

THE CONTINUUM LIMIT OF FOLLOW-THE-LEADER MODELS — A SHORT PROOF

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We dedicate this paper to the memory of Hans Petter Langtangen (1962–2016)

ABSTRACT. We offer a simple and self-contained proof that the Follow-the-Leader model converges to the Lighthill–Whitham–Richards model for traffic flow.

1. INTRODUCTION

The problem of convergence of particle models to continuum models is fundamental. We here study it in the context of traffic flow. In this case there are two fundamentally different models: The first one is based on individual vehicles whose dynamics is determined by the behavior of the vehicle immediately in front of it. This gives the Follow-the-Leader (FtL) model, which constitutes a system of ordinary differential equations describing the dynamics of individual vehicles. The other model is based on the assumption of heavy traffic where the individual vehicles are represented by a density. Assuming that the number of vehicles is conserved, we get the classical Lighthill–Whitham–Richards (LWR) model [11, 12], which is nothing but a scalar hyperbolic conservation law. The question that we address in this paper is in what sense the FtL model approaches or approximates the LWR model in the case of dense traffic.

The principal assumption in FtL models is that the velocity V of any given vehicle is a function of the distance to the vehicle in front of it. We shall write this function as

$$V\left(\frac{\Delta Z}{\ell}\right),$$

where ΔZ denotes the distance to nearest vehicle in front, and ℓ the length of each vehicle. For obvious reasons, $\Delta Z \geq \ell$. It is commonly assumed that V is an increasing positive function defined in $[1, \infty)$, such that $\lim_{y \rightarrow \infty} V(y) = v_{\max} < \infty$. Consider N vehicles with length ℓ and position $Z_1(t) < \dots < Z_N(t)$ on the real axis with dynamics given by

$$(1.1) \quad \frac{d}{dt} Z_i = V\left(\frac{Z_{i+1} - Z_i}{\ell}\right) \quad \text{for } i = 1, \dots, N - 1.$$

To close this system, we must prescribe the velocity of the first vehicle at Z_N . It is natural to model this by letting $\dot{Z}_N = v_{\max}$.

Date: September 29, 2017.

2010 Mathematics Subject Classification. Primary: 35L02; Secondary: 35Q35, 82B21.

Key words and phrases. Follow-the-Leader model, Lighthill–Whitham–Richards model, traffic flow, continuum limit.

Research was supported by the grant *Waves and Nonlinear Phenomena (WaNP)* from the Research Council of Norway. The research was done while the authors were at Institut Mittag-Leffler, Stockholm.

In this paper we analyze the limit of this system of ordinary differential equations when $N \rightarrow \infty$ and $\ell \rightarrow 0$. We show that

$$\frac{\ell}{Z_{i+1}(t) - Z_i(t)} \rightarrow \rho(t, z),$$

where intuitively $Z_{i+1}, Z_i \rightarrow z$, and where ρ is an entropy solution to the scalar conservation law

$$(1.2) \quad \rho_t + f(\rho)_z = 0, \quad f(\rho) = \rho V\left(\frac{1}{\rho}\right).$$

This problem has also been addressed by several other researchers. We here mention [1, 2, 4, 5, 7, 8]. The long and technically demanding paper [6] shows this convergence, while in [3, 13], the convergence of the discrete system is assumed rather than proved. The approach here resembles [10] where FtL models are viewed as a numerical approximation of the LWR model, and the proof of convergence depends on classical results by Crandall–Majda and Wagner for a grid approximation.

Here we offer a simple and straightforward proof of the continuum limit.

Solutions to scalar conservation laws are in general not continuous, and (1.2) must be considered in the weak sense; furthermore weak solutions to the Cauchy problem are not unique, and in order for the Cauchy problem to have a unique solution, one must impose the Kruřkov entropy condition [9]: A function $\rho \in C([0, \infty); L^1(\mathbb{R}))$ is called an *entropy solution* to the Cauchy problem for (1.2) if for all constants $k \in \mathbb{R}$ and all non-negative test functions $\varphi \in C_0^1([0, \infty) \times \mathbb{R})$, one has

$$(1.3) \quad \int_0^\infty \int_{\mathbb{R}} (|\rho - k| \varphi_t + \text{sign}(\rho - k)(f(\rho) - f(k))\varphi_z) dz dt + \int_{\mathbb{R}} |\rho(0, z) - k| \varphi(0, z) dz \geq 0.$$

More precisely, we show the following result. Assume that the velocity function satisfies the reasonable assumptions (2.1), and the initial data $\rho(0, \cdot) \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$. Let $\rho_\ell(t, z)$ be the density of vehicles as defined by the FtL model, see (2.14). Then we show that $\lim_{\ell \rightarrow 0} \rho_\ell = \rho \in C([0, \infty); L^1(\mathbb{R}))$, where ρ is the unique solution to (1.2) satisfying the entropy condition (1.3) such that $\rho(0, z) = \rho_0(z)$.

The rest of this note is organized as follows: In Section 2 we define the discrete model and prove some simple bounds on its solutions, and in Section 3 we give the elementary proof of convergence.

2. THE MODEL

We use units such that $v_{\max} = 1$. Let $v(\rho)$ be a continuously differentiable function $v: [0, 1] \rightarrow [0, 1]$, such that $v' \leq 0$, $v(0) = 1$ and $v(1) = 0$. We use the notation $V(y) = v(1/y)$, assume that

$$(2.1a) \quad V(y) \geq 1 - \frac{1}{y^{\sigma-1}}, \quad \text{for some constant } \sigma > 1,$$

$$(2.1b) \quad y^2 V'(y) \leq M, \quad \text{for } y \geq 1, \text{ and for some constant } M.$$

Define the forward difference

$$D_+ h_i = \frac{1}{\ell} (h_{i+1} - h_i).$$

Let $\{y_i(t)\}_{i=1}^{N-1}$ satisfy

$$(2.2) \quad \dot{y}_i = D_+ V_i, \quad i = 1, \dots, N-1, \quad t > 0,$$

where $V_i = V(y_i)$ and $V_N = 1$. Later we will also need $y_N = \infty$. Regarding the initial values, we assume that there is a function $\rho_0: \mathbb{R} \rightarrow [0, 1]$ normalized such that $\int_{\mathbb{R}} \rho_0 dz = 1$. Define $\{z_{i+1/2}(0)\}_{i=0}^{N-1}$ inductively as

$$(2.3) \quad \int_{z_{i-1/2}(0)}^{z_{i+1/2}(0)} \rho_0(z) dz = \frac{1}{N+1} = \ell, \quad i = 0, \dots, N-1.$$

Thus with the current scaling, the length ℓ of each vehicle is $\ell = 1/(N+1)$. We will also need $z_{-1/2}(0) = -\infty$. Here we choose the infimum of possible values for $z_{i+1/2}(0)$ satisfying (2.3). Set

$$(2.4) \quad y_i(0) = \frac{1}{\ell} (z_{i+1/2}(0) - z_{i-1/2}(0)), \quad i = 1, \dots, N-1.$$

Observe that it follows from (2.3) that $y_i(0) \geq 1$ for $i = 1, \dots, N-1$, since $\rho_0 \in [0, 1]$.

Lemma 2.1. *Assume that V satisfies (2.1a) and that $\{y_i\}_{i=1}^{N-1}$ solves the system (2.2) with initial values (2.4). Then*

$$(2.5) \quad 1 \leq y_i(t) \leq \left(y_i(0)^\sigma + \frac{\sigma t}{\ell} \right)^{1/\sigma}, \quad i = 1, \dots, N-1.$$

In particular,

$$(2.6) \quad \lim_{\ell \rightarrow 0} (\ell^\kappa y_i(t)) = 0, \quad t \in (0, \infty), \quad \kappa > 1/\sigma, \quad i = 1, \dots, N-1.$$

Proof. If $y_i(t) = 1$, then $V(y_i(t)) = 0$ and hence $\dot{y}_i(t) \geq 0$. This gives the lower bound on y_i .

Using (2.1a) and the bound $V_{i+1} \leq 1$, we get

$$\dot{y}_i = \frac{1}{\ell} (V_{i+1} - V_i) \leq \frac{1}{\ell y_i^{\sigma-1}}.$$

By integrating this inequality, we see that the estimate (2.5) holds, and the limit (2.6) then follows trivially. \square

Lemma 2.2. *Define $\rho_i(t) = 1/y_i(t)$. Write $V_i(t) = V(y_i(t))$. We have that*

$$(2.7) \quad \sum_{i=1}^{N-1} |V_{i+1}(t) - V_i(t)| \leq \sum_{i=1}^{N-1} |V_{i+1}(0) - V_i(0)|$$

and

$$(2.8) \quad \sum_{i=1}^{N-1} |\rho_{i+1}(t) - \rho_i(t)| \leq \sum_{i=1}^{N-1} |\rho_{i+1}(0) - \rho_i(0)|.$$

Proof. We find that

$$\begin{aligned} \frac{d}{dt} |V_{i+1} - V_i| &= \text{sign}(V_{i+1} - V_i) V'(y_{i+1}) D_+ V_{i+1} - V'(y_i) |D_+ V_i| \\ &\leq V'(y_{i+1}) |D_+ V_{i+1}| - V'(y_i) |D_+ V_i|, \end{aligned}$$

since $V' \geq 0$. For $i = N$ we recall the conventions that $y_N = \infty$ and $V_N = 1$, and thus $V'(y_N) = 0$. We have that $0 \leq V_i \leq 1$. Then

$$\begin{aligned} \frac{d}{dt} |V_{i+1} - V_i| &= \text{sign}(V_{i+1} - V_i) V'(y_{i+1}) D_+ V_{i+1} - V'(y_i) |D_+ V_i| \\ &\leq V'(y_{i+1}) |D_+ V_{i+1}| - V'(y_i) |D_+ V_i|, \end{aligned}$$

since $V' \geq 0$. This means that

$$(2.9) \quad \frac{d}{dt} \sum_{i=1}^{N-1} |V_{i+1} - V_i| \leq \frac{1}{\ell} (V'(y_N) |1 - 1| - V'(y_1) |V_2 - V_1|)$$

$$(2.10) \quad = -\frac{1}{\ell} V'(y_1) |V_2 - V_1| \leq 0,$$

which shows (2.7). Since $\text{sign}(V_{i+1} - V_i) = -\text{sign}(\rho_{i+1} - \rho_i)$, we could also carry out these estimates for ρ_i , proving (2.8). \square

Note that

$$\sum_i |\rho_{i+1}(0) - \rho_i(0)| \leq |\rho_0|_{BV},$$

where $|\cdot|_{BV}$ denotes the bounded variation norm. For $t > 0$, define $z_{i+1/2}(t)$ by

$$(2.11) \quad \dot{z}_{i-1/2} = V_i, \quad i = 1, \dots, N,$$

with initial values given by (2.3). Combining (2.2), (2.4), and (2.11) we conclude that

$$(2.12) \quad y_i(t) = \frac{1}{\ell} (z_{i+1/2}(t) - z_{i-1/2}(t)), \quad t \in [0, \infty), i = 1, \dots, N-1.$$

In particular,

$$(2.13) \quad (z_{i+1/2}(t) - z_{i-1/2}(t)) \rho_i(t) = \ell, \quad i = 1, \dots, N-1.$$

Note that $z_{i-1/2}$ coincides with the position of the i th vehicle from the left, given by (1.1). Thus $Z_i = z_{i+1/2}$.

Furthermore, define the functions

$$(2.14) \quad \begin{aligned} \rho_\ell(t, z) &= \sum_{i=1}^{N-1} \rho_i(t) \chi_{[z_{i-1/2}(t), z_{i+1/2}(t)]}(z), \\ V_\ell(t, z) &= \sum_{i=1}^{N-1} V_i(t) \chi_{[z_{i-1/2}(t), z_{i+1/2}(t)]}(z), \end{aligned}$$

where χ_I denotes the characteristic function of an interval I . Observe that Lemma 2.2 implies that

$$(2.15) \quad |\rho_\ell(t)|_{BV} \leq |\rho_\ell(0)|_{BV}, \quad |V_\ell(t)|_{BV} \leq |V_\ell(0)|_{BV}.$$

3. THE CONTINUUM LIMIT

Theorem 3.1. *Assume that the function V satisfies (2.1), and that $\rho_0 \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$. Let ρ_ℓ be as defined above. Then $\lim_{\ell \rightarrow 0} \rho_\ell = \rho \in C([0, \infty); L^1(\mathbb{R}))$, where ρ is the unique entropy solution to (1.2) such that $\rho(0, z) = \rho_0(z)$.*

Proof. Observe that ρ_ℓ is in $L^1(\mathbb{R})$, since it is positive and

$$\frac{d}{dt} \int_{\mathbb{R}} \rho_\ell(t, z) dz = \sum_{i=1}^{N-1} \frac{d}{dt} \int_{z_{i-1/2}}^{z_{i+1/2}} \rho_i dz = \sum_{i=1}^{N-1} \frac{d}{dt} ((z_{i+1/2} - z_{i-1/2}) \rho_i) = 0.$$

Hence $\|\rho_\ell(t)\|_{L^1} \leq 1$. For any $\{h_i\}$ define

$$D_+^z h_i = \frac{h_{i+1} - h_i}{z_{i+1/2} - z_{i-1/2}} = \rho_i D_+ h_i.$$

Let $\varphi = \varphi(t, z)$ be a smooth test function with compact support in $\mathbb{R} \times (0, \infty)$. We calculate

$$\begin{aligned} \int_0^\infty \int_{\mathbb{R}} \rho_\ell \varphi_t dz dt &= \int_0^\infty \sum_i \rho_i \int_{z_{i-1/2}}^{z_{i+1/2}} \varphi_t dz dt \\ &= \int_0^\infty \sum_i \left[\rho_i \frac{\partial}{\partial t} \left(\int_{z_{i-1/2}}^{z_{i+1/2}} \varphi dz \right) - \rho_i \ell D_+ (\dot{z}_{i-1/2} \varphi_{i-1/2}) \right] dt \\ &= - \int_0^\infty \sum_i \left[\int_{z_{i-1/2}}^{z_{i+1/2}} \dot{\rho}_i \varphi dz - \ell D_+ (\rho_i) V_{i+1} \varphi_{i+1/2} \right] dt \end{aligned}$$

$$\begin{aligned}
&= \int_0^\infty \sum_i \int_{z_{i-1/2}}^{z_{i+1/2}} \left(\rho_i^2 D_+(V_i) \varphi + D_+^z(\rho_i) V_{i+1} \varphi_{i+1/2} \right) dz dt \\
(3.1) \quad &= \int_0^\infty \sum_i \int_{z_{i-1/2}}^{z_{i+1/2}} \left(\rho_i D_+^z(V_i) \varphi + D_+^z(\rho_i) V_{i+1} \varphi_{i+1/2} \right) dz dt,
\end{aligned}$$

where we have used that $\dot{\rho}_i = -\rho_i^2 D_+ V_i$, and introduced the notation $\varphi_{i+1/2} = \varphi(t, z_{i+1/2})$. Similarly,

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{R}} \rho_\ell V_\ell \varphi_z dz dt &= \int_0^\infty \sum_i \rho_i V_i \int_{z_{i-1/2}}^{z_{i+1/2}} \varphi_z dz dt \\
&= - \int_0^\infty \sum_i \ell D_+(\rho_i V_i) \varphi_{i+1/2} dt \\
(3.2) \quad &= - \int_0^\infty \sum_i \int_{z_{i-1/2}}^{z_{i+1/2}} \left(\rho_i D_+^z(V_i) + D_+^z(\rho_i) V_{i+1} \right) \varphi_{i+1/2} dz dt.
\end{aligned}$$

Using (3.1) with $\varphi(t, z) = \chi_{[t_1, t_2]}(t) \psi(z)$ for $0 < t_1 < t_2 < \infty$ and a smooth test function ψ with $|\psi| \leq 1$, we formally get with $\psi_{i+1/2} = \psi(z_{i+1/2})$ that

$$\begin{aligned}
&\left| \int_{\mathbb{R}} (\rho_\ell(t_2, z) - \rho_\ell(t_1, z)) \psi(z) dz \right| \\
&= \left| \int_{t_1}^{t_2} \sum_i \int_{z_{i-1/2}}^{z_{i+1/2}} \left(\rho_i D_+^z(V_i) \psi + D_+^z(\rho_i) V_{i+1} \psi_{i+1/2} \right) dz dt \right| \\
&\leq \int_{t_1}^{t_2} \sum_i \int_{z_{i-1/2}}^{z_{i+1/2}} \left(|\rho_i| |D_+^z(V_i)| |\psi| + |D_+^z(\rho_i)| |V_{i+1}| |\psi_{i+1/2}| \right) dz dt \\
&\leq \ell \int_{t_1}^{t_2} \sum_i \left(|\rho_i| |D_+(V_i)| + |D_+(\rho_i)| |V_{i+1}| \right) dt \\
&\leq \ell \int_{t_1}^{t_2} \sum_i \left(|D_+(V_i)| + |D_+(\rho_i)| \right) dt \\
&\leq (t_2 - t_1) \sum_i \left(|V_{i+1}(0) - V_i(0)| + |\rho_{i+1}(0) - \rho_i(0)| \right) \\
&\leq (t_2 - t_1) \left(|V_\ell(0)|_{BV} + |\rho_\ell(0)|_{BV} \right),
\end{aligned}$$

using first that $|\psi|, |\rho_i|, |V_i| \leq 1$ and subsequently Lemma 2.2. By approximating the characteristic function $\chi_{[t_1, t_2]}$ with a smooth function, and taking the limit, we still obtain the above estimate. This implies

$$\begin{aligned}
\|\rho_\ell(t_2) - \rho_\ell(t_1)\|_{L^1} &= \sup_{|\psi| \leq 1} \int (\rho_\ell(t_2, z) - \rho_\ell(t_1, z)) \psi(z) dz \\
&\leq (t_2 - t_1) \left(|\rho_\ell(0)|_{BV} + |V_\ell(0)|_{BV} \right).
\end{aligned}$$

Thus, recalling (2.15), we can apply [9, Theorem A.11] to conclude that the set $\{\rho_\ell\}_{\ell > 0}$ is compact in $C([0, \infty); L^1(\mathbb{R}))$, and there exists a sequence $\{\ell_j\}_{j=1}^\infty$, $\ell_j \rightarrow 0$ as $j \rightarrow \infty$, and a function ρ such that

$$\rho_{\ell_j} \rightarrow \rho \quad \text{in } C([0, \infty); L^1(\mathbb{R})), \text{ as } j \rightarrow \infty.$$

To simplify the notation, we henceforth write $\ell = \ell_j$. Furthermore, since $v(\rho_\ell) = V_\ell$, $V_\ell \rightarrow v(\rho)$. Adding (3.1) and (3.2), we get

$$\left| \int_0^\infty \int_{\mathbb{R}} (\rho_\ell \varphi_t + \rho_\ell V_\ell \varphi_z) dz dt \right| = \left| \int_0^\infty \sum_i \rho_i (V_{i+1} - V_i) \right|$$

$$\begin{aligned}
& \times \int_{z_{i-1/2}}^{z_{i+1/2}} (\varphi(t, z) - \varphi(t, z_{i+1/2})) dz dt \Big| \\
& \leq \frac{1}{2} \int_0^\infty \sup_i (\ell y_i(t))^2 \|\varphi_z(t, \cdot)\|_{L^\infty} \sum_i |V_{i+1} - V_i| dt \\
& \rightarrow 0, \text{ as } \ell \text{ to } 0,
\end{aligned}$$

and thus ρ is a weak solution. To show that ρ is an entropy solution, let η be a twice differentiable convex function. Since $V' \geq 0$, we get

$$\begin{aligned}
\frac{d}{dt} \eta(y_i) &= \eta'(y_i) D_+ V_i = \frac{1}{\ell} \eta'(y_i) \int_{y_i}^{y_{i+1}} V'(y) dy \\
&\leq \frac{1}{\ell} \int_{y_i}^{y_{i+1}} \eta'(y) V'(y) dy = \frac{1}{\ell} \int_{y_i}^{y_{i+1}} Q'(y) dy = D_+ Q_i,
\end{aligned}$$

where $Q' = \eta' V'$ and $Q_i = Q(y_i)$. Introduce $q(\rho) = Q(1/\rho)$ with $q_i = q(\rho_i)$. Define $\mu = \mu(\rho)$ by $\mu(\rho) = \rho \eta(1/\rho)$. As usual we write $\mu_i = \mu(\rho_i)$ and $\eta_i = \eta(y_i)$. Then μ is a convex function of ρ , and if μ is a convex function of ρ , then η is a convex function of y . We have that

$$\frac{d}{dt} \mu(\rho_i) = -\rho_i^2 (D_+ V_i) \eta_i + \rho_i \frac{d}{dt} \eta_i.$$

Set $\mu_\ell(t, z) = \sum_i \mu_i(t) \chi_{[z_{i-1/2}(t), z_{i+1/2}(t))}(z)$ with $\mu_i(t) = \mu(\rho_i(t))$, and define $q_\ell(t, z)$ similarly. As when establishing (3.1), we find for a non-negative test function φ with support in $\mathbb{R} \times (0, \infty)$ that

$$\begin{aligned}
& \int_0^\infty \int_{\mathbb{R}} \mu_\ell \varphi_t dz dt \\
&= \int_0^\infty \sum_i \mu_i \int_{z_{i-1/2}}^{z_{i+1/2}} \varphi_t dz dt \\
&= \int_0^\infty \sum_i \left[\mu_i \frac{\partial}{\partial t} \left(\int_{z_{i-1/2}}^{z_{i+1/2}} \varphi dz \right) - \mu_i \ell D_+ (z_{i-1/2} \varphi_{i-1/2}) \right] dt \\
&= - \int_0^\infty \sum_i \left[\int_{z_{i-1/2}}^{z_{i+1/2}} \dot{\mu}_i \varphi dz - \ell D_+ (\mu_i) V_{i+1} \varphi_{i+1/2} \right] dt \\
&\geq \int_0^\infty \sum_i \int_{z_{i-1/2}}^{z_{i+1/2}} \left[(\eta_i \rho_i^2 D_+ (V_i) - \rho_i D_+ (q_i)) \varphi + D_+^z (\mu_i) V_{i+1} \varphi_{i+1/2} \right] dz dt \\
&= \int_0^\infty \sum_i \int_{z_{i-1/2}}^{z_{i+1/2}} \left[(\mu_i D_+^z (V_i) - D_+^z (q_i)) \varphi + D_+^z (\mu_i) V_{i+1} \varphi_{i+1/2} \right] dz dt.
\end{aligned}$$

Similarly

$$\begin{aligned}
& \int_0^\infty \int_{\mathbb{R}} (V_\ell \mu_\ell - q_\ell) \varphi_z dz dt \\
&= \int_0^\infty \sum_i (V_i \mu_i - q_i) \int_{z_{i-1/2}}^{z_{i+1/2}} \varphi_z dz dt \\
&= - \int_0^\infty \sum_i \ell [D_+ (\mu_i V_i) - D_+ (q_i)] \varphi_{i+1/2} dt \\
&= - \int_0^\infty \sum_i \int_{z_{i-1/2}}^{z_{i+1/2}} [\mu_i D_+^z (V_i) + V_{i+1} D_+^z (\mu_i) - D_+^z (q_i)] \varphi_{i+1/2} dz dt.
\end{aligned}$$

Therefore,

$$\int_0^\infty \int_{\mathbb{R}} (\mu_\ell \varphi_t + (\mu_\ell V_\ell - q_\ell) \varphi_z) dz dt \geq r_\ell,$$

where

$$\begin{aligned}
|r_\ell| &= \left| \int_0^T \sum_i \int_{z_{i-1/2}}^{z_{i+1/2}} (\mu_i D_+^z(V_i) - D_+^z(q_i)) (\varphi - \varphi_{i+1/2}) dz dt \right| \\
&\leq \int_0^\infty \sum_i (|\mu_i| |V_{i+1} - V_i| + |q_{i+1} - q_i|) \int_{z_{i-1/2}}^{z_{i+1/2}} |\varphi(t, z) - \varphi(t, z_{i+1/2})| dz dt \\
&\leq C \int_0^\infty \sup_i (\ell y_i(t))^2 \|\varphi_z(t, \cdot)\|_{L^\infty} \sum_i (|V_{i+1} - V_i| + |q_{i+1} - q_i|) dt.
\end{aligned}$$

If $q_\ell(t, \cdot)$ is of bounded variation, then $r_\ell \rightarrow 0$. We now assume that μ is (a smooth approximation to) the Kruřkov entropy $\mu(\rho) = |\rho - k|$. A short computation yields that

$$\mu(\rho)V\left(\frac{1}{\rho}\right) - q(\rho) = \text{sign}(\rho - k) \left(\rho V\left(\frac{1}{\rho}\right) - kV\left(\frac{1}{k}\right) \right),$$

which is consistent with (1.3). Then $\eta(y) = y|1/y - k|$, and $|\eta'(y)| \leq |k|$. If V satisfies (2.1b), the mapping $\rho \mapsto q(\rho)$ is Lipschitz, since

$$|q'(\rho)| = \left| \frac{d}{d\rho} Q\left(\frac{1}{\rho}\right) \right| = \left| \eta'\left(\frac{1}{\rho}\right) V'\left(\frac{1}{\rho}\right) \frac{1}{\rho^2} \right| \leq M|k|.$$

Hence $q_\ell(t, \cdot)$ is of bounded variation, since ρ_ℓ is in BV .

A similar argument with a test function whose support include the initial data on $t = 0$, will show (1.3). We conclude that ρ is an entropy solution. Since the entropy solution is unique, we also conclude that the whole sequence, rather than just a subsequence, converges. \square

REFERENCES

- [1] B. Argall, E. Cheleshkin, J. M. Greenberg, C. Hinde, and P.-J. Lin. A rigorous treatment of a follow-the-leader traffic model with traffic lights present. *SIAM J. Appl. Math.* 63(19): 149–168, 2002.
- [2] A. Aw, A. Klar, T. Materne, and M. Rascle. Derivation of continuum traffic flow models from microscopic follow-the-leader models. *SIAM J. Appl. Math.*, 63(1): 259–278, 2002.
- [3] R. M. Colombo and E. Rossi. On the micro-macro limit in traffic flow. *Rend. Sem. Math. Univ. Padova* 131:217–235, 2014.
- [4] E. Cristiani and S. Sahu. On the micro-to-macro limit for first-order traffic flow models on networks. *Networks and Heterogeneous Media* 11(3):395–413, 2016. doi:10.3934/nhm.2016002.
- [5] M. Di Francesco, S. Fagioli, and M. D. Rosini. Deterministic particle approximation of scalar conservation laws. Preprint, [arXiv:1605.05883v1](https://arxiv.org/abs/1605.05883v1), 2016.
- [6] M. Di Francesco and M. D. Rosini. Rigorous derivation of nonlinear scalar conservation laws from follow-the-leader type models via many particle limit. *Arch. Ration. Mech. Anal.*, 217(3):831–871, 2015.
- [7] P. Goatin and F. Rossi. A traffic flow model with non-smooth metric interaction: well-posedness and micro-macro limit. *Commun. Math. Sci.* 15(1):261–287, 2017.
- [8] Ke Han, Tao Yaob, and T. L. Friesz. Lagrangian-based hydrodynamic model: Freeway traffic estimation. Preprint [arXiv:1211.4619v1](https://arxiv.org/abs/1211.4619v1), 2012.
- [9] H. Holden and N. H. Risebro. *Front Tracking for Hyperbolic Conservation Laws*. Springer-Verlag, New York, 2015, Second edition.
- [10] H. Holden and N. H. Risebro. Follow-the-leader models can be viewed as a numerical approximation to the Lighthill–Whitham–Richards model for traffic flow. Preprint, [arXiv:1702.01718](https://arxiv.org/abs/1702.01718), 2017.
- [11] M. J. Lighthill and G. B. Whitham. Kinematic waves. II. A theory of traffic flow on long crowded roads. *Proc. Roy. Soc. (London), Series A*, 229(1178):317–345, 1955.
- [12] P. I. Richards. Shockwaves on the highway. *Operations Research*, 4(1): 42–51, 1956.
- [13] E. Rossi. A justification of a LWR model based on a follow the leader description. *Discrete Cont. Dyn. Syst. Series S* 7(3): 579–591, 2014.

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