 and V_2 are closed varieties having trivial intersection, then L decomposes as $L = L_1 \oplus L_2$ where the variety of L_i is V_i. Our proof follows closely that of Carlson, but with some adjustments.

Theorem 3.1. *If for an A-module M we have $\mathrm{V}(M) = V_1 \cup V_2$ where V_1 and V_2 are closed homogeneous varieties having trivial intersection, then the completion $\widehat{C}_M$ of the minimal MCM-approximation of M decomposes as $\widehat{C}_M = C_1 \oplus C_2$ with $\mathrm{V}(C_i) = V_i$.*

Proof. Let

$$0 \to Y \to C \to M \to 0$$

be the minimal MCM-approximation of M. Since Y has finite injective dimension (or equivalently, finite projective dimension), it follows from [AvB, Theorem 5.6] that $\mathrm{V}(Y)$ is trivial and that we therefore have $\mathrm{V}(M) = \mathrm{V}(C)$. Moreover, by definition the equality $\mathrm{V}(X) = \mathrm{V}(\widehat{X})$ holds for every A-module X, and therefore we may suppose that A is complete.

We argue by induction on the integer $\dim V_1 + \dim V_2$. If one of V_1 and V_2, say V_2, is zero dimensional, then V_2 is trivial, and the decomposition $C = C' \oplus P$, with P being the maximal projective summand of C, satisfies the conclusion of the theorem. Suppose therefore that $\dim V_i$ is nonzero for $i = 1, 2$.

Let $\mathfrak{a}_1$ and $\mathfrak{a}_2$ be homogeneous ideals of $H = k[\chi_1, \dots, \chi_c]$ defining the varieties V_1 and V_2, i.e. V_i is the algebraic set $\mathrm{V}_H(\mathfrak{a}_i)$ in $\tilde{k}^c$ defined by $\mathfrak{a}_i$ for $i = 1, 2$. We then have equalities

$$\{0\} = V_1 \cap V_2 = \mathrm{V}_H(\mathfrak{a}_1) \cap \mathrm{V}_H(\mathfrak{a}_2) = \mathrm{V}_H(\mathfrak{a}_1 + \mathfrak{a}_2),$$

and so it follows from Hilbert's Nullstellensatz that for each $1 \leq i \leq c$ we have $\chi_i \in \sqrt{\mathfrak{a}_1 + \mathfrak{a}_2}$. Therefore $\sqrt{\mathfrak{a}_1 + \mathfrak{a}_2}$ is the graded maximal ideal H^+ of H, i.e. $\sqrt{\mathfrak{a}_1 + \mathfrak{a}_2} = (\chi_1, \dots, \chi_c)$.

Pick a homogeneous element $\theta \in H^+$ with the property that $\dim H/(\mathfrak{a}_2, \theta) < \dim H/\mathfrak{a}_2$ (this is possible since $\dim H/\mathfrak{a}_2 = \dim V_2 > 0$). By the above there is an integer $n \geq 1$ such that θ^n belongs to $\mathfrak{a}_1 + \mathfrak{a}_2$, i.e. $\theta^n = \theta_1 + \eta$ where $\theta_1 \in \mathfrak{a}_1$ and $\eta \in \mathfrak{a}_2$. Then $\dim H/(\mathfrak{a}_2, \theta_1) < \dim H/\mathfrak{a}_2$, which translates to the language of varieties as $\dim (\mathrm{V}_H(\mathfrak{a}_2) \cap \mathrm{V}_H(\theta_1)) = \dim \mathrm{V}_H(\mathfrak{a}_2 + (\theta_1)) < \dim \mathrm{V}_H(\mathfrak{a}_2)$. Similarly we can find an element $\theta_2 \in \mathfrak{a}_2$ having the property that it "cuts down" the variety defined by $\mathfrak{a}_1$. Hence the two homogeneous elements θ_1 and θ_2 satisfy

$$\theta_1 \in \mathfrak{a}_1, \quad \dim (V_2 \cap \mathrm{V}_H(\theta_1)) < \dim V_2,$$
$$\theta_2 \in \mathfrak{a}_2, \quad \dim (V_1 \cap \mathrm{V}_H(\theta_2)) < \dim V_1.$$

Now since $\mathrm{V}_H(\theta_1\theta_2) = \mathrm{V}_H(\theta_1) \cup \mathrm{V}_H(\theta_2) \supseteq V_1 \cup V_2 = \mathrm{V}(C)$, it follows once more from Hilbert's Nullstellensatz that $\theta_1\theta_2 \in \sqrt{\operatorname{Ann}_H E(C,C)}$, where $E(C,C) = \operatorname{Ext}_A^*(C,C) \otimes_A k$. Replacing θ_1 and θ_2 by suitable powers, we may assume that $\theta_1\theta_2 \in \operatorname{Ann}_H E(C,C)$. Viewed as elements in $A[\chi_1,\dots,\chi_c]\otimes_A k$ we have $\theta_i = \overline{\theta}_i \otimes 1$, where $\overline{\theta}_1$ and $\overline{\theta}_2$ are homogeneous elements of positive degrees in $A[\chi_1,\dots,\chi_c]$ with the property that $\overline{\theta}_1\overline{\theta}_2 \in \operatorname{Ann}_{A[\chi_1,\dots,\chi_c]} \operatorname{Ext}_A^*(C,C)$. To see the latter, note that $0 = \theta_1\theta_2 \left(\operatorname{Ext}_A^i(C,C) \otimes_A k\right) = \overline{\theta}_1\overline{\theta}_2 \operatorname{Ext}_A^i(C,C) \otimes_A k$ for every $i \geq 0$, and since $\overline{\theta}_1\overline{\theta}_2 \operatorname{Ext}_A^i(C,C)$ is a finitely generated A-module ($\overline{\theta}_1\overline{\theta}_2$ commutes with elements in A), the claim follows.

Now consider the images θ_1^C and θ_2^C of $\overline{\theta}_1$ and $\overline{\theta}_2$ in $\operatorname{Ext}_A^*(C,C)$. Since $\theta_1^C\theta_2^C = 0$, the bottom exact sequence in the exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega_A^{|\theta_1^C|+|\theta_2^C|}(C) \arrow[r] \arrow[d, "f_{\theta_1^C\theta_2^C}"] & Q_{|\theta_1^C|+|\theta_2^C|-1} \arrow[r] \arrow[d] & \Omega_A^{|\theta_1^C|+|\theta_2^C|-1}(C) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & C \arrow[r] & K_{\theta_1^C\theta_2^C} \arrow[r] & \Omega_A^{|\theta_1^C|+|\theta_2^C|-1}(C) \arrow[r] & 0
\end{tikzcd}$$

splits, where Q_n denotes the n'th module in the minimal free resolution of C. Therefore $K_{\theta_1^C\theta_2^C}$ is isomorphic to $C \oplus \Omega_A^{|\theta_1^C|+|\theta_2^C|-1}(C)$, and from Lemma 2.1 we see that there exists an exact sequence

$$(\dagger) \qquad 0 \to \Omega_A^{|\theta_1^C|}(K_{\theta_2^C}) \to C \oplus \Omega_A^{|\theta_1^C|+|\theta_2^C|-1}(C) \oplus F \to K_{\theta_1^C} \to 0$$

for some free module F. From Theorem 2.2 we have $\mathrm{V}(K_{\theta_i^C}) = \mathrm{V}(C) \cap \mathrm{V}_H(\theta_i)$, hence the equality $\mathrm{V}(C) = V_1 \cup V_2$ and the inclusion $V_i \subseteq \mathrm{V}_H(\theta_i)$ give the equalities

$$\begin{aligned}
\mathrm{V}(K_{\theta_1^C}) &= V_1 \cup (V_2 \cap \mathrm{V}_H(\theta_1)), \\
\mathrm{V}(K_{\theta_2^C}) &= V_2 \cup (V_1 \cap \mathrm{V}_H(\theta_2)).
\end{aligned}$$

By induction there exist A-modules X_1, X_2, Y_1 and Y_2 such that $K_{\theta_1^C} = X_1 \oplus X_2$ and $\Omega_A^{|\theta_1^C|}(K_{\theta_2^C}) = Y_1 \oplus Y_2$, and such that

$$\begin{aligned}
\mathrm{V}(X_1) &= V_1, \\
\mathrm{V}(X_2) &= V_2 \cap \mathrm{V}_H(\theta_1), \\
\mathrm{V}(Y_1) &= V_1 \cap \mathrm{V}_H(\theta_2), \\
\mathrm{V}(Y_2) &= V_2.
\end{aligned}$$

Now since $\mathrm{V}(X_1) \cap \mathrm{V}(Y_2)$ and $\mathrm{V}(X_2) \cap \mathrm{V}(Y_1)$ are contained in $V_1 \cap V_2$, which is trivial, we see from [AvB, Theorem 5.6] that $\operatorname{Ext}_A^i(X_1,Y_2)$ and $\operatorname{Ext}_A^i(X_2,Y_1)$ vanish for $i \gg 0$. But $K_{\theta_1^C}$ is MCM, implying X_1 and X_2 are both MCM, and so it follows from [ArY, Theorem 4.2] that $\operatorname{Ext}_A^i(X_1,Y_2)$ and $\operatorname{Ext}_A^i(X_2,Y_1)$ vanish for $i \geq 1$. Therefore

$$\operatorname{Ext}_A^1(K_{\theta_1^C}, \Omega_A^{|\theta_1^C|}(K_{\theta_2^C})) = \operatorname{Ext}_A^1(X_1,Y_1) \oplus \operatorname{Ext}_A^1(X_2,Y_2),$$

and this implies that the exact sequence $(\dagger)$ is equivalent to the direct sum of two sequences of the form

$$0 \to Y_i \to Z_i \to X_i \to 0$$

for $i = 1,2$, where Z_i is an A-module. Then $C \oplus \Omega_A^{|\theta_1^C|+|\theta_2^C|-1}(C) \oplus F$ must be isomorphic to $Z_1 \oplus Z_2$, and since $\mathrm{V}(Z_i) \subseteq \mathrm{V}(X_i) \cup \mathrm{V}(Y_i) \subseteq V_i$ and the Krull-Schmidt property holds for the category of (finitely generated) modules over a complete local ring, there must exist A-modules C_1 and C_2 such that $C = C_1 \oplus C_2$ and $\mathrm{V}(C_i) = \mathrm{V}(Z_i)$. Since

$$V = \mathrm{V}(C_1) \cup \mathrm{V}(C_2) \subseteq V_1 \cup V_2 = V$$

we must have $\mathrm{V}(C_i) = V_i$, and the proof is complete. $\square$

Corollary 3.2. *The projective variety of a complete indecomposable MCM A-module is connected.*

Acknowledgements

I would like to thank Dave Jorgensen and my supervisor Øyvind Solberg for valuable suggestions and comments on this paper.

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