Two-Sided Boundary Control and State Estimation of 2×2 Semilinear Hyperbolic Systems

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***Abstract*— We solve the problem of controlling a class of one-dimensional semilinear 2×2 hyperbolic systems to the origin in minimum time using actuation at both boundaries of the domain. The control method can also be used to solve a class of tracking problems. For the special case of time-invariant linear systems, the state-feedback control law can be written explicitly as the inner product of kernels with the state. We further design an observer to estimate the distributed state from measurements at both boundaries, also in minimum time. The state-feedback controller and observer are combined to solve the output-feedback control problem. A numerical example is given to demonstrate the controller performance.**

I. INTRODUCTION

The control of one-dimensional hyperbolic partial differential equations(PDEs) has received significant attention since they model many relevant systems, such as water channels [1], [2], [3], gas pipelines [4], road traffic [5], and oil wells [6]. One approach of stabilizing such systems is to design dissipative boundary conditions, see e.g. [2], [3]. An alternative approach is to design the control input to drive the system to a desired state, as it is done in e.g. [1], [6].

There exist several results on the exact controllability of one-dimensional hyperbolic systems, see e.g. [7] for linear systems, [8] for a class of semilinear systems, and [9], [10] for local results for quasilinear systems. However, these papers discuss only the *existence* of *open-loop* control signals driving the state to the origin. Constructive methods for feedback control and state estimation have been developed recently in the form of backstepping for linear systems, see e.g. [11] and subsequent papers, and in [12] for semilinear systems. These papers consider actuation and sensing at one boundary of the domain. However, if actuation and measurements are available at both boundaries of the domain, the minimum times to control the system and to estimate the state are shorter [10]. This motivated the developed of explicit state-feedback laws based on backstepping transformations in [13] and [14]. However, these results are only for linear systems and under the somewhat restrictive assumption of constant and in the latter case equal transport speeds. They also do not consider the estimation problem. Another motivation for using actuation at both boundaries is that two objectives can be tracked simultaneously. Moreover, redundancy can be exploited to design e.g. fault-tolerant designs.

In this paper, we develop a state-feedback law to stabilize the origin of a semilinear hyperbolic system in minimum time using actuation at both boundaries. The controller design method can also be used to solve a class of tracking problems. Moreover, we solve the problem of estimating the state of a semilinear system from measurements at both boundaries in minimum time.

The paper is organized as follows. The class of systems is described in Section II. In Section III, a state-feedback controller is designed to solve the tracking (Section III-C) and stabilization (Section III-D) problems. For linear systems, an explicit state-feedback (i.e. the product of (infinite-dimensional) gains with the state) is derived in Section IV. An observer is designed in Section V which, combined with the state-feedback controller, is used to solve the output-feedback control problem in Section VI. Finally, the controller performance is demonstrated in a numerical example in Section VII.

II. SYSTEM DESCRIPTION

We consider systems of the form

$$u_t(x,t) = -\varepsilon_u(x) u_x(x,t) + F_u((u,v)(x,t),x,t), \quad (1)$$

$$v_t(x,t) = \varepsilon_v(x) v_x(x,t) + F_v((u,v)(x,t),x,t), \quad (2)$$

$$u(0,t) = U_1(t), \qquad v(1,t) = U_2(t), \quad (3)$$

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \quad (4)$$

where $x \in [0,1]$ and $t \geq 0$, the subscripts $_t$ and $_x$ denote partial derivatives with respect to t and x, respectively, and $U_1(t)$ and $U_2(t)$ are control inputs. The notation $(u,v)(x,t)$ in F_u and F_v is used to denote the state, (u,v) evaluated at (x,t). We consider the state space of bounded functions on $[0,1]$, $\mathscr{X} = \{f : [0,1] \to \mathbb{R} : |f(x)| < \infty \quad \forall x \in [0,1]\}$, and denote the spatial supremum norm by $\|f\| = \max_{x\in[0,1]} |f(x)|$ for $f \in \mathscr{X}$. The initial conditions u_0, v_0 are assumed to lie in $\mathscr{X}$. We also use the notation $\mathscr{X}_{[a,b]}$ to denote the space of bounded functions on the interval $[a,b]$.

We make the following assumptions on the system coefficients: There exist uniform bounds $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ such that $\varepsilon_1 < \varepsilon_u(x), \varepsilon_v(x) < \varepsilon_2$ for all $x \in [0,1]$. Moreover, there exists a Lipschitz constant L such that

$$|F_u((u_1,v_1),x,t) - F_u((u_2,v_2),x,t)| \leq L(|u_1 - u_2| + |v_1 - v_2|), \quad (5)$$

$$|F_v((u_1,v_1),x,t) - F_v((u_2,v_2),x,t)| \leq L(|u_1 - u_2| + |v_1 - v_2|), \quad (6)$$

for all (u_1,v_1), (u_2,v_2), x, t. The Lipschitz condition prevents finite-time blowup of the state. Thus, global existence of a solution of (1)-(4) can be guaranteed. We allow the nonlinearities to be time-varying, but need to assume that at every

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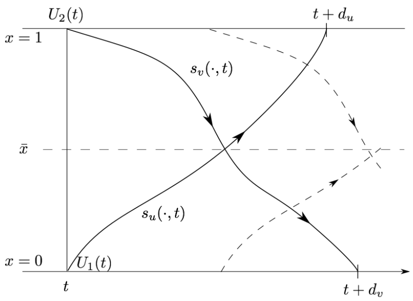

Fig. 1. Characteristic lines of u ("upwards") and v ("downwards"), as well as the characteristic lines s_u and s_v used in controller design.

time F_u and F_v (as functions of (u,v)) are exactly known $\phi_v(\bar{x})$ into the future (see the next section for definitions of ϕ_v and $\bar{x}$). For estimation, F_u and F_v must also be known $\hat{\phi}_v(\hat{x})$ into the past. Moreover, ε_u and ε_v must be measureable in x (that is, discontinuous ε_u and ε_v are possible), and F_u and F_v must be measureable in x and t. For the stabilization problem, we also need to assume that

$$F_u((0,0),x,t)=0, \qquad F_v((0,0),x,t)=0. \tag{7}$$

for all $x\in[0,1]$ and $t\geq 0$. Note that (7) implies that the origin is an equilibrium. A system can be controlled to any equilibrium by applying a state transformation to align the to-be stabilized equilibrium with the origin. For tracking, we do not need (7). Moreover, we consider the problem of estimating the distributed state from the boundary measurements

$$Y_1(t)=v(0,t), \qquad Y_2(t)=u(1,t). \tag{8}$$

III. STATE-FEEDBACK CONTROL

A. Preliminaries

Our analysis is based on the method of characteristics. Since the propagation speeds ε_u and ε_v are state independent, the characteristic lines are known a priori. We denote the characteristic lines along which $U_1(t)$ and $U_2(t)$ propagate by $s_u(\cdot,t)$ and $s_v(\cdot,t)$, respectively. They are depicted in Figure 1. We make the following definition

$$\phi_u(x)=\int_0^x \frac{1}{\varepsilon_u(\xi)}d\xi, \qquad \phi_v(x)=\int_x^1 \frac{1}{\varepsilon_v(\xi)}d\xi. \tag{9}$$

Since ε_u and ε_v are positive, ϕ_u and ϕ_v are well defined, strictly monotonically increasing and decreasing, respectively, and invertible. The characteristic lines can be written as

$$s_u(x,t)=(x,t+\phi_u(x)), \quad s_v(x,t)=(x,t+\phi_v(x)). \tag{10}$$

We denote the location at which $s_u(\cdot,t)$ and $s_v(\cdot,t)$ intersect by $\bar{x}$. At $\bar{x}$,

$$t+\phi_u(\bar{x})=t+\phi_v(\bar{x}). \tag{11}$$

Since the left-hand side of (11) is strictly monotonically increasing in $\bar{x}$ and the right-hand side of (11) is strictly monotonically decreasing in $\bar{x}$, (11) has a unique solution $\bar{x}$ for given ε_u and ε_v. Finally, we define the delays

$$d_u=\phi_u(1),\ d_v=\phi_v(0),\ d=\max\{d_u,d_v\},\ \bar{t}=\phi_v(\bar{x}). \tag{12}$$

Due to the finite propagation speed, the actuation does not affect the state in the whole domain immediately. Therefore, it is only possible to control the state in the interior of the domain some time into the future. More precisely, the input $U_1(t)$ at time t affects the state at some location $x\in[0,1]$ only at the future time $t+\phi_u(x)$. All states "before" $s_u(\cdot,t)$ are independent of $U_1(t)$. Analogously, $U_2(t)$ affects the state only at the future time $t+\phi_v(x)$. Therefore, we base our analysis on the dynamics on the characteristic lines s_u and s_v.

Definition 1: We define the states on $s_u(x,t)$ and on $s_v(x,t)$ as

$$\bar{u}^1(x,t)=u(x,t+\phi_u(x)), \quad \bar{v}^1(x,t)=v(x,t+\phi_u(x)), \tag{13}$$

$$\bar{u}^2(x,t)=u(x,t+\phi_v(x)), \quad \bar{v}^2(x,t)=v(x,t+\phi_v(x)). \tag{14}$$

Note that $u(x,t)=\bar{u}^1(x,t-\phi_u(x))=\bar{u}^2(x,t-\phi_v(x)$ and the same for v.

Theorem 2: For every t, there exists a continuous, bounded operator $\Phi^t:\mathscr{X}_{[0,1]}\times\mathscr{X}_{[0,1]}\to\mathscr{X}_{[0,\bar{x}]}\times\mathscr{X}_{[\bar{x},1]}$, independent of $U_1(t)$ and $U_2(t)$, such that

$$\left(\bar{v}^1([0,\bar{x}],t),\bar{u}^2([\bar{x},1],t)\right)=\Phi^t\left(u([0,1],t),v([0,1],t)\right). \tag{15}$$

Moreover, $(\bar{u}^1,\bar{v}^1,\bar{u}^2,\bar{v}^2)$ satisfies the PDE-ODE systems

$$\bar{u}^1_x(x,t)=\frac{1}{\varepsilon_u(x)}F_u((\bar{u}^1,\bar{v}^1)(x,t),x,t+\phi_u(x)), \tag{16}$$

$$\bar{v}^1_t(x,t)=\frac{\varepsilon_u(x)\varepsilon_v(x)}{\varepsilon_u(x)+\varepsilon_v(x)}\bar{v}^1_x+\frac{\varepsilon_u(x)}{\varepsilon_u(x)+\varepsilon_v(x)} \times F_v((\bar{u}^1,\bar{v}^1)(x,t),x,t+\phi_u(x)), \tag{17}$$

$$\bar{u}^2_t(x,t)=-\frac{\varepsilon_u(x)\varepsilon_v(x)}{\varepsilon_u(x)+\varepsilon_v(x)}\bar{u}^2_x+\frac{\varepsilon_v(x)}{\varepsilon_u(x)+\varepsilon_v(x)} \times F_u((\bar{u}^2,\bar{v}^2)(x,t),x,t+\phi_v(x)), \tag{18}$$

$$\bar{v}^2_x(x,t)=-\frac{1}{\varepsilon_v(x)}F_v((\bar{u}^2,\bar{v}^2)(x,t),x,t+\phi_v(x)), \tag{19}$$

$$\bar{u}^1(0,t)=U_1(t), \qquad \bar{v}^2(1,t)=U_2(t), \tag{20}$$

$$\bar{v}^1(\bar{x},t)=\bar{v}^2(\bar{x},t), \quad \bar{u}^2(\bar{x},t)=\bar{u}^1(\bar{x},t), \tag{21}$$

$$\bar{v}^1(x,0)=\bar{v}^1_0(x), \qquad \bar{u}^2(x,0)=\bar{u}^2_0(x) \tag{22}$$

where $\left(\bar{v}^1_0,\bar{u}^2_0\right)=\Phi^0\left(u_0,v_0\right)$.

Proof: We only sketch the proof of existence and continuity of Φ^t. A detailed proof for a similar case can be found in [12]. See also Figure 2. First, for any small $\delta>0$ existence of a solution of (1)-(2) with the input arguments of Φ^t as initial condition in the domain

$$\mathscr{T}^\delta=\left\{(x,\theta):x\in[\delta,1-\delta],\theta\in[t,t+\phi^\delta(x)]\right\}, \tag{23}$$

where $\phi^\delta(x)=\min\{\phi_v(x)-\phi_v(1-\delta),\phi_u(x)-\phi_u(\delta)\}$, is proven by transforming the PDEs into integral equations by the method of characteristics, and applying a successive approximation argument. The integration paths are sketched in Figure 2. Note that the solution in this domain is independent of $U_1(t)$ and $U_2(t)$ because the points $(0,t)$ and

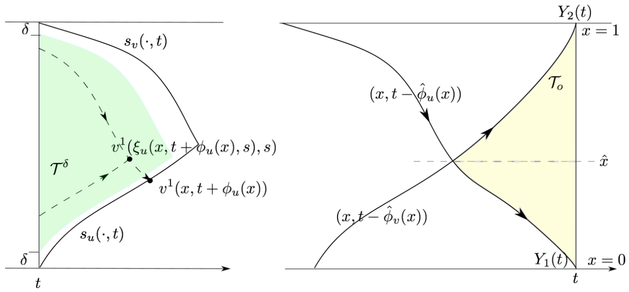

Fig. 2. Left: Illustration of the domain $\mathscr{T}^\delta$ (green), the characteristic lines along which the PDEs are integrated, and of the continuity argument used in the proof of Theorem 2. Right: Illustration of the characteristic lines used in observer design and the domain $\mathscr{T}_o$ (yellow, see also Theorem 10.

$(1,t)$ lie outside $\mathscr{T}^\delta$ for all $\delta > 0$. Moreover, the solution is Lipschitz continuous in the input arguments of Φ^t. Then, $\bar{v}^1$ on s_u and $\bar{u}^2$ on s_v, i.e. the output of Φ^t, is obtained by uniform continuity of $\bar{v}^1$ and $\bar{u}^2$ along their characteristic lines. One can derive appropriate parametrizations of the characteristic lines, denoted $\xi_u(x,\tau,s)$ and $\xi_v(x,\tau,s)$, such that $v^1(\xi_v(x,\theta,s),s)$ and $u^2(\xi_u(x,\theta,s),s)$ satisfy ODEs in s and

$$\bar{v}^1(x,t) = v(x,t+\phi_u(x)) = \lim_{s\to t+\phi_u(x)} v(\xi_u(x,t+\phi_u(x),s),s), \tag{24}$$

$$\bar{u}^2(x,t) = u(x,t+\phi_v(x)) = \lim_{s\to t+\phi_v(x)} u(\xi_u(x,t+\phi_v(x),s),s). \tag{25}$$

Moreover, for every $s < t+\phi_u(x)$ (with s sufficiently close to $t+\phi_u(x)$) there exists a $\delta > 0$ such that $(\xi_u(x,t+\phi_u(x),s),s) \in \mathscr{T}^\delta$. Thus, the right-hand side of (24) exists, is Lipschitz continuous in the input arguments of Φ^t. Since the limit is attained uniformly, the same holds for $\bar{v}^1(x,t)$. Applying the same arguments to (25) finishes the first part of the proof.

For the second statement, we use $\frac{d}{dt}$ and $\frac{d}{dx}$ to denote the total derivative with respect to t and x, respectively, while $_t$ and $_x$ are partial derivatives w.r.t. time and space. We have

$$\begin{aligned}\bar{u}^1_x(x,t) =& \frac{d}{dx}u(x,t+\phi_u(x))\\ =& u_x(x,t+\phi_u(x)) + u_t(x,t+\phi_u(x))\phi_u'(x)\\ =& -\frac{1}{\varepsilon_u(x)}\left[u_t(x,t+\phi_u(x))\right.\\ & \left.-F_u((u,v)(x,t+\phi_u(x)),x,t+\phi_u(x))\right]\\ & +\frac{1}{\varepsilon_u(x)}u_t(x,t+\phi_u(x))\\ =& \frac{1}{\varepsilon_u(x)}F_u((\bar{u}^1,\bar{v}^1)(x,t),x,t+\phi_u(x)).\end{aligned} \tag{26}$$

For $\bar{v}^1$,

$$\bar{v}^1_t(x,t) = \frac{d}{dt}v(x,t+\phi_u(x)) = v_t(x,t+\phi_u(x)), \tag{27}$$

$$\begin{aligned}\bar{v}^1_x(x,t) =& \frac{d}{dx}v(x,t+\phi_u(x))\\ =& v_x(x,t+\phi_u(x)) + v_t(x,t+\phi_u(x))\phi_u'(x)\\ =& v_x(x,t+\phi_u(x)) + \frac{1}{\varepsilon_u(x)}v_t(x,t+\phi_u(x))\\ =& \frac{1}{\varepsilon_v(x)}\left[v_t(x,t+\phi_u(x))\right.\\ & \left.-F_v((u,v)(x,t+\phi_u(x)),x,t+\phi_u(x))\right]\\ & +\frac{1}{\varepsilon_u(x)}v_t(x,t+\phi_u(x))\\ =& \frac{\varepsilon_u(x)+\varepsilon_v(x)}{\varepsilon_u(x)\varepsilon_v(x)}v_t(x,t+\phi_u(x))\\ & -\frac{1}{\varepsilon_v(x)}F_v((\bar{u}^1,\bar{v}^1)(x,t),x,t+\phi_u(x)).\end{aligned} \tag{28}$$

Replacing $v_t(x,t+\phi_u(x))$ in the latter equation by (27) yields (17). Repeating the same steps for $(\bar{u}^2,\bar{v}^2)$ gives (18)-(19). The boundary, coupling, and initial conditions (20)-(22) follow directly from the definitions of ϕ_u and ϕ_v, and Definition 1. ■

Remark 3: The operator Φ^t can be implemented by solving (1)-(2) with the input arguments of Φ^t as initial condition in the domain $\mathscr{T}^\delta$ for small $\delta > 0$, and getting $\bar{v}^1$ and $\bar{u}^2$ by continuity as in (24)-(25). Since Φ^t will be used in the control law this requires the online solution of a PDE system which makes evaluating the control law computationally expensive.

B. Dynamics with virtual actuation

The central idea of our controller design method is to virtually move the control inputs U_1 and U_2 to a desired location inside the domain. The first step is to design locations $x_1^* \in [0,\bar{x}]$ and $x_2^* \in [\bar{x},1]$, as well as virtual actuations $U_1^*(t)$ and $U_2^*(t)$. Exploiting the fact that (16) and (19) are ODEs in space without dynamics in time, $U_1(t)$ can be constructed such that $\bar{u}^1(x_1^*,t)$ becomes $U_1^*(t)$, and $U_2(t)$ such that $\bar{v}^2(x_2^*,t)$ becomes $U_2^*(t)$. This is made precise in the following theorem.

Theorem 4: For given $t \geq 0$, $x_1^* \in [0,\bar{x}]$ and $x_2^* \in [\bar{x},1]$, consider the operator $\Psi^1_{t,x_1^*} : \mathscr{X}_{[0,\bar{x}]} \times \mathbb{R} \to \mathbb{R}$, mapping $\phi \in \mathscr{X}_{[0,\bar{x}]}$ and $U_1^*(t)$ to $\varphi(0)$, where φ is the solution of the Cauchy problem

$$\begin{aligned}&\varphi_x = \frac{1}{\varepsilon_u(x)}F_u((\varphi,\phi)(x),x,t+\phi_u(x)), \quad x \in [0,x_1^*],\\ &\varphi(x_1^*) = U_1^*(t),\end{aligned} \tag{29}$$

and the operator $\Psi^2_{t,x_2^*} : \mathscr{X}_{[\bar{x},1]} \times \mathbb{R} \to \mathbb{R}$, mapping $\phi \in \mathscr{X}_{[\bar{x},1]}$ and $U_2^*(t)$ to $\varphi(1)$, where φ is the solution of the Cauchy problem

$$\begin{aligned}&\varphi_x = -\frac{1}{\varepsilon_v(x)}F_v((\phi,\varphi)(x),x,t+\phi_v(x)), \quad x \in [x_2^*,1],\\ &\varphi(x_2^*) = U_2^*(t).\end{aligned} \tag{30}$$

The system consisting of (16)-(22) in closed loop with $U_1(t) = \Psi^1_{t,x_1^*}(\bar{v}^1[0,\bar{x}],U_1^*(t))$ and $U_2(t) =$

$\Psi^2_{t,x_2^*}(\bar{u}^2[\bar{x},t],U_2^*(t))$ satisfies

$$\bar{u}^1_x(x,t) = \frac{1}{\varepsilon_u(x)} F_u((\bar{u}^1,\bar{v}^1)(x,t),x,t+\phi_u(x)), \tag{31}$$

$$\bar{v}^1_t(x,t) = \frac{\varepsilon_u(x)\varepsilon_v(x)}{\varepsilon_u(x)+\varepsilon_v(x)}\bar{v}^1_x + \frac{\varepsilon_u(x)}{\varepsilon_u(x)+\varepsilon_v(x)} \times F_v((\bar{u}^1,\bar{v}^1)(x,t),x,t+\phi_u(x)), \tag{32}$$

$$\bar{u}^2_t(x,t) = -\frac{\varepsilon_u(x)\varepsilon_v(x)}{\varepsilon_u(x)+\varepsilon_v(x)}\bar{u}^2_x + \frac{\varepsilon_v(x)}{\varepsilon_u(x)+\varepsilon_v(x)} \times F_u((\bar{u}^2,\bar{v}^2)(x,t),x,t+\phi_v(x)), \tag{33}$$

$$\bar{v}^2_x(x,t) = -\frac{1}{\varepsilon_v(x)} F_v((\bar{u}^2,\bar{v}^2)(x,t),x,t+\phi_v(x)), \tag{34}$$

$$\bar{u}^1(x_1^*,t) = U_1^*(t), \qquad \bar{v}^2(x_2^*,t) = U_2^*(t), \tag{35}$$

$$\bar{v}^1(\bar{x},t) = \bar{v}^2(\bar{x},t), \quad \bar{u}^2(\bar{x},t) = \bar{u}^1(\bar{x},t), \tag{36}$$

$$\bar{v}^1(\cdot,0) = v_0^1, \qquad \bar{u}^2(\cdot,0) = u_0^2. \tag{37}$$

Proof: The Carathéodory theorem and the Lipschitz condition (5) ensure that the ODE (29) has a unique solution for any given initial condition [15]. Therefore, if two solutions φ and $\bar{\varphi}$ both satisfy (29), then

$$\varphi(\bar{x}) = \bar{\varphi}(x_1^*) \Leftrightarrow \varphi(0) = \bar{\varphi}(0). \tag{38}$$

Since (29) is a copy of (16) for $\phi = \bar{v}^1(\cdot,t)$, this is equivalent to

$$\bar{u}^1(0,t) = \Psi^1_{t,x_1^*}(\bar{v}^1,U_1^*(t)) \Leftrightarrow \bar{u}^1(x_1^*,t) = U_1^*(t). \tag{39}$$

The same arguments can be utilized to show that

$$\bar{v}^2(1,t) = \Psi^2_{t,x_2^*}(\bar{u}^2,U_2^*(t)) \Leftrightarrow \bar{v}^2(x_2^*,t) = U_2^*(t). \tag{40}$$

Thus, (20) using the given feedback law is equivalent to (35). The other equations remain unchanged. ■

C. Tracking

Theorem 4 can directly be used to solve a tracking problem of the form

$$u^1(x_t^1,t) = g^1(v^1(x_t^1,t),t), \quad v^2(x_t^2,t) = g^2(u^2(x_t^2,t),t), \tag{41}$$

where $x_t^1 \in [0,\bar{x}]$, $x_t^2 \in [\bar{x},1]$, and $g^1\,g^2 : \mathbb{R}^2 \to \mathbb{R}$. We do not need to assume (7). Hence, F_u and F_v can include disturbance terms for which exact short-term predictions ($\bar{t}$ into the future) are available.

Theorem 5: The system (1)-(4) in closed loop with

$$\begin{aligned} &\left(\bar{v}^1([0,\bar{x}],t),\bar{u}^2([\bar{x},1],t)\right) = \Phi^t\left(u([0,1],t),v([0,1],t)\right)\\ &U_1^*(t) = g^1\left(\bar{v}^1(x_t^1,t),t+\phi_u(x_t^1)\right)\\ &U_2^*(t) = g^2\left(\bar{u}^2(x_t^2,t),t+\phi_v(x_t^2)\right)\\ &U_1(t) = \Psi^1_{t,x_t^1}\left(\bar{v}^1([0,\bar{x}],t),U_1^*(t)\right)\\ &U_2(t) = \Psi^2_{t,x_t^2}\left(\bar{u}^2([\bar{x},1],t),U_2^*(t)\right) \end{aligned} \tag{42}$$

satisfies (41) for all $t \geq \bar{t}$, where $\bar{t}$ was defined in (12).

Proof: The feedback law (42) ensures that $(\bar{u}^1,\bar{v}^1)$ and $(\bar{u}^2,\bar{v}^2)$ satisfy the tracking objective for all $t \geq 0$. Hence, Definition 1 implies that (u,v) satisfies (41) for all $t \geq \bar{t}$, where we also exploited that $\phi_u(x_t^1) \leq \bar{t}$ since $x_t^1 \leq \bar{x}$ and $\phi_v(x_t^2) \leq \bar{t}$ since $x_t^2 \geq \bar{x}$. ■

D. Stabilization

In order to stabilize the origin, we choose $x_1^* = x_2^* = \bar{x}$ and $U_1^*(t) = U_2^*(t) = 0$. This way, the actuation drives the state to zero at $x = \bar{x}$, and this zero 'propagates' towards the boundaries of the domain by the closed-loop dynamics.

Theorem 6: The system (1)-(4) in closed loop with

$$\begin{aligned} &\left(\bar{v}^1([0,\bar{x}],t),\bar{u}^2([\bar{x},1],t)\right) = \Phi^t\left(u([0,1],t),v([0,1],t)\right)\\ &U_1(t) = \Psi^1_{t,\bar{x}}\left(\bar{v}^1([0,\bar{x}],t),0\right)\\ &U_2(t) = \Psi^2_{t,\bar{x}}\left(\bar{u}^2([\bar{x},1],t),0\right) \end{aligned} \tag{43}$$

satisfies $u(\cdot,t) = v(\cdot,t) = 0$ for all $t \geq d$, where d was defined in (12).

Proof: Choosing $x_1^* = x_2^* = \bar{x}$ and $U_1^*(t) = U_2^*(t) = 0$ ensures that $\bar{u}^1(\bar{x},t) = \bar{v}^2(\bar{x},t) = 0$ and, by (36), $\bar{v}^1(\bar{x},t) = \bar{u}^2(\bar{x},t) = 0$. We first prove that the solution of (31)-(37) satisfies

$$u^1(x,t) = v^1(x,t) = 0 \text{ for all } (x,t) \in \mathscr{A}^1, \tag{44}$$

$$u^2(x,t) = v^2(x,t) = 0 \text{ for all } (x,t) \in \mathscr{A}^2, \tag{45}$$

where

$$\mathscr{A}^1 = \left\{(x,t) : x \in [0,\bar{x}], t \geq \int_x^{\bar{x}} \frac{1}{\varepsilon_u(\xi)} + \frac{1}{\varepsilon_v(\xi)} d\xi\right\}, \tag{46}$$

$$\mathscr{A}^2 = \left\{(x,t) : x \in [\bar{x},1], t \geq \int_{\bar{x}}^{x} \frac{1}{\varepsilon_u(\xi)} + \frac{1}{\varepsilon_v(\xi)} d\xi\right\}. \tag{47}$$

For this purpose, we transform the PDEs into integral equations. We define

$$\phi_1(x) = \int_x^{\bar{x}} \frac{\varepsilon_u(\xi)+\varepsilon_v(\xi)}{\varepsilon_u(\xi)\varepsilon_v(\xi)} d\xi \tag{48}$$

$$\xi^1(x,t,s) = \phi_1^{-1}\left(\phi_1(x)+s-t\right), \tag{49}$$

$$s_1^0(x,t) = t - \phi_1(x). \tag{50}$$

and integrate (31)-(32) along its characteristic lines to obtain, for $(x,t) \in \mathscr{A}^1$,

$$\bar{u}^1(x,t) = \bar{u}^1(\bar{x},t) + \int_{\bar{x}}^{x} \frac{1}{\varepsilon_u(\xi)} F_u((\bar{u}^1,\bar{v}^1)(\xi,t), \xi, t+\phi_u(\xi)) d\xi, \tag{51}$$

$$\bar{v}^1(x,t) = \bar{v}^1(\bar{x},s_1^0) + \int_{s_1^0}^{t} \frac{\varepsilon_u}{\varepsilon_u+\varepsilon_v} F_v((\bar{u}^1,\bar{v}^1)(\xi^1(x,t,s),s), \xi^1(x,t,s),s), s+\phi_u(\xi^1(x,t,s),s)))ds. \tag{52}$$

$(x,t) \in \mathscr{A}^1$ ensures that $s_1^0 \geq 0$, hence $\bar{v}^1(\bar{x},s_1^0) = 0$. $\bar{u}^1(\bar{x},t) =$ follows from (36). For $(x,t) \in \mathscr{A}^1$, we also have that $\left(\xi^1(x,t,s),s\right) \in \mathscr{A}^1$ for all $s \in [s_1^0,t]$, and that $(\xi,t) \in \mathscr{A}^1$ for all $\xi \in [x,\bar{x}]$. Therefore, inserting (44) into (51)-(52) and exploiting (7), the right-hand sides become zero. That is, (44) solves (51)-(52). Since the solution is unique (which can be shown by exploiting the Lipschitz assumption on F_u and F_v), we can reverse the statement, i.e. the solution of (51)-(52) must satisfy (44), and thus the original PDEs.

Performing the same steps for $(\bar{u}^2,\bar{v}^2)$ gives (45).

By definitions (46) and (12), $(x,t) \in \mathscr{A}_1$ if and only if $x \in [0,\bar{x}]$ and

$$t \geq (\phi_u(\bar{x}) - \phi_u(x)) + (\phi_v(x) - \phi_v(\bar{x})) = \phi_v(x) - \phi_u(x), \tag{53}$$

where (11) was used in the last equation. (53) is equivalent to $t-\phi_u(x)\geq\phi_v(x)$. Hence, if $x\in[0,\bar{x}]$ and $t\geq d$, then $t\geq d\geq\phi_v(0)\geq\phi_v(x)$, i.e. $(x,t-\phi_u(x))\in\mathscr{A}_1$. Thus, $u(x,t)=\bar{u}^1(x,t-\phi_u(x))=0$ by (44), and the same for v. By the same steps, $x\in[\bar{x},1]$ and $t\geq d$ implies $u(x,t)=\bar{u}^2(x,t-\phi_v(x))=0$. Summarizing, $t\geq d$ implies $u(x,t)=v(x,t)=0$ for all $x\in[0,1]$, which finishes the proof. ∎

Remark 7: In [12], a Lyapunov function was constructed to prove exponential stability of the closed-loop system in the spatial supremum norm. Since the controller design methods are similar, we conjecture that a function of the form

$$V\left(\bar{v}^1([0,\bar{x}],t),\bar{u}^2([\bar{x},1],t)\right)=\sup_{x\in[0,\bar{x}]}\left|\bar{v}^1(x,t)e^{k(x-\bar{x})}\right| + \sup_{x\in[\bar{x},1]}\left|\bar{u}^2(x,t)e^{-k(x-\bar{x})}\right| \tag{54}$$

for sufficiently large k can serve as a Lyapunov function. However, pursuing this idea is beyond the scope of this paper.

IV. Semi-explicit state feedback law for linear systems

In this section we consider the stabilization of the origin for linear time-invariant systems, which we without loss of generality assume to be written in the form

$$u_t(x,t)=-\varepsilon_u(x)u_x(x,t)+c_u(x)v(x,t), \tag{55}$$

$$v_t(x,t)=\quad\varepsilon_v(x)v_x(x,t)+c_v(x)u(x,t), \tag{56}$$

where additionally piecewise differentiability of ε_u and ε_v is required. The boundary and initial conditions are as in (3)-(4). For this class of systems, it is possible to write the state-feedback law "explicit" as the inner product of kernels with the state, i.e. without evaluating the operators Φ^t and Ψ^1,Ψ^2. There might not be an explicit expression for the kernels (hence "semi") but they can be precomputed numerically.

Since the system is linear, the state-feedback law must be a linear functional of the state. Therefore, we make the following ansatz for $\bar{u}(x,t)$ for $x\in[0,\bar{x}]$ and $\bar{v}(x,t)$ for $x\in[\bar{x},1]$ when using the state-feedback law (43):

$$\bar{u}^1(x,t)=\int_x^{\phi_v^{-1}(\phi_u(x))}K^{uu}(x,\xi)u(\xi,t+\phi_u(x)) + K^{uv}(x,\xi)v(\xi,t+\phi_u(x))d\xi, \tag{57}$$

$$\bar{v}^2(x,t)=\int_{\phi_u^{-1}(\phi_v(x))}^{x}K^{vu}(x,\xi)u(\xi,t+\phi_v(x)) + K^{vv}(x,\xi)v(\xi,t+\phi_v(x))d\xi, \tag{58}$$

to derive a condition for the control inputs $U_1(t)=\bar{u}(0,t)$ and $U_2(t)=\bar{v}(1,t)$. This ansatz is motivated by the observation that the actuation is entirely determined by the states in the domain $\mathscr{T}^\delta$ for $\delta\to 0$. Note that the integrals in (57) and (58) are taken over the intersection of $\mathscr{T}^0$ with the lines $[0,1]\times\{\phi_u(x)\}$ and $[0,1]\times\{\phi_v(x)\}$, respectively.

In order to shorten notation, we abbreviate $\zeta_u(x)=\phi_v^{-1}(\phi_u(x))$ and, for fixed t, $\tau_u(x)=t+\phi_u(x)$. It can be verified that $\zeta_u'(x)=-\frac{\varepsilon_v(\zeta_u(x))}{\varepsilon_u(x)}$. Differentiating the right-hand side of (57) with respect to x gives

$$\begin{aligned}\bar{u}_x^1(x,t)=&\left[K^{uu}(x,\zeta_u(x))u(\zeta_u(x),\tau_u(x))\right.\\&\left.+K^{uv}(x,\zeta_u(x))v(\zeta_u(x),\tau_u(x))\right]\zeta_u'(x)\\&+\int_x^{\zeta_u(x)}K_x^{uu}(x,\xi)u(\xi,\tau_u(x))+K_x^{uv}(x,\xi)v(\xi,\tau_u(x))\\&+\left[K^{uu}(x,\xi)u_t(\xi,\tau_u(x))+K^{uv}(x,\xi)v_t(\xi,\tau_u(x))\right]\tau_u'(x)d\xi\\&-\left[K^{uu}(x,x)u(x,\tau_u(x))+K^{uv}(x,x)v(x,\tau_u(x))\right]\end{aligned} \tag{59}$$

Inserting the dynamics (55)-(56) into the integral term and integrating by parts gives (note that $_x$ and $_t$ denote partial derivatives of $\bar{u}^1$ wrt space and time, respectively, not total derivatives wrt x or t)

$$\begin{aligned}&\int_x^{\zeta_u(x)}K_x^{uu}(x,\xi)u(\xi,\tau_u(x))+K_x^{uv}(x,\xi)v(\xi,\tau_u(x))\\&\quad+\{K^{uu}(x,\xi)[-\varepsilon_u(\xi)u_x(\xi,\tau_u(x))+c_u(\xi)v(\xi,\tau_u(x))]\\&\quad+K^{uv}(x,\xi)[\varepsilon_v(\xi)v_x(\xi,\tau_u(x))+c_v(\xi)u(\xi,\tau_u(x))]\}\frac{1}{\varepsilon_u(x)}d\xi\\&=\int_x^{\zeta_u(x)}\left\{K_x^{uu}(x,\xi)+\frac{1}{\varepsilon_u(x)}\left[K_\xi^{uu}(x,\xi)\varepsilon_u(\xi)\right.\right.\\&\quad\left.\left.+K^{uu}(x,\xi)\varepsilon_u'(\xi)+K^{uv}(x,\xi)c_v(\xi)\right]\right\}u(\xi,\tau_u(x))\\&\quad+\left\{K_x^{uv}(x,\xi)+\frac{1}{\varepsilon_u(x)}\left[K^{uu}(x,\xi)c_u(\xi)\right.\right.\\&\quad\left.\left.-K^{uv}(x,\xi)\varepsilon_v'(\xi)-K_\xi^{uv}(x,\xi)\varepsilon_v(\xi)\right]\right\}v(\xi,\tau_u(x))d\xi\\&+\frac{-\varepsilon_u(\zeta_u(x))}{\varepsilon_u(x)}K^{uu}(x,\zeta_u(x))u(\zeta_u(x),\tau_u(x))\\&+\frac{\varepsilon_v(\zeta_u(x))}{\varepsilon_u(x)}K^{uv}(x,\zeta_u(x))v(\zeta_u(x),\tau_u(x))\\&-\left[-K^{uu}(x,x)u(x,\tau_u(x))+\frac{\varepsilon_v(x)}{\varepsilon_u(x)}K^{uv}(x,x)v(x,\tau_u(x))\right].\end{aligned} \tag{60}$$

Inserting (60) into (59), rearranging, and equating the results with $\bar{u}_x(x,t)=\frac{c_u(x)}{\varepsilon_u(x)}v(x,\tau_u(x))$ (see (31)) for all u and v, K^{uu} and K^{uv} must satisfy

$$\varepsilon_u(x)K_x^{uu}(x,\xi)+K_\xi^{uu}(x,\xi)\varepsilon_u(\xi)+K^{uu}(x,\xi)\varepsilon_u'(\xi) +K^{uv}(x,\xi)c_v(\xi)=0, \tag{61}$$

$$\varepsilon_u(x)K_x^{uv}(x,\xi)+K^{uu}(x,\xi)c_u(\xi)-K^{uv}(x,\xi)\varepsilon_v'(\xi) -K_\xi^{uv}(x,\xi)\varepsilon_v(\xi)=0 \tag{62}$$

in the domain $\mathscr{S}_u=\{(x,\xi):x\in[0,\bar{x}],\xi\in[x,\zeta_u(x)]\}$, and

$$-\left(1+\frac{\varepsilon_v(x)}{\varepsilon_u(x)}\right)K^{uv}(x,x)=\frac{c_u(x)}{\varepsilon_u(x)}, \tag{63}$$

$$K^{uu}(x,\zeta_u(x))=0 \tag{64}$$

for $x\in[0,\bar{x}]$. Well-posedness of this linear hyperbolic system can be proven by integrating (61)-(62) along its characteristic lines and showing existence of a unique solution by a successive approximation argument, similarly as it is done in the appendix in [16]. Thereby, the fact that all points in $\mathscr{S}_u$ lie on a characteristic line of (61) originating in $(x,\zeta_u(x))$ for some $x\in[0,\bar{x}]$ (where K^{uu} is determined by (64)), and on a characteristic line of (62) originating in (x,x) for some

$x \in [0,\bar{x}]$, and that these characteristic lines lie completely in $\mathscr{S}_u$, must be exploited.
With $\zeta_v(x) = \phi_u^{-1}(\phi_v(x))$, performing the same steps for (58) gives

$$\varepsilon_v(x)K_x^{vv}(x,\xi) + K_\xi^{vv}(x,\xi)\varepsilon_v(\xi) + K^{vv}(x,\xi)\varepsilon_v'(\xi) - K^{vu}(x,\xi)c_u(\xi) = 0, \quad (65)$$

$$\varepsilon_v(x)K_x^{vu}(x,\xi) - K^{vv}(x,\xi)c_v(\xi) - K^{vu}(x,\xi)\varepsilon_u'(\xi) - K_\xi^{vu}(x,\xi)\varepsilon_u(\xi) = 0 \quad (66)$$

in the domain $\mathscr{S}_v = \{(x,\xi) : x \in [0,\bar{x}], \xi \in [\zeta_v(x), x]\}$ and

$$-\left(1 + \frac{\varepsilon_u(x)}{\varepsilon_v(x)}\right) K^{vu}(x,x) = \frac{c_v(x)}{\varepsilon_v(x)}, \quad (67)$$

$$K^{vv}(x,\zeta_v(x)) = 0 \quad (68)$$

for $x \in [0,\bar{x}]$.

Remark 8: For constant transport speeds, the state-feedback law is the same as the one in [13], which can be verified by comparing the kernel equations.

V. Estimation

For estimation, we assume that the state can be measured at both boundaries as given in (8). Due to the finite propagation speeds, information from within the domain cannot be sensed at the boundaries immediately. Therefore, we base our observer design on the dynamics on the characteristic lines along which the measurements evolve.

A. Preliminaries

The following definition will be needed

$$\hat{\phi}_u(x) = \int_x^1 \frac{1}{\varepsilon_u(\xi)} d\xi, \qquad \hat{\phi}_v(x) = \int_0^x \frac{1}{\varepsilon_v(\xi)} d\xi. \quad (69)$$

We denote the location at which the characteristic lines of Y_1 and Y_2 intersect by $\hat{x}$, which is implicitly defined by the unique solution of

$$t - \phi_u(\hat{x}) = t - \hat{\phi}_v(\hat{x}). \quad (70)$$

The following definition and theorem will be central for observer design.

Definition 9: We define the states on the characteristic lines along which Y_1 and Y_2 evolve as

$$\check{u}^1(x,t) = u(x, t - \hat{\phi}_v(x)), \quad \check{v}^1(x,t) = v(x, t - \hat{\phi}_v(x)), \quad (71)$$

$$\check{u}^2(x,t) = u(x, t - \hat{\phi}_u(x)), \quad \check{v}^2(x,t) = v(x, t - \hat{\phi}_u(x)). \quad (72)$$

Theorem 10: $(\check{u}^1, \check{v}^1, \check{u}^2, \check{v}^2)$ satisfies the PDE-ODE system

$$\check{u}_t^1(x,t) = -\frac{\varepsilon_u(x)\varepsilon_v(x)}{\varepsilon_u(x)+\varepsilon_v(x)}\check{u}_x^1 + \frac{\varepsilon_v(x)}{\varepsilon_u(x)+\varepsilon_v(x)} \times F_v((\check{u}^1,\check{v}^1)(x,t), x, t - \hat{\phi}_v(x)), \quad (73)$$

$$\check{v}_x^1(x,t) = -\frac{1}{\varepsilon_v(x)} F_v((\check{u}^1,\check{v}^1)(x,t), x, t - \hat{\phi}_v(x)), \quad (74)$$

$$\check{u}_x^2(x,t) = \frac{1}{\varepsilon_u(x)} F_u((\check{u}^2,\check{v}^2)(x,t), x, t - \hat{\phi}_u(x)), \quad (75)$$

$$\check{v}_t^2(x,t) = \frac{\varepsilon_u(x)\varepsilon_v(x)}{\varepsilon_u(x)+\varepsilon_v(x)}\check{v}_x^2 + \frac{\varepsilon_u(x)}{\varepsilon_u(x)+\varepsilon_v(x)} \times F_v((\check{u}^2,\check{v}^2)(x,t), x, t - \hat{\phi}_u(x)), \quad (76)$$

$$\check{u}^1(0,t) = U_1(t), \qquad \check{v}^2(1,t) = U_2(t), \quad (77)$$

$$\check{v}^1(\hat{x},t) = \check{v}^2(\hat{x},t), \quad \check{u}^2(\bar{x},t) = \check{u}^1(\bar{x},t), \quad (78)$$

$$\check{u}^1(x,\hat{\phi}_v(x)) = u_0(x), \qquad \check{v}^2(x,\hat{\phi}_u(x)) = v_0(x). \quad (79)$$

Moreover, for every t there exists a continuous, bounded operator $\Lambda^t : \left(\mathscr{X}_{[0,\hat{x}]}\right)^2 \times \left(\mathscr{X}_{[\hat{x},1]}\right)^2 \to \mathscr{X}_{[0,1]} \times \mathscr{X}_{[0,1]}$, independent of $U_1(t)$ and $U_2(t)$, such that

$$(u([0,1],t), v([0,1],t)) = \Lambda^t\left(\check{u}^1([0,\hat{x}],t), \check{v}^1([0,\hat{x}],t), \check{u}^2([\hat{x},1],t), \check{v}^2([\hat{x},1],t)\right). \quad (80)$$

Proof: The derivation of (73)-(76) follows by the same steps as the derivation of (16)-(19), and the boundary, coupling and initial conditions follow directly from the definitions.
Existence and continuity of Λ^t can be proven by showing that for given t, (1)-(2) with the input arguments of Λ^t as 'initial' condition has a unique solution in the domain $\mathscr{T}_o = \{(x,\theta) : \theta \in [t - \hat{\phi}_u(\hat{x}), t], x \in [\hat{\phi}_v^{-1}(t-\theta), \hat{\phi}_v^{-1}(t-\theta)]\}$ that is Lipschitz-continuous in the input arguments of Λ^t. See also Figure 2. As usual, this can be done by transforming the PDEs into integral equations and applying a successive approximation argument. ■

B. Observer design

Since (74) and (75) are simple ODEs in space without dynamics in time, the coupling conditions (78) can be replaced by the measurements (8). Therefore, we design the observer as a copy of (73)-(79) with (78) replaced by the measurements and the initial condition (79) replaced by some initial guess $(\hat{u}_0^1, \hat{v}_0^2)$:

$$\hat{u}_t^1(x,t) = -\frac{\varepsilon_u(x)\varepsilon_v(x)}{\varepsilon_u(x)+\varepsilon_v(x)}\hat{u}_x^1 + \frac{\varepsilon_v(x)}{\varepsilon_u(x)+\varepsilon_v(x)} \times F_v((\hat{u}^1,\hat{v}^1)(x,t), x, t - \hat{\phi}_v(x)), \quad (81)$$

$$\hat{v}_x^1(x,t) = -\frac{1}{\varepsilon_v(x)} F_v((\hat{u}^1,\hat{v}^1)(x,t), x, t - \hat{\phi}_v(x)), \quad (82)$$

$$\hat{u}_x^2(x,t) = \frac{1}{\varepsilon_u(x)} F_u((\hat{u}^2,\hat{v}^2)(x,t), x, t - \hat{\phi}_u(x)), \quad (83)$$

$$\hat{v}_t^2(x,t) = \frac{\varepsilon_u(x)\varepsilon_v(x)}{\varepsilon_u(x)+\varepsilon_v(x)}\hat{v}_x^2 + \frac{\varepsilon_u(x)}{\varepsilon_u(x)+\varepsilon_v(x)} \times F_v((\hat{u}^2,\hat{v}^2)(x,t), x, t - \hat{\phi}_u(x)), \quad (84)$$

$$\hat{u}^1(0,t) = U_1(t), \qquad \hat{v}^2(1,t) = U_2(t), \quad (85)$$

$$\hat{v}^1(0,t) = Y_1(t), \qquad \hat{u}^2(1,t) = Y_2(t), \quad (86)$$

$$\hat{u}^1(x,0) = \hat{u}_0^1(x), \qquad \hat{v}^2(x,0) = \hat{v}_0^2(x). \quad (87)$$

Theorem 11: The observer (81)-(87) yields exact state estimates

$$(u_{est}(\cdot,t), v_{est}(\cdot,t)) = \Lambda^t\left(\hat{u}^1([0,\hat{x}],t), \hat{v}^1([0,\hat{x}],t), \hat{u}^2([\hat{x},1],t), \hat{v}^2([\hat{x},1],t)\right)$$

i.e. $u_{est}(\cdot,t) = u(\cdot,t)$ and $v_{est}(\cdot,t) = v(\cdot,t)$, for all $t \geq d$, where d was defined in (12).

Proof: We form error equations by subtracting (73)-(79) with (78) replaced by the measurements, i.e. $\check{u}^2(1,t) = Y_2(t)$ and $\check{v}^1(0,t) = Y_1(t)$, from (81)-(87). In order to shorten the presentation, only the steps for the first subsystem in the interval $[0,\hat{x}]$ are shown.

$$e_t^{u_1}(x,t) = -\hat{\varepsilon}(x)e_x^{u_1}(x,t) + E_u^1(\hat{u}^1,\hat{v}^1,\check{u}^1,\check{v}^1,x,t), \tag{88}$$
$$e_x^{v_1}(x,t) = E_v^1(\hat{u}^1,\hat{v}^1,\check{u}^1,\check{v}^1,x,t), \tag{89}$$
$$e_t^{u_1}(0,t) = 0, \tag{90}$$
$$e_t^{v_1}(0,t) = 0, \tag{91}$$

where $e^{u_1} = \hat{u}^1 - \check{u}^1$, $e^{v_1} = \hat{v}^1 - \check{v}^1$, $\hat{\varepsilon} = \frac{\varepsilon_u(x)\varepsilon_v(x)}{\varepsilon_u(x)+\varepsilon_v(x)}$, and

$$E_u^1(\hat{u}^1,\hat{v}^1,\check{u}^1,\check{v}^1,x,t) = \frac{\varepsilon_v(x)}{\varepsilon_u(x)+\varepsilon_v(x)}\left(F_v((\hat{u}^1,\hat{v}^1)(x,t),\right.$$
$$\left. x,t-\hat{\phi}_v(x)) - F_v((\check{u}^1,\check{v}^1)(x,t),x,t-\hat{\phi}_v(x))\right), \tag{92}$$
$$E_v^1(\hat{u}^1,\hat{v}^1,\check{u}^1,\check{v}^1,x,t) = -\frac{1}{\varepsilon_v(x)}\left(F_v((\hat{u}^1,\hat{v}^1)(x,t),x,t-\hat{\phi}_v(x))\right.$$
$$\left. -F_v((\check{u}^1,\check{v}^1)(x,t),x,t-\hat{\phi}_v(x))\right). \tag{93}$$

Note that $e^{u_1} = 0$ implies $E_u^1(\hat{u}^1,\hat{v}^1,\check{u}^1,\check{v}^1,x,t) = 0$, and the same for E_v^1, E_u^2 and E_v^2. This is a similar condition as (7), which was central in the proof of Theorem 6. Therefore, by similar steps as in the proof of Theorem 6, it can be shown that

$$e^{u_1}(x,t) = e^{v_1}(x,t) = 0 \text{ for all } (x,t) \in \mathscr{B}^1 \tag{94}$$

for

$$\mathscr{B}^1 = \left\{(x,t) : x \in [0,\bar{x}], t \geq \int_0^x \frac{1}{\varepsilon_u(\xi)} + \frac{1}{\varepsilon_v(\xi)} d\xi\right\}. \tag{95}$$

Performing the same steps for the second system yields

$$e^{u_2}(x,t) = e^{v_2}(x,t) = 0 \text{ for all } (x,t) \in \mathscr{B}^2 \tag{96}$$

for $e^{u_2} = \hat{u}^2 - \check{u}^2$, $e^{v_2} = \hat{v}^2 - \check{v}^2$, and

$$\mathscr{B}^2 = \left\{(x,t) : x \in [\bar{x},1], t \geq \int_x^1 \frac{1}{\varepsilon_u(\xi)} + \frac{1}{\varepsilon_v(\xi)} d\xi\right\}. \tag{97}$$

Then, the claim follows from Definition 9. ■

Remark 12: The operator Λ^t can be implemented by solving (1)-(2) with the input arguments of Λ^t as 'initial' condition in the domain $\{(x,\theta) : \theta \in [t-\hat{\phi}_u(\hat{x}),t], x \in [\hat{\phi}_v^{-1}(t-\theta), \hat{\phi}_v^{-1}(t-\theta)]\}$. Since the estimation and output feedback control laws involve Λ^t this requires to solve a PDE system online, which is computationally expensive.

VI. Output-feedback control

The output feedback control problem can be solved by combining the controller from Section III with the Observer from Section V.

Theorem 13: The system (1)-(4) in closed loop with the output feedback controller consisting of the observer (81)-(87) and the feedback law

$$\begin{aligned}\left(\bar{v}^1([0,\bar{x}],t),\bar{u}^2([\bar{x},1],t)\right) &= \Phi^t\left(\Lambda^t\left(\hat{u}^1([0,\hat{x}],t),\hat{v}^1([0,\hat{x}],t),\right.\right.\\ &\qquad \left.\left.\hat{u}^2([\hat{x},1],t),\hat{v}^2([\hat{x},1],t)\right)\right)\\ U_1(t) &= \Psi^1_{t,\bar{x}}\left(\bar{v}^1([0,\bar{x}],t),0\right)\\ U_2(t) &= \Psi^2_{t,\bar{x}}\left(\bar{u}^2([\bar{x},1],t),0\right)\end{aligned} \tag{98}$$

with $x_1^* = x_2^* = \bar{x}$ and $U_1^*(t) = U_2^*(t) = 0$, reaches the origin within $2d$, or, with x_1^*, x_2^*, $U_1^*(t)$ and $U_2^*(t)$ as in (42), satisfies the tracking objective (41) for all $t \geq d + \bar{t}$.

Proof: The theorem follows directly by combining Theorems 5 or 6, respectively, with Theorem 11. ■

Remark 14: The output feedback law (98) requires knowledge of all observer states $(\hat{u}^1,\hat{v}^1,\hat{u}^2,\hat{v}^2)$. However, the state-feedback laws (42) and (43) are *decentralized* in the sense that $U_1(t)$ is independent of $(\bar{u}^2([\bar{x},1],t),\bar{v}^2([\bar{x},1],t))$ and $U_2(t)$ is independent of $(\bar{u}^1([0,\bar{x}],t),\bar{v}^1([0,\bar{x}],t))$. Alternatively, it is possible to design two observers, one for estimating the state from $Y_2(t)$ (this is exactly the observer from [12]) and the other for estimating the state from $Y_1(t)$ (which is the same as the one from [12] when making a coordinate change from x to $1-x$). Both these observers can estimate the full state exactly, although in a larger time $(d_u + d_v)$. Then, a *decentralized* output feedback control law, i.e. without communication between the boundaries, can be obtained by using the state estimate obtained from $Y_1(t)$ to determine $U_1(t)$ and the state estimated from $Y_2(t)$ to determine $U_2(t)$.

VII. Example

We illustrate the performance of the controller in an example with

$$\varepsilon_u(x) = \begin{cases} 0.2 & \text{if } x < 0.5, \\ 2-x & \text{if } x \geq 0.5, \end{cases} \tag{99}$$
$$\varepsilon_v(x) = 0.2 \times (1+x), \tag{100}$$
$$F_u((u,v),x,t) = \frac{1}{3-x}\sin(u+v), \tag{101}$$
$$F_v((u,v),x,t) = \sin(v-u), \tag{102}$$

and initial condition $u_0 = v_0 = 1$. With these propagation speeds, the delay times are $d_u \approx 2.9$ and $d_v \approx 3.5$, $\bar{x} \approx 0.37$, and $\hat{x} \approx 0.31$. The initial condition of the observer is set to zero. The operators Φ^t and Λ^t are implemented as sketched in Remarks 3 and 12, and Ψ^1 and Ψ^2 are implemented by solving the Cauchy problems (29) and (30), respectively.

First, we stabilize the origin using output feedback. In order to illustrate the open loop behavior, the controller is switched on at $t = 20$. For $t < 20$, the inputs are set to $U_1(t) = U_2(t) = 0$. The resulting state trajectories and the error between true and estimated state $(u_{est},v_{est}) = \Lambda(\hat{u},\hat{v})$ are depicted in Figure 3. Due to the coupling terms F_u and F_v, the state oscillates wildly even when setting the controlled boundary values to zero. As predicted by theory, the observer manages to estimate the state within $d \approx 3.5$, up to numerical errors. Once switched on, the controller drives the system to the origin also within d.

Second, we consider a tracking example where the objectives are $u(0.2,t) - v(0.2,t) = 0$ and $u(0.7,t) + v(0.7,t) = 0$. Using the notation of (41), this corresponds to $x_t^1 = 0.2$, $x_t^2 = 0.7$, $g^1(v,t) = v$, and $g^2(u,t) = -u$. The resulting trajectories are also depicted in Figure 3. Again, the simulations confirms the theoretical result of exact tracking after $d + \bar{t} \approx 4.8$, up to numerical errors.

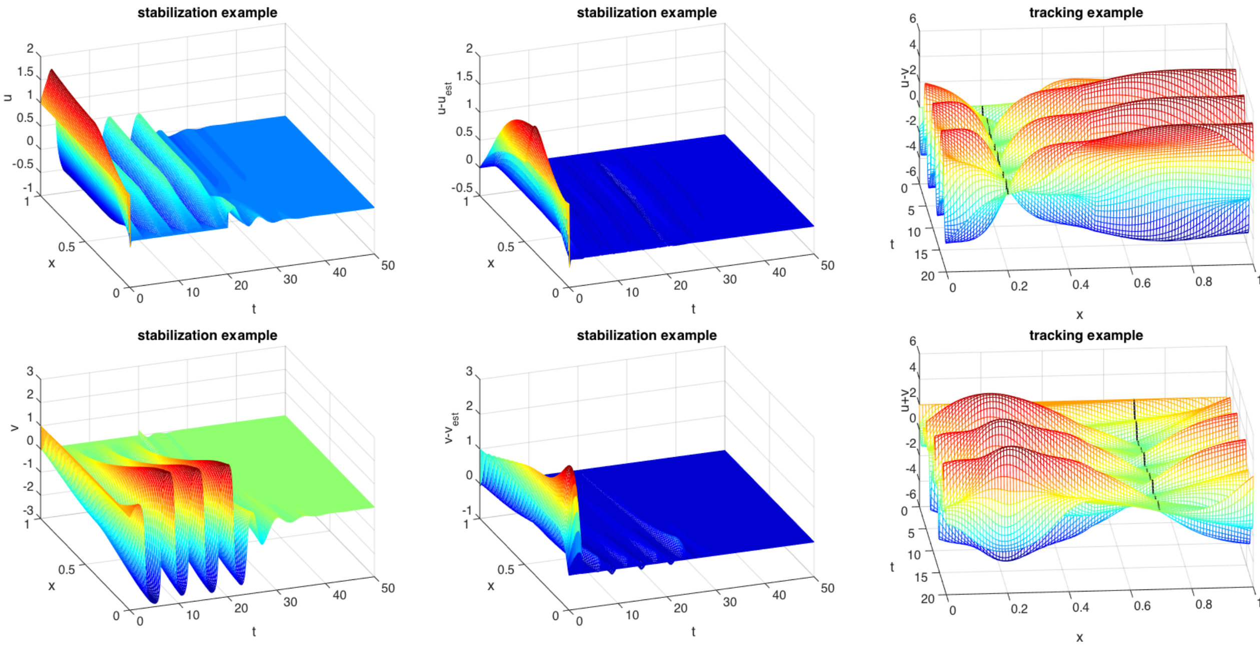

Fig. 3. System trajectories and estimation errors.

VIII. Conclusions

We derived feedback control and estimation laws for the minimum-time control and state estimation of a class of semilinear hyperbolic systems using actuation and sensing at both boundaries of the one-dimensional domain. The control law works for a more general system class than previous results, notably that it allows nonlinear coupling terms between the counter-vecting states and non-smooth transport speeds. The approach for controller design is to derive the dynamics on the characteristic lines along which the control input evolves, and solving these dynamics backwards in time to determine the required actuation. Likewise, the state is estimated via reconstructing the past state on the characteristic line along which the measurements evolve, and is to the best of our knowledge the first constructive observer design for such systems using sensing at both boundaries. The present paper is a continuation of recent results using this method [12], [17], and it would be interesting to see which other cases this approach can be applied to, such as general heterodirectional systems with variable numbers of actuators and sensors at each boundary.

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