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Some Improved Estimates in the Dirichlet Divisor Problem from Bourgain's Exponent Pair

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Abstract

The thesis work is a survey of recent developments on the famous error terms in the Dirichlet divisor problem. We consider the power moments of the Riemann zeta- function $\zeta(s)$ in the critical strip and we managed to obtain some new bound estimates for power moments using a recently obtained exponent pair by *Jean Bourgain*. Thus, applying the slight improvements on bounds of the power moment estimates and the order of the zeta-function in the critical strip, we obtain new improved bounds for the order of the error term in the Dirichlet divisor problem.

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Contents

Overview of the Thesis

This thesis in Analytic Number Theory consists of four chapters and one Appendix. The first chapter presents some preliminary concepts of the Riemann zeta function. In chapter two, we discuss the basic theory of exponent pairs for estimating exponential sums, particularly, for obtaining an upper bound estimate for the Riemann zeta-function $\zeta(\sigma + it)$. In this chapter, we discuss what is known about the order of the Riemann zeta-function in the critical strip and the connection with the theory of exponent pairs. The third chapter is about the higher power moments which are used to estimate the maximal order of the Riemann zeta-function $\zeta(s)$ in the critical strip. New bounds for the higher power moment estimate are presented, relying on an exponent pair that was recently computed by *Jean Bourgain*. The fourth chapter applies these bounds to the Dirichlet divisor problem. As a result, we obtain improved estimates for the order and average order of the error term in the Dirichlet divisor problem.

Chapter 1

Some Preliminaries

1.1 Definition and Some Properties of $\zeta(s)$

The Riemann zeta function is one of the most important and fascinating functions in mathematics. It is very natural as it deals with the series of powers of natural numbers

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \sum_{n=1}^{\infty} \frac{1}{n^3}, \quad \sum_{n=1}^{\infty} \frac{1}{n^4}, \quad \text{etc.}$$

Originally the function was defined for real numbers as

$$\sum_{n=1}^{\infty} \frac{1}{n^\sigma} \quad \text{for} \quad \sigma > 1. \tag{1.1}$$

It was Bernhard Riemann (1826−1866), who recognized the importance of viewing $\zeta(s)$ as a function of a complex variable $s = \sigma + it$ rather than a real variable σ.

Definition 1.1.1. The Riemann zeta-function is defined on $\{s \in \mathbb{C}\colon \Re(s) > 1\}$, replacing the real variable σ in (1.1) by complex $s = \sigma + it$ as

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \tag{1.2}$$

Note that from the integral test the series in (1.2) converges absolutely in the region described; moreover if $\Re(s) \geq a > 1$ then the series is dominated term by term by the absolutely convergent serious $\sum_{n\geq 1} n^{-a}$ so that, by Weierstrass's criterion it converges uniformly in $\{s \in \mathbb{C}\colon \Re(s) > 1\}$. It therefore defines an analytic function $\zeta(s)$, regular for $\Re(s) > 1$.

The prime numbers are a fundamental object of interest in number theory. The connection between the zeta function and the distribution of the primes are not obvious. The zeta-function is defined in terms of $\frac{1}{n^s}$, for all $n \in \mathbb{N}$. Since each n has a unique prime factorization, we might hope to express $\zeta(s)$ only in terms of $\frac{1}{p^s}$, for p prime. Leonhard Euler found a beautiful relationship between prime numbers and $\zeta(s)$ as follows.

Theorem 1.1.1. ***(Euler Product)***: *For $s = \sigma + it$ and $\sigma > 1$, we have*

$$\zeta(s) = \prod_{p} \frac{1}{1 - p^{-s}} \tag{1.3}$$

where the product is taken over all prime numbers p.

Proof. We shall prove now that the representation holds, and at the same time explicate the nature of the convergence of the product. For any prime p, we have

$$\frac{1}{1 - p^{-s}} = \sum_{k=0}^{\infty} \frac{1}{p^{ks}}.$$

Since we can rearrange and multiply out a finite product of absolutely convergent series, we see by unique prime factorization that

$$\prod_{p\leq P} \frac{1}{1 - p^{-s}} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

where $a_n = 1$ if all prime factors of n are at most P, and 0 otherwise. Therefore,

$$\Big| \zeta(s) - \prod_{p\leq P} \Big(\frac{1}{1 - p^{-s}}\Big) \Big| \leq \sum_{n\geq P} \Big| \frac{1}{n^{-s}} \Big| = \sum_{n>P} \frac{1}{n^{-\sigma}},$$

which vanishes as $P \to \infty$. □

The Euler identity in (1.3) can be taken as a definition of $\zeta(s)$. The Euler product representation of $\zeta(s)$ plays a fundamental role in the application of zeta-function theory in number theory.

Theorem 1.1.2. ***(Analytic Continuation)***: *The function $\zeta(s)$, which is defined by* (1.2) *for $\sigma >$ 1, extends to a meromorphic function to the half-plane $\sigma > 0$ with only one pole at $s = 1$, which is a simple pole with residue 1, and that $\zeta(s)$ is negative on the segment $0 < \sigma < 1$, t=0.*

Proof. For $x > 1$, we have

$$s\int_n^{n+1} x^{-s-1}dx = \frac{1}{n^s} - \frac{1}{(n+1)^s}.$$

Hence

$$s\int_n^{n+1} \frac{[x]}{x^{s+1}}dx = n\Big(\frac{1}{n^s} - \frac{1}{(n+1)^s}\Big).$$

Summing over $n = 1, 2, \ldots$ we obtain

$$\zeta(s) = s\int_n^{\infty} \frac{[x]}{x^{s+1}}dx = \frac{s}{s-1} - \frac{1}{2} - s\int_1^{\infty} \frac{\psi(x)}{x^{s+1}}dx \tag{1.4}$$

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} - s(s+1)\int_1^{\infty}\Big(\int_1^x \psi(y)dy\Big)\frac{x}{x^{s+2}}dx$$

where $\psi(s) = x - [x] - \frac{1}{2}$. Since the integral of $\psi(y)$ is bounded, the last integral in x converges absolutely if $\sigma > -1$ giving the analytic continuation of $\zeta(s)$ to the half plane $\sigma > -1$. Note that

$$\zeta(0) = -\frac{1}{2}.$$

Moreover,

$$\begin{aligned}\lim_{s\to 1}\Big(\zeta(s) - \frac{1}{s-1}\Big) &= \frac{1}{2} - \int_{n=1}^{\infty} \frac{\psi(x)}{x^2}dx = \frac{1}{2} - \lim_{N\to\infty}\int_1^2 \frac{x - [x] - \frac{1}{2}}{x^2}dx \\ &= \frac{1}{2} - \lim_{N\to\infty}\Big[\log N - \sum_{n=1}^{N}\frac{1}{n} + \frac{N-1}{N} - \frac{1}{2} + \frac{1}{2N}\Big] \\ &= \lim_{N\to\infty}\Big(\sum_{n=1}^{N}\frac{1}{n} - \log N\Big) = \gamma\end{aligned}$$

where $\gamma = \gamma_0 = 0.577\ldots$ is the *Euler constant.* Hence

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(\mid s-1 \mid), \quad \text{as} \quad s \to 1. \tag{1.5}$$

If we start summing from $n = N, N+1, \ldots$, we get the formula

$$\zeta(s) = \sum_1^N n^{-s} + \frac{N^{1-s}}{s-1} - \frac{1}{2}N^{-s} - s(s+1)\int_N^{\infty}\Big(\psi(y)dy\Big)\frac{dx}{x^{s+2}}. \tag{1.6}$$

Hence we get the approximation

$$\zeta(s) = \sum_{n\leq N} n^{-s} + \frac{N^{1-s}}{s-1} + O\Big(\frac{\mid s(s+1)\mid}{\sigma+1} N^{-\sigma-1}\Big), \tag{1.7}$$

which is valid for $\sigma > -1$ and $N \geq 1$, the implied constant being absolute. Suppose $s = \sigma + it$ with $\sigma \leq 0$ and $\mid t \mid \leq 2T$. If $N \geq T \geq 1$, then we can evaluate the partial sum in (1.7) by applying (A.4) with $h(n) = n^{-\sigma}$ and $g(n) = (t/2\pi)\log n$. We get

$$\int_T^N x^{-s}dx + O(T^{-\sigma}) = \frac{N^{1-s} - T^{1-s}}{1-s} + O(T^{-\sigma}).$$

Hence (1.7) follows. □

Theorem 1.1.3. ***(The Functional Equation)****[15] The function $\zeta(s)$ is regular for all values of s except s=1, where there is a simple pole with residue 1. Then for all complex s the functional equation for the Riemann zeta-function is given by*

$$\zeta(s) = 2^s\pi^{s-1}\sin(\frac{1}{2}s\pi)\Gamma(1-s)\zeta(1-s). \tag{1.8}$$

where $\Gamma(s)$ is a Gamma-function [see A.10].

Proof. There are several ways of proving this Theorem in [15]. Let us choose one, we have the fundamental formula for $\zeta(s)$ in integral form, which is

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x-1}dx \quad (\sigma > 1). \tag{1.9}$$

For $\sigma \geq 1$, (1.9) may be written

$$\zeta(s)\Gamma(s) = \int_0^1 \Big(\frac{1}{e^x-1} - \frac{1}{x}\Big)x^{s-1}dx + \frac{1}{s-1} + \int_1^\infty \frac{x^{s-1}}{e^x-1},$$

and this holds by analytic continuation for $\sigma > 0$. Also for $0 < \sigma < 1$

$$\frac{1}{s-1} = -\int_1^\infty \frac{x^{s-1}}{x}dx.$$

Hence

$$\zeta(s)\Gamma(s) = \int_0^\infty \Big(\frac{1}{e^x-1} - \frac{1}{x}\Big)x^{s-1}dx \quad (0 < \sigma < 1). \tag{1.10}$$

and (1.10) gives

$$\Gamma(s)\zeta(s) = \int_0^1 \Big(\frac{1}{e^x-1} - \frac{1}{x} + \frac{1}{2}\Big)x^{s-1}dx - \frac{1}{2s} + \int_1^\infty \Big(\frac{1}{e^x-1} - \frac{1}{x} + \frac{1}{2}\Big)x^{s-1}dx,$$

and this holds by analytic continuation for $\sigma > -1$. But

$$\int_1^\infty \frac{1}{2}x^{s-1}dx = -\frac{1}{2s} \quad (-1 < \infty < 0).$$

Hence

$$\Gamma(s)\zeta(s) = \int_0^\infty \Big(\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2}\Big)x^{s-1}dx \quad (-1 < \sigma < 0). \tag{1.11}$$

Now

$$\frac{1}{e^x - 1} = \frac{1}{x} - \frac{1}{2} + 2x\sum_{n=1}^{\infty}\frac{1}{4n^2\pi^2 + x^2}.$$

Hence,

$$\begin{aligned}\Gamma(s)\zeta(s) = \int_0^\infty 2x\sum_0^\infty \frac{1}{4n^2\pi^2 + x^2}x^{s-1}dx &= 2\sum_{n=1}^\infty \int_0^\infty \frac{x^s}{4n^2\pi^2 + x^2}dx \\ &= 2\sum_{n=1}^\infty (2n\pi)^{s-1}\frac{\pi}{2\cos\frac{1}{2}s\pi} = \frac{2^{s-1}\pi^s}{\cos\frac{1}{2}s\pi}\zeta(1-s),\end{aligned} \tag{1.12}$$

we can be rewrite as

$$\zeta(1-s) = 2^{1-s}\pi^{-s}\cos\frac{1}{2}s\pi\Gamma(s)\zeta(s). \tag{1.13}$$

the functional equation. The inversion is justified by absolute convergence if $-1 < \sigma < 0$. □

The functional equation (1.8) can be written in several equivalent ways using some standard properties of the gamma-function (A.10). It may be written

$$\zeta(s) = \chi(s)\zeta(1-s), \tag{1.14}$$

where,

$$\chi(s) = 2^{s-1}\pi^s\sec\frac{1}{2}s\pi/\Gamma(s).$$

In any fixed strip $\alpha \le \sigma \le \beta$, as $t \to \infty$

$$\log\Gamma(\sigma + it) = (\sigma + it - \frac{1}{2})\log{(it)} - it + \frac{1}{2}\log 2\pi + O(\frac{1}{2}). \tag{1.15}$$

Hence

$$\Gamma(\sigma + it) = t^{\sigma+it-\frac{1}{2}}e^{-\frac{1}{2}\pi t - it + \frac{1}{2}i\pi(\sigma-\frac{1}{2})}(2\pi)^{\frac{1}{2}}\Big\{1 + O\Big(\frac{1}{t}\Big)\Big\}, \tag{1.16}$$

Thus,

$$\chi(s) = \Big(\frac{2\pi}{t}\Big)^{\sigma+it-\frac{1}{2}} e^{i(t+\frac{1}{4}\pi)}\Big\{1+O\Big(\frac{1}{t}\Big)\Big\}. \tag{1.17}$$

Writing

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{1}{2}s}\Gamma(\frac{1}{2}s)\zeta(s) \tag{1.18}$$

it is at once verified from (1.13) and (1.14) that

$$\xi(s) = \xi(1-s). \tag{1.19}$$

The functional equation (1.8) takes the more symmetric form

$$\pi^{\frac{-s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{\frac{-(1-s)}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s). \tag{1.20}$$

This follows immediately from the previously stated functional equation for $\zeta(s)$ and from the well-known duplication formula

$$\Gamma(s)\Gamma(s+\frac{1}{2}) = 2^{1-2s}\sqrt{\pi}\Gamma(2s).$$

The functional equation allows the properties of $\zeta(s)$ for $\sigma < 0$ to be inferred from its properties for $\sigma > 1$. In particular, the only zeros of $\zeta(s)$ for $\sigma < 0$ are at the poles of $\Gamma(\frac{1}{2}s)$, that is, at the points $s = -2, -4, -6, ...$ These are called the trivial zeros. The remainder of the plane, where $0 \leq \sigma \leq 1$, is called the critical strip.

Theorem 1.1.4. *(Factorization Formulae) $\xi(s)$ is an integral function of order 1.*

Proof. It follows from (1.18) and what we have proved in the previous Theorem about $\zeta(s)$ that $\xi(s)$ is regular for $\sigma > 0$, $(s-1)\zeta(s)$ being regular at $s = 1$. Since $\xi(s) = \xi(1-s)$, $\xi(s)$ is also regular for $\sigma < 1$. Hence $\xi(s)$ is an integral function. Also

$$\mid \Gamma(\frac{1}{2}s) \mid = \mid\int_0^\infty e^{-u}u^{\frac{1}{2}s-1}du\mid \leq \int_0^\infty e^{-u}u^{\frac{1}{2}\sigma-1}du = \Gamma(\frac{1}{2}\sigma) = O(e^{A\sigma\log\sigma}) \quad (\sigma > 0),$$

and (1.4) gives for $\sigma \geq \frac{1}{2}$, $\mid s-1 \mid > A$,

$$\zeta(s) = O\Big(\mid s \mid \int_1^\infty \frac{du}{u^{\frac{3}{2}}}\Big) + O(1) = O(\mid s \mid). \tag{1.21}$$

Hence (1.18) gives $\xi(s) = O(e^{A|s|\log|s|})$ for $\sigma \geq \frac{1}{2}$, $\mid s \mid > A$. By (1.19) this holds for $\sigma \leq \frac{1}{2}$ also. Hence $\xi(s)$ is of order 1 at most. The order is exactly 1 since as $s \to \infty$ by real values $\log\zeta(s) \sim 2^{-s}$, $\log\xi(s) \sim \frac{1}{2}s\log s$. □

1.2 Dirichlet Series Connected with $\zeta(s)$

Definition 1.2.1. A *Dirichlet series* is a series of the form

$$F(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, \tag{1.22}$$

where $\{a_n\}$ is a sequence of complex numbers.

A Dirichlet series can be regarded as a pure formal infinite series, or as a function of the complex variable s, defined in the region in which the series converges. If a Dirichlet series converges for some $s = s_0 \in \mathbb{C}$, then $\frac{a_n}{n^s} \to 0$ as $n \to \infty$, so $a_n = O(n^{\Re(s_0)})$, and it follows that $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges absolutely for $\Re(s) > \Re(s_0) + 1$ and defines an analytic function in that region. If only finitely many of the a_n are non-zero, then the resulting finite sum $\sum_{n \leq N} \frac{a_n}{n^s}$ is called a *Dirichlet polynomial.*
A Dirichlet series generated by an arithmetical function $f(n)$ is an infinite series of the form

$$F(s) = \sum_{n=1}^{\infty} f(n) n^{-s}, \tag{1.23}$$

provided that a series converges for some $s = s_0$.
Theorem 1.2.1. *For every Dirichlet series of the form* (1.23) *there exists a number $\sigma_a \in \mathbb{R} \cup \{\pm\infty\}$, called the abscissa of absolute convergence, such that for all s with $\sigma > \sigma_a$ the series converges absolutely, and for all s with $\sigma < \sigma_a$, the series does not converge absolutely.*

Proof. Let A be the set of complex numbers s at which $F(s)$ converges absolutely. If the set A is empty, the conclusion of the theorem holds with $\sigma_a = \infty$. Otherwise, set $\sigma_a = \inf\{\Re(s) : s \in A\} \in \mathbb{R} \cup \{-\infty\}$. By the definition of σ_a, the series $F(s)$ does not converge absolutely if $\sigma < \sigma_a$. On the other hand, if $s = \sigma + it$ and $s' = \sigma' + it'$ with $\sigma' \geq \sigma$, then

$$\sum_{n=1}^{\infty} \left| \frac{f(n)}{n^{s'}} \right| = \sum_{n=1}^{\infty} \frac{| f(n) |}{n^{\sigma'}} \leq \sum_{n=1}^{\infty} \frac{| f(n) |}{n^{\sigma}} = \sum_{n=1}^{\infty} \left| \frac{f(n)}{n^{s}} \right|.$$

hence, if $F(s)$ converges absolutely at some point s, then it also converges absolutely at any point s' with $\Re(s') \geq \Re(s)$. Since, by the definition of σ_a, there exist point s with σ arbitrarily close to σ_a at which the Dirichlet series $F(s)$ converges absolutely, it follows that the series converges absolutely at every point s with $\sigma > \sigma_a$. This completes the proof of the theorem. □

Theorem 1.2.2. *Suppose that the Dirichlet series in* (1.23) *converges at the point* $s_0 = \sigma_0 - it_0$. *Then* $F(s)$ *converges uniformly in the sector* $| t - t_0 | \leq \frac{1}{\varepsilon}(\sigma - \sigma_0)$, *representing consequently a regular function in the half-plane* $\sigma \geq \sigma_0 + \varepsilon$.

Proof. Let M and N, M<N be two natural numbers, using the partial summation formula (A.5) to obtain

$$\sum_{M \leq n \leq N} f(n)n^{-s} = \sum_{M \leq n \leq N} f(n)n^{-s_0}n^{s_0-s}$$

$$= N^{s_0-s} \sum_{M \leq n \leq N} f(n)n^{-s_0} - \int_M^N (\sum_{M \leq n \leq t} f(n)n^{-s_0})(s_0 - s)t^{s_0-s-1}dt$$

$$\ll N^{\sigma_0-\sigma} + \frac{| s - s_0 |}{\sigma - \sigma_0} M^{\sigma_0-\sigma} \to 0 \quad \text{uniformly as} \quad M \to \infty$$

proving uniform convergence of $F(s)$ for $\sigma \geq \sigma_0 + \varepsilon$, where "$\ll$" is the Vinogradov symbol, and $f(x) \ll g(x)$ means the same as $f(x) = O(g(x))$. □

Theorem 1.2.3. *For* $\Re(s) > 1$ *and a given integer* $k \geq 1$, *we have*

$$\zeta^k(s) = \sum_{n=1}^{\infty} d_k(n)n^{-s} \tag{1.24}$$

where $d_k(n) = \sum_{v_1 v_2 \ldots v_k = n} 1$.

Proof. In the first place, let us look at the case $k = 2$,

$$\zeta^2(s) = \sum_{n=1}^{\infty} \frac{d(n)}{n^s} \quad (\sigma > 1), \tag{1.25}$$

where $d(n)$ denotes the number of divisors of n (including 1 and n itself). For

$$\zeta^2(s) = \sum_{\mu=1}^{\infty} \frac{1}{\mu^s} \sum_{v=1}^{\infty} \frac{1}{v^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{\mu v = n} 1, \tag{1.26}$$

and the number of terms in the last sum is $d(n)$. And generally

$$\zeta^k(s) = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} \quad (\sigma > 1), \tag{1.27}$$

where $k = 2, 3, 4, ...$
Thus,

$$\zeta^k(s) = \sum_{v_1=1}^{\infty} \frac{1}{{v_1}^s} ... \sum_{v_k=1}^{\infty} \frac{1}{{v_k}^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{v_1...v_k=n} 1, \tag{1.28}$$

and the last sum is $d_k(n)$. □

1.3 Inversion Formula for Dirichlet Series

The philosophy of analytic number theory is to study the asymptotic behaviour of a counting function $\sum_{n \le x} a_n$ through the generating function $A(s) = \sum_{n=1}^{\infty} a_n n^{-s}$. The main tool to facilitate this is a trick from complex analysis: For any $y > 0$, $c > 0$,

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^s \frac{ds}{s} = \begin{cases} 1, & y > 1, \\ \frac{1}{2}, & y = 1, \\ 0, & y < 1. \end{cases} \tag{1.29}$$

From this, we have that

$$\sum\nolimits'_{n \le x} a_n = \sum_{n<x} a_n + \begin{cases} \frac{a_x}{2}, & x \in \mathbb{N}, \\ 0, & \text{otherwise} \end{cases} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} A(s) x^s s^{-1} ds, \tag{1.30}$$

where $c > 0$ is such a number that $A(s)$ is absolutely convergent for $\Re(s) = c$. This formula (1.30) is known as *Perron's Formula.*
Let $A(s) = \sum_{n=1}^{\infty}$ converge absolutely for $\sigma = \Re(s) > 1$ and let $| a_n | < C\Phi(n)$, where $C > 0$ and for $x \ge x_0$ $\Phi(x)$ is monotonically increasing. Let further

$$\sum_{n=1}^{\infty} | a_n | n^{-\sigma} \ll (\sigma - 1)^{-\alpha}$$

as $\sigma \to 1 + 0$ for some $\alpha > 0$. If $w = u + iv$ $(u, v$ real) is arbitrary, $b > 0$, $T > 0$, $u + b > 1$, then

$$\begin{aligned} \sum_{n<x} a_n n^{-w} = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} & A(s+w) x^s s^{-1} ds + O(x^b T^{-1} (u+b-1)^{-\alpha}) \\ & + O(T^{-1} \Phi(2x) x^{1-u} \log 2x) + O(\Phi(2x) x^{-u}), \end{aligned} \tag{1.31}$$

and the estimate is uniform in $x, T, b,$ and u provided that b and u are bounded. Thus the above equation (1.31) is the *Inversion Formula* for Dirichlet series.

1.4 Approximate Formulae

Let us look at the simplest theorem on the approximation to the Riemann zeta function $\zeta(s)$ in the critical strip by a partial sum of its Dirichlet series (see [4]) which we present as

Theorem 1.4.1. *Given $s = \sigma + it$, then we have*

$$\zeta(s) = \sum_{n \le x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma}) \tag{1.32}$$

uniformly for $\sigma \ge \sigma_0 > 0$, $\mid t \mid < 2\pi x/C$, when $C > 1$ is a given constant.

Proof. From the Euler-Maclaurin formula (A.9), letting $f(n) = n^{-s}$, where $s \ne 1$, we obtain

$$\sum_{n=a+1} \frac{1}{n^s} = \frac{b^{1-s} - a^{1-s}}{1-s} - s\int_a^b \frac{x - [x] - \frac{1}{2}}{x^{s+1}} dx + \frac{1}{2}(b^{-s} - a^{-s}) \tag{1.33}$$

In (1.33), take $\sigma > 1$, $a = N$, and make $b \to \infty$. We obtain

$$\zeta(s) - \sum_{n=a+1} \frac{1}{n^s} = s\int_a^b \frac{x - [x] - \frac{1}{2}}{x^{s+1}} dx + \frac{N^{1-s}}{s-1} - \frac{1}{2}N^{-s}, \tag{1.34}$$

We have, by (1.34),

$$\begin{aligned} \zeta(s) &= \sum_{n=1}^{N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} + s\int_N^\infty \frac{[u] - u + \frac{1}{2}}{u^{s+1}} du - \frac{1}{2}N^{-s} \\ &= \sum_{n=1}^{N} \frac{1}{n^s} - \frac{N^{1-s}}{1-s} + O\Big(\frac{\mid s \mid}{N^\sigma}\Big) + O(N^{-\sigma}). \end{aligned} \tag{1.35}$$

The sum

$$\sum_{x<n\le N} \frac{1}{n^s} = \sum_{x<n\le N} \frac{n^{-it}}{n^\sigma}$$

is of the form considered in (A.15), with $g(u) = u^{-\sigma}$, and

$$f(x) = -\frac{t\log u}{2\pi}, \quad f'(u) = -\frac{t}{2\pi u}.$$

Thus

$$| f'(u) | \leq \frac{t}{2\pi x} < \frac{1}{C}. \tag{1.36}$$

Hence

$$\sum_{x<n\leq N} \frac{1}{n^s} = \int_x^N \frac{du}{u^s} + O(x^{-\sigma}) = \frac{N^{1-s} - x^{1-s}}{1-s} + O(x^{-\sigma}).$$

Therefore,

$$\zeta(s) = \sum_{n\leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma}) + O\Big(\frac{| s | +1}{N^\sigma}\Big).$$

Making $N \to \infty$, the result follows. □

The sum involved in Theorem 1.4.1 contains too many terms to be of use. We therefore consider the result of taking smaller values of x in the above formulae. The form of the result is given on the appendix (A.14), with an extra factor $g(n)$ in the sum. If we ignore error terms for the moment, this gives

$$\sum_{a<n\leq b} g(n)e^{2\pi i f(n)} \sim e^{-\frac{1}{4}\pi i} \sum_{\alpha<v\leq\beta} \frac{e^{2\pi i\{f(x_v)-vx_v\}}}{| f''(x_v) |^{\frac{1}{2}}} g(x_v).$$

Taking

$$g(u) = u^{-\sigma}, \quad f(u) = \frac{t\log u}{2\pi}, \quad f'(u) = \frac{t}{2\pi u},$$

$$f''(u) = -\frac{t}{2\pi u^2}, \quad x_v = \frac{t}{2\pi v}, \quad f''(x_v) = -\frac{2\pi v^2}{t},$$

and replacing a, b by x, N, and i by $-i$, we obtain

$$\sum_{x<n\leq N} \frac{1}{n^s} \sim e^{\frac{1}{4}\pi i} \sum_{t/2\pi N<v\leq t/2\pi x} \frac{e^{-2\pi i\{(t/2\pi)\log(t/2\pi v)-(t/2\pi)\}}}{(t/2\pi v)^\sigma(2\pi v^2/t)^{\frac{1}{2}}}$$

$$= \Big(\frac{t}{2\pi}\Big)^{\frac{1}{2}-\sigma} e^{\frac{1}{4}\pi i - it\log(t/2\pi e)} \sum_{t/2\pi N<v\leq t/2\pi x} \frac{1}{v^{1-s}}.$$

Now the functional equation is

$$\zeta(s) = \chi(s)\zeta(1-s),$$

where $\quad \chi(s) = 2^{s-1}\pi^s \sec\frac{1}{2}s\pi/\Gamma(s)$.

In any fixed strip $\alpha \leq \sigma \leq \beta$, as $t \to \infty$ can be easily shown as follows

$$\log\Gamma(\sigma + it) = (\sigma + it - \frac{1}{2})\log(it) - it + \frac{1}{2}\log 2\pi + \mathcal{O}(\frac{1}{t}).$$

Hence
$$\Gamma(\sigma+it) = t^{(\sigma+it-\frac{1}{2})}e^{-\frac{1}{2}\pi t-it+\frac{1}{2}2\pi(\sigma-\frac{1}{2})}(2\pi)^{\frac{1}{2}}\{1+\mathcal{O}(\tfrac{1}{t})\},$$
$$\chi(s) = (\frac{2\pi}{t})^{\sigma+it-\frac{1}{2}}e^{i(t+\frac{\pi}{4})}(1+\mathcal{O}(\frac{1}{t})) \tag{1.37}$$
Hence the above relation is equivalent to

$$\sum_{x<n\leq N}\frac{1}{n^s} \sim \chi(s)\sum_{\frac{t}{2\pi}N<v\leq\frac{t}{2\pi}x}\frac{1}{v^{1-s}}.$$

The formula therefore suggests that, with some suitable error terms,

$$\zeta(s) \sim \sum_{n\leq x}\frac{1}{n^s} + \chi(s)\sum_{v\leq y}\frac{1}{v^{1-s}},$$

Where $2\pi xy =| t |$.
By the functional equation for $\zeta(s)$ and (1.37) the result is that

$$\zeta(s) = \sum_{n\leq x}\frac{1}{n^s} + \chi(s)\sum_{n\leq y}\frac{1}{n^{1-s}} + O(x^{-\sigma}) + O(y^{\sigma-1}| t |^{\frac{1}{2}-\sigma}) \tag{1.38}$$

for $0 < \sigma < 1$. This is known as the *Approximate functional equation* ([8]; Ch.4). The purpose of the 'approximate functional equation' is to facilitate the study of $\zeta(s)$ in the "critical strip strip" $0 \leq \sigma \leq 1$. The function is represented for $\sigma > 1$ by the formula $\zeta(s) = \sum n^{-s}$, and for $\sigma < 0$ by the formula $\zeta(s) = \chi(s)\zeta(1-s) = \chi(s)\sum n^{s-1}$. The formula (1.38) is, so to say, a compromise between the two, and it seems to have many important applications.

Chapter 2

The Theory of Exponent Pairs

2.1 Introduction and Definition

The method of exponent pairs was introduced by Van der Corput for estimating certain bounds for exponential sums that arise in number-theoretic problems. We write, as is customary, $e(x)$ for $e^{2\pi ix}$. In this chapter, we will briefly look at the application of exponent pairs in estimating upper bounds for the sum of the form

$$S = \sum_{n\in(N,2N]} e(f(n)) \tag{2.1}$$

where $N > 0$ and $f(x)$ is a many times differentiable real-valued function. In the 1920's J.G. Van der Corput introduced a powerful new method of estimating (2.1), which enabled him in [16] to prove that the error term in Dirichlet's divisor problem is $O(x^{33/100+\varepsilon})$. His method of exponent pairs, simplified later by E. C. Titchmarsh [13] and E. Phillips [11] which brought on remarkable improvements of error terms in many divisor problems and problems connected with the estimation of the zeta-function of Riemann in the critical strip. E. Phillips proved

$$\sum_{a<n\leq b} n^{-1/2-it} \ll t^k a^{l-k-1/2} \tag{2.2}$$

for $1 \leq a < b \leq 2a < t/\pi$ and any pair (k, l) called exponent pair, satisfying $l - k \geq 1/2$. As shown by R.A.Rankin [12], the best result concerning the order of $\zeta(1/2+it)$ and obtainable from (2.2) is

$$\zeta(1/2+it) \ll t^{\alpha/2+\varepsilon}, \quad t \geq t_0, \quad \alpha = 0.3290213568... \tag{2.3}$$

Rankin's paper contains a good account of applications of Van der Corput's method to various other problems. The sharpest known result concerning the order $\zeta(1/2+it)$ is due to *Jean Bourgain* who proved in his recent work [1] that

$$\zeta(1/2+it) \ll t^{13/84+\varepsilon}, \qquad t \geq t_0. \tag{2.4}$$

The proof of inequality (2.4) will be presented at the end chapter as proved in J. Bourgain's paper.

Definition 2.1.1. A pair (k,l) of real numbers is called an *exponent pair* if $0 \leq k \leq \frac{1}{2} \leq l \leq 1$, and if for each $s > 0$ there exists an integer $r > 4$ and real $c \in (0, \frac{1}{2})$ depending only on s such that the inequality

$$\sum_{a<n\leq b} e(f(n)) \ll z^k a^l \tag{2.5}$$

holds with respect to s and u when the following conditions are satisfied:

$$u > 0, \quad 1 \leq a < b < au, \quad y > 0, \quad z = ya^{-s} > 1; \tag{2.6}$$

$f(n)$ a real function with differential coefficients of the first r orders in (a,b) and

$$\mid f^{(p+1)}(n) - y\frac{d^p}{dt^p}t^{-s} \mid < (-1)^p cy \frac{d^p}{dt^p}t^{-s} \tag{2.7}$$

for $a \leq n \leq b$ and $0 \leq p \leq r-1$, where ya^{-s} is the order of magnitude of f'.

It follows immediately that $(0,1)$ is an exponent pair since $\sum_{a<n\leq b} e(f(n)) \leq b - a < au = uz^0 a$ which is (2.5) with $k=0, l=1$.

In most applications of exponent pairs, the ultimate result is expressed as a function of k and l of the form

$$\theta(k,l) = \frac{ak+bl+c}{dk+el+f} \tag{2.8}$$

where $a,...,f$ are real numbers and (k,l) is an exponent pair.

For example, one can show that

$$\zeta(\frac{1}{2}+it) \ll t^{\theta(k,l)} \log t \quad (t \geq 2), \quad \text{where} \quad \theta(k,l) = \frac{1}{4}(2k+2l-1)$$

In this chapter, we will briefly discuss the upper bounds for $\zeta(s)$ applying the method of exponent pairs. The purpose of estimating exponential sums using exponent pairs will be seen best in the next chapter to obtain general power moment estimates for $\zeta(s)$.

2.2 The A-B Processes and Some Lemmas

The construction of new exponent pairs from the old ones is achieved by the following two Theorems (see [2])

Theorem 2.2.1. (***Weyl's process***) *If $f(x)$ is a real-valued function and Q is an integer not exceeding $b-a$, then*

$$\sum_{a<n\le b} e^{(f(n))} \ll (b-a)Q^{-\frac{1}{2}} + \Big\{(b-a)Q^{-1}\sum_{r=1}^{Q-1}\Big|\sum_{a<n\le b-r} e^{(f(n+r)-f(n))}\Big|\Big\}^{\frac{1}{2}} \tag{2.9}$$

Proof. For convenience in the proof, let $e^{(f(n))}$ denote 0 if $n\le a$ or $n>b$. Then

$$\sum_{a<n\le b} e^{(f(n))} = \frac{1}{Q}\sum_{n}\sum_{m=1}^{Q} e^{(f(m+n))},$$

the inner sum vanishing if $n\le -Q$ or $n>b-1$. Hence

$$\Big|\sum_{n} e^{(f(n))}\Big| \le \frac{1}{Q}\sum_{n}\Big|\sum_{m=1}^{Q} e^{(f(m+n))}\Big| \le \frac{1}{Q}\Big\{\sum_{n} 1\Big|\sum_{m=1}^{Q} e^{(f(m+n))}\Big|\Big\}^{\frac{1}{2}}.$$

Since there are at most $b-a+q\le 2(b-a)$ values of n for which the inner sum does not vanish, this does not exceed

$$\frac{1}{Q}\Big\{2(b-a)\sum_{n}\Big|\sum_{m=1}^{Q} e^{(f(m+n))}\Big|\Big\}^{\frac{1}{2}}.$$

Now

$$\begin{aligned}\Big|\sum_{m=1}^{Q} e^{(f(m+n))}\Big|^{2} &= \sum_{m=1}^{Q}\sum_{\mu=1}^{Q} e^{(f(m+n)-f(\mu+n))}\\ &= Q + \sum\sum_{\mu<m} e^{(f(m+n)-f(\mu+n))} + \sum\sum_{m<\mu} e^{(f(m+n)-f(\mu+n))}\end{aligned}$$

Hence

$$\sum_{n}\Big|\sum_{m=1}^{Q} e^{(f(m+n))}\Big|^{2} \le 2(b-a)Q + 2\Big|\sum_{n}\sum_{\mu<m}\sum e^{(f(m+n)-f(\mu+n))}\Big|.$$

In the last sum, $f(m+n)-f(\mu+n)=f(v+r)-f(v)$, for given values of v and r, $1\le r\le q-1$, just $q-r$ times, namely $\mu=1, m=r+1$, up to $\mu=q-r, m=q$, with a consequent value of n in each case. Hence the modulus of this sum is equal to

$$\mid \sum_{r=1}^{Q-1}(Q-r)\sum_{v} e^{(f(v+r)-f(v))} \mid\le Q\sum_{r=1}^{Q-1}\mid \sum_{v} e^{(f(v+r)-f(v))}\mid .$$

Hence

$$\mid \sum_{n} e^{(f(n))}\mid\le \frac{1}{Q}\Big\{4(b-a)^2Q+4(b-a)Q\sum_{r=1}^{Q-1}\mid \sum_{v} e^{(f(v+r)-f(v))}\mid \Big\}^{\frac{1}{2}}.$$

and the result stated follows. □

Theorem 2.2.2. (***Van der Corput's process***) *If f is a real-valued function which is five times differentiable in (a,b) such that $f''(x)<0$ in (a,b) and for some $C>1$*

$$m_2\le\mid f''(x)\mid< Cm_2,\quad \mid f^{(3)}(x)\mid< Cm_3,\quad \mid f^{(4)}(x)\mid< Cm_4,\quad {m_3}^2=m_2m_4,$$

Then

$$\sum_{a<n\le b} e^{(f(n))}=e^{-\frac{1}{8}}\sum_{\alpha\le v\le\beta}\mid f''(n_v)\mid^{-\frac{1}{2}}e^{(f(n_v)-vn_v)}+O({m_2}^{-\frac{1}{2}})+ \\ O(\log{(2+(b-a)m_2)})+O((b-a){m_3}^{\frac{1}{3}}), \tag{2.10}$$

$$\textit{where}\quad f'(b)=\alpha,\quad f'(a)=\beta,\quad f'(n_v)=v\quad \textit{for}\quad \alpha\le v\le\beta.$$

Proof. We begin from a result which transforms an exponent sum into a sum of exponential integrals, which are easier to estimate. By Lemma 2.4 from [8] for any η with $0<\eta<1$ we have that

$$\sum_{a<n\le b} e(f(n))=\sum_{\alpha-\eta<m<\beta+\eta}\int_a^b e(f(x)-mx)dx+O(\log(\beta-\alpha+2)). \tag{2.11}$$

By mean value theorem

$$\beta-\alpha\ll(b-a)m_2. \tag{2.12}$$

We have a result for estimating exponential integrals from (Lemma 2.2; [8]) that if $m\le\mid f''(x)\mid< Cm$, then

$$\Big|\int_a^b e^{if(x)}dx\Big|\le 8m^{-1/2}. \tag{2.13}$$

Then by equation (2.13) the limits of summation coming from equation (2.11) may be replaced by $\alpha+1$ and $\beta-1$ with an error $\ll {m_2}^{-1/2}$. we also have another result from [8] presented in a form of theorem called "saddle-point" Theorem 2.1 from [8], which shows that the main contribution to the exponential integral comes from its saddle point, provided that the conditions in Theorem 2.2.2 are satisfied. If $f'(c)=0$ for some $a \le c \le b$, then

$$\begin{aligned}\int_a^b e(f(x))dx = e(f(c)-\frac{1}{8})\mid f''(c)\mid^{-1/2} + O({m_2}^{-1}{m_3}^{1/3})\\ +O(\min({m_2}^{-1/2}, \mid f'(a)\mid^{-1})) + O(\min({m_2}^{-1/2}, \mid f'(b)\mid^{-1})).\end{aligned} \tag{2.14}$$

An application of (2.14) then gives

$$\begin{aligned}\sum_{\alpha+1<v<\beta-1}\int_a^b e(f(x)-vx)dx = e^{(-\frac{1}{8})}\sum_{\alpha+1<v<\beta-1} f''(x_v)^{-1/2}e(f(x_v)-vx_v)\\ +O(\sum_{\alpha+1<v<\beta-1} {m_2}^{-1}{m_3}^{1/3}) + (\sum_{\alpha+1<v<\beta-1}((v-\alpha)^{-1}+(\beta-v)^{-1})).\end{aligned}$$

In view of (2.12) the first O-term above is $O((b-a){m_3}^{1/3})$, and the second is $O(\log(\beta-\alpha+2)) = O(\log((b-a)m_2+2))$, which ends the proof of (2.10), since again by (2.13) the limits of summation $(\alpha+1, \beta-1)$ may be changed to (α, β) with an error which is $\ll {m_2}^{-1/2}$. □

Note that if $f''(x)>0$, but the remaining hypotheses of Theorem 2.2.2 are satisfied, then one can apply this Theorem with $-f(x)$ instead of $f(x)$. With the notation introduced earlier (2.10) becomes

$$\begin{aligned}S = e^{-\frac{1}{8}}\sum_{\alpha\le v\le\beta}\mid f''(n_v)\mid^{-\frac{1}{2}} e^{(f(n_v)-vn_v)} + O(A^{-\frac{1}{2}}B^{\frac{1}{2}})+\\ O(\log(2+A)) + O((AB)^{\frac{1}{3}}),\end{aligned} \tag{2.15}$$

Trivially, $S \ll B = A^0B^1$, and likewise (2.15) gives

$$S \ll (\beta-\alpha)(AB^{-1})^{-\frac{1}{2}} + A^{-\frac{1}{2}}B^{\frac{1}{2}} + (AB)^{\frac{1}{3}} \ll (AB)^{\frac{1}{2}}, \tag{2.16}$$

which means that $(0,1)$ and $(\frac{1}{2},\frac{1}{2})$ are exponent pairs. Exponent pairs form a convex set, as shown by

Lemma 2.2.3. *If* (χ_1, λ_1) *and* (χ_2, λ_2) *are arbitrary exponent pairs and* $0 \le t \le 1$ *is arbitrary, then* $(\chi_1 t + (1-t)\chi_2, \lambda_1 t + (1-t)\lambda_2)$ *is also an exponent pair.*

Proof. Let us start from the exponential sum

$$S = \sum_{B<n\leq B+h} e(f(n)) \qquad (B \geq 1, 1 < h \leq B).$$

We know from Definition 2.1.1 and the above Theorem that the estimation of exponential sum S depends on the number of summands, which is $\leq B$, and on the order of the first derivative of f. Therefore we shall suppose that

$$A \ll \mid f'(x) \mid \ll A \qquad (A > \frac{1}{2})$$

when $B \leq x \leq 2B$, and we seek an upper bound for $\mid S \mid$ of the form

$$S \ll A^{\chi} B^{\lambda}.$$

$$S = S^t S^{1-t} \ll (A^{\chi_1} B^{\lambda_1})^t (A^{\chi_2} B^{\lambda_2})^{(1-t)},$$
$$S \ll A^{\chi_1 t+(1-t)\chi_2} B^{\lambda_1 t+(1-t)\lambda_2},$$

which implies that $(\chi_1 t + (1-t)\chi_2, \lambda_1 t + (1-t)\lambda_2)$ is also an exponent pair. □

Using (2.9) it is seen that S is transformed into sums of the same type, only now $f(x)$ is replaced by $g(x) = f(x+r) - f(x)$, giving $rAB^{-1} \ll \mid g'(x) \mid \ll rAB^{-1}$. The optimal choice of Q leads then to

Lemma 2.2.4. *If (k, l) is an exponent pair, then $(\frac{k}{2k+2}, \frac{k+l+1}{2k+2})$ is also an exponent pair.*

Proof. We shall prove this Lemma based on some Lemmas from [2]. Let $\chi = \frac{k}{2k+2}$ and $\lambda = \frac{k+l+1}{2k+2}$. Since $0 \leq k \leq \frac{1}{2}$, $\frac{1}{2} \leq l \leq 1$, we have

$$0 \leq \chi = \frac{k}{2k+2} < \frac{k+1}{2k+2} = \frac{1}{2} \quad \text{and} \quad \frac{1}{2} \leq \lambda = \frac{1}{2} + \frac{l}{2k+2} \leq 1.$$

Now, let $y, N,$ and s be positive. We need to show that there exists $P_1 > 0$ and $\epsilon_1 > 0$ such that if $f \in F(N, P_1, s, y, \epsilon_1)$ and $L = yN^{-s} \geq 1$ then

$$\sum_{n\in I} e(f(n)) \ll L^{\chi} N^{\lambda}.$$

where f is defined on $I = [a, b] \subseteq [N, 2N]$. Our proof breaks into two cases: (i) $L \geq \log N$, (ii) $1 \leq L < \log N$. We begin with case (i).

Since (k, l) is an exponent pair, we know that there exists $P > 0$ and $\epsilon(0 < \epsilon < 1/2)$ such that if $f \in F(N, P, s, y, \epsilon)$ then

$$\sum_{n \in I} e(f(n)) \ll (yN^{-s})^k N^l + y^{-1} N^s.$$

We will show that we may take $P_1 = P + 1$ and $\epsilon_1 = \epsilon/3$.
Assume that $H \leq \min(b-a, 2\epsilon N/(s+P))$. Let S be as defined in (2.1). Weyl-van der Corput inequality on [Lemma 2.5; [2]] states if $H \leq N$ then

$$| S |^2 \ll \frac{| N |^2}{H} + \frac{| N |}{H} \sum_{1 \leq h \leq H} | S_1(h) | .$$

where

$$S_1(h) = \sum_{\{a<n\leq b, \quad a<n+h\leq b\}} e(f_1(n; h)), \tag{2.17}$$

and $f_1(n; h) = f(n+h) - f(n)$. By Lemma 3.7 from [2], $f_1 \in F(N, P, shy, s+1, \epsilon)$, so the exponent pair (k, l) may applied to $S_1(h)$, we obtain

$$\begin{aligned} | S |^2 &\ll H^{-1}N^2 + H^{-1}N \sum_{1 \leq h \leq H} | \{(hLN^{-1})^k N^l + h^{-1}L^{-1}N\} \\ &\ll H^{-1}N^2 + H^k L^k N^{l-k+1} + H^{-1}L^{-1}N^2 \log N. \end{aligned}$$

Since we are assuming that $L \geq \log N$, the first term dominates the third. Applying (Lemma 2.4; [2]] and using the upper bound $H \leq \min(b-a, 2\epsilon N/(s+P))$ the above equation gives

$$S \ll L^\chi N^\lambda + N(b-a)^{-1/2}.$$

If the first term dominates, we are done. Otherwise, we employ the trivial estimate to get

$$S \ll \min(N(b-a)^{-1/2}, b-a) \ll N^{2/3}.$$

Since $k \leq 1/2$ and $l \geq 1/2$, we have

$$\lambda = \frac{1}{2} + \frac{l}{2k+2} \geq \frac{2}{3},$$

and the desired estimate $S \ll N^\chi L^\lambda$ follows.
The remaining case where $1 \leq L \leq \log N$ is easily dispatched. By Lemma 2.2 from [2],

$$S \ll N^{1/2}(\log N)^{1/2} \ll N^{2/3} \ll L^\chi N^\lambda.$$

□

Relation (2.15) offers another possibility for the construction of exponent pairs. The sum on the right-hand side of (2.15) (after removing f'' by partial summation) is of the same type as S, only now A and B are interchanged. This leads to an involutory process for the construction of exponent pairs, given by

Lemma 2.2.5. *If (k,l) is an exponent pair, then $(\chi,\lambda)=(l-\frac{1}{2},k+\frac{1}{2})$ is also an exponent pair, provided that $2k+l\geq 1$.*

Proof. First, we observe that $0\leq\chi\leq 1/2\leq\lambda\leq l\leq 1$ follows immediately from $0\leq k\leq l\leq 1$. Moreover, if $l=1/2$, then, by the remarks at the end of section 3.3 on [2], we have $k=1/2$. It follows that $(\chi,\lambda)=(0,1)$ is an exponent pair by the trivial estimate. We may henceforth assume that $l\geq 1/2$ and $\chi>0$.
Assume that $y>0,s>0,N>0$, and that $L=yN^{-s}\geq 1$. We want to find P_1 and ϵ_1 such that if $f\in F(N,P_1,s,y,\epsilon_1)$ then

$$S=\sum_{n\in I}e^{(f(n))}\ll L^{\chi}N^{\lambda}.$$

Since (k,l) is an exponent pair, we know that there exist $P>0$ and $\epsilon(0<\epsilon<1/2)$ such that if $f\in F(N,P,s,y,\epsilon)$ then

$$\sum_{a<n\leq b}e(f(n))\ll(yN^{-s})^{k}N^{l}+y^{-1}N^{s},\tag{2.18}$$

holds. We will show that we may take $P_1=P$ and $\epsilon_1=\epsilon/C$, where $C=C(s,P)$ is the constant occurring in (Lemma 3.9; [2]). Since f satisfies the hypothesis of (Lemma 3.6; [2]) with $F=LN$, we may write

$$S=\sum_{\alpha\leq v\leq\beta}\frac{e(-\phi(v)-1/8)}{\mid f''(x_v)\mid^{1/2}}+O(\log(2L)+L^{-1/2}N^{1/2}).\tag{2.19}$$

By Lemma 3.9 from [2],

$$T(w)=\sum_{\alpha<v\leq w}e^{(\phi(v))}\ll(\eta J^{-\sigma})^{k}J^{l}+\eta^{-1}J^{\sigma}\ll N^{k}J^{l}+N^{-1}.$$

Consequently, the sum in (2.19) is

$$\begin{aligned}\int_{\alpha}^{\beta}\mid f''(x_w)\mid^{-1/2}d\overline{T(w)}&=\overline{T(w)}\mid f''(x_w)\mid^{-1/2}]_{\alpha}^{\beta}-\int_{\alpha}^{\beta}\overline{T(w)}\frac{d}{dw}\mid f''(x_w)\mid^{-1/2}dw\\&\ll(N^{k}L^{l}+N^{-1})\{(LN^{-1})^{-1/2}+\int_{\alpha}^{\beta}\mid\frac{d}{dw}\mid f''(x_w)\mid^{-1/2}\mid dw\}\\&\ll L^{\chi}N^{\lambda}+L^{-1/2}N^{-1/2}.\end{aligned}$$

All together, we get

$$S \ll L^{\chi} N^{\lambda} + \log(2L) + L^{-1/2} N^{1/2}.$$

Since we are assuming that $\chi > 0$ and $L \geq 1$, the first term dominates and the result is proved. □

Classically most exponent pairs are usually produced from $(0,1)$, $(\frac{1}{2}, \frac{1}{2})$ and the processes described in Lemmas 2.2.3, 2.2.4 and 2.2.5, in which case the condition $2\chi + \lambda \geq 1$ is always satisfied. Moreover the condition (2.6) may be replaced by the less stringent one $AB^{1-r} \ll | f^{(r)} | \ll AB^{1-r}$, $r = 1, 2, ...$

Lemma 2.2.6. *If (χ, λ) is an exponent pair, then so is (χ_q, λ_q), where*

$$\chi_q = \frac{\chi}{Q + 2(Q-1)\chi}, \quad \lambda_q = 1 - \frac{1 - \lambda + q\chi}{Q + 2(Q-1)\chi}$$

and q is an integer ≥ 1, $Q = 2^q$.

Proof. In the case $q = 1$ this lemma reduces to Lemma 2.2.4. we prove the general case by induction. We suppose the theorem is true for a particular value of q, that is, we suppose that (χ_q, λ_q) is an exponent pair. Then, by Lemma 2.2.4, so is the pair

$$\Big(\frac{\chi_q}{2 + 2\chi_q}, 1 - \frac{1 - \lambda_q + \chi_q}{2 + 2\chi_q}\Big),$$

i.e.

$$\Big(\frac{\chi}{2Q + 2(2Q-1)\chi}, 1 - \frac{Q + (2Q-2)\chi - Q + 1 - \lambda - (2Q - q - 2)\chi + \chi}{2Q + 2(2Q-1)\chi}\Big),$$

i.e.

$$\Big(\frac{\chi}{2Q + 2(2Q-1)\chi}, 1 - \frac{1 - \lambda + (q+1)\chi}{2Q - 2(2Q-1)\chi}\Big),$$

That is

$$(\chi_{q+1}, \lambda_{q+1}).$$

Thus the Lemma follows by induction. □

Lemma 2.2.7. *If (χ, λ) is an exponent pair, then so is (k, l), where*

$$k = \frac{1}{2} - \frac{1 - \lambda + q\chi}{Q + 2(Q-1)\chi}, \quad l = \frac{1}{2} + \frac{\chi}{Q + 2(Q-1)\chi}$$

and q is an integer ≥ 1, $Q = 2^q$.

Proof. By Lemma 2.2.6

$$\Big(\frac{\chi}{Q+2(Q-1)\chi}, 1-\frac{1-\lambda+q\chi}{Q+2(Q-1)\chi}\Big)$$

is an exponent pair. Hence, by Lemma 2.2.5,

$$\Big(\frac{1}{2}-\frac{1-\lambda+q\chi}{Q+2(Q-1)\chi}, \frac{1}{2}+\frac{\chi}{Q+2(Q-1)\chi}\Big),$$

i.e. (k,l) is an exponent pair, provided that $2k+l \geq 1$. This condition is equivalent to

$$1-2\frac{1-\lambda+q\chi}{Q+2(Q-1)\chi}+\frac{1}{2}+\frac{\chi}{Q+2(Q-1)\chi}\Big) \geq 1,$$

i.e

$$2\{2-2\lambda+(2q-1)\chi\} \leq Q+2(Q-1)\chi,$$

and this is true because

$$2(Q-1)\chi \geq 2(2q-1)\chi, \quad \text{since} \quad \chi \geq 0,$$

$$\text{and} \qquad Q \geq 2 \geq 4(1-\lambda), \quad \text{since} \quad \lambda \geq \frac{1}{2}.$$

Therefore the condition is satisfied for all values of q and the Lemma is proved. □

Let P denote the set of all exponent pairs generated from $(0,1)$ by A- and B-processes. Many asymptotic questions of number theory (especially in the area of divisor problems) come to the optimization problem of the form

$$\inf_{(k,l)\in P}\{\theta(k,l)|R_i(\alpha_i k+\beta_i l+\gamma_i), \qquad i=1,...,j\},$$

where $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}$, $R_i \in \{R_>, R_\geq\}$, the symbol $R_>$ checks whether its argument is a positive value and $R_\geq$ checks whether its argument is non-negative, $i=1,2,...,j$. Graham [3] gave an effective method of computing $\inf \theta(k,l)$, which in many cases is able to determine

$$\inf_{(k,l)\in P}\{\theta(k,l)\}$$

with a given precision (for certain θ-even exactly), where

$$\theta \in \Theta := \Big\{(k,l) \mapsto \frac{ak+bl+c}{dk+el+f}\,\Big|\, a,b,c,d,e,f \in \mathbb{R}, \quad dk+el+f>0 \quad \text{for} \quad (k,l)\in P\Big\}.$$

This method of computing the infimum of θ over P is called the *Graham Algorithm* (See [2]). Later another effective algorithm of computation was developed (see [10]). The output of these computing algorithms led to some improved estimates in the Dirichlet divisor problems.

2.3 The order of $\zeta(s)$ in the critical strip

In this section we will briefly discuss the order of $\zeta(s)$ as $t \to \infty$ in the interval $0 \leq \sigma \leq 1$. The problem of the order of $\zeta(s)$ in the critical strip is yet unsolved. But it is clear from the definition (1.2) that $\zeta(s)$ is bounded in any half-plane $\sigma \geq 1 + \delta > 1$; and we have proved in (1.21) that

$$\zeta(s) = O(\mid t \mid) \quad (\sigma \geq \frac{1}{2}).$$

For $\sigma < \frac{1}{2}$, the order of $\zeta(s)$ follows from the functional equation (1.14)

$$\zeta(s) = \chi(s)\zeta(1-s)$$

In any fixed strip $\alpha \leq \sigma \leq \beta$, as $t \to \infty$, by (1.17) we have

$$\mid \chi(s) \mid \sim (\frac{t}{2\pi})^{\frac{1}{2}-\sigma}$$

Hence,

$$\zeta(s) \ll t^{\frac{1}{2}-\sigma} \quad (\sigma \leq -\delta < 0) \quad \text{and} \quad \zeta(s) \ll t^{\frac{3}{2}+\sigma} \quad (\sigma \geq -\delta) \tag{2.20}$$

Thus in any half plane $\sigma \geq \sigma_0$ the zeta-function $\zeta(s) \ll t^c$, where $c = c(\sigma_0)$
Definition 2.3.1. The order for $\zeta(s)$ in the "critical strip" $0 \leq \sigma \leq 1$ is a function $\mu(\sigma)$ such that for each real σ in the interval $\mu(\sigma) = \inf\{c \geq 0 : \zeta(\sigma + it) \ll t^c\}$, or alternatively as

$$\mu(\sigma) = \limsup_{t \to \infty} \frac{\log \mid \zeta(\sigma + it) \mid}{\log t} \tag{2.21}$$

It follows from the general theory of Dirichlet series that the function $\mu(\sigma)$ is continuous, non-increasing, and convex downwards in the sense that no arc of the curve $y = \mu(\sigma)$ has any point above its chord; also it is never negative. Furthermore, for $\sigma_1 \leq \sigma \leq \sigma_2$, we have

$$\mu(\sigma) \leq \mu(\sigma_1)\frac{\sigma_2 - \sigma}{\sigma_2 - \sigma_1} + \mu(\sigma_2)\frac{\sigma - \sigma_1}{\sigma_2 - \sigma_1} \tag{2.22}$$

Since $\zeta(s)$ is bounded for $\sigma \geq 1 + \delta(\delta > 0)$, it follows that

$$\mu(\sigma) = 0 \quad \text{for} \quad \sigma > 1 \tag{2.23}$$

and then from the functional equation that

$$\mu(\sigma) = \frac{1}{2} - \sigma, \quad \text{for} \quad \sigma < 0 \tag{2.24}$$

Equation (2.23) and (2.24) also hold by continuity for $\sigma = 1$ and $\sigma = 0$ respectively. The chord joining the points $(0, \frac{1}{2})$ and $(1, 0)$ on the curve $y = \mu(\sigma)$ is $y = \frac{1}{2} - \frac{1}{2}\sigma$. It therefore follows from the convexity property that

$$\mu(\sigma) \leq \frac{1}{2} - \frac{1}{2}\sigma \quad \text{for} \quad 0 < \sigma < 1 \tag{2.25}$$

In particular, $\mu(\frac{1}{2}) \leq \frac{1}{4}$, i.e

$$\zeta(\frac{1}{2} + it) \ll t^{\frac{1}{4}+\epsilon} \tag{2.26}$$

for every positive ϵ. The exact value of $\mu(\sigma)$ for any $0 < \sigma < 1$ remains unknown to this date. The simplest possible hypothesis is that the graph of $\mu(\sigma)$ consists of two straight lines

$$\mu(\sigma) = \frac{1}{2} - \sigma \quad (\sigma \leq \frac{1}{2}), \quad 0 \quad (\sigma > \frac{1}{2}). \tag{2.27}$$

This is known as *Lindelöf's Hypothesis.* It is equivalent to the statement that

$$\zeta(\frac{1}{2} + it) = O(t^{\varepsilon}) \qquad \text{for every positive} \quad \varepsilon. \tag{2.28}$$

The approximate functional equation gives a slight refinement on the above results. For example taking $\sigma = \frac{1}{2}$, $x = y = \sqrt{\frac{t}{2\pi}}$ in the approximate functional equation (1.38), we obtain

$$\zeta(\frac{1}{2} + it) \ll \sum_{n \leq \sqrt{\frac{t}{2\pi}}} \frac{1}{n^{\frac{1}{2}+it}} + \sum_{n \leq \sqrt{\frac{t}{2\pi}}} \frac{1}{n^{\frac{1}{2}-it}} + t^{-\frac{1}{4}} \ll \sum_{n \leq \sqrt{\frac{t}{2\pi}}} \frac{1}{n^{\frac{1}{2}}} + t^{-\frac{1}{4}} \ll t^{\frac{1}{4}} \tag{2.29}$$

2.4 Zeta-Function and Exponent Pairs

In this section, we shall discuss the connection of the theory of exponent pairs with the problem of the order of the Riemann zeta-function. Here we shall bound $\zeta(s)$ in the range $\frac{1}{2} \leq \sigma \leq 1$ and $t \geq 3$. Let us first present two lemmas that reduce the upper bound problem to estimates of finite exponential sums.

Lemma 2.4.1. *If $\frac{1}{2} \leq \sigma \leq 1$ and $N \leq M$ then*

$$\sum_{N<n\leq M} n^{-\sigma-it} \ll N^{-\sigma} \max_{N<u\leq M} \mid \sum_{N<n\leq u} n^{-it} \mid$$

Proof. Let

$$S(u) = \sum_{N<n\leq u} n^{-it}$$

Then

$$\sum_{N<n\leq M} n^{-\sigma-it} = \int_N^M u^{-\sigma} dS(u) = S(M)M^{-\sigma} + \sigma \int_N^M S(u)u^{-\sigma-1} du$$
$$\ll N^{-\sigma} \max_{N<u\leq M} \mid S(u) \mid,$$

and the desired result follows. □

Lemma 2.4.2. *If $\frac{1}{2} \leq \sigma \leq 1$ and $t \geq 3$ then*

$$\mid \zeta(\sigma+it) \mid \ll \mid \sum_{n\leq t} n^{-\sigma-it} \mid + t^{1-2\sigma} \log t$$

Proof. If $\sigma > 1$ and $M \geq 1$ then

$$\zeta(s) = \sum_{n\leq M} n^{-s} + \int_M^\infty u^{-s} d[u] = \sum_{n\leq M} n^{-s} + \frac{M^{1-s}}{s-1} + s\int_M^\infty \frac{[u]-u}{u^{s+1}} du.$$

The last integral converges for $\sigma > 0$, so this gives an analytic continuation of $\zeta(s)$ to the region $\sigma > 0$, $s \neq 1$. We set $M = t^2$ and use the inequality $\mid [u] - u \mid \leq 1$ to get

$$\zeta(s) = \sum_{n\leq t^2} n^{-s} + O(t^{1-2\sigma}).$$

The sum over the range $t < n \leq t^2$ may be divided into $\log t$ sub-sums of the form

$$\sum_{N<n\leq N_1} n^{-\sigma-it}.$$

where $N_1 = \min(2N, t^2)$. From Lemma 2.4.1 and (Theorem 2.1; [2]), we see that each of the above sub-sums is $\ll N^{1-\sigma}t^{-1} \ll t^{1-2\sigma}$, and the result follows. □

Applying the above two Lemmas we can show the problem $\mid \zeta(\frac{1}{2}+it) \mid \ll t^{\theta(k,l)} \log t$ where $\theta(k,l)$ a function of exponent pair (k,l). Let us now look at the role of exponent pairs in the order estimates of the zeta-function.

Theorem 2.4.3.

$$\zeta(\frac{1}{2}+it) = O(t^{\frac{229}{1392}})$$

Proof. We begin by proving that

$$\zeta(\frac{1}{2}+it) = O(t^{\frac{1}{2}(k+l-\frac{1}{2})}), \tag{2.30}$$

where (k,l) is any exponent pair such that $l-k > \frac{1}{2}$. From the approximate functional equation, we have

$$\zeta(\frac{1}{2}+it) = \sum_{n<\sqrt{(t/2\pi)}} n^{-\frac{1}{2}+it} + \chi \sum_{n<\sqrt{(t/2\pi)}} n^{-\frac{1}{2}-it} + O(t^{-\frac{1}{4}}),$$

where $\chi = O(1)$. The inequality (2.30) will follow from this if we can prove that

$$\sum_{n<\sqrt{(t/2\pi)}} n^{-\frac{1}{2}+it} = O(t^{\frac{1}{2}(l+k-\frac{1}{2})}). \tag{2.31}$$

Now the function $f(n) = (t/2\pi)\log n$ satisfies the condition (2.7) in the definition of an exponent pair, if we take $y = t/2\pi$, $s = 1$, $u = 2$, $c = \frac{1}{3}$, and $r = 5$. And if further $1 \leq a < b < 2a$ and $z = t/2\pi a$ the conditions (2.6) are satisfied. Therefore, since (k,l) is an exponent pair, we have

$$\sum_{a\leq n\leq b} e^{2\pi i(t/2\pi)\log n} = O(z^k a^l),$$

i.e

$$\sum_{a\leq n\leq b} n^{it} = O\{(t/a)^k a^l\} = O(t^t a^{l-k})$$

for $1 \leq a < b < 2a < t/\pi$. Hence by the partial summation formula (A.5), we have

$$\sum_{a\leq n\leq b} n^{-\frac{1}{2}+it} = O(t^k a^{l-k-\frac{1}{2}}).$$

If t is large enough to ensure $1 < \sqrt{(t/2\pi)} < t/\pi$, we can apply this with $a = 1, b = 1$; $a = 2, b = 3$;... $a = 2^m$, $b = 2^{m+1}-1$;...; the last value of b being $[\sqrt{(t/2\pi)}]$; and then adding we have, since $l-k-\frac{1}{2} > 0$,

$$\sum_{n<\sqrt{(t/2\pi)}} n^{-\frac{1}{2}+it} = O\{t^k t^{\frac{1}{2}(l-k-\frac{1}{2})}\} = O\{t^{\frac{1}{2}(l+k-\frac{1}{2})}\},$$

which proves (2.31), and therefore (2.30).
Now from the definition 2.1.1 we know that $(0,1)$ is an exponent pair. Hence, by Lemma 2.2.5, so is $(\frac{1}{2},\frac{1}{2})$. Applying Lemma 2.2.6 with $q=2$ we see that

$$\Big(\frac{1}{2}-\frac{1-\frac{1}{2}+2\cdot\frac{1}{2}}{4+6\cdot\frac{1}{2}},\quad \frac{1}{2}+\frac{\frac{1}{2}}{4+6\cdot\frac{1}{2}}\Big),$$

i.e. $(\frac{2}{7},\frac{4}{7})$, is an exponent pair. Applying Lemma 2.2.6 with $q=2$ to this last pair we see that

$$\Big(\frac{1}{2}-\frac{1-\frac{4}{7}+2\cdot\frac{2}{7}}{4+6\cdot\frac{2}{7}},\quad \frac{1}{2}+\frac{\frac{2}{7}}{4+6\cdot\frac{2}{7}}\Big),$$

i.e. $(\frac{13}{40},\frac{22}{40})$, is an exponent pair. Again applying Lemma 2.2.6 with $q=3$ we see that

$$\Big(\frac{1}{2}-\frac{1-\frac{22}{40}+3\cdot\frac{13}{40}}{8+14\cdot\frac{13}{40}},\quad \frac{1}{2}+\frac{\frac{13}{40}}{8+14\cdot\frac{13}{40}}\Big),$$

i.e. $(\frac{97}{251},\frac{132}{251})$, is an exponent pair. Finally applying Lemma 2.2.4 to this last pair we see that

$$\Big(\frac{\frac{97}{251}}{2(1+\frac{97}{251})},\quad \frac{1}{2}+\frac{\frac{132}{251}}{2(1+\frac{97}{251})}\Big),$$

i.e. $(\frac{97}{696},\frac{480}{696})$, is an exponent pair. Since $\frac{480}{696}-\frac{97}{696}>\frac{1}{2}$, we can use this pair in (2.30), and we obtain

$$\zeta(\frac{1}{2}+it)=O(t^{\frac{229}{1392}})$$

This completes the proof of the Theorem. □

Theorem 2.4.4.

$$\zeta(\sigma+it)=O\Big[t^{\frac{1}{4R-2}\{\frac{240Rr-16R+128}{240Rr-15R+128}\}}\Big]$$

where $R=2^{r-1}$ *on each of the lines* $\sigma=1-\frac{r+1}{4R-2}$; $r=3,4,5,...$

Proof. We start from the simplest approximation $\zeta(s)$ in the critical strip (1.32)

$$\zeta(\sigma+it)=\sum_{n<t/\pi}n^{-\sigma-it}+O(t^{-\sigma})\qquad(t>1).\tag{2.32}$$

As in the proof Theorem 2.4.3 we have, for any exponent pair (k,l),

$$\sum_{a\le n\le b}n^{-it}=O(t^k a^{l-k})\qquad(1\le a<b<2a<t/\pi).\tag{2.33}$$

We now choose a set (k_q,l_q) of exponent pairs as follows: Applying Lemma 2.2.6 with $q=4$ to the pair $(\frac{1}{2},\frac{1}{2})$ we obtain the pair $(\frac{13}{31},\frac{16}{31})$. Then applying Lemma

2.2.6 with $q=1$ to this we obtain the pair $(\frac{16}{88},\frac{57}{88})$. Finally, applying Lemma 2.2.5, we obtain the set of pairs

$$(k_q,l_q)=\Big(\frac{16}{120Q-32},1-\frac{16q+31}{120Q-32}\Big)\quad(Q=2^q).$$

If we put $x_q=l_q-k_q$ and $\sigma_q=1-\frac{q+2}{4Q-2}$, then for every $q\geq 2$ we have $x_q>\sigma_q>x_{q-1}$. Using (k_q,l_q) in (2.33) we have, by partial summation,

$$\sum_{a\leq n\leq b} n^{-\sigma_q-it}=O(t^{k_q}a^{x_q-\sigma_q}),\tag{2.34}$$

and using (k_{q-1},l_{q-1}) similarly we have

$$\sum_{a\leq n\leq b} n^{-\sigma_q-it}=O(t^{k_q}a^{-(\sigma_q-x_{q-1})})\tag{2.35}$$

for $1\leq a<b<2a<t/\pi$.
We divide the sum on the right-hand side of (2.32) into two parts according as $n<t^\lambda$ or $n\geq t^\lambda\quad(0<\lambda<1)$. For the first part we use (2.34) and for the second part we use (2.35). We thus have

$$\begin{aligned}\sum_{n<t^\lambda} n^{-\sigma_q-it}&=\sum_{\frac{1}{2}t^\lambda\leq n<t^\lambda} n^{-\sigma_q-it}+\sum_{\frac{1}{4}t^\lambda\leq n<\frac{1}{2}t^\lambda} n^{-\sigma_q-it}+...\\&=O[t^{k_q+\lambda(x_q-\sigma_q)}\{(\frac{1}{2})^{x_q-\sigma_q}+(\frac{1}{4})^{x_q-\sigma_q}+...\}]\\&=O[t^{k_q+\lambda(x_q-\sigma_q)}],\qquad\text{since}\quad x_q-\sigma_q>0.\end{aligned}\tag{2.36}$$

And

$$\begin{aligned}\sum_{t^\lambda\leq n<t/\pi} n^{-\sigma_q-it}&=\sum_{t^\lambda\leq n<2t^\lambda} n^{-\sigma_q-it}+\sum_{2t^\lambda\leq n<4t^\lambda} n^{-\sigma_q-it}+...\\&=O[t^{k_{q-1}-\lambda(\sigma_q-x_{q-1})}\{1+2^{-(\sigma-x_{q-1}}+4^{-(\sigma_q-x_{q-1})}+...\}]\\&=O[t^{k_{q-1}-\lambda(\sigma_q-x_{q-1})}],\quad\text{since}\quad\sigma_q-x_{q-1}>0.\end{aligned}\tag{2.37}$$

The right-hand sides of (2.36) and (2.37) are of the same order if

$$k_q+\lambda(x_q-\sigma_q)=k_{q-1}-\lambda(\sigma_q-x_{q-1})$$

i.e. if

$$\lambda=\frac{k_{q-1}-k_q}{x_q-x_{q-1}}.$$

It is easily seen on substitution for k_q, etc, that this value of λ lies between 0 and 1. Hence, putting it into (2.36) and (2.37) and adding, we have

$$\sum_{n<t/\pi} n^{-\sigma_q - it} = O[t^{\mu_q}], \tag{2.38}$$

where

$$\mu_q = \frac{k_q(\sigma_q - x_{q-1}) + k_{q-1}(x_q - \sigma_q)}{x_q - x_{q-1}}.$$

Putting in the values of k_q, x_q, σ_q,.. etc., we obtain

$$\mu_q = \frac{1}{4Q-2}\left\{\frac{240Qq + 224Q + 128}{240Qq + 225Q + 128}\right\}. \tag{2.39}$$

We have proved this for each $q \geq 2$. If we put $q = r-1$, $R = 2^{r-1} = Q$, σ_q becomes $1 - \frac{r+1}{4R-2}$ and μ_q becomes

$$\mu_{r-1} = \frac{1}{4Q-2}\left\{\frac{240Rr - 16R + 128}{240Rr - 15R + 128}\right\}.$$

Putting these into (2.38) and substituting in (2.32) we finish the proof of the Theorem. □

Remark. The results given in the above two theorems are not the best that can be obtained. For instance, the original Bombieri-Iwaniec [2] argument provided the estimate $\mu(\frac{1}{2}) = \frac{9}{56}$ and Huxley in [6] produced the better exponent $\mu(\frac{1}{2}) = \frac{89}{570}$.

Let us look at a new estimate of $\zeta(s)$ on the critical line which was recently obtained by *Jean Bourgain* using the classical approximate functional equation and some exponential sum bounds. Due to the new estimate for $\zeta(s)$ a new exponent pair which can not be obtained from the trivial exponent pair $(0, 1)$ using the usual A-B processes is also stated. This new exponent pair plays very great role in improving our power moment estimates of the Riemann zeta-function, and consequently desired improved results on the error terms in the Dirichlet divisor problem. Now, let us state a new exponential sum bound in a form of Lemma without proof (Refer [1]).

Lemma 2.4.5. *Let F be a smooth function on $[\frac{1}{2}, 1]$ satisfying, for some constant $c \in (0, 1]$, the condition*

$$\min\{\mid F''(x) \mid, \mid F'''(x) \mid, \mid F''''(x) \mid\} > c. \tag{2.40}$$

Given T sufficiently large, $1 \leq M \leq \sqrt{T}$, $f(u) = TF(u/M)$ with $\frac{M}{2} \leq u \leq M$ and

$$S = \sum_{M<m\leq 2M} e^{(f(m))} \tag{2.41}$$

Then, we have

$$| S | \ll M^{1/2} T^{13/84+\varepsilon} \quad if \quad \frac{1}{2} \geq u = \frac{\log M}{\log T} \geq \frac{17}{42}. \tag{2.42}$$

Theorem 2.4.6. *[Jean Bourgain's new estimate]*

$$| \zeta(\frac{1}{2} + it) | \ll | t |^{\frac{13}{84}+\varepsilon} \tag{2.43}$$

Proof. To find the upper bound estimate of $\zeta(1/2 + it)$ one wants to show that (2.42) also holds when $17/42 > u \geq 0$. The cases with $0 \leq u \leq 13/42$ are trivial (there one can just use $| S | \leq M$), so all that remains to be done is establishing that (2.42) holds when α lies in the interval $(13/42, 17/42)$. To achieve this one can employ the bound

$$| S | \ll T^{\frac{1}{128}(4+103u)+\varepsilon} \qquad (12/31 < u \leq 1), \tag{2.44}$$

which is [6], Theorem 3, in combination with the exponent pair estimate

$$| S | \ll \left(\frac{T}{M}\right)^{1/9} M^{13/18} = M^{11/18} T^{1/9} \qquad (0 \leq u \leq 1), \tag{2.45}$$

which corresponds to the exponent pair $(\frac{1}{9}, \frac{13}{18}) = ABA^2B(0,1)$ mentioned in [14], section 5.20. It should be noted that (2.45) (and also (2.44)) assume additional hypotheses concerning the function F, beyond condition (2.40). This, however, is not an obstacle to the application to $| \zeta(1/2 + it) |$, since that only requires consideration of cases in which $F(x) = \log x$ (a function that does satisfy all the unmentioned conditions attached to (2.44) and (2.45)). Assume henceforth that F is "a suitable function" such that (2.44) and (2.45) are applicable. A calculation shows that (2.42) is implied by (2.44) for all u in the interval $(12/31, 332/819]$, and is implied by (2.45) for all u in the interval $[0, 11/28]$: noting that $11/28 = 0.39285... > 0.38709... = 12/31$, we find that the union of these two intervals is $[0, 332/819] = [0, 0.40537...] \supset (0.30952..., 0.40476...) = (13/42, 17/42)$.
By the preceding one has the bound (2.42) whenever $0 < u \leq 1/2$ (at least this is so in the case $F(x) = \log x$). It follows from the "approximate functional equation" for $\zeta(s)$ in the critical strip that

$$| \zeta(\frac{1}{2} + it) | \leq 2 \, | \sum_{n \leq \sqrt{t/2\pi}} n^{-\frac{1}{2}+it} | + O(1) \quad (t \to \infty). \tag{2.46}$$

From partial summation and dyadic dissection, Theorem 2.4.6 follows. $\square$

Further Comments

Recalling (2.41), Lemma 2.4.5 and Theorem 2.4.6 shows that one has the estimate

$$| S | \ll M^{\frac{1}{2}} T^{\frac{13}{84}+\varepsilon} = (T/M)^{\frac{13}{84}+\varepsilon} T^{\frac{55}{84}+\varepsilon} \quad \text{if} \quad \frac{1}{2} \geq \alpha = \frac{\log M}{\log T} > 0 \tag{2.47}$$

provided f is in the class of functions to which the exponent pair theory applies. (see for instance [2], chapter 3 for details).

Theorem 2.4.7. *[Jean Bourgain's exponent pair]*

$$\left(\frac{13}{84} + \varepsilon, \frac{55}{84} + \varepsilon\right) \tag{2.48}$$

is an exponent pair, for any $\varepsilon > 0$*.*

Chapter 3

The Higher Power Moments

3.1 Introduction and Definitions

The power moment problem in analytic number theory is related to the investigations of value-distribution of the Riemann zeta-function $\zeta(\sigma + it)$. Estimates of integrals of the form $I_k = \int_0^T | \zeta(\sigma + it) |^{2k} dt$ $(\sigma \geq \frac{1}{2}, k \geq 0)$ play a prominent role in many parts of zeta-function theory. In this chapter, we will discuss power moments that help to estimate the maximal order of the zeta–function $\zeta(s)$ in the critical strip, $\frac{1}{2} \leq \sigma < 1$. The applications of upper bounds for I_k to divisor problems at relatively large heights will be considered in the last chapter of this paper. A classical problem in zeta-function theory is the investigation of the asymptotic behaviour of the integral I_k for $k = 1$, i.e.

$$I = \int_0^T \left| \zeta(\frac{1}{2} + it) \right|^2 dt, \tag{3.1}$$

and the first nontrivial result has been obtained by G. H. Hardy and J. E. Littlewood in 1918, who showed that

$$I = \int_0^T \left| \zeta(\frac{1}{2} + it) \right|^2 dt = (1 + o(1))T \log T \quad (T \to \infty). \tag{3.2}$$

The first significant results in the field of power moment problem of zeta-function for $k = 2$ were also obtained by the famous G.H. Hardy and J.E. Littlewood (an asymptotic formula for the mean square of the Riemann zeta-function), i.e.

$$\int_0^T \left|\zeta(\frac{1}{2}+it)\right|^4 dt = O(T\log T^4) \tag{3.3}$$

Later A. E. Ingham proved an asymptotic formula for the fourth power moment,

$$\int_0^T \left|\zeta(\frac{1}{2}+it)\right|^4 dt = \frac{1}{2\pi^2}T\log^4 T + O(T\log T^3). \tag{3.4}$$

which remained the best-known mean value estimate of the zeta-function for a very long time. The investigation of moments of zeta-functions is a very complicated but interesting problem of analytic number theory.
Absolute moments of the Riemann zeta-function on the critical line have been the subject of intense theoretical investigations by Hardy, Littlewood, Heath-Brown, M. Jutila, A. Ivic, A. Selberg and many others. It has long been conjectured that the $2k$-th moment of $\mid \zeta(1/2+it) \mid$ should grow like $c_k \cdot T(\log T)^{k^2}$ for some constant $c_k > 0$. We may define two numbers, $M(A)$ and $m(\delta)$, which characterize power moments when $\delta = \frac{1}{2}$ and $\frac{1}{2} < \delta \leq 1$, respectively, as follows

Definition 3.1.1. For any fixed number $A \geq 4$ the number $M(A)(\geq 1)$ is defined as the infimum of all numbers $M(\geq 1)$ such that

$$\int_1^T \left|\zeta(\frac{1}{2}+it\right|^A dt \ll T^{M+\epsilon} \qquad \text{for any} \quad \epsilon > 0. \tag{3.5}$$

Definition 3.1.2. For $\frac{1}{2} < \sigma < 1$ fixed we define $m(\sigma)(\geq 4)$ as the supremum of all numbers $m(\geq 4)$ such that

$$\int_1^T \left|\zeta(\sigma+it\right|^m dt \ll T^{1+\epsilon} \qquad \text{for any} \quad \epsilon > 0. \tag{3.6}$$

Naturally, we seek upper bounds for $M(A)$ and lower bounds for $m(\sigma)$.
An important feature of power moments for $\zeta(s)$ is that (3.5) and (3.6) with $M = M(A)$, $m = m(\sigma)$ are respectively equivalent to

$$\sum_{r\leq R} \left|\zeta(\frac{1}{2}+it_r)\right|^A \ll T^{M(A)+\varepsilon} \tag{3.7}$$

and

$$\sum_{r\leq R} \left|\zeta(\sigma+it_r)\right|^{m(\sigma)} \ll T^{1+\varepsilon}, \tag{3.8}$$

where

$$T \leq t_r \leq 2T \quad r = 1, 2, ..., R; \quad | t_r - t_s | \geq \log^C T \tag{3.9}$$

for $1 \leq r \neq s \leq R$ and $C \geq 0$ is fixed.
More generally, we can avoid the terminology of both discrete sums [(3.7), (3.8)] and integrals [(3.5), (3.6)] and use large values of $| \zeta(\sigma + it) |$ instead. Namely, let $a(\sigma)(\geq 1)$ and $b(\sigma)(\geq 4)$ be such a function of σ ($\frac{1}{2} \leq \sigma < 1$ fixed) for which

$$R \ll T^{a(\sigma)+\varepsilon} V^{-b(\sigma)}, \tag{3.10}$$

where for t_r defined by (3.9) we have

$$| \zeta(\sigma + it_r) | \geq V \geq T^{\varepsilon} \qquad (r = 1, 2, ..., R). \tag{3.11}$$

Then this is equivalent to

$$\sum_{r \leq R} \left| \zeta(\sigma + it_r) \right|^{b(\sigma)} dt \ll T^{a(\sigma)+\varepsilon} \tag{3.12}$$

or

$$\int_1^T \left| \zeta(\sigma + it) \right|^{b(\sigma)} dt \ll T^{a(\sigma)+\varepsilon} \tag{3.13}$$

In the next section we shall derive some convexity estimates for $a(\sigma)$, $b(\sigma)$ by an argument which gives also the convexity of the function $\mu(\sigma)$, defined by (2.21).

3.2 The Convexity of Power Moments

Let $s = \sigma + it$, $T \leq t \leq 2T$, $0 < \sigma < 1$. From Theorem 1.4.1 we have

$$\zeta(s) = \sum_{n \leq 2T} n^{-s} + O(T^{-\sigma}), \tag{3.14}$$

which means that $\zeta(s)$ is approximated by $O(\log T)$ sums of the form $\sum_{N<n\leq 2N} n^{-s}$, where $N \leq T$. For $\sigma_0 > 0$ the partial summation formula gives

$$\sum_{N<n\leq 2N} n^{-\sigma-it} \ll N^{\sigma_0-\sigma} \mid \sum_{N<n\leq 2N} n^{-\sigma_0-it} \mid + N^{\sigma_0-\sigma-1} \int_N^{2N} \mid \sum_{N<n\leq x} n^{-\sigma_0-it} \mid dx. \tag{3.15}$$

Take now $N \leq N_1 < N_2 \leq 2N \leq 2T$. Then by the inversion formula (1.31) we have

$$\sum_{N_1<n\leq N_2} n^{-s} = \frac{1}{2\pi i}\int_{1-\sigma+\varepsilon-iT}^{1-\sigma+\varepsilon+iT} \zeta(s+w)({N_2}^w - {N_1}^w)w^{-1}dw + O(N^{1-\sigma+\varepsilon}T^{-1})$$

$$= \frac{1}{2\pi i}\int_{-iT}^{iT} \zeta(s+w)({N_2}^w - {N_1}^w)w^{-1}dw$$

$$+O\Big(\int_0^{1-\sigma+\varepsilon} \mid \zeta(\sigma+u+iT+it) \mid N^uT^{-1}du\Big) + O(T^{\varepsilon-\sigma}). \tag{3.16}$$

Now observe that

$$({N_2}^w - {N_1}^w)w^{-1} = \int_{N_1}^{N_2} z^{w-1}dw \ll \min(1, \mid v \mid^{-1}) \quad (w = iv)$$

and

$$\int_0^{1-\sigma+\varepsilon} \mid \zeta(\sigma+u+iT+it) \mid N^uT^{-1}du \ll T^{\varepsilon-\sigma}\int_0^{1-\sigma+\varepsilon}(N/T)^u du \ll T^{\varepsilon-\sigma},$$

since trivially $\zeta(\sigma+u+iT+it) \ll T^{1-\sigma-u+\varepsilon}$. Therefore, we have uniformly in t, N_1, N_2 and $0 < \sigma_0 < 1$

$$\sum_{N_1<n\leq N_2} n^{-\sigma_0-it} \ll \int_0^T \mid \zeta(\sigma_0+it+iv) \mid \frac{dv}{v+1} + T^{\varepsilon-\sigma_0}, \tag{3.17}$$

and (3.15) gives

$$\sum_{N<n\leq 2N} n^{-\sigma_0-it} \ll N^{\sigma_0-\sigma}T^{\varepsilon-\sigma_0} + N^{\sigma_0-\sigma}\int_0^T \mid \zeta(\sigma_0+it+iv) \mid \frac{dv}{v+1}. \tag{3.18}$$

We proceed now to a convexity estimate for the functions $a(\sigma)$ and $b(\sigma)$, defined by (3.10) and (3.11), where the notation is the same as in Section 3.1. Suppose that for $N \leq T$ we have

$$T^\varepsilon \leq V \leq \mid \sum_{N<n\leq 2N} n^{-\sigma-it_r} \mid \quad (r = 1, 2, ..., R),$$

where the t_r's satisfy (3.9). Using (3.18) with $\sigma_0 = \sigma_1$ we obtain

$$RV \leq \sum_{r\leq R} \mid \sum_{N<n\leq 2N} n^{-\sigma-it_r} \mid \ll N^{\sigma_1-\sigma}\sum_{r\leq R}\Big(T^{\varepsilon-\sigma_1} + \int_0^T \mid \zeta(\sigma_1+it_r+iv) \mid \frac{dv}{v+1}\Big),$$

whence

$$RV \ll N^{\sigma_1-\sigma}\int_0^T \sum_{r\le R} \mid \zeta(\sigma_1+it_r+iv) \mid \frac{dv}{v+1} = N^{\sigma_1-\sigma}R^{1-1/b(\sigma_1)(T^{a(\sigma_1)+\varepsilon})^{1/b(\sigma_1)}}\log T,$$

where we used Holder's inequality. Hence

$$R \ll T^{a(\sigma_1)+\varepsilon}V^{-b(\sigma_1)}N^{b(\sigma_1)(\sigma_1-\sigma)}, \tag{3.19}$$

and analogously it follows

$$R \ll T^{a(\sigma_2)+\varepsilon}V^{-b(\sigma_2)}N^{b(\sigma_2)(\sigma_2-\sigma)}. \tag{3.20}$$

However if (3.11) holds, then in view of (3.14) there must exist a subset $\{t_{r_1}, t_{r_2}, ...\}$ of $\{t_r\}$ which contains at least $CR/\log T$ of all R points, so that for some $N \le T$ and points of this subset

$$V \le \mid \sum_{N<n\le 2N} n^{-\sigma-it_r} \mid \quad (r = r_1, r_2, ...).$$

Therefore using (3.19) and (3.20) we have, for $0 \le \alpha \le 1$,

$$R = R^{\alpha}R^{1-\alpha} \ll T^{\alpha a(\sigma_1)+(1-\alpha)a(\sigma_2)+\varepsilon}V^{-\alpha b(\sigma_1)-(1-\alpha)b(\sigma_2)}N^{\alpha b(\sigma_1)(\sigma_1-\sigma)+(1-\alpha)b(\sigma_2)(\sigma_2-\sigma)}. \tag{3.21}$$

If we choose

$$\alpha = \frac{b(\sigma_2)(\sigma_2-\sigma)}{b(\sigma_2)(\sigma_2-\sigma)+b(\sigma_1)(\sigma-\sigma_1)}.$$

then the exponent of N in (3.21) vanishes, and we obtain

$$RT^{-\varepsilon} \ll T^{\frac{a(\sigma_1)b(\sigma_2)(\sigma_2-\sigma)+a(\sigma_2)b(\sigma_1)(\sigma-\sigma_1)}{b(\sigma_2)(\sigma_2-\sigma)+b(\sigma)(\sigma-\sigma_1)}}V^{\frac{b(\sigma_1)b(\sigma_2)(\sigma_2-\sigma_1)}{b(\sigma_2)(\sigma_2-\sigma)+b(\sigma)(\sigma-\sigma_1)}} \tag{3.22}$$

If we consider $a(\sigma)$ and $b(\sigma)$ as lower and upper bounds of the numbers in (3.10) respectively, then (3.22) gives

$$a(\sigma) \le \frac{a(\sigma_1)b(\sigma_2)(\sigma_2-\sigma)+a(\sigma_2)b(\sigma_1)(\sigma-\sigma_1)}{b(\sigma_2)(\sigma_2-\sigma)+b(\sigma)(\sigma-\sigma_1)}, \tag{3.23}$$

$$b(\sigma) \ge \frac{b(\sigma_1)b(\sigma_2)(\sigma_2-\sigma_1)}{b(\sigma_2)(\sigma_2-\sigma)+b(\sigma)(\sigma-\sigma_1)}, \tag{3.24}$$

if $\frac{1}{2} \le \sigma_1 < \sigma < \sigma_2 < 1$. Taking $a(\sigma_1) = a(\sigma_2) = 1$, we obtain as a special case of (3.24) a sort of a convexity property of the function $m(\sigma)$, which we state as
Theorem 3.2.1. *Let $m(\sigma)$ be defined by (3.6). Then for $\frac{1}{2} \le \sigma_1 < \sigma < \sigma_2 < 1$*

$$m(\sigma) \ge \frac{m(\sigma_1)m(\sigma_2)(\sigma_2-\sigma_1)}{m(\sigma_2)(\sigma_2-\sigma)+m(\sigma_1)(\sigma-\sigma_1)}. \tag{3.25}$$

3.3 Power Moment of $\zeta(\sigma)$ on $\sigma = \frac{1}{2}$

In this section the aim is to derive upper bounds for $R = R(V)$ of the type (3.10), which will then lead to estimates of $M(A)$. Suppose $t_1 < ... < t_R$ are real numbers which satisfy

$$\mid t_r \mid \leq T \quad \text{for} \quad r = 1, ..., R; \quad \mid t_r - t_s \mid \geq 1 \quad \text{for} \quad 1 \leq r \neq s \leq R \tag{3.26}$$

$$\text{and} \quad \mid \zeta(\frac{1}{2} + it_r) \mid \geq V > 0 \quad \text{for} \quad r = 1, 2, ..., R. \tag{3.27}$$

By Theorem 2 from [5], we have

$$R \ll TV^{-4}\log^5 T \quad (V > 0) \tag{3.28}$$

$$R \ll T^2 V^{-12}\log^{16} T, \quad (V > 0) \tag{3.29}$$

$$R \ll T^5 V^{-32}\log^{40} T, \quad (T^{2/13}\log^{16/13} T \geq V \geq T^{3/20}\log^{6/5} T) \tag{3.30}$$

$$R \ll TV^{-6}\log^8 T \quad (V > T^{2/13}\log^{16/13} T). \tag{3.31}$$

Theorem 3.3.1. *Let* (χ, λ) *be any exponent pair with* $\chi > 0$, *and let* $t_1 < ... < t_R$ *satisfy* (3.26) *and* (3.27). *Then*

$$R \ll TV^{-6}\log^8 T + T^{\frac{\chi+\lambda}{\chi}} V^{-2\frac{(1+2\chi+2\lambda)}{\chi}} (\log T)^{\frac{(3+6\chi+4\lambda)}{\chi}}. \tag{3.32}$$

Proof. Let A_3 be a set of points t_r satisfying the spacing condition (3.26), but with $\frac{2T}{3} \leq t_r \leq \frac{5T}{6}$. We shall divide the interval $[\frac{2T}{3}, \frac{5T}{6}]$ into N sub-intervals of length at most $J = \frac{T}{6N}$, and denote by $A_{1,k}$ $(k = 1, 2, ..., N)$ the set of points t_r in the k-th of these intervals. The points of each $A_{1,k}$ lie in $[T_0, T_0 + J]$ for some T_0 which satisfies $\frac{2T}{3} \leq T_0 \leq \frac{5T}{6} - J$. We shall first estimate $\mid A_{1,k} \mid$ by taking as in [5]

$$BG \log^2 T = V^2 \tag{3.33}$$

for some suitable $B > 0$ and defining $A'_{1,k} = A'_{1,k}(\tau) = A_{1,k} \cap [\tau - G/2, \tau + \tau - G/2]$, where we assume $\frac{7T}{12} \leq \tau \leq \frac{11T}{12}$. By Lemma 3.1 from [7]

$$\zeta(\frac{1}{2} + it_r) \ll \log^{\frac{1}{2}} t_r \ll \log^{\frac{1}{2}} T \tag{3.34}$$

or

$$\mid \zeta(\frac{1}{2} + it_r) \mid^2 \ll \log t_r \int_{-\log^2 t_r}^{\log^2 t_r} e^{-|u|} \mid \zeta(\frac{1}{2} + it_r + iu) \mid^2 du \tag{3.35}$$

Taking in Theorem 3.3.1, $\log^{\frac{1}{2}} T \ll V \leq \mid \zeta(\frac{1}{2}+it_r) \mid$ we may consider only those t_r for which (3.35) holds, and thus

$$\mid A'_{1,k} \mid V^2 \leq C_1 \log T \int_{\tau-G}^{\tau+G} \mid \zeta(\frac{1}{2}+it) \mid^2 \sum_{t_r \in A'_{1,k}} e^{-|t-t_r|} dt, \qquad (3.36)$$

provided that $[t_r - \log^2 t_r, t_r + \log^2 t_r] \subseteq [\tau - G, \tau + G]$ for $t_r \in [\tau - G/2, \tau + G/2]$. This inclusion is certainly true for $G \gg \log^3 T$, which in view of (3.33) follows if we take $V \gg \log^{\frac{5}{2}} T$. Since $\mid t_r - t_s \mid \geq 1$ for $r \neq s$ we have that the sum in (3.36) is bounded, and for $\log^3 T \ll G \ll T^{\frac{5}{12}}$ we may apply lemma 3.2 from [7] to obtain

$$\mid A'_{1,k} \mid BG \log^2 T \leq C_2 G \log^2 T + C_2 G \log T \sum_{T^{\frac{1}{3}} \leq K = 2^k \leq TG^{-2} \log^3 T} (TK)^{\frac{-1}{4}} e^{-G^2 \frac{K}{T}} \Big(\mid S(K,K,\tau) \mid + \frac{1}{K} \int_0^K \mid S(K,K,\tau) \mid dx \Big) \qquad (3.37)$$

For $B = 2C_2$ this simplifies to

$$\mid A'_{1,k} \mid \ll \log^{-1} T \sum_K (TK)^{\frac{-1}{4}} e^{-G^2 \frac{K}{T}} \Big(\mid S(K,K,\tau) \mid + \frac{1}{K} \int_0^K \mid S(K,K,\tau) \mid dx \Big) \qquad (3.38)$$

Let now $A_{2,k}$ denote the set numbers $\tau = T_0 + G/2 + nG$ such that $A'_{1,k}(\tau) \neq \emptyset$ and n is such an integer that $T_0 \leq \tau \leq T_0 + J + G/2$. Then if τ_r and τ_s are two different elements of $A_{2,k}$ we have $G \leq \mid \tau_r - \tau_s \mid \leq J$, and application of Lemma 3.5 on [7] gives

$$\sum_{\tau \in A_{2,k}} \mid A'_{1,k}(\tau) \mid \ll \log^{\frac{1}{2}} T \sum_K (TK)^{\frac{1}{4}} e^{-G^2 \frac{K}{T}} \{ (K + K^{\frac{3}{4}} T^{\frac{1}{4}} G^{\frac{-1}{2}} \log^{\frac{1}{2}} T) \mid A_{2,k} \mid^{\frac{1}{2}} + \mid A_{2,k} \mid J^{\frac{p}{2}} T^{\frac{-p}{4}} K^{\frac{(2\Lambda - \chi + 2)}{4}} \}. \qquad (3.39)$$

summing over $K = 2^k$, using equation (3.13) from (Lemma 3; [7]) and

$$\sum_{\tau \in A_{2,k}} \mid A'_{1,k}(\tau) \mid \geq \mid A_{1,k} \mid, \qquad \sum_{\tau \in A_{2,k}} \mid A'_{1,k}(\tau) \mid \geq \mid A_{2,k} \mid \qquad (3.40)$$

we obtain

$$\sum_{\tau \in A_{2,k}} \mid A'_{1,k}(\tau) \mid \ll T^{-\frac{1}{2}} \log T \sum_K K^{\frac{3}{2}} e^{-G^2 \frac{K}{T}} + G^{-1} \log^2 T \sum_K K e^{-G^2 \frac{K}{T}} + J^{\frac{\chi}{2}} T^{-\frac{\chi+1}{4}} \log^{\frac{1}{2}} T \mid A_{2,k} \mid \sum_K K^{\frac{(2\lambda - \chi + 1)}{4}} e^{\frac{-G^2 K}{T}} \qquad (3.41)$$

$$TG^{-3} \log^2 T + J^{\frac{\chi}{2}} G^{\frac{(\chi - 1 - 2\lambda)}{2}} T^{\frac{(\lambda - \chi)}{2}} \log^{\frac{1}{2}} T \mid A_{2,k} \mid .$$

In view of (3.40) we have

$$| A_{1,k} |\leq \sum_{\tau\in A_{2,k}} | A'_{1,k}(\tau) |\ll TG^{-3}\log^2 T \tag{3.42}$$

if for some suitable $C_3 > 0$

$$J \leq C_3 G^{\frac{(2\lambda-\chi+1)}{\chi}} T^{\frac{(\chi-\lambda)}{\chi}} \log^{\frac{-1}{\chi}} T. \tag{3.43}$$

Now choose N such that

$$J = \frac{T}{6N} \leq C_3 T^{\frac{(\chi-\lambda+1)}{\chi}} G^{\frac{(2\lambda-\chi+1)}{\chi}} \log^{\frac{-1}{\chi}} T < \frac{T}{6N-6}. \tag{3.44}$$

Then we have

$$N \ll 1 + T^{\frac{\lambda}{\chi}} G^{\frac{-(2\lambda-\chi+1)}{\chi}} \log^{\frac{1}{\chi}} T \tag{3.45}$$

$$\begin{aligned} | A_3 |= \sum_{k\leq N} | A_{1,k} |\ll NTG^{-3}\log^2 T \\ \ll TG^{-6}\log^8 T + T^{\frac{\chi+\lambda}{\chi}} V^{-2(\frac{1+2\chi+2\lambda}{\chi})} (\log T)^{\frac{(3+6\chi+4\lambda)}{\chi}} \end{aligned} \tag{3.46}$$

if $G \leq J$. This is certainly satisfied for

$$G \leq C_4 T^{\frac{\chi-\lambda}{\chi}} G^{\frac{(2\lambda-\chi+1)}{\chi}} \log^{-\frac{1}{\chi}} T \tag{3.47}$$

or

$$V > T_1 = C_5 T^{\frac{\chi-\lambda}{(2+4\lambda-4\chi)}} (\log T)^{\frac{(3-4\chi+4\lambda)}{(2+4\lambda-4\chi)}}, \tag{3.48}$$

Considering intervals of the form $[T(5/4)^{-n-1}, T(5/4)^{-n}]$ we obtain then

$$R \ll TV^{-6}\log^8 T + T^{\frac{(\chi+\lambda)}{\chi}} V^{\frac{-2(1+2\chi+2\lambda)}{\chi}} (\log T)^{\frac{(3+6\chi+4\lambda)}{\chi}}, \tag{3.49}$$

provided that (3.48) holds. choosing first $(\chi, \lambda) = (1/2, 1/2)$ we obtain in view of (2.3) that

$$R \ll TV^{-6}\log^8 T + T^2 V^{-12}\log^{16} T \ll T^2 V^{-12}\log^{16} T \tag{3.50}$$

for $V \gg \log^{\frac{5}{2}} T$, and this is trivial for other values of V by (3.28). Likewise

$$R \ll TV^{-4}\log^5 T \ll T^{\frac{(\chi+\lambda)}{\chi}} V^{\frac{-2(1+2\chi+2\lambda)}{\chi}} (\log T)^{\frac{(3+6\chi+4\lambda)}{\chi}}, \tag{3.51}$$

for $V < T^{\frac{\lambda}{2+4\lambda}} \log^C T$. But for T_1 given by (3.48) we have $T_1 < T^{\frac{\lambda}{2+4\lambda}} \log^C T$ for any fixed real $C, \chi \geq 0$ and T sufficiently large, which completes the proof. □

For special choices of the exponent pair (χ, λ) we obtain from (3.32)

Corollary 3.3.1. *Under the hypotheses of Theorem 3.3.1 we have*

$$R \ll TV^{-6}\log^8 T + T^{\frac{29}{13}}V^{\frac{-178}{13}}\log^{\frac{235}{13}} T, \tag{3.52}$$

$$R \ll TV^{-6}\log^8 T + T^{\frac{5}{2}}V^{\frac{-31}{2}}\log^{\frac{81}{4}} T, \tag{3.53}$$

$$R \ll TV^{-6}\log^8 T + T^{3}V^{-19}\log^{\frac{49}{2}} T, \tag{3.54}$$

$$R \ll TV^{-6}\log^8 T + T^{4}V^{\frac{-128}{5}}\log^{\frac{162}{5}} T, \tag{3.55}$$

$$R \ll TV^{-6}\log^8 T + T^{\frac{15}{4}}V^{-24}\log^{\frac{61}{2}} T. \tag{3.56}$$

Proof. The result of this corollary follows from Theorem 3.3.1 with the exponent pairs $(\chi, \lambda) = (\frac{13}{31}, \frac{16}{31}), (\frac{4}{11}, \frac{6}{11}), (\frac{2}{7}, \frac{4}{7})$ and $(\frac{5}{24}, \frac{15}{24})$, respectively, while the pair $(\frac{1}{6}, \frac{2}{3})$ gives (3.30) for the wider range $V \le T^{\frac{2}{13}}\log^{\frac{16}{13}} T$. With

$$H(T) = T^{\lambda/(2-2\chi+4\lambda)}(\log T)^{(3-2\chi+4\lambda)} \tag{3.57}$$

we deduce from Theorem 3.3.1

$$R \ll \begin{cases} TV^{-6}\log^8 T, & V \ge H(T) \\ T^{\frac{(\chi+\lambda)}{\chi}}V^{\frac{-2(1+2\chi+2\lambda)}{\chi}}(\log T)^{\frac{(3+6\chi+4\lambda)}{\chi}}, & V \le H(T). \end{cases} \tag{3.58}$$

Choosing $(\alpha/2 + \varepsilon, \alpha/2 + 1/2 + \varepsilon)$ as an exponent pair we obtain

$$R \ll TV^{-6}\log^8 T \qquad V \ge T^{0.153501...}, \tag{3.59}$$

$$R \ll T^{5.0393165...}V^{-32.314532...}\log^8 T \qquad V \le T^{0.153501...}, \tag{3.60}$$

Since 2/13=0.153846...>0.153501..., it is seen that (3.59) improves (3.31). Taking $V <\mid \zeta(\frac{1}{2} + it_r) \mid \le 2V$ and summing over $O(\log T)$ values $V = 2^k$ we obtain for t_r satisfying (3.26)

$$\sum_{|t_r|\le T, |\zeta| \ge H(T)} \mid \zeta(1/2 + it_r) \mid^6 \ll T\log^9 T, \tag{3.61}$$

$$\sum_{|t_r|\le T, |\zeta| \le H(T)} \mid \zeta(1/2 + it_r) \mid^{2(1+2\chi+2\lambda)/\chi} \ll T^{(\chi+\lambda)/\chi}(\log T)^{(3+7\chi+4\lambda)/\chi}. \tag{3.62}$$

One may crudely deduce from (3.61) and (3.62) that in accordance with (3.7) we have in a certain sense either M(6)=1 or $M((2 + 4\chi + 4\lambda)/\chi)$. □

Corollary 3.3.2. *If $A \ge 4$ is a fixed number and $M(A)$ is defined by* (3.7)*, then*

$$M(A) = \begin{cases} 1 + \frac{(A-4)}{8}, & 4 \le A \le 12, \\ 2 + \frac{3(A-12)}{22}, & 12 \le A \le \frac{178}{13}, \\ 1 + \frac{35(A-6)}{216}, & A \ge \frac{178}{13}. \end{cases} \tag{3.63}$$

Proof. To prove this corollary let us first assume $4 \le A \le 12$ and use (3.28) for $V \le T^{\frac{1}{8}}$ and (3.29) for $V > T^{\frac{1}{8}}$, thus obtaining $M(A) \le 1 + (A-4)/8$. Similarly, for $12 \le A \le \frac{178}{13}$ we use (3.29) when $V \le T^{\frac{3}{22}}$ and (3.52) when $V > T^{\frac{3}{22}}$ to obtain

$$M(A) \le 2 + 3(A-12)/22$$

Finally, we prove the last part of this corollary in a somewhat more general form, viz.

$$S = \sum_{r \le R} \mid \zeta(\frac{1}{2} + it_r) \mid^A \ll T^{(A-6)c+1+\varepsilon}, \quad A \ge 178/13, \quad c \ge 4/25, \tag{3.64}$$

where the $t_r's$ satisfy (3.26) and $\zeta(\frac{1}{2}+it) \ll t^{c+\varepsilon}$, so that Kolesnik's value $c = 35/216$ gives (3.63). To see that (3.64) holds write

$$S = S_1 + S_2.$$

In S_1 we suppose $V \le \mid \zeta(1/2 + it_r) \mid < 2V$ and $V \ge T^{\frac{4}{25}}$, so that (3.59) gives $R \ll T^{1+\varepsilon}V^{-6}$ and we obtain after summing over $O(\log T)$ values of V that

$$\begin{aligned} S_1 = \sum_V \sum_{V \le |\zeta| < 2V} & \mid \zeta(1/2 + it_r) \mid^{A-6} \mid \zeta(1/2 + it_r) \mid^6 \\ & \ll \sum_V T^{(A-6)c+\varepsilon} R V^6 \ll T^{(A-6)c+1+\varepsilon}. \end{aligned} \tag{3.65}$$

□

In S_2 we suppose $V \le \mid \zeta(1/2 + it_r) \mid < 2V$ and $V \le T^{4/25}$. Then (3.52) gives $R \ll T^{29/13+\varepsilon} V^{-178/13}$ and we obtain

$$S_2 \ll T^{(A-178/13)c+29/13+\varepsilon} \ll T^{(A-6)c+1+\epsilon}, \tag{3.66}$$

provided that $c \ge 4/25$. Combining (3.65) and (3.66) we obtain (3.64)

3.4 Power Moment of $\zeta(\sigma)$ on $\frac{1}{2} < \sigma < 1$

Suppose that we have given real numbers $t_1, ..., t_R$ which satisfy

$$\log^2 T \le \mid t_r \mid \ll T \quad \text{for} \quad r \le R; \quad \mid t_r - t_s \mid \ge \log^4 T \quad \text{for} \quad r \ne s \le R, \tag{3.67}$$

$$| \zeta(\sigma + it_r) | \geq V > T^{\varepsilon}, \tag{3.68}$$

for $\frac{1}{2} < \sigma < 1$ fixed. An upper bound for R will lead to estimates of the type

$$\sum_{r \leq R} | \zeta(\sigma + it_r) |^{m(\sigma)} \ll T^{1+\varepsilon} \tag{3.69}$$

which is equivalent to the upper bound of R written as

$$R \ll T^{1+\varepsilon} V^{-m(\sigma)} \tag{3.70}$$

by collecting $O(\log T)$ sub-sums in (3.69), where $V \leq | \zeta(\sigma + it_r) | < 2V < T^{\frac{1}{6}}$. If we choose the $t_r{}'s$ so that

$$| \zeta(\sigma + it_r) | = \max_{r \log^4 T \leq t \leq (r+1) \log^4 T} | \zeta(\sigma + it) |, \quad r = 1, 2, ...,$$

and then consider separately $t_1, t_3, ...$ and $t_2, t_4, ...$, it is seen that (3.69) gives

$$\int_1^T | \zeta(\sigma + it |^{m(\sigma)} dt \ll T^{1+\varepsilon}, \tag{3.71}$$

which is also an estimate with a large number of applications, particularly to various divisor problems.

The next Lemma is a large values estimate for Dirichlet polynomials, which will enable us to prove Theorem 3.4.2, the result may be stated as

Lemma 3.4.1. *Let $t_1 < ... < t_R$ be real numbers such that $T \leq t_r \leq 2T$ for $r = 1, ..., R$ and $| t_r - t_s | \geq \log^4 T \quad$ for $\quad 1 \leq r \neq s \leq R$. If*

$$T^{\epsilon} < V \leq | \sum_{M < n \leq 2M} a(n) n^{-\sigma - it_r} |$$

where $a(n) \ll M^{\epsilon}$ for $M < n \leq 2M$, $1 \ll M \ll T^C$ ($C > 0$ a fixed number), then

$$R \ll T^{\epsilon}(M^{2-2\sigma} V^{-2} + T V^{-f(\sigma)}), \tag{3.72}$$

where

$$f(\sigma) = \begin{cases} \frac{2}{(3-4\sigma)}, & \text{for } \frac{1}{2} < \sigma \leq \frac{2}{3}, \\ \frac{10}{(7-8\sigma)}, & \text{for } \frac{2}{3} < \sigma \leq \frac{11}{14}, \\ \frac{34}{(15-16\sigma)}, & \text{for } \frac{11}{14} < \sigma \leq \frac{13}{15}, \\ \frac{98}{(31-32\sigma)}, & \text{for } \frac{13}{15} < \sigma \leq \frac{57}{62}, \\ \frac{5}{(1-\sigma)}, & \text{for } \frac{57}{62} < \sigma \leq 1 - \epsilon. \end{cases} \tag{3.73}$$

Proof. The expected bound in (3.72) is $R \ll T^{\epsilon} M^{2-2\sigma} V^{-2}$, and $TV^{-f(\sigma)}$ is the extra term which may be thought of as an error term. We start from the well-known Halasz- Montgomery inequality (A.21)

$$\sum_{r \le R} \mid (\xi, \varphi_r) \mid \le \| \xi \| \Big(\sum_{r,s \le R} \mid (\varphi_r, \varphi_s) \mid \Big)^{\frac{1}{2}}, \tag{3.74}$$

which is valid for vectors $\xi, \varphi_1, ..., \varphi_R$ in any inner product vector space. For ξ we take the vector $\xi = \{\xi_n\}_{n=1}^{\infty}$, where $\xi_n = a(n) b^{-\frac{1}{2}}(n) n^{-\sigma}$ for $M < n \le 2M$ and zero otherwise, and $\varphi_r = \sum_{n=1}^{\infty} b^{\frac{1}{2}}(n) n^{-it_r}$, whence by the standard inner product we have $(\varphi_r, \varphi_s) = H(it_r - it_s)$, where with $b(n) = e^{-(\frac{n}{2M})^h} - e^{-(\frac{n}{M})^h}$, $h = \log^2 T$ we have

$$\begin{aligned} H(it) &= \sum_{n=1}^{\infty} b(n) n^{-it} \\ &= \frac{1}{2\pi i} \int_{2-\infty}^{2+\infty} \zeta(w+it) \Gamma(1+\frac{w}{h})((2M)^w - M^w) \frac{dw}{w}, \end{aligned} \tag{3.75}$$

which follows after easy transformations from the Mellin integral

$$e^{-Y} = \frac{1}{2\pi i} \sum_{2-i\infty}^{2+i\infty} \Gamma(w) Y^{-w} dw, \quad Y > 0. \tag{3.76}$$

Note that for $M < n \le 2M$ we have $1 \ll b(n) \ll 1$, $H(0) \ll M$, $\| \xi \|^2 \ll T^{\varepsilon} M^{1-2\sigma}$, and that the integrand in (3.75) is regular for $\Re(w) > -h$, except for a simple pole at $w = 1 - it$ with residue $o(1)$ for $\mid t \mid \gg \log^3 T$. Now lets define $c(\theta)$ to be an upper bound function for $\mu(\sigma)$ in the interval $(-\infty, \infty)$ for real θ for which

$$\zeta(\theta + it) \ll t^{c(\theta)+\varepsilon}, \quad t \ge t_0. \tag{3.77}$$

From the theory of the zeta-function (see 2.3) it is known that $c(\theta)$ is a non-negative, non-increasing, convex function of θ such that $c(\theta) = \frac{1}{2} - \theta$ for $\theta \le 0$, $c(\frac{1}{2}) < \frac{1}{6}$, $c(\theta) = 0$ for $\theta \ge 1$. If $L = 2^{l-1}$, $l \ge 3$ then one may take $c(\theta) \le \frac{1}{2L-2}$ for $\theta = 1 - \frac{l}{(2L-2)}$ and $c(\theta) \le \frac{1}{L(l+1)}$ for $\theta = 1 - \frac{1}{L}$. These are the classical estimates due to vander Corput and Hardy and Littlewood, respectively (see [15], Ch.V).

From these estimates and convexity it follows that one may take

$$\begin{aligned}
c(\sigma) &= \frac{1}{2} - \theta && \text{for} \quad \theta \le 0, \\
c(\sigma) &= \frac{(3-4\theta)}{6} && \text{for} \quad 0 \le \theta \le \frac{1}{2}, \\
c(\sigma) &= \frac{7-8\theta}{18} && \text{for} \quad \frac{1}{2} \le \theta \le \frac{5}{7}, \\
c(\sigma) &= \frac{15-16\theta}{50} && \text{for} \quad \frac{5}{7} \le \theta \le \frac{5}{6}, \\
c(\sigma) &= \frac{1-\theta}{5} && \text{for} \quad \frac{5}{6} \le \theta \le 1,
\end{aligned} \tag{3.78}$$

To estimate $H(it)$ in (3.75) we move the line of integration to $\Re(w) = 0$, where

$$\begin{aligned}
\theta &= \frac{(3\sigma-2)}{(2\sigma-1)} && \text{for} \quad \frac{1}{2} < \sigma_0 \le \sigma \le \frac{2}{3}, \\
\theta &= \frac{(9\sigma-6)}{(4\sigma-1)} && \text{for} \quad \frac{2}{3} \le \sigma \le \frac{11}{14}, \\
\theta &= \frac{(25\sigma-16)}{(8\sigma+1)} && \text{for} \quad \frac{11}{14} \le \sigma \le \frac{13}{15}, \\
\theta &= \frac{(65\sigma-40)}{(16\sigma+9)} && \text{for} \quad \frac{13}{15} \le \sigma \le \frac{57}{62}, \\
\theta &= \frac{(12\sigma-7)}{(2\sigma+3)} && \text{for} \quad \frac{57}{62} \le \sigma \le 1-\varepsilon,
\end{aligned} \tag{3.79}$$

so that the values of θ lie in the range $\theta \le 0$, $0 \le \theta \le \frac{1}{2}$, $\frac{1}{2} \le \frac{5}{7}$, $\frac{5}{7} \le \theta \le \frac{5}{6}$, $\frac{5}{6} \le \theta \le 1$, respectively, and therefore (3.78) may be used. Using stirling's Formula, we obtain for $r \ne s$

$$\begin{aligned}
H(it_r - it_s) &\ll T^{\varepsilon} \int_{-\infty}^{\infty} \mid \zeta(\theta + iv + it_r - it_s) \mid e^{\frac{-|v|}{h}} M^{\theta} dv + o(1) \\
&\ll T^{c(\theta)+\varepsilon} M^{\theta} + o(1).
\end{aligned} \tag{3.80}$$

From (3.74) we infer

$$R \ll (M^{2-2\sigma} V^{-2} + R M^{\theta+1-2\sigma} T^{c(\theta)} V^{-2}) T^{\varepsilon} \ll T^{\varepsilon} M^{2-2\sigma} V^{-2}, \tag{3.81}$$

provided that

$$T = T_0 = V^{\frac{2-\varepsilon}{c(\theta)}} M^{\frac{(2\sigma-1-\theta)}{c(\theta)}}, \tag{3.82}$$

since $V > T^\varepsilon$ by hypothesis of the Lemma. If we divide T into sub-intervals of length at most T_0, where T_0 is given by (3.82), then the upper bound for the number of $t_r's$ in each of these intervals is given by (3.81), and so

$$\begin{aligned} R &\ll T^\varepsilon M^{2-2\sigma}V^{-2}(1+T/T_0) \\ &\ll T^\varepsilon (M^{2-2\sigma}V^{-2} + TM^{\frac{(2c(\theta)+1+\theta-2(1+c(\theta))\sigma)}{c(\theta)}} V^{\frac{-2(1+c(\theta))}{c(\theta)}}. \end{aligned} \tag{3.83}$$

With $c(\theta)$ and θ given by (3.78) and (3.79) it is readily checked that

$$2c(\theta)+1+\theta-2(1+c(\theta))\sigma = 0, \quad \frac{(1+c(\theta))}{c(\theta)} = f(\sigma),$$

where $f(\sigma)$ is given by (3.73), which completes the proof of the Lemma. □

Estimates for $m(\sigma)$ are furnished by

Theorem 3.4.2. *[8] Let $m(\sigma)$ for each fixed $\frac{1}{2} < \sigma < 1$ be defined by (3.71), then*

$$\begin{aligned} m(\sigma) &\geq \frac{4}{(3-4\sigma)} && \textit{for} \quad \frac{1}{2} < \sigma \leq \frac{5}{8}, \\ m(\sigma) &\geq \frac{10}{(5-6\sigma)} && \textit{for} \quad \frac{5}{8} \leq \sigma \leq \frac{35}{54}, \\ m(\sigma) &\geq \frac{19}{(6-6\sigma)} && \textit{for} \quad \frac{35}{54} \leq \sigma \leq \frac{41}{60}, \\ m(\sigma) &\geq \frac{2112}{(859-948\sigma)} && \textit{for} \quad \frac{41}{60} \leq \sigma \leq \frac{3}{4}, \\ m(\sigma) &\geq \frac{12408}{(4537-4890\sigma)} && \textit{for} \quad \frac{3}{4} \leq \sigma \leq \frac{5}{6}, \\ m(\sigma) &\geq \frac{4324}{(1031-1044\sigma)} && \textit{for} \quad \frac{5}{6} \leq \sigma \leq \frac{7}{8}, \\ m(\sigma) &\geq \frac{98}{(31-32\sigma)} && \textit{for} \quad \frac{7}{8} \leq \sigma \leq 0.91591..., \\ m(\sigma) &\geq \frac{(24\sigma-9)}{(4\sigma-1)(1-\sigma)} && \textit{for} \quad 0.91591... \leq \sigma \leq 1-\varepsilon. \end{aligned} \tag{3.84}$$

In addition to this; we have $m(\frac{35}{54}) \geq 9$, $m(\frac{41}{60}) \geq 10$, $m(\frac{7}{10}) \geq 11$, $m(\frac{5}{7}) \geq 12$, $m(\frac{2}{3}) \geq 9.6187...$, $m(\frac{3}{4}) \geq 14.270270...$, and $m(\frac{5}{6}) \geq \frac{188}{7}$, $m(\frac{7}{8}) \geq 36.8$.

Proof. Our starting point is the relation

$$\sum_{n=1}^{\infty} d_k(n) e^{-\frac{n}{Y}} n^{-s} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} Y^w \zeta^k(s+w)\Gamma(w)dw, \tag{3.85}$$

where $s = \sigma + it_r$, $\frac{1}{2} < \sigma < 1$, $k \geq 1$ is an integer, $1 \ll Y \ll T^C$. The above formula follows from (3.76), and we shall use it first with k=2. Moving the line of integration to $\Re(w) = \frac{1}{2} - \sigma$ we encounter a double pole at $w = 1 - s$ with residue $O(1)$, and simple pole at $w = 0$ with residue $\zeta^2(s)$. Therefore

$$\sum_{n\leq Y} d(n)e^{-\frac{n}{Y}}n^{-s} = \zeta^2(s) + O(1) + \frac{1}{2\pi i}\int_{\Re(w)=\frac{1}{2}-\sigma} \zeta^2(s+w)\Gamma(w)Y^w dw, \quad (3.86)$$

By Stirling's formula the above integral is $O(1)$ for $\mid \Im(w) \mid \geq \log^2 T$, and so for each $s = \sigma + it_r$ satisfying (3.67) we have

$$\zeta^2(\sigma + it_r) \ll 1 + \mid \sum_{n\leq Y} d(n)e^{-\frac{n}{Y}}n^{-\sigma-it_r} \mid + \int_{-\log^2 T}^{\log^2 T} \mid \zeta(\frac{1}{2} + it_r + iv) \mid^2 Y^{\frac{1}{2}-\sigma}e^{-|v|}dv. \quad (3.87)$$

Taking into account (3.68) the above equation implies either

$$V^2 \ll \mid \sum_{n\leq Y} d(n)e^{-\frac{n}{Y}}n^{-\sigma-it_r} \mid \ll \log T \max_{M\leq \frac{Y}{2}} \mid \sum_{M<n\leq 2M} d(n)e^{-\frac{n}{Y}}n^{-\sigma-it_r} \mid, \quad (3.88)$$

or

$$V^2 \ll Y^{\frac{1}{2}-\sigma} \mid \zeta(\frac{1}{2} + it'_r) \mid^2, \quad (3.89)$$

where

$$\mid \zeta(\frac{1}{2} + it'_r) \mid = \max_{-\log^2 T\leq v\leq \log^2 T} \mid \zeta(\frac{1}{2} + it_r + iv) \mid. \quad (3.90)$$

Here one may choose $Y = Y(r)$ as a function of r which satisfies $1 \ll Y \ll T^C$. To prove that $m(\sigma) \geq \frac{4}{(3-4\sigma)}$ for $\frac{1}{2} < \sigma < \frac{5}{8}$ (by (3.28) this holds also for $\sigma = \frac{1}{2}$) it will be sufficient to prove

$$R \ll T^{1+\varepsilon}V^{-\frac{4}{(3-4\sigma)}}. \quad (3.91)$$

We shall omit in the rest of this proof factors like $T^\varepsilon \log^C T$ on the right-hand sides of inequalities implied by $\ll$ for simplicity of writing.
To obtain (3.91) we consider separately subsets A and B of $\{t_r\}$ such that $t_r \in A$ if V in (3.68) satisfies $V \leq T^{\frac{(3-4\sigma)}{8}}$ and $t_r \in B$ if $V > T^{\frac{(3-4\sigma)}{8}}$. If $R_1 = \mid A \mid$ and $R_2 = \mid B \mid$, then $R = R_1 + R_2$ and

$$R_1 \ll Y_1^{2-2\sigma}V^{-4} + TV^{-\frac{4}{(3-4\sigma)}} + Y_1^{\frac{1}{2}-\sigma}V^{-2}\sum_{t_r\in A} \mid \zeta(\frac{1}{2} + it'_r) \mid^2, \quad (3.92)$$

when one applies Lemma 3.4.1 with V^2 in place of V because of (3.88), and where $M \leq \frac{Y}{2} = \frac{Y_1}{2}$ in (3.88) is chosen in such a way that $\gg \frac{R_1}{\log T}$ numbers $t_r \in A$

satisfy (3.88) with that particular M. Using $M(4) = 1$ and the Cauchy-Schwarz inequality we have

$$R_1 \ll {Y_1}^{2-2\sigma}V^{-4} + TV^{-\frac{4}{(3-4\sigma)}} + TV^{-4}{Y_1}^{1-2\sigma}, \tag{3.93}$$

and in view of $V \le T^{\frac{(3-4\sigma)}{8}}$ the choice $Y_1 = T$ gives

$$R_1 \ll TV^{-\frac{4}{(3-4\sigma)}} + T^{2-2\sigma}V^{-4} \ll TV^{-\frac{4}{(3-4\sigma)}}. \tag{3.94}$$

To bound R_2 we reason analogously, only now we use Holder's inequality to obtain

$$R_2 \ll {Y_2}^{2-2\sigma}V^{-4} + TV^{-\frac{4}{(3-4\sigma)}} + {Y_2}^{\frac{1}{2}-\sigma}V^{-2}{R_2}^{\frac{5}{6}}\Big(\sum_{t_r\in B} \mid \zeta(\frac{1}{2} + it'_r) \mid^{12}\Big)^{\frac{1}{6}}, \tag{3.95}$$

whence using $M(12) \le 2$ and simplifying we have

$$R_2 \ll {Y_2}^{2-2\sigma}V^{-4} + TV^{-\frac{4}{(3-4\sigma)}} + {Y_2}^{3-6\sigma}V^{-12}. \tag{3.96}$$

Choosing $Y_2 = T^{\frac{2}{4\sigma-1}}V^{-\frac{8}{(4\sigma-1)}} \gg 1$ we have

$$R_2 \ll TV^{-\frac{4}{(3-4\sigma)}} + T^{\frac{(4-4\sigma)}{(4\sigma-1)}}V^{\frac{-12}{(4\sigma-1)}}. \tag{3.97}$$

The second term in the above equation does not exceed the first if

$$T^{\frac{5-8\sigma}{(4\sigma-1)}} \le V^{\frac{8(5-8\sigma)}{(4\sigma-1)(3-4\sigma)}}.$$

and this condition is satisfied since $\frac{1}{2} < \sigma \le \frac{5}{8}$ and $V > T^{\frac{(3-4\sigma)}{8}}$. Thus from (3.94) and (3.97) we obtain

$$R = R_1 + R_2 \ll TV^{-\frac{4}{(3-4\sigma)}}.$$

as asserted, implying $m(\sigma) \ge \frac{4}{(3-4\sigma)}$ for $\frac{1}{2} < \sigma \le \frac{5}{8}$.
Except for the last two bounds in (3.84), all the other bounds will follow from Theorem 3.2.1 (convexity) and the bounds

$$\begin{gathered} m(\sigma) \ge 9, \quad m(\frac{41}{60}) \ge 10, \quad m(\frac{3}{4}) \ge \frac{528}{37} = 14.27027..., \\ m(\frac{5}{6}) \ge \frac{188}{7} = 26.85714..., \quad m(\frac{7}{8}) \ge \frac{184}{5} = 36.8. \end{gathered} \tag{3.98}$$

The bounds in (3.98), and consequently the corresponding bounds in Theorem 3.4.2, can be somewhat improved by a more elaborate choice of the exponent pairs which will appear in the course of the proof.
We consider now the range $\frac{5}{8} \le \sigma \le \frac{2}{3}$, and let A and B denote subsets of $\{t'_r\}$

(see (3.89) and (3.90)) for which the first and the second inequality, respectively, holds in (3.32), with V replaced by $VY^{\frac{(2\sigma-1)}{4}}$. If $R_1 =\mid A\mid$, $R_2 =\mid B\mid$, then by Lemma 3.4.1 and (3.32)

$$R_1 \ll {Y_1}^{2-2\sigma} V^{-4} + TV^{-\frac{4}{(3-4\sigma)}} + {Y_1}^{3-6\sigma} TV^{-6}. \tag{3.99}$$

with the choice $Y_1 = (TV^{-2})^{\frac{2}{(1+2\sigma)}}$ this becomes

$$R_1 \ll TV^{-\frac{4}{(3-4\sigma)}} + T^{\frac{4(1-\sigma)}{(1+2\sigma)}} V^{\frac{-12}{(1+2\sigma)}}. \tag{3.100}$$

To estimate R_2 we use again Lemma 3.4.1 and (3.32) to obtain

$$R_2 \ll {Y_2}^{2-2\sigma} V^{-4} + T^{\frac{\chi+\lambda}{\chi}} V^{\frac{-2(1+2\chi+2\lambda)}{\chi}} Y_2^{\frac{(\frac{1}{2}-\sigma)(1+2\chi+2\lambda)}{\chi}} + TV^{\frac{-4}{(3-4\sigma)}}. \tag{3.101}$$

The first two terms above are equal if

$$Y_2 = T^{\frac{2(\chi+\lambda)}{((2+4\lambda)\sigma-1+2\chi-2\lambda)}} V^{\frac{-4(1+2\lambda)}{((2+4\lambda)\sigma-1+2\chi-2\lambda)}}, \tag{3.102}$$

and since by (3.78) $c(\theta) \le \frac{1}{8}$ for $\theta \ge \frac{5}{8}$ and $2\frac{(\chi+\lambda)}{4(1+2\lambda)} \ge \frac{1}{8}$ we have $Y_2 \gg 1$ and thus

$$R \ll TV^{-\frac{4}{(3-4\sigma)}} + T^{\frac{4-4\sigma}{1+2\sigma}} V^{\frac{-12}{1+2\sigma}} + T^{\frac{4(1-\sigma)(\chi+\lambda)}{((2+4\lambda)\sigma-1+2\chi-2\lambda)}} V^{\frac{-4(1+2\chi+2\lambda)}{((2+4\lambda)\sigma-1+2\chi-2\lambda)}}. \tag{3.103}$$

The exponent of T of the last term in (3.103) equals unity for

$$\sigma = \frac{1+2\chi+6\lambda}{2+4\chi+8\lambda}, \tag{3.104}$$

giving

$$R \ll TV^{-\frac{4}{(3-4\sigma)}} + T^{\frac{(4-4\sigma)}{(1+2\sigma)}} V^{\frac{-12}{(1+2\sigma)}} + TV^{-F}, \tag{3.105}$$

where

$$F = \frac{2(1+2\chi+2\lambda)(1+2\chi+4\lambda)}{\chi+\lambda+4\chi\lambda+2\chi^2+2\lambda^2}.$$

In (3.105) the term TV^{-F} is the largest, which will be shown now for $\sigma = \frac{2}{3}$. In that case (3.104) reduces to $\lambda - \chi = \frac{1}{2}$, and so with the exponent pair $(\chi, \lambda) = (\frac{\alpha}{2}+\epsilon, \frac{\alpha}{2}+\epsilon)$, $\alpha = 0.3290213568...$ one obtains from (3.105)

$$R \ll TV^{-9.61872...} + (TV^{-9})^{\frac{4}{7}}. \tag{3.106}$$

and one has $(TV^{-9})^{\frac{4}{7}} \le TV^{-x}$ for

$$V \le T^{\frac{3}{(7\chi-36)}}. \tag{3.107}$$

Since by Lemma 3.4.1 one has $c(\frac{2}{3}) = \frac{5}{54}$, it is seen that (3.90) is certainly satisfied for

$$\frac{5}{54} \le \frac{3}{(7x-36)} \quad \text{or} \quad x \le \frac{342}{35} = 9.7714...,$$

this proves $m(\frac{2}{3}) \ge 9.61872...$, which is the optimal value this method allows. With $(\chi, \lambda) = (\frac{2}{7}, \frac{4}{7})$ in (3.104) we obtain $\sigma = \frac{35}{54} = 0.6481481...$, and a similar calculation as the one above gives $m(\sigma) \ge 9$. The above procedure may be also used when $\sigma \ge \frac{2}{3}$, only in view of Lemma 3.4.1 the first term in (3.103) is to be replaced by $TV^{-2f(\sigma)}$, namely, we obtain

$$R \ll TV^{-2f(\sigma)} + T^{\frac{4-4\sigma}{1+2\sigma}} V^{\frac{-12}{1+2\sigma}} + T^{\frac{4(1-\sigma)(\chi+\lambda)}{((2+4\lambda)\sigma-1+2\chi-2\lambda)}} V^{\frac{-4(1+2\chi+2\lambda)}{((2+4\lambda)\sigma-1+2\chi-2\lambda)}}. \tag{3.108}$$

Calculations for $\sigma \ge \frac{2}{3}$ are carried out in the manner described above. The term $TV^{-2f(\sigma)}$ is always the smallest one, and the second and third term in (3.108) do not exceed TV^x and TV^y, respectively, for values of x and y which will depend on $c(\theta)$, where for $c(\theta)$ we use the values given by (3.78). With the exponent pair $(\chi, \lambda) = (\frac{1}{14}, \frac{11}{14})$ the last term in (3.108) is TV^{-10} for $\sigma = \frac{41}{60} = 0.68333...$, and then we obtain $m(\frac{41}{60}) \ge 10$. Using $(\chi, \lambda) = (\frac{2}{7}, \frac{4}{7})$ and $\sigma = \frac{7}{10}$, $\sigma = \frac{5}{7}$ we obtain likewise $m(\frac{7}{10}) > 11$ and $m(\frac{5}{7}) > 12$. For $\sigma \ge \frac{3}{4}$ we have from (3.108) that $R \ll TV^{-x}$ for

$$x \le \frac{8(3+6\chi+2\lambda)}{1+4\chi+2\lambda}, \tag{3.109}$$

where we used $c(\frac{3}{4}) \le \frac{1}{16}$. The choice $(\chi, \lambda) = (\frac{5}{24}, \frac{15}{24})$ gives $x \le \frac{528}{37} = 14.270270...$, so that $m(\frac{3}{4}) \ge \frac{528}{37}$, since the middle term in (3.108) turns out to be $T^{\frac{2}{5}} V^{\frac{-24}{5}} \le TV^{-y}$ for $y \le \frac{72}{5} = 14.4$. Similarly one obtains $m(\frac{5}{6}) \ge \frac{188}{7} = 26.857142...$ and $m(\frac{7}{8}) \ge \frac{184}{5}$ for $(\chi, \lambda) = (\frac{2}{7}, \frac{4}{7})$.
To finish the proof of Theorem 3.4.2 it remains to prove the general estimate for $m(\sigma)$ when $\sigma > \frac{5}{8}$, as given by (3.84). For $\frac{5}{8} \le \sigma \le \frac{13}{15}$ we use Lemma 3.4.1 and $M(12) \le 2$ to obtain as before

$$R \ll TV^{-2f(\sigma)} + Y^{2-2\sigma}V^{-4} + Y^{3-6\sigma}T^2V^{-12} \ll TV^{-2f(\sigma)} + T^{\frac{4-4\sigma}{4\sigma-1}} V^{\frac{-12}{(4\sigma-1)}} \tag{3.110}$$

for $Y = T^{\frac{2}{(4\sigma-1)}} V^{\frac{-8}{(4\sigma-1)}}$. Using estimates for $c(\theta)$ when $\frac{5}{8} \le \theta \le \frac{13}{15}$ (as given in (3.78)) it is seen that the last term in (3.110) is $\ll TV^{-x}$ for values of $x = m(\sigma)$ given by (3.84), while the term is of a lower order of magnitude than $TV^{-2f(\sigma)}$ is of a lower order of magnitude than $TV^{m(\sigma)}$. To obtain estimates for $m(\sigma)$ when $\sigma \ge \frac{13}{15}$ we shall use (3.85) with $k = 1$, since for that range the values of $f(\sigma)$ are large enough and $TV^{-f(\sigma)}$ suffices, whereas for smaller values of σ it was necessary to use $k = 2$ in (3.85), with the effect that V in Lemma 3.4.1 is replaced by V^2. Therefore we shall have

$$V \le \mid \zeta(\sigma + it_r) \mid \ll \log T \max_{M \le \frac{Y}{2}} \mid \sum_{M < n \le 2M} e^{\frac{-n}{Y}} n^{-\sigma - it_r} \mid, \tag{3.111}$$

or

$$V \leq| \zeta(\sigma + it_r) |\ll Y^{\frac{1}{2}-\sigma} \mid \zeta(\frac{1}{2} + it'_r) \mid, \tag{3.112}$$

where t'_r is given by (3.90). Now we take $(\chi, \lambda) = (\frac{2}{7}, \frac{4}{7})$ in (3.32) and let A and B be subsets of $\{t'_r\}$ for which $TV^{-6}\log^8 T$ and $T^3V^{-19}\log^{\frac{49}{2}} T$, respectively, dominate in size in (3.32). Then applying as before Lemma 3.4.1 we have from (3.111) and (3.112) with $R_1 =| A |$, $R_2 =| B |$

$$R_1 \ll {Y_1}^{2-2\sigma}V^{-2} + {Y_1}^{3-6\sigma}TV^{-6} + TV^{-f(\sigma)}, \tag{3.113}$$

$$R_2 \ll {Y_2}^{2-2\sigma}V^{-2} + {Y_2}^{19(\frac{1}{2}-\sigma)}T^3V^{-19} + TV^{-f(\sigma)}. \tag{3.114}$$

If we choose $Y_1 = (T^6V^{-4})^{\frac{1}{(4\sigma-1)}} \gg 1$ and $Y_2 = (T^6V^{-34})^{\frac{1}{(34\sigma-15)}} \gg 1$, then

$$R_1 \ll T^{\frac{2-2\sigma}{4\sigma-1}}V^{\frac{-6}{4\sigma-1}} + TV^{-f(\sigma)}, \tag{3.115}$$

$$R_2 \ll T^{\frac{12-12\sigma}{34\sigma-15}}V^{\frac{-38}{31\sigma-15}} + TV^{-f(\sigma)}. \tag{3.116}$$

With $c(\theta) \leq \frac{(1-\theta)}{5}$ for $\theta \geq \frac{5}{6}$ we obtain $R_1 + R_2 \ll TV^{-x}$ for

$$x = \min\Big(f(\sigma), \frac{24\sigma - 9}{(4\sigma - 1)(1 - \sigma)}, \frac{192\sigma - 97}{(43\sigma - 15)(1 - \sigma)}\Big), \tag{3.117}$$

where $f(\sigma) = \frac{98}{(31-32\sigma)}$ for $\frac{13}{15} \leq \sigma \leq \frac{57}{62} = 0.91935...$ and $f(\sigma) = \frac{5}{(1-\sigma)}$ for $\frac{57}{62} \leq \sigma \leq 1 - \varepsilon$. Now for $\frac{13}{15} \leq \sigma \leq 1$ we have $\frac{24\sigma-9}{4\sigma-1} \leq 5$ and the second term in (3.117) does not exceed the third. For $\frac{57}{62} \geq \sigma \geq 0.91591...$ we have $\frac{24\sigma-9}{(4\sigma-1)(1-\sigma)} \leq \frac{98}{(31-32\sigma)} = f(\sigma)$, hence the last part of the Theorem. In particular we have

$$m(\sigma) \geq \frac{4.873}{(1 - \sigma)} \quad \text{for} \quad \sigma \geq 0.91591... \tag{3.118}$$

□

3.5 New Bounds for $m(\sigma)$ on $\frac{1}{2} < \sigma < 1$

In this section we shall derive some new bounds for the function $m(\sigma)$ applying some suitably chosen exponent pairs, *J. Bourgain's* exponent pair, which will lead then to new bounds for α_k and β_k in the next chapter. We shall refine the method which is exploited in section 3.4. Therein one of the ingredients in estimating $m(\sigma)$ was Lemma 3.4.1. To find new estimates, we shall indicate how for σ relatively close to 1 the last expression for $f(\sigma)$ in Lemma 3.4.1 equation (3.73) may be replaced by a better one. According to [9], one can take

$$f(\sigma) = \frac{2^l(l-2)+2}{2^l - 1 - 2^l\sigma} \quad \text{for} \quad 1 - \frac{l-1}{2^l-2} \leq \sigma \leq 1 - \frac{l}{2^{l+1}-2} \tag{3.119}$$

for any $l = 3, 4, ...,$ and also for $k \geq 3$

$$f(\sigma) = \frac{k}{1-\sigma} \quad \text{for} \quad 1 - \frac{k}{2^{k+1}-2} \leq \sigma \leq 1 - \epsilon \tag{3.120}$$

for any fixed $\epsilon > 0$. Therefore the last value of $f(\sigma)$ in (3.73) may be replaced by an arbitrary number of values furnished by (3.119) for $l \geq 6$, plus a value of $f(\sigma)$ furnished by (3.120) with a suitable k. The proof is analogous to the proof of (3.73) given in [8], and therefore the details will be omitted. As defined earlier in (2.21),

$$\mu(\sigma) = \inf\{c \geq 0 : \zeta(\sigma + it) \ll t^c\}$$

for a given real σ, and $c(\sigma)$ is an upper bound for $\mu(\sigma)$, it was shown in Lemma 3.4.1 that $f(\sigma)$ may be determined by the following two equations

$$2c(\theta) + 1 + \theta - 2(1 + c(\theta))\sigma = 0, \tag{3.121}$$

$$f(\sigma) = \frac{(1 + c(\theta)}{c(\theta)}. \tag{3.122}$$

Using the classical estimates (see [14]) $\mu(\sigma) \leq \frac{1}{(2L-2)}$ for $\sigma = 1 - \frac{l}{(2L-2)}$, $L = 2^{l-1}$, $l \geq 3$, and convexity of $\mu(\sigma)$ it follows that one may take

$$c(\theta) = \frac{2^{l-1} - 1 - 2^{l-1}\theta}{l \cdot 2^{l-1} - 2^l + 2} \quad \text{for} \quad 1 - \frac{l-1}{2^{l-1}-2} \leq \theta \leq 1 - \frac{l}{2^l - 2}, \tag{3.123}$$

and similarly one can take

$$c(\theta) = \frac{1-\theta}{k} \qquad \text{for} \qquad 1 - \frac{k}{2^k - 2} \leq \theta \leq 1. \tag{3.124}$$

Substituting (3.123) and (3.124) in (3.121) and (3.122), we obtain (3.119) and (3.120), respectively.
We are now going to bound the function $f(\sigma)$ for certain values of σ in the interval $\frac{1}{2} < \sigma < 1$, let us take $\sigma = \frac{27}{40}, \frac{5}{7}, \frac{5}{6}, \frac{7}{8}$, and $\frac{14}{15}$. It was shown in section 3.4 (3.70) that to obtain bounds for $m(\sigma)$ it suffices to obtain bounds of the form $R \ll T^{1+\epsilon}V^{-m(\sigma)}$, where R is the number of points $t_r(r = 1, 2, .., R)$ such that $| t_r |\leq T$, $| t_r - t_s |\geq \log^4 T$ for $1 \leq r \neq s \leq R$ and $| \zeta(\sigma + it_r) |\geq V > 0$ for any given V. Moreover, by (3.108) we have (with omitted T^ϵ for brevity)

$$R \ll TV^{-2f(\sigma)} + T^{\frac{(4-4\sigma)}{1+2\sigma}}V^{\frac{-12}{1+2\sigma}} + T^{\frac{4(1-\sigma)(\chi+\lambda)}{((2+4\lambda)\sigma-1+2\chi-2\lambda)}}V^{\frac{-4(1+2\chi+2\lambda)}{((2+4\lambda)\sigma-1+2\chi-2\lambda)}} = R_1 + R_2 + R_3, \tag{3.125}$$

say, Here $f(\sigma)$ has the same meaning as in Lemma 3.4.1, and (χ, λ) is an exponent pair. We shall now plug-in the recently obtained exponent pair by *Jean Bourgain* $(l_\epsilon, k_\epsilon) = (\frac{13}{84} + \epsilon, \frac{55}{84} + \epsilon)$ in the above equation (3.125), as $\epsilon \to 0$ we have $(\chi, \lambda) = (\frac{13}{84}, \frac{55}{84})$. This new exponent pair plays a decisive role in the improvement of the error term estimates in Dirichlet divisor problem.

For $\sigma = \frac{27}{40}$ we obtain $f(\frac{27}{40}) = \frac{25}{4}$, $c(\frac{27}{40}) = \frac{4}{45}$, hence $R_1 = TV^{\frac{-25}{2}}$, $R_2 = T^{\frac{26}{47}}V^{\frac{-240}{47}} \ll TV^{-x}$ for $V \ll T^{\frac{21}{47x-240}}$, which is certainly satisfied for

$$c\left(\frac{27}{40}\right) = \frac{4}{45} \leq \frac{21}{47x - 240}, \quad x \leq \frac{1905}{188} = 10.1329..., \quad \text{whence} \quad R_2 = TV^{-10.132978...}$$

with $(\chi, \lambda) = (\frac{13}{84}, \frac{55}{84})$ we obtain $R_3 = T^{\frac{884}{939}}V^{\frac{-8800}{939}} \ll TV^{-y}$ for $V \ll T^{\frac{55}{939x-8800}}$ this gives

$$c\left(\frac{27}{40}\right) = \frac{4}{45} \leq \frac{55}{939x - 8800}, \quad \text{which gives} \quad y \leq \frac{37675}{3756} = 10.03061767...,$$

Now, we will take the lower bound for $m(\sigma)$ the minimum of $\{2f(\sigma), x, y\}$, and proves that

$$m(\frac{27}{40}) \geq \frac{37675}{3756} = 10.03061767... \tag{3.126}$$

By a similar procedure we obtain for $\sigma = \frac{5}{7}$ using $c(\frac{5}{7}) = \frac{1}{14}$ that $m(\frac{5}{7}) \geq x$ for

$$x = \min\left(\frac{210}{17}, \frac{14(5 + 10\chi + 2\lambda)}{3 + 14\chi + 6\lambda}\right)$$

The above exponent pair $(\chi, \lambda) = (\frac{13}{84}, \frac{55}{84})$ can be transformed by applying Theorem 2.2.1 (Weyl A-process) followed by Theorem 2.2.2 (Van der corput B-process).

Thus the new exponent pair $BA(\chi, \lambda)$ is of the form $(\frac{55}{194}, \frac{110}{194})$, substituting this exponent pair in above equation we obtain,

$$m(\frac{5}{7}) \geq \frac{6090}{503} = 12.10735586... \tag{3.127}$$

and for $\sigma = \frac{3}{4}$ (using $c(\frac{3}{4}) = \frac{1}{16}$) that $m(\frac{3}{4}) \geq x$ for

$$x = \min\left(\frac{72}{5}, \frac{8(3+6\chi+2\lambda)}{1+4\chi+2\lambda}\right)$$

Substituting $(\chi, \lambda) = (\frac{13}{84}, \frac{55}{84})$ in the above equation we obtain,

$$m(\frac{3}{4}) \geq \frac{1760}{123} = 14.30894308... \tag{3.128}$$

Similar calculations with $(\chi, \lambda) = (\frac{13}{84}, \frac{55}{84})$ yield

$$m(\frac{5}{6}) \geq \frac{131670}{4893} = 26.90987124... \tag{3.129}$$

$$m(\frac{7}{8}) \geq \frac{41030}{1029} = 39.87366375... \tag{3.130}$$

$$m(\frac{9}{10}) \geq \frac{274450}{4983} = 55.07726269... \tag{3.131}$$

$$m(\frac{14}{15}) \geq \frac{119625}{1274} = 93.89717425... \tag{3.132}$$

All these values of $m(\sigma)$ listed from equation (3.126) up to equation (3.132) slightly improve the corresponding values in Theorem 3.4.2 and the results in [9], and for the intermediate values of σ one may use the properties of $m(\sigma)$. Namely, by equation (3.25) one has, for $\frac{1}{2} \leq \sigma_1 < \sigma < \sigma_2 < 1$,

$$m(\sigma) \geq \frac{m(\sigma_1)m(\sigma_2)(\sigma_2 - \sigma_1)}{m(\sigma_2)(\sigma_2 - \sigma) + m(\sigma_1)(\sigma - \sigma_1)} \tag{3.133}$$

For values of σ between $\frac{14}{15}$ and 1, we can use the bound

$$c(\sigma) = \frac{1}{6}(1-\sigma) \quad (\frac{28}{31} \leq \sigma \leq 1)$$

and from (3.115) and (3.116) we have,

$$R \ll TV^{-f(\sigma)} + T^{\frac{(2-2\sigma)}{(4\sigma-1)}} V^{\frac{-6}{4\sigma-1}} + T^{\frac{(12-12\sigma)}{(34\sigma-15)}} V^{\frac{-38)}{(34\sigma-15)}}$$

and it gives $R \ll TV^{-x}$ for

$$x = \min\left(f(\sigma), \frac{30\sigma - 12}{(4\sigma-1)(1-\sigma)}, \frac{238\sigma - 124}{(34\sigma - 15)(1-\sigma)}\right)$$

Hence using (3.119) with $l = 6$ and (3.128) with $k = 6$ we obtain

$$m(\sigma) \geq \begin{cases} \frac{258}{(63-64\sigma)} & \text{for} \quad \frac{14}{15} \leq \sigma \leq c_0, \\ \frac{30\sigma-12}{(4\sigma-1)(1-\sigma)} & \text{for} \quad c_0 \leq \sigma \leq 1-\varepsilon, \end{cases} \tag{3.134}$$

where $c_0 = \frac{1}{222}(171 + \sqrt{1602}) = 0.95056302...$

Theorem 3.5.1. *Let $m(\sigma)$ for each fixed $\frac{1}{2} < \sigma < 1$ be defined by (3.6). Then*

$$\begin{aligned} m(\tfrac{27}{40}) \geq 10.030617... \quad m(\tfrac{5}{7}) \geq 12.107355... \quad m(\tfrac{3}{4}) \geq 14.308943..., \\ m(\tfrac{5}{6}) \geq 26.909871... \quad m(\tfrac{7}{8}) \geq 39.873663... \quad m(\tfrac{9}{10}) \geq 55.077262..., \\ m(\tfrac{14}{15}) \geq 93.897174... \end{aligned} \tag{3.135}$$

$$m(\sigma) \geq \begin{cases} \frac{258}{(63-64\sigma)} & \textit{for} \quad \frac{14}{15} \leq \sigma \leq c_0, \\ \frac{30\sigma-12}{(4\sigma-1)(1-\sigma)} & \textit{for} \quad c_0 \leq \sigma \leq 1-\varepsilon, \quad \textit{where} \quad c_0 = 0.950563... \end{cases}$$

Chapter 4

The Dirichlet Divisor Problem

4.1 Introduction

Let $d(n)$ denote the number of divisors of n, and consider the summatory function $D(x) = \sum_{n \le x} d(n)$ which occurs in the study of many important problems in number theory, such as the asymptotic behaviour of the Riemann zeta-function. The *Dirichlet Divisor problem*, or briefly, the *Divisor problem*, arises from estimating the sum $\sum_{n \le x} d(n)$. The problem is to estimate the number of lattice points (u, v) lying in the first quadrant under the hyperbola $x = uv$. In 1849, Dirichlet proved the asymptotic formula in an elementary way using Dirichlet Hyperbola, for $x \ge 1$,

$$D(x) = x \log x + (2\gamma - 1)x + \Delta(x). \tag{4.1}$$

where $\Delta(x) = O(\sqrt{x})$ is the error term, and $\gamma = 0.5772...$ is Euler's constant. Subsequently, in 1904, Voronoi employed the summation formula ([8]; chap 3) named after him and proved that $\Delta(x) = O(x^{\frac{1}{3}} \log x)$, and the estimate has been continually improved since, although the precise result is still unknown. Our objective in this chapter is to apply the theory of zeta-function to find the sharpest values of the error terms in the general divisor problem.
The Dirichlet divisor problem is closely related to that of the Riemann zeta-function. By (1.24) with $k = 2$, we have

$$D(x) = {\sum_{n \le x}}' d(n) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta^2(s) x^s s^{-1} ds \qquad (c > 1), \tag{4.2}$$

where the prime on the summation sign on the left-hand side indicates that if x is an integer then only $\frac{1}{2}d(x)$ is counted.
On moving the line of integration in (4.2) to the left, we encounter a double pole at $w = 1$, the residue being $x \log x + (2\gamma - 1)x$, by (A.3). Thus

$$\Delta(x) = \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \zeta^2(s)x^s s^{-1} ds \qquad (\frac{1}{2} < c' < 1). \tag{4.3}$$

The more general divisor problem is a problem of determining the asymptotic behavior of the divisor summatory function, is defined as $x \to \infty$ of the sum

$$D_k(x) = \sum_{n\le x} d_k(n) \quad \text{for} \quad k \ge 2 \tag{4.4}$$

where $d_k(n) = \sum_{m_1 m_2 \dots m_k = n} 1$ is the divisor function which represents the number of ways n may be written as a product of $k(\ge 2$, fixed) factors, was also considered by Dirichlet. We have

$$D_k(x) = {\sum_{n\le x}}' d_k(n) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \zeta^k(s)x^s s^{-1} ds \qquad (c > 1). \tag{4.5}$$

Here there is a pole of order k at $s = 1$, one has by the residue theorem

$$ {\sum_{n\le x}}' d_k(n) = xP_k(\log x) + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \zeta^k(s)x^s s^{-1} ds \qquad (\frac{1}{2} < c < 1), \tag{4.6}$$

where $P_k(t)$ is a polynomial of degree $k - 1$. We write

$$\Delta_k(x) = \sum_{n\le x}{}' d_k(n) - xP_{k-1}(\log x). \tag{4.7}$$

The coefficients of P_k may be evaluated by using

$$P_k(\log x) = \operatorname{Res}_{s=1} x^{s-1}\zeta(s)^k s^{-1} \tag{4.8}$$

In fact from the Laurent expansion (A.3), i.e $\zeta(s) = \frac{1}{(s-1)} + \gamma_0 + \sum_{k=1}^{\infty} \gamma_k (s-1)^k$ and (4.8) one may calculate explicitly the coefficients of P_k as functions of the γ_k's.

4.2 The Order of $\Delta_k(x)$

Definition 4.2.1. Let $\Delta_k(x)$ be as given in (4.7), we define the order α_k of $\Delta_k(x)$ as the least number possessing the property that, as $x \to \infty$,

$$\Delta_k(x) \ll x^{\alpha_k + \epsilon} \qquad \text{for every} \quad \varepsilon > 0. \tag{4.9}$$

4.2.1 Estimates of α_k for $k = 2, 3$

As the problem of determining α_k is notoriously difficult, so far, there is no simple closed expression for $\Delta_2(x)$ known. At the beginning of the last century, G. F. Voronoi [17] proved the remarkable formula that

$$\Delta_2(x) = -\frac{2}{\pi}\sqrt{x}\sum_{n=1}^{\infty}\frac{d(n)}{\sqrt{n}}(K_1(4\pi\sqrt{nx}) + \frac{\pi}{2}Y_1(4\pi\sqrt{nx})), \tag{4.10}$$

where K_1, Y_1 are the Bessel functions, and the series on the right hand side is bounded and convergent for x lying in each fixed closed interval. G. F. Voronoi then showed that $\Delta_2(x) \ll x^{35/108+\varepsilon}$ holds for any $\varepsilon > 0$. Although (4.10) is an exact formula, in many applications the most convenient way of obtaining estimates for $\Delta_2(x)$ and $\Delta_3(x)$ seems to be the use of the Truncated Voronoi formula (see [8]; Chap 3)

$$\Delta_k(x) \ll x^{\frac{(k-1)}{2k}} \mid \sum_{n\leq N} d_k(n)n^{-\frac{(k+1)}{2k}} e(k(nx)^{\frac{1}{k}}) \mid + x^{\frac{(k-1+\epsilon)}{k}} N^{\frac{-1}{k}} + x^{\epsilon}, \tag{4.11}$$

in the above equation (4.11) will be transformed into a multiple exponential sum, inserting k=2 in the above formula, we obtain

$$\Delta_k(x) \ll x^{\frac{1}{4}} \mid \sum_{n\leq N}(mn)^{-\frac{(3)}{4}} e(2(mnx)^{\frac{1}{2}}) \mid + x^{\frac{1}{2}} N^{\frac{-1}{2}} + x^{\epsilon}; \tag{4.12}$$

hence an application of Lemma 7.3 from [8] gives, similarly as in the proof of Theorem 7.3 on [8],

$$\begin{aligned}\Delta_2(x) &\ll x^{\epsilon} + \log^2 x\{\max_{M\leq N}(x^{\frac{3}{16}}M^{\frac{173}{152}-\frac{3}{4}} + x^{\frac{5}{16}}M^{\frac{119}{152}-\frac{3}{4}}) + x^{\frac{1}{2}}N^{-\frac{1}{2}}\} \\ &\ll x^{\epsilon} + x^{\frac{3}{16}}N^{\frac{5}{152}}\log^2 x + x^{\frac{3}{16}}N^{\frac{59}{152}}\log^2 x + x^{\frac{1}{2}}N^{-\frac{1}{2}}\log^2 x.\end{aligned} \tag{4.13}$$

Theorem 4.2.1. *If we choose $N = x^{19/54}$ in the above equation (4.13), We obtain*

$$\Delta_2(x) \ll x^{35/108}\log^2 x. \tag{4.14}$$

Th estimation of $\Delta_3(x)$ is naturally more complicated than the estimation of $\Delta_2(x)$, and is carried out via (4.11) with k=3. The best result is

$$\Delta_3(x) \ll x^{43/96+\varepsilon}. \tag{4.15}$$

The proof is long and complicated and will not be presented here.

4.2.2 Estimates of α_k by Power Moments

Let us start from the Perron's inversion formula (1.31), with $c = 1+\varepsilon$ we have

$$\sum_{n\le x} d_k(n) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \zeta^k(s)x^s s^{-1}ds + O(x^c T^{-1}) \quad (T \le x). \tag{4.16}$$

Deforming the path of integration we obtain by the residue theorem

$$\Delta_k(x) = \sum_{n\le x} d_k(n) - Res_{s=1}\{\zeta^k(s)x^s s^{-1}\} = I_1 + I_2 + I_3 + O(x^{1+\varepsilon}T^{-1}), \tag{4.17}$$

say, where for σ fixed satisfying $\frac{1}{2} \le \sigma < 1$

$$I_1 = \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma+iT} \zeta^k(s)x^s s^{-1}ds \ll x^\sigma + x^\sigma \int_1^T \mid \zeta(\sigma+iv) \mid^k \tag{4.18}$$

and,

$$\begin{aligned} I_2 + I_3 \ll \int_\sigma^{1+\varepsilon} x^\theta \mid \zeta(\theta+iT) \mid^k T^{-1} d\theta &\ll \max_{\sigma\le\theta\le 1+\varepsilon} x^c T^{k\mu(\theta)-1+\varepsilon} \\ &\ll x^{1+\varepsilon}T^{-1} + x^\sigma T^{kc(\sigma)-1+k\varepsilon}, \end{aligned} \tag{4.19}$$

where $\mu(\sigma)$ is defined by (2.21) and $c(\theta)$ is given by (3.78). From (4.18) it is immediately seen that estimates for power moments of the zeta-function lead to estimates of $\Delta_k(x)$. The estimates of the error term in the general divisor problem will be as follows

Theorem 4.2.2. *Let α_k be the infimum of numbers a_k such that $\Delta_k(x) \ll x^{a_k+\varepsilon}$ for any $\varepsilon > 0$. Then*

$$\begin{aligned} \alpha_k &\le \frac{(3k-4)}{4k} \qquad (4 \le k \le 8), \\ \alpha_9 \le \frac{35}{54} \qquad &\alpha_{10} \le \frac{41}{60} \qquad \alpha_{11} \le \frac{7}{10}, \\ \alpha_k &\le \frac{(k-2)}{(k+2)} \qquad (14 \le k \le 25), \\ \alpha_k &\le \frac{(k-1)}{(k+4)} \qquad (26 \le k \le 50), \\ \alpha_k &\le \frac{(7k-34)}{7k} \qquad (k \ge 58). \end{aligned} \tag{4.20}$$

Proof. The proof is based on estimates for $m(\sigma)$, as furnished by Theorem 3.4.2. For a fixed k we choose σ in such a way that $m(\sigma) = k$, where for $m(\sigma)$ we take the estimates given by Theorem 3.4.2. For $\mu(\sigma)$ in (2.21) we use the bound $\mu(\sigma) \le c(\sigma)$, where $c(\sigma)$ is the piece-wise linear function given by (3.78). Note that by Theorem 3.4.2 and (3.78) we have $m(\sigma) \le 1/c(\sigma)$; so that $T = x^{1-\sigma}$ gives

$$I_2 + I_3 \ll x^{1+\varepsilon}T^{-1} + x^{\sigma}T^{kc(\sigma)-1+k\varepsilon} \ll x^{\sigma+\varepsilon},$$

and therefore (4.18) gives

$$\Delta_k(x) \ll x^{\sigma+\varepsilon}.$$

In this fashion estimates for $9 \le k \le 11$ given by Theorem 3.4.2 follow at once, and for $4 \le k \le 8$ we use $m(\sigma) \ge 4/(3-4\sigma)$ $(\frac{1}{2} \le \sigma \le \frac{5}{8})$, so that $k = 4/(3-4\sigma)$ gives $\sigma = (3k-4)/4k$. For $4 \le k \le 8$ this value of σ satisfies $\frac{1}{2} \le \sigma \le \frac{5}{8}$, and $\alpha_k \le (3k-4)/4k$ follows for $4 \le k \le 8$. Next, we take $\sigma = \frac{5}{7}$ in (4.18) and (4.19): with $m(\frac{5}{7}) \ge 12$, $c(\frac{5}{7}) = \frac{1}{14}$ we have

$$\begin{aligned} I_1 \ll x^{\frac{5}{7}} + x^{\frac{5}{7}} \int_1^T \left| \zeta(\frac{5}{7}+it) \right|^{12} t^{-1} \left| \zeta(\frac{5}{7}+it) \right|^{k-12} dt \\ \ll x^{\frac{5}{7}} T^{(k-12+\varepsilon)/14} \end{aligned}$$

for $k \ge 12$, and therefore

$$\Delta_k(x) \ll x^{1+\varepsilon}T^{-1} + x^{5/7}T^{(k-12+\varepsilon)/14} \ll x^{(k-2)/(k+2)+\varepsilon}$$

for $12 \le k \le 25$ if $T = x^{4/(k+2)}$.
A similar argument gives $\alpha_k \le (k-1)/(k+4)$ for $k \ge 26$ by using $m(\frac{5}{6}) \ge 26$, $c(\frac{5}{6}) = \frac{1}{30}$. Also by Theorem 3.4.2 we have $m(\sigma) \ge 98/(31-32\sigma) = k$ for $\frac{13}{15} \le \sigma = (31k-98)/32k \le 0.91591...$, which is satisfied for $30 \le k \le 57$. From the last estimate in (3.71) we have $m(\sigma) \ge 34/(7-7\sigma) = k$ for $\sigma = (7k-34)/7k \le 0.91591...$ for $k \ge 57$. On comparing then $(k-1)/(k+4)$ with $(31k-98)/32k$ we obtain the full assertion of Theorem 4.2.2. □

For each particular $k \ge 10$ the bounds for Theorem 4.2.2 can be slightly improved by a more careful choice of exponent pairs in the proof of Theorem 3.4.2 and more care one could also derive bounds of the type $\Delta_k(x) \ll x^{\alpha_k} \log^{d_k} x$ for some $d_k \ge 0$.

4.2.3 Some New Estimates of α_k

In section 3.5 we have discussed some slight improvements for power moment estimates on the critical strip $\frac{1}{2} < \sigma < 1$. Now using these improved power moments we shall present some new estimates for the order of the error terms in the Dirichlet divisor problem which improves the results of Theorem 4.2.2.

Theorem 4.2.3. $\alpha_{10} < 0.674295,\ \alpha_{11} < 0.695290,\ \alpha_{12} < 0.712786,\ \alpha_{13} < 0.730363,\ \alpha_{14} < 0.7456962,\ \alpha_{15} < 0.758199,\ \alpha_{16} < 0.768809,\ \alpha_{17} < 0.778171,\ \alpha_{18} < 0.786493,\ \alpha_{19} < 0.793938,\ \alpha_{20} < 0.800640,\ \alpha_{26} < 0.828854,\ \alpha_{56} < 0.901329,...$

Proof. To obtain the improved bounds for α_k where $k = 10, 11, 12, ...$ in Theorem 4.2.3 we use

$$\Delta_k \ll x^{\sigma+\varepsilon} \tag{4.21}$$

which is the estimate proved in section 4.2.2. Here $\frac{1}{2} < \sigma < 1$ is a constant for which $m(\sigma) = k$, where for $m(\sigma)$ one may take lower bounds for this function, such as those furnished above from equation (3.126) up to (3.132) and convexity. All the latter are easily seen to satisfy $m(\sigma) \leq 1/c(\sigma)$, where $c(\sigma)$ is given by (3.123) and (3.124), and this condition is necessary for (4.21) to hold.

Letting $m(\sigma) = k$ and then solving for σ in the equation (3.133) will lead us to find $\sigma = \alpha_k$ in the interval $\frac{1}{2} < \sigma = \alpha_k < 1$. That is from (3.133) we have,

$$m(\sigma) = k \geq \frac{m(\sigma_1)m(\sigma_2)(\sigma_2 - \sigma_1)}{m(\sigma_2)(\sigma_2 - \sigma) + m(\sigma_1)(\sigma - \sigma_1)}$$

$$k\{m(\sigma_2)(\sigma_2 - \sigma) + m(\sigma_1)(\sigma - \sigma_1)\} \geq m(\sigma_1)m(\sigma_2)(\sigma_2 - \sigma_1)$$

$$km(\sigma_2)\sigma_2 - km(\sigma_2)\sigma + km(\sigma_1)\sigma - km(\sigma_1)\sigma_1) \geq m(\sigma_1)m(\sigma_2)(\sigma_2 - \sigma_1)$$

$$k\sigma(m(\sigma_1) - m(\sigma_2)) \geq m(\sigma_1)m(\sigma_2)(\sigma_2 - \sigma_1) - km(\sigma_2)\sigma_2 + km(\sigma_1)\sigma_1)$$

Dividing both sides by $k(m(\sigma_2) - m(\sigma_1))$, we obtain

$$\alpha_k = \sigma \leq \frac{k(m(\sigma_2)\sigma_2 - m(\sigma_1)\sigma_1) - m(\sigma_1)m(\sigma_2)(\sigma_2 - \sigma_1)}{k(m(\sigma_2) - m(\sigma_1))} \tag{4.22}$$

Now to find α_k for $k = 10, 11, 12$ we are going to take $\sigma_1 = \frac{27}{40}$ and $\sigma_2 = \frac{5}{7}$. Thus, from (3.126) and (3.127) we have $m(\frac{27}{40}) \geq \frac{37675}{3756} = 10.03061767...$ and $m(\frac{5}{7}) \geq$

$\frac{2310}{191} = 12.09424083...$ substituting all in (4.22) we obtain

$$\alpha_{10} \leq \frac{10(m(\frac{5}{7})(\frac{5}{7}) - m(\frac{27}{40})(\frac{27}{40})) - m(\frac{27}{40})m(\frac{5}{7})(\frac{5}{7} - \frac{27}{40})}{10(m(\frac{5}{7}) - m(\frac{27}{40}))} \leq 0.674336... \quad (4.23)$$

$$\alpha_{11} \leq \frac{11(m(\frac{5}{7})(\frac{5}{7}) - m(\frac{27}{40})(\frac{27}{40})) - m(\frac{27}{40})m(\frac{5}{7})(\frac{5}{7} - \frac{27}{40})}{11(m(\frac{5}{7}) - m(\frac{27}{40}))} \leq 0.695184... \quad (4.24)$$

$$\alpha_{12} \leq \frac{12(m(\frac{5}{7})(\frac{5}{7}) - m(\frac{27}{40})(\frac{27}{40})) - m(\frac{27}{40})m(\frac{5}{7})(\frac{5}{7} - \frac{27}{40})}{12(m(\frac{5}{7}) - m(\frac{27}{40}))} \leq 0.712619... \quad (4.25)$$

Similarly, to find α_{13} and α_{14}, Let us pick $\sigma_1 = \frac{5}{7}$, $m(\frac{5}{7}) \geq \frac{6090}{503} = 12.10735586...$ and $\sigma_2 = \frac{3}{4}$, $m(\frac{3}{4}) \geq \frac{21760}{123} = 14.30894308...$ substituting in (4.22) we obtain

$$\alpha_{13} \leq \frac{13(m(\frac{3}{4})(\frac{3}{4}) - m(\frac{5}{7})(\frac{5}{7})) - m(\frac{5}{7})m(\frac{3}{4})(\frac{3}{4} - \frac{5}{7})}{13(m(\frac{3}{4}) - m(\frac{5}{7}))} \leq 0.730224... \quad (4.26)$$

$$\alpha_{14} \leq \frac{14(m(\frac{3}{4})(\frac{3}{4}) - m(\frac{5}{7})(\frac{5}{7})) - m(\frac{5}{7})m(\frac{3}{4})(\frac{3}{4} - \frac{5}{7})}{14(m(\frac{3}{4}) - m(\frac{5}{7}))} \leq 0.745666... \quad (4.27)$$

Let us find the value of α_{15} and α_{16} taking $\sigma_1 = \frac{3}{4}$, $m(\frac{3}{4}) \geq \frac{1760}{123} = 14.30894308...$ and $\sigma_2 = \frac{5}{6}$, $m(\frac{5}{6}) \geq \frac{131670}{4893} = 26.90987124...$ substituting in (4.22) we obtain

$$\alpha_{15} \leq \frac{15(m(\frac{5}{6})(\frac{5}{6}) - m(\frac{3}{4})(\frac{3}{4})) - m(\frac{3}{4})m(\frac{5}{6})(\frac{5}{6} - \frac{3}{4})}{15(m(\frac{5}{6}) - m(\frac{3}{4}))} \leq 0.758199... \quad (4.28)$$

$$\alpha_{16} \leq \frac{16(m(\frac{5}{6})(\frac{5}{6}) - m(\frac{3}{4})(\frac{3}{4})) - m(\frac{3}{4})m(\frac{5}{6})(\frac{5}{6} - \frac{3}{4})}{16(m(\frac{5}{6}) - m(\frac{3}{4}))} \leq 0.768809... \quad (4.29)$$

In a similarly way, substituting $k = 17, 18, 19, 20$ in the equation

$$\alpha_k \leq \frac{k(m(\frac{5}{6})(\frac{5}{6}) - m(\frac{3}{4})(\frac{3}{4})) - m(\frac{3}{4})m(\frac{5}{6})(\frac{5}{6} - \frac{3}{4})}{k(m(\frac{5}{6}) - m(\frac{3}{4}))} \leq 0.768809... \quad (4.30)$$

we have the following values

$$\alpha_{17} \leq 0.778171..., \alpha_{18} \leq 0.786493..., \alpha_{19} \leq 0.793938..., \alpha_{20} \leq 0.800639... \quad (4.31)$$

By a similar calculation with the use of *Jean Bourgain's* exponent pair $(\chi, \lambda) = (\frac{13}{84}, \frac{55}{84})$ we have from (3.131) and (3.132), that is, $m(\frac{9}{10}) \geq 55.07726269...$ and $m(\frac{14}{15}) \geq 93.89717425...$, letting $\sigma_1 = \frac{9}{10}$ and $\sigma_2 = \frac{14}{15}$ we obtain,

$$\alpha_{56} \leq \frac{56(m(\frac{14}{15})(\frac{14}{15}) - m(\frac{9}{10})(\frac{9}{10})) - m(\frac{9}{10})m(\frac{14}{15})(\frac{14}{15} - \frac{9}{10})}{56(m(\frac{14}{15}) - m(\frac{9}{10}))} \leq 0.901328... \quad (4.32)$$

From the first bound in (3.134) one has

$$m(\sigma) \geq \frac{258}{63 - 64\sigma} \quad \text{for} \quad \frac{14}{15} \leq \sigma \leq c_0$$

implying by (4.21), we have

$$\alpha_k \leq \frac{63k - 258}{64k} \quad \text{for} \quad (79 \leq k \leq 119). \quad (4.33)$$

Likewise for $\sigma \geq \frac{19}{20} = 0.95$ we have $(30\sigma - 12)/(4\sigma - 1) \geq 165/28$, hence

$$m(\sigma) \geq \frac{165}{28(1 - \sigma)} \quad (c_0 \leq \sigma \leq 1 - \varepsilon),$$

implying by (4.21)

$$\alpha_k \leq \frac{28k - 165}{28k} \quad \text{for} \quad (k \geq 120). \quad (4.34)$$

The bounds in (4.33) and (4.34) complete the proof of the Theorem. □

4.3 The Average Order of $\Delta_k(x)$

Definition 4.3.1. The average order of $\Delta_k(x)$ denoted by β_k, is the infimum of b_k for which

$$\int_0^x \Delta_k^2(y)dy \ll x^{1+2b_k+\varepsilon} \quad \text{for} \quad (\varepsilon > 0). \quad (4.35)$$

So that β_k may be thought of as the exponent of the average order of $| \Delta_k(y) |$. Since

$$\frac{1}{x}\int_0^\infty {\Delta_k}^2(y)dy = \frac{1}{x}\int_0^x O(y^{2\alpha_k+\varepsilon})dy = O(x^{\alpha_k+\varepsilon}), \quad (4.36)$$

we have $\beta_k \leq \alpha_k$ for each k, which indicates that we have a set of upper bounds for β_k.

4.3.1 Estimates of β_k by Power Moments

The problem of average order is easier than that of order, and we can prove more about the β_k than about α_k. The classical elementary results concerning the estimation of β_k are embodied in the following three lemmas, while some specific estimates are given by Theorem 4.3.4 as provided on [8]. We shall first prove the following Lemmas.

Lemma 4.3.1. *Let γ_k be the infimum of positive numbers σ for which*

$$\int_{-\infty}^{\infty} \mid \zeta(\sigma+it) \mid^{2k} \mid \sigma+it \mid^{-2} dt \ll 1. \tag{4.37}$$

Then $\beta_k = \gamma_k$; and for $\sigma > \beta_k$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mid \zeta(\sigma+it) \mid^{2k}}{\mid \sigma+it \mid^{2}} dt = \int_{0}^{\infty} \Delta_k^2(x) x^{-2\sigma-1} dx \tag{4.38}$$

Proof. We have

$$D_k(x) = \frac{1}{2\pi i}\int_{b-iT}^{b-iT} \zeta^k(s) x^s s^{-1} ds \qquad (b>1).$$

Applying Cauchy's theorem to the rectangle $c-iT$, $b-iT$, $b+iT$, $c+iT$, where c is less than, but sufficiently close to 1, and allowing for the residue at $s=1$, we obtain

$$\Delta_k(x) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT} \frac{\zeta^k(s)}{s} x^s ds \tag{4.39}$$

Actually equation (4.39) holds for $\gamma_k < c < 1$. Since $\zeta(s)^k s^{-1} \to 0$ uniformly in the strip as $t \to \pm\infty$ in the strip, if we integrate the integrand of (4.39) over the rectangle $c' \pm iT$, $c \pm iT$, where $\gamma_k < c' < c < 1$ and making $T \to \infty$, we obtain the same result with c' instead of c. If we replace x by $\frac{1}{x}$, (4.39) expresses the relation between the Mellin transforms (A.16)

$$f(x) = \Delta_k(1/x), \quad \mathcal{F}(s) = \frac{\zeta^k(s)}{s},$$

the relevant integrals holding also in the mean-square sense. Hence taking $c > \gamma_k$, and using Parseval's identity (A.19) we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mid \zeta(c+it) \mid^{2k}}{\mid c+it \mid^{2}} dt = \int_{0}^{\infty} \Delta_k^2(1/x) x^{2c-1} dx = \int_{0}^{\infty} \Delta_k^2(x) x^{-2c-1} dx \tag{4.40}$$

It follows that, if $\gamma_k < c < 1$,

$$\int_N^{2N} \Delta_k^2(x) x^{-2c-1} dx \ll 1, \quad \int_N^{2N} \Delta_k^2(y) dy \ll N^{2c+1},$$

and therefore replacing N by $\frac{1}{2}N$, $\frac{1}{4}N$,..., and adding,

$$\int_0^x \Delta_k^2(y) dy < \sum_{N=x2^{-j}, j \geq 1} \int_N^N \Delta_k^2(y) dy \ll x^{2c+1},$$

hence $\beta_k \leq c$, and so $\beta_k \leq \gamma_k$.
The other inequality, namely $\beta_k \geq \gamma_k$, may be obtained by observing that from (4.39) and using the inverse of Mellin formula (A.17) one has

$$\frac{\zeta(s)^k}{s} = \int_0^\infty \Delta_k(1/x) x^{s-1} dx = \int_0^\infty \Delta_k(x) x^{-s-1} dx, \tag{4.41}$$

The integral in (4.41) is absolutely and uniformly convergent in any region interior to the strip $\beta_k < \sigma < 1$, since by the Cauchy-Schwarz inequality

$$\int_N^{2N} \mid \Delta_k(x) \mid x^{-\sigma-1} dx \leq \Big(\int_N^{2N} \Delta_k^2(x) dx \Big)^{\frac{1}{2}} \Big(\int_N^{2N} x^{-2\sigma-2} dx \Big)^{\frac{1}{2}} \ll N^{\beta_k - \sigma + \epsilon},$$

and on putting $N = 1, 2, 3, \ldots$, and adding integrals over various intervals $[N, 2N]$ it is seen that the right-hand side of (4.41) is regular for $\beta_k < \sigma < 1$, so that (4.41) therefore holds by analytic continuation in the strip $\beta_k \leq \sigma \leq 1$. Also(by the same argument just given) the right-hand side of (4.40) is finite for $\beta_k < \sigma < 1$, hence so is the left-hand side, and the formula holds in the same strip, giving $\beta_k \geq \gamma_k$, which combined with $\beta_k \leq \gamma_k$ yields finally $\beta_k = \gamma_k$. This proves the Lemma. □

Lemma 4.3.2. *For $k = 2, 3, \ldots$*

$$\beta_k \geq \frac{k-1}{2k}$$

Proof. For $\frac{1}{2} < \sigma < 1$, by Theorem 7.2 from [15],

$$C_\sigma T < \int_{\frac{1}{2}T}^T \mid \zeta(\sigma + it) \mid^2 dt \leq \Big(\int_{\frac{1}{2}T}^T \mid \zeta(\sigma + it) \mid^{2k} dt \Big)^{\frac{1}{k}} \Big(\int_{\frac{1}{2}T}^T dt \Big)^{1-1/k}.$$

Hence,

$$\int_{\frac{1}{2}T}^T \mid \zeta(\sigma + it) \mid^{2k} dt \geq 2^{k-1} C_\sigma^k T.$$

For $0 < \sigma < \frac{1}{2}$, we have,

$$\int_{-\infty}^{\infty} \frac{|\zeta(\sigma+it)|^{2k}}{|\sigma+it|^2} dt \geq \int_{\frac{T}{2}}^{T} \frac{|\zeta(\sigma+it)|^{2k}}{|\sigma+it|^2} dt \gg \frac{1}{T^2} \int_{\frac{T}{2}}^{T} |\zeta(\sigma+it)|^{2k} dt$$
$$\gg T^{k(1-2\sigma)-2} \int_{\frac{T}{2}}^{T} |\zeta(1-\sigma-it)|^{2k} dt \quad \text{by (1.8)}$$
$$\gg T^{k(1-2\sigma)-1}$$

This can be made as large as we please by choice of T if $\sigma < (k-1)/2k$ the last expression above remains unbounded when $T \to \infty$, hence $\gamma_k \geq (k-1)/2k$, and the theorem follows from $\beta_k = \gamma_k$. □

The above Lemma shows that α_k and γ_k can not be smaller than $(k-1)/2k$. A classical conjecture is that $\alpha_k = \beta_k = \frac{(k-1)}{2k}$ holds for all $k \geq 2$. This conjecture is very strong, since $\beta_k = \frac{(k-1)}{2k} (k \geq 2)$ is equivalent of the *Lindelof Hypothesis.*
Lemma 4.3.3. *For each integer $k \geq 2$, a necessary an sufficient condition that $\beta_k = \frac{(k-1)}{2k}$ is that $m(\frac{k+1}{2k}) \geq 2k$, where $m(\sigma)$ is defined by* (3.6)

Proof. Suppose first that $m(\frac{k-1}{2k}) \geq 2k$. Then for $\sigma < \frac{(k-1)}{2k}$ we have by the functional equation (1.14) and (1.37)

$$\int_1^T |\zeta(\sigma+it)|^{2k} dt \ll T^{k(1-2\sigma)} \int_1^T |\zeta(1-\sigma-it)|^{2k} dt \ll T^{k(1-2\sigma)+1+\epsilon}.$$

Therefore for $\frac{(k-1-\epsilon)}{2k} < \sigma < \frac{(k+1+\epsilon)}{2k}$ by convexity of mean values [Lemma 8.3 of [8]] we have

$$\int_1^T |\zeta(\sigma+it)|^{2k} dt \ll T^{1+\epsilon+(\frac{1}{2}+\frac{1}{2k}-\sigma)k}$$

and the exponent of T is less than 2 if $\sigma > \frac{k-1+\epsilon}{2k}$, giving

$$\int_{\frac{T}{2}}^{T} |\zeta(\sigma+it)|^{2k} |\sigma+it|^{-2} dt \ll T^{-\delta} \tag{4.42}$$

for some $\delta = \delta(\epsilon) > 0$. Replacing in (4.42) T by $T2^{-j}$ and summing over $j \geq 1$ it follows that $\gamma_k \leq \frac{(k-1)}{2k}$; therefore by Lemma 4.3.1 we have also $\beta_k \leq \frac{(k-1)}{2k}$, and Lemma 4.3.2 finally gives $\beta_k = \frac{(k-1)}{2k}$.

In the other direction, if $\beta_k = \frac{(k-1)}{2k}$, then by (4.37) we have

$$\int_{\frac{T}{2}}^{T} \mid \zeta(\sigma + it) \mid^{2k} dt \ll T^{2+\epsilon}$$

for $\sigma = \frac{(k-1)}{2k}$. Hence by the functional equation for $\zeta(s)$ and convexity of mean values we obtain $m((k+1)/2k) \geq 2k$ by following the foregoing argument. □

Finally, we prove some explicit estimates for β_k, contained in
Theorem 4.3.4. $\beta_k = \frac{(k-1)}{2k}$ *for* $k = 2, 3, 4,$ *and* $\beta_5 \leq \frac{119}{260} = 0.45769...$, $\beta_6 \leq \frac{1}{2}$, $\beta_7 \leq \frac{39}{70} = 0.55714...$

Proof. By Theorem 3.4.2 we have $m(\sigma) \geq \frac{4}{(3-4\sigma)}$ for $\frac{1}{2} \leq \sigma \leq \frac{5}{8}$, hence $m(\frac{5}{8}) \geq 8$. By Lemma 4.3.3 we have obtained at once that $\beta_k = \frac{(k-1)}{2k}$ for $k = 2, 3, 4$, which in view of Lemma 4.3.2 shows that this is best possible, while for other values of k the estimate $\beta_k = \frac{(k-1)}{2k}$ seems to be beyond reach at present.
Consider now the case $k = 5$. By Lemma 4.3.1 it will suffice to show

$$\int_{T}^{2T} \mid \zeta(\sigma + it) \mid^{10} dt \ll T^{2-\delta}$$

for $\sigma > \frac{119}{260}$ and any fixed $\delta > 0$. From the estimate $m(\frac{41}{60}) \geq 10$, furnished by Theorem 3.4.2, the functional equation (1.14), and (1.37) we have, for $\frac{19}{60} \leq \sigma \leq \frac{1}{2}$,

$$\int_{T}^{2T} \mid \zeta(\sigma + it) \mid^{10} dt \ll T^{\frac{(207-260\delta)}{44}+\epsilon},$$

where we also used convexity and the estimate $M(10) \leq \frac{7}{4}$ from Corollary 3.3.2. Since $\frac{(207-260\sigma)}{44} < 2$ for $\sigma > \frac{119}{260}$, we obtain $\beta_5 \leq \frac{119}{260}$ as asserted. Similarly, from $M(12) \leq 2$ it follows as once that $\beta_6 \leq \frac{1}{2}$, while for β_7 we use $M(14) \leq \frac{62}{27}$ (Corollary 3.3.2) and $m(\frac{3}{4}) \geq 14$ (Theorem 3.4.2). Thus by convexity

$$\int_{T}^{2T} \mid \zeta(\sigma + it) \mid^{14} dt \ll T^{\frac{132-140\sigma}{27}+\epsilon}$$

for $\frac{1}{2} \leq \sigma \leq \frac{3}{4}$, and $\frac{132-140\sigma}{27} \leq 2$ for $\sigma \geq \frac{39}{70}$, proving the last part of the theorem. Other values of β_k for $k \geq 8$ may be calculated analogously, but the present form of estimates for $m(\sigma)$ and $M(A)$ would render a general formula for $\beta_k (k \geq 8)$ too complicated, and for this reason only estimates for small values of k are explicitly stated here. We shall discuss some improvements in the next section. □

4.3.2 Some New Estimates of β_k

The new results announced here are slight improvements of some β_k which are similar to what we have discussed about α_k in section 4.2.3 due the improved bounds of power moment estimates, and the use of the new order estimate of $\zeta(\frac{1}{2}+it)$ obtained by *Jean Bourgain* in Theorem 2.4.6. We can state the result as follows

Theorem 4.3.5. *$\beta_7 < 0.547249$, $\beta_8 < 0.595117$, $\beta_9 < 0.632227$, $\beta_{10} < 0.661883$.*

Proof. To obtain upper bounds for β_k one may note that $\beta_k \leq \sigma_1 = \sigma_1(k)$, if σ_1 satisfies

$$\int_T^{2T} \mid \zeta(\sigma_1 + it) \mid^{2k} dt \ll T^{2-\delta} \tag{4.43}$$

for some $\delta = \delta(k) > 0$.

For $k \geq 7$ one can improve all the existing upper bounds for β_k by using the improved estimates for $m(\sigma)$, which were derived above (3.120) up to (3.132) and the recently obtained estimate provided on Theorem 2.4.6. For k fixed let $c = c(k)$ be such a constant for which $M(2k) \leq 1 + c$, and let $\sigma_0 = \sigma_0(k) \geq \frac{1}{2}$ satisfy $m(\sigma_0) \geq 2k$. Then we can show that

$$\beta_k \leq \frac{(c-1)\sigma_0 + \frac{1}{2}}{c} \tag{4.44}$$

Indeed if

$$F(\sigma) = \frac{2c(\sigma_0 - \sigma) + \sigma_0 - 1}{2\sigma_0 - 1},$$

then $F(\frac{1}{2}) = 1 + c$ and $F(\sigma_0) = 1$. Hence by convexity

$$\int_T^{2T} \mid \zeta(\sigma + it) \mid^{2k} dt \ll T^{F(\sigma)+\varepsilon} \qquad (\frac{1}{2} \leq \sigma \leq \sigma_0), \tag{4.45}$$

and $F(\sigma) < 2$ for $\sigma > \frac{(c\sigma_0 - \sigma_0 + \frac{1}{2})}{c}$, so that (4.44) follows from (4.43).

Following the proof of corollary 3.3.2 and using the new bound $\mu(\frac{1}{2}) \leq \frac{13}{84} = 0.15476190...$ provided by Jean Bourgain, Theorem 2.4.6, we obtain

$$M(2k) \leq 1 + 2\mu(\frac{1}{2})(k-3) \quad (k \geq 7)$$

$$M(2k) \leq 1 + \frac{13}{42}(k-3) \quad (k \geq 7) \tag{4.46}$$

Whence $c = c(k) = \frac{13}{42}(k-3)$ From the proof of the upper bounds for α_k we readily find that

$$\sigma_0(7) = \alpha_{14} \le 0.745666..., \quad \sigma_0(8) = \alpha_{16} \le 0.768809...,$$
$$\sigma_0(9) = \alpha_{18} \le 0.786493..., \quad \sigma_0(10) = \alpha_{20} \le 0.800639...$$

It follows then immediately from (4.43) and (4.44) that

$$\beta_7 \le \frac{(c(7)-1)(\sigma_0(7)+\frac{1}{2})}{c(7)} = \frac{(\frac{52}{42}-1)\sigma_0(7)+\frac{1}{2}}{\frac{52}{42}} = 0.54724346... \tag{4.47}$$

$$\beta_8 \le \frac{(c(8)-1)(\sigma_0(8)+\frac{1}{2})}{c(8)} = \frac{(\frac{65}{42}-1)\sigma_0(8)+\frac{1}{2}}{\frac{65}{42}} = 0.59511703... \tag{4.48}$$

$$\beta_9 \le \frac{(c(9)-1)(\sigma_0(9)+\frac{1}{2})}{c(9)} = \frac{(\frac{78}{42}-1)\sigma_0(9)+\frac{1}{2}}{\frac{78}{42}} = 0.63222740... \tag{4.49}$$

$$\beta_{10} \le \frac{(c(10)-1)(\sigma_0(10)+\frac{1}{2})}{c(10)} = \frac{(\frac{91}{42}-1)\sigma_0(10)+\frac{1}{2}}{\frac{91}{42}} = 0.66188289... \tag{4.50}$$

and upper bounds for β_k when $k \ge 11$ may be calculated analogously.

□

The new results mentioned in section 4.2.3 and section 4.3.2 shows that a suitable choice of exponent is indeed useful in estimating the order of error terms on the Dirichlet divisor problem the zeta-function theory.

4.4 Estimates of α_k and β_k when k is large

The estimates of α_k and β_k when k is large do not depend on the power moment estimates for $\zeta(s)$, but only on the order estimate for $\zeta(s)$ provided in [8], the last bound given for α_k in the same book is

$$\alpha_k \le 1 - \frac{1}{2}(Dk)^{-2/3}, \tag{4.51}$$

where $D > 0$ is such a constant for which

$$\zeta(\sigma+it) \ll t^{D(1-\sigma)^{\frac{3}{2}}}(\log t)^{\frac{2}{3}} \qquad (0 < D \le 100, t \ge t_0, \frac{1}{2} \le \sigma \le 1) \tag{4.52}$$

Theorem 4.4.1. *For k very large and if* (4.52) *holds, then from [9] we have*

$$\alpha_k \leq 1 - \frac{1}{3} \cdot 2^{\frac{2}{3}} (Dk)^{\frac{-2}{3}}. \tag{4.53}$$

Proof. We shall start from the standard Perron's inversion formula (1.31) applied to

$$A(s) = \zeta^k(s) = \sum_{n=1}^{\infty} d_k(n) n^{-s} \quad \text{for} \quad \sigma = \Re(s) > 1.$$

we have, for $X^{\varepsilon} \ll T \ll X^{1-\varepsilon}$, $\frac{1}{2}X \leq x \leq X$, $b = 1 + \varepsilon$,

$$\sum_{n \leq x} d_k(n) = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} \zeta^k(s) x^s s^{-1} ds + O(X^{1+\varepsilon} T^{-1}).$$

Now we replace the segment of integration in the above formula by the segment $[\sigma - iT, \sigma + iT]$, where $1/2 < \sigma < 1$ will be suitably chosen later. We pass over the pole $s = 1$ of the integrand, which gives rise to the main term in (4.7). Writing $G = XT^{-1}$ it follows that

$$\Delta_k(x) = \frac{1}{2\pi} \int_{\frac{-X}{G}}^{\frac{X}{G}} \zeta^k(\sigma + it) \frac{x^{\sigma + it}}{\sigma + it} dt + O(GX^{\varepsilon} + G \int_{\sigma}^{1+\varepsilon} \mid \zeta(\alpha + iXG^{-1}) \mid^k x^{\alpha - 1} d\alpha). \tag{4.54}$$

Suppose now that G satisfies, besides $X^{\varepsilon} \ll G \ll X^{1-\varepsilon}$, the additional condition

$$\int_{\sigma}^{1+\varepsilon} \mid \zeta(\alpha + iXG^{-1}) \mid^k x^{\alpha - 1} d\alpha \ll X^{\varepsilon}. \tag{4.55}$$

We use then (4.54) to obtain from (4.55)

$$\Delta_k(x) \ll X^{\varepsilon}(G + X^{\sigma} \int_1^{\frac{X}{G}} t^{kD(1-\sigma)^{\frac{3}{2}} - 1}) dt \ll X^{\varepsilon}(G + X^{\sigma}(X/G)^{kD(1-\sigma)^{\frac{3}{2}}}) \tag{4.56}$$

We choose G so that the last two terms in (4.56) are equal. Thus $G = X^{1-f(\sigma)}$, where

$$f(\sigma) = (1 - \sigma)/(1 + kD(1 - \sigma)^{3/2}),$$

hence $f'(\sigma) = 0$ for $\sigma = \sigma_0 = 1 - 2^{2/3}(Dk)^{-2/3}$. We have

$$1 - f(\sigma_0) = 1 - \frac{1}{3} 2^{2/3} (Dk)^{-2/3},$$

hence (4.53) follows with $\sigma = \sigma_0$ in (4.54), provided that (4.55) holds. To see this note that $\zeta(\sigma + it) \ll \log^{2/3} \mid t \mid$ uniformly for $\sigma \geq 1$, and it follows from (4.52) that

$$\max_{\sigma_0 \leq \alpha \leq 1} \mid \zeta(\alpha + iXG^{-1}) \mid^k X^{\alpha - 1} \ll \max_{\sigma_0 \leq \alpha \leq 1} \{(X/G)^{kD(1-\alpha)^{3/2}} X^{\alpha-1}\} \log^k x \ll X^{\varepsilon}.$$

This is because

$$\max_{\sigma_0 \leq \alpha \leq 1} \exp\{(\frac{1}{3} \cdot 2^{2/3}(Dk)^{1/3}(1-\alpha)^{3/2} + \alpha - 1) \log X\} \leq 1,$$

since

$$\frac{1}{3} \cdot 2^{2/3}(Dk)^{1/3}(1-\alpha)^{3/2} + \alpha - 1 \leq 0$$

reduces to

$$1 \geq \alpha \geq 1 - 9 \cdot 2^{-4/3}(Dk)^{-2/3},$$

and we have

$$\alpha \geq \sigma_0 = 1 - 2^{2/3} \cdot (Dk)^{-2/3} > 1 - 9 \cdot 2^{-4/3}(Dk)^{-2/3}.$$

This proves (4.53). □

Theorem 4.4.2. *For k very large and if (4.54) holds, then from [9] we have*

$$\beta_k \leq 1 - \frac{2}{3}(Dk)^{\frac{-2}{3}} \tag{4.57}$$

Proof. The bound for β_k given by (4.57) will follow from

$$I = \int_{X/2}^{X} {\Delta_k}^2(x)dx \ll X^{1+2\eta+\varepsilon}, \quad \eta = 1 - \frac{2}{3}(Dk)^{-2/3}, \tag{4.58}$$

on replacing X by $X2^{-j}$ and summing over $j = 0, 1, 2, \ldots$ We use (4.54), supposing again that (4.55) holds. This gives

$$I \ll X^{1+\varepsilon}G^2 + \int_{X/2}^{X} \Big| \int_{-X/G}^{X/G} \zeta^k(\sigma + it) \frac{x^{\sigma+it}}{\sigma + itdt} \Big|^2 dx$$

$$= X^{1+\varepsilon}G^2 + \int_{-X/G}^{X/G} \int_{-X/G}^{X/G} \frac{\zeta^k(\sigma+it)\zeta^k(\sigma-iu)}{(\sigma+it)(\sigma-iu)} \Big(\int_{X/2}^{X} x^{2\sigma+it-iu} dx \Big) dtdu.$$

Using $\mid ab \mid \leq \frac{1}{2}(\mid a \mid^2 + \mid a \mid^2)$, it further follows that

$$I \ll X^{1+\varepsilon}G^2 + X^{1+2\sigma} \int_{-X/G}^{X/G} \mid \zeta(\sigma+it) \mid^{2k} \frac{1}{\sigma^2 + t^2} \Big(\int_{-X/G}^{X/G} \frac{du}{1 + \mid t - u \mid} \Big) dt \tag{4.59}$$

$$\ll X^{1+\varepsilon}G^2 + X^{1+2\sigma}\log X(1+\int_2^{X/G} |\,\zeta(\sigma+it)\,|^{2k}t^{-2}dt)$$
$$\ll X^{1+\varepsilon}G^2 + X^{1+2\sigma+\varepsilon}(1+\int_2^{X/G} t^{2Dk(1-\sigma)^{3/2}-2}dt)$$
$$\ll X^{\varepsilon}(XG^2 + X^{1+2\sigma} + X^{2\sigma+2Dk(1-\sigma)^{3/2}-2}G^{1-2Dk(1-\sigma)^{3/2}}),$$

provided that

$$2Dk(1-\sigma)^{3/2} > 1. \tag{4.60}$$

This time we choose G to make the first and the third term in the above estimate equal. We obtain

$$G = X^{1-g(\sigma)}, \qquad g(\sigma) = 2(1-\sigma)/(1+2Dk(1-\sigma)^{3/2}),$$

so that $g'(\sigma) = 0$ for $\sigma = \sigma_1 = 1-(Dk)^{-2/3}$, and (4.60) holds. Hence we choose $G = X^{1-g(\sigma_1)}$, where $1-g(\sigma_1) = \eta$, as given by (4.58). Since $\sigma_1 < 1-g(\sigma_1)$, (4.57) follows from (4.59), provided that (4.55) holds. This will in turn follow from

$$\max_{\sigma_1\le\alpha\le1} |\,(XG^{-1})^{Dk(1-\alpha)^{3/2}}X^{\alpha-1}\,| = \max_{\sigma_1\le\alpha\le1} \exp|\,(\frac{2}{3}(Dk)^{1/3}(1-\alpha)^{3/2}+\alpha-1)\log X\,| \le 1.$$

The inequality

$$\frac{2}{3}(Dk)^{1/3}(1-\alpha)^{3/2}+\alpha-1 \le 0.$$

reduces to $1 \ge \alpha \ge 1-\frac{9}{4}(Dk)^{-2/3}$, and we have

$$1 \ge \alpha \ge \sigma_1 = 1-(Dk)^{-2/3} > 1-\frac{9}{4}(Dk)^{-2/3},$$

so that (4.57) is proved. □

Appendix A

Additional results

A.1 Laurent Series Expansion of $\zeta(s)$

We have a useful representation of $\zeta(s)$ for $s = \sigma + it$ and $\Re(s) = \sigma > 1$,

$$1 - 2^{-s} + 3^{-3} - 4^{-s} + 5^{-s} - ... = \zeta(s) - 2\sum_{n=1}^{\infty}(2n)^{-s} = (1 - 2^{1-s})\zeta(s), \quad (A.1)$$

so that

$$\zeta(s) = (1 - 2^{1-s})^{-1}\sum_{n=1}^{\infty}(-1)^{n-1}n^{-s} \quad (\sigma > 0), \quad (A.2)$$

since the alternating series in (A.1) converges for $\sigma > 0$. Thus (A.2) also provides the analytic continuation of $\zeta(s)$ for $\sigma > 0$, and shows in particular that $\sigma < 0$ for $0 < \sigma < 1$. The Laurent series expansion of $\zeta(s)$ about the pole $s = 1$ is

$$\zeta(s) = \frac{1}{s-1} + \gamma_0 + \sum_{n=1}^{\infty}\gamma_k(s-1)^k \quad (A.3)$$

with

$$\gamma_k = \frac{(-1)^k}{k!}\lim_{N\to\infty}(\sum_{m\leq N}\frac{1}{m}\log^k m - \frac{\log^{k+1}N}{k+1}),$$

where γ_0 is Euler's constant, for $k \geq 1$, γ_k is sometimes called generalized Euler constants, and $\log^0 m$ mean 1 for all m including $m = 1$. particularly,

$$\gamma_0 = \lim_{N\to\infty}(1 + \frac{1}{2} + ... + \frac{1}{N} - \log N) = 0.5772157... \quad (A.4)$$

A.2 Partial Summation Formula

Classically one calls Abel's partial summation the process whereby one transforms a fine sum of products of two terms by means of the partial sums of one of them. Thus, by letting A_0, $\{a_n\}_{n=1}^{\infty}$ a sequence of complex numbers and $\{a_n\}_{n=1}^{\infty}$ a sequence of real numbers, we have $A_n = \sum_{m=1}^{n} a_m$ $(n \geq 1)$,

$$\sum_{n=1}^{N} a_n b_n = \sum_{n=1}^{N}(A_n - A_{n-1})b_n = \sum_{n=1}^{N} A_n b_n - \sum_{n=1}^{N-1} A_n b_{n+1} = \sum_{n=1}^{N-1} A_n(b_n - b_{n+1}) + A_N b_N. \tag{A.5}$$

In the setting of the Stieltjes integral, Abel summation takes the innocuous form of partial integration, and therefore it is referred to as *Partial Summation Formula.* Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of complex numbers. Set $A(t) = \sum_{n \leq t} a_n \quad (t \leq 0)$ and let $\phi(t)$ be a continuously differentiable function on the interval $[1, x]$. Then we have

$$\sum_{1 \leq n \leq x} a_n \phi(n) = A(x)\phi(x) - \int_1^x A(t)\phi'(t)dt. \tag{A.6}$$

More generally, we have

$$\sum_{x \leq n \leq y} a_n \phi(n) = A(y)\phi(y) - A(x)\phi(x) - \int_x^y A(t)\phi'(t)dt. \tag{A.7}$$

A.3 Poisson Summation Formula

Poisson summation formula is an equation that relates the Fourier series coefficients of the periodic summation of a function to values of the function's continuous Fourier transform. There exist several variants of this useful formula. We shall state the following version: let a, b be integers and let $f(x)$ be a function of real variable x with bounded first derivatives on $[a, b]$. Then

$$\sum\nolimits'_{a \leq n \leq b} f(x) = \int_a^b f(x)dx + 2\sum_{n=1}^{\infty} \int_a^b f(x)\cos(2n\pi x)dx. \tag{A.8}$$

Here as usual $\sum'$ means that $\frac{1}{2}f(a)$ and $\frac{1}{2}f(b)$ are to be taken instead of $f(a)$ and $f(b)$, respectively.

A.4 The Euler–Maclaurin Formula

Let us take a continuous arithmetic functions $f : [a, b] \to \mathbb{C}$, which has relatively slow variation, then one should expect that the sum $\sum_{a<n\leq b} f(n)$ is well approximated by the corresponding integral. We have the following exact formula

$$\sum_{a<n\leq b} f(n) = \int_a^b (f(x) + \psi(x)f'(x))dx + \frac{1}{2}(f(b) - f(a)) \tag{A.9}$$

provided $a < b$, $a, b \in \mathbb{Z}$, and f is of class $C^1[a, b]$. Here $\psi(x)$ is the Saw function

$$\psi(x) = x - [x] - \frac{1}{2}$$

This classical formula of Euler-Maclaurin follows easily by partial integration.
Let $g(x)$ and $h(x)$ be real-valued smooth functions on $[a, b]$ with $g''(x) \neq 0$ and $g'(x)$ satisfying $| g'(x) |\leq \theta$, $0 < \theta < 1$. Then

$$\sum_{a<n\leq b} h(n)e(g(n)) = \int_a^b h(x)e(g(x))dx + O(H(1 - \theta)^{-1})$$

where the implied constant is absolute.

A.5 The Gamma-Function

Here are a few basic properties of the gamma function and the proofs are readily found in books on analysis. For $s = \sigma + it$ and $\Re(s) > 0$ the gamma-function is defined as

$$\Gamma(s) = \int_0^\infty e^{-x}x^{s-1}dx. \qquad \text{if} \quad \sigma > 0, \tag{A.10}$$

$$\Gamma(s) = \int_1^\infty e^{-y}y^{s-1}dy + \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(s+m)}, \quad \text{for all s.} \tag{A.11}$$

This shows that $\Gamma(s)$ is meromorphic in the whole plane, except for the points $s = 0, -1, -2, ..., -n, ...$, which are poles of the first order with residues $\frac{(-1)^n}{n!}$ $(n = 0, 1, 2, ...)$. The gamma-function satisfies the functional equation

$$\Gamma(s + 1) = s\Gamma(s), \tag{A.12}$$

and the functional equation and duplication formula, respectively,

$$\Gamma(s)\Gamma(1 - s) = \pi / \sin \pi s, \qquad \Gamma(s)\Gamma(s + \frac{1}{2}) = s\sqrt{\pi}2^{-2s}\Gamma(2s). \tag{A.13}$$

A.6 Exponential Sum Estimates

Let us consider exponential sums of the form $\sum e^{2\pi i f(n)}$ where $f(n)$ is a real function. If the numbers $f(n)$ are the values taken by a function $f(x)$ of a simple kind, we can approximate to such a sum by an integral, or a sum of integrals.
Let $f(x)$ be a real function with derivatives up to the third order. Let $f'(x)$ be steadily decreasing in $a \leq x \leq b$, and $f'(b) = \alpha$, $f'(a) = \beta$. Let x_v be defined by $f'(x_v) = v$ $(\alpha < v \leq \beta)$. Let $\lambda_2 \leq\mid f''(x) \mid< A\lambda_2$, $\mid f'''(x) \mid< A\lambda_3$. Then

$$\begin{aligned}\sum_{a<n\leq b} e^{2\pi i f(n)} = e^{-\frac{1}{4}\pi i} \sum_{\alpha<v\leq\beta} \frac{e^{2\pi i\{f(x_v)-vx_v\}}}{\mid f''(x_v) \mid^{\frac{1}{2}}} + O({\lambda_2}^{-\frac{1}{2}})+ \\ O[\log\{2+(b-a)\lambda_2\}] + O\{(b-a){\lambda_2}^{\frac{1}{5}}{\lambda_3}^{\frac{1}{5}}\}.\end{aligned} \tag{A.14}$$

Let $f(x)$ be a real function with a continuous and steadily decreasing derivative $f'(x)$ in (a,b), and $f'(b) = \alpha$, $f'(a) = \beta$ and let $g(x)$ be a real positive decreasing function, with a continuous derivative $g'(x)$, and let $\mid g'(x) \mid$ be steadily decreasing. Then

$$\begin{aligned}\sum_{a<n\leq b} g(n)e^{2\pi i f(n)} = \sum_{\alpha-\eta<v<\beta+\eta} \int_a^b g(x)e^{2\pi i\{f(x)-vx\}}dx \\ +O\{g(a)\log(\beta-\alpha-2)\} + O\{\mid g'(a) \mid\}.\end{aligned} \tag{A.15}$$

A.7 The Mellin Transform

Let $f(x)x^{\sigma-1}$ belong to $L(0,\infty)$ and let $f(x)$ have bounded variation on every finite x-interval. Then the *Mellin Transform* of a function $f(x)$ is defined as

$$\mathcal{M}\{f(s)\} = F(s) = \int_0^\infty x^{s-1}f(x)dx, \qquad s = \sigma + it \quad (\sigma, t \quad \text{real}). \tag{A.16}$$

For (A.16) we can recover $f(x)$ in terms of $F(s)$ by *Mellin's Inversion Formula*

$$\mathcal{M}^{-1}\{F(x)\} = f(x) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\sigma-iT}^{\sigma+iT} F(s)x^{-s}ds. \tag{A.17}$$

A.8 Parseval's Identity

An equality expressing the square of the norm of an element in a vector space with a scalar product in terms of the square of the moduli of the Fourier coefficients of this element in some orthogonal system. Thus, if X is a normed separable vector space with a scalar product $(,)$, if $\|\cdot\|$ is the corresponding norm and if $\{e_n\}$ is an orthogonal system in X, $e_n \neq 0, n = 1, 2, ...$, then Parseval's equality for an element $x \in X$ is

$$\|x\|^2 = \sum_{v \in B} | a_n |^2 \|e_n\|^2, \tag{A.18}$$

where $a_n = \frac{(x,e_n)}{(e_n,e_n)}$ are the Fourier coefficients of x in the system $\{e_n\}$.
Parseval's identity is a fundamental result on the summability of the Fourier series of a function. Informally, the identity asserts that the sum of the squares of the Fourier coefficients of a function is equal to the integral of the square of the function,

$$\sum_{-\infty}^{\infty} | c_n |^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} | f(x) |^2 dx, \tag{A.19}$$

where the Fourier coefficients c_n of f are given by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

More formally, the result holds as stated provided f is square-integrable.

A.9 Halasz-Montgomery Inequalities

Suppose that $\xi, \varphi_1, \varphi_2 ..., \varphi_R$ are arbitrary vectors in an inner product space over the field of complex numbers, where (a, b) will be the notation for the inner product and $\| a \| = (a, a)$. Then

$$\sum_{r \le R} | (\xi, \varphi_r) |^2 \le \|\xi\|^2 \max_{r \le R} \sum_{s \le R} | (\varphi_r, \varphi_s) |, \tag{A.20}$$

$$\sum_{r \le R} | (\xi, \varphi_r) | \le \|\xi\| \Big(\sum_{r,s \le R} | (\varphi_r, \varphi_s) | \Big)^{1/2}. \tag{A.21}$$

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