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ORIGINAL RESEARCH ARTICLE

Some probabilistic properties of deep water wave steepness

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Summary This paper provides some probabilistic properties of the deep water wave steepness and the spectral wave steepness by using distributions based on data from the Norwegian continental shelf. Here the average statistical properties represented by the mean value and the standard deviation of the two steepness parameters are considered. Examples of results for the wave steepness are given for a Phillips spectrum and a family of JONSWAP spectra for wind sea, and for sea states described by a joint frequency table of significant wave height and mean zero-crossing wave period for combined wind sea and swell. The results for the spectral wave steepness are obtained by using a joint distribution of significant wave height and spectral wave steepness, and the average statistical features are given for joint frequency tables of significant wave height and mean zero-crossing wave period from three locations on the Norwegian continental shelf.

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1. Introduction

The wave steepness for individual waves as well as the spectral wave steepness for a sea state are parameters which are frequently used to represent the random wave characteristics at sea, relevant e.g. for a design of offshore and coastal structures. The wave steepness for individual waves is defined in terms of the wave height H and the wave period T, while the spectral wave steepness is defined in terms of the significant wave height H_s and the mean zero-crossing wave period T_z (or the spectral peak period T_p). Various aspects of wave steepness statistics, also jointly with the wave height, have been discussed by Myrhaug and Kjeldsen (1984), Myrhaug and Kvålsvold (1995). Further details of the relevance and the literature on wave steepness statistics are given in e.g. Myrhaug and Fouques (2007). The spectral wave steepness is also a random variable and has been addressed in e.g. Guedes Soares et al. (2001), Bitner-Gregersen and Guedes Soares (2007), Myrhaug and Fouques (2008). Guedes Soares et al. (2001) analyzed the statistics of total ship losses in the North Atlantic and found that areas with high mean wave steepness coincided with areas with many accidents. Bitner-Gregersen and Guedes Soares (2007) used five databases from the North Atlantic to investigate the uncertainty of predicting the characteristic wave steepness from joint distributions of H_s and T_z, in addition to give a review of the literature. Myrhaug and Fouques (2008) provided a joint distribution of H_s and spectral wave steepness defined in terms of H_s and T_p based on data from the northern North Sea.

In the present paper, some average probabilistic properties of the deep water wave steepness and the spectral wave steepness are considered. The average statistical properties are given in terms of the mean value and the standard deviation. The Myrhaug and Fouques (2007) wave steepness distribution is used to estimate the statistical values of the wave steepness for individual waves within a sea state; i.e. for a given value of the spectral wave steepness for a sea state. Examples of results are given for a Phillips spectrum, for a family of JONSWAP spectra, and for sea states described by a joint frequency table of H_s and T_z. The statistical properties of the spectral wave steepness defined in terms of H_s and T_z are provided by using a joint distribution of significant wave height and spectral wave steepness, obtained by transforming the Mathisen and Bitner-Gregersen (1990) joint distribution of H_s and T_z.

The paper is organized as follows. The introduction is followed by Section 2 giving the background by presenting the Myrhaug and Fouques (2007) wave steepness distribution (Section 2.1) and the joint distribution of significant wave height and spectral wave steepness (Section 2.2). Section 3 presents examples of estimates of wave steepness (Section 3.1) and spectral wave steepness (Section 3.2). Summary and conclusions are given in Section 4.

2. Background

2.1. Myrhaug and Fouques (2007) *pdf* of wave steepness

According to Myrhaug and Fouques (2007) the probability density function (*pdf*) of the normalized deep water wave steepness $s = S S_{rms}^{-1}$ (where $S = H/((g/2\pi)T^2)$ is the wave steepness) is given by the following combined lognormal and Weibull *pdfs*

$$p(s) = \begin{cases} p_1(s) = \dfrac{1}{\sqrt{2\pi}\sigma s}\exp\left[-\dfrac{1}{2}\left(\dfrac{\ln s - \mu}{\sigma}\right)^2\right]; & s \le s_1 = 1.2 \\ p_2(s) = \theta\dfrac{s^{\theta-1}}{\zeta^\theta}\exp\left[-\left(\dfrac{s}{\zeta}\right)^\theta\right]; & s > s_1 = 1.2 \end{cases} \tag{1}$$

Here μ =−0.257, σ = 0.523 are the mean value and the standard deviation, respectively, of $\ln s$, and ζ = 0.84, θ = 1.40 are the Weibull parameters. Furthermore, S_{rms} = 0.7 s_m where $s_m = H_s/((g/2\pi)T_z^2)$ is the spectral wave steepness, $H_s = 4\sqrt{m_0}$ is the significant wave height, $T_z = 2\pi\sqrt{m_0/m_2}$ is the mean zero-crossing wave period, $m_n = \int_0^\infty \omega^n S(\omega)d\omega; \quad n = 0, 1, 2, \ldots$ are the spectral moments, $S(\omega)$ is the single-sided wave spectrum, $\omega = 2\pi/T$ is the wave frequency, and g = 9.81 ms^{-2} is the acceleration of gravity. Thus, the *pdf* of s is strictly a conditional distribution of S for a given sea state, i.e. for given values of H_s and T_z. Consequently, statistical values of the wave steepness can be obtained for a given sea state. The data upon which Eq. (1) is based, were taken from a larger database measured with the Waverider buoys located at Utsira, Halten and Tromsøflaket during the period 1974—1978: see Myrhaug and Fouques (2007) and the references therein for more details.

In the following the expected (mean) value, $E[S]$, and the variance, $Var[S] = E[S^2] - (E[S])^2$ of the wave steepness will be considered. Thus, this requires the calculation of $E[s^n]$ for n = 1 and n = 2, i.e.

$$E[s^n] = \int_0^\infty s^n p(s)ds = \int_0^{s_1} s^n p_1(s)ds + \int_{s_1}^\infty s^n p_2(s)ds. \tag{2}$$

Here (Bury, 1975)

$$\int_0^{s_1} s^n p_1(s)ds = e^{n\mu+\frac{1}{2}n^2\sigma^2}\Phi\left[\frac{\ln s_1 - (\mu + n\sigma^2)}{\sigma}\right], \tag{3}$$

where Φ is the standard Gaussian cumulative distribution function (*cdf*)

$$\Phi(\nu) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\nu} e^{-t^2/2}dt. \tag{4}$$

Furthermore (Abramowitz and Stegun, 1972; Chs. 6.5 and 26.4)

$$\int_{s_1}^\infty s^n p_2(s)ds = \zeta^n \Gamma\left[1 + \frac{n}{\theta}, \left(\frac{s_1}{\zeta}\right)^\theta\right], \tag{5}$$

where $\Gamma(r, t)$ is the incomplete gamma function; $\Gamma(r, 0) = \Gamma(r)$ where Γ is the gamma function, and $\Gamma(r, \infty)$ =0.

Then, by using Eqs. (1)—(5), the results are

$$E[s] = 0.873, \tag{6}$$

$$Var[s] = 0.238. \tag{7}$$

This gives the ratio between the standard deviation of the wave steepness and the mean value of the wave steepness, i.e. the coefficient of variation = $\sqrt{Var[s]}/E[s] = 0.559$. Now it follows that

$$E[S] = E[s]\cdot S_{rms} = E[s]\cdot 0.7\cdot\frac{2\pi}{g}\frac{H_s}{T_z^2} = 0.391\frac{H_s}{T_z^2} \tag{8}$$

with the standard deviation to the mean value ratio of S equal to 0.559.

2.2. Joint *pdf* of significant wave height and spectral wave steepness

Here the joint *pdf* of H_s and s_m is obtained from the joint *pdf* of H_s and T_z given by Mathisen and Bitner-Gregersen (1990) as summarized in Appendix B, representing wave data from the three locations Utsira, Halten and Tromsøflaket on the Norwegian continental shelf. The Utsira data are the same as those referred to in Section 3.

The joint *pdf* of H_s and s_m is obtained from Eq. (B1) by following the same procedure as in Myrhaug and Fouques (2008), i.e. by a change of variables from H_s, T_z to H_s, s_m which takes the form

$$p(H_s, s_m) = p(s_m|H_s)p(H_s), \tag{9}$$

where $p(H_s)$ is given in Eq. (B2). The change of variable only affects $p(T_z|H_s)$ since $T_z = [H_s/(g/2\pi)]^{1/2}s_m{}^{-1/2}$. By using the Jacobian $|dT_z/ds_m|$, this yields the lognormal *pdf* of s_m given H_s as

$$p(s_m|H_s) = \frac{1}{\sqrt{2\pi}\sigma_{s_m}s_m}\exp\left[-\frac{1}{2}\left(\frac{\ln s_m - \mu_{s_m}}{\sigma_{s_m}}\right)^2\right], \tag{10}$$

where μ_{s_m} and $\sigma_{s_m}^2$ are the mean value and the variance, respectively, of $\ln s_m$, given as

$$\mu_{s_m} = \ln\left(\frac{H_s}{g/2\pi}\right) - 2\mu, \tag{11}$$

$$\sigma_{s_m}^2 = 4\sigma^2, \tag{12}$$

where $\mu = \mu_t$ and $\sigma = \sigma_t$ are given in Eqs. (B4) and (B5), respectively.

In Section 3 the results will be exemplified by considering the statistical quantities $E[H_s]$, $E[s_m|H_s]$ and the coefficient of variation $R = \sigma[s_m|H_s]/E[s_m|H_s]$, given by (Bury, 1975)

$$E[H_s] = \varepsilon_h + \rho_h\Gamma\left(1 + \frac{1}{\theta_h}\right), \tag{13}$$

$$E[s_m|H_s] = \exp\left(\mu_{s_m} + \frac{1}{2}\sigma_{s_m}^2\right), \tag{14}$$

$$R = \left(e^{\sigma_{s_m}^2} - 1\right)^{1/2}. \tag{15}$$

3. Examples of estimates of wave steepness and spectral wave steepness

3.1. Estimates of wave steepness

Estimates of the wave steepness will now be exemplified by using a Phillips spectrum, by a family of JONSWAP spectra, and by a joint frequency table of H_s and T_z.

Table 1 Conditional mean value plus and minus one standard deviation for given sea states at Utsira.

H_s(m)/T_z(s)	2.5	3.5	4.5	5.5	6.5	7.5	8.5	9.5	10.5
1	0.0626 ± 0.0350	0.0319 ± 0.0178	0.0193 ± 0.0108	0.0129 ± 0.0072	0.0093 ± 0.0052	0.0070 ± 0.0039	0.0054 ± 0.0030		
2	0.125 ± 0.0699	0.0638 ± 0.0357	0.0386 ± 0.0216	0.0259 ± 0.0145	0.0185 ± 0.0103	0.0139 ± 0.0078	0.0108 ± 0.0060	0.0087 ± 0.0049	
3			0.0579 ± 0.0324	0.0388 ± 0.0217	0.0278 ± 0.0155	0.0209 ± 0.0117	0.0162 ± 0.0091	0.0130 ± 0.0073	0.0106 ± 0.0059
4			0.0772 ± 0.0432	0.0517 ± 0.0289	0.0370 ± 0.0207	0.0278 ± 0.0155	0.0216 ± 0.0121	0.0173 ± 0.0097	0.0142 ± 0.0079
5				0.0646 ± 0.0361	0.0463 ± 0.0259	0.0348 ± 0.0195	0.0271 ± 0.0151	0.0217 ± 0.0121	0.0177 ± 0.0099
6					0.0555 ± 0.0310	0.0417 ± 0.0233	0.0325 ± 0.0182	0.0260 ± 0.0145	0.0213 ± 0.0119
7						0.0487 ± 0.0272	0.0379 ± 0.0212	0.0303 ± 0.0169	0.0248 ± 0.0139
8							0.0433 ± 0.0242	0.0347 ± 0.0194	0.0284 ± 0.0159
9							0.0487 ± 0.0272	0.0390 ± 0.0218	0.0319 ± 0.0178
10								0.0433 ± 0.0242	0.0355 ± 0.0198
11								0.0477 ± 0.0266	0.0390 ± 0.0218
12									0.0426 ± 0.0238

First, the Phillips spectrum is given as (see e.g. Tucker and Pitt, 2001)

$$S(\omega) = \alpha \frac{g^2}{\omega^5}; \quad \omega \geq \omega_p, \tag{16}$$

$$S(\omega) = 0; \quad \omega < \omega_p, \tag{17}$$

where $\alpha = 0.0081$ is the Phillips constant, $\omega_p = 2\pi/T_p$ is the spectral peak frequency, and T_p is the spectral peak period. Then, according to the definition of the spectral moments m_n, $H_s/T_z^2 = m_2/(\pi^2\sqrt{m_0}) = g\sqrt{\alpha}/\pi^2 = 0.0895$, and thus from Eq. (8)

$$E[S] = 0.0350. \tag{18}$$

Then, in order to give an impression of the variation of the parameter, the mean value plus and minus ($\pm$) one standard deviation is provided; given by

$$\text{mean value}\pm\text{one standard deviation} = (0.0154, 0.0546). \tag{19}$$

Second, by using a family of JONSWAP spectra the following results are obtained by using Eq. (8) and Eqs. (A5)—(A7) in Appendix A:

$$\gamma = 1: \quad E[S] = 0.0307, \tag{20}$$

$$\text{mean value}\pm\text{one standard deviation} = (0.0135, 0.0479), \tag{21}$$

$$\gamma = 3: E[S] = 0.0407, \tag{22}$$

$$\text{mean value}\pm\text{one standard deviation} = (0.0180, 0.0634), \tag{23}$$

$$\gamma = 5: E[S] = 0.0464, \tag{24}$$

$$\text{mean value}\pm\text{one standard deviation} = (0.0205, 0.0725). \tag{25}$$

Here γ is the spectral peakedness parameter as given in Appendix A. It should be noted that $\gamma = 1$ corresponds to the Pierson-Moskowitz spectrum. It is also noticed that the wave steepness increases as γ increases. Furthermore, it appears that $E[S]$ is larger for the Phillips spectrum than for the Pierson-Moskowitz spectrum which is due to that the Phillips spectrum contains waves with higher frequencies than the Pierson-Moskowitz spectrum, i.e. overall these sea states contain steeper waves than those described by a Pierson-Moskowitz spectrum. However, $E[S]$ for $\gamma = 3$ and $\gamma = 5$ are larger than for the Phillips spectrum, reflecting that overall these sea states contain steeper waves than those described by a Phillips spectrum. One should also note that all these mean values are lower than the wave steepness values in the range 0.05—0.13 as given in Myrhaug and Kjeldsen (1986, Fig. 3), belonging to the same database upon which the wave steepness distribution in Eq. (1) is based. However, these latter values are associated with steep and high waves in the database (see Myrhaug and Kjeldsen (1986) for more details).

Till now wind sea has been considered. However, if swell and combined wind sea and swell are considered then other spectral formulations have to be used, e.g. the Torsethaugen spectrum (Torsethaugen, 1996), or to use a joint frequency table of H_s and T_z given in Mathisen and Bitner-Gregersen (1990), which represents data obtained by a Waverider buoy covering the period 1974—1986 at the Utsira location on the Norwegian continental shelf. Thus, these data and the data upon which the wave steepness distribution are based represent partly the same area and period. Now estimates of $E[S]$ according to Eq. (8) and the mean value plus and minus one standard deviation are given in Table 1 for the sea states H_s, T_z corresponding to those for which there are data in Mathisen and Bitner-Gregersen (1990). The results in Table 1 exhibit the following expected features: $E[S]$ decreases as T_z increases for a given value of H_s, i.e. as the spectral wave steepness decreases; $E[S]$ increases as H_s increases for a given value of T_z, i.e. as the spectral wave steepness increases. It is also noted that the values of the wave steepness cover the wide range 0.0054—0.125.

3.2. Estimates of spectral wave steepness

Here estimates of the spectral wave steepness are exemplified by using the joint frequency tables of H_s and T_z from the three locations Utsira, Halten and Tromsøflaket given by Mathisen and Bitner-Gregersen (1990). First, $E[H_s]$ is calculated according to Eq. (13) and the Weibull parameters in Table B1 for each class of H_s and T_z at the three locations. Second, $E[s_m|E[H_s]]$ and the coefficient of variation are calculated according to Eqs. (14) and (15), respectively. The results are given in Table 2, showing that there are small

Table 2 Statistical results for H_s and s_m at Utsira, Halten and Tromsøflaket based on the Mathisen and Bitner-Gregersen (1990) joint *pdf* of H_s and T_z.

Location	Utsira	Halten	Tromsøflaket
$E[H_s]$(m), Eq. (13)	2.11	2.30	2.34
$E[s_m\|E[H_s]]$, Eq. (14)	0.0464	0.0450	0.0454
R, Eq. (15)	0.313	0.316	0.315
Mean value plus and minus one standard deviation	(0.0319, 0.0609)	(0.0308, 0.0592)	(0.0311, 0.0597)

differences between the results obtained for the three locations; the values of $E[H_s]$ are in the range 2.11—2.34 m, and the mean value of $E[s_m|E[H_s]]$ is about 0.045 with the standard deviation to mean value ratio of about 0.32. Overall, one should note that this value of $E[s_m|E[H_s]]$ is lower than the spectral wave steepness values in the range 0.048—0.082, as given in Myrhaug and Kjeldsen (1984), which are based on data representing partly the same area and period as those from Mathisen and Bitner-Gregersen (1990). However, these latter values represent sea states which are selected based on that they contain at least one steep and high wave (see Myrhaug and Kjeldsen (1984) for more details).

4. Conclusions

Some average probabilistic features of the wave steepness and the spectral wave steepness in terms of the mean values and the standard deviations are presented. The results are exemplified by using distributions representing deep water waves based on data from the Norwegian continental shelf.

Estimates of the wave steepness are obtained by using the Myrhaug and Fouques (2007) distribution together with a Phillips spectrum and a family of JONSWAP spectra for wind sea, and for combined wind sea and swell the sea states are described by a joint frequency table of H_s and T_z. For wind sea the mean values of the wave steepness are in the range 0.031—0.046; for combined wind sea and swell the mean values are in the wider range 0.0054—0.125. For both wind sea as well as wind sea and swell the standard deviation to mean value ratio is 0.56.

Estimates of the spectral wave steepness are obtained by using a joint distribution of significant wave height and spectral wave steepness. The results are exemplified for joint frequency tables of H_s and T_z from Utsira, Halten and Tromsøflaket on the Norwegian continental shelf. There are small differences between the results for the three locations; the mean value of the spectral wave steepness for given values of the mean significant wave height in the range 2.11—2.34 m is about 0.045, with the standard deviation to mean value ratio of about 0.32.

Appendix A. A family of JONSWAP spectra for wind sea

Here a brief summary of a family of JONSWAP spectra for wind sea as given in Myrhaug and Kjeldsen (1987) is provided. It should be noted that the spectrum in Myrhaug and Kjeldsen (1987) is given in terms of the frequency $f = \omega/2\pi$ in $Hz = s^{-1}$, and thus $S(\omega) = S(f)/2\pi$, giving the JONSWAP spectrum

$$S(\omega)=\alpha\frac{g^2}{\omega^5}\exp\left[-\frac{5}{4}\left(\frac{\omega_p}{\omega}\right)^4\right]\gamma^{\exp\left[-\frac{1}{2}\left(\frac{\omega-\omega_p}{\sigma_b\omega_p}\right)^2\right]}, \quad \text{(A1)}$$

where $\sigma_b = 0.08$ is the width of the spectral peak, α is an equilibrium range parameter determining the high frequency part of the spectrum, and γ is the spectral peakedness parameter. By considering wind sea this spectral formulation is valid in a subspace of the whole H_s, T_p (or H_s, T_z) space, i.e.

$$3.6\sqrt{H_s}\le T_p\le 5\sqrt{H_s}, \quad \text{(A2)}$$

where H_s and T_p are in meters and seconds, respectively; α is taken as

$$\alpha = 0.036-0.0056\frac{T_p}{\sqrt{H_s}} \quad \text{(A3)}$$

and γ is given by

$$\gamma=\exp\left[3.484\left(1-0.1975\alpha\frac{T_p^4}{H_s^2}\right)\right]. \quad \text{(A4)}$$

Then, for given values of H_s and T_p according to Eq. (A2), the corresponding values of α and γ can be determined and accordingly the wave spectrum in Eq. (A1). Furthermore, for a JONSWAP spectrum the ratio T_p/T_z depends on γ (see Fig. 11 in Myrhaug and Kjeldsen (1987)). In this article $\gamma = 1$, 3, 5 are considered, where $\gamma = 1$ corresponds to the Pierson-Moskowitz spectrum. Then, for:

$$\gamma=1;\ \ \alpha=0.0081,\ \ T_p=5\sqrt{H_s},\ \ T_p=1.40T_z \quad \text{(A5)}$$

$$\gamma=3;\ \ \alpha=0.0136,\ \ T_p=4\sqrt{H_s},\ \ T_p=1.29T_z \quad \text{(A6)}$$

$$\gamma=5,\ \ \alpha=0.016,\ \ T_p=3.6\sqrt{H_s},\ \ T_p=1.24T_z \quad \text{(A7)}$$

More details about this JONSWAP formulation are given in Myrhaug and Kjeldsen (1987).

Appendix B. Mathisen and Bitner-Gregersen (1990) joint *pdf* of H_s and T_z

The joint *pdf* of H_s and T_z used by Mathisen and Bitner-Gregersen (1990) is given as

$$p(H_s,T_z)=p(T_z|H_s)p(H_s), \quad \text{(B1)}$$

where $p(H_s)$ is the marginal *pdf* of H_s given by the following three-parameter Weibull *pdf*

$$p(H_s)=\frac{\theta_h}{\zeta_h}\left(\frac{H_s-\varepsilon_h}{\zeta_h}\right)^{\theta_h-1}\exp\left[-\left(\frac{H_s-\varepsilon_h}{\zeta_h}\right)^{\theta_h}\right];\quad H_s\ge\varepsilon_h, \quad \text{(B2)}$$

where θ_h, ζ_h, ε_h are the Weibull parameters. $p(T_z|H_s)$ is the conditional *pdf* of T_z given H_s, given by the following lognormal *pdf*

$$p(T_z|H_s)=\frac{1}{\sqrt{2\pi}\sigma_tT_z}\left[-\frac{1}{2}\left(\frac{\ln T_z-\mu_t}{\sigma_t}\right)^2\right], \quad \text{(B3)}$$

where μ_t and σ_t are the mean value and the standard deviation, respectively, of $\ln T_z$, given by

$$\mu_t=a_1+a_2H_s^{a_3}, \quad \text{(B4)}$$

$$\sigma_t=b_1+b_2e^{b_3H_s}. \quad \text{(B5)}$$

Here the same values of the parameters as used by Mathisen and Bitner-Gregersen (1990) are adopted; see Table B1. These data represent wave conditions at three deep water locations on the Norwegian continental shelf. The data were obtained by Waverider buoys located at Utsira

Table B1 Parameters in the Mathisen and Bitner-Gregersen (1990) joint *pdf* of H_s and T_z.

Location	Utsira	Halten	Tromsøflaket
Weibull parameters for H_s, Eq. (B2)			
ρ_h (m)	1.50	1.91	1.41
θ_h	1.15	1.27	1.12
ε_h (m)	0.679	0.532	0.987
Parameters for mean of $\ln T_z$, Eq. (B4)			
a_1	0.933	1.09	1.24
a_2	0.578	0.479	0.337
a_3	0.395	0.417	0.538
Parameters for standard deviation of $\ln T_z$, Eq. (B5)			
b_1	0.0550	0.0407	0.0728
b_2	0.336	0.221	0.383
b_3	−0.585	−0.289	−0.665

(1974—1986), Halten (1974—1984), Tromsøflaket (1977—1983) covering the years given in the parenthesis; see Mathisen and Bitner-Gregersen (1990) for more details. As mentioned in Section 3, these data and the data upon which the wave steepness distribution is based, are from the same locations and represent partly the same periods.

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