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Mathias Nikolai Arnesen

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On Nonlocal Dispersive Equations and Water Waves

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Preface

This thesis is submitted for the degree of Philosophiae Doctor (PhD) at the Norwegian University of Science and Technology (NTNU) in Trondheim, Norway.

First and foremost I want to thank my advisor Mats Ehrnström. He was the one to encourage me to do a PhD in the first place and he has been a great advisor throughout my PhD both in regards to the mathematical research itself and academic life in general, and although in my stubbornness I took less advantage of it than I should, I always felt welcome to discuss with him and for all this I am very grateful. I am also grateful to my co-advisor Boris Buffoni, in particular for his hospitality and patience in working with me during my stay at the department of mathematics at EPFL in Lausanne.

Mathias Nikolai Arnesen
Trondheim, November 2017

Introduction

This thesis includes a collection of four articles on topics within the mathematical theory of water waves, of which one is published, one is submitted to a journal and two are in preparation for publication. Each article deals with different equations and topics within the field, but as a common point they all concern two-dimensional flows in an inviscid fluid (which one may well assume water is) or equations related to the modelling of waves in such a setting. The governing equations of the water-wave problem are the Euler equations, but as they are very complex one is impelled to also consider simpler equations that approximate the water-wave problem in various settings, which is the context of the first three articles. As equations that suppose to model water waves, much of the research in the field naturally concerns existence and (in)stability of various types of physically relevant waves, such as travelling waves, both periodic and solitary, and peaked or cusped waves. In this thesis we will consider solitary waves and their stability; peaked/cusped waves; and issues related to well-posedness for various model equations, as well as existence and stability of solitary waves for a variant of the Euler equations with non-constant vorticity.

The model equations under consideration are nonlinear dispersive equations of the form

$$u_t + (f(u))_x - (Lu)_x = 0, \quad t \in \mathbb{R} \text{ and } x \in \mathbb{R}, \tag{0.1}$$

where u and f takes real values, f is a nonlinear function, typically a power function, and L is a pseudo-differential operator. As the water wave problem is both nonlinear and dispersive this is a natural form for a model equation to take and, while not exhaustive, many of the most well known model equations are of this form. For instance, to mention a few, the KdV and BO [3] equations, and, rising to greater prominence as a model for shallow water in recent years, the Whitham equation [28]. Allowing for nonlinear dispersive terms, the Camassa-Holm equation [7] and Degasperis-Procesi equation [10] can also be cast in this form. For a survey of the mathematical theory of water waves and how the various model equations are derived from the water-wave problem, we refer to e.g. [21] and references therein.

Generally L will be a Fourier multiplier operator with symbol $m(\xi)$, that is,

$$\widehat{Lf}(\xi) = m(\xi)\widehat{f}(\xi),$$

where $\widehat{f}$ denotes the Fourier transform of f. If m is a polynomial, then by basic properties of the Fourier transform L will be a classical differentiation operator written fancily. In general, however, L is a nonlocal operator. Of particular interest for this thesis are nonlocal equations with weak dispersion; that is,

$$|m(\xi)| \lesssim (1 + |\xi|)^s, \quad \xi \in \mathbb{R},$$

for $s \in \mathbb{R}$ small, say $s < 1$, and potentially even negative, in which case the operator is smoothing rather than differentiating. The Whitham equation [28] was introduced half a century ago and nonlocal, weakly dispersive equations are as such nothing new, but they have become the object of more attention in recent years. This is in no small part due to new results on the Whitham equation such as wave-breaking [18] and the existence of solitary waves [11] and cusped waves [13]. Weaker dispersion potentially allows for the equations to capture concepts such as wave-breaking and cusped/peaked waves which equations with strong dispersion, such as the KdV equation, do not feature.

The study of nonlocal equations is also of intrinsic mathematical interest, as the nonlocal nature of the problem requires different methods and approaches than the more classical differential equations. Indeed, as physically tangible as water waves may be, this thesis is first and foremost the work of a theoretical mathematician and it is from this perspective the concepts are studied. Hence, in the papers concerning model equations, the focus is on understanding how the mathematical structure of equations of the form (0.1) gives rise to the properties considered, rather than seeking to validate any particular one equation as a model for water waves.

Below we give an overview of the results and methods used in each paper, as well as discussing how it relates to previous results on the topics considered.

Paper I: Existence of solitary-wave solutions to nonlocal equations.

This paper considers existence and stability of solitary-wave solutions to a class of equations of the form (0.1) and also equations of the form

$$u_t + (f(u))_x + (Lu)_t = 0. \tag{0.2}$$

The nonlinearity f is assumed to be homogeneous, either $f(u) = cu|u|^{p-1}$ or $f(u) = c|u|^p$ where c is a non-zero constant and $p > 1$. For equations of the form (0.2), nonlinearities $g(u) = u + f(u)$ are also considered; for (0.1) the additional term has only a trivial impact. The symbol m is assumed to satisfy $A_1|\xi|^s \leq m(\xi) \leq A_2|\xi|^s$ when $|\xi| \geq 1$ for some constants A_1, A_2 and some $s \in \mathbb{R}$. The main result establishes under very mild regularity assumptions on $m(\xi)$ that for any $s > 0$, there exists non-empty sets of solitary-wave solutions to both (0.1) and (0.2) for all $p \in (1, \frac{1+s}{1-s})$ and all positive wave-speeds. When $s \geq 1$, the upper range for p is interpreted as ∞. For (0.2) there are sets of solutions that are conditionally energetically stable in $H^{s/2}(\mathbb{R})$ for all $p \in (1, \frac{1+s}{1-s})$, whilst for (0.1) stable sets of solutions are found only for $p \in (1, 2s+1)$. Stability is known to fail at $p = 2s + 1$ for some equations of the form (0.1) covered by our assumptions (for instance the generalized KdV equation [2]) and existence is known to be impossible when $p > \frac{1+s}{1-s}$ for both equations (cf. [23]), hence the ranges of p considered are optimal.

The proof follows a general variational approach to finding existence of solitary waves that has been successfully employed for some equations of the forms (0.1) and (0.2) before. The idea is to relate solitary-wave solutions to minimizers of a constrained variational problem in terms of preserved quantities for (0.1) and (0.2) and use the concentration-compactness principle of Lions [24] to establish the existence of minimizers. Letting F be the primitive of f with no constant term, minimizers of

$$\inf\{\frac{1}{2}\int_{\mathbb{R}} uLu \,\mathrm{d}x - \int_{\mathbb{R}} F(u)\,\mathrm{d}x : u \in H^{s/2}(\mathbb{R}),\ \frac{1}{2}\int_{\mathbb{R}} u^2\,\mathrm{d}x = q\} \tag{0.3}$$

and

$$\inf\{\frac{1}{2}\int_{\mathbb{R}} uLu + u^2\,\mathrm{d}x : u \in H^{s/2}(\mathbb{R}),\ \int_{\mathbb{R}} F(u)\,\mathrm{d}x = \lambda\} \tag{0.4}$$

for $q > 0$ and λ fixed and non-zero are seen to be solitary-wave solutions to (0.1) and (0.2) respectively. The wave-speed is given by the Lagrange multiplier in the first case and its reciprocal in the second. The advantage of treating both equation (0.1) and (0.2) together is that minimizers of (0.4) can be scaled into solitary-wave solutions of (0.1) and by multiplying the term u^2 in (0.4) by a positive constant, which clearly does not influence the existence of minimizers, one can obtain solitary-wave solutions of (0.1) of all positive wave-speeds by varying this constant and scaling.

In order to establish the existence of minimizers the general approach of [1] and [29], dealing with (0.1) and (0.2), respectively, is followed. Their results cover the case $s \geq 1$ under fairly general assumptions on $m(\xi)$, but rely on arguments involving abstract commutator results for the estimates needed to apply the concentration-compactness principle, and there is no straightforward extension of these arguments to the case $0 < s < 1$. For $0 < s < 1$ the only result prior to this paper was [16], dealing only with (0.1) and with L being the fractional Laplace operator. In paper I the required estimates are calculated working directly with the nonlocal operator, allowing more general symbols $m(\xi)$ to be considered, in particular inhomogeneous symbols also for $0 < s < 1$, covering for instance the Capillary Whitham equation.

Paper II: Non-uniform dependence on initial data for equations of Whitham type.

This paper considers the initial value problem for equations of the form (0.1) on the real line and on the torus with $f(u) = \frac{1}{2}u^2$; the nonlinearity of the Euler equations, and investigates the regularity of the data to solution map, or flow map, and how the regularity depends on the strength of the dispersion. The symbol $m(\xi)$ is assumed to be even, locally bounded and of at most polynomial growth in the limit, i.e., $|m(\xi)| \lesssim |\xi|^p$ for some $p \geq 0$ for $|\xi| \ggg 1$. From [12] (0.1) is known to be well-posed in H^s for $s > \frac{3}{2}$ on $\mathbb{R}$ and $\mathbb{T}$ under these assumptions with continuous flow map and it therefore makes sense to speak of the regularity of the flow map. The main result is that the flow map is not uniformly continuous on $H^s(\mathbb{R})$ for $s > \frac{3}{2}$ if the strength of the dispersion is less than that of the KdV, in the sense that $p < 2$ in the bound on $|m(\xi)|$ above. As the flow map of the KdV equation is known to be uniformly continuous on these spaces, and in fact locally Lipschitz for $s > -\frac{3}{4}$ (cf. [19]), this upper bound on the strength of the dispersion is optimal. On the torus the flow map is shown to not be uniformly continuous in H^s for any $s > 0$ (provided that the flow map exists - which is not known in general when $s \leq \frac{3}{2}$) regardless of the strength of the dispersion.

The fundamental idea of the proof is based on the approach used in [20] for proving non-uniform continuity for the Benjamin-Ono equation. This approach has also been used to prove equivalent results, for instance for the Camassa-Holm equation ([17]) and the fractional KdV equation ([27]). The idea is to construct two sequences of solutions such that their difference at time zero vanishes, while being uniformly bounded below for some later time t. This is done by considering high-frequency waves and adding a low-frequency perturbation and seeing how the solution evolves in time. As explicit solutions are greatly lacking in general for equations of the form (0.1), approximate solutions displaying the desired behaviour are constructed and the challenge is to prove that these are sufficiently close to real solutions in the limit. In the periodic case, the arguments used for fKdV in [26] are straightforwardly extended to general symbols m as the approximate solutions can be constructed by sin and cos, which are in $H^s(\mathbb{T})$, and the action of L on these functions can be quite readily calculated for very general symbols m. On the real line, however, previous results have only considered homogeneous symbols m (except for the CH equation [17], but this has very specific properties of its own) and used scaling arguments to prove that the sequence of approximate solutions converge to actual solutions, but when m is inhomogeneous, as it is for instance for the Whitham equation, the equation (0.1) has no scaling properties. To remedy the situation, low-frequency solutions are shown to satisfy an "approximate" scaling result in the sense that there is a long-wave scaling of such solutions that are close enough to being solutions to get the required estimates.

Paper III: A nonlocal approach to waves of maximal height for the Degasperis-Procesi equation.

In this paper we consider travelling waves of maximal height for the Degasperis-Procesi equation

$$u_t - u_{xxt} + 4uu_x - 3u_x u_{xx} - u_{xxx} = 0. \tag{0.5}$$

Although the DP equation is, obviously, a local equation, it can also be written in nonlocal form as

$$u_t + uu_x + \left(L(\frac{3}{2}u^2)\right)_x = 0, \tag{0.6}$$

where $L = (1-\partial_x^2)^{-1}$, which on the Fourier side has symbol $m(\xi) = (1+\xi^2)^{-1}$. Working with the nonlocal formulation allows the basic ideas of the approach used to show the existence of cusped waves for the Whitham equation in [13] to be employed. The purpose of the paper is twofold: To provide novel information about travelling waves of maximal height for the Degasperis-Procesi equation, and to give some indication of the validity of the approach used as a general method for investigating travelling waves of maximal height for nonlocal equations of the form (0.6) in general. All results are therefore proved in the general framework of this approach without reference to previous works, even though several of the properties of the DP equation that are established in order to prove the main results are already explicitly known or easily deduced from past work based on the local formulation (see, for instance, [22]).

Letting $K(x) = \mathscr{F}^{-1}(m)(x)$, the inverse Fourier transform of m, the action of L on a function can be expressed as a convolution with the kernel K. Firstly, using the properties of this kernel and the structure of the equation, it is shown that at any point where the height of a non-constant L^∞ travelling-wave solution is equal to its wave-speed, the solution has a peak. That is, the wave is Lipschitz continuous at that point, but not C^1; at all points where the wave-height is below the wave-speed it is smooth.

Secondly it is established that there are non-constant travelling waves for which the maximal height is achieved, and which are thus peakons. Using global bifurcation, these are found as the limiting case along the main bifurcation branch for P-periodic solutions for all sufficiently small periods. The bifurcation curve consists of pairs $(\varphi(s), \mu(s))$ of P-periodic solutions φ with wave-speed μ and in order to conclude that $\varphi(s)$ approaches a peaked solution in the limit, it is necessary to preclude, among other things, that $\lim_{s\to\infty} \mu(s) = \infty$. This is done by showing that for sufficiently small P there is an upper bound on μ above which there only exists constant solutions. Whether this is true for all $P > 0$ is not known. In [22] it is proved that for all $\mu > 0$ there exists smooth, periodic solutions for the DP equation, but that result doesn't say what the period is and how it depends on μ.

Explicit soliton peakons for the DP equation are known [10] as is the existence of periodic peaked solutions [22], but the latter are found by studying the local formulation and for what periods is not explicitly known. Moreover, the existence of cusped solutions to the local equation (0.5) has been claimed by several authors (e.g. [22], [30]), which at first glance seem to be a direct contradiction to our result that all waves of maximal height are peaked. However, as discussed in detail in the introduction of the paper, the cuspons are not weak solutions to (0.5): they solve the equation pointwise everywhere except at the cusps, but treating it as a distributional solution and applying the left-hand side to a test function one finds that they are in fact weak solutions to (0.5) not with zero right-hand side, but with point mass distributions at the cusps. This is overlooked in the papers dealing with cuspons as they require the travelling waves to solve the (local) equation only on the open intervals between the points of maximal height.

Paper IV: On conditional energetic stability of gravity-capillary solitary water waves with non-constant vorticity function.

Note that this is a work in preparation. Here we consider solitary waves with vorticity of the Euler equations on finite depth with surface tension. That is, for $\eta \in H^2(\mathbb{R})$ such that $\inf \eta > -1$ we consider, for domains $\Omega \subset \mathbb{R} \times \mathbb{R}_+$ with lower and upper boundary given by $\mathbb{R} \times \{0\}$ and $\{(x, 1+\eta(x)) : x \in \mathbb{R}\}$, respectively, and $\psi \in H^2_{loc}(\Omega)$ such that $\psi(\cdot, 0) = 0$, the equations

$$\begin{aligned} -\alpha\psi_y(x, 1+\eta(x)) + \frac{1}{2}|\nabla\psi(x, 1+\eta(x))|^2 - F_\alpha(-\Delta\psi(x, 1+\eta(x))) \\ + g\eta(x) + (-\psi_0(1) + \alpha)\Delta\psi(x, 1+\eta(x)) - T\sigma = 0, \end{aligned} \tag{0.7}$$

$$\psi - \alpha y = f_\alpha(-\Delta\psi) \quad \text{on } \Omega, \tag{0.8}$$

and

$$\psi(x, 1+\eta(x)) - \alpha(1+\eta(x)) = \psi_0(1) - \alpha \quad \text{on } \mathbb{R}, \tag{0.9}$$

where $T > 0$ is the surface tension, σ is the curvature, g is gravity, F_α is a primitive of f_α, and ψ_0 is a prescribed parallel flow satisfying certain assumptions on the sign of the derivatives; any function satisfying those assumptions is admissible. The functions f_α and F_α are chosen such that (Ω_0, ψ_0) solves the above equations, where $\Omega_0 = \mathbb{R} \times (0, 1)$. If f_α is invertible the pair (Ω, ψ) defines a solitary wave with vorticity $\zeta = -\Delta\psi$; the parameter α can be interpreted as the speed of the travelling wave. These equations arise from a functional $\mathcal{J}_\alpha(\eta, \psi)$ consisting of two parts: $\mathcal{J}_\alpha = \mathcal{J}_0 - \alpha\mathcal{I}$, where $\mathcal{J}_0$ is the energy and $\mathcal{I}$ is a combination of preserved quantities called "generalized horizontal impulse". This variational formulation is inspired by the variational formulation developed for periodic waves in [8], [9] and [6] (see also [5]). As we are here working on an unbounded domain, we subtract from the functionals the energy and preserved quantities of the flat solution so they will take finite values "near" (Ω_0, ψ_0) - this is one of the purposes of first fixing a flat solution.

For $\alpha < 0$ small we find solutions (Ω, ψ) close to the parallel solution (Ω_0, ψ_0) that, under additional assumptions on their regularity, satisfy an energetic stability result. The proofs are highly technical and we give here only a shallow description of the main ideas.

Writing $\psi = \psi_0 + w$ where $w \in H^2(\Omega)$, we get from (0.8) the equation

$$\Delta w = -f_\alpha^{-1}(w + f_\alpha(-\psi_0'')) - \psi_0'' \text{ on } \Omega \tag{0.10}$$

with boundary conditions derived from (0.9), and in essence the idea is to solve this for fixed η and α, which defines a map $\eta \to w(\eta)$, and then use (0.7) to find solutions η. The essential part of this approach is to do a nonlocal transformation of the fluid domain $\Omega_0 \ni (\bar{x}, \bar{y}) \to (x, y) \in \Omega$ and work on the rectangle and define η through a new function $\bar{\eta} \in H^2(\mathbb{R})$ by $\eta(x) = \bar{\eta}(\bar{x})$. Setting $w(x, y) = \bar{w}(\bar{x}, \bar{y}) \in H^2(\Omega_0)$, (0.10) gives an equation for $\bar{w}$. Assuming $\alpha \in (-2\epsilon, -\epsilon/2)$, $\epsilon > 0$ small and $\|\bar{\eta}\|_{H^2(\mathbb{R})}$, the right-hand side of (0.10) can expanded and explicitly expressed up to third order in ϵ, $\bar{w}$ and $\bar{\eta}$, where the remainder will be small. The equation is then solved in several steps and an explicit expression for the solution up to sufficient order in ϵ and $\bar{\eta}$ is found. Inserting this into (0.7), now in the new variables, gives an equation in $\bar{\eta}$ only, which in the limit case as $\epsilon \to 0^+$ gives the same result as in [15], and we conclude that there are solutions for $\epsilon > 0$ sufficiently small.

The stability is investigated by considering the variational formulation in terms of the functional $\mathcal{J}_\alpha(\eta, \psi)$. Assuming further regularity the solutions $(\bar{\eta}, \bar{w})$ will be critical points of $\bar{\mathcal{J}}_\alpha(\bar{\eta}, \bar{w}) := \mathcal{J}_\alpha(\eta, w)$ and building on ideas from [14] and following the methodology of [25], we show conditional energetic stability of the solutions obtained (see also [4]). As it stands, the stability is proved under some additional assumptions on the solutions obtained; these assumptions are expected to hold true for the solutions obtained through our method, but this is something that remains to be proved.

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Paper I

Paper I

Existence of solitary-wave solutions to nonlocal equations

Mathias Nikolai Arnesen

Published in *Discrete and Continuous Dynamical Systems- series A*

Note: this version differs from the published version, as some mistakes were found which are fixed in the present version

EXISTENCE OF SOLITARY-WAVE SOLUTIONS TO NONLOCAL EQUATIONS

MATHIAS NIKOLAI ARNESEN

ABSTRACT. We prove existence and conditional energetic stability of solitary-wave solutions for the two classes of pseudodifferential equations

$$u_t + (f(u))_x - (Lu)_x = 0$$

and

$$u_t + (f(u))_x + (Lu)_t = 0,$$

where f is a nonlinear term, typically of the form $c|u|^p$ or $cu|u|^{p-1}$, and L is a Fourier multiplier operator of positive order. The former class includes for instance the Whitham equation with capillary effects and the generalized Korteweg-de Vries equation, and the latter the Benjamin-Bona-Mahony equation. Existence and conditional energetic stability results have earlier been established using the method of concentration-compactness for a class of operators with symbol of order $s \geq 1$. We extend these results to symbols of order $0 < s < 1$, thereby improving upon the results for general operators with symbol of order $s \geq 1$ by enlarging both the class of linear operators and nonlinearities admitting existence of solitary waves. Instead of using abstract operator theory, the new results are obtained by direct calculations involving the nonlocal operator L, something that gives us the bounds and estimates needed for the method of concentration-compactness.

1. INTRODUCTION

In this paper we discuss solitary-wave solutions of pseudodifferential equations of the form

$$u_t + (f(u))_x - (Lu)_x = 0 \tag{1.1}$$

or

$$u_t + (f(u))_x + (Lu)_t = 0, \tag{1.2}$$

where u and f are real-valued functions, and L is a Fourier multiplier operator with symbol m of order $s > 0$. That is,

$$\widehat{Lu}(\xi) = m(\xi)\widehat{u}(\xi),$$

The author gratefully acknowledges the support of the project Nonlinear water waves (Grant No. 231668) from the Research Council of Norway.

where the hat denotes the Fourier transform $\widehat{f}(\xi) = \int_{\mathbb{R}} \mathrm{e}^{-2\pi i x\xi} f(x)\,\mathrm{d}x$ with respect to the spatial coordinate, and m is a function satisfying

$$\begin{aligned} &A_1|\xi|^s \leq m(\xi) \leq A_2|\xi|^s, \quad |\xi| \geq 1,\\ &0 \leq m(\xi) \leq A_2, \quad |\xi| \leq 1, \end{aligned}$$

for some constants $A_1, A_2 > 0$. Our inspiration comes from [8] and a series of recent papers on nonlinear dispersive equations with weak [15] or very weak [14] dispersion. This includes investigations into existence [7], stability [12, 13] and travelling waves [9] of the Whitham equation. We mention here that our results yield existence of solitary-waves to the capillary Whitham equation (see [14]), a case not earlier covered in the literature [1, 19, 11].

A solitary-wave is a travelling wave of the form $u(x,t) = u(x-ct)$, where $c > 0$ is the speed of the wave moving from left to right, that vanishes as $x - ct \to \pm\infty$. Assuming that u is a solitary-wave solution of (1.1) or (1.2), we obtain the following equations by integrating (1.1) or (1.2), respectively, with respect to the spatial variable:

$$Lu + cu - f(u) = 0 \tag{1.3}$$

and

$$c(Lu + u) - f(u) = 0. \tag{1.4}$$

For studying existence and stability of solutions to (1.3) and (1.4) by variational methods one can consider constrained variational problems (see equations (2.5) and (2.6)). The loss of compactness that results from working in the unbounded domain $\mathbb{R}$ is overcome by the method of concentration-compactness as introduced in [16]. The main challenge in applying the concentration-compactness method is usually, particularly in the nonlocal case, to preclude dichotomy (cf. Lemma 2.6), for which one needs a result like Theorem 3.6 to hold for the operator L.

Albert, Bona and Saut [3] prove existence of solitary-wave solutions to the Kubota-Ko-Dobbs equation, which belongs to the class of equations (1.1) with an operator of order $s = 1$, and their approach is presented in a more general form in [1]. The results are remarked to hold for any nonlinearity $f(u) = |u|^p$, $p \in (1, 2s+1)$ or $f(u) = u^p$, $p \in \mathbb{N} \cap (1, 2s+1)$ and any operator L with symbol m of order $s \geq 1$ satisfying

$$\|L(\theta f) - \theta L(f)\|_{L^2} \leq C\|\theta'\|_{L^\infty}\|f\|_{L^2}, \tag{1.5}$$

for any function θ and $f \in C_0^\infty$. Using general commutator estimates ([6, Theorem 35]), (1.5) leads the authors of [3] and [1] to impose the condition

$$\left|\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\right)^n \left(\frac{m(\xi)}{\xi}\right)\right| \leq C|\xi|^{-n} \quad \text{for all } \xi \in \mathbb{R} \text{ and } n \in \mathbb{N}, \tag{1.6}$$

for some constant $C > 0$. This condition is never satisfied when $s > 1$ or $0 < s < 1$. By a splitting argument, Zeng [19] establishes a similar inequality to (1.5) for all operators with symbol m such that (1.6) is satisfied for $n \in \{0, 1, 2, 3, 4\}$ with $m(\xi)/\xi$ replaced by $(m(\xi) - m(0))/\xi$ when $|\xi| \leq 1$ and by $\sqrt{m(\xi)}/|\xi|^{s/2}$ when $|\xi| \geq 1$. This also excludes symbols of order $0 < s < 1$, but allows one to consider operators of order $s > 1$, for instance the fractional Laplace operator $(-\Delta)^{s/2}$ where $m(\xi) = |\xi|^s$. Zeng [19] does this for equations of the form (1.4) for nonlinearities satisfying Assumption (B) (see the Assumptions below), but this argument can easily be implement in the method of [1] to extend the results of that paper for (1.3) to operators satisfying the assumptions of [19].

For pseudodifferential operators of order $0 < s < 1$, however, the only known result is, to the author's knowledge, the recent publication [11], which proves existence of solitary-wave solutions to (1.1) for $L = (-\Delta)^{s/2}$ ($m(\xi) = |\xi|^s$) and $f(u) = c_p u|u|^{p-1}$, where $p \in (1, 2s + 1)$. That result was achieved by using a commutator estimate that has only been established for the fractional Laplace operator. The authors of [10] remark that the method of Weinstein [17], used to prove solitary-wave solutions of (1.1) and (1.2) when $s \geq 1$, holds equally well when $0 < s < 1$. While it is true that the method in [17] can be modified to prove the existence for $s \in (0, 1)$ (which is one of the results in this paper), as noted in [3] and proved in the Appendix, further care needs to be taken to the nonlocal part of the problem than what is done in [17]; in particular, equation (3.20) in [17] does not hold in general. In this paper we will establish Theorem 3.6 by direct calculation without any reference to general results on commutator estimates. This allows us to treat all operators of any order $s > 0$ that satisfies natural and easy to check assumptions (see Assumption (A)). Moreover, in the Appendix we prove that our assumptions are, almost if not completely, as weak as they can be: under weaker assumptions, the method of concentration compactness cannot be applied.

The structure of the paper is as follows: In Section 2 we state and describe our assumptions and results in detail. The main result on the existence of solitary-wave solutions, Theorem 2.1, will be proved in three parts, using the method of concentration-compactness, in Sections 3, 4 and 5. In Section 6, Theorem 2.3 concerning the stability of the sets of solutions is proved, as well as a result on the regularity of solutions. Lastly, in the Appendix, we prove by counter-example the necessity of a continuity assumption on the symbol m in order to obtain compactness from the concentration-compactness method. The general outline of the procedure in Sections 3, 4 and 5 is inspired primarily by [19], and also by [1]. While [19] works only with (1.2) and [1] with (1.1), we will relate the variational formulation (2.6) of (1.4) to solutions of (1.3) using a scaling argument from [17]. This allows us to extend the range of nonlinearities for which we have existence of solutions to (1.3).

For $1 \leq p \leq \infty$ and measurable sets $\Omega \subseteq \mathbb{R}$ we will write $L^p(\Omega)$ for the usual Banach spaces with norm $\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f|^p \, \mathrm{d}x\right)^{1/p}$ if $1 \leq p < \infty$, and

$\|f\|_{L^\infty(\Omega)} = \operatorname{ess\,supp}_{x\in\Omega}|f(x)|$. The ambient space is always $\mathbb{R}$ and we will for convenience write L^p for $L^p(\mathbb{R})$. Similarly, we denote by H^s the Hilbert space $H^s(\mathbb{R})$ with norm $\|f\|_{H^s} = \left(\int_{\mathbb{R}}(1+|\xi|^2)^s|\widehat{f}(\xi)|^2\,\mathrm{d}\xi\right)^{1/2}$.

2. Assumptions and main results

In this section we will state our fundamental assumptions and describe our results. The precise, technical details of our results are contained in Theorems 2.1 and 2.3, while a simpler summary of which nonlinearities we have existence and stability of solitary-wave solutions for is given in Table 1.

(A) The operator L is a Fourier multiplier with symbol m of order $s > 0$. That is,

$$\widehat{Lu}(\xi) = m(\xi)\widehat{u}(\xi),$$

where m satisfies

$$\begin{aligned} &A_1|\xi|^s \leq m(\xi) \leq A_2|\xi|^s \text{ for } |\xi| \geq 1,\\ &0 \leq m(\xi) \leq A_2 \text{ for } |\xi| \leq 1, \end{aligned} \tag{2.1}$$

for some constants $A_1, A_2 > 0$. Furthermore, we assume that m is piecewise continuous with finitely many discontinuities and that there exists a $K > 0$ such that for all $|\xi| > K$ and $|t| \ll 1$ such that m is continuous on $(\xi - t, \xi)$,

$$|m(\xi) - m(\xi - t)| \leq |k(t)||\xi|^s, \tag{2.2}$$

where $\lim_{t\to 0} k(t) = 0$.

(B) The nonlinearity f is of one of the forms:

(B1) $f(u) = c_p u|u|^{p-1}$ where $c_p > 0$,

(B2) $f(u) = c_p|u|^p$ where $c_p \neq 0$,

where either $p \in (1, 2s+1)$ or $p \in (1, \frac{1+s}{1-s})$. When $s \geq 1$, $p \in (1, \frac{1+s}{1-s})$ should be interpreted as $p \in (1, \infty)$.

The assumption (2.1) in (A) is to ensure that $\left(\int_{\mathbb{R}} uLu + u^2\,\mathrm{d}x\right)^{1/2}$ is an equivalent norm to the standard norm on $H^{s/2}$. The continuity assumption is essential for proving Theorem 3.6 which is necessary in order to exclude dichotomy, and in the Appendix we will show that a continuity assumption is necessary. The two different forms (B1) and (B2) of the nonlinearity are considered to cover both the case when the sign of u does affect the sign of f and when it does not, generalizing the cases when p is, respectively, an odd or an even integer. The two ranges of p are related to stability and existence. For e.g. the generalized Korteweg-de Vries equation (where $s = 1$), it is known that $p = 2s + 1$ is the critical exponent beyond which one loses stability, while one has existence for all $p \in (1, \infty)$ (see, for instance, [4]).

To state our results, let F be the primitive of f. That is,

$$F(x) := \begin{cases} c_p \frac{|x|^{p+1}}{p+1}, & \text{if } f(x) = c_p x|x|^{p-1}, \\ c_p \frac{x|x|^p}{p+1}, & \text{if } f(x) = c_p |x|^p. \end{cases} \tag{2.3}$$

As one can check (or see e.g. Lemma 1 in [2]), if u solves equation (1.1) with initial condition $u(x, 0) = \psi(x)$ for all $x \in \mathbb{R}$ where $\psi \in H^r$, $r \geq s/2$, the functionals

$$\mathcal{E}(u) = \frac{1}{2} \int_{\mathbb{R}} uLu \,\mathrm{d}x - \int_{\mathbb{R}} F(u) \,\mathrm{d}x \tag{2.4}$$

and

$$\mathcal{Q}(u) = \frac{1}{2} \int_{\mathbb{R}} u^2 \,\mathrm{d}x$$

are independent of t. Likewise, the functionals

$$\mathcal{J}(u) = \frac{1}{2} \int_{\mathbb{R}} \left(uLu + u^2\right) \,\mathrm{d}x$$

and

$$\mathcal{U}(u) = \int_{\mathbb{R}} F(u) \,\mathrm{d}x$$

are invariant in time for solutions of equation (1.2). Furthermore, the Lagrange multiplier principle (cf. [18]) implies that, for every $q > 0$, minimizers of the constrained variational problem

$$I_q := \inf\{\mathcal{E}(w) : w \in H^{s/2} \text{ and } \mathcal{Q}(w) = q\} \tag{2.5}$$

solve equation (1.3) with c being the Lagrange multiplier. We denote by D_q the set of minimizers of I_q. Equation (2.5) is the variational problem studied in [1] for a class of symbols with $s = 1$, and as we shall show the results of [1] can be extended to hold for all operators satisfying assumption (A). This formulation, however, has the disadvantage that I_q is unbounded below when $p \geq 2s + 1$, and so the range of p for which one can find minimizers is restricted to $(1, 2s+1)$. One would expect a change in behaviour at the critical exponent $p = 2s + 1$, as with the GKdV equation as mentioned above, but one would also expect existence, if not stability, beyond the critical exponent. This is indeed the case, as we will show. For any $\lambda > 0$, equation (1.4) is the Euler-Lagrange equation of the constrained variational problem

$$\Gamma_\lambda = \inf\{\mathcal{J}(w) : w \in H^{s/2} \text{ and } \mathcal{U}(w) = \lambda\}. \tag{2.6}$$

For all $p \in (1, \frac{1+s}{1-s})$, one can show that Γ_λ is well defined. The Lagrange multiplier principle implies that if u is a minimizer of Γ_λ, then there exists a γ such that

$$Lu + u - \gamma f(u) = 0$$

in a weak sense, here meaning that

$$\int_{\mathbb{R}} (Lu + u + \gamma f(u))\, \varphi \,\mathrm{d}x = 0$$

for all $\varphi \in H^{s/2}$. Hence u solves equation (1.4) with $c = 1/\gamma$. If we define

$$\mathcal{J}_\kappa = \frac{1}{2} \int_{\mathbb{R}} uLu + \kappa u^2 \,\mathrm{d}x, \tag{2.7}$$

where $\kappa > 0$, we get that minimizers of $\Gamma_\lambda = \Gamma_\lambda(\kappa)$, which now depends on κ (we will generally omit this from the notation where it is clear from the context), solve

$$Lu + \kappa u - \gamma f(u) = 0.$$

Letting

$$\beta^{-1} v = u, \quad \beta^{p-1} = \gamma \tag{2.8}$$

one gets that v solves (1.3) with wave-speed $c = \kappa > 0$. We will denote by $G_\lambda(\kappa)$ the set of minimizers of $\Gamma_\lambda(\kappa)$. For equation (1.4) one can consider $\mathcal{J}(= \mathcal{J}_1)$ again and let

$$\beta^{-1} v = u, \quad \beta^{p-1} = \kappa\gamma \tag{2.9}$$

for $\kappa > 0$. Then v will be a solution to (1.4) with wave speed $c = \kappa$. This is equivalent to consider $\kappa\mathcal{J}$ instead of $\mathcal{J}_\kappa$ in (2.6), which in turn is equivalent to scaling λ by some factor. Thus every wave speed c can be attained as (the reciprocal of) the Lagrange multiplier by varying λ.

As in [19], we will also consider inhomogeneous nonlinearities of the form $g(u) = u + f(u)$, where f satisfies Assumption (B). For solitary-wave solutions of (1.1), the difference between homogeneous nonlinearities f and inhomogeneous nonlinearities $g(u) = u + f(u)$ is trivial. A variational formulation in terms of conserved quantities is given by minimizing $\mathcal{E} - \mathcal{Q}$ in place of $\mathcal{E}$ in (2.5), which clearly makes no difference for the existence of minimizers. And if $\kappa > 0$, every element of $G_\lambda(\kappa)$ will be a solitary-wave solution with wave speed $c = \kappa + 1$ upon scaling as in (2.8). For (1.2) it is more complicated. Equation (1.2) in this case becomes

$$u_t + u_x + f(u)_x + (Lu)_t = 0 \tag{2.10}$$

and solitary-wave solutions satisfy

$$cLu + (c-1)u - f(u) = 0. \tag{2.11}$$

For $\kappa > 1$, every element of $G_\lambda(1 - 1/\kappa)$ will be a solution to (2.11) with wave speed $c = \kappa$ upon scaling as in (2.9). The functional $\mathcal{J}_{1-1/\kappa}$ is, however, not a preserved quantity for (1.2), nor is the functional $\mathcal{U}$, and we are therefore not able to prove stability of the set of minimizers. We consider instead $\mathcal{J}(= \mathcal{J}_1)$, set

$$\tilde{\mathcal{U}}(u) = \int_{\mathbb{R}} \left(\frac{u^2}{2} + F(u) \right) \mathrm{d}x$$

and look for minimizers of

$$\tilde{\Gamma}_\lambda = \inf\{\mathcal{J}(w) : w \in H^{s/2} \text{ and } \tilde{\mathcal{U}}(w) = \lambda\}.$$

We denote by $\tilde{G}_\lambda$ the set of minimizers of $\tilde{\Gamma}_\lambda$. By the Lagrange multiplier principle, any element of $\tilde{G}_\lambda$ will be a solution of (2.11) with $c = 1/\gamma$, where γ is the Lagrange multiplier. Furthermore, the functionals $\mathcal{J}$ and $\tilde{\mathcal{U}}$ are preserved quantities for (2.10) and we can therefore prove stability for the set of minimizers of $\tilde{\Gamma}_\lambda$ (cf. Theorem 2.3 and Section 6). Note that since $\tilde{\mathcal{U}}$ is inhomogeneous, the scaling arguments performed on the elements of G_λ in order to choose the wave speed cannot be performed for the minimizers of $\tilde{\Gamma}_\lambda$; we will only get the wave speeds given by the Lagrange multiplier principle. Moreover, existence of minimizers of $\tilde{\Gamma}_\lambda$ can only be established for $\lambda > \lambda_0$ for some $\lambda_0 \geq 0$ whose precise value is unknown. Equation (2.10) can thus be said to have more in common with (1.1) than (1.2) in that fixing the wave speed comes at the cost of stability. The precise details of our main results on existence and stability of solitary-wave solutions are contained in Theorem 2.1 and Theorem 2.3 below.

Theorem 2.1 (Existence of solitary-wave solutions)**.** *Assume L satisfies Assumption (A) and f satisfies Assumption (B). Then:*

(i) *If $p \in (1, 2s+1)$, there is a number $q_0 \geq 0$ such that set D_q of minimizers of I_q is non-empty for any $q > q_0$, and every element of D_q is a solution to* (1.3) *with the wave speed c being the Lagrange multiplier in this constrained variational problem. If, in addition to (A), $m(\xi)$ satisfies $0 \leq m(\xi) \leq A_2|\xi|^s$ for $|\xi| \leq 1$, then $q_0 = 0$.*

(ii) *If $p \in (1, \frac{1+s}{1-s})$, the set $G_\lambda = G_\lambda(\kappa)$ of minimizers of $\Gamma_\lambda = \Gamma_\lambda(\kappa)$ is non-empty for any $\lambda, \kappa > 0$, and if f satisfies (B2), then this is true also for $\lambda < 0$. If $\kappa = 1$, then every element of G_λ solves* (1.4) *with the wave speed c being the reciprocal of the Lagrange multiplier in this constrained variational problem, and by varying the parameter λ one can get any wave speed $c > 0$. Moreover, scaling the set $G_\lambda(\kappa)$ as in* (2.8) *for any $\kappa > 0$, or the set $G_\lambda(1 - 1/\kappa)$ as in* (2.9) *for any $\kappa > 1$, every element will be a solution to* (1.3) *or* (2.11)*, respectively, with wave speed $c = \kappa$.*

(iii) *If $p \in (1, \frac{1+s}{1-s})$, there exists a $\lambda_0 \geq 0$ such that the set $\tilde{G}_\lambda$ of minimizers of $\tilde{\Gamma}_\lambda$ is non-empty for any $\lambda > \lambda_0$, and every element of $\tilde{G}_\lambda$ is a solution to* (2.11) *with the wave speed c being the reciprocal of the Lagrange multiplier in this constrained variational problem. If, in addition to (A), $m(\xi)$ satisfies $0 \leq m(\xi) \leq A_2|\xi|^s$ for $|\xi| \leq 1$, then $\lambda_0 = 0$ for $p \in (1, 2s+1)$.*

Moreover, if $\{u_n\}_n \subset H^{s/2}$ is a minimizing sequence of I_q, $\Gamma_\lambda(\kappa)$ or $\tilde{\Gamma}_\lambda$, under the conditions of (i), (ii) or (iii), respectively, then there exists a sequence $\{y_n\} \subset \mathbb{R}$

such that a subsequence of $\{u_n(\cdot + y_n)\}_n$ *converges in* $H^{s/2}$ *to an element of* D_q, $G_\lambda(\kappa)$ *or* $\tilde{G}_\lambda$, *respectively. Furthermore,* $D_q, G_\lambda(\kappa), \tilde{G}_\lambda \subset H^s$.

Remark 2.2. Scaling elements of $G_\lambda(\kappa)$ in order to choose the wave speed comes at the cost of losing information about the quantity $\mathcal{U}(u)$ for solutions u. For given energy $\mathcal{U}(u) = \lambda$ one faces the opposite problem, that the wave speed c is given as the reciprocal of the Lagrange multiplier which one cannot directly control. However, the Lagrange multiplier γ, which is the unknown factor in the scalings (2.8) and (2.9), can be expressed in terms of the quantities λ, p and Γ_λ as follows (see Section 3):

$$\gamma = \frac{2\Gamma_\lambda}{(p+1)\lambda}. \tag{2.12}$$

This expression illustrates at least the relationship between the different quantities c, λ and Γ_λ.

Theorem 2.3 (Conditional energetic stability). *The sets* D_q, *(any positive scaling of)* $G_\lambda(1)$ *with* $\kappa = 1$ *and* $\tilde{G}_\lambda$ *are, under the conditions in Theorem 2.1 (i), (ii) and (iii) respectively, stable sets for the initial value problems of* (1.1), (1.2) *and* (2.10), *respectively, in the following sense as described for* D_q*: For every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if*

$$\inf_{w \in D_q} \|u_0 - w\|_{H^{s/2}} < \delta,$$

where $u(x,t)$ *solves* (1.1) *with* $u(x,0) = u_0(x)$, *then*

$$\inf_{w \in D_q} \|u(\cdot,t) - w\|_{H^{s/2}} < \varepsilon$$

for all $t \in \mathbb{R}$.

Remark 2.4. While the upper bounds $2s+1$ and $\frac{1+s}{1-s}$ on p appear in the proof of Theorem 2.1 by appealing to Sobolev embedding and interpolation theorems rather than from the equations themselves, they are, in fact, strictly related to existence and stability of solitary-wave solutions. In [15], it is proven that for $m(\xi) = |\xi|^s$ and $p = 2$, there are no non-trivial solutions to (1.3) if $s < 1/3$. If $s = 1/3$, then $p = 2$ is the upper bound $\frac{1+s}{1-s}$. Their arguments can easily be generalized to show that if $p > \frac{1+s}{1-s}$, for any $s > 0$, there are no solutions to (1.3). Similarly, as already mentioned one has instability for $p > 2s+1$ for equations like the GKdV, and this limitation on p is therefore also not due to any limitations of the proofs presented in this paper.

The following table summarizes the essential content of Theorems 2.1 and 2.3 in terms of which nonlinearities one has existence for, and for which one has stability, for equations (1.1), (1.2) and (2.10) (here L. multiplier is short for Lagrange multiplier).

Equation	Wave speed	Existence	Stability
(1.1)	any $c > 0$	$p \in (1, \frac{1+s}{1-s})$	
	L. multiplier	$p \in (1, 2s+1)$	$p \in (1, 2s+1)$
(1.2)	any $c > 0$/ L. multiplier	$p \in (1, \frac{1+s}{1-s})$	$p \in (1, \frac{1+s}{1-s})$
(2.10)	any $c > 1$	$p \in (1, \frac{1+s}{1-s})$	
	L. multiplier	$p \in (1, \frac{1+s}{1-s})$	$p \in (1, \frac{1+s}{1-s})$

TABLE 1. Ranges of existence and stability of solitary-wave solutions of (1.1), (1.2) and (2.10) in terms of the exponent p of the nonlinearity. The Lagrange multipliers come from variational formulations in terms of conserved quantities, while "any" c is obtained through scaling arguments.

Remark 2.5. Recall that existence and stability of solitary-waves for equation $u_t + u_x + (f(u))_x - (Lu)_x = 0$ is equivalent to that of $u_t + (f(u))_x - (Lu)_x = 0$ (see the discussion leading up to Theorem 2.1).

We end the section by stating the concentration-compactness lemma that will be the main ingredient in the sequel:

Lemma 2.6 (Lions [16]). *Let $\{\rho_n\}_n \subset L^1$ be a sequence that satisfies*

$$\rho_n \geq 0 \ a.e. \ on\ \mathbb{R},$$
$$\int_{\mathbb{R}} \rho_n \,\mathrm{d}x = \mu$$

for a fixed $\mu > 0$ and all $n \in \mathbb{N}$. Then there exists a subsequence $\{\rho_{n_k}\}_k$ that satisfies one of the three following properties:

(1) *(Compactness). There exists a sequence $\{y_k\}_k \subset \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $r < \infty$ satisfying for all $k \in \mathbb{N}$:*

$$\int_{y_k - r}^{y_k + r} \rho_{n_k}(x) \,\mathrm{d}x \geq \mu - \varepsilon.$$

(2) *(Vanishing). For all $r < \infty$,*

$$\lim_{k\to\infty} \sup_{y\in\mathbb{R}} \int_{y-r}^{y+r} \rho_{n_k} \,\mathrm{d}x = 0$$

(3) *(Dichotomy). There exists $\bar{\mu} \in (0, \mu)$ such that for every $\varepsilon > 0$ there exists a natural number $k_0 \geq 1$ and two sequences of positive L^1 functions*

$\{\rho_k^{(1)}\}_k, \{\rho_k^{(2)}\}_k$ *satisfying for* $k \geq k_0$,

$$\begin{aligned} &\|\rho_{n_k} - (\rho_k^{(1)} + \rho_k^{(2)})\|_{L^1} \leq \varepsilon, \\ &|\int_{\mathbb{R}} \rho_k^{(1)} \,\mathrm{d}x - \bar{\mu}| \leq \varepsilon, \\ &|\int_{\mathbb{R}} \rho_k^{(2)} \,\mathrm{d}x - (\mu - \bar{\mu})| \leq \varepsilon, \\ &\mathrm{dist}(\mathrm{supp}(\rho_k^{(1)}), \mathrm{supp}(\rho_k^{(2)})) \to \infty. \end{aligned} \tag{2.13}$$

Remark 2.7. The condition $\int_{\mathbb{R}} \rho_n \,\mathrm{d}x = \mu$ can be replaced by $\int_{\mathbb{R}} \rho_n \,\mathrm{d}x = \mu_n$ where $\mu_n \to \mu$ (see [5]).

3. Concentration-compactness for (2.6)

The variational problem

$$\Gamma_\lambda = \inf\{\mathcal{J}_\kappa(w) : w \in H^{s/2} \text{ and } \mathcal{U}(w) = \lambda\}. \tag{3.1}$$

is equivalent to the one considered in [17], where it was arrived at by first considering the functional

$$J(u) = \frac{\frac{1}{2}\int_{\mathbb{R}} \left(uLu + \kappa u^2\right) \,\mathrm{d}x}{\left(\int_{\mathbb{R}} F(u) \,\mathrm{d}x\right)^{\frac{2}{p+1}}},$$

for some constant $\kappa > 0$ and noting that it is invariant under the scaling $u \mapsto \theta u$ for $\theta \neq 0$. As minimizers of the constrained variational problem then also minimize the unconstrained functional over $H^{s/2}$, one can ascertain some a-priori information about the sign and size of the wave speed $1/\gamma$ in terms of the quantities p, λ and Γ_λ. We henceforth assume $p \in (1, \frac{1+s}{1-s})$, so that by the Sobolev embedding theorem, $\int_{\mathbb{R}} F(u) \,\mathrm{d}x$ is finite for all $u \in H^{s/2}$.

Assume now that u is a minimizer of Γ_λ. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} J(u + t\varphi)|_{t=0} = 0,$$

for all $\varphi \in H^{s/2}$. Calculating the derivative, we get

$$\begin{aligned} &\int_{\mathbb{R}} \varphi Lu + \kappa\varphi u \,\mathrm{d}x \left(\int_{\mathbb{R}} F(u) \,\mathrm{d}x\right)^{\frac{2}{p+1}} \\ &\quad - \frac{1}{p+1} \int_{\mathbb{R}} uLu + \kappa u^2 \,\mathrm{d}x \left(\int_{\mathbb{R}} F(u) \,\mathrm{d}x\right)^{\frac{1-p}{p+1}} \int_{\mathbb{R}} \varphi f(u) \,\mathrm{d}x = 0. \end{aligned}$$

That is,

$$Lu + \kappa u - \frac{1}{p+1} \int_{\mathbb{R}} uLu + \kappa u^2 \,\mathrm{d}x \left(\int_{\mathbb{R}} F(u) \,\mathrm{d}x\right)^{-1} f(u) = 0.$$

Thus, if $\kappa = 1$ and $\lambda > 0$, minimizers of Γ_λ will be solutions of equation (1.4) with wave speed $\frac{(p+1)\lambda}{2\Gamma_\lambda} > 0$. Moreover, this establishes the expression for γ given in Remark 2.2.

Now we turn to the existence of minimizers of (2.6), which we will prove using concentration-compactness arguments. As $u \in H^{s/2}$ implies $F(u) \in L^1$ and we fix $\mathcal{U}(u) = \lambda$, it would be natural for a minimizing sequence $\{u_n\}_n$ of Γ_λ to apply Lemma 2.6 to $\{F(u_n)\}_n$ as in [17]. Unfortunately, $F(u)$ does not satisfy the non-negativity criterion for all nonlinearities f we would like to consider. Nor does the other natural candidate $uLu + \kappa u^2$. We therefore replace uLu with a non-negative term the integral of which (over $\mathbb{R}$) is equal to that of uLu. We define the operator $L^{\frac{1}{2}}$ by replacing m with $\sqrt{m}$ in Assumption (A), and let

$$\rho_n = \kappa u_n^2 + \left(L^{\frac{1}{2}} u_n\right)^2,$$
$$\mu_n = \int_{\mathbb{R}} \rho_n \,\mathrm{d}x.$$

Thus $\rho_n \geq 0$ and there exist $k_1, k_2 > 0$ depending on κ such that

$$k_1 \|u_n\|^2_{H^{s/2}} \leq \int_{\mathbb{R}} \rho_n \,\mathrm{d}x \leq k_2 \|u_n\|^2_{H^{s/2}}. \tag{3.2}$$

In order to apply Lemma 2.6 we will need the following lemma.

Lemma 3.1. *If $\{u_n\}_n$ is a minimizing sequence of Γ_λ, then there exists $M > 0$ and $N > 0$ such that $N \leq \|u_n\|_{H^{s/2}} \leq M$ for all n. Furthermore, $\Gamma_\lambda > 0$.*

Proof. Noting that Assumption (A) implies that for any $\kappa > 0$, $(\mathcal{J}_\kappa(\cdot))^{1/2}$ defines a norm on $H^{s/2}$ equivalent to the standard norm, the upper bound follows trivially from the boundedness of $\{\mathcal{J}_\kappa(u_n)\}_n \subset \mathbb{R}$. Similarly, the lower bound is a consequence of $\int_{\mathbb{R}} |F(u)| \,\mathrm{d}x = \lambda$ and the Sobolev embedding theorem. That $\Gamma_\lambda > 0$ is an immediate consequence of the lower bound. $\square$

By (3.2) and Lemma 3.1, for any minimizing sequence $\{u_n\}_n \subset H^{s/2}$ of Γ_λ, the sequence $\{\mu_n\}_n \subset \mathbb{R}$ as defined above will be bounded. Moreover, $\mu_n > 0$ for all n. Thus there exists a number $\mu > 0$ and a subsequence of $\{\rho_n\}_n$, still denoted by $\{\rho_n\}_n$, such that $\int_{\mathbb{R}} \rho_n \to \mu$. By Remark 2.7, Lemma 2.6 then applies to the sequence $\{\rho_n\}_n$ and there exists a subsequence, still denoted by $\{\rho_n\}_n$, for which either compactness, vanishing or dichotomy holds. In what follows we will eliminate vanishing and dichotomy. To this purpose, we will first establish some structural properties of Γ_λ considered as a function of λ, as well as some general properties of minimizing sequences for Γ_λ.

We start with the following Lemma from [19] (Lemma 2.9).

Lemma 3.2. *If $\lambda_2 > \lambda_1 > 0$, then $\Gamma_{\lambda_2} \geq \Gamma_{\lambda_1}$.*

Proof. For any $\varepsilon > 0$, there exists a function $\varphi \in H^{s/2}$ such that $\mathcal{U}(\varphi) = \lambda_2$ and $\mathcal{J}_\kappa(\varphi) \leq \Gamma_{\lambda_2} + \varepsilon$. Since $\mathcal{U}(a\varphi)$ is a continuous function of $a \in \mathbb{R}$, then by the intermediate value theorem we can find $C \in (0,1)$ such that $\mathcal{U}(C\varphi) = \lambda_1$. Hence

$$\Gamma_{\lambda_1} \leq \mathcal{J}_\kappa(C\varphi) = C^2 \mathcal{J}_\kappa(\varphi) < \mathcal{J}_\kappa(\varphi) < \Gamma_{\lambda_2} + \varepsilon.$$

This proves the result. □

Lemma 3.3. *For $\lambda > 0$ and any $\alpha \in (0, \lambda)$,*

$$\Gamma_\lambda < \Gamma_{\lambda-\alpha} + \Gamma_\alpha.$$

Proof. Let $\theta \in (1, \lambda\alpha^{-1})$. Then,

$$\begin{aligned}
\Gamma_{\theta\alpha} &= \inf\{\mathcal{J}_\kappa(u) : u \in H^{s/2}, \int_{\mathbb{R}} F(u)\,\mathrm{d}x = \alpha\theta\} \\
&= \inf\{\mathcal{J}_\kappa(\theta^{\frac{1}{p+1}} v) : v \in H^{s/2}, \int_{\mathbb{R}} F(v)\,\mathrm{d}x = \alpha\} \\
&= \theta^{\frac{2}{p+1}} \Gamma_\alpha \\
&< \theta\Gamma_\alpha,
\end{aligned}$$

where the last inequality follows from that $\theta > 1$ and, by Assumption (B), $p > 1$. Now if $\alpha \geq \lambda - \alpha$,

$$\begin{aligned}
\Gamma_\lambda = \Gamma_{\lambda-\alpha+\alpha} = \Gamma_{\alpha\left(1+\frac{\lambda-\alpha}{\alpha}\right)} &< \left(1 + \frac{\lambda-\alpha}{\alpha}\right)\Gamma_\alpha \\
&= \Gamma_\alpha + \frac{\lambda-\alpha}{\alpha}\Gamma_{\frac{\alpha}{\lambda-\alpha}(\lambda-\alpha)} < \Gamma_\alpha + \Gamma_{\lambda-\alpha}.
\end{aligned} \tag{3.3}$$

For $\alpha \leq \lambda - \alpha$ one can derive the same inequality in a similar manner. □

To exclude vanishing, we will need the following result from [1]:

Lemma 3.4. *Given $K > 0$ and $\delta > 0$, there exists $\eta = \eta(K, \delta) > 0$ such that if $v \in H^{s/2}$ with $\|v\|_{H^{s/2}} \leq K$ and $\|v\|_{L^{p+1}} \geq \delta$, then*

$$\sup_{y\in\mathbb{R}} \int_{y-2}^{y+2} |v(x)|^{p+1}\,\mathrm{d}x \geq \eta.$$

Proof. Without loss of generality we may assume $s/2 \leq 1$; if $s/2 > 1$ then $\|v\|_{H^1} \leq K$ and the argumentation that follows can be carried out for H^1. Choose a smooth function $\zeta : \mathbb{R} \to [0,1]$ with support in $[-2,2]$ and satisfying $\sum_{j\in\mathbb{Z}} \zeta(x-j) = 1$ for all $x \in \mathbb{R}$, and define $\zeta_j(x) = \zeta(x-j)$ for $j \in \mathbb{Z}$. The map $T : H^r \to l_2(H^r)$ defined by

$$Tv = \{\zeta_j v\}_{j\in\mathbb{Z}}$$

is easily seen to be bounded for $r = 0$ and $r = 1$. For $r = 0$,

$$\|Tv\|_{l_2(L^2)}^2 = \sum_{j\in\mathbb{Z}} \|\zeta_j v\|_{L^2}^2 \leq \sum_{j\in\mathbb{Z}} \int_{-2+j}^{2+j} v^2 \, \mathrm{d}x = 4\|v\|_{L^2}^2,$$

and one can argue similarly when $r = 1$, recalling that ζ is a smooth function. By interpolation the map T is therefore also bounded for $r = s/2$. That is, there exists a constant C_0 such that for all $v \in H^{s/2}$,

$$\sum_{j\in\mathbb{Z}} \|\zeta_j v\|_{H^{s/2}}^2 \leq C_0 \|v\|_{H^{s/2}}^2.$$

Since $l_{p+1} \hookrightarrow l_1$, there exists a positive number C_1 such that $\sum_{j\in\mathbb{Z}} |\zeta(x-j)|^{p+1} \geq C_1$ for all $x \in \mathbb{R}$. We claim that for every $v \in H^{s/2}$ that is not identically zero, there exist an integer j_0 such that

$$\|\zeta_{j_0} v\|_{H^{s/2}}^2 \leq \left(1 + C_2 \|v\|_{L^{p+1}}^{-p-1}\right) \|\zeta_{j_0} v\|_{L^{p+1}}^{p+1}, \tag{3.4}$$

where $C_2 = C_0 K^2 / C_1$. To see this, assume to the contrary that

$$\|\zeta_j v\|_{H^{s/2}}^2 > \left(1 + C_2 \|v\|_{L^{p+1}}^{-p-1}\right) \|\zeta_j v\|_{L^{p+1}}^{p+1},$$

for every $j \in \mathbb{Z}$. Summing over j, we obtain

$$C_0 \|v\|_{H^{s/2}}^2 > \left(1 + C_2 \|v\|_{L^{p+1}}^{-p-1}\right) \sum_{j\in\mathbb{Z}} \|\zeta_j v\|_{L^{p+1}}^{p+1}$$

and hence by our choice of C_2

$$C_0 K^2 > \left(1 + C_2 \|v\|_{L^{p+1}}^{-p-1}\right) C_1 \|v\|_{L^{p+1}}^{p+1} = C_1 \|v\|_{L^{p+1}}^{p+1} + C_0 K^2,$$

for $\|v\|_{H^{s/2}} \leq K$, which is a contradiction. This proves (3.4).

Observe now that from (3.4) and the assumptions of the lemma it follows that

$$\|\zeta_{j_0} v\|_{H^{s/2}}^2 \leq \left(1 + C_2/\delta^{p+1}\right) \|\zeta_{j_0} v\|_{L^{p+1}}^{p+1}.$$

For $p \leq \frac{1+s}{1-s}$, we have by the Sobolev embedding theorem that

$$\|\zeta_{j_0} v\|_{L^{p+1}} \leq C \|\zeta_{j_0} v\|_{H^{s/2}}$$

where C is independent of v. Combining the above two inequalities we get that

$$\|\zeta_{j_0} v\|_{L^{p+1}} \geq \left[C^2 \left(1 + C_2/\delta^3\right)\right]^{1/(1-p)},$$

and since

$$\int_{j_0-2}^{j_0+2} |v|^{p+1} \, \mathrm{d}x \geq \|\zeta_{j_0} v\|_{L^{p+1}}^{p+1}$$

the result follows, with $\eta = \left[C^2 \left(1 + C_2/\delta^3\right)\right]^{(p+1)/(1-p)}$. ◻

We may now exclude vanishing.

Lemma 3.5. *Vanishing does not occur.*

Proof. Let $\{u_n\}_n$ be a minimizing sequence for Γ_λ. By Assumption (B), the constraint $\int_{\mathbb{R}} F(u_n)\,\mathrm{d}x = \lambda > 0$ for all $n \in \mathbb{N}$ implies that $\|u_n\|_{L^{p+1}} \geq \delta$ for $\delta = (|c_p^{-1}|\lambda)^{1/(p+1)}$. By Lemma 3.1, we additionally have that there is a K such that $\|u_n\|_{H^{s/2}} \leq K$ for all $n \in \mathbb{N}$. The criteria for Lemma 3.4 are therefore satisfied for all $n \in \mathbb{N}$ and there exists an $\eta(K,\delta) > 0$ such that $\sup_{y\in\mathbb{R}} \int_{y-2}^{y+2} |u_n|^{p+1}\,\mathrm{d}x \geq \eta$ for all $n \in \mathbb{N}$. The result now follows from the Sobolev embedding theorem:

$$C\eta^{2/p+1} \leq \sup_{y\in\mathbb{R}} C \left(\int_{y-2}^{y+2} |u_n|^{p+1}\,\mathrm{d}x \right)^{2/p+1} \leq \sup_{y\in\mathbb{R}} \int_{y-2}^{y+2} \rho_n\,\mathrm{d}x,$$

for all $n \in \mathbb{N}$, where $C > 0$ is an embedding constant. $\square$

Now it only remains to preclude dichotomy. The following theorem is the key result in order to do so.

Theorem 3.6. *Assume that L satisfies Assumption (A). Let $u \in H^{s/2}$ and $\varphi, \psi \in C^\infty$ satisfy $0 \leq \varphi \leq 1$, $0 \leq \psi \leq 1$,*

$$\varphi(x) = \begin{cases} 1, & \text{if } |x| < 1, \\ 0, & \text{if } |x| > 2, \end{cases}$$

and

$$\psi(x) = \begin{cases} 0, & \text{if } |x| < 1, \\ 1, & \text{if } |x| > 2. \end{cases}$$

Define $\varphi_r(x) = \varphi(x/r)$ and $\psi_r(x) = \psi(x/r)$ for all $x \in \mathbb{R}$. Then for all $r > 0$ sufficiently large,

$$\left| \int_{\mathbb{R}} \varphi_r u(L(\varphi_r u) - \varphi_r Lu)\,\mathrm{d}x \right| \leq \beta(r)\|u\|_{H^{s/2}}^2$$

and

$$\left| \int_{\mathbb{R}} \psi_r u(L(\psi_r u) - \psi_r Lu)\,\mathrm{d}x \right| \leq \beta(r)\|u\|_{H^{s/2}}^2,$$

where $\beta(r) \to 0$ as $r \to \infty$. In particular, the integrals above converge to 0 as $r \to \infty$ uniformly in $u \in H^{s/2}$.

Proof. By Plancherel's theorem, basic properties of the Fourier transform, and Fubini's theorem,

$$\begin{aligned}
&\int_{\mathbb{R}} \varphi_r u(L(\varphi_r u) - \varphi_r Lu)\,\mathrm{d}x \\
&\qquad = \int_{\mathbb{R}} \overline{\widehat{\varphi_r u}}(\xi)\,[m(\xi)(\widehat{\varphi_r} * \widehat{u})(\xi) - (\widehat{\varphi_r} * (m\widehat{u}))(\xi)]\,\mathrm{d}\xi \\
&\qquad = \int_{\mathbb{R}} \overline{\widehat{\varphi_r u}}(\xi) \int_{\mathbb{R}} \widehat{\varphi_r}(t)\widehat{u}(\xi - t)(m(\xi) - m(\xi - t))\,\mathrm{d}t\,\mathrm{d}\xi \\
&\qquad = \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{\widehat{\varphi_r u}}(\xi)\widehat{\varphi_r}(t)\widehat{u}(\xi - t)(m(\xi) - m(\xi - t))\,\mathrm{d}t\,\mathrm{d}\xi \\
&\qquad = \int_{\mathbb{R}} \widehat{\varphi_r}(t) \int_{\mathbb{R}} \overline{\widehat{\varphi_r u}}(\xi)\widehat{u}(\xi - t)(m(\xi) - m(\xi - t))\,\mathrm{d}\xi\,\mathrm{d}t. \qquad (3.5)
\end{aligned}$$

We can write $m(\xi - t) = (1 + \sqrt{m(\xi)})\sqrt{m(\xi - t)}\frac{\sqrt{m(\xi-t)}}{1+\sqrt{m(\xi)}}$. By assumption,

$$\frac{\sqrt{m(\xi - t)}}{1 + \sqrt{m(\xi)}} \leq \frac{A_2^{1/2}|\xi - t|^{s/2}}{1 + A_1^{1/2}|\xi|^{s/2}} \leq C(1 + |t|^{s/2})$$

for some C independent of t and ξ. Hence, by Hölder's inequality,

$$\begin{aligned}
&\Big|\int_{\mathbb{R}} \overline{\widehat{\varphi_r u}}(\xi)\widehat{u}(\xi - t)m(\xi - t)\,\mathrm{d}\xi\Big| \\
&= \left|\int_{\mathbb{R}} \overline{\widehat{\varphi_r u}}(\xi)(1 + \sqrt{m(\xi)})\widehat{u}(\xi - t)\sqrt{m(\xi - t)}\frac{\sqrt{m(\xi - t)}}{1 + \sqrt{m(\xi)}}\,\mathrm{d}\xi\right| \\
&\leq C(1 + |t|^{s/2})\|\overline{\widehat{\varphi_r u}}(1 + \sqrt{m})\|_{L^2}\|\widehat{u}\sqrt{m}\|_{L^2} \\
&\leq C_1(1 + |t|^{s/2})\|u\|_{H^{s/2}}^2,
\end{aligned}$$

where C_1 is independent of t and r. Arguing in the same way, we find that

$$\left|\int_{\mathbb{R}} \overline{\widehat{\varphi_r u}}(\xi)\widehat{u}(\xi - t)m(\xi)\,\mathrm{d}\xi\right| \leq C_2(1 + |t|^{s/2})\|u\|_{H^{s/2}}^2,$$

where C_2 is independent of t and r. Hence

$$\left|\int_{\mathbb{R}} \overline{\widehat{\varphi_r u}}(\xi)\widehat{u}(\xi - t)(m(\xi) - m(\xi - t))\,\mathrm{d}\xi\right| \leq C(1 + |t|^{s/2})\|u\|_{H^{s/2}}^2$$

for some $C > 0$ independently of r. For any $\alpha < 1$, we have that

$$\begin{aligned}
&\Big|\int_{|t|>r^{-\alpha}} \widehat{\varphi_r}(t) \int_{\mathbb{R}} \overline{\widehat{\varphi_r u}}(\xi)\widehat{u}(\xi - t)(m(\xi) - m(\xi - t))\,\mathrm{d}\xi\,\mathrm{d}t\Big| \\
&\qquad\qquad \leq C\|u\|_{H^{s/2}}^2 \int_{|t|>r^{-\alpha}} |\widehat{\varphi_r}(t)|(1 + |t|^{s/2})\,\mathrm{d}t
\end{aligned}$$

As φ is a Schwartz function $\widehat{\varphi_r}$ approximates unity as $r \to \infty$ and

$$\int_{|t|>r^{-\alpha}} |\widehat{\varphi_r}(t)|(1+|t|^{s/2})\,\mathrm{d}t \to 0$$

as $r \to \infty$ for any $\alpha < 1$. It remains to consider

$$\int_{|t|<r^{-\alpha}} \widehat{\varphi_r}(t) \int_{\mathbb{R}} \overline{\widehat{\varphi_r u}}(\xi)\widehat{u}(\xi-t)(m(\xi)-m(\xi-t))\,\mathrm{d}\xi\,\mathrm{d}t.$$

Let $\varepsilon > 0$ be given, and let us first assume that $m(\xi)$ has no discontinuities. Then $m(\xi)$ is uniformly continuous on any bounded domain and hence there exists a number $R = R(r)$ such that $\lim_{r\to\infty} R(r) = \infty$ and $|m(\xi) - m(\xi - t)| < \varepsilon$ for all $|\xi| \le R$ and $|t| < r^{-\alpha}$. Thus

$$\begin{aligned} \int_{|t|<r^{-\alpha}} \widehat{\varphi_r}(t) \left| \int_{\mathbb{R}} \overline{\widehat{\varphi_r u}}(\xi)\widehat{u}(\xi-t)(m(\xi)-m(\xi-t))\,\mathrm{d}\xi \right| \mathrm{d}t \le \varepsilon C \|u\|_{L^2}^2 \\ + \int_{|t|<r^{-\alpha}} \widehat{\varphi_r}(t) \left| \int_{|\xi|>R} \overline{\widehat{\varphi_r u}}(\xi)\widehat{u}(\xi-t)(m(\xi)-m(\xi-t))\,\mathrm{d}\xi \right| \mathrm{d}t. \end{aligned} \tag{3.6}$$

Note that ε can be made arbitrarily small by taking r sufficiently large independently of $\|u\|_{H^{s/2}}$. Hence it remains only to show that the second term also converges to 0 uniformly in $u \in H^{s/2}$. Assumption (2.2) implies that

$$\begin{aligned} &\int_{|t|<r^{-\alpha}} \widehat{\varphi_r}(t) \left| \int_{|\xi|>R} \overline{\widehat{\varphi_r u}}(\xi)\widehat{u}(\xi-t)(m(\xi)-m(\xi-t))\,\mathrm{d}\xi \right| \mathrm{d}t \\ &\qquad \le \int_{|t|<r^{-\alpha}} \widehat{\varphi_r}(t) \int_{|\xi|>R} |\overline{\widehat{\varphi_r u}}(\xi)\widehat{u}(\xi-t)||m(\xi)-m(\xi-t)|\,\mathrm{d}\xi\,\mathrm{d}t \\ &\qquad \le \int_{|t|<r^{-\alpha}} \widehat{\varphi_r}(t)k(t) \int_{|\xi|>R} |\overline{\widehat{\varphi_r u}}(\xi)\widehat{u}(\xi-t)||\xi|^s\,\mathrm{d}\xi\,\mathrm{d}t \\ &\qquad \le \|u\|_{H^{s/2}}^2 \int_{|t|<r^{-\alpha}} \widehat{\varphi_r}(t)k(t)\,\mathrm{d}t \\ &\qquad \le \left(\sup_{|t|<r^{-\alpha}} k(t) \right) \|\widehat{\varphi}\|_{L^1} \|u\|_{H^{s/2}}^2. \end{aligned} \tag{3.7}$$

By assumption, $\lim_{r\to\infty} \sup_{|t|<r^{-\alpha}} k(t) = 0$. This proves the first part when m is continuous. Now let m have a finite number of discontinuities. The inequalities (3.6) and (3.7) fail in an interval of length $2r^{-\alpha}$ around each discontinuity. As the number of discontinuities is finite, the total measure of the set where the inequalities fail therefore goes to 0 as $r \to \infty$. Hence the result holds also in this case.

To prove the result for $\left|\int_{\mathbb{R}} \psi_r u(L(\psi_r u) - \psi_r Lu)\,\mathrm{d}x\right|$, note that without loss of generality it can be assumed that $\psi_r = 1 - \varphi_r$. Then

$$\begin{aligned}\int_{\mathbb{R}} \psi_r u(L(\psi_r u) - \psi_r Lu)\,\mathrm{d}x = &\int_{\mathbb{R}} \varphi_r u(L(\varphi_r u) - \varphi_r Lu)\,\mathrm{d}x \\ &- \int_{\mathbb{R}} u(L(\varphi_r u) - \varphi_r Lu)\,\mathrm{d}x.\end{aligned}$$

The first integral on the right-hand side is exactly what we had above, while the second integral can be written as in (3.5) with $\overline{\widehat{\varphi_r u}}(\xi)$ replaced by $\overline{\widehat{u}}(\xi)$, which does not change the estimates. □

Using Theorem 3.6 we are able to prove the equivalent of Lemma 2.15 in [19] for a much larger class of operators L, in particular extending the result to operators of order $0 < s < 1$.

Lemma 3.7. *Assume the dichotomy alternative holds for ρ_n. Then for each $\varepsilon > 0$ there is a subsequence of $\{u_n\}_n$, still denoted $\{u_n\}_n$, a real number $\bar{\lambda} = \bar{\lambda}(\varepsilon)$, $N \in \mathbb{N}$ and two sequences $\{u_n^{(1)}\}_n, \{u_n^{(2)}\}_n \subset H^{s/2}$ satisfying for all $n \geq N$:*

$$|\mathcal{U}(u_n^{(1)}) - \bar{\lambda}| \leq \varepsilon, \tag{3.8a}$$

$$|\mathcal{U}(u_n^{(2)}) - (\lambda - \bar{\lambda})| < \varepsilon, \tag{3.8b}$$

$$|\mathcal{J}_\kappa(u_n) - \mathcal{J}_\kappa(u_n^{(1)}) - \mathcal{J}_\kappa(u_n^{(2)})| < \varepsilon. \tag{3.8c}$$

Furthermore,

$$|\mathcal{J}_\kappa(u_n^{(1)}) - \bar{\mu}| \leq \varepsilon \tag{3.9}$$

and

$$|\mathcal{J}_\kappa(u_n^{(2)}) - (\mu - \bar{\mu})| \leq \varepsilon, \tag{3.10}$$

where $\bar{\mu}$ is defined as in Lemma 2.6.

Proof. By assumption we can for every $\varepsilon > 0$ find a number $N \in \mathbb{N}$ and sequences of positive functions $\{\rho_n^{(1)}\}_n$ and $\{\rho_n^{(2)}\}_n$ satisfying the properties (2.13). In addition, we may assume (see [16]) that $\{\rho_n^{(1)}\}_n$ and $\{\rho_n^{(2)}\}_n$ satisfy

$$\begin{aligned}&\operatorname{supp} \rho_n^{(1)} \subset (y_n - R_n, y_n + R_n), \\ &\operatorname{supp} \rho_n^{(2)} \subset (-\infty, y_n - 2R_n) \cup (y_n + 2R_n, \infty),\end{aligned}$$

where $y_n \in \mathbb{R}$ and $R_n \to \infty$. Then

$$\int_{R_n \leq |x - y_n| \leq 2R_n} \rho_n\,\mathrm{d}x \leq \varepsilon. \tag{3.11}$$

Choose φ, ψ as in Theorem 3.6, satisfying $\varphi^2 + \psi^2 = 1$ in addition, and define

$$\begin{aligned}\varphi_n(x) &= \varphi((x - y_n)/R_n), \\ \psi_n(x) &= \psi((x - y_n)/R_n)\end{aligned}$$

and set $u_n^{(1)} = \varphi_n u_n$, $u_n^{(2)} = \psi_n u_n$. Since $\mathcal{U}(u_n^{(1)})$ is uniformly bounded for all $n \in \mathbb{N}$, there exists a subsequence of $\{u_n^{(1)}\}_n$, still denoted $\{u_n^{(1)}\}_n$, and a real number $\bar{\lambda} = \bar{\lambda}(\varepsilon)$ such that $\mathcal{U}(u_n^{(1)}) \to \bar{\lambda}$. This implies that (3.8a) holds for sufficiently large n.

To prove (3.8b), we write

$$\begin{aligned}
\int_{\mathbb{R}} F(u_n)\,\mathrm{d}x &= \int_{|x-y_n|\leq R_n} F(u_n)\,\mathrm{d}x + \int_{|x-y_n|\geq 2R_n} F(u_n)\,\mathrm{d}x \\
&\quad + \int_{R_n\leq|x-y_n|\leq 2R_n} F(u_n)\,\mathrm{d}x \\
&= \int_{|x-y_n|\leq R_n} F(u_n^{(1)})\,\mathrm{d}x + \int_{|x-y_n|\geq 2R_n} F(u_n^{(2)})\,\mathrm{d}x \\
&\quad + \int_{R_n\leq|x-y_n|\leq 2R_n} F(u_n)\,\mathrm{d}x \\
&= \int_{\mathbb{R}} F(u_n^{(1)})\,\mathrm{d}x + \int_{\mathbb{R}} F(u_n^{(2)})\,\mathrm{d}x \\
&\quad + \int_{R_n\leq|x-y_n|\leq 2R_n} F(u_n) - F(u_n^{(1)}) - F(u_n^{(2)})\,\mathrm{d}x.
\end{aligned} \tag{3.12}$$

By the Sobolev embedding theorem, (3.11) implies

$$\int_{R_n\leq|x-y_n|\leq 2R_n} |u_n|^{p+1}\,\mathrm{d}x \leq C\varepsilon^{2/(p+1)},$$

where $C > 0$ is independent of n. This implies that the last line of (3.12) can be made less than ε by taking n large enough. Thus $|\mathcal{U}(u_n) - \mathcal{U}(u_n^{(1)}) - \mathcal{U}(u_n^{(2)})| \leq \varepsilon$ and (3.8b) follows. To prove (3.8c), note that

$$\begin{aligned}
&\mathcal{J}_\kappa(u_n^{(1)}) + \mathcal{J}_\kappa(u_n^{(2)}) = \\
&= \int_{\mathbb{R}} \varphi_n^2 u_n L u_n\,\mathrm{d}x + \int_{\mathbb{R}} \varphi_n u_n (L(\varphi_n u_n) - \varphi_n L u_n)\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}} \psi_n^2 u_n L u_n\,\mathrm{d}x + \int_{\mathbb{R}} \psi_n u_n (L(\psi_n u_n) - \psi_n L u_n)\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}} (\varphi_n^2 + \psi_n^2)\kappa u_n^2\,\mathrm{d}x.
\end{aligned}$$

By Theorem 3.6 and Lemma 3.1,

$$\left| \int_{\mathbb{R}} \varphi_n u_n (L(\varphi_n u_n) - \varphi_n L u_n)\,\mathrm{d}x \right| \leq \beta(R_n)\|u_n\|_{H^{s/2}}^2 \leq \beta(R_n) M,$$

with an equivalent bound for $\left|\int_{\mathbb{R}} \psi_n u_n(L(\psi_n u_n) - \psi_n L u_n)\,\mathrm{d}x\right|$. As $\beta(R_n) \to 0$, N can therefore be chosen sufficiently big so that

$$\mathcal{J}_\kappa(u_n) - \varepsilon \leq \mathcal{J}_\kappa(u_n^{(1)}) + \mathcal{J}_\kappa(u_n^{(2)}) \leq \mathcal{J}_\kappa(u_n) + \varepsilon,$$

for all $n \geq N$.

To prove (3.9), we first write

$$\begin{aligned}\left(L^{\frac{1}{2}} u_n^{(1)}\right)^2 &= \left(L^{\frac{1}{2}}(\varphi_n u_n) - \varphi_n L^{\frac{1}{2}} u_n\right)^2 \\ &\quad + 2\varphi_n L^{\frac{1}{2}} u_n \left(L^{\frac{1}{2}}(\varphi_n u_n) - \varphi_n L^{\frac{1}{2}} u_n\right) + \varphi_n^2 \left(L^{\frac{1}{2}} u_n\right)^2.\end{aligned} \tag{3.13}$$

Theorem 3.6 holds equally well for $L^{\frac{1}{2}}$, and so N can be taken sufficiently large so that

$$\int_{\mathbb{R}} \left(L^{\frac{1}{2}} u_n^{(1)}\right)^2 \mathrm{d}x = \int_{\mathbb{R}} \varphi_n^2 \left(L^{\frac{1}{2}} u_n\right)^2 \mathrm{d}x + \mathcal{O}(\varepsilon),$$

for all $n \geq N$. Thus we can write

$$\begin{aligned}\mathcal{J}_\kappa(u_n^{(1)}) &\geq \int_{\mathbb{R}} \kappa (u_n^{(1)})^2 + \left(L^{\frac{1}{2}} u_n^{(1)}\right)^2 \mathrm{d}x \\ &= \int_{\mathbb{R}} \varphi_n^2 \rho_n \,\mathrm{d}x + \mathcal{O}(\varepsilon) \\ &= \int_{|x-y_n|\leq R_n} \rho_n \,\mathrm{d}x + \int_{R_n \leq |x-y_n| \leq 2R_n} \varphi_n^2 \rho_n \,\mathrm{d}x + \mathcal{O}(\varepsilon) \\ &= \int_{\mathbb{R}} \rho_n^{(1)} + \mathcal{O}(\varepsilon) \\ &\geq \bar{\mu} + \mathcal{O}(\varepsilon),\end{aligned}$$

where the fourth line follows from (2.13) and our assumptions on the support of $\rho_n^{(1)}$. This proves (3.9). To prove (3.10) we proceed similarly, noting that (3.13) holds for $u_n^{(2)}$ and ψ_n, and that Theorem 3.6 still applies in this case. We get

$$\begin{aligned}\mathcal{J}_\kappa(u_n^{(2)}) &\geq \int_{\mathbb{R}} \kappa (u_n^{(2)})^2 + \left(L^{\frac{1}{2}} u_n^{(2)}\right)^2 \mathrm{d}x \\ &= \int_{\mathbb{R}} \psi_n^2 \rho_n \,\mathrm{d}x + \mathcal{O}(\varepsilon) \\ &= \int_{|x-y_n|\geq 2R_n} \rho_n \,\mathrm{d}x + \int_{R_n \leq |x-y_n| \leq 2R_n} \psi_n^2 \rho_n \,\mathrm{d}x + \mathcal{O}(\varepsilon) \\ &= \int_{\mathbb{R}} \rho_n^{(2)} + \mathcal{O}(\varepsilon) \\ &\geq \mu - \bar{\mu} + \mathcal{O}(\varepsilon).\end{aligned}$$

□

As a consequence of Lemma 3.7, we have the following result from [19]:

Lemma 3.8. *Assume that dichotomy holds for ρ_n. Then there exists $\lambda_1 \in (0, \lambda)$ such that*

$$\Gamma_\lambda \geq \Gamma_{\lambda_1} + \Gamma_{\lambda-\lambda_1}.$$

Proof. This is Lemma 2.16 in [19], and having established Lemma 3.7 and Lemma 3.2, the proof presented there holds without modification in the present case. □

As an immediate consequence of 3.8 we can preclude dichotomy.

Corollary 3.9. *Dichotomy does not occur.*

Proof. This follows directly from Lemma 3.3 and Lemma 3.8. □

With vanishing and dichotomy excluded we are able to state the main existence result of this section:

Lemma 3.10. *The set G_λ is non-empty for every $\lambda > 0$. Moreover, for every minimizing sequence $\{u_n\}_n$, there exists a sequence of real numbers $\{y_n\}_n$ such that $\{u_n(\cdot + y_n)\}_n$ has a subsequence that converges in $H^{s/2}$ to an element $w \in G_\lambda$.*

Proof. Let $\{u_n\}_n$ be a minimizing sequence. From Lemmas 2.6, 3.5 and Corollary 3.9 we know that compactness occurs. That is, there exists a subsequence of $\{u_n\}_n$, which we denote by $\{u_n\}_n$, and a sequence of real numbers $\{y_n\}_n$ such that for any $\varepsilon > 0$, one can find $R > 0$ for which

$$\int_{|x-y_n|\leq R} \rho_n \, \mathrm{d}x \geq \mu - \varepsilon$$

for all n, which implies

$$\varepsilon \geq \int_{|x-y_n|\geq R} \rho_n \, \mathrm{d}x \geq \int_{|x-y_n|\geq R} \kappa u_n^2 \, \mathrm{d}x.$$

We define $\tilde{u}_n(x) = u_n(x + y_n)$. Then

$$\kappa \int_{|x|\geq R} \tilde{u}_n^2 \, \mathrm{d}x \leq \varepsilon. \tag{3.14}$$

Thus, for every $k \in \mathbb{N}$, there exists an R_k such that

$$\int_{|x|\geq R_k} \tilde{u}_n^2 \, \mathrm{d}x \leq \frac{1}{k}.$$

Furthermore $\{\tilde{u}_n\}_n$ is bounded in $H^{s/2}$, so for every $k \in \mathbb{N}$ there exists a $w_k \in L^2([-R_k, R_k]^c)$ and a subsequence of $\{\tilde{u}_n\}_n$, denoted $\{\tilde{u}_{k,n}\}_n$, such that $\tilde{u}_{k,n} \to w_k$ in $L^2([-R_k, R_k]^c)$ and

$$\int_{|x|\geq R_k} \tilde{u}_{k,n}^2 \, \mathrm{d}x \leq \frac{1}{k} \tag{3.15}$$

for all n. A Cantor diagonalization argument on the sequences $\{\tilde{u}_{k,n}\}_n$ yields a subsequence of $\{\tilde{u}_n\}_n$, still denoted by $\{\tilde{u}_n\}_n$, that converges strongly in L^2 to some function $w \in L^2$. Furthermore, $\{\tilde{u}_n\}_n$ converges weakly in $H^{s/2}$ by the Banach-Alaoglu theorem, and so $w \in H^{s/2}$ as well. The L^2 and weak $H^{s/2}$ convergence implies L^{p+1} convergence:

$$\begin{aligned}\|\tilde{u}_n - w\|_{L^{p+1}} &\leq C\|\tilde{u}_n - w\|_{H^{(p-1)/2(p+1)}} \\ &\leq C\|\tilde{u}_n - w\|_{H^0}^{((p+1)s-p+1)/(s(p+1))}\|\tilde{u}_n - w\|_{H^{s/2}}^{(p-1)/(s(p+1)} \\ &\leq C'\|\tilde{u}_n - w\|_{L^2}^{((p+1)s-p+1)/(s(p+1))}.\end{aligned} \tag{3.16}$$

Recall that we are assuming $1 < p < \frac{1+s}{1-s}$, and so $\frac{p-1}{2(p+1)} < \frac{s}{2}$, which makes the above use of the Sobolev interpolation inequality valid.

Thus

$$\mathcal{U}(w) = \lambda. \tag{3.17}$$

By weak lower semi-continuity of the Hilbert norm we have

$$\mathcal{J}_\kappa(w) \leq \liminf \mathcal{J}_\kappa(\tilde{u}_n) = \Gamma_\lambda,$$

hence, by (3.17) and the definition of Γ_λ,

$$\mathcal{J}_\kappa(w) = \Gamma_\lambda$$

and $w \in G_\lambda$. The above equations and remarks imply $\tilde{u}_n \to w$ in $H^{s/2}$. □

Now observe that if f satisfies (B2), then $\{u_n\}_n$ is minimizing sequence for Γ_λ if and only if $\{-u_n\}_n$ is a minimizing sequence for $\Gamma_{-\lambda}$. Recalling the calculations regarding the Lagrange multiplier at the beginning of the section and the scalings (2.8) and (2.9), we have proven Theorem 2.1 (ii).

4. Concentration-compactness for (2.5)

In this section we will prove existence of minimizers of

$$I_q := \inf\{\mathcal{E}(w) : w \in H^{s/2} \text{ and } \mathcal{Q}(w) = q\},$$

for $q > 0$. The basic approach to finding minimizers of I_q is the same as for Γ_λ, and many of the arguments above can be reused and/or modified to the present case. In this section we assume that $p \in (1, 2s+1)$.

Lemma 4.1. *For all $q > 0$, one has $-\infty < I_q$. Moreover, there exists a number $q_0 \geq 0$ such that for all $q > q_0$,*

$$-\infty < I_q < 0.$$

If $0 \leq m(\xi) \leq A_2|\xi|^s$ for $|\xi| \leq 1$, then the statement holds for $q_0 = 0$.

Proof. To prove $I_q > -\infty$, we use the Sobolev embedding and interpolation theorems to obtain

$$\left|\int_{\mathbb{R}} F(\varphi)\,\mathrm{d}x\right| \leq C\|\varphi\|_{H^{(p-1)/(2(p+1))}}^{p+1} \leq C\|\varphi\|_{H^0}^{((p+1)s-p+1)/s}\|\varphi\|_{H^{s/2}}^{(p-1)/s}.$$

Thus

$$\begin{aligned}\mathcal{E}(\varphi) &= \mathcal{E}(\varphi) + \mathcal{Q}(\varphi) - \mathcal{Q}(\varphi)\\ &= \frac{1}{2}\int_{\mathbb{R}} \varphi L\varphi + \varphi^2\,\mathrm{d}x - \int_{\mathbb{R}} F(\varphi)\,\mathrm{d}x - q \qquad (4.1)\\ &\geq C_1\|\varphi\|_{H^{s/2}}^2 - C_2 q^{((p+1)s-p+1)/s}\|\varphi\|_{H^{s/2}}^{(p-1)/s} - q\end{aligned}$$

where $C_1, C_2 > 0$ depend only on the symbol m and the Sobolev embedding constant, respectively. By assumption $(p-1)/s < 2$, hence the growth of the term with negative sign in the last line of (4.1) is bounded by the growth of the positive term, and it follows that $I_q > -\infty$.

To prove that $I_q < 0$ for all q big enough, choose $\varphi \in H^{s/2}$ such that $F(\varphi)$ is non-negative. This can be done by taking φ to be non-positive if $c_p < 0$, and φ non-negative if $c_p > 0$. For each $q > 0$, there exists a number $a = a(q) > 0$ such that $\mathcal{Q}(a(q)\varphi) = q$. Then

$$I_q \leq \mathcal{E}(a(q)\varphi) = \frac{a(q)^2}{2}\int_{\mathbb{R}} \varphi L\varphi\,\mathrm{d}x - a(q)^{p+1}\int_{\mathbb{R}} F(\varphi)\,\mathrm{d}x.$$

Note that $a(q) \to \infty$ as $q \to \infty$ and that $p+1 > 2$. Hence the right-hand side in the equation above will be negative for all q large enough. This proves the existence of a $q_0 \geq 0$ as stated.

Assume now that, in addition to (A), $0 \leq m(\xi) \leq A_2|\xi|^s$ for $|\xi| \leq 1$. Let $q > 0$ and again choose $\varphi \in H^{s/2}$ such that $F(\varphi)$ is non-negative and $\mathcal{Q}(\varphi) = q$. For $t > 0$, set $\varphi_t(x) = \sqrt{t}\varphi(tx)$. Then $\mathcal{Q}(\varphi_t) = q$ for all $t > 0$,

$$\int_{\mathbb{R}} F(\varphi_t)\,\mathrm{d}x = t^{(p-1)/2}\int_{\mathbb{R}} F(\varphi)\,\mathrm{d}x,$$

and

$$\int_{\mathbb{R}} \varphi_t L\varphi_t\,\mathrm{d}x = \int_{\mathbb{R}} m(t\xi)|\widehat{\varphi}(\xi)|^2\,\mathrm{d}\xi \leq t^s A_2\int_{\mathbb{R}} |\xi|^s|\widehat{\varphi}(\xi)|^2\,\mathrm{d}\xi \leq t^s A_2\|\varphi\|_{H^{s/2}}^2.$$

As $p \in (1, 2s+1)$ by assumption, we get $(p-1)/2 < s$, so that $t^s \to 0$ faster than $t^{(p-1)/2} \to 0$ as $t \to 0^+$. As $F(\varphi)$ is non-negative, it follows from the calculations above that for $t > 0$ sufficiently small, $\int_{\mathbb{R}} F(\varphi_t)\,\mathrm{d}x > \int_{\mathbb{R}} \varphi_t L\varphi_t\,\mathrm{d}x$, which implies that $I_q \leq \mathcal{E}(\varphi_t) < 0$. As $q > 0$ was arbitrary, this proves the statement. $\square$

Lemma 4.2. *If $\{u_n\}_n$ is a minimizing sequence for I_q, then*

(i) $\|u_n\|_{H^{s/2}} \leq K$ *for some constant $K > 0$ and all n,*

(ii) *if* $I_q < 0$*, then* $\|u_n\|_{p+1} \geq \delta$ *for some constant* $\delta > 0$ *and all sufficiently large* n*.*

In particular, (ii) holds if $q > q_0$ *for* q_0 *as in Lemma 4.1.*

Proof. By Assumption (A), Pareseval's inequality and the Sobolev embedding and interpolation theorems, we have, for some constant θ depending only on m, that

$$\begin{aligned}\theta\|u_n\|^2_{H^{s/2}} &\leq \mathcal{E}(u_n) + Q(u_n) + \int_{\mathbb{R}} F(u_n)\,\mathrm{d}x \\ &\leq \sup_n \mathcal{E}(u_n) + q + C\|u_n\|^{p+1}_{H^{(p-1)/(2(p+1))}} \\ &\leq C' + Cq^{((p+1)s-p+1)/s}\|u_n\|^{(p-1)/s}_{H^{s/2}}.\end{aligned}$$

By assumption, $(p-1)/2 < 2$, and so we have bounded the square of $\|u_n\|_{H^{s/2}}$ by a smaller power, and the existence of a bound K follows. To prove statement 2 we argue by contradiction. If no such constant δ exists, then

$$\liminf_{n\to\infty} \int_{\mathbb{R}} F(u_n)\,\mathrm{d}x \leq 0,$$

which implies

$$\begin{aligned}I_q &= \lim_{n\to\infty}\left(\frac{1}{2}\int_{\mathbb{R}} u_n L u_n\,\mathrm{d}x - \int_{\mathbb{R}} F(u_n)\,\mathrm{d}x\right) \\ &\geq \liminf_{n\to\infty}\left(-\int_{\mathbb{R}} F(u_n)\,\mathrm{d}x\right) \geq 0,\end{aligned}$$

contradicting the assumption that $I_q < 0$. □

Lemma 4.3. *For all* $q_1, q_2 > 0$ *such that* $I_{q_1+q_2} < 0$*, one has*

$$I_{(q_1+q_2)} < I_{q_1} + I_{q_2}.$$

Proof. We start by claiming that if $q > 0$ and $I_q < 0$, then for $t > 1$

$$I_{tq} < tI_q.$$

To see this let $\{u_n\}$ be a minimizing sequence for I_q and define $\tilde{u}_n = \sqrt{t}u_n$ for all n, so that $\mathcal{Q}(\tilde{u}_n) = tq$ and hence $\mathcal{E}(\tilde{u}_n) \geq I_{tq}$ for all n. Then for all n we have

$$I_{tq} \leq \frac{1}{2}\int_{\mathbb{R}} \tilde{u}_n L\tilde{u}_n\,\mathrm{d}x - \int_{\mathbb{R}} F(\tilde{u}_n)\,\mathrm{d}x = t\mathcal{E}(u_n) + (t - t^{(p+1)/2})\int_{\mathbb{R}} F(u_n)\,\mathrm{d}x.$$

Now taking $n \to \infty$ and using Lemma 4.2, we obtain

$$I_{tq} \leq tI_q + \frac{1}{2}(t - t^{(p+1)/2})\delta < tI_q$$

since $p + 1 > 2$ and $t > 1$. If $I_{q_1}, I_{q_2} \geq 0$ the statement is trivial. Assume therefore that, say, $I_{q_1} < 0$. Then the claim above holds for I_{q_1} and we can argue as in the proof of Lemma 3.3. □

We note that for any minimizing sequence $\{u_n\}_n$ of I_q, the sequence $\{\frac{1}{2}u_n^2\}_n$ satisfies the conditions for Lemma 2.6 with $\mu = q$. We will now preclude vanishing and dichotomy, and thereby prove that compactness occurs.

Lemma 4.4. *If $I_q < 0$, vanishing does not occur.*

Proof. From Lemmas 4.2 and 3.4 we conclude that there exists a number $\eta > 0$ and an integer $N \in \mathbb{N}$ such that

$$\sup_{y\in\mathbb{R}} \int_{y-2}^{y+2} |u_n|^{p+1} \,\mathrm{d}x \geq \eta$$

for all $n > N$. The result now follows from standard embedding and interpolation arguments. ☐

Lemma 4.5. *Assume dichotomy occurs. Then for each $\varepsilon > 0$ there is a subsequence of $\{u_n\}_n$, still denoted $\{u_n\}_n$, a real number $\bar{q} \in (0, q)$, $N \in \mathbb{N}$ and two sequences $\{u_n^{(1)}\}_n, \{u_n^{(2)}\}_n \subset H^{s/2}$ satisfying for all $n \geq N$:*

$$|\mathcal{Q}(u_n^{(1)}) - \bar{q}| < \varepsilon, \tag{4.2a}$$

$$|\mathcal{Q}(u_n^{(2)}) - (q - \bar{q})| < \varepsilon, \tag{4.2b}$$

$$\mathcal{E}(u_n) \geq \mathcal{E}(u_n^{(1)}) + \mathcal{E}(u_n^{(2)}) + \varepsilon. \tag{4.2c}$$

Proof. This proof follows along the lines of the proof of Lemma 3.7, but we present it in detail for the sake of clarity as some different argumentation are needed.

As noted in Lemma 3.7, we can, by assumption, for $\varepsilon > 0$ find a number $N \in \mathbb{N}$ and sequences $\{\rho_n^{(1)}\}_n$ and $\{\rho_n^{(2)}\}_n$ of positive functions satisfying the properties (2.13), where $\rho_n = \frac{1}{2}u_n^2$, $\mu = q$ and $\bar{q} = \bar{\mu}$. We may assume that $\{\rho_n^{(1)}\}_n$ and $\{\rho_n^{(2)}\}_n$ satisfy

$$\begin{aligned} &\operatorname{supp} \rho_n^{(1)} \subset (y_n - R_n, y_n + R_n), \\ &\operatorname{supp} \rho_n^{(2)} \subset (-\infty, y_n - 2R_n) \cup (y_n + 2R_n, \infty), \end{aligned}$$

where $y_n \in \mathbb{R}$ and $R_n \to \infty$. Then

$$\frac{1}{2} \int_{R_n \leq |x-y_n| \leq 2R_n} u_n^2 \,\mathrm{d}x \leq \varepsilon, \tag{4.3}$$

for all $n \geq N$. Now choose φ, ψ as in Theorem 3.6, satisfying $\varphi^2 + \psi^2 = 1$ in addition, and define $\varphi_n(x) = \varphi((x - y_n)/R_n)$, $\psi_n(x) = \psi((x - y_n)/R_n)$,

$u_n^{(1)} = \varphi_n u_n$ and $u_n^{(2)} = \psi_n u_n$. By the definitions of $u_n^{(1)}$ and $\rho_n^{(1)}$,

$$\begin{aligned}\left|\mathcal{Q}\left(u_n^{(1)}\right) - \int_{\mathbb{R}} \rho_n^{(1)}\,\mathrm{d}x\right| =& \int_{|x-y_n|\le R_n} |\frac{1}{2}u_n^2 - \rho_n^{(1)}|\,\mathrm{d}x \\ &+ \frac{1}{2}\int_{R_n\le|x-y_n|\le 2R_n} \varphi_n^2 u_n^2\,\mathrm{d}x \\ \le& \varepsilon + \frac{1}{2}\int_{R_n\le|x-y_n|\le 2R_n} u_n^2\,\mathrm{d}x \le 2\varepsilon,\end{aligned}$$

for all $n \ge N$, where the last inequality follows from (4.3). By definition $\left|\int_{\mathbb{R}} \rho_n^{(1)}\,\mathrm{d}x - \bar{q}\right| \le \varepsilon$ and we get (4.2a) by repeating the procedure for $\varepsilon/3$. Comparing $\mathcal{Q}\left(u_n^{(2)}\right)$ to $\int_{\mathbb{R}} \rho_n^{(2)}\,\mathrm{d}x$, we obtain (4.2b) by exactly the same arguments. To prove (4.2c), we write

$$\begin{aligned}&\mathcal{E}(u_n^{(1)}) + \mathcal{E}(u_n^{(2)}) = \\ &=\frac{1}{2}\left[\int_{\mathbb{R}} \varphi_n^2 u_n L u_n\,\mathrm{d}x + \int_{\mathbb{R}} \varphi_n u_n (L(\varphi_n u_n) - \varphi_n L u_n)\,\mathrm{d}x\right] \\ &\quad + \frac{1}{2}\left[\int_{\mathbb{R}} \psi_n^2 u_n L u_n\,\mathrm{d}x + \int_{\mathbb{R}} \psi_n u_n (L(\psi_n u_n) - \psi_n L u_n)\,\mathrm{d}x\right] \\ &\quad - \int_{\mathbb{R}} \left(\varphi_n^2 + \psi_n^2\right) F(u_n)\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}} \left[(\varphi_n^2 - \varphi_n^{p+1}) + (\psi_n^2 - \psi_n^{p+1})\right] F(u_n)\,\mathrm{d}x.\end{aligned}$$

It follows from Theorem 3.6 and Lemma 4.2 that by taking n sufficiently large (so that R_n is large enough), we get

$$\mathcal{E}(u_n^{(1)}) + \mathcal{E}(u_n^{(2)}) \le \mathcal{E}(u_n) + \varepsilon + \int_{\mathbb{R}} \left[(\varphi_n^2 - \varphi_n^{p+1}) + (\psi_n^2 - \psi_n^{p+1})\right] F(u_n)\,\mathrm{d}x.$$

For $|x - y_n| \notin (R_n, 2R_n)$ we have, by our choice of φ and ψ, that $\varphi_n^2 = \varphi_r^{p+1}$ and $\psi_r^2 = \psi_r^{p+1}$. Thus

$$\begin{aligned}\left|\int_{\mathbb{R}} \left[(\varphi_n^2 - \varphi_n^{p+1}) + (\psi_n^2 - \psi_n^{p+1})\right] F(u_n)\,\mathrm{d}x\right| &\le \int_{R_n\le|x-y_n|\le 2R_n} 2\,|F(u_n)|\,\mathrm{d}x \\ &\le CK^{(p-1)/s}\varepsilon^{((p+1)s-p+1)/s},\end{aligned}$$

where we used the usual Sobolev embedding and interpolation theorems, combined with the boundedness of u_n in $H^{s/2}$ and (4.3). As $((p+1)s - p + 1)/s > 0$, this proves that there is an N such that (4.2c) holds for all $n \ge N$. □

Corollary 4.6. *Dichotomy does not occur.*

Proof. Assume to the contrary that dichotomy occurs for a minimizing sequence $\{u_n\}_n$. Then by Lemma 4.5 there is for every $\varepsilon > 0$, a subsequence of $\{u_n\}_n$, still denoted $\{u_n\}_n$, a number $\bar{q} \in (0, q)$, $N \in \mathbb{N}$ and two sequences $\{u_n^{(1)}\}_n, \{u_n^{(2)}\}_n \subset H^{s/2}$ such that (4.2a)-(4.2c) are satisfied for all $n \geq N$. Then

$$\begin{aligned} I_q &= \liminf_{n\to\infty} \mathcal{E}(u_n) \\ &\geq \liminf_{n\to\infty} \mathcal{E}(u_n^{(1)}) + \mathcal{E}(u_n^{(2)}) + \varepsilon \\ &\geq I_{\bar{q}\pm\varepsilon} + I_{(q-\bar{q})\pm\varepsilon} + \varepsilon. \end{aligned}$$

Taking $\varepsilon \to 0^+$, we get a contradiction with Lemma 4.3. □

Now we are able to present the main existence result of this section.

Lemma 4.7. *Let $\{u_n\}_n$ be a minimizing sequence for I_q. If $I_q < 0$, then there exists a sequence $\{y_n\}_n \subset \mathbb{R}$ such that the sequence $\{\tilde{u}_n\}_n$ defined by $\tilde{u}_n(x) = u_n(x + y_n)$ has a subsequence that converges in $H^{s/2}$ to a minimizer of I_q. In particular, there is a $q_0 \geq 0$ such that for all $q > q_0$ the set of minimizers is non-empty.*

Proof. Let $\{u_n\}_n$ be a minimizing sequence for I_q. By Lemma 4.4 and Corollary 4.6 we know that compactness occurs. That is, there is a subsequence of $\{u_n\}_n$, denoted $\{u_n\}_n$, and a sequence $\{y_n\} \subset \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $0 < r < \infty$ satisfying for all $n \in \mathbb{N}$:

$$\frac{1}{2}\int_{|x-y_n|\leq r} u_n^2 \,\mathrm{d}x \geq q - \varepsilon.$$

This implies that for every $k \in \mathbb{N}$ we can find $r_k \in \mathbb{R}_+$ so that

$$\frac{1}{2}\int_{|x|\leq r_k} \tilde{u}_n^2 \,\mathrm{d}x \geq q - \frac{1}{k}.$$

Furthermore, by Lemma 4.2 and the Rellich-Kondrachov theorem, for every $k \in \mathbb{N}$, there is a subsequence of $\{\tilde{u}_n\}_n$, denoted $\{\tilde{u}_{k,n}\}_n$, and a function $w_k \in L^2([-r_k, r_k])$ such that $\tilde{u}_{k,n} \to w_k$ in $L^2([-r_k, r_k])$. From the inequalities above, we deduce that $\mathcal{Q}(w_k) \geq q - \frac{1}{k}$. Now the arguments in the proof of Lemma 3.10, involving a Cantor diagonalization argument, can be straightforwardly be applied to the present case.

Lemma 4.1 guarantees the existence of a $q_0 \geq 0$ such that the assumption $I_q < 0$ is satisfied for all $q > q_0$. □

It only remains to prove that minimizers of I_q solve (1.3) with positive wave speed c.

Lemma 4.8. *If $I_q < 0$, any minimizer of I_q is a solution to* (1.3) *with the wave speed $c > 0$ being the Lagrange multiplier.*

Proof. Let $w \in H^{s/2}$ be a minimizer of I_q. Then by the Lagrange multiplier principle there exists $\gamma \in \mathbb{R}$ such that

$$\mathcal{E}'(w) + \gamma \mathcal{Q}'(w) = 0,$$

where $\mathcal{E}'(w)$ and $\mathcal{Q}'(w)$ denote the Fréchet derivatives of $\mathcal{E}$ and $\mathcal{Q}$ at w. The the Fréchet derivatives $\mathcal{E}'(w)$ and $\mathcal{Q}'(w)$ are given by

$$\begin{aligned}\mathcal{E}'(w) &= Lw - f(w)\\ \mathcal{Q}'(w) &= w.\end{aligned}$$

Thus w solves (1.3) with $c = \gamma$. Now it remains to prove $c = \gamma > 0$. Note first that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{E}(w\theta)|_{\theta=1} &= \int_{\mathbb{R}} wLw \,\mathrm{d}x - (p+1)\int_{\mathbb{R}} F(w)\,\mathrm{d}x\\ &= 2\mathcal{E}(w) - (p-1)\int_{\mathbb{R}} F(w)\,\mathrm{d}x.\end{aligned}$$

But $\mathcal{E}(w) = I_q < 0$ and $\int_{\mathbb{R}} F(w)\,\mathrm{d}x > 0$, so that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{E}(w\theta)|_{\theta=1} < 0.$$

By the definition of Fréchet derivative,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{E}(w\theta)|_{\theta=1} &= \int_{\mathbb{R}} \mathcal{E}'(w)\cdot \frac{\mathrm{d}}{\mathrm{d}\theta}[w\theta]|_{\theta=1}\,\mathrm{d}x\\ &= -\gamma\int_{\mathbb{R}} \mathcal{Q}'(w)w\,\mathrm{d}x\\ &= -\gamma\int_{\mathbb{R}} w^2\,\mathrm{d}x,\end{aligned}$$

and thus $\gamma > 0$. □

This concludes the proof of Theorem 2.1 (i).

5. Concentration-compactness for inhomogeneous nonlinearities

In this section we will prove Theorem 2.1 (iii). That is, we will consider nonlinearities of the form $g(u) = u + f(u)$, where f is as in (B). Recall that we have set $\tilde{\mathcal{U}}(u) = \int_{\mathbb{R}} \frac{1}{2}u^2 + F(u)\,\mathrm{d}x$ and

$$\tilde{\Gamma}_\lambda = \inf\{\mathcal{J}(w) : w \in H^{s/2} \text{ and } \tilde{\mathcal{U}}(w) = \lambda\}.$$

For a minimizing sequence $\{u_n\}_n$ of $\tilde{\Gamma}_\lambda$ we will define the sequence $\{\rho_n\}_n$ as in Section 3. As before, we will show existence of minimizers by precluding vanishing and dichotomy and appeal to Lemma 2.6. To do so, we need the following Lemma from [19]:

Lemma 5.1. *There exists a real number $\lambda_0 \geq 0$ such that for all $\lambda > \lambda_0$, any minimizing sequence $\{u_n\}_n$ of $\tilde{\Gamma}_\lambda$ satisfies for sufficiently large n:*

$$\int_{\mathbb{R}} |u_n|^{p+1} \,\mathrm{d}x \geq \delta,$$

for some $\delta > 0$. If $0 \leq m(\xi) \leq A_2|\xi|^s$ for $|\xi| \leq 1$ and $p \in (1, 2s+1)$, then the statement holds for $\lambda_0 = 0$.

Proof. Observe that $\mathcal{J}(u) - \tilde{\mathcal{U}}(u) = \mathcal{E}(u)$ and any minimizing sequence for $\tilde{\Gamma}_\lambda$ is also a minimizing sequence for $\bar{\Gamma}_\lambda = \inf\{\mathcal{E}(u) : u \in H^{s/2} \text{ and } \tilde{\mathcal{U}}(u) = \lambda\}$. If $\bar{\Gamma}_\lambda < 0$ then the result follows from the proof of Lemma 4.2 (ii). That there exists $\lambda_0 \geq 0$ such that $\bar{\Gamma}_\lambda < 0$ for all $\lambda > \lambda_0$ can be proved exactly in the same way as $I_q < 0$ for $q > q_0$ was proved in Lemma 4.1: Choose $\varphi \in H^{s/2}$ such that $F(\varphi)$ is non-negative and let $a(\lambda) > 0$ be such that $\tilde{\mathcal{U}}(a(\lambda)\varphi) = \lambda$ for $\lambda > 0$. Then $a(\lambda) \to \infty$ as $\lambda \to \infty$. This proves the existence of a $\lambda_0 \geq 0$ as in the statement.

Assume now that $0 \leq m(\xi) \leq A_2|\xi|^s$ and let $1 < p < 2s+1$. Let $q > 0$. From the proof of Lemma 4.1, we know that for any $\varepsilon > 0$, we can find $\varphi \in H^{s/2}$ such that $\mathcal{Q}(\varphi) = q$, $F(\varphi)$ is non-negative, $\mathcal{E}(\varphi) < 0$ and $\int_{\mathbb{R}} F(\varphi) \,\mathrm{d}x < \varepsilon$. The latter inequality implies that $|\tilde{\mathcal{U}}(\varphi) - q| < \varepsilon$. As $\varepsilon > 0$ and $q > 0$ were arbitrary, this shows that $\bar{\Gamma}_\lambda < 0$ for every $\lambda > 0$ when $1 < p < 2s+1$. □

Lemma 5.1 illustrates the difference between homogeneous and inhomogeneous nonlinearities in (1.2), and that there is a change in behaviour for the inhomogeneous case at the critical exponent $p = 2s+1$.

Lemma 5.2. *Vanishing does not occur.*

Proof. Having established Lemma 5.1 the conditions in Lemma 3.4 are satisfied and the result follows from the proof of Lemma 3.5. □

Lemma 5.3. *If $\lambda > \lambda_0$ and $\theta > 1$ then $\tilde{\Gamma}_{\theta\lambda} < \theta\tilde{\Gamma}_\lambda$*

Proof. Let $\{u_n\}_n$ be a minimizing sequence for $\tilde{\Gamma}_\lambda$. Choose $\alpha_n > 0$ such that $\tilde{\mathcal{U}}(\alpha_n u_n) = \theta\lambda$. Since $\tilde{\mathcal{U}}(u_n) = \lambda$ we find

$$\alpha_n^2 = \theta - \frac{\alpha_n^2(\alpha_n^{p-1} - 1)}{\lambda} \int_{\mathbb{R}} F(u_n) \,\mathrm{d}x.$$

Thus

$$\begin{aligned} \tilde{\Gamma}_{\theta\lambda} &\leq \mathcal{J}(\alpha_n u_n) = \alpha_n^2 \mathcal{J}(u_n) \\ &= \left(\theta - \frac{\alpha_n^2(\alpha_n^{p-1} - 1)}{\lambda} \int_{\mathbb{R}} F(u_n) \,\mathrm{d}x\right) \mathcal{J}(u_n). \end{aligned}$$

Considering the proof of Lemma 5.1 we see that the statement of that lemma is true with $F(u_n)$ replacing $|u|^{p+1}$. Thus for there is a $\delta > 0$ such that $\int_{\mathbb{R}} F(u_n) \,\mathrm{d}x \geq \delta$ for all sufficiently large n. Furthermore, since $\theta > 1$ and

$\tilde{\mathcal{U}}(\alpha_n u_n) = \theta\lambda$ it is clear that $\alpha_n \geq 1 + \varepsilon$ for some $\varepsilon > 0$ for all sufficiently large n. It follows that

$$\theta - \frac{\alpha_n^2(\alpha_n^{p-1} - 1)}{\lambda} \int_{\mathbb{R}} F(u_n)\,\mathrm{d}x < \theta,$$

and so

$$\tilde{\Gamma}_{\theta\lambda} \leq \alpha_n^2 \mathcal{J}(u_n) < \theta\tilde{\Gamma}_\lambda.$$

□

Lemma 5.4. *If* $\lambda > \lambda_0$*,* $\lambda_1 > 0$*,* $\lambda_2 > 0$ *and* $\lambda_1 + \lambda_2 = \lambda$*, then* $\tilde{\Gamma}_\lambda < \tilde{\Gamma}_{\lambda_1} + \tilde{\Gamma}_{\lambda_2}$.

Proof. Observe that $\bar{\Gamma}_\lambda = \tilde{\Gamma}_\lambda - \lambda$ and it is sufficient to prove that $\bar{\Gamma}_\lambda < \bar{\Gamma}_{\lambda_1} + \bar{\Gamma}_{\lambda_2}$.

From the proof of Lemma 5.1 we conclude that $\bar{\Gamma}_\lambda < 0$. Hence if $\bar{\Gamma}_{\lambda_1}, \bar{\Gamma}_{\lambda_2} \geq 0$ the claim is trivial. Assume therefore that, say, $\bar{\Gamma}_{\lambda_1} < 0$. Then Lemma 5.3 holds for $\bar{\Gamma}_{\lambda_1}$. From here the result can be proven in the same fashion as in Lemma 4.3 or equation (3.3). □

Lemma 3.7 and Lemma 3.8 carry over to the present case straightforwardly, and from Lemma 5.4 we then conclude that dichotomy does not occur. The arguments in Lemma 3.10 then gives the existence of minimizers. This concludes the proof of Theorem 2.1 (iii).

6. Stability and Regularity

In this section we will prove Theorem 2.3 and discuss the regularity of solutions to equations (1.3), (1.4) and (2.11).

Proof of Theorem 2.3. We prove it only for D_q; the proofs for G_λ and $\tilde{G}_\lambda$ are equivalent. Assume the statement is false. That is, there exist a number $\varepsilon > 0$, a sequence $\{\varphi_n\}_n \subset H^{s/2}$, and a sequence of times $\{t_n\}_n \subset \mathbb{R}$ such that

$$\inf_{w \in D_q} \|\varphi_n - w\|_{H^{s/2}} < \frac{1}{n}$$

and

$$\inf_{w \in D_q} \|u_n(\cdot, t_n) - w\|_{H^{s/2}} > \varepsilon$$

for all n, where $u_n(x,t)$ solves (1.1) with $u_n(x,0) = \varphi_n(x)$. Since $\varphi_n \to D_q$ in $H^{s/2}$, $\mathcal{E}(w) = I_q$ and $\mathcal{Q}(w) = q$, we have $\mathcal{E}(\varphi_n) \to I_q$ and $\mathcal{Q}(\varphi_n) \to q$.

Choose a sequence $\{\alpha_n\}_n \subset \mathbb{R}$ such that $\mathcal{Q}(\alpha_n \varphi_n) = q$ for all $n \in \mathbb{N}$. Then $\alpha_n \to 1$. As $\mathcal{E}(u)$ and $\mathcal{Q}(u)$ are independent of t if u solves (1.1), the sequence $f_n := \alpha_n u_n(\cdot, t_n)$ satisfies $\mathcal{Q}(f_n) = q$ for all n and

$$\lim_{n\to\infty} \mathcal{E}(f_n) = \lim_{n\to\infty} \mathcal{E}(u_n(\cdot, t_n)) = \lim_{n\to\infty} \mathcal{E}(\varphi_n) = I_q.$$

The sequence $\{f_n\}_n$ is therefore a minimizing sequence for $\mathcal{E}$ and from Theorem 2.1 and the translation invariance of the functionals $\mathcal{E}$ and $\mathcal{Q}$ we deduce that there is an $N \in \mathbb{N}$ such that for $n \geq N$, there exists $w_n \in D_q$ satisfying

$$\|f_n - w_n\|_{H^{s/2}} \leq \frac{\varepsilon}{2}.$$

So for $n \geq N$ we have

$$\begin{aligned} \varepsilon &\leq \|u_n(\cdot, t_n) - w_n\|_{H^{s/2}} \\ &\leq \|u_n(\cdot, t_n) - f_n\|_{H^{s/2}} + \|f_n - w_n\|_{H^{s/2}} \\ &\leq |1 - \alpha_n| \|u_n(\cdot, t_n)\|_{H^{s/2}} + \frac{\varepsilon}{2}. \end{aligned}$$

Taking $n \to \infty$ we get $\varepsilon \leq \varepsilon/2$, a contradiction. □

Next we consider regularity of the solutions. Since $L : H^r \to H^{r-s}$ for all $r \in \mathbb{R}$, $Lu \in H^{-s/2}$ for $u \in H^{s/2}$ and the solutions we have found may be only distributional solutions of (1.3) and (1.4). However, the solutions inherit regularity from the equations themselves.

Lemma 6.1. *If f satisfies (B) with $p \in (1, \frac{1+s}{1-s})$, then any solution $u \in H^{s/2}$ of* (1.3)*,* (1.4) *or* (2.11) *is in L^∞.*

Proof. Let $p \in (1, \frac{1+s}{1-s})$ be given. Applying the Fourier transform on both sides of (1.3), (1.4) or (2.11) yields

$$(k + m(\xi))\widehat{u} = \widehat{f(u)},$$

for some constant $k > 0$ depending on the wave speed c. Without loss of generality, we may assume $k = 1$, and so

$$\widehat{u} = \frac{\widehat{f(u)}}{1 + m(\xi)}.$$

Since $u \in H^{s/2}$ we have $u \in L^q$ and $f(u) \in L^{\frac{q}{p}}$ for all $q \in [2, \frac{2}{1-s}]$. It follows that $\widehat{f(u)} \in L^q$ for all $\frac{2}{2-p(1-s)} \leq q \leq \infty$. By assumption (A), $(1 + m(\cdot))^{-1} \in L^q$ for all $q > 1/s$. Let $\varepsilon > 0$ be a number such that $\frac{2}{1+s+\varepsilon} \geq 1$. We have

$$\|\widehat{u}\|_{L^{2/(1+s+\varepsilon)}}^{2/(1+s+\varepsilon)} = \|\widehat{u}^{\frac{2}{1+s+\varepsilon}}\|_{L^1} \leq \|\widehat{f(u)}^{\frac{2}{1+s+\varepsilon}}\|_{L^r} \|(1 + m(\cdot))^{-\frac{2}{1+s+\varepsilon}}\|_{L^{r'}}, \tag{6.1}$$

where $r, r' > 0$ are such that $1/r + 1/r' = 1$. We choose the smallest r for which we can guarantee the first term on the right-hand side is finite. That is $\frac{2r}{1+s+\varepsilon} = \frac{2}{2-p(1-s)}$, which gives $r = \frac{1+s+\varepsilon}{2-p(1-s)}$. Then $r' = \frac{1+s+\varepsilon}{(1-s)(p-1)+\varepsilon}$. In order for $\|(1 + m(\cdot))^{-\frac{2}{1+s+\varepsilon}}\|_{L^{r'}}$ to be finite we need

$$\frac{2}{(1-s)(p-1)+\varepsilon} > \frac{1}{s},$$

which gives the inequality $(1-s)(p-1)+\varepsilon < 2s$ to be satisfied. We claim that $\varepsilon > 0$ can always be chosen such that this holds. For p minimal, i.e. close to 1, $\varepsilon \leq s$ is sufficient. If p is large, let δ be the distance from p to $\frac{1+s}{1-s}$. Then we get $2s+\varepsilon-(1-s)\delta < 2s$, which is satisfied when $\varepsilon < (1-s)\delta$. Assuming $\varepsilon > 0$ is appropriately chosen we get $\widehat{u} \in L^{2/(1+s+\varepsilon)}$ which implies $u \in L^{2/(1-s-\varepsilon)}$, and so $f(u) \in L^q$ for $1 \leq q \leq \frac{2}{p(1-s-\varepsilon)}$ and $\widehat{f(u)} \in L^q$ for $\frac{2}{2-p(1-s-\varepsilon)} \leq q \leq \infty$. Inserting 2ε instead of ε in (6.1) and repeating the procedure by choosing r minimal with respect to the new lower bound on q for the $\widehat{f(u)} \in L^q$ we get the inequality $(1-s)(p-1)+\varepsilon(2-p) < 2s$ to be satisfied, which is already guaranteed by the first step since $2-p < 1$. In general we get the inequality $(1-s)(p-1)+\varepsilon(n+1-np) < 2s$ to be satisfied after n iterations. By iterating the procedure enough times, we get $\widehat{u} \in L^1$ and thus $u \in L^\infty$. □

Theorem 6.2. *If f satisfies (B) with $p \in (1, \frac{1+s}{1-s})$, then any solution $u \in H^{s/2}$ of* (1.3)*,* (1.4) *or* (2.11) *is in H^s.*

Proof. We apply the Fourier transform on both sides of (1.3), (1.4) or (2.11) and obtain

$$(k+m(\xi))\widehat{u} = \widehat{f(u)}, \tag{6.2}$$

for some $k > 0$. As in Lemma 6.1 we assume, without loss of generality, that $k = 1$ and furthermore that $c_p = 1$. By Plancherel's Theorem and Lemma 6.1 we have

$$\|\widehat{f(u)}\|_{L^2} = \|f(u)\|_{L^2} \leq \|u\|_{L^\infty}^{p-1}\|u\|_{L^2} < \infty,$$

and so, by (6.2), $(1+m(\xi))\widehat{u} \in L^2$ which implies $u \in H^s$. □

Appendix: The necessity of a continuity assumption on the symbol m

In this appendix we prove that assuming only that m satisfies (2.1), the existence of solitary wave solutions of either (1.1) or (1.2) cannot in general be proved using the method of concentration compactness as in the previous sections. This will be done by providing a counter example, where Dichotomy cannot be precluded for minimizing sequences.

Let $E \subset \mathbb{R}_+$ be a closed nowhere dense set of non-zero measure such that every point of E is a limit point of the set. Such a set can be made by constructing a fat Cantor set $C \subset [0,1]$ with measure $1-\alpha \in (0,1)$ and let E be a union of disjoint translates of C. To be precise, we first remove from $[0,1]$ the open interval with centre $1/2$ and length $\alpha/2$. In the next step, we remove from each of the two remaining closed intervals the middle open interval of length $\alpha/8$. At the n-th step one removes from each remaining closed interval the middle open intervals of length $\alpha/(2^{2n-1})$. What remains in the limit is called the (fat) Cantor set of measure $1-\alpha$. If we define m by

$$m(\xi) := \begin{cases} A_2|\xi|^s, & \text{if } \xi \in \mathbb{R}\setminus E, \\ A_1|\xi|^s, & \text{if } \xi \in E \end{cases}$$

for any $s > 0$, then m satisfies (2.1). Assume that $u \in \mathcal{S}(\mathbb{R})$, the space of smooth, rapidly decreasing functions. Let φ_r be as in Theorem 3.6. Then $L(\varphi_r u) - \varphi_r L u \in L^2$ and thus

$$\int_{\mathbb{R}} \varphi_r u (L(\varphi_r u) - \varphi_r L u)\,\mathrm{d}x \\ = \int_{\mathbb{R}} (m(\xi)(\widehat{\varphi_r} * \widehat{u})(\xi) - (\widehat{\varphi_r} * m\widehat{u})(\xi))\,\overline{\widehat{\varphi_r u}}(\xi)\,\mathrm{d}\xi.$$

Using standard properties of the Fourier transform and convolutions, the right hand side can be written as

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \widehat{\varphi_r}(t)\widehat{u}(\xi - t)(m(\xi) - m(\xi - t))\,\mathrm{d}t\overline{\widehat{\varphi_r u}}(\xi)\,\mathrm{d}\xi.$$

Consider now $\xi \in E$. For every $r > 0$, the set $E^c \cap [\xi - 1/r, \xi + 1/r]$ has non-zero measure, where $E^c = \mathbb{R}\setminus E$. This can be seen by taking any point $x \in E^c \cap [\xi - 1/r, \xi + 1/r]$ (such a point must exist since the closure of E has empty interior); if there exists no interval

$$(x - \varepsilon, x + \varepsilon) \subset E^c \cap [\xi - 1/r, \xi + 1/r]$$

then x is in the closure of E and, moreover, $(\xi - 1/r, \xi + 1/r) \subset \overline{E}$, which is a contradiction since E is nowhere dense. Let $\mathcal{L}$ be the Lebesgue measure on the real line. We claim that there exists a constant $k = k(\alpha) \in (0, 2)$ independent of ξ and r such that

$$\mathcal{L}\left(E^c \cap [\xi - 1/r, \xi + 1/r]\right) \geq k(\alpha)/r \text{ for all } \xi \in E \text{ and all } r > 1.$$

This means the ratio

$$\frac{\mathcal{L}\left(E^c \cap [\xi - 1/r, \xi + 1/r]\right)}{\mathcal{L}\left([\xi - 1/r, \xi + 1/r)\right.}$$

is bounded from below; i.e. a minimum portion of the interval in the sense of measure always belongs to E^c. To see that this claim is true, let $\mathcal{P}$ be the collection of all open intervals deleted from $[0, 1]$ in the construction of the Cantor set and their translates according the definition of E. Let $\mathcal{P}'$ denote the corresponding collection of closed sets remaining at each step of the construction. Let ξ be given. If there is a $P \in \mathcal{P}$ such that, say, $\mathcal{L}\left(P \cap [\xi - 1/r, \xi + 1/r]\right) > 1/(2r)$ the result is satisfied by $k(\alpha) = 1/2$. Assume therefore that this is not the case. By assumption there exists a point $\tau_1 \in [\xi + 1/(2r), \xi + 1/r]$ that is a left endpoint of some $P' \in \mathcal{P}'$. Similarly, there is a point $\tau_2 \in [\xi - 1/r, \xi - 1/(2r)]$ that is the right endpoint of some $P' \in \mathcal{P}'$. Now $[\tau_1, \tau_2] \subseteq [\xi - 1/r, \xi + 1/r]$ and $\mathcal{L}\left([\tau_1, \tau_2]\right) > 1/r$.

Furthermore, $[\tau_1, \tau_2]$ can by construction be written as a union of disjoint elements of $\mathcal{P}'$ and $\mathcal{P}$. Clearly, $\mathcal{L}(E^c \cap P) = \mathcal{L}(P)$ and by the construction of the Cantor set, $\mathcal{L}(E^c \cap P') = \alpha\mathcal{L}(P')$. Thus $\mathcal{L}(E^c \cap [\tau_1, \tau_2]) > \alpha/r$ and therefore our claim holds with $k(\alpha) = \alpha$.

Observe next that $m(\xi) - m(\xi - t) \leq -(A_2 - A_1)|\xi|^s + A_2|1/r|^s$ when $t \in (E^c - \xi) \cap [-1/r, 1/r]$. Furthermore, we may assume that $\widehat{u} \geq M$ for some $M > 0$ in all intervals $[\xi - 1/r, \xi + 1/r]$ where $\xi \in E$ for r sufficiently large. Thus

$$\begin{aligned}
\int_{(E^c-\xi)\cap[-1/r,1/r]} &\widehat{\varphi_r}(t)\widehat{u}(\xi - t)(m(\xi - t) - m(\xi))\,\mathrm{d}t \\
&\geq M(A_2 - A_1)|\xi|^s \int_{(E^c-\xi)\cap[-1/r,1/r]} \widehat{\varphi_r}(t)\,\mathrm{d}t \\
&= M(A_2 - A_1)|\xi|^s \int_{r((E^c-\xi)\cap[-1/r,1/r])} \widehat{\varphi}(x)\,\mathrm{d}x \\
&\geq N(A_2 - A_1)|\xi|^s,
\end{aligned}$$

for some constant $N > 0$ that does not depend on r. This follows since the measure of the area of integration in line three is greater than $k(\alpha)$ and so the integral of $\widehat{\varphi}$ over the set is non-zero. Hence

$$\begin{aligned}
\liminf_{r\to\infty} &\left| \int_E \int_{(E^c-\xi)\cap[-1/r,1/r]} \widehat{\varphi_r}(t)\widehat{u}(\xi - t)(m(\xi) - m(\xi - t))\,\mathrm{d}t\overline{\widehat{\varphi_r u}}(\xi)\,\mathrm{d}\xi \right| \\
&\geq N(A_2 - A_1) \left| \int_E |\xi|^s \overline{\widehat{\varphi_r u}}(\xi)\,\mathrm{d}\xi \right| > 0.
\end{aligned}$$

Assume now that dichotomy occurs for a minimizing sequence $\{u_n\}_n$. In Lions' [16] construction of the sequences in the case of dichotomy $u_n^{(1)} = \varphi_{R_1} u_n$ and $u_n^{(2)} = \psi_{r_n} u_n$ for suitably large R_1 and $r_n \to \infty$, where φ_r, ψ_r are defined as in Theorem 3.6. We have

$$\begin{aligned}
\mathcal{J}(u_n) =& \mathcal{J}(u_n^{(1)}) + \mathcal{J}(u_n^{(2)}) + \frac{1}{2}\int_{\mathbb{R}} \eta_{r_n}\left(uLu + \kappa u^2\right)\,\mathrm{d}x \\
&- \int_{\mathbb{R}} \varphi_{R_1} u_n (L(\varphi_{R_1} u_n) - \varphi_{R_1} L u_n)\,\mathrm{d}x \\
&- \int_{\mathbb{R}} \psi_{r_n} u_n (L(\psi_{r_n} u_n) - \psi_{r_n} L u_n)\,\mathrm{d}x,
\end{aligned}$$

where $\eta_{r_n} = \chi_{\mathbb{R}} - \psi_{r_n}^2 - \varphi_{R_1}^2$. For any $\varepsilon > 0$ the last term can be taken to be smaller than ε in absolute value by choosing r_n big enough according to Theorem 3.6, as continuity of m was not used in that part of the proof. By assumption, $\int_{\mathbb{R}} \eta_{r_n}|u|^{p+1}\,\mathrm{d}x \leq \varepsilon$ for n sufficiently large. If u_n is, say, a rapidly decreasing function so that $Lu \in L^{(p+1)/p}$ then Hölder's inequality implies that

$\frac{1}{2}\int_{\mathbb{R}} \eta_{r_n}\left(c_1 uLu + c_2 u^2\right)\,\mathrm{d}x$ goes to zero as ε goes to zero. However, as our calculations above show, the term $\int_{\mathbb{R}} \varphi_{R_1} u_n (L(\varphi_{R_1} u_n) - \varphi_{R_1} L u_n)\,\mathrm{d}x$ can in general be bounded away from zero for all choices of R_1 for elements $u_n \in H^{s/2}$.

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Paper II

Paper II

Non-uniform dependence on initial data for equations of Whitham type

Mathias Nikolai Arnesen

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Paper III

Paper III

A nonlocal approach to waves of maximal height for the Degasperis-Procesi equation

Mathias Nikolai Arnesen

In preparation

A NON-LOCAL APPROACH TO WAVES OF MAXIMAL HEIGHT FOR THE DEGASPERIS-PROCESI EQUATION

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Abstract. We consider the non-local formulation of the Degasperis-Procesi equation $u_t + uu_x + L(\frac{3}{2}u^2)_x = 0$, where L is the non-local Fourier multiplier operator with symbol $m(\xi) = (1+\xi^2)^{-1}$. We show that all L^∞, pointwise travelling-wave solutions are bounded above by the wave-speed and that if the maximal height is achieved they are peaked at those points, otherwise they are smooth. For sufficiently small periods we find the highest, peaked, travelling-wave solution as the limiting case at the end of the main bifurcation curve of P-periodic solutions. The results imply that the Degasperis-Procesi equation does not admit cuspon solutions in L^∞.

1. Introduction

The Degasperis-Procesi (DP) equation

$$u_t - u_{xxt} + 4uu_x - 3u_x u_{xx} - uu_{xxx} = 0$$

was discovered in [4] as one of three equations within a certain class of third order PDEs satisfying an asymptotic integrability condition up to third order, the other two being the KdV and the Camassa-Holm equations. In the subsequent work [3] it was shown to be completely integrable, as the other two aforementioned equations are. Obviously, the Degasperis-Procesi equation is local, but like its relative Camassa-Holm it can also be written as a nonlocal equation: Introducing the nonlocal operator $L = (1 - \partial_x^2)^{-1}$, which on the Fourier side has symbol $m(\xi) = (1+\xi^2)^{-1}$, the equation can be rewritten as

$$u_t + uu_x + (L(\frac{3}{2}u^2))_x = 0. \tag{1.1}$$

The operator L can also be expressed trough convolution with the kernel $K = \mathscr{F}^{-1} m$: $Lf = K * f$. For travelling waves $u(x,t) = \varphi(x - \mu t)$, where $\mu \in \mathbb{R}$ is the wave-speed, (1.1) takes the form

$$-\mu\varphi + \frac{1}{2}\varphi^2 + \frac{3}{2}L(\varphi^2) = a, \tag{1.2}$$

where $a \in \mathbb{R}$ is a constant of integration. By a Galilean change of variables this is equivalent to $-\mu\varphi + \frac{1}{2}\varphi^2 + \frac{3}{2}L(\varphi^2 + k\varphi) = 0$, where k depends on μ and a;

in particular, $k \neq 0$ for $a \neq 0$. Hence there is no Galilean change of variables that removes a while preserving the form of the equation. We will work with the equation in the form (1.2).

From the structure of the equation it is readily deducible that all non-constant solutions to (1.2) are smooth except potentially at points where the wave-height equals the wave-speed (cf. Theorem 3.4 or [8]) and singularities can only occur in the form of sharp crests with height equal to the wave-speed. Explicit peaked soliton solutions, as well as multipeakon solutions which are not travelling waves, to (1.1) are known [3]. These are of the same form as the ones for CH [2], and indeed every equation in the so-called 'b-family' of equations that DP and CH belong to has such solutions [3]. In this paper we consider periodic travelling waves of maximal height more generally.

The motivation of this paper is two-fold: to provide novel information about waves of maximal height for the DP equation specifically and to better understand the formation of highest waves and their singularities for nonlinear dispersive equations more generally. We therefore consider the non-local formulation and follow the general framework of [7] and [5]. We show firstly that any even, non-constant L^∞ solution of (1.2) is peaked wherever the maximal height is achieved. That is, it is Lipschitz continuous at the crest(s), but not C^1. In particular this means that there are no cuspon solutions of (1.2) in L^∞. The restriction to bounded to solutions is quite natural as while equation (1.2) makes sense for any $\varphi \in H^{-2}(\mathbb{R})$, if we exclude purely distributional solutions, any function solving (1.2) a.e. clearly belongs to L^∞. Secondly, for sufficiently small periods peaked solutions of (1.2) are found as the limiting case at the end of the main bifurcation curve of $C^\alpha_{\text{even}}(\mathbb{S}_P)$ solutions for $\alpha \in (1,2)$.

As L^∞ cuspon solutions to (1) has been established by several authors, our claim that they do not exists requires some comment. The cuspons are invariably found as solutions to the local equation, as they cannot appear in the non-local formulation as we show in this paper, and they are strong solutions in all points except the cusps. They are, however, not weak solutions. Consider for instance the stationary cusped soliton $u(x) = \sqrt{1 - \mathrm{e}^{-2|x|}}$ discovered in [9], which is a solution to (1) at all points except 0, where the function has a cusp. However, for any test function $\varphi \in C_0^\infty(\mathbb{R})$, treating the left-hand side of (1) as a distribution (note that u is independent of time), one can with basic calculus show that

$$
\begin{aligned}
\langle 4uu_x - 3u_xu_{xx} - uu_{xxx}, \varphi\rangle &= \langle u^2, \frac{1}{2}\varphi_{xxx} - 2\varphi_x\rangle \\
&= \int_{\mathbb{R}} u^2 \left(\frac{1}{2}\varphi_{xxx} - 2\varphi_x\right) \,\mathrm{d}x \\
&= 2\varphi_x(0)
\end{aligned}
$$

and hence it is not a weak solution to (1), but rather to

$$u_t - u_{xxt} + 4uu_x - 3u_x u_{xx} - uu_{xxx} = -2\delta',$$

where δ is the usual delta-distribution. This is the case with all cuspons of the DP equation - there are point mass distributions at the cusps. To accept functions that solve the equation pointwise at all but a countable number of points as solutions is equivalent to claiming that the sawtooth function $u(x) = x - \text{floor}(x)$, or indeed any piece-wise linear function, is a solution to the equation

$$u''(x) = 0, \quad x \in \mathbb{R},$$

which is clearly absurd. Hence we think it more correct to call the cuspons solutions not of (1) with 0 right-hand side, but with some point mass distributions.

The paper is structured as follows: first some essential properties of the operator L and its kernel K are recounted in Section 2. In Section 3 we establish some general results about solutions to (1.2) and, in particular, using the properties of K, study the behaviour around points of critical height and prove Theorem 3.6, stating that any even, nonconstant solution is peaked at points where $\varphi = \mu$. Lastly, in Section 4 we use the bifurcation Theory of [1] to construct a global bifurcation curve of even, periodic solutions in C^α for $\alpha \in (1,2)$. Using the properties of solutions established in Section 3, we show that for sufficiently small periods the solutions along the curve converge to an even, non-constant solution that achieves the maximal height and must therefore be a peakon.

2. The operator L and its kernel

As $\widehat{Lf}(\xi) = (1+\xi^2)^{-1}\widehat{f}(\xi)$, Lf can formally be expressed as a convolution

$$Lf(x) = K * f(x) = \int_{\mathbb{R}} K(x-y)f(y)\,\mathrm{d}y,$$

where $K(x)$ is the inverse Fourier transform of $m(\xi)$. In this case, an explicit expression is well known from virtually any textbook on Fourier analysis:

$$K(x) = \mathscr{F}^{-1}((1+\xi^2)^{-1}) = \frac{1}{2}\mathrm{e}^{-|x|}. \tag{2.1}$$

In particular, we note that K is completely monotone on $(0,\infty)$; it is positive, strictly decreasing and strictly convex for $x > 0$.

The periodic kernel is

$$K_P(x) = \sum_{n\in\mathbb{Z}} K(x+nP),$$

for $P \in (0, \infty)$. For $x \in (-P/2, P/2)$, $K(x+nP) = \frac{1}{2}\mathrm{e}^{-|x+nP|} = \frac{1}{2}\mathrm{e}^{-x}\mathrm{e}^{-nP}$ for $n \geq 1$, and $K(x+nP) = \frac{1}{2}\mathrm{e}^{x}\mathrm{e}^{nP}$ for $n \leq -1$. Thus

$$\begin{aligned} K_P(x) &= \sum_{n\in\mathbb{Z}} K(x+nP) \\ &= \frac{1}{2}\mathrm{e}^{-|x|} + \frac{1}{2}(\mathrm{e}^{x} + \mathrm{e}^{-x})\sum_{n=1}^{\infty}\mathrm{e}^{-nP} \\ &= \frac{1}{2}\mathrm{e}^{-|x|} + \cosh(x)\frac{1}{\mathrm{e}^{P}-1}. \end{aligned} \tag{2.2}$$

For periodic functions the operator L is given by $Lf(x) = \int_{-P/2}^{P/2} K_P(x-y)f(y)\,\mathrm{d}y$.

We note that K and K_P has all the properties used in the proofs of Lemmata 3.5 and 3.6 in [7] and those results therefore hold for L. We repeat them here:

Lemma 2.1. *L is strictly monotone: $Lf > Lg$ if f and g are bounded and continuous functions with $f \gneq g$.*

Lemma 2.2. *The operator L is parity-preserving on any period $P \in (0, \infty]$, and $Lf(x) > 0$ on $(-P/2, 0)$ for f P-periodic, odd and continuous with $f \gneq 0$ on $(-P/2, 0)$.*

For periodic functions, it is sometimes more convenient to describe the action of L through its action on Fourier coefficients. An even, P-periodic function f can be written as

$$f(x) = \sum_{k=0}^{\infty} \widehat{f}_k \cos\left(\frac{2\pi k}{P}x\right),$$

where

$$\begin{aligned} \widehat{f}_0 &= \frac{1}{P}\int_{-P/2}^{P/2} f(x)\,\mathrm{d}x, \\ \widehat{f}_k &= \frac{2}{P}\int_{-P/2}^{P/2} f(x)\mathrm{e}^{-\frac{2\pi k i x}{P}}\,\mathrm{d}x. \end{aligned}$$

Then

$$L(f)(x) = \sum_{k=0}^{\infty} \widehat{f}_k \frac{1}{1+(\frac{2\pi k}{P})^2}\cos\left(\frac{2\pi k}{P}x\right).$$

3. Periodic travelling waves

First we investigate how the parameter a in (1.2) influences the behaviour/existence of solutions.

Theorem 3.1. *Fix $\mu > 0$ and $P < \infty$. For all values of $a \in \mathbb{R}$, non-constant P-periodic solutions to* (1.2) *(if they exist) satisfy*

$$\min \varphi < \frac{\mu + \sqrt{\mu^2 + 8a}}{4} < \max \varphi.$$

Moreover,

(i) *For $a \le 0$, all solutions are non-negative. When $a < -\frac{\mu^2}{8}$ there are no real solutions and for $a = -\frac{\mu^2}{8}$ there is only the constant solution $\varphi = \frac{\mu}{4}$.*
(ii) *There are only constant solutions when $a \ge \mu^2$.*
(iii) *When $0 < a < \mu^2$, all solutions satisfy* $\max \varphi > |\min \varphi|$.

Proof. At any point x where $\varphi(x)^2 = L(\varphi^2)(x) =: R$, (1.2) reduces to

$$R(2R - \mu) = a,$$

which has the positive solution $R = \frac{\mu + \sqrt{\mu^2+8a}}{4}$. As $L(c) = c$ for constants and L is strictly monotone (Lemma 2.1), there has to exist points where $\varphi^2 < L(\varphi^2)$ and points where $\varphi^2 > L(\varphi^2)$ for non-constant P-periodic solutions φ. Thus the first inequality has to hold if $\max \varphi > |\min \varphi|$. If $a \le 0$, then φ cannot be negative in any point as then the left-hand side of (1.2) would be strictly positive in that point (Lf is non-negative if f is non-negative). Let $m = \min \varphi$. Then $L(\varphi^2) \ge m^2$ with equality if and only if $\varphi \equiv m$. Hence, if φ is a solution to (1.2), we get

$$m(2m - \mu) \le a.$$

For $a < -\frac{\mu^2}{8}$ this has no real solutions, and for $a = -\frac{\mu^2}{8}$ this has only the constant solution $\varphi = \frac{\mu}{4}$. This proves (i).

Now let $a > 0$. Assume that $\varphi < 0$ on some intervals. Clearly φ is bounded below, so there is a point x_0 such that $\varphi(x_0) = \min \varphi$. Then $L(\varphi\varphi')(x_0) = 0$ and $L(\varphi^2)$ attains it minimum at x_0. This implies that φ also has to be positive at some point, and $M := \max \varphi > |\min \varphi|$. Thus the first inequality holds and $M > \frac{\mu+\sqrt{\mu^2+8a}}{4}$. In particular, this means that $\max \varphi \ge \frac{\mu}{2}$ for all $a \ge 0$ and $M > \sqrt{a}$ if $a < \mu^2$. We have that

$$(\varphi - \mu)^2 = \mu^2 + 2a - 3L(\varphi^2). \tag{3.1}$$

Assume $a \ge \mu^2$. Note that $3L(\varphi^2) = \mu^2 + 2a \ge 3\mu^2$ at all points where $\varphi = \mu$. If $a = \mu^2$, then the constant solution $\varphi \equiv \mu$ is a valid solution, otherwise Lemma 2.1 implies that φ must also take values above μ. Assume $\varphi \gneq \mu$ is a non-constant solution. Then the left-hand side of (3.1) attains its minimum where φ is attains its minimum, while the right-hand side attains its minimum where $L(\varphi^2)$ attains its maximum. This is a contradiction. As both K and K_P are even and completely monotone on $(0, \infty)$ and $(0, P/2)$, respectively, $L(\varphi^2)$ cannot be maximal where φ^2 is minimal.

Assume now that φ takes values both above and below μ. Then $L(\varphi^2)(x)$ is maximal whenever $\varphi(x) = \mu$ and $3L(\varphi^2)(x) = \mu^2 + 2a$ these points. Moreover, $3L(\varphi^2) < \mu^2 + 2a$ when $\varphi > \mu$. This implies that there are infinitely many disjoint intervals, each of finite length, where $\varphi > \mu$, and that $L(\varphi^2)$ has its minimum on each interval at the points where φ is maximal. This is again not possible. $\square$

Henceforth we will assume that a is such that non-constant solutions exists, i.e. that $-\mu^2/8 < a < \mu^2$.

Theorem 3.2. *Let $P(0,\infty]$. Any P-periodic, non-constant and even solution $\varphi \in BUC^1(\mathbb{R})$ of (1.2) satisfies $\varphi \geq \frac{\mu-\sqrt{\mu^2+8a}}{4}$. Moreover, if φ is non-decreasing on $(-P/2,0)$, then $\varphi' > 0$ at all points in $(-P/2,0)$ where $\varphi > \frac{\mu-\sqrt{\mu^2+8a}}{4}$, which must necessarily form an open interval with 0 as the right endpoint, and $\varphi < \mu$ on $(-P/2,0)$. If $\varphi \in BUC^2(\mathbb{R})$, then*

$$\varphi''(0) < 0, \quad \textit{and} \quad \varphi(0) < \mu.$$

Proof. We can rewrite (1.2) as $(\mu - \varphi)^2 = \mu^2 + 2a - 3L(\varphi^2)$, and if $\varphi \in BC^1(\mathbb{R})$ we can differentiate on each side to get

$$(\mu - \varphi(x))\varphi'(x) = \frac{3}{2}L(\varphi^2)'(x), \tag{3.2}$$

and if $\varphi \in BUC^2(\mathbb{R})$ we can differentiate each side again:

$$(\mu - \varphi(x))\varphi''(x) - (\varphi'(x))^2 = \frac{3}{2}L(\varphi^2)''(x). \tag{3.3}$$

As φ^2 is even, $(\varphi^2)'$ will be odd and using the evenness of K_P we get

$$L(\varphi^2)'(x) = 2\int_{-P/2}^{0} (K_P(x-y) - K_P(x+y))\varphi'(y)\varphi(y)\,\mathrm{d}y,$$

and hence

$$L(\varphi^2)''(x) = 2\int_{-P/2}^{0} (K_P'(x-y) - K_P'(x+y))\varphi'(y)\varphi(y)\,\mathrm{d}y. \tag{3.4}$$

Note that this is valid for $\varphi \in BC^1(\mathbb{R})$. Clearly $K_P'(x)$ is smooth away from the origin, where it is not defined. However, the left and right derivatives exist at the origin, and for any $x \in (-P/2,0)$,

$$\left(\lim_{y\to x^-} K_P'(x-y) - K_P'(x+y)\right) - \left(\lim_{y\to x^+} K_P'(x-y) - K_P'(x+y)\right) = -1.$$

Splitting the integral in (3.4) into two integrals, one over $(-P/2,x)$ and one over $(x,0)$, and using integration by parts in each part gives, considering the limit above,

$$\begin{aligned}
L(\varphi^2)''(x) &= \int_{-P/2}^{0} (K_P''(x-y) + K_P''(x+y))(\varphi(y))^2 \, \mathrm{d}y - (\varphi(x))^2 \\
&= \int_{-P/2}^{0} (K_P(x-y) + K_P(x+y))(\varphi(y))^2 \, \mathrm{d}y - (\varphi(x))^2 \\
&= L(\varphi^2)(x) - (\varphi(x))^2. \qquad (3.5)
\end{aligned}$$

As $\varphi(x)^2 \leq \varphi(0)^2$ for all $x \in (-P/2, 0)$ (recall that $|\min \varphi| < \max \varphi$) and φ is non-constant, we have that $0 < L(\varphi^2)(x) < \varphi(0)^2$ for all $x \in (-P/2, 0)$ and by (3.5) that $L(\varphi^2)''(0) < 0$. If φ is non-decreasing on $(-P/2, 0)$, then $\varphi''(0)$ cannot be positive. From (3.3) we therefore conclude that $\varphi''(0) < 0$ and $\varphi(0) < \mu$. This proves the part of the statement concerning $\varphi \in BUC^2(\mathbb{R})$.

Assume now that $\varphi \in BUC^1(\mathbb{R})$. From (1.2) we get that

$$L(\varphi^2)(x) > \varphi(x)^2, \quad \frac{\mu - \sqrt{\mu^2 + 8a}}{4} < \varphi(x) < \frac{\mu + \sqrt{\mu^2 + 8a}}{4},$$

with equality at the endpoints and the opposite inequality outside. Note that (3.5) is valid also if φ is only $BUC^1(\mathbb{R})$, hence $L(\varphi^2)'' < 0$ when $\varphi < \frac{\mu - \sqrt{\mu^2+8a}}{4}$. As $L(\varphi^2)'(-P/2) = 0$, it must be the case that $L(\varphi^2)' \leq 0$ at some point where $\varphi < \frac{\mu - \sqrt{\mu^2+8a}}{4}$, and thus, from (3.2) and (3.5) φ, $L(\varphi^2)$ and $L(\varphi^2)'$ will be decreasing from that point, leading to a contradiction with $L(\varphi^2)'(0) = 0$. This implies that $\varphi \geq \frac{\mu - \sqrt{\mu^2+8a}}{4}$. Assume now that φ is also non-decreasing. The evenness of K_P and φ gives

$$L(\varphi^2)'(x) = 2 \int_{-P/2}^{0} (K_P(x-y) - K_P(x+y))\varphi'(y)\varphi(y) \, \mathrm{d}y. \qquad (3.6)$$

For $x, y \in (-P/2, 0)$, $K_P(x-y) - K_P(x+y) > 0$, so if φ is non-negative and non-constant, the right hand side of (3.6) is clearly positive for all $x \in (-P/2, 0)$ and the conclusion follows from (3.2). However, when $0 < a < \mu^2$ we do not know whether φ is non-negative or not. By assumption, the left-hand side of (3.2) is non-negative when $\varphi < \mu$, and as we know that $\min \varphi < \mu$, we get that $L(\varphi^2)'(x)$ is non-negative in some interval with left endpoint $-P/2$. Thus

$$L(\varphi^2)''(-P/2) = L(\varphi^2)(-P/2) - (\varphi(-P/2))^2 \geq 0.$$

As φ is non-decreasing, it follows that $L(\varphi^2)''(x)$ can change sign only once on $(-P/2, 0)$. As $L(\varphi^2)'(-P/2) = L(\varphi^2)'(0) = 0$ and $L(\varphi^2)''(0) < 0$, if follows that if $L(\varphi^2)'(x) > 0$ for $x \in (-P/2, 0)$, then $L(\varphi^2)' > 0$ on $(x, 0)$. This implies that there is a maximal $x_0 \in [-P/2, 0)$ such that $L(\varphi^2)' = 0$ on $[-P/2, x_0]$, and hence φ is constant on this interval and strictly increasing on $(x_0, 0)$. This proves the result. $\square$

Remark 3.3. The proof also implies that any even and non-constant solution that satisfies $\varphi = \frac{\mu - \sqrt{\mu^2+8a}}{4}$ at some point in $(-P/2, 0)$ will be monotonically increasing from that point at least until $\varphi = \frac{\mu + \sqrt{\mu^2+8a}}{4}$.

3.1. **Singularity at $\varphi = \mu$.**

Theorem 3.4. *Let $\varphi \leq \mu$ be a solution of* (1.2). *Then:*

(i) *If $\varphi < \mu$ uniformly on $\mathbb{R}$, then $\varphi \in C^\infty(\mathbb{R})$ and all of its derivatives are uniformly bounded on $\mathbb{R}$.*
(ii) *If $\varphi < \mu$ uniformly on $\mathbb{R}$ and $\varphi \in L^2(\mathbb{R})$, then $\varphi \in H^\infty(\mathbb{R})$.*
(iii) *φ is smooth on any open set where $\varphi < \mu$.*

Proof. Assume first that $\varphi < \mu$ uniformly on $\mathbb{R}$. Note that as $\varphi \to -\infty$, the left-hand side of (1.2) goes to ∞, hence φ must be bounded below as well. Clearly, $|m^{(n)}(\xi)| \lesssim (1+\xi^2)^{-2-n}$ (that is, m is a S^{-2}-multiplier) and L is therefore continuous from the Besov space $B^s_{p,q}(\mathbb{R})$ to $B^{s+2}_{p,q}(\mathbb{R})$ for all $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. Denoting by $\mathcal{C}^s(\mathbb{R})$, $s \in \mathbb{R}$ the Zygmund space $B^s_{\infty,\infty}(\mathbb{R})$, we have in particular that L maps $L^\infty(\mathbb{R}) \subset B^0_{\infty,\infty}(\mathbb{R})$ into $\mathcal{C}^2(\mathbb{R})$, and therefore $\varphi \mapsto L(\varphi^2)$ maps $L^\infty(\mathbb{R})$ into $\mathcal{C}^2(\mathbb{R})$. Recall that if $s \in \mathbb{R}_+ \setminus \mathbb{N}$, then $\mathcal{C}^s(\mathbb{R}) = C^s(\mathbb{R})$, the ordinary Hölder space, and if $s \in \mathbb{N}$ then $W^{s,\infty}(\mathbb{R}) \subsetneq \mathcal{C}^s(\mathbb{R})$.

As φ solves (1.2) we have

$$(\varphi - \mu)^2 = \mu^2 + 2a - 3L(\varphi^2).$$

The assumption $\varphi < \mu$ therefore implies that $3L(\varphi^2) < \mu^2 + 2a$, and the operator $L(\varphi^2) \mapsto \mu - \sqrt{\mu^2 + 2a - 3L(\varphi^2)}$ therefore maps $B^s_{p,q}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ into itself for $s > 0$. Since $\varphi < \mu$, we also get that $\mu - \sqrt{\mu^2 + 2a - 3L(\varphi^2)} = \varphi$. Combining this map with $\varphi \mapsto L(\varphi^2)$ and iterating, we get (i). When $p = q = 2$, $B^s_{p,q}(\mathbb{R})$ can be identified with $H^s(\mathbb{R})$. Assume now that $\varphi \in L^2(\mathbb{R})$ in addition. As φ is also bounded, we get that $\varphi^2 \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and in general $\varphi^2 \in H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$ if $\varphi \in H^s(\mathbb{R}) \cap L^\infty$, and thus $\varphi \mapsto L(\varphi^2)$ maps $H^s(\mathbb{R}) \cap L^\infty(\mathbb{R})$ to $H^{s+2}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and we can apply the above iteration argument again. This proves (ii).

Lastly, to prove (iii), we note that if $\varphi \in L^\infty(\mathbb{R})$ and $\mathcal{C}^s_{loc}$ on an open set U in the sense that $\psi\varphi \in \mathcal{C}^s(\mathbb{R})$ for any $\psi \in C_0^\infty(U)$, we still get that $L(\varphi)$ is $\mathcal{C}^{s+2}_{loc}$ in U (the proof of this is the same as in Theorem 5.1 [7]). Thus we can apply the same iteration argument as above again. □

Lemma 3.5. *Let $P < \infty$, and let φ be an even, non-constant solution of* (1.2) *that is non-decreasing on $(-P/2, 0)$ with $\varphi \leq \mu$. Then there exists a universal constant $C_{K,P,\mu}$, depending only on the kernel K and the period P and $\mu > 0$, such that*

$$\mu - \varphi\left(\frac{P}{2}\right) \geq C_{K,P,\mu}.$$

Proof. If $\varphi(-P/2) = \varphi(P/2) < 0$, the statement is true with $C_{K,P,\mu} = \mu$. Assume therefore that φ is non-negative. From the evenness and periodicity of K_P and φ, we get the formula

$$\begin{aligned} &L(\varphi^2)(x+h) - L(\varphi^2)(x-h) \\ &\qquad = \int_{-P/2}^{0} (K_P(x-y) - K_P(x+y))(\varphi(y+h)^2 - \varphi(y-h)^2)\,\mathrm{d}y. \end{aligned} \tag{3.7}$$

As $\varphi \geq 0$, both factors in the integrand are non-negative for $x \in (-P/2, 0)$ and $h \in (0, P/2)$. We also have the equality

$$(2\mu - \varphi(x) - \varphi(y))(\varphi(x) - \varphi(y)) = 3\left(L(\varphi^2)(x) - L(\varphi^2)(y)\right), \tag{3.8}$$

which shows that $L(\varphi^2)(x) = L(\varphi^2)(y)$ whenever $\varphi(x) = \varphi(y)$. As φ is assumed to be non-constant and non-negative, this identity together with (3.7) implies that φ is strictly increasing on $(-P/2, 0)$, and it therefore follows from Theorem 3.4 that φ is smooth away from $x = kP$, $k \in \mathbb{Z}$. Let $x \in \left[-\frac{3P}{8}, -\frac{P}{8}\right]$. Then for a solution φ as in the assumptions,

$$(\mu - \varphi(\frac{P}{2}))\varphi'(x) \geq (\mu - \varphi(x))\varphi'(x) = \frac{3}{2}\lim_{h\to 0} \frac{L(\varphi^2)(x+h) - L(\varphi^2)(x-h)}{4h}.$$

As the integrand in (3.7) is non-negative for $h \in (0, P/2)$ and non-positive for $h \in (-P/2, 0)$, we can apply Fatou's lemma to the limit above and we get

$$\begin{aligned} (\mu - \varphi(\frac{P}{2}))\varphi'(x) &\geq 3\int_{-P/2}^{P/2} K_P(x-y)\varphi(y)\varphi'(y)\,\mathrm{d}y \\ &= 3\int_{-P/2}^{0} (K_P(x-y) - K_P(x+y))\varphi(y)\varphi'(y)\,\mathrm{d}y. \end{aligned}$$

Assume for a contradiction that the statement is not true. Then for all $k < \mu$ there must exist a solution φ satisfying the assumptions and such that $k \leq \varphi \leq \mu$. Then $\mu - \varphi(P/2) < \mu - k$. On the other hand, as $K_P(x-y) > K_P(x+y)$ for $x, y \in (-P/2, 0)$, we get that

$$\begin{aligned} (\mu - \varphi(\frac{P}{2}))\varphi'(x) &\geq 3\int_{-P/2}^{0} (K_P(x-y) - K_P(x+y))\varphi(y)\varphi'(y)\,\mathrm{d}y \\ &\geq 3k\int_{-3P/8}^{-P/8} (K_P(x-y) - K_P(x+y))\varphi'(y)\,\mathrm{d}y. \end{aligned}$$

There is a universal constant $\tilde{\lambda}_{K,P} > 0$ depending only on K_P and $P < \infty$ such that

$$\min\{K_P(x-y) - K_P(x+y) : x, y \in \left[-\frac{3P}{8}, -\frac{P}{8}\right]\} \geq \tilde{\lambda}_{K,P}.$$

Integrating both sides above over $x \in \left(-\frac{3P}{8}, -\frac{P}{8}\right)$, we get that

$$(\mu - \varphi(\frac{P}{2}))(\varphi(-P/8) - \varphi(-3P/8)) \geq 3k\frac{P}{8}\tilde{\lambda}_{K,P}(\varphi(-P/8) - \varphi(-3P/8)).$$

As shown above φ is strictly increasing on $(-P/2, 0)$, so $\varphi(-P/8) > \varphi(-3P/8)$ and we may divide out $(\varphi(-P/8) - \varphi(-3P/8))$ on both sides to get

$$(\mu - \varphi(\frac{P}{2})) \geq 3k\frac{P}{8}\tilde{\lambda}_{K,P}.$$

This implies that $\mu - k \geq 3k\frac{P}{8}\tilde{\lambda}_{K,P}$ for all $k < \mu$. Taking the limit $k \nearrow \mu$, we get a contradiction. □

Theorem 3.6. *Let $\varphi \leq \mu$ be a solution of* (1.2) *which is even, non-constant, and non-decreasing on $(-P/2, 0)$ with $\varphi(0) = \mu$. Then:*

(i) *φ is smooth on $(-P, 0)$.*
(ii) *$\varphi \in C^{0,1}(\mathbb{R})$, i.e. φ is Lipschitz.*
(iii) *φ is exactly Lipschitz at $x = 0$; that is, there exists constants $0 < c_1 < c_2$ such that*

$$c_1|x| \leq |\mu - \varphi(x)| \leq c_2|x|$$

for $|x| \ll 1$.

Proof. Part (i) will follow directly from Theorem 3.4 if we can show that $\varphi < \mu$ on $(-P/2, 0)$. Assume that $x_0 \in (-P/2, 0]$ is the smallest number such that $\varphi(x_0) = \mu$; as φ is assumed to be non-constant, it must be the case that $x_0 > -P/2$. Then $\varphi(x) = \mu$ and $L(\varphi^2)'(x) = 0$ for $x \in [x_0, 0]$. That is,

$$\int_{-P/2}^{0} (K_P'(x-y) + K_P'(x+y))\,(\varphi(y))^2\,\mathrm{d}y = 0, \quad x \in [x_0, 0].$$

Clearly $\int_{-P/2}^{0} (K_P'(x-y) + K_P'(x+y))\,\mathrm{d}y = 0$ and as

$$\begin{aligned} K_P'(x-y) + K_P'(x+y) &< 0, \quad -P/2 < y < x < 0, \\ K_P'(x-y) + K_P'(x+y) &> 0, \quad -P/2 < x < y < 0, \end{aligned}$$

we get that

$$\int_{-P/2}^{x} (K_P'(x-y) + K_P'(x+y))\,\mathrm{d}y = -\int_{x}^{0} (K_P'(x-y) + K_P'(x+y))\,\mathrm{d}y.$$

Hence, by the mean value theorem for integrals,

$$\begin{aligned}
L(\varphi^2)'(x_0) = &\int_{-P/2}^{0} \left(K_P'(x_0-y)+K_P'(x_0+y)\right)(\varphi(y))^2\,\mathrm{d}y\\
=&\varphi(c)^2\int_{-P/2}^{x_0}\left(K_P'(x_0-y)+K_P'(x_0+y)\right)\mathrm{d}y\\
&+\mu^2\int_{x_0}^{0}\left(K_P'(x_0-y)+K_P'(x_0+y)\right)\mathrm{d}y\\
=&\int_{x_0}^{0}\left(K_P'(x_0-y)+K_P'(x_0+y)\right)\mathrm{d}y(\mu^2-\varphi(c)^2),
\end{aligned}$$

for some $c\in(-P/2,x_0)$. As $-\mu<\varphi<\mu$ on $(-P/2,0)$, $(\mu^2-\varphi(c)^2)>0$, which is contradiction unless $\int_{x_0}^{0}\left(K_P'(x_0-y)+K_P'(x_0+y)\right)\mathrm{d}y=0$. That can only happen if $x_0=0$. This proves part (i).

At any point x_0 where $\varphi(x_0)=\mu$, (3.8) reduces to

$$(\varphi(x_0)-\varphi(x))^2=3\left((L(\varphi^2)(x_0)-L(\varphi^2)(x)\right). \tag{3.9}$$

From (3.9), we get in the real line case that

$$\begin{aligned}
(\varphi(0)-\varphi(x))^2=&\frac{3}{2}\int_{\mathbb{R}}(2K(y)-K(x+y)-K(x-y))(\varphi(y))^2\,\mathrm{d}y\\
\le&\frac{3}{2}\int_{|y|<|x|}(2K(y)-K(x+y)-K(x-y))(\varphi(y))^2\,\mathrm{d}y\\
\le&\frac{3}{2}\|\varphi\|^2_{L^\infty(\mathbb{R})}\int_{|y|<|x|}|2K(y)-K(x+y)-K(x-y)|\,\mathrm{d}y, \quad (3.10)
\end{aligned}$$

where we used that the first integral on the right-hand side is clearly non-negative, while $2K(y)-K(x+y)-K(x-y)<0$ when $|y|\ge|x|$. Moreover, there is a constant C that can be chosen independently of x such that

$$|2K(y)-K(x+y)-K(x-y)|\le C|x|, \tag{3.11}$$

for all $y\in\mathbb{R}$. Taking the square root on each side of (3.10) we then get that

$$|\varphi(0)-\varphi(x)|\le C'\|\varphi\|_{L^\infty(\mathbb{R})}|x|=C'\mu|x|.$$

This proves that φ is Lipschitz at 0. For the periodic kernel, we have that $2K_P(y)-K_P(x+y)-K_P(x-y)<0$ when $|x|\le|y|\le P/2-|x|$ (we are only interested in x close to 0, so we can assume $|x|<P/2-|x|$). In the intervals $|y|<|x|$ and $P/2-|x|<|y|\le P/2$, (3.11) holds for K_P and we therefore get the same result.

It remains to show the opposite inequality, i.e. that $|\mu-\varphi(x)|\gtrsim|x|$ near $x=0$; in particular this implies that $\varphi\notin C^1$. As φ is smooth on $(-P/2,0)$ and (at least) Lipschitz in 0, we can use integration by parts for $x \in (-P/2, 0)$ to get

$$\begin{aligned}(\mu-\varphi(x))\varphi'(x) =&\frac{3}{2}L(\varphi^2)'(x)\\ =&\frac{3}{2}\int_{-P/2}^{0}\left(K_P'(x-y)+K_P'(x+y)\right)(\varphi(y))^2\,\mathrm{d}y\\ =&3\int_{-P/2}^{0}\left(K_P(x-y)-K_P(x+y)\right)\varphi(y)\varphi'(y)\,\mathrm{d}y.\end{aligned}$$

As $\mu-\varphi(x)\leq C'\mu|x|$ for $x\in(-P/2,0)$ as shown above, we divide out $\mu-\varphi(x)$:

$$\varphi'(x)\geq C\int_{-P/2}^{0}\frac{K_P(x-y)-K_P(x+y)}{|x|}\varphi'(y)\varphi(y)\,\mathrm{d}y,$$

for some constant $C>0$ independent of x. Let $x\in(-P/2,0)$. By the mean value theorem,

$$\frac{|\mu-\varphi(x)|}{|x|}=\varphi'(\xi)\geq C\int_{-P/2}^{0}\frac{K_P(\xi-y)-K_P(\xi+y)}{|\xi|}\varphi'(y)\varphi(y)\,\mathrm{d}y \tag{3.12}$$

for some $\xi\in(x,0)$. It suffices to show that this is bounded below by a positive constant as $x\nearrow 0$, but while φ' is defined for all $x\in(-P/2,0)$, the limit may not exist. We therefore consider the limit infimum. On the other hand, the limit of the integral on the right hand side exists. Indeed, by direct calculation, one finds that

$$\lim_{\xi\nearrow 0}\frac{K_P(\xi-y)-K_P(\xi+y)}{|\xi|}=\frac{\mathrm{e}^{2y}\mathrm{e}^{P}-1}{\mathrm{e}^{y}\mathrm{e}^{P}-\mathrm{e}^{y}}.$$

This function is strictly monotonically increasing on $(-P/2,0)$, going from 0 to 1, and as φ is non-decreasing on this interval, we get by Lebesgue's dominated convergence theorem that for any sequence $\{\xi_n\}_n\subset(-P/2,0)$ such that $\xi_n\to 0$,

$$\begin{aligned}&\lim_{n\to\infty}C\int_{-P/2}^{0}\frac{K_P(\xi_n-y)-K_P(\xi_n+y)}{|\xi_n|}\varphi'(y)\varphi(y)\,\mathrm{d}y\\ =&C\int_{-P/2}^{0}\lim_{n\to\infty}\frac{K_P(\xi_n-y)-K_P(\xi_n+y)}{|\xi_n|}\varphi'(y)\varphi(y)\,\mathrm{d}y\\ \geq&C''\int_{-P/2}^{0}\varphi'(y)\varphi(y)\,\mathrm{d}y\\ =&\frac{C''}{2}(\mu^2-(\varphi(-P/2))^2)>0.\end{aligned}$$

In particular the limit exists and therefore equals the limit infimum and from (3.12) it follows that for any sequence $\{x_n\}_n\subset(-P/2,0)$, and by symmetry indeed any sequence in $(-P/2, P/2)$, such that $x_n \to 0$,

$$\liminf_{n\to\infty} \frac{|\mu - \varphi(x_n)|}{|x_n|} \gtrsim 1.$$

As the sequence was arbitrary this proves (iii).

Since $\varphi \in L^\infty(\mathbb{R})$ is symmetric and $\varphi' \geq 0$, and therefore also $L(\varphi^2)' \geq 0$, on $(-P/2, 0)$, we have that for $x < 0$

$$\begin{aligned}\left(L(\varphi^2)\right)'(x) &= \int_{\mathbb{R}} K'(x-y)(\varphi(y))^2 \,\mathrm{d}y \\ &= \int_{-\infty}^{0} (K'(x-y) + K'(x+y))(\varphi(y))^2 \,\mathrm{d}y \\ &\leq \int_{x}^{0} (K'(x-y) + K'(x+y))(\varphi(y))^2 \,\mathrm{d}y \\ &\leq C|x|,\end{aligned}$$

for some constant $C > 0$, where we used that K is completely monotone on $(0, \infty)$ and that the integrand is L^∞. The results above imply that $(\mu - \varphi(x)) \geq C'|x|$ for some constant C' independent of x when $\varphi(x) > \frac{\mu}{4}$ and from the equation

$$(\mu - \varphi(x))\varphi'(x) = 3\left(L(\varphi^2)\right)'(x) \leq \min(L(\varphi^2)(0), C|x|),$$

which holds for $x \leq 0$, we then see that φ' is uniformly bounded on the closed interval $[-P/2, 0]$ and therefore Lipschitz. This proves (ii). □

Remark 3.7 (On cuspons). The equality (3.9) holds when $\varphi(x_0) = \mu$ for any solution of (1.2), regardless of the integration constant a, and we have therefore shown that any L^∞ solution of (1.2) is at least Lipschitz continuous. We have not yet proved that a solution that touches the line μ exists, but any that do will be Lipschitz. This means that there are no L^∞ cuspons for the DP equation.

4. Global bifurcation

We now fix $\alpha \in (1, 2)$ and consider $C^\alpha_{\text{even}}(\mathbb{S}_P)$, the space of even, real-valued functions on the circle $\mathbb{S}_P$ of finite circumference $P > 0$ that are $\lfloor\alpha\rfloor$-times differentiable with the $\lfloor\alpha\rfloor$ derivative being $\alpha - \lfloor\alpha\rfloor$-Hölder continuous.

According to [8] one does not have periodic peakons when $a = 0$ in (1.2), only a one-parameter family of smooth periodic solutions and a peaked solitary wave. As our final goal is to find a bifurcation curve that converges to a peaked solution, the case $a \leq 0$ is not interesting for finite periods. Peaked periodic waves can only occur when $0 < a < \mu^2$.

Remark 4.1. As one can easily check (following the procedure below), for $a = 0$ one can only do local bifurcation form the curve $(\varphi, \mu) = (\mu/2, \mu)$ of constant solutions only when the period is $\sqrt{2}\pi$, but this curve cannot be extended to a global one. When $-\mu^2/8 < a < 0$ all the results regarding bifurcation below holds for periods $0 < P < \sqrt{2}\pi$ and we get global bifurcation curves. However, in this case $\sqrt{-8a} < \mu < \infty$ and the equivalent of Lemma 4.8 does not hold. That is, we cannot preclude that alternative (ii) in Theorem 4.5 occurs by $\mu(s)$ approaching $\sqrt{-8a}$.

Fix $a > 0$ and let $F : C^\alpha_{\text{even}}(\mathbb{S}_P) \times \mathbb{R} \to C^\alpha_{\text{even}}(\mathbb{S}_P)$ be the operator defined by

$$F(\varphi, \mu) = \mu\varphi - \frac{3}{2}L(\varphi^2) - \frac{1}{2}\varphi^2 + a. \tag{4.1}$$

Then $F(\varphi(s), \mu(s)) = 0$ on the curve $(\varphi(s), \mu(s)) = (\frac{s}{4} + \frac{\sqrt{s^2+8a}}{4}, s)$ for all $s \in \mathbb{R}$. Let

$$\lambda(\mu) := \frac{\mu}{4} + \frac{\sqrt{\mu^2 + 8a}}{4}$$

and set

$$\tilde{F}(\varphi, \mu) = F(\lambda(\mu) - \varphi, \mu) = (\lambda - \mu)\varphi + 3\lambda L(\varphi) - \frac{3}{2}L(\varphi^2) - \frac{1}{2}\varphi^2. \tag{4.2}$$

Then $\tilde{F}(0, \mu) = 0$ for all $\mu \in \mathbb{R}$ and

$$\mathrm{D}_\varphi \tilde{F}[0, \mu] = (\lambda(\mu) - \mu)\,\mathrm{id} + 3\lambda(\mu)L.$$

When $\mu^2 > a$ we have that $4\lambda > \mu$ while $\mu > \lambda$, and we get that

$$\ker \mathrm{D}_\varphi \tilde{F}[0, \mu] = \{C \cos\left(\sqrt{\frac{4\lambda - \mu}{\mu - \lambda}}x\right) : C \in \mathbb{R}\}.$$

Restricting to P-periodic functions, the kernel is one-dimensional if and only if $\sqrt{\frac{4\lambda-\mu}{\mu-\lambda}} = \frac{2k\pi}{P}$ for some $k \in \mathbb{N}$. Clearly, $\sqrt{\frac{4\lambda(\mu)-\mu}{\mu-\lambda(\mu)}}$ is continuous in μ for $\mu \in (\sqrt{a}, \infty)$, strictly monotone on this interval, bounded below by $\sqrt{2}$, the bound being achieved in the limit as $\mu \to \infty$, and unbounded above as $\mu^2 \searrow a$. This means that for every $P > 0$, for each $k \in \mathbb{N}$ such that $\frac{2k\pi}{P} > \sqrt{2}$ there exists a unique $\mu > \sqrt{a}$ such that $\cos\left(\sqrt{\frac{4\lambda-\mu}{\mu-\lambda}}x\right) \in C^\alpha_{\text{even}}(\mathbb{S}_P)$. When $P \geq \sqrt{2}\pi$, we get that $k > 1$.

Theorem 4.2 (Local bifurcation)**.** *Fix $a > 0$ and $P > 0$, and let F and $\tilde{F}$ be defined as in* (4.1) *and* (4.2)*, respectively. Then for each $k \in \mathbb{N}$ such that $\frac{2k\pi}{P} > \sqrt{2}$, there exists a unique $\mu_k \in (\sqrt{a}, \infty)$ such that $(0, \mu_k)$ is a bifurcation point for $\tilde{F}$. That is, there exists $\varepsilon > 0$ and an analytic curve*

$$s \mapsto (\varphi(s), \mu(s)) \subset C^\alpha_{even}(\mathbb{S}_P) \times (\sqrt{a}, \infty), \quad |s| < \varepsilon,$$

of nontrivial P/k-periodic solutions, where $\mu(0) = \mu_k$ and

$$\mathrm{D}_s\varphi(0) = \cos\left(\sqrt{\frac{4\lambda(\mu_k) - \mu_k}{\mu_k - \lambda(\mu_k)}}x\right).$$

Proof. It is sufficient to consider only the case $k = 1$ and $P < \sqrt{2}\pi$. As shown above, there exists a unique $\mu \in (\sqrt{a}, \infty)$ such that $\ker \mathrm{D}_\varphi \tilde{F}[0,\mu]$ is one-dimensional. The space $C^\alpha_{\text{even}}(\mathbb{S}_P)$ has basis $\{\cos(\frac{2\pi}{P}k\cdot) : k \in \mathbb{N}\}$ and by straightforward calculation one finds that $\mathrm{D}_\varphi \tilde{F}[0,\mu]$ maps the basis element $k = 1$ to zero while all others are preserved modulo a constant. In other words, $\operatorname{codim} \operatorname{range} \mathrm{D}_\varphi \tilde{F}[0,\mu] = 1$ and $\mathrm{D}_\varphi \tilde{F}[0,\mu]$ is Friedholm of index zero. The result now follows from Theorem 8.3.1 in [1]. □

We want to extend these bifurcation curves globally. Let

$$U := \{(\varphi,\mu) \in C^\alpha_{\text{even}}(\mathbb{S}_P) \times (\sqrt{a}, \infty) : \varphi < \mu\}.$$

and

$$S := \{(\varphi,\mu) \in U : \tilde{F}(\varphi,\mu) = 0\}.$$

Lemma 4.3. *Whenever $(\varphi,\mu) \in S$ the function φ is smooth, and bounded and closed subsets of S are compact in $C^\alpha_{even}(\mathbb{S}_P) \times (\sqrt{a}, \infty)$.*

Proof. Can be proved in the same way as Lemma 4.3 in [6]. □

For simplicity we consider the case $P < \sqrt{2}\pi$ and $k = 1$. Let $\mu^* := \mu_1$ and

$$\varphi^*(x) := \cos\left(\frac{2\pi}{P}x\right), \tag{4.3}$$

and let furthermore

$$M := \{\sum_{k\neq 1} a_k \cos\left(\frac{2\pi k x}{P}\right) \in C^\alpha_{\text{even}}(\mathbb{S}_P)\},$$

and

$$N := \ker \mathrm{D}_\varphi \tilde{F}[0,\mu^*] = \operatorname{span}(\varphi^*).$$

Then $C^\alpha_{\text{even}}(\mathbb{S}_P) = M \oplus N$ and we can use the canonical embedding $C^\alpha(\mathbb{S}_P) \hookrightarrow L^2(\mathbb{S}_P)$ to define a continuous projection

$$\Pi\varphi = \langle \varphi, \varphi^* \rangle_{L^2(\mathbb{S}_P)} \varphi^*, \tag{4.4}$$

where $\langle u, v\rangle_{L^2(\mathbb{S}_P)} = \frac{2}{P}\int_{-P/2}^{P/2} uv \,\mathrm{d}x$.

Theorem 4.4 (Lyapunov-Schmidt reduction)**.** *There exists a neighbourhood $O \times Y \subset U$ around $(0,\mu^*)$ in which the problem*

$$\tilde{F}(\varphi,\mu) = 0 \tag{4.5}$$

is equivalent to

$$\Phi(\varepsilon\varphi^*,\mu) := \Pi\tilde{F}(\varepsilon\varphi^* + \psi(\varepsilon\varphi^*,\mu),\mu) = 0 \tag{4.6}$$

for functions $\psi \in C^\infty(O_N \times Y, M)$, $\Phi \in C^\infty(O_N \times Y, N)$, and $O_N \subset N$ an open neighbourhood of the zero function in N. One has $\Phi(0,\mu^*) = 0$, $\psi(0,\mu^*) =$

0, $\mathrm{D}_\varphi\psi(0,\mu^*)=0$, *and solving the finite dimensional problem* (4.6) *provides a solution* $\varphi=\varepsilon\varphi^*+\psi(\varepsilon\varphi^*,\mu)$ *to the infinite dimensional problem* (4.5).

We want to show that $\mu(\varepsilon)$ is not constant around 0. We calculate

$$\begin{aligned}\mathrm{D}^2_{\varphi\varphi}\tilde{F}[0,\mu^*](\varphi^*,\varphi^*) &= -(\varphi^*)^2-3L((\varphi^*)^2),\\ \mathrm{D}^2_{\mu\varphi}\tilde{F}[0,\mu^*]\varphi^* &= (\lambda'(\mu^*)-1)\varphi^*+3\lambda'(\mu^*)L(\varphi^*).\end{aligned}$$

As $L(\cos(p\cdot))(x)=\frac{1}{1+p^2}\cos(px)$ for $p\neq 0$, we get that

$$\mathrm{D}^2_{\mu\varphi}\tilde{F}[0,\mu^*]\varphi^*=\left(\lambda'(\mu^*)(1+\frac{3}{1+(2\pi/P)^2})-1\right)\varphi^*.$$

By choice $\sqrt{\frac{4\lambda(\mu^*)-\mu^*}{\mu^*-\lambda(\mu^*)}}=\frac{2\pi}{P}$, so that the coefficient of φ^* above is zero if and only if

$$\lambda'(\mu^*)=\frac{\lambda(\mu^*)}{\mu^*}.$$

This is impossible, as the left-hand side lies in $(\frac{1}{3},\frac{1}{2})$ when $\mu^*\in(\sqrt{a},\infty)$, while the right-hand side lies in $(\frac{1}{2},1)$.

Using bifurcation formulas, we readily calculate $\dot{\mu}(0)$:

$$\dot{\mu}(0)=-\frac{1}{2}\frac{\langle \mathrm{D}^2_{\varphi\varphi}\tilde{F}[0,\mu^*](\varphi^*,\varphi^*),\varphi^*\rangle_{L^2(\mathbb{S}_P)}}{\langle \mathrm{D}^2_{\mu\varphi}\tilde{F}[0,\mu^*]\varphi^*,\varphi^*\rangle_{L^2(\mathbb{S}_P)}}=0.$$

When $\dot{\mu}(0)=0$, one has that

$$\ddot{\mu}(0)=-\frac{1}{3}\frac{\langle \mathrm{D}^3_{\varphi\varphi\varphi}\Phi[0,\mu^*](\varphi^*,\varphi^*,\varphi^*),\varphi^*\rangle_{L^2(\mathbb{S}_P)}}{\langle \mathrm{D}^2_{\mu\varphi}\tilde{F}[0,\mu^*]\varphi^*,\varphi^*\rangle_{L^2(\mathbb{S}_P)}}.$$

The denominator equals $\left(\lambda'(\mu^*)(1+\frac{1}{1+(2\pi/P)^2})-1\right)\neq 0$. Using that F is quadratic in φ, one can calculate that

$$\begin{aligned}&\mathrm{D}^3_{\varphi\varphi\varphi}\Phi[\varphi,\mu](\varphi^*,\varphi^*,\varphi^*)\\ &=3\,\Pi\,\mathrm{D}^2_{\varphi\varphi}\tilde{F}[\varphi+\psi(\varphi,\mu),\mu](\varphi^*+\mathrm{D}_\varphi\psi[\varphi,\mu]\varphi^*,\mathrm{D}^2_{\varphi\varphi}\psi[\varphi,\mu](\varphi^*,\varphi^*))\\ &\quad+\Pi\,\mathrm{D}_\varphi\tilde{F}[\varphi+\psi(\varphi,\mu),\mu]\mathrm{D}^3_{\varphi\varphi\varphi}\psi[\varphi,\mu](\varphi^*,\varphi^*,\varphi^*).\end{aligned}$$

As $N=\ker\mathrm{D}_\varphi\tilde{F}[0,\mu^*]$, we get that the projection $\Pi\,\mathrm{D}_\varphi\tilde{F}[0,\mu^*]=0$. Using that $\psi(0,\mu^*)=\mathrm{D}_\varphi\psi[0,\mu^*]=0$ and the expression for $\mathrm{D}^2_{\varphi\varphi}\tilde{F}[0,\mu^*]$ above, one finds that

$$\begin{aligned}&\mathrm{D}^3_{\varphi\varphi\varphi}\Phi[0,\mu^*](\varphi^*,\varphi^*,\varphi^*)\\ &=-\Pi\left(\varphi^*\mathrm{D}^2_{\varphi\varphi}\psi[0,\mu^*](\varphi^*,\varphi^*)+3L(\varphi^*\mathrm{D}^2_{\varphi\varphi}\psi[0,\mu^*](\varphi^*,\varphi^*))\right).\end{aligned}\tag{4.7}$$

We can rewrite $\mathrm{D}^2_{\varphi\varphi}\psi[0,\mu^*](\varphi^*,\varphi^*)$ as

$$\begin{aligned}
\mathrm{D}^2_{\varphi\varphi}\psi[0,\mu^*](\varphi^*,\varphi^*) &= -\left(\mathrm{D}_\varphi\tilde{F}[0,\mu^*]\right)^{-1}(\mathrm{id}-\Pi)\mathrm{D}^2_{\varphi\varphi}\tilde{F}[0,\mu^*](\varphi^*,\varphi^*) \\
&= \left(\mathrm{D}_\varphi\tilde{F}[0,\mu^*]\right)^{-1}\left((\varphi^*)^2+3L((\varphi^*)^2)\right) \\
&= \left(\mathrm{D}_\varphi\tilde{F}[0,\mu^*]\right)^{-1}\left(2+(\frac{1}{2}+\frac{3P^2}{16\pi^2+P^2})\cos\left(\frac{4\pi}{P}x\right)\right) \\
&= \frac{2}{\lambda(\mu^*)-\mu^*} \\
&\quad+\frac{16\pi^2+7P^2}{2((4\lambda(\mu^*)-\mu^*)P^2+16\pi^2(\lambda(\mu^*)-\mu^*)}\cos\left(\frac{4\pi}{P}x\right),
\end{aligned}$$

where we used that $L(\cos(p\cdot))(x)=\frac{1}{1+p^2}\cos(px)$ for $p\neq 0$. Multiplying with $\varphi^*(x)=\cos\left(\frac{2\pi}{P}x\right)$ and using double and triple angle formulas, we get

$$\begin{aligned}
\frac{2\cos(2\pi x/P)}{\lambda(\mu^*)-\mu^*}&+\frac{1}{2}\frac{16\pi^2+7P^2}{2((4\lambda(\mu^*)-\mu^*)P^2+16\pi^2(\lambda(\mu^*)-\mu^*)}\cos\left(\frac{2\pi}{P}x\right) \\
&+\frac{1}{2}\frac{16\pi^2+7P^2}{2((4\lambda(\mu^*)-\mu^*)P^2+16\pi^2(\lambda(\mu^*)-\mu^*)}\cos\left(\frac{6\pi}{P}x\right).
\end{aligned}$$

Denoting by C be the coefficient of $\cos\left(\frac{2\pi}{P}x\right)=\varphi^*(x)$ in the above expression, we see from (4.7) that

$$\mathrm{D}^3_{\varphi\varphi\varphi}\Phi[0,\mu^*](\varphi^*,\varphi^*,\varphi^*)=-C\left(1+\frac{3P^2}{P^2+4\pi^2}\right)\varphi^*.$$

Hence $\ddot{\mu}(0)\neq 0$ and $\dot{\mu}\not\equiv 0$ on $(-\varepsilon,\varepsilon)$. Lemma 4.3 and the calculations above show that the conditions of Theorem 9.1.1 in [1] are fulfilled and we have the following result:

Theorem 4.5 (Global bifurcation)**.** *The local bifurcation curves $s\mapsto(\varphi(s),\mu(s))$ of solutions to the Degasperis-Procesi equation from Theorem 4.2 extend to global continuous curves $\mathfrak{R}$ of solutions $\mathbb{R}_{\geq 0}\to S$. One of the following alternatives hold:*

(i) $\|(\varphi(s),\mu(s))\|_{C^\alpha(\mathbb{S}_P)\times\mathbb{R}}\to\infty$ *as* $s\to\infty$.
(ii) $(\varphi(s),\mu(s))$ *approaches the boundary of* U *as* $s\to\infty$.
(iii) *The function* $s\mapsto(\varphi(s),\mu(s))$ *is (finitely) periodic.*

Theorem 4.6. *Alternative (iii) in Theorem 4.5 cannot occur.*

Proof. Let

$$\mathcal{K}:=\{\varphi\in C^\alpha_{\mathrm{even}}(\mathbb{S}_P):\varphi\text{ is non-decreasing on }(-P/2,0)\},$$

which is a closed cone in $C^\alpha(\mathbb{S}_P)$, and let $\mathfrak{R}^1$ and S^1 denote the φ parts of $\mathfrak{R}$ and S respectively. The result follows from Theorem 9.2.2 in [1] if we can show that if $\varphi \in \mathrm{R}^1 \cap \mathcal{K}$ is non-constant, then φ is an interior point of $S^1 \cap \mathcal{K}$. To see this, let φ be a non-constant solution that is non-decreasing on $(-P/2, 0)$. By Theorem 3.4, φ is smooth and we can apply Theorem 3.2 to conclude that $\varphi''(0) < 0$ and either $\varphi > \frac{\mu - \sqrt{\mu^2+8a}}{4}$ and $\varphi' > 0$ on $(-P/2, 0)$, or $\varphi = \frac{\mu - \sqrt{\mu^2+8a}}{4}$ on $[-P/2, x_0]$, for some $x_0 \in [-P/2, 0)$, and $\varphi' > 0$ on $(x_0, 0)$. Let ϕ be a solution within $\delta \ll 1$ distance of φ in C^α, with δ small enough that $\phi < \mu$. Iterating as in the proof of Theorem 3.4, we get that $\|\varphi - \phi\|_{C^2} < \tilde{\delta}$, where $\tilde{\delta}$ can be made arbitrarily small by taking δ smaller. This implies that ϕ also is non-decreasing on $(x_0, 0)$, and by Theorem 3.2 and the subsequent remark we conclude that it must be non-decreasing on $(-P/2, x_0)$ as well. Hence $\phi \in S^1 \cap \mathcal{K}$. $\square$

Lemma 4.7. *Any sequence $\{(\varphi_n, \mu_n)\}_n \subset S$ of solutions to (1.2) with $\{\mu_n\}_n$ bounded has a subsequence that converges uniformly to a solution φ.*

Proof. From (1.2) we have that

$$\frac{1}{2}\varphi^2 = a + \mu\varphi - \frac{3}{2}L(\varphi^2) < a + \mu\varphi,$$

which implies that

$$\|\varphi\|_{L^\infty}^2 \leq 2a + 2\mu\|\varphi\|_{L^\infty}.$$

Hence $\{\varphi_n\}_n$ is bounded whenever $\{\mu_n\}_n$ is. We have that

$$\begin{aligned} |L(\varphi_n^2)(x+h) - L(\varphi_n^2)(x)| &= \left| \int_{\mathbb{R}} (K(x+h-y) - K(x-y))\varphi_n(y)^2 \,\mathrm{d}y \right| \\ &\leq \|\varphi_n\|_{L^\infty}^2 \int_{\mathbb{R}} |K(x+h-y) - K(x-y)| \,\mathrm{d}y. \end{aligned}$$

As K is continuous and integrable, the final integral can be made arbitrarily small by taking h sufficiently small. This shows that $\{L(\varphi_n^2)\}_n$ is equicontinuous. Arzela-Ascoli's theorem then implies the existence of a uniformly convergent subsequence. $\square$

Lemma 4.8. *For fixed $a > 0$ and $P > 0$, $\mu(s)$ does not approach $\sqrt{a}$ as $s \to \infty$.*

Proof. Assume for a contradiction that there is a sequence $\{\mu_n\}_n$ such that $\mu_n \to \sqrt{a}$ as $n \to \infty$, while at the same time $\varphi_n = \varphi_{\mu_n}$ is a sequence along the global bifurcation curve in Theorem 4.5. According to Lemma 4.7 a subsequence $\{\varphi_{n_k}\}_k$ converges to a solution φ_0 of (1.2). From Theorem 3.1 we have that $\max \varphi_{n_k} > \frac{\mu_{n_k} + \sqrt{\mu_{n_k}^2 + 8a}}{4} > \sqrt{a}$, while $\max \varphi_{n_k} < \mu_{n_k} \to \sqrt{a}$. It follows that $\max \varphi_0 = \sqrt{a}$ and hence $\max L(\varphi_0^2) = a$. However, $\max L(\varphi^2) \leq \max \varphi^2$ with equality if and only if φ is constant. Hence $\varphi_0 \equiv \sqrt{a}$. This leads to a contradiction with Lemma 3.5, noting that the constant $C_{K,P,\mu}$ is positive for all positive μ, as we get that

$$0 = \lim_{k\to\infty} \mu_{n_k} - \varphi_{n_k}(P/2) \geq \lim_{k\to\infty} C_{K,P,\mu_{n_k}} > 0.$$

□

Lemma 4.9. *Let $a > 0$ and $P > 0$. If $\sup_{s\geq 0} \mu(s) < \infty$, then alternatives (i) and (ii) in Theorem 4.5 both occur.*

Proof. We already know from Theorem 4.6 that alternative (iii) cannot occur, thus either (i), (ii), or both has to occur. Theorem 3.6 implies that alternative (i) happens if $\lim_{s\to\infty} \mu(s) - \varphi(s)(0) = 0$. From

$$(\mu - \varphi)\varphi' = \frac{3}{2}\left(L(\varphi^2)\right)' \leq \frac{3}{2}L(\varphi^2),$$

we see that φ' is bounded in μ. Similarly, it is easy to see that if $\varphi(0) < \mu$, then $\|\varphi\|_{C^2(\mathbb{S}_P)}$ is bounded in μ. Hence, if $\sup_{s\geq 0} \mu(s) < \infty$, alternative (i) happens if and only if $\lim_{s\to\infty} \mu(s) - \varphi(s)(0) = 0$, which implies that (ii) occurs as well.

From Lemma 4.8 we know that $\inf_{s\geq 0} \mu(s) > \sqrt{a}$ and the assumption that $\mu(s)$ is bounded above uniformly in s then implies that $\mu(s)$ does not approach the boundary of $(\sqrt{a}, \infty)$. Thus alternative (ii) can only happen if $\lim_{s\to\infty} \mu(s) - \varphi(s)(0) = 0$, which in turn implies (i). □

Proposition 4.10. *For fixed $a > 0$, there is a number $C > 0$ such that if $P < C$, there is an upper bound on μ above which there are no smooth solutions to* (1.2) *except constant solutions.*

Proof. Assume φ is a smooth solution to (1.2) which is even and non-decreasing on $(-P/2, 0)$ (recall that φ is smooth if $\varphi(0) < \mu$, and a peakon if $\varphi(0) = \mu$; no other possibilities exists). We know that φ' has a maximum on $(-P/2, 0)$, say $\varphi'(x_0) = \max \varphi$. Then $\varphi''(x_0) = 0$. As

$$(\mu - \varphi(x))\varphi''(x) = (\varphi'(x))^2 + 3\int_{-P/2}^{0} (K_P'(x-y) - K_P'(x+y))\varphi(y)\varphi'(y)\,\mathrm{d}y,$$

and $K_P'(x-y) - K_P'(x+y) > 0$ for $x < y < 0$, we then get that

$$\begin{aligned}(\varphi'(x_0))^2 &= -3\int_{-P/2}^{0} (K_P'(x_0-y) - K_P'(x_0+y))\varphi(y)\varphi'(y)\,\mathrm{d}y \\ &\leq -3\int_{-P/2}^{x_0} (K_P'(x_0-y) - K_P'(x_0+y))\varphi(y)\varphi'(y)\,\mathrm{d}y \\ &= \varphi(c_0)\varphi'(c_0)3\left|\int_{-P/2}^{x_0} (K_P'(x_0-y) - K_P'(x_0+y))\,\mathrm{d}y\right|,\end{aligned}$$

where $-P/2 < c_0 < x_0$. As $\varphi'(c_0) < \varphi'(x_0)$ and $\varphi(c_0) < \mu$, it follows that $\max \varphi' < C_P\mu$, where the constant C_P depends on P through the final integral above. As $K_P(x) = \frac{1}{2}\mathrm{e}^{-|x|} + \frac{\cosh(x)}{\mathrm{e}^P - 1}$, the derivative is bounded by $1/2$ and the final integral above, hence also C_P, therefore goes to 0 as $P \to 0$. From Theorem 3.1, we know that a solution φ satisfies

$$\min \varphi < \frac{\mu + \sqrt{\mu^2 + 8a}}{4} < \max \varphi.$$

If $\mu \ggg a$, then $\frac{\mu + \sqrt{\mu^2+8a}}{4} = \frac{\mu}{2} + \mathcal{O}(\mu^{-1})$. Hence there exists a point $x_1 \in (-P/2, 0)$ such that $\varphi(x_1) = \frac{\mu}{2} + \mathcal{O}(\mu^{-1})$. Trivially, for every $x \in (-P/2, 0)$ we have the bounds

$$\varphi(x_1) - (P/2) \max \varphi' < \varphi(x) < \varphi(x_1) + (P/2) \max \varphi'.$$

Combining this with the bound on the derivative above, we get

$$\max \varphi < \frac{\mu}{2} + \frac{P}{2} C_P \mu + \mathcal{O}(\mu^{-1}).$$

For any $c \in (\frac{1}{2}, 1)$ we can take $P > 0$ sufficiently small independently of μ such that

$$\max \varphi \leq c\mu + \mathcal{O}(\mu^{-1}). \tag{4.8}$$

By the mean value theorem,

$$\begin{aligned}
(\mu - \varphi(x))\varphi'(x) &= \frac{3}{2} \left(L(\varphi^2)\right)'(x) \\
&= 3 \int_{-P/2}^{0} (K_P(x-y) - K_P(x+y))\varphi'(y)\varphi(y) \,\mathrm{d}y \\
&= 3\varphi'(c_x)\varphi(c_x) \int_{-P/2}^{0} (K_P(x-y) - K_P(x+y)) \,\mathrm{d}y,
\end{aligned}$$

for some constant c_x that depends on x. From (4.8) we get that there is a constant C independent of μ and decreasing in P such that $\varphi(c_x)/(\mu - \varphi(x)) \leq C + \mathcal{O}(\mu^{-2})$ for all $x \in (-P/2, 0)$. We therefore get that

$$\varphi'(x) \leq \varphi'(c_x) C \int_{-P/2}^{0} (K_p(x-y) - K_P(x-y)) \,\mathrm{d}y + \mathcal{O}(\mu^{-1}), \tag{4.9}$$

where C is independent of μ and decreases with P. The integral on the right hand side goes to 0 for all $x \in (-P/2, 0)$ as $P \to 0$. For P sufficiently small, (4.9) implies that $\varphi' \equiv 0$ for all sufficiently large μ. $\square$

Theorem 4.11. *Let $a > 0$ be fixed. For all $P > 0$ sufficiently small, alternatives (i) and (ii) in Theorem 4.5 both occur. Given any unbounded sequence of positive numbers s_n, a subsequence of $\{\varphi(s_n)\}_n$ converges uniformly to a limiting wave φ that solves* (1.2) *and satisfies*

$$\varphi(0) = \mu, \quad \varphi \in C^{0,1}(\mathbb{R}).$$

The limiting wave is even, strictly increasing on $(-P/2, 0)$ *and is exactly Lipschitz at* $x \in P\mathbb{Z}$.

Proof. From Theorem 4.6, we know that alternative (iii) cannot occur. The proof of Theorem 4.6 also implies that the curve $(\varphi(s), \mu(s))$ cannot reconnect to the curve of constant solutions we bifurcated from for any finite s. Hence Proposition 4.10 implies that for all $P > 0$ sufficiently small, $\sup_{s\geq 0} \mu(s) < \infty$, and by Lemma 4.9 we get that alternatives (i) and (ii) both occur. Moreover, as $\{\mu(s_n)\}_n$ is bounded, Lemma 4.7 gives that a subsequence of $\{\varphi(s_n)\}_n$ converges uniformly to a solution φ. As alternatives (i) and (ii) both occur, this solution must necessarily have the stated properties. $\square$

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Paper IV

On conditional energetic stability of gravity-capillary solitary water waves with non-constant vorticity function

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In preparation

Paper IV

This article is awaiting publication and is therefore not included.