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Output feedback boundary control of series interconnections of 2×2 semilinear hyperbolic systems

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Abstract: In this paper, we present an output feedback controller for systems consisting of n 2×2 semilinear hyperbolic systems in series interconnection where actuation and sensing are restricted to one boundary. The output-feedback control law consists of a state-feedback controller combined with an observer. The control and estimation laws are based on the dynamics on the characteristic lines of the hyperbolic system, and achieve stabilization of the origin or tracking at one location, and full state estimation, respectively, globally and in minimum time. We demonstrate the controller performance in a numerical example, and apply the controller to a relevant disturbance rejection problem in oil well drilling.

Keywords: Distributed-parameter systems, Boundary control, Minimum-time control, Stabilization, Regulation, Estimation

1. INTRODUCTION

Many physical systems are described by 2×2 hyperbolic partial differential equations (PDEs), such as open water channels (de Halleux et al., 2003; Dos Santos and Prieur, 2008), pipelines (Gugat and Herty, 2011) and oil wells (Aamo, 2013; Di Meglio and Aarsnes, 2015). Often, actuation and sensing are restricted to one boundary of the domain. An interesting property of hyperbolic systems is that they can be controlled exactly; that is, the system can be driven exactly to an equilibrium, or track a reference signal, within a certain minimum time, as shown by Russell (1978) for linear systems, by Zuazua (1993) for a semilinear wave equation, and by Fu et al. (2007) for a very general class of semilinear hyperbolic systems. The minimum time for exact one-sided boundary controllability is given, e.g., by Li and Rao (2010). For quasilinear systems, local results exist (Li and Rao, 2003). However, these papers discuss only the existence of an open-loop control signal, which limits their practical applicability. Over the last years, the backstepping method has become a popular tool to constructively design state-feedback controllers and observers for linear hyperbolic PDEs (Vazquez et al., 2011; Aamo, 2013; Hu et al., 2015). However, backstepping is (still) limited to the linear case. Recently, we presented a constructive method for the controller and observer design for semilinear systems (Strecker and Aamo, 2016), and applied it to the so-called heave problem in drilling (Strecker and Aamo, 2017a). In the present paper, we generalize this method to n semilinear systems in series interconnection. The development is mainly motivated by a special case of the heave problem where the bit is far from the bottom of the well.

The remainder of this paper is organized as follows. The precise problem statement is given in Section 1.1. Preliminary preparations are done in Section 2, which are used for state-feedback controller, observer and output-feedback controller design in Sections 3, 4 and 5, respectively. The controller performance is demonstrated in a numerical example in Section 6 and the heave problem in Section 7, before concluding remarks are given in Section 8.

1.1 Problem statement

For a positive integer n, we consider a system consisting of n 2×2 semilinear hyperbolic systems in series interconnection. Without loss of generality, we consider the spatial interval $[0, n]$ and assume that the i-th subsystem evolves in the interval $\mathcal{I}^i = [i-1, i]$. For $i = 1, \dots, n$, the system is governed by

$$u_t^i(x,t) = -\lambda_u^i(x) u_x^i(x,t) + f_u^i(u^i(x,t), v^i(x,t), x, t), \quad (1)$$

$$v_t^i(x,t) = \lambda_v^i(x) v_x(x,t) + f_v^i(u^i(x,t), v^i(x,t), x, t), \quad (2)$$

for $x \in \mathcal{I}^i$ and $t \geq 0$. The n subsystems are coupled through

$$u^1(0,t) = g_u^1(v^1(0,t), t), \quad (3)$$

$$v^i(i,t) = g_v^i(v^{i+1}(i,t), u^i(i,t), t) \qquad \Big|_{i=1}^{n-1}, \quad (4)$$

$$u^{i+1}(i,t) = g_u^{i+1}(u^i(i,t), v^{i+1}(i,t), t) \qquad \Big|_{i=1}^{n-1}, \quad (5)$$

$$v^n(n,t) = U(t), \quad (6)$$

where we use $\big|_{i=i_1}^{i_2}$ to denote that an equation holds for all $i = i_1, \dots, i_2$. We assume the nonlinear functions f_u^i, f_v^i, g_u^i and g_v^i to be uniformly Lipschitz continuous in the state arguments and uniformly bounded and measureable in x and t. $U(t)$ is the actuation. Moreover, we assume that g_u^i and g_v^i are invertible in the first argument in the sense that there exist functions $g_u^{i\dagger}$ for $i = 2, \dots, n$ and $g_v^{i\dagger}$ for $i = 1, \dots, n-1$ such that

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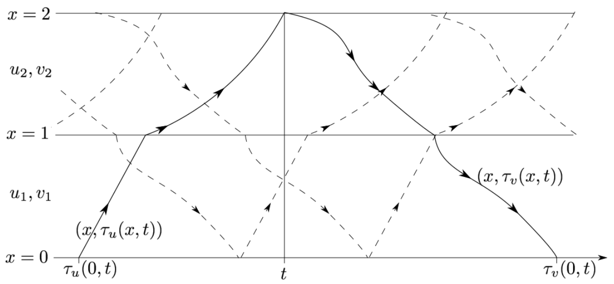

Fig. 1. The characteristic lines along which the states u^i (upwards) and v^i (downwards) evolve for $n = 2$. The bold lines represent the characteristic lines along which $Y(t)$ and $U(t)$ propagate.

$$u^+ = g_u^i(u^-, v^+, t) \quad \Leftrightarrow \quad u^- = g_u^{i\dagger}(u^+, v^+, t), \tag{7}$$
$$v^- = g_v^i(v^+, u^-, t), \quad \Leftrightarrow \quad v^+ = g_v^{i\dagger}(v^-, u^-, t) \tag{8}$$

for all $u^+, u^-, v^+, v^- \in \mathbb{R}$ and $t \geq 0$. We consider the state space of bounded functions on the respective intervals, denoted by $\mathcal{X}_{\mathcal{I}_i}$, equipped with the spatial supremum norm, and assume the initial conditions $u^i(x, 0) = u_0^i(x)$ and $v^i(x, 0) = v_0^i(x)$ to lie in these spaces. Note that the states might be discontinuous and satisfy (1)-(2) only almost everywhere, i.e. not in the classical sense. Weak solutions can be defined as the solution of the integral equations that are obtained by integrating (1)-(2) along its characteristic lines.

Finally, we assume there exist positive bounds $\underline{\lambda}$ and $\bar{\lambda}$ on the transport speeds such that $\underline{\lambda} \leq \lambda_u^i(x), \lambda_v^i(x) \leq \bar{\lambda}$ for all x and i, and that the λ's are measureable.

We consider the following two control problems

- for $i^* \in \{1, \ldots, n\}$ and $x^* \in \mathcal{I}^{i^*}$, achieve
$$v^{i^*}(x^*, t) = h(u^{i^*}(x^*, t), t), \tag{9}$$
where $h : \mathbb{R}^2 \to \mathbb{R}$ defines a tracking problem. Note that in general f_u^i, f_v^i, g_u^i and g_v^i can include disturbance terms (for which short-term predictions are needed). The tracking problem must be well posed in the sense that the solution satisfying (9) remains bounded for the given initial condition.
- stabilization of the origin in minimum time (as given in Li and Rao (2010)) under the additional assumption
$$f_u^i(0, 0, x, t) = f_v^i(0, 0, x, t) = 0 \qquad \forall x, t, i, \tag{10}$$
$$g_u^i(0, 0, t) = g_v^{i\dagger}(0, 0, t) = 0 \qquad \forall t, i. \tag{11}$$
This problem will be solved globally and the effect of the initial condition is removed in minimum time.

Moreover, we design an observer to estimate the infinite-dimensional state from the measurement $Y(t) = u^n(n, t)$ in minimum time, and combine the observer with the state-feedback controller to achieve stabilization or tracking using output-feedback control.

2. PRELIMINARIES

The characteristic lines of the system are sketched in Figure 1. Due to the hyperbolic nature of (1)-(2), the input $U(t)$ propagates with finite speed λ_v from $x = n$ throughout the domain. Therefore, we base the controller design on the dynamics on the characteristic line along which $U(t)$ propagates. Similarly, the measurement $Y(t)$ evolves along a characteristic line corresponding to the transport speed λ_u. We require the following preliminary definitions.

$$\phi_v(x) = \int_x^n \frac{1}{\lambda_v(\xi)} d\xi, \qquad \tau_v(x, t) = t + \phi_v(x), \tag{12}$$
$$\phi_u(x) = \int_x^n \frac{1}{\lambda_u(\xi)} d\xi, \qquad \tau_u(x, t) = t - \phi_u(x), \tag{13}$$

where we used the short notation $\lambda_{u/v} = \lambda_{u/v}^i$ if $x \in (i-1, i]$. The integrals are not affected by the value of λ at the finitely many points $x = 1, \ldots, n-1$ where both $\lambda_{u/v}^i$ and $\lambda_{u/v}^{i-1}$ are defined. The following definition will be central for controller design.

Definition 1. We define the state on the characteristic line along which the actuation $U(t)$ propagates by

$$\bar{u}^i(x, t) = u^i(x, \tau_v(x, t)), \tag{14}$$
$$\bar{v}^i(x, t) = v^i(x, \tau_v(x, t)), \tag{15}$$

for $x \in \mathcal{I}^i$ and $i = 1, \ldots, n$.

For observer design, we require the following definition.

Definition 2. We define the state on the characteristic line along which the measurement $Y(t)$ propagates by

$$\check{u}^i(x, t) = u^i(x, \tau_u(x, t)), \tag{16}$$
$$\check{v}^i(x, t) = v^i(x, \tau_u(x, t)), \tag{17}$$

for $x \in \mathcal{I}^i$ and $t \geq \tau_u(x, t)$, $i = 1, \ldots, n$.

For later use, we also introduce the notation

$$\left\{w^i(\cdot)\right\}_{i=i_1}^{i_2} = \left\{w^{i_1}(\cdot), \ldots, w^{i_2}(\cdot)\right\} \tag{18}$$

to denote sets of functions, where the $w^i(\cdot)$ do not necessarily share their domain of definition.

2.1 Dynamics on the characteristic line $(x, \tau_v(x, t))$

Due to the finite propagation speeds, the states $u^i(x, \theta)$ for $x \in \mathcal{I}^i$ and $\theta \in [t, \tau_v(x, t)]$ are independent of the input $U(t)$. Likewise, $v^i(x, \theta)$ in the open interval $\theta \in [t, \tau_v(x, t))$ is independent of $U(t)$. Therefore, it is possible to predict $\bar{u}$ from the state at time t as stated by the following theorem.

Theorem 3. For every t, there exists a continuous operator $\Phi^t : (\mathcal{X}_{\mathcal{I}_1})^2 \times \ldots \times (\mathcal{X}_{\mathcal{I}_n})^2 \to \mathcal{X}_{\mathcal{I}_1} \times \ldots \times \mathcal{X}_{\mathcal{I}_n}$ such that, independent of $U(t)$,

$$\left\{\bar{u}^i(\cdot, t)\right\}_{i=1}^n = \Phi^t\left(\left\{u^i(\cdot, t), v^i(\cdot, t)\right\}_{i=1}^n\right). \tag{19}$$

Moreover, $\left(\bar{u}^i, \bar{v}^i\right)$ for $i = 1, \ldots, n$ satisfy the PDE-ODE system

$$\bar{u}_t^i(x, t) = -\frac{\lambda_u^i(x)\lambda_v^i(x)}{\lambda_u^i(x) + \lambda_v^i(x)} \bar{u}_x^i(x, t) + \frac{\lambda_v^i(x)}{\lambda_u^i(x) + \lambda_v^i(x)} f_u^i(\bar{u}^i(x, t), \bar{v}^i(x, t), x, \tau_v(x, t)), \tag{20}$$
$$\bar{v}_x^i(x, t) = -\frac{1}{\lambda_v^i(x)} f_v^i(\bar{u}^i(x, t), \bar{v}^i(x, t), x, \tau_v(x, t)), \tag{21}$$

with the coupling conditions

$$\bar{u}^1(0, t) = g_u^1(\bar{v}^1(0, t), \tau_v(0, t)), \tag{22}$$
$$\bar{v}^i(i, t) = g_v^i(\bar{v}^{i+1}(i, t), \bar{u}^i(i, t), \tau_v(i, t)) \quad \Big|_{i=1}^{n-1}, \tag{23}$$
$$\bar{u}^{i+1}(i, t) = g_u^{i+1}(\bar{u}^i(i, t), \bar{v}^{i+1}(i, t), \tau_v(i, t)) \quad \Big|_{i=1}^{n-1}, \tag{24}$$
$$\bar{v}^n(n, t) = U(t), \tag{25}$$

and initial condition

$$\left\{\bar{u}^i(\cdot,0)\right\}_{i=1}^n = \Phi^0\left(\left\{u_0^i(\cdot),v_0^i(\cdot)\right\}_{i=1}^n\right). \tag{26}$$

Proof. A sketch of the proof of existence and continuity of Φ^t is given in Appendix A.
To prove the second statement, we denote the total derivatives with respect to t and x by $\frac{d}{dt}$ and $\frac{d}{dx}$, whereas $_t$ and $_x$ are partial derivatives with respect to time and space. For $\bar{u}^i$, $x \in \mathcal{I}^i$, $i = 1,\ldots,n$, we have

$$\begin{aligned}\bar{u}_t^i(x,t) &= \frac{d}{dt}u^i(x,\tau_v(x,t))\\ &= u_t^i(x,\tau_v(x,t))\frac{d\tau_v(x,t)}{dt} = u_t^i(x,\tau_v(x,t)),\end{aligned} \tag{27}$$

$$\begin{aligned}\bar{u}_x^i(x,t) &= \frac{d}{dx}u^i(x,\tau_v(x,t))\\ &= u_x^i(x,\tau_v(x,t)) + u_t^i(x,\tau_v(x,t))\frac{d\tau_v(x,t)}{dx}\\ &= u_x^i(x,\tau_v(x,t)) - \frac{1}{\lambda_v^i(x)}u_t^i(x,\tau_v(x,t))\\ &= -\frac{1}{\lambda_v^i(x)}u_t^i(x,\tau_v(x,t)) - \frac{1}{\lambda_u^i(x)}\left[u_t^i(x,\tau_v(x,t))\right.\\ &\quad \left.-f_u^i(u^i(x,\tau_v(x,t),v^i(x,\tau_v(x,t)),x,\tau_v(x,t))\right]\\ &= -\frac{\lambda_u^i+\lambda_v^i}{\lambda_u^i\lambda_v^i}u_t^i(x,\tau_v) + \frac{1}{\lambda_u^i}f_u^i(\bar{u}^i(x,t),\bar{v}^i(x,t),x,\tau_v(x,t)).\end{aligned} \tag{28}$$

Inserting (27) into the latter equation gives (20). Analogously for $\bar{v}^i$,

$$\begin{aligned}\bar{v}_x^i(x,t) &= \frac{d}{dx}v^i(x,\tau_v(x,t))\\ &= v_x^i(x,\tau_v(x,t)) - \frac{1}{\lambda_v^i(x)}v_t^i(x,\tau_v(x,t))\\ &= -\frac{1}{\lambda_v^i(x)}v_t^i(x,\tau_v(x,t)) + \frac{1}{\lambda_v^i(x)}\left[v_t^i(x,\tau_v(x,t))\right.\\ &\quad \left.-f_v^i(u(x,\tau_v(x,t)),v(x,\tau_v(x,t)),x,\tau_v(x,t))\right]\\ &= -\frac{1}{\lambda_v^i(x)}f_v^i((\bar{u}^i(x,t),\bar{v}^i(x,t),x,\tau_v(x,t)).\end{aligned} \tag{29}$$

The coupling conditions follow directly from

$$\begin{aligned}\bar{u}^{i+1}(i,t) &= u^{i+1}(i,\tau_v(i,t))\\ &= g_u^{i+1}(u^i(i,\tau_v(i,t)),v^{i+1}(i,\tau_v(i,t)),\tau_v(i,t))\\ &= g_u^{i+1}(\bar{u}^i(i,t),\bar{v}^{i+1}(i,t),\tau_v(i,t))\end{aligned} \tag{30}$$

and

$$\begin{aligned}\bar{v}^i(i,t) &= v^i(i,\tau_v(i,t))\\ &= g_v^i(v^{i+1}(i,\tau_v(i,t)),u^i(i,\tau_v(i,t)),\tau_v(i,t))\\ &= g_v^i(\bar{v}^{i+1}(i,t),\bar{u}^i(i,t),\tau_v(i,t)).\end{aligned} \tag{31}$$

Equation (25) follows from $\tau_v(n,t) = t$.

Remark 4. The operator Φ^t can be implemented by solving (1)-(6) in the domain $\{(x,\theta) : x \in [0,n], \theta \in [t,\tau_v(x,t)]\}$.

2.2 Dynamics on the characteristic line $(x,\tau_u(x,t))$

Due to the finite propagation speeds, the states $u^i(x,\theta)$ and $v^i(x,\theta)$ for $x \in \mathcal{I}^i$ and θ in the open interval $\theta \in (\tau_u(x,t),t]$ have no influence on the measurement $Y(t)$. Therefore, it is possible to predict the state at time t, $\left(u^i(\cdot,t),v^i(\cdot,t)\right)$, $i = 1,\ldots,n$, from the past state $\left(\breve{u}^i(\cdot,t),\breve{v}^i(\cdot,t)\right)$, $i = 1,\ldots,n$.

Theorem 5. For $t \geq \tau_u(0)$, there exists a continuous operator Λ^t such that, independent of $U(t)$,

$$\left\{u^i(\cdot,t),v^i(\cdot,t)\right\}_{i=1}^n = \Lambda^t\left(\left\{\breve{u}^i(\cdot,t),\breve{v}^i(\cdot,t)\right\}_{i=1}^n\right). \tag{32}$$

Moreover, $\left(\breve{u}^i,\breve{v}^i\right)$, $i = 1,\ldots,n$, satisfy the PDE-ODE system

$$\breve{u}_x^i(x,t) = \frac{1}{\lambda_u^i(x)}f_u^i(\breve{u}^i(x,t),\breve{v}^i(x,t),x,\tau_u(x,t)), \tag{33}$$

$$\begin{aligned}\breve{v}_t^i(x,t) &= \frac{\lambda_u^i(x)\lambda_v^i(x)}{\lambda_u^i(x)+\lambda_v^i(x)}\breve{v}_x^i(x,t)\\ &\quad +\frac{\lambda_u^i(x)}{\lambda_u^i(x)+\lambda_v^i(x)}f_v^i(\breve{u}^i(x,t),\breve{v}^i(x,t),x,\tau_u(x,t)),\end{aligned} \tag{34}$$

with the coupling conditions

$$\breve{u}^1(0,t) = g_u^1(\breve{v}^1(0,t),\tau_u(0,t)), \tag{35}$$

$$\breve{v}^i(i,t) = g_v^i(\breve{v}^{i+1}(i,t),\breve{u}^i(i,t),\tau_u(i,t)) \quad \Big|_{i=1}^{n-1}, \tag{36}$$

$$\breve{u}^{i+1}(i,t) = g_u^{i+1}(\breve{u}^i(i,t),\breve{v}^{i+1}(i,t),\tau_u(i,t)) \quad \Big|_{i=1}^{n-1}, \tag{37}$$

$$\breve{v}^n(n,t) = U(t), \tag{38}$$

and $\breve{v}^i(x,0) = v^i\left(x,\phi_u(x)\right)$ for $i = 1,\ldots,n$.

Proof. Proving existence and continuity of Λ^t is done by transforming (1)-(5) with "initial" conditions $u^i(x,\tau_u(x,t)) = \breve{u}^i(x,t)$ and $v^i(x,\tau_u(x,t)) = \breve{v}^i(x,t)$ for $x \in \mathcal{I}^i$ and $i = 1,\ldots,n$ into integral equations and proving existence of a solution in the domains

$$\mathcal{S}_\delta^i = \{(x,\theta) : x \in \mathcal{I}^i, \theta \in [\tau_u(x,t),t]\} \setminus \{(n,t)\}, \tag{39}$$

where the point $\{(n,t)\}$ is omitted in order to remove the effect of the actuation $U(t)$.
The derivation of the PDE-ODE system follows the same steps as in the proof of Theorem 3.

Remark 6. The operator Λ^t can be implemented by solving (1)-(6) in the domain $\{(x,\theta) : x \in [0,n], \theta \in [\tau_u(x,t),t]\}$.

3. STATE-FEEDBACK CONTROLLER DESIGN

The actuation $U(t)$ enters the system at $x = n$, while the tracking objective (9) is located at some general $x^* \in [0,n]$. Since $\bar{u}^{i^*}(x^*,t)$ is predictable from the current state, the desired value of $\bar{v}^{i^*}(x^*,t)$ can be computed. The idea is to consider the desired value of $\bar{v}^{i^*}(x^*,t)$ as a virtual control input, which we denote by $U^*(t)$, and control the system by $U^*(t)$. Exploiting the fact that (21) are ODEs in space without dynamics in time, we can design $U(t)$ such that $\bar{v}^{i^*}(x^*,t)$ becomes $U^*(t)$. In Section 3.3, we show how the same approach can be used to stabilize the system at the origin under the additional assumptions (10)-(11).

3.1 Dynamics with virtual actuation

For given time t, predicted states $\{\bar{u}^i(\cdot,t)\}_{i=1}^n$, and virtual actuation $U^*(t)$ for fixed i^* and x^*, the required actuation $U(t)$ in order to achieve $\bar{v}^{i^*}(x^*,t) = U^*(t)$ can be constructed by solving the ODEs (21) backwards in space (as seen from the propagation direction of $U(t)$) and inverting the coupling conditions (23). This construction can be written in algorithmic form as follows.

- solve the Cauchy problem
$$\varphi_x^{i^*}(x) = -\frac{1}{\lambda_v^{i^*}(x)} f_v^{i^*}(\bar{u}^{i^*}(x,t), \varphi^{i^*}(x), x, \tau_v^{i^*}(x,t)) \tag{40}$$
for $x \in [x^*, i^*]$ with initial value $\varphi^{i^*}(x^*) = U^*(t)$.
- for $i = i^* + 1, \ldots, n$:
  - step from $i-1$ to i by inverting the coupling condition:
$$\varphi_0^i = g_v^{i-1\dagger}(\varphi^{i-1}(i-1), \bar{u}^i(i-1,t), \tau_v^i(i-1,t). \tag{41}$$
  - solve the Cauchy problem
$$\varphi_x^i(x) = -\frac{1}{\lambda_v^i(x)} f_v^i(\bar{u}^i(x,t), \varphi^i(x), x, \tau_v^i(x,t)), \tag{42}$$
$x \in [i-1, i]$, with initial value $\varphi^i(i-1) = \varphi_0^i$.
- set $U(t) = \varphi^n(n)$.

Finally, we define the operator $\Psi^t_{i^*,x^*}$ by
$$\Psi^t_{i^*,x^*} : \left(\{\bar{u}^i(\cdot,t))\}_{i=1}^n, U^*(t)\right) \mapsto U(t). \tag{43}$$

Theorem 7. The system (20)-(25) in closed loop with $U(t) = \Psi^t_{i^*,x^*}\left(\{\bar{u}^i(\cdot,t))\}_{i=1}^n, U^*(t)\right)$ satisfies
$$\bar{u}_t^i(x,t) = -\frac{\lambda_u^i(x)\lambda_v^i(x)}{\lambda_u^i(x)+\lambda_v^i(x)}\bar{u}_x^i(x,t) + \frac{\lambda_v^i(x)}{\lambda_u^i(x)+\lambda_v^i(x)} f_u^i(\bar{u}^i(x,t), \bar{v}^i(x,t), x, \tau_v(x,t)), \tag{44}$$
$$\bar{v}_x^i(x,t) = -\frac{1}{\lambda_v^i(x)} f_v^i(\bar{u}^i(x,t), \bar{v}^i(x,t), x, \tau_v(x,t)), \tag{45}$$
with the coupling conditions
$$\bar{u}^1(0,t) = g_u^1(\bar{v}^1(0,t), \tau_v(0,t)), \tag{46}$$
$$\bar{u}^{i+1}(i,t) = g_u^{i+1}(\bar{u}^i(i,t), \bar{v}^{i+1}(i,t), \tau_v(i,t)) \quad \Big|_{i=1}^{n-1}, \tag{47}$$
$$\bar{v}^i(i,t) = g_v^i(\bar{v}^{i+1}(i,t), \bar{u}^i(i,t), t) \quad \Big|_{i=1}^{i^*-1}, \tag{48}$$
$$\bar{v}^{i^*}(x^*,t) = U^*(t) \tag{49}$$
$$\bar{v}^{i+1}(i,t) = g_v^{i\dagger}(\bar{v}^i(i,t), \bar{u}^i(i,t), \tau_v(i,t)) \quad \Big|_{i=i^*}^{n-1}. \tag{50}$$

Proof. Due to the Lipschitz conditions on f_v^i, g_v^i and $g_v^{i\dagger}$, and the uniform bounds on the transport speeds λ_v^i, the system consisting of the ODEs (42) and the coupling conditions, for given t and $\bar{u}^i(\cdot,t)$, has a unique solution for any given initial conditions. Due to (8), stepping from the $i+1$-st to the i-th subsystem by g_v^i is equivalent to stepping from the i-th to the $i+1$-st subsystem by $g_v^{i\dagger}$. Hence, the system can be solved both in positive and negative x-direction. Therefore, the trajectories of φ^i for all $i = 1, \ldots, n$ are uniquely defined by $\varphi^{i^*}(x^*) = U^*(t)$. Hence, $\varphi^n(n) = U(t)$ if and only if $\varphi^{i^*}(x^*) = U^*(t)$. Since (42) is a copy of (21) for $\varphi^i(\cdot) = \bar{v}^i(\cdot,t)$, this is equivalent to
$$\bar{v}^{i^*}(x^*,t) = U^*(t) \Leftrightarrow \bar{v}^n(n,t) = \Psi^t\left(\{\bar{u}^i(\cdot,t))\}_{i=1}^n, U^*(t)\right).$$

3.2 Tracking

We can now prove the following tracking result.

Theorem 8. The system (1)-(6) in closed loop with the feedback law
$$\begin{aligned} \left\{\bar{u}^i(\cdot,t)\right\}_{i=1}^n &= \Phi^t\left(\left\{u^i(\cdot,t), v^i(\cdot,t)\right\}_{i=1}^n\right), \\ U^*(t) &= h(\bar{u}^{i^*}(x^*,t), \tau_v(x^*,t)), \\ U(t) &= \Psi^t_{i^*,x^*}\left(\{\bar{u}^i(\cdot,t))\}_{i=1}^n, U^*(t)\right), \end{aligned} \tag{51}$$
satisfies the tracking objective (9) for all $t \geq \phi_v(x^*)$.

Proof. Using the feedback law (51), the system (20)-(25) satisfies the tracking objective for all $t \geq 0$ by Theorem 7. Hence, the claim follows directly from Definition 1.

3.3 Stabilization

Above, we established that we can move the location where the virtual actuation enters the $(\bar{u}^i, \bar{v}^i)$ system to a desired location. In order to stabilize the origin, the idea is to force the inflow boundary of the $\bar{u}^i$-subsystems, i.e. $\bar{u}^1(0,t)$, to zero.

Theorem 9. The system (1)-(6) in closed loop with the state-feedback law
$$\begin{aligned} \left\{\bar{u}^i(\cdot,t)\right\}_{i=1}^n &= \Phi^t\left(\left\{u^i(\cdot,t), v^i(\cdot,t)\right\}_{i=1}^n\right), \\ U(t) &= \Psi^t_{1,0}\left(\{\bar{u}^i(\cdot,t))\}_{i=1}^n, 0\right), \end{aligned} \tag{52}$$
satisfies $u^i(x,t) = v^i(x,t) = 0$ for all $i = 1, \ldots, n$, $x \in \mathcal{I}^i$ and $t \geq \phi_v(0) + \phi_u(0)$.

Proof. First, we prove that using (52),
$$\bar{u}^i(x,t) = \bar{v}^i(x,t) = 0 \text{ for all } (x,t,i) \in \mathcal{A}, \tag{53}$$
where
$$\mathcal{A} = \{(x,t,i : i \in \{i,\ldots,n\}, x \in \mathcal{I}^i, t \geq \bar{\phi}(x))\}, \tag{54}$$
$$\bar{\phi}(x) = \int_0^x \frac{\lambda_v(x)+\lambda_u(x)}{\lambda_v(x)\lambda_u(x)} d\xi. \tag{55}$$

For this purpose, we transform the coupled PDE-ODE system (20)-(25) into a system of integral equations. We fix $(x,t,i) \in \mathcal{A}$ and define
$$\bar{\xi}(s) = \bar{\phi}^{-1}\left(\bar{\phi}(x) - t + s\right), \quad \nu(j) = t - \bar{\phi}(x) + \bar{\phi}(j). \tag{56}$$

Since the transport speeds are positive, $\bar{\phi}$ is strictly monotonically increasing, hence invertible. Thus, $\bar{\xi}$ is well defined. Integrating (45) from $\xi = 0$ to $\xi = x$ and inserting the coupling conditions (50) gives the following recursive expression for $\bar{v}^i(x,t)$. For $j = 1, \ldots, i-1$,
$$\begin{aligned} \bar{v}^j(j,t) =& \bar{v}^j(j-1,t) - \int_{j-1}^j \frac{1}{\lambda_v^j(\xi)} \\ & \times f_v^j(\bar{u}^j(\xi,t), \bar{v}^j(\xi,t), \xi, \tau_v(\xi,t)) d\xi, \end{aligned} \tag{57}$$
$$\bar{v}^{j+1}(j,t) = g_v^{j\dagger}\left(\bar{v}^j(j,t), \bar{u}^j(j,t), \tau_v(j,t)\right), \tag{58}$$
and
$$\begin{aligned} \bar{v}^i(x,t) =& \bar{v}^i(i-1,t) - \int_{i-1}^x \frac{1}{\lambda_v^i(\xi)} \\ & \times f_v^i(\bar{u}^i(\xi,t), \bar{v}^i(\xi,t), \xi, \tau_v(\xi,t)) d\xi. \end{aligned} \tag{59}$$

Integrating (44) along its characteristic line and inserting the coupling conditions gives recursively for $j = 1, \ldots, i-1$
$$\begin{aligned} \bar{u}^j(j,\nu(j)) =& \bar{u}^j(j-1,\nu(j-1)) + \int_{\nu(j-1)}^{\nu(j)} \tilde{\lambda}_v^j(\bar{\xi}(s)) \\ & \times f_u^j(\bar{u}^j(\bar{\xi}(s),s), \bar{v}^j(\bar{\xi}(s),s), \bar{\xi}(s), \tau_v^j(\bar{\xi}(s),s)) ds, \end{aligned} \tag{60}$$
$$\begin{aligned} \bar{u}^{j+1}(j,\nu(j)) =& g_u^{j+1}\left(\bar{u}^j(j,\nu(j)),\right. \\ & \left. \bar{v}^{j+1}(j,\nu(j)), \tau_v(j,\nu(j))\right), \end{aligned} \tag{61}$$

where we abbreviated $\tilde{\lambda}_v^j(x) = \frac{\lambda_v^j(x)}{\lambda_u^j(x)+\lambda_v^j(x)}$, and

$$\bar{u}^i(x,t) = \bar{u}^i(i-1,\nu(i-1)) + \int_{\nu(j-1)}^{t} \tilde{\lambda}_v^j(\bar{\xi}(s)) \times f_u^j(\bar{u}^j(\bar{\xi}(s),s), \bar{v}^j(\bar{\xi}(s),s), \bar{\xi}(s), \tau_v^j(\bar{\xi}(s),s))ds. \quad (62)$$

The choice $U^*(t) = 0$ ensures $\bar{v}^1(0,\theta) = 0$ and, due to assumption (11), $\bar{u}^1(0,\theta) = 0$ for all $\theta \geq 0$. Next, we note that if $(x,t,i) \in \mathcal{A}$, then $(\xi,t,j) \in \mathcal{A}$ for all $j \leq i$ and $\xi \in [0,x]$ (due to the monotonicity of $\bar{\phi}$), $(\bar{\xi}(s),s,j) \in \mathcal{A}$ for all $j \leq i$ and $s \in [\nu(0),t]$ (because, if $(x,t,i) \in \mathcal{A}$, then $\bar{\phi}(x) \leq t$, hence $\bar{\phi}(\bar{\xi}(s)) \leq s$ due to the monotonicity of $\bar{\phi}$, hence $\bar{\phi}(\bar{\xi}(s)) \leq t$ for $s \leq t$), and $(j,\nu(j),j) \in \mathcal{A}$ for all $j < i$. Thus, the evaluations of $\bar{u}$ and $\bar{v}$ in the right-hand sides of (57)-(62) lie in $\mathcal{A}$. Inserting (53) into the right-hand sides of (57)-(62), we see that, due to the assumptions (10)-(11), all right-hand sides become zero. That is, (53) solves the integral equations (57)-(62). Because of the Lipschitz conditions and the uniform bounds on the transport speeds, it is also possible to show that the solution of (57)-(62) is unique. Thus, we can reverse the statement and say that the solution of (57)-(62) must satisfy (53), and the same must hold for the original PDE-ODE system (20)-(25).

Finally, the claim for (1)-(6) follows directly from (53), Definition 1 and the equality $\bar{\phi}(n) = \phi_v(0) + \phi_u(0)$.

Remark 10. Exponential Lyapunov stability of the origin can be proven similarly as in (Strecker and Aamo, 2016).

4. OBSERVER DESIGN

Next, we assume we can measure $Y(t) = u^n(n,t)$. Written in the form of (33) with (35)/(37), the equations are to be solved in positive x-direction from $x = 0$ to $x = n$. Since (33) is a set of coupled ODEs in space without dynamics in time, and since we know the boundary value at $x = n$, we can instead solve (33) with (37) in negative x-direction from $x = n$ to $x = 0$. Therefore, we design the observer as a copy of (33)-(38) with (35) replaced by the measurement and (37) replaced by its inverse:

$$\hat{u}_x^i(x,t) = \frac{1}{\lambda_u^i(x)} f_u^i(\hat{u}^i(x,t), \hat{v}^i(x,t), x, \tau_u(x,t)), \quad (63)$$

$$\hat{v}_t^i(x,t) = \frac{\lambda_u^i(x)\lambda_v^i(x)}{\lambda_u^i(x)+\lambda_v^i(x)} \hat{v}_x^i(x,t) + \frac{\lambda_u^i(x)}{\lambda_u^i(x)+\lambda_v^i(x)} f_v^i(\hat{u}^i(x,t), \hat{v}^i(x,t), x, \tau_u(x,t)), \quad (64)$$

with the coupling conditions

$$\hat{u}^i(i,t) = g_u^{i+1\dagger}(\hat{u}^{i+1}(i,t), \hat{v}^{i+1}(i,t), \tau_u(i,t)) \quad \Big|_{i=1}^{n-1}, \quad (65)$$

$$\hat{v}^i(i,t) = g_v^i(\hat{v}^{i+1}(i,t), \hat{u}^i(i,t), \tau_u(i,t)) \quad \Big|_{i=1}^{n-1}, \quad (66)$$

$$\hat{u}^n(n,t) = Y(t), \quad (67)$$

$$\hat{v}^n(n,t) = U(t), \quad (68)$$

and some initial guess $\hat{v}(x,0) = \hat{v}_0(x)$.

Theorem 11. Consider the observer (63)-(68). The state estimates

$$\left\{u_{est}^i(\cdot,t), v_{est}^i(\cdot,t)\right\}_{i=1}^n = \Lambda^t\left(\left\{\hat{u}^i(\cdot,t), \hat{v}^i(\cdot,t)\right\}_{i=1}^n\right) \quad (69)$$

satisfy $u_{est}^i(x,t) = u^i(x,t)$ and $v_{est}(x,t) = v^i(x,t)$ for all $i = 1,\ldots,n$, $x \in \mathcal{I}^i$ and $t \geq \phi_v(0) + \phi_u(0)$.

Proof. The idea of the proof is to establish that the estimator errors $e_u^i = \hat{u}^i - \check{u}^i$ and $e_v^i = \hat{v}^i - \check{v}^i$ satisfy

$$e_u^i(x,t) = e_v^i(x,t) = 0 \text{ for all } (x,t,i) \in \mathcal{B}, \quad (70)$$

where

$$\mathcal{B} = \{(x,t,i : i \in \{i,\ldots,n\}, x \in \mathcal{I}^i, t \geq \hat{\phi}(x))\}, \quad (71)$$

$$\hat{\phi}(x) = \int_x^n \frac{\lambda_v^i(x)+\lambda_u^i(x)}{\lambda_v^i(x)\lambda_u^i(x)} d\xi. \quad (72)$$

Then, the claim follows directly from Definition 2, Theorem 5 and the equality $\hat{\phi}(0) = \phi_v(0) + \phi_u(0)$.

Subtracting (33)-(38) with (37) replaced by its inverse and $\check{u}^n(n,t) = Y(t)$ instead of (35) from (63)-(68) gives

$$e_{u,x}^i(x,t) = \tilde{f}_u(\hat{u}^i, \hat{v}^i, \check{u}^i, \check{v}^i, x, t), \quad (73)$$

$$e_{v,t}^i(x,t) = \frac{\lambda_u^i\lambda_v^i}{\lambda_u^i+\lambda_v^i} e_{v,x}^i(x,t) + \tilde{f}_v(\hat{u}^i, \hat{v}^i, \check{u}^i, \check{v}^i, x, t), \quad (74)$$

with the coupling conditions

$$e_u^i(i,t) = \tilde{g}_u^{i+1\dagger}(\hat{u}^{i+1}, \hat{v}^{i+1}, \check{u}^{i+1}, \check{v}^{i+1}) \quad \Big|_{i=1}^{n-1}, \quad (75)$$

$$e_v^i(i,t) = \tilde{g}_v^i(\hat{v}^{i+1}, \hat{u}^i, \check{v}^{i+1}, \check{u}^i) \quad \Big|_{i=1}^{n-1}, \quad (76)$$

$$e_u^n(n,t) = 0, \quad (77)$$

$$e_v^n(n,t) = 0, \quad (78)$$

where

$$\tilde{f}_u^i(\hat{u}^i, \hat{v}^i, \check{u}^i, \check{v}^i, x, t) = \frac{1}{\lambda_u^i(x)}\left(f_u^i(\hat{u}^i(x,t), \hat{v}^i(x,t), x, \tau_u(x,t)) - f_u^i(\check{u}^i(x,t), \check{v}^i(x,t), x, \tau_u(x,t))\right), \quad (79)$$

$$\tilde{f}_v^i(\hat{u}^i, \hat{v}^i, \check{u}^i, \check{v}^i, x, t) = \frac{\lambda_u^i(x)}{\lambda_u^i(x)+\lambda_v^i(x)}\left(f_v^i(\hat{u}^i(x,t), \hat{v}^i(x,t), x, \tau_u(x,t)) - f_v^i(\check{u}^i(x,t), \check{v}^i(x,t), x, \tau_u(x,t))\right), \quad (80)$$

$$\tilde{g}_u^{i+1\dagger}(\hat{u}^{i+1}, \hat{v}^{i+1}, \check{u}^{i+1}, \check{v}^{i+1}, t) = g_u^{i+1\dagger}(\hat{u}^{i+1}(i,t), \hat{v}^{i+1}(i,t), \tau_u(i,t)) - g_u^{i+1\dagger}(\check{u}^{i+1}(i,t), \check{v}^{i+1}(i,t), \tau_u(i,t)), \quad (81)$$

$$\tilde{g}_v^i(\hat{v}^{i+1}, \hat{u}^i, \check{v}^{i+1}, \check{u}^i, t) = g_v^i(\hat{v}^{i+1}(i,t), \hat{u}^i(i,t), \tau_u(i,t)) - g_v^i(\check{v}^{i+1}(i,t), \check{u}^i(i,t), \tau_u(i,t)). \quad (82)$$

The boundary values (77)-(78) follow directly from $\hat{u}^n(n,t) = \check{u}^n(n,t) = Y(t)$ and $\hat{v}^n(n,t) = \check{v}^n(n,t) = U(t)$. Next, we note that $e_u^i(x,t) = 0$, i.e. $\hat{u}^i(x,t) = \check{u}^i(x,t)$, implies $\tilde{f}_u^i(\hat{u}^i, \hat{v}^i, \check{u}^i, \check{v}^i, x, t) = 0$, and that the analogue holds for $\tilde{f}_v^i$, $\tilde{g}_u^{i+1\dagger}$ and $\tilde{g}_v^i$. Thus, $\tilde{f}_u^i$, $\tilde{f}_v^i$, $\tilde{g}_u^{i+1\dagger}$ and $\tilde{g}_v^i$ satisfy conditions of the form (10)-(11). Therefore, if we make the coordinate change from x to $z = n-x$ and i to $k = n+1-i$, the error system (73)-(78) has exactly the same structure as (44)-(50) for $U^*(t) = 0$, $x^* = 0$ and $i^* = 1$. In that sense, the estimation problem is dual to the stabilization problem from Section 3.3. Thus, the proof can be finished by following the proof of Theorem 9.

5. OUTPUT-FEEDBACK CONTROL

We combine the observer from Section 4 with the state feedback laws from Section 3 to get an output feedback controller.

Corollary 12. The system (1)-(6) in closed loop with the output feedback controller consisting of the observer (63)-(68) and the feedback law

$$\left\{u_{est}^i(\cdot,t), v_{est}^i(\cdot,t)\right\}_{i=1}^n = \Lambda^t\left(\left\{\hat{u}^i(\cdot,t), \hat{v}^i(\cdot,t)\right\}_{i=1}^n\right),$$
$$\left\{\bar{u}_{est}^i(\cdot,t)\right\}_{i=1}^n = \Phi^t\left(\left\{u_{est}^i(\cdot,t), v_{est}^i(\cdot,t)\right\}_{i=1}^n\right),$$
$$U(t) = \Psi_{i^*,x^*}^t\left(\{\bar{u}_{est}^i(\cdot,t)\}_{i=1}^n, U^*(t)\right), \quad (83)$$

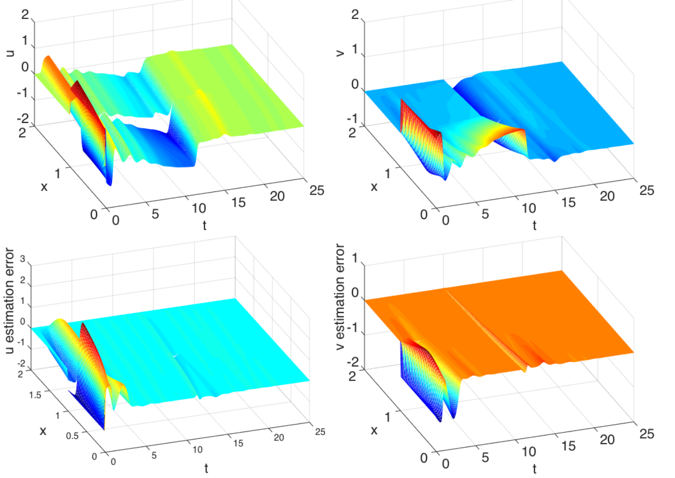

Fig. 2. System state and estimation error trajectories.

satisfies one of the following control objectives.

- If i^* and x^* are as defined by the tracking problem and $U^*(t) = h(\bar{u}^{i^*}(x^*, t), \tau_v(x^*, t))$, (9) is satisfied for all $t \geq \phi_v(0) + \phi_u(0) + \phi_v(x^*)$.
- If $i^* = 1$, $x^* = 0$, $U^*(t) = 0$ and the additional assumptions (10)-(11) hold, then $u^i(x,t) = v^i(x,t) = 0$ for all i, $x \in \mathcal{I}^i$ and $t \geq 2(\phi_v(0) + \phi_u(0))$.

6. EXAMPLE

We demonstrate the controller performance in an example with $n = 2$ and

$$\begin{aligned}
&\lambda_u^1(x) = 0.5\exp(x), && f_u^1(u,v,x,t) = \frac{\sin(u+v)}{3-x}, \\
&\lambda_u^2(x) = 1, && f_u^2(u,v,x,t) = v/3, \\
&\lambda_v^1(x) = 1 + 0.5x, && f_v^1(u,v,x,t) = \sin(v-u), \\
&\lambda_v^2(x) = 3\exp(-x), && f_v^2(u,v,x,t) = -0.25u, \\
&g_u^1(v,t) = -v, && g_u^2(u,v,t) = 0.5(u-v), \\
&g_v^1(v,u,t) = v + 0.5\sin(v).
\end{aligned} \tag{84}$$

In open-loop ($U(t) = 0$), the origin is an unstable equilibrium and the system slowly converges to a stable non-zero equilibrium. The initial states are set to $u_0^1(x) = v_0^1(x) = 1$ and $u_0^2(x) = v_0^2(x) = 0$, and all initial states of the observer are set to zero. We stabilize the system using output feedback. In order to demonstrate the open-loop behavior, the controller is switched on only for $t \geq 10$, before that we use $U(t) = 0$. The resulting trajectories are depicted in Figure 2. Note that the respective trajectories for $i = 1, 2$ are plotted together. As predicted by theory, the estimation error becomes zero within $\phi_u(0) + \phi_v(0) \approx 3.6$, up to minor numerical errors. Once the controller is switched on, the same holds for the system states.

7. APPLICATION TO OIL WELL DRILLING

We apply the output feedback controller to the heave problem in offshore managed pressure drilling (MPD). The heave problem occurs when the drill string in an oil well oscillates with the wave-induced heave motion of the rig, and has been described in more detail elsewhere ((Aamo, 2013; Strecker and Aamo, 2017b) and references therein).

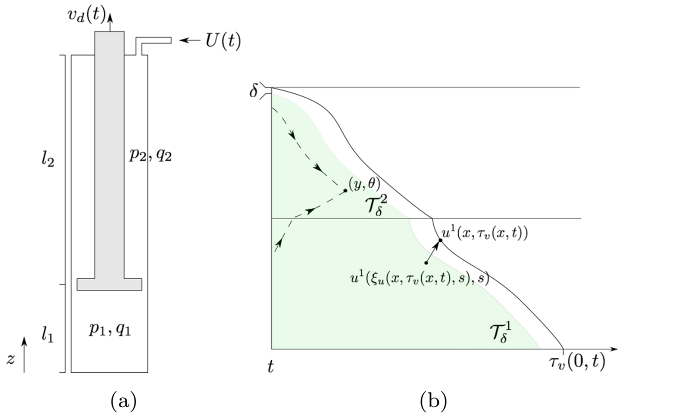

Fig. 3. Left: Sketch of an oil well. Right: Illustration of the domains $\mathcal{T}_\delta^i$ (green), the integration paths to obtain u^2 and v^2 at $(y,\theta) \in \mathcal{T}_\delta^2$ (dashed lines), and how to obtain $u^1(x, \tau_v(x,t))$ by continuity (right arrow), see also Appendix A.

A sketch of an oil well is also depicted in Figure 3a. Briefly speaking, an oil well is filled with a fluid called drilling mud, which is designed to keep the pressure in the well within tolerable margins. However, movement of the drill string induces pressure oscillations that can violate these margins. A typical control objective is to keep the pressure at the bottom of the well at a setpoint. We consider the problem of attenuating such pressure oscillations by controlling the topside annular pressure and flow rate at the rig via the opening of an outflow choke. In this paper, we are concerned with the case that the drill bit is not at the bottom of the well, which occurs for instance when the drill string is pulled out of the borehole. That is, the drill string is only in the upper part of the well, and there is a significant mud column below the bit. In this case, the well can be modeled be two coupled hydraulic transmission lines describing the mud dynamics in the section below the bit and in the annular space around the drill string, respectively. We assume the drill string to be rigid, which is a reasonable assumptions in approximately vertical wells up to 5000 m length. The governing equations are (see also (Strecker and Aamo, 2017b; Aamo, 2013)

$$p_t^1(z,t) = -\frac{\beta}{A_1} q_z^1(z,t) \tag{85}$$

$$q_t^1(z,t) = -\frac{A_1}{\rho} p_z^1(z,t) - \frac{1}{\rho} F^1(q^1(z,t)) - A_1 g \tag{86}$$

$$p_t^2(z,t) = -\frac{\beta}{A_2} q_z^2(z,t) \tag{87}$$

$$q_t^2(z,t) = -\frac{A_2}{\rho} p_z^2(z,t) - \frac{1}{\rho} F^2(q^2(z,t), v_d(t)) - A_2 g \tag{88}$$

where the index 1 represents the subsystem below the bit, 2 is the mud in the annular space around the drill string, $z \in [0, l_1 + l_2]$ is the position measured from the bottom, l_i is the length of the respective section, p is pressure, q is the volumetric flow rate, the subscripts $_z$ and $_t$ denote partial derivatives with respect to space and time, respectively, $v_d(t)$ is the drill string velocity, A denotes the respective cross sectional area, β the bulk modulus, ρ the mud density, and g the gravitational acceleration. F^1 and F^2 are nonlinear functions representing friction, which are constructed as described in (Strecker and Aamo, 2017b) depending on the mud rheology and well geometry. There is no flow at the bottom, hence

$$q^1(0,t) = 0. \tag{89}$$

The two subsystems are coupled at the (very short) drill bit, where we have

$$p^2(l_1,t) = p^1(l_1,t), \qquad q^2(l_1,t) = q^1(l_1,t) + d(t) \tag{90}$$

for $d(t) = -A_d v_d(t)$, where $A_d = A_1 - A_2$ is the cross sectional area displaced by the drill string. The boundary condition at the top, $z = l_1 + l_2$, is left as the actuation. The objective is to control the pressure at the bottom of the well to a setpoint pressure p_{sp}, i.e.

$$p^1(0,t) = p_{sp}. \tag{91}$$

7.1 State transformation

In order to apply the output feedback controller, we have to transform the well model into the form (1)-(6). The transformation

$$u^i(x,t) = \frac{1}{2}\left(q^i(z_i(x),t) + \frac{A_i}{\sqrt{\beta\rho}}(p^i(z_i(x),t) - p_{sp} + \rho g z_i(x))\right), \tag{92}$$

$$v^i(x,t) = \frac{1}{2}\left(q^i(z_i(x),t) - \frac{A_i}{\sqrt{\beta\rho}}(p^i(z_i(x),t) - p_{sp} + \rho g z_i(x))\right), \tag{93}$$

for $i = 1,2$ with $z_1(x) = l_1 x$ and $z_2(x) = l_2(x-1) + l_1$ maps (85)-(88) into (1)-(2) with

$$\lambda_u^i(x) = \lambda_v^i(x) = \frac{1}{l_i}\sqrt{\frac{\beta}{\rho}}, \tag{94}$$

$$f_u^1(u,v,x,t) = -\frac{1}{2\rho}F^1(u+v), \tag{95}$$

$$f_u^2(u,v,x,t) = -\frac{1}{2\rho}F^2(u+v, v_d(t)), \tag{96}$$

which can be verified by differentiating (92)-(93) w.r.t. t, inserting (85)-(88), and substituting q^i and p^i by u^i and v^i by inverting (92)-(93). Inserting the inverse of (92)-(93) into the coupling conditions (90) gives

$$(u^2(1,t) - v^2(1,t)) = \frac{A_2}{A_1}(u^1(1,t) - v^1(1,t)), \tag{97}$$

$$u^2(1,t) + v^2(1,t) = u^1(1,t) + v^1(1,t) + d(t). \tag{98}$$

Solving this linear system for the unknowns $v^1(1,t)$ and $u^2(1,t)$ gives

$$g_v^1(v,u,t) = -\frac{1}{1+\frac{A_2}{A_1}}\left(\left(1 - \frac{A_2}{A_1}\right)u - 2v + d(t)\right) \tag{99}$$

$$g_u^2(u,v,t) = \frac{1}{1+\frac{A_2}{A_1}}\left(2\frac{A_2}{A_1}u - \left(\frac{A_2}{A_1} - 1\right)v + \frac{A_2}{A_1}d(t)\right) \tag{100}$$

Inserting the inverse of (92)-(93) into (89) gives

$$g_u^1(v,t) = -v. \tag{101}$$

7.2 Simulations

We simulate a 4000 m deep vertical well where the bit is 1000 m above the bottom, hence $l_1 = 1000\,\mathrm{m}$ and $l_2 = 3000\,\mathrm{m}$. The well and drill string outer diameters are 216 mm and 127 mm, respectively, hence $A_1 = 0.0366\,\mathrm{m}^2$, $A_2 = 0.0239\,\mathrm{m}^2$ and $A_d = 0.0127\,\mathrm{m}^2$. We use a mud with $\rho = 1500\,\mathrm{kg/m}^3$, $\beta = 16000\,\mathrm{bar}$, and a Bingham-type rheology with plastic viscosity 20 mPas and yield point 5 Pa. With this rheology and well geometry, the parameter fitting procedure from (Strecker and Aamo, 2017b) returns the following friction terms:

$$F^1(q^1) = \left(c_0^1 + c_K^1|q^1/A_1|^{n^1}\right)s\left(q^1/A_1\right), \tag{102}$$

$$F^2(q^2, v_d) = \sum_{j=2}^{3}\left(c_0^j + c_K^j|v_{eff}^j|^{n^j}\right)s\left(v_{eff}^j\right), \tag{103}$$

where $v_{eff}^j(q^2, v_d) = q^2/A_2 - k^j v_d$ for $j = 2,3$,

$$c_0^1 = 3.4, \quad c_K^1 = 1.1, \quad n^1 = 0.85, \tag{104}$$

$$c_0^2 = 2, \quad c_K^2 = 2, \quad n^2 = 1, \quad k^2 = 0.8, \tag{105}$$

$$c_0^3 = 3.4, \quad c_K^3 = 3, \quad n^3 = 0.95, \quad k^3 = 0.07, \tag{106}$$

and $s(v) = \frac{v}{\sqrt{v^2+0.01}}$ is a smooth approximation of the sign function.

For simplicity, we use a simple sinusoidal heave motion with amplitude $a = 1\,\mathrm{m}$ and wave period $T = 12\,\mathrm{s}$ (a typical dominant wave period in the North Sea), i.e. $v_d(t) = a\omega\sin(\omega t)$ for $\omega = 2\pi/T$. We set $p_{sp} = 600\,\mathrm{bar}$. The pressure and flow rate trajectories are depicted in Figure 4. The controller is switched on only after 40 s. Before that, we use $U(t) = 0$. In the uncontrolled case, the pressure at the bottom of the well oscillates by approximately $\pm 5\,\mathrm{bar}$ around p_{sp}, whereas the output feedback controller attenuates the pressure oscillations up to minor numerical errors. Since no disturbance terms enter in the section below the bit, the pressure oscillations become zero in the whole section $z \in [0, l_1]$.

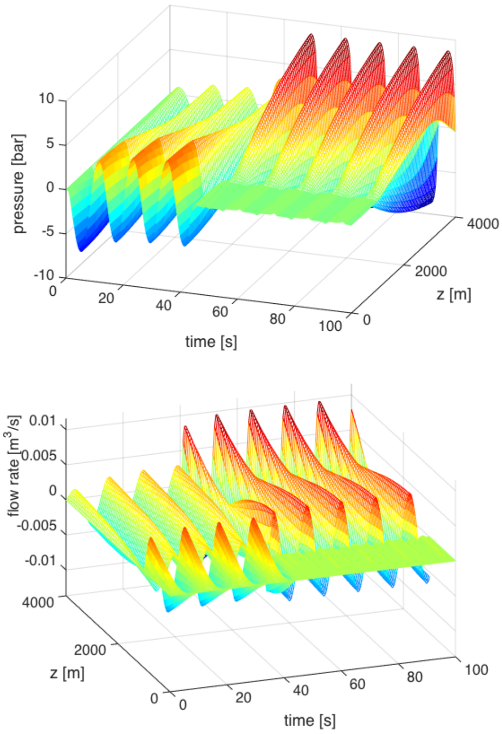

Fig. 4. Pressure deviation from steady state, $p(z,t) - p_{sp} + \rho g z$, and flow rate trajectories using output feedback control.

8. CONCLUSION

We generalized recent results on the controller and observer design of 2×2 semilinear systems to n such systems in serious interconnection. The controller was applied to the heave-problem in MPD and successfully attenuates pressure oscillations at the bottom of an oil well modeled by two coupled hydraulic transmission lines. It would be interesting to see if the approach in this paper can be generalized to more general network structures, perhaps with actuation and sensing at more locations. In Gugat et al. (2011), for instance, tree-like networks with one actuator for each tracking objective were considered, and it should be possible to extend the methods in the present paper to this case.

Evaluation of the presented control and estimation laws requires solving PDE systems. For linear systems, explicit formulas for the state-feedback control law could be derived in (Strecker and Aamo, 2016) for the case $n = 1$. Such explicit formulas are desirable from a computational point of view, and it should be investigated if the same is possible for $n > 1$.

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Appendix A. EXISTENCE OF Φ^t

Due to page limitations, we provide only a sketch of the proof of existence and continuity of Φ^t in Theorem 3. A detailed proof for the case $n = 1$ can be found in (Strecker and Aamo, 2016), which needs to be modified by including the coupling conditions (4)-(5). First, the PDE system (1)-(5) (note that we omit the actuation (6)) is transformed into integral equations by integrating them along their characteristic lines, similarly as it is done in the proof of Theorem 9. See also Figure 3b. Then, for fixed t, existence of a unique solution that is Lipschitz continuous in the state $\left(u^i(\cdot,t), v^i(\cdot,t)\right)$ in the domain

$$\begin{aligned}\mathcal{T}^i_\delta = \{(x,\theta) : x \in \mathcal{I}^i \setminus [n-\delta, 1],\\ \theta \in [t, \tau_v(x,t) - \phi_v(n-\delta)]\}\end{aligned} \tag{A.1}$$

for small $\delta > 0$ and $i = 1, \ldots, n$ can be proven by a successive approximation method. The domains $\mathcal{T}^i_\delta$ are designed such that for every i and $(x,\theta) \in \mathcal{T}^i_\delta$, the resulting integral equations involve only terms of u^j and v^j evaluated at points inside the respective $\mathcal{T}^j_\delta$ for $j \in \{1, \ldots, n\}$. In particular, the integral equations are independent of $U(t)$.

Finally, the output of the Φ^t, $\left(\left\{\bar{u}^i(\cdot,t)\right\}_{i=1}^n\right)$, is obtained by uniform continuity of u^i along its characteristic lines. More precisely, one can derive an appropriate $\xi_u(x,\theta,s)$ with $\xi_u(x,\theta,s) < x$ for all θ and $s < \theta$, and $\xi_u(x,\theta,\theta) = x$ for all x, such that $u^i(\xi_u(x,\theta,s),s)$ satisfies an ODE in s. Thus, we have

$$\bar{u}^i(x,t) = u^i(x,\tau_v(x,t)) = \lim_{s\to\tau_v(x,t)} u^i(\xi_u(x,\tau_v(x,t),s),s), \tag{A.2}$$

where the limit is taken over $s \in [\underline{s}, \tau_v(x,t))$ for some appropriate $\underline{s}$. Since the characteristic line $(\xi_u(x,\tau_v(x,t),s),s)$ intersects $(\xi,\tau_v(\xi,t))$, $\xi \in [0,n]$, in a non-zero angle, there exists for every $s \in [\underline{s}, \tau_v(x,t))$ a $\delta > 0$ such that $(\xi_u(x,\tau_v(x,t),s),s) \in \mathcal{T}^i_\delta$. Thus, the right-hand side of (A.2) is well defined and Lipschitz continuous in the state at time t. Since the limit is attained uniformly, the same holds for $\bar{u}^i(x,t)$.