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Iterative Algorithms for Computing the Feedback Nash Equilibrium Point for Positive Systems

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N-player linear quadratic differential games on an infinite time horizon with deterministic feedback information structure have been studied. Two iterative methods (the Newton method and its acceleration modification) are introduced to compute the stabilizing solution of a set of generalized algebraic Riccati equations which is related to the Nash equilibrium point of the considered game model. Moreover, the sufficient conditions for convergence of the proposed methods are derived. Finally, we discuss two numerical examples so as to illustrate both algorithms.

Keywords: H_∞ optimal control problem, feedback Nash equilibrium, generalized Riccati equation, stabilizing solution, positive system.

1. Introduction

Recently, a linear quadratic (LQ) differential game theory based on Riccati equations, both for deterministic and stochastic systems, has been studied. It is an important research field in control theory and some interesting applications have been developed Azevedo-Perdicoulis & Jank (2005); Bazar & Olsder (1999); van den Broek (2001); van den Broek et al. (2003); Dragan et al. (2007); Jank & Kremer (2004); Li & Gajic (1995). We investigate the problem of finding a deterministic feedback Nash equilibrium for an N-player infinite-horizon linear-quadratic differential game. This equilibrium is defined as an N-tuple of linear time-invariant state feedback strategies stabilizing the closed-loop system. The issue has been investigated in van den Broek (2001); van den Broek et al. (2003).

Consider the dynamic system

$$\dot{x} = Ax + \sum_{j=1}^{N} B_j\, u_j\,, \quad x(0) = x_0 \tag{1}$$

where x is the state vector, $x_0 \in \mathbb{R}^{n\times 1}$, $A \in \mathbb{R}^{n\times n}$, $B_j \in \mathbb{R}^{n\times m_j}$ and u_j is the control vector, chosen by player $j, j = 1, \ldots, N$. The controls u_j are of the type $u_j = F_j x$ and $F_j \in \mathbb{R}^{m_j\times n}$. We define $\mathcal{F}$

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through the following set of matrices:

$$\mathcal{F} = \{\mathbf{F} = (F_1, \ldots, F_n) \text{ such that } F_j \in \mathbb{R}^{m_j \times n} \text{ and } A + \sum_{j=1}^{N} B_j\, F_j \text{ is asymptotically stable}\}\,.$$

The aim of each player i $(i = 1, \ldots, N)$ is to maximize the own cost function, which is a quadratic functional J_i defined by

$$J_i(F_1, \ldots, F_N, x_0) = \int_0^\infty x^T \left(Q_i + \sum_{j=1}^{N} F_j^T\, R_{ij}\, F_j \right) x\, dt\,, \tag{2}$$

where Q_i and R_{ij} are symmetric matrices with $Q_i \in \mathbb{R}^{n\times n}$ and $R_{ij} \in \mathbb{R}^{m_j \times m_j}$ and $i, j = 1, \ldots, N$. Additional requirements to these matrices are:
(a) the matrices Q_i and $R_{ij}, (i \neq j)$ are symmetric and nonnegative;
(b) the matrix R_{ii}^{-1} is nonpositive, $i = 1, \ldots, N$.
Further on, we want to apply the above N-player infinite-horizon linear-quadratic differential game to a positive system defined by (1). For this purpose, we introduce a definition, some facts and notations for nonnegative matrices and positive systems, , with the text that follows.

Definition 1: The system (1) is said to be positive if for all initial nonnegative x_0 and for non-negative controls $u_j, j = 1 \ldots, N$, the state trajectory $x(t)$ takes only nonnegative values.

There are many examples and applications for positive systems in economics (do Amaral et al. , 2006; Metzler , 1945), and financial modelling (Filipović et al. , 2010). The specific properties to reset the control of positive linear systems are established and problems of reset stabilization are commented in (Zhao et al. , 2015). A necessary and sufficient condition to guarantee the admissibility of the stability of positive descriptor systems via linear matrix inequalities (LMIs) is derived in (Zhang et al. , 2013).

An $n \times n$ matrix A is called a Z-matrix if it has nonpositive off-diagonal entries. Any Z-matrix A can be presented as $A = \alpha I - N$ with N being a nonnegative matrix, and it is called a nonsingular M-matrix if $\alpha > \rho(N)$, where $\rho(N)$ is the spectral radius of N. In addition, a matrix is called nonnegative (nonpositive) if all of its entries are nonnegative (nonpositive).

The next lemma gives equivalent statements about when a Z-matrix is an M-matrix.

Lemma 1.1: *(Berman & Plemmons , 1994). For a Z-matrix A, the following items are equivalent:*
(a) A is a nonsingular M-matrix;
(b) $A^{-1} \geq 0$;
(c) $A\,u > 0$ for some vector $u > 0$;
(d) All eigenvalues of A have positive real parts.

In our investigation we exploit the fact that the following statements are equivalent for a Z-matrix (-A):
(a) $-A$ is a nonsingular M-matrix;
(b) $I_n \otimes (-A^T) + (-A^T) \otimes I_n$ is a nonsingular M-matrix;
(c) A is asymptotically stable.

Lemma 1.2: *(Guo & Laub , 2000). Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be an M-matrix. If the elements of $B = (b_{ij}) \in \mathbb{R}^{n\times n}$ satisfy the relations $b_{ii} \geq a_{ii}\,,\ a_{ij} \leq b_{ij} \leq 0\,,\ \ i \neq j\,,\ i, j = 1, \ldots, N$ then B also is an M-matrix.*

Lemma 1.3: *Consider the linear matrix equation $-A^T X - X A = Q$ for a real symmetric matrix*

Q and a Z-matrix (-A). The following statements are equivalent:
(a) A is asymptotically stable;
(b) for any nonnegative Q the equation $-A^TX - XA = Q$ has a unique nonnegative solution X .

Proof. (a) Assume A is asymptotically stable and -A is a Z-matrix. Then $(-A^T) \otimes I_n + I_n \otimes (-A^T)$ is a nonsingular M-matrix and $vec(X) = [(-A^T) \otimes I_n + I_n \otimes (-A^T)]^{-1} vec(Q) > 0$ when Q is nonnegative. Thus (a) implies (b).

(b) Let X and Q be symmetric nonnegative matrices and X satisfies the equation $-A^TX - XA = Q$. Thus $[(-A^T) \otimes I_n + I_n \otimes (-A^T)] \, vec(X) = vec(Q) > 0$. By (1.1), $(-A^T) \otimes I_n + I_n \otimes (-A^T)$ is a nonsingular M-matrix and (a) holds. □

The following property of the positive systems are well known (see (Farina & Rinaldi , 2000)).

Proposition 1.4: *The system (1) is positive if and only if $B_j, j = 1, \dots, N$ are nonnegative matrices and the matrix $-A$ is a Z-matrix.*

The concept of a Nash equilibrium in games with feedback information structure has been introduced (van den Broek , 2001; van den Broek et al. , 2003). Following their findings we refer that the deterministic feedback Nash equilibria are characterized by the solutions of a set of coupled algebraic Riccati equations with a stability property. We consider and investigate the linear quadratic differential games for positive systems and more specially we attract how to compute the Nash equilibrium point of the game defined on a positive system. Further on, we refer two investigations where the stabilizing solution and its properties of a set of coupled algebraic Riccati equations are studied. A Newton iteration to approximate the stabilizing solution of a system of coupled generalized Riccati equations associated to deterministic feedback Nash equilibria in feedback information structure is proposed (Azevedo-Perdicoulis & Jank , 2005). The existence of a stabilizing solution of a system of coupled nonlinear matrix differential equations arising in connection with the computation of the deterministic feedback Nash equilibrium strategy in the case of linear quadratic two player Nash game is proved in (Dragan et al. , 2005).

Let introduce the following equilibrium definition:

Definition 2: An N-tuple of matrices $\mathbf{F}^* = (F_1^*, \dots, F_N^*)$ is called a deterministic feedback Nash equilibrium on the positive system (1) if the following inequalities hold:

$$J_i(F_1^*, \dots, F_N^*, x_0) \geq J_i(F_1^*, \dots, F_{i-1}^*, F_i, F_{i+1}^*, \dots, F_N^*, x_0) \,, \quad i = 1, \dots, N \,,$$

for all initial nonnegative states x_0, all $F_i \in \mathbb{R}^{m_i \times n}$ such that
$(F_1^*, \dots, F_{i-1}^*, F_i, F_{i+1}^*, \dots, F_N^*) \in \mathcal{F}$ and nonnegative strategies $u_i = F_i x, (F_i \geq 0), i = 1, \dots, N$.

Then, the deterministic feedback Nash equilibrium strategy is $u_i^* = F_i^* \, x(t)$ for the player i and $i = 1, \dots, N$. Moreover, the state $x(t)$ is a solution to the following equation:

$$\bar{x} = \left(A + \sum_{j=1}^{N} B_j \, F_j^* \right) x \,, \quad x(0) = x_0 \geq 0 \,, \quad x \in [0, \infty) \,.$$

Considering Definition 2, it might be deduced that every player wants to maximize its utility function $J_i(\mathbf{F}, x_0)$.

The following result for the existence of a deterministic feedback Nash equilibrium for an N-player linear quadratic differential game with feedback information structure described by a general system (1) was also obtained in (van den Broek , 2001).

Theorem 1.5: *(Theorem 5.3.2 (van den Broek , 2001)) Assume that the matrices Q_i and R_{ij} are symmetric and that $-R_{ii}$ is positive definite for $i, j = 1, \dots, N$. A deterministic feedback Nash equilibrium exists if and only if there exist N real symmetric $n \times n$ solutions X_i^* to the following set of equations (i=1,..., N):*

$$0 = \mathcal{R}_i(\mathbf{X}) := -A^T X_i - X_i A - Q_i + X_i S_i X_i + \textstyle\sum_{j \neq i} (X_i S_j X_j + X_j S_j X_i - X_j S_{ij} X_j)\,, \quad (3)$$

with the matrix $A - \sum_{j=1}^{N} S_j X_j^*$ is asymptotically stable. Moreover, the N-tuple of feedback matrices $(F_1^*, \dots, F_N^*)$ with $F_i^* = -R_{ii}^{-1} B_i^T X_i^*$ is a deterministic feedback Nash equilibrium and

$$J_i(F_1^*, \dots, F_N^*, x_0) = x_0^T X_i^* x_0\,, \qquad i = 1, \dots, N.$$

Here, the matrix coefficients S_i and S_{ij} are $S_i = B_i R_{ii}^{-1} B_i^T, i = 1, \dots, N; S_{ij} = B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j^T,\ i, j = 1, \dots, N,\ i \neq j$.

Based on conclusions derived in (Starr & Ho , 1969, Section 3) as well as on the definition for a positive system we find that the equilibrium strategy is the $u_i^* = F_i^* x$, with $F_i^* = -R_{ii}^{-1} B_i^T X_i$, where X_i is nonnegative solution described in the above theorem. Since F_i^* has to be nonnegative, i.e. R_{ii}^{-1} has to be nonpositive, i.e. $R_{ii}^{-1} \leq 0$. Moreover, based on the assumptions (a) and (b) we conclude that the matrix coefficient S_i is nonpositive for $i = 1, \dots, N$ and S_{ij} is nonnegative for $i, j = 1, \dots, N,\ i \neq j$. Our investigation is derived under the above properties of the matrices S_i and S_ij.

In order to find a deterministic feedback Nash equilibrium point one has to solve the set of Riccati equations (3). This system is equivalent to a system of $Nn(n+1)/2$ quadratic scalar equations in $Nn(n+1)/2$ real scalar unknowns. Hence, there exist at most $(Nn(n+1)/2)^2$ different solutions and the stability condition needs to be verified for each of them (van den Broek , 2001). The Newton method for the computation of a stabilizing solution to (3) in case $N = 2$ have been considered by Azevedo-Perdicoulis and Jank in (Azevedo-Perdicoulis & Jank , 2005). Here, we extend their approach to obtain the Newton method for in the general case $N > 2$.

In this paper we introduce two iterative methods for finding of the stabilizing solution to the set of Riccati equations (3). We are going to prove the convergence properties of the proposed iterations under new assumptions, which can be considered as sufficient conditions for the existence of the stabilizing solution to (3). The Newton method (the first one) is described in terms of the solution to a system of linear equations with high dimensional structure. The Lyapunov matrix equation has to be solved at each iterative step of the second method. Numerical examples have been introduced so as to demonstrate the effectiveness of the proposed algorithms.

In this paper we use the following notations: $\mathbb{R}^{n \times s}$ stands for $n \times s$ real matrices. The inequality $X \geq 0$ ($X > 0$) means that all elements of the matrix (or vector) X are real nonnegative (positive) and we call the matrix X nonnegative (positive). For the matrices $A = (a_{ij}), B = (b_{ij})$, we write $A \geq B (A > B)$ if $a_{ij} \geq b_{ij} (a_{ij} > b_{ij})$ hold for all indexes i and j. The notation $\mathbf{X} \geq \mathbf{Y}$ with $\mathbf{X} = (X_1, \dots, X_N)$ means that $X_i \geq Y_i\,, i = 1, \dots, N$. A matrix A is called asymptotically stable (or Hurwitz) if the eigenvalues of A have a negative real part. A symmetric matrix A is called positive definite (semidefinite) matrix if all eigenvalues are positive (nonnegative). The matrix Y is a stabilizing solution to (3) if the matrix $A - \sum_{j=1}^{N} S_j Y_j$ is asymptotically stable.

We use the fact that the matrix equation $AXB = C$ is equivalent to the linear system $(B^T \otimes A)\, vec(X) = vec(C)$, where the sign $\otimes$ denotes the Kronecker matrix product and the *vec* operator arranges the columns of a matrix into a column vector. A usual Gaussian elimination technique for solving this system requires $O(n^6)$ operations.

2. Iterative Methods

2.1. *Preliminary Statements*

We consider the matrix function $\mathcal{R}_i(\mathbf{X}), i = 1, \ldots, N$ defined as in (3).

Lemma 2.1: *For the matrix function $\mathcal{R}_i(\mathbf{X}), i = 1, \ldots, N$ the following identities hold:*

$$\text{(i)} \;\; \mathcal{R}_i(\mathbf{X}) \;=\; {A_{\mathbf{X}}}^T X_i - X_i\, A_{\mathbf{X}} - Q_i - X_i\, S_i\, X_i - \textstyle\sum_{j\neq i}^N X_j\, S_{ij}\, X_j\,, \tag{4}$$

with $A_{\mathbf{X}} = A - \sum_j S_j\, X_j$, and

$$\begin{aligned}\text{(ii)} \;\; \mathcal{R}_i(\mathbf{X}) \;=\; \mathcal{R}_i(\mathbf{Z}, \mathbf{X}) \;:=\; & -{A_{\mathbf{Z}}}^T X_i - X_i\, A_{\mathbf{Z}} - Q_i - Z_i\, S_i\, Z_i + (X_i - Z_i) S_i (X_i - Z_i) \\ & + \textstyle\sum_{j\neq i} \left[(X_j - Z_j)\, S_j\, X_i + X_i\, S_j (X_j - Z_j) \right] - \sum_{j\neq i} X_j\, S_{ij}\, X_j\,,\end{aligned} \tag{5}$$

where $Z_i = Z_i^T,\ i = 1, \ldots, N$.

Proof. The statements of Lemma 2.1 are verified by direct manipulations. □

We denote $\mathcal{R}_i(\mathbf{Z}, \mathbf{X})$ the presentation of $\mathcal{R}_i(\mathbf{X})$ through a symmetric matrix $\mathbf{Z}$.

In order to consider the Newton method for the set of Riccati equations $\mathcal{R}_i(\mathbf{X}) = 0,\ i = 1, \ldots, N$, we need the Fréchet derivative of $\mathcal{R}(\mathbf{X}) = \begin{pmatrix} \mathcal{R}_1(X_1, \ldots, X_N) \\ \vdots \\ \mathcal{R}_N(X_1, \ldots, X_N) \end{pmatrix}$. Following the results of Kantorovich & Akilov (1964) we find the Fréchet derivative (i=1,..., N):

$$\mathcal{R}'_{i,\mathbf{X}}(\mathbf{H}) \;=\; -{A_{\mathbf{X}}}^T H_i - H_i A_{\mathbf{X}} + \textstyle\sum_{j\neq i} (W_{ij,\mathbf{X}}\, H_j + H_j\, {W_{ij,\mathbf{X}}}^T)\,, \tag{6}$$

where $W_{ij,\mathbf{X}} = X_i S_j - X_j S_{ij}\ , i, j = 1, \ldots, N$.

2.2. *The Newton Method*

In this section, we derive a convergence result for the Newton method applied to a set of Riccati equations $\mathcal{R}_i(\mathbf{X}) = 0, i = 1, \ldots, N$.

The Newton method is given by the formula (Damm & Hinrichsen, 2001):

$$\mathbf{X}^{(k+1)} = \mathbf{X}^{(k)} - \left(\mathcal{R}'_{\mathbf{X}^{(k)}}\right)^{-1} (\, \mathcal{R}(\mathbf{X}^{(k)})\,)\,,$$

and it can be written down

$$\mathcal{R}'_{i,\mathbf{X}^{(k)}}(\mathbf{X}^{(k+1)}) = \mathcal{R}'_{i,\mathbf{X}^{(k)}}(\mathbf{X}^{(k)}) - \mathcal{R}_i(\mathbf{X}^{(k)})\,, \qquad i = 1, \ldots, N\,, \tag{7}$$

which is equivalent to the following set of recursive equations ($i = 1, \ldots, N$):

$$-{A^{(k)}}^T X_i^{(k+1)} - X_i^{(k+1)}\, A^{(k)} + \sum_{j\neq i} \left[W_{ij}^{(k)}\, X_j^{(k+1)} + X_j^{(k+1)}\, {W_{ij}^{(k)}}^T \right] = Q_i^{(k)}\,, \tag{8}$$

where

$$
\begin{array}{l}
A^{(k)} = A - \sum_j S_j X_j^{(k)}, \quad W_{ij}^{(k)} = X_i^{(k)} S_j - X_j^{(k)} S_{ij}, \quad i, j = 1, \ldots, N, \\
Q_i^{(k)} = Q_i + X_i^{(k)} S_i X_i^{(k)} + \sum_{j \neq i} [X_i^{(k)} S_j X_j^{(k)} + X_j^{(k)} S_j X_i^{(k)} - X_j^{(k)} S_{ij} X_j^{(k)}].
\end{array} \tag{9}
$$

The set of matrix equations (8) is equivalent to the linear system:

$$
L^{(k)} \; vec(X_1^{(k+1)}, \ldots, X_N^{(k+1)}) \; = \; vec(Q_1^{(k)}, \ldots, Q_N^{(k)}),
$$

where

$$
L^{(k)} = \left(L_{ij}^{(k)} \right)_{i,j=1}^{N} = \begin{cases} L_{ii}^{(k)} = -I_n \otimes A^{(k)^T} - A^{(k)^T} \otimes I_n \\ L_{ij}^{(k)} = -I_n \otimes W_{ij}^{(k)^T} - W_{ij}^{(k)^T} \otimes I_n, \quad i \neq j. \end{cases} \tag{10}
$$

We use the notation $\mathcal{N}_i^{(k)}(\mathbf{X}^{(k+1)})$ for the left-hand side of (8). Thus, iteration (8) has the form $\mathcal{N}_i^{(k)}(\mathbf{X}^{(k+1)}) = Q_i^{(k)}$.

Lemma 2.2: *Let $\{\mathbf{X}^{(s)}\}_{s=0}^{\infty}$ be a matrix sequence created by iteration (8). The following identities hold ($i = 1, \ldots, N$):*

$$
\text{(i)} \quad \mathcal{R}_i(\mathbf{X}^{(s)}) = \mathcal{N}_i^{(s)}(\mathbf{X}^{(s)} - \mathbf{X}^{(s+1)}), \tag{11}
$$

and

$$
\begin{array}{rcl}
\text{(ii)} \qquad \mathcal{N}_i^{(s)}(\mathbf{X}^{(s+1)} - \hat{\mathbf{X}}) & = & -\mathcal{R}_i(\hat{\mathbf{X}}) - \sum_{j \neq i}[(\hat{X}_j - X_j^{(s)}) \, S_{ij} \, (\hat{X}_j - X_j^{(s)}) \\
+(\hat{X}_i - X_i^{(s)}) S_i (\hat{X}_i - X_i^{(s)}) + \sum_{j \neq i}[(\hat{X}_i - X_i^{(s)}) S_j (\hat{X}_j - X_j^{(s)}) & + & (\hat{X}_j - X_j^{(s)}) \, S_j (\hat{X}_i - X_i^{(s)})].
\end{array} \tag{12}
$$

Proof. We apply the iteration equation (8) with $k = s : \mathcal{N}_i^{(s)}(\mathbf{X}^{(s+1)}) = Q_i^{(s)}$ and identity (5) with $\mathbf{Z} = \mathbf{X} = \mathbf{X}^{(s)}$:

$$
\begin{array}{rcl}
\mathcal{R}_i(\mathbf{X}^{(s)}) & = & -A^{(s)^T} X_i^{(s)} - X_i^{(s)} A^{(s)} - Q_i - X_i^{(s)} S_i X_i^{(s)} - \sum_{j \neq i} X_j^{(s)} S_{ij} X_j^{(s)}, \\
-A^{(s)^T} X_i^{(s)} - X_i^{(s)} A^{(s)} & = & \mathcal{R}_i(\mathbf{X}^{(s)}) + Q_i + X_i^{(s)} S_i X_i^{(s)} + \sum_{j \neq i} X_j^{(s)} S_{ij} X_j^{(s)}.
\end{array}
$$

After some matrix manipulations we obtain:

$$
\begin{array}{rl}
\mathcal{N}_i^{(s)}(X_i^{(s)} - X_i^{(s+1)}) = & \mathcal{R}_i(\mathbf{X}^{(s)}) + \sum_{j \neq i}^{N} [X_j^{(s)} S_{ij} X_j^{(s)} + X_j^{(s)} S_{ij} X_j^{(s)}], \\
-\sum_{j \neq i}(X_i^{(s)} S_j X_j^{(s)} + X_j^{(s)} S_j X_i^{(s)}) & + \sum_{j \neq i} \left[W_{ij}^{(s)} X_j^{(s)} + X_j^{(s)} W_{ij}^{(s)^T} \right].
\end{array}
$$

Using the notation for $W_{ij}^{(s)}$ we transform the right-hand side of the above equation and equality (11) yields.

Now, we will prove equality (12). Consider the difference $\mathcal{N}_i^{(s)}(\mathbf{X}^{(s+1)}) - \mathcal{R}_i(\mathbf{X}^{(s)}, \hat{\mathbf{X}})$ and it is derived:

$$\begin{aligned}\mathcal{N}_i^{(s)}(X_i^{(s+1)} - \hat{X}_i) = \quad & -\mathcal{R}_i(\hat{\mathbf{X}}) + \sum_{j\neq i}(X_i^{(s)} S_j X_j^{(s)} + X_j^{(s)} S_j X_i^{(s)}) \\ & -\sum_{j\neq i}[X_j^{(s)} S_{ij} X_j^{(s)} + \hat{X}_j\, S_{ij}\, \hat{X}_j] + \sum_{j\neq i}(\hat{X}_j - X_j^{(s)})\, S_j\, \hat{X}_i + \hat{X}_i\, S_j (\hat{X}_j - X_j^{(s)}), \\ & +(\hat{X}_i - X_i^{(s)}) S_i (\hat{X}_i - X_i^{(s)}) - \sum_{j\neq i}\left[W_{ij}^{(s)}\, \hat{X}_j + \hat{X}_j\, W_{ij}^{(s)^T}\right].\end{aligned}$$

We transform the right-hand side and we obtain

$$\begin{aligned}\mathcal{N}_i^{(s)}(X_i^{(s+1)} - \hat{X}_i) = \quad & -\mathcal{R}_i(\hat{\mathbf{X}}) - \sum_{j\neq i}(\hat{X}_j - X_j^{(s)})\, S_{ij}\, (\hat{X}_j - X_j^{(s)}) \\ & +(\hat{X}_i - X_i^{(s)}) S_i (\hat{X}_i - X_i^{(s)}) + \sum_{j\neq i}[(X_i^{(s)} - \hat{X}_i) S_j (X_j^{(s)} - \hat{X}_j) + (\hat{X}_j - X_j^{(s)})\, S_j (\hat{X}_i - X_i^{(s)})].\end{aligned}$$

Equality (12) is proved.

□

According to previous investigations on the convergence of iterative methods (Damm & Hinrichsen, 2001; Freiling & Hochhaus , 2004; Ivanov , 2008) we prove the following theorem:

Theorem 2.3: *Assume there exist symmetric nonnegative matrices* $\hat{\mathbf{X}}$ *and* $\mathbf{X}^{(0)}$ *such that* $\mathbf{X}^{(0)} \leq \hat{\mathbf{X}}$*, and* $\mathcal{R}_i(\mathbf{X}^{(0)}) \leq 0$ *and* $\mathcal{R}_i(\hat{\mathbf{X}}) \geq 0$*, and* $L^{(0)}$ *is an M-matrix. Then, the matrix sequence* $\{\mathbf{X}^{(k)}\}_{k=0}^{\infty}$ *defined by (8) satisfies:*

(i) $\hat{\mathbf{X}} \geq \mathbf{X}^{(k+1)} \geq \mathbf{X}^{(k)}$ *and* $\mathcal{R}_i(\mathbf{X}^{(k)}) \leq \ldots \geq 0$ *for* $k = 0, 1, \ldots$*;*

(ii) *The matrix* $L^{(k)}$ *is an M-matrix for* $k = 0, 1, \ldots$*;*

(iii) *The matrix sequence* $\{\mathbf{X}^{(k)}\}_{k=0}^{\infty}$ *converges to the nonpositive solution* $\tilde{\mathbf{X}} = (\tilde{X}_1, \ldots, \tilde{X}_N)$ *to the set of Riccati equations (3) with* $\tilde{\mathbf{X}} \leq \hat{\mathbf{X}}$ *and the matrix* $\tilde{L}$ *is an M-matrix.*

(iv) *In addition, if the matrix* $-(A - \sum_{j=1}^{N} S_j\, \hat{X}_j)$ *is an M-matrix, then the matrix* $\tilde{A} = -(A - \sum_{j=1}^{N} S_j\, \tilde{X}_j)$ *is also the M-matrix , i.e. the matrix* $\tilde{A}$ *is asymptotically stable.*

Proof. Let us choose a matrix $\mathbf{X}^{(0)}$ such that $L^{(0)}$, given by (10), is an M-matrix and compute $\mathbf{X}^{(1)}$ via (8) for $k = 0$. We already have $\hat{\mathbf{X}} \geq \mathbf{X}^{(0)}$. Applying (11) for $s = 0$ and $i = 1, \ldots, N$ we find that $\mathcal{N}_i^{(0)}(\mathbf{X}^{(1)} - \mathbf{X}^{(0)}) = -\mathcal{R}_i(\mathbf{X}^{(0)})$. Since $-\mathcal{R}_i(\mathbf{X}^{(0)}) \geq 0$, there exists a unique nonnegative solution $X_i^{(1)} - X_i^{(0)}, i = 1, \ldots, N$ because

$$vec(X_1^{(1)} - X_1^{(0)}, \ldots, X_N^{(1)} - X_N^{(0)}) \;=\; L^{(0)^{-1}}\, vec\,(-\mathcal{R}_1(\mathbf{X}^{(0)}), \ldots, -\mathcal{R}_N(\mathbf{X}^{(0)}))\,.$$

Thus $\mathbf{X}^{(1)} \geq \mathbf{X}^{(0)}$.

Using the iteration (8) we construct the matrix sequence $\mathbf{X}^{(0)}, \mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(r)}$. We will prove by induction the following statements for $r = 0, \ldots$:

(A) $\mathcal{R}_i(\mathbf{X}^{(r)}) \leq 0$ and the matrix $L^{(r)}$ is an M-matrix;

(B) $\mathbf{X}^{(r+1)} \geq \mathbf{X}^{(r)}$;

(C) $\hat{\mathbf{X}} \geq \mathbf{X}^{(r+1)}$.

Assume that $\mathcal{R}_i(\mathbf{X}^{(k-1)}) \leq 0$ and the matrix $L^{(k-1)}$ is an M-matrix and $\hat{\mathbf{X}} \geq \mathbf{X}^{(k)} \geq \mathbf{X}^{(k-1)}$. We will prove the statements (A)-(B)-(C) for $r = k$.

First, we would prove $\mathcal{R}_i(\mathbf{X}^{(k)}) \;\leq\; 0$ and $L^{(k)}$ is an M-matrix. Secondly, we would compute $\mathbf{X}^{(k+1)}$ as a unique solution of (8). Third, we would prove that $\hat{\mathbf{X}} \geq \mathbf{X}^{(k+1)} \geq \mathbf{X}^{(k)}$.

We will prove $\mathcal{R}_i(\mathbf{X}^{(k)}) \leq 0,\ i = 1, \ldots, N$. We perform $\mathcal{R}_i(\mathbf{X}^{(k)}) = \mathcal{R}_i(\mathbf{X}^{(k-1)}, \mathbf{X}^{(k)})$ and according to formula (8) for computing $X_i^{(k)}$ we write down

$$-A^{(k-1)^T} X_i^{(k)} - X_i^{(k)} A^{(k-1)} = Q_i^{(k-1)} - \sum_{j\neq i} \left[W_{ij}^{(k-1)} X_j^{(k)} + X_j^{(k)} W_{ij}^{(k-1)^T} \right].$$

After short calculations for $\mathcal{R}_i(\mathbf{X}^{(k)})$ we obtain:

$$\begin{aligned}\mathcal{R}_i(\mathbf{X}^{(k)}) \; &= \; +(X_i^{(k)} - X_i^{(k-1)}) S_i (X_i^{(k)} - X_i^{(k-1)}) - \textstyle\sum_{j\neq i} (X_j^{(k-1)} - X_j^{(k)})\, S_{ij} (X_j^{(k-1)} - X_j^{(k)}) \\ &+ \textstyle\sum_{j\neq i} [(X_i^{(k-1)} - X_i^{(k)})\, S_j (X_j^{(k-1)} - X_j^{(k)}) + (X_j^{(k-1)} - X_j^{(k)}) S_j (X_i^{(k-1)} - X_i^{(k)})].\end{aligned}$$

Hence $\mathcal{R}_i(\mathbf{X}^{(k)}) \; \leq \; 0 \;, i = 1, \ldots, N$.

Next, we will prove that $L^{(k)}$ is an M-matrix. We apply the presentation $\mathcal{R}_i(\hat{\mathbf{X}}) = \mathcal{R}_i(\mathbf{X}^{(k)}, \hat{\mathbf{X}})$ and $\mathcal{R}_i(\mathbf{X}^{(k)})$ through (4). Next, we consider the difference $\mathcal{R}_i(\mathbf{X}^{(k)}) - \mathcal{R}_i(\hat{\mathbf{X}})$ and we obtain $\mathcal{N}_i^{(k)}(\mathbf{X}^{(k)} - \hat{\mathbf{X}}) = V_i^{(k)}$, where $(i = 1, \ldots, N)$

$$\begin{aligned}V_i^{(k)} \; &= \; +\mathcal{R}_i(\mathbf{X}^{(k)}) - \mathcal{R}_i(\hat{\mathbf{X}}) + (\hat{X}_i - X_i^{(k)}) S_i (\hat{X}_i - X_i^{(k)}) \\ &- \textstyle\sum_{j\neq i}^{N} [(\hat{X}_j - X_j^{(k)})\, S_{ij}\, (\hat{X}_j - X_j^{(k)})] \\ &+ \textstyle\sum_{j\neq i} [(\hat{X}_j - X_j^{(k)})\, S_j\, (\hat{X}_i - X_i^{(k)}) + (\hat{X}_i - X_i^{(k)})\, S_j (\hat{X}_j - X_j^{(k)})].\end{aligned}$$

Since $\mathcal{R}_i(\hat{\mathbf{X}}) \geq 0, \mathcal{R}_i(\mathbf{X}^{(k)}) \leq 0$ and $S_i \leq 0, S_{ij} \geq 0, i \neq j$ and hence, together with $\hat{\mathbf{X}} \geq \mathbf{X}^{(k)} \geq 0$ we infer $V_i^{(k)} \leq 0 \,, i, j = 1, \ldots, N$. By the relation

$$L^{(k)}\, vec((X_1^{(k)} - \hat{X}_1), \ldots, (X_N^{(k)} - \hat{X}_N)) \; = \; vec\; (V_1^{(k)}, \ldots, V_N^{(k)}),$$

we have that $L^{(k)}$ is an M-matrix by Lemma 1.1. Therefore, the matrix $L^{(k)}$, defined in (10), is an M-matrix.

Thus, we can apply the recursive equation (8) to find the matrix $\mathbf{X}^{(k+1)}$. We will prove $\hat{\mathbf{X}} \geq \mathbf{X}^{(k+1)}$. We apply equality (12) with $s = k$. Then $(i = 1, \ldots, N)$

$$\begin{aligned}\mathcal{N}_i^{(k)}(\mathbf{X}^{(k+1)} - \hat{\mathbf{X}}) = \quad & -\mathcal{R}_i(\hat{\mathbf{X}}) + (\hat{X}_i - X_i^{(k)}) S_i (\hat{X}_i - X_i^{(k)}) \\ + \textstyle\sum_{j\neq i} & \left[(\hat{X}_j - X_j^{(k)}) S_j (\hat{X}_j - X_j^{(k)}) + (\hat{X}_j - X_j^{(k)})\, S_j (\hat{X}_i - X_i^{(k)}) - (\hat{X}_j - X_j^{(k)})\, S_{ij}\, (\hat{X}_j - X_j^{(k)}) \right].\end{aligned}$$

Now let us analyze the last set of matrix equations. The matrix $L^{(k)}$ is an M-matrix. The right-hand side of each equation is nonpositive. Thus $X_i^{(k+1)} - \hat{X}_i \leq 0 \,, i = 1, \ldots, N$ and $\hat{\mathbf{X}} \geq \mathbf{X}^{(k+1)}$.

For proving $\mathbf{X}^{(k+1)} \geq \mathbf{X}^{(k)}$ we consider (11) for $s = k$, which is $\mathcal{N}_i^{(k)}(\mathbf{X}^{(k)} - \mathbf{X}^{(k+1)}) = \mathcal{R}_i(\mathbf{X}^{(k)})$ for $i = 1, \ldots, N$. Since $\mathcal{R}_i(\mathbf{X}^{(k)})$ is a nonpositive matrix and $L^{(k)}$ is an M-matrix we obtain $X_i^{(k)} - X_i^{(k+1)} \leq 0, i = 1, \ldots, N$. Thus $\mathbf{X}^{(k+1)} \geq \mathbf{X}^{(k)}$. Hence, the induction process has been completed.

Thus the matrix sequence $\{\mathbf{X}^{(k)}\}_{k=0}^{\infty}$ is monotonically increasing and bounded above by $\hat{\mathbf{X}}$ (in the elementwise ordering). We denote $\lim_{k\to\infty} \mathbf{X}^{(k)} = \tilde{\mathbf{X}}$. By taking the limits in (8) it follows that $\tilde{\mathbf{X}}$ is a solution of $\mathcal{R}_i(\mathbf{X}) = 0, i = 1, \ldots, N$ with the property $\tilde{\mathbf{X}} \leq \hat{\mathbf{X}}$.

Further on, assume that the matrix $A - \sum_{j=1}^{N} S_j \hat{X}_j$ is an M-matrix. Since $S_j \leq 0, \hat{X}_j \geq \tilde{X}_j \geq 0, j = 1, \ldots, N$, then $-(A - \sum_{j=1}^{N} S_j \hat{X}_j) \leq -(A - \sum_{j=1}^{N} S_j \tilde{X}_j) = \tilde{A}$. According to Lemma 1.2, it follows that $\tilde{A}$ is an M-matrix and therefore $-\tilde{A}$ is asymptotically stable. The proof is complete. □

Corollary 2.4: *Assume than* $\mathbf{Y}$ *and* $\tilde{\mathbf{X}}$ *are nonnegative solutions to* $\mathcal{R}_i(\mathbf{X}) = 0, i = 1, \ldots, N$ *and* $\tilde{\mathbf{X}}$ *is the stabilizing solution. Then* $\mathbf{Y} \geq \tilde{\mathbf{X}}$.

Proof. The stabilizing solution $\tilde{\mathbf{X}}$ is obtained via iteration (8). Assume $\mathbf{Y} \leq \tilde{\mathbf{X}}$. There exists an index q such that $\mathbf{X}^{(q)} \geq \mathbf{Y}$. Applying (12) with $k = q$ and $\hat{\mathbf{X}} = \mathbf{Y}$. We obtain $(i = 1, \ldots, N)$

$$\begin{array}{rl} \mathcal{N}_i^{(q)}(\mathbf{X}^{(q+1)} - \mathbf{Y}) = & -\mathcal{R}_i(\mathbf{Y}) + (Y_i - X_i^{(q)})S_i(Y_i - X_i^{(q)}) \\ + \sum_{j\neq i} \Big[(Y_j - X_j^{(q)})S_j(Y_j - X_j^{(q)}) + (Y_j - X_j^{(q)})\, S_j(Y_i - X_i^{(q)}) - (Y_j - X_j^{(q)})\, S_{ij}\,(Y_j - X_j^{(q)}) \Big] . & \end{array}$$

The right-hand side of each equation is nonpositive $(\mathcal{R}_i(\mathbf{Y}) = 0)$. The matrix $L^{(q)}$ is an M-matrix. Thus $X_i^{(q+1)} - Y_i \leq 0\,, i = 1, \ldots, N$ and therefore $\mathbf{Y} \geq \mathbf{X}^{(k+1)}$, which is a contradiction. □

Remark 1: Very often the initial point $\mathbf{X}^{(0)}$ for the Newton iteration is chosen to be a zero matrix. Thus the matrix $L^{(0)}$ is the diagonal matrix $L_0 = diag[-(I_n \otimes A^T), \ldots, -(I_n \otimes A^T)]$, i.e. it is necessary the matrix $-A$ to be an M-matrix to start the Newton iteration.

2.3. *The accelerated Newton method*

In order to apply Newton iteration (8) we have to solve a high dimensional linear system on each iteration step. To avoid the computational work for solving these linear systems we introduce a new iterative scheme named the accelerated Newton method (AN). Consider the set of iteration equations (8). We put $X_2^{(k+1)} = X_2^{(k)}, \ldots, X_N^{(k+1)} = X_N^{(k)}$ in the first equation $(i = 1)$ of set (8). It becomes a Lyapunov equation relating to $X_1^{(k+1)}$:

$$-{A^{(k)}}^T X_1^{(k+1)} - X_1^{(k+1)}\, A^{(k)} = Q_1^{(k)} - \sum_{j>1} \left[W_{1j}^{(k)}\, X_j^{(k)} + X_j^{(k)}\, {W_{1j}^{(k)}}^T \right] .$$

Let us assume that $X_1^{(k+1)}, \ldots, X_{s-1}^{(k+1)}$ are computed. In order to compute $X_s^{(k+1)}, s = 2, \ldots, N$ we apply the following equation:

$$\begin{array}{rl} & -{A^{(k)}}^T X_s^{(k+1)} - X_s^{(k+1)}\, A^{(k)} \\ = & Q_s^{(k)} - \sum_{j<s} \left[W_{sj}^{(k)}\, X_j^{(k+1)} + X_j^{(k+1)}\, {W_{sj}^{(k)}}^T \right] - \sum_{j>s} \left[W_{sj}^{(k)}\, X_j^{(k)} + X_j^{(k)}\, {W_{sj}^{(k)}}^T \right] . \end{array}$$

Thus, we obtain the accelerated Newton method:

$$-{A^{(k)}}^T X_i^{(k+1)} - X_i^{(k+1)}\, A^{(k)} = \tilde{Q}_i^{(k)}, \qquad i = 1, \ldots, N\,, \tag{13}$$

where

$$\tilde{Q}_i^{(k)} \;=\; Q_i^{(k)} - \sum_{j<i} \left[W_{ij}^{(k)}\, X_j^{(k+1)} + X_j^{(k+1)}\, {W_{ij}^{(k)}}^T \right] - \sum_{j>i} \left[W_{ij}^{(k)}\, X_j^{(k)} + X_j^{(k)}\, {W_{ij}^{(k)}}^T \right] . \tag{14}$$

We will prove several properties of the sequences of Lyapunov algebraic equations (13), whose solutions construct monotone increasing matrix sequences bounded by above with an upper bound. The limits of these sequences complete a symmetric stabilizing solution to the set of Riccati equations (3). The convergence properties of iteration (13) are proved in the following theorem.

Theorem 2.5: *Assume there exist symmetric nonnegative matrices* $\hat{\mathbf{X}}$ *and* $\mathbf{X}^{(0)}$ *such that* $\mathbf{X}^{(0)}) \leq \hat{\mathbf{X}}$*, and* $\mathcal{R}_i(\mathbf{X}^{(0)}) \leq 0$ *and* $\mathcal{R}_i(\hat{\mathbf{X}}) \geq 0$*, and* $A - \sum_{j=1}^N S_j\, X_j^{(0)}$ *is asymptotically stable. Then, the matrix sequence* $\{\mathbf{X}^{(k)}\}_{k=0}^{\infty}$ *defined by (13) satisfies:*

(i) $\hat{\mathbf{X}} \geq \mathbf{X}^{(k+1)} \geq \mathbf{X}^{(k)}$ *and* $\mathcal{R}_i(\mathbf{X}^{(k)}) \leq 0$ *for* $k = 0, 1, \ldots$;

(ii) *The matrix* $A - \sum_{j=1}^N S_j\, X_j^{(k)}$ *is asymptotically stable* $k = 0, 1, \ldots$;

(iii) *The matrix sequence* $\{\mathbf{X}^{(k)}\}_{k=0}^{\infty}$ *converges to the nonnegative solution* $\tilde{\mathbf{X}} = (\tilde{X}_1, \ldots, \tilde{X}_N)$ *to the set of Riccati equations (3) with* $\tilde{\mathbf{X}} \leq \hat{\mathbf{X}}$ *and the matrix* $A - \sum_{j=1}^N S_j\, \tilde{X}_j$ *is asymptotically stable.*

Proof. Consider iteration (13) for $k = 0$:$-{A^{(0)}}^T X_i^{(1)} - X_i^{(1)}\, A^{(0)} = \tilde{Q}_i^{(0)}$. Moreover,

$$-{A^{(0)}}^T X_1^{(0)} - X_1^{(0)}\, A^{(0)} = \mathcal{R}_1(\mathbf{X}^{(0)}) + Q_1 + X_1^{(0)}\, S_1\, X_1^{(0)} + \textstyle\sum_{j\neq 1}^N X_j^{(0)}\, S_{1j}\, X_j^{(0)}\,.$$

We obtain that $-{A^{(0)}}^T(X_1^{(1)} - X_1^{(0)}) - (X_1^{(1)} - X_1^{(0)})A^{(0)} = -\mathcal{R}_1(\mathbf{X}^{(0)})$. Since $\mathcal{R}_1(\mathbf{X}^{(0)}) \leq 0$ and $A^{(0)} = A - \sum_{j=1}^N S_j\, X_j^{(0)}$ is asymptotically stable, then $X_1^{(1)} - X_1^{(0)} \geq 0$ or $X_1^{(1)} \geq X_1^{(0)} \geq 0$, i.e. the matrices $(X_1^{(1)} - X_1^{(0)})$ and $X_1^{(1)}$ are nonnegative. Assume that for the matrices $X_1^{(1)}, \ldots, X_{s-1}^{(1)}$ the inequalities $(X_r^{(1)} - X_r^{(0)}) \geq 0$ and $X_r^{(1)} \geq 0$ for $r = 1, \ldots, s-1$ are satisfied. We will prove that the matrix $X_s^{(1)}$ satisfies the inequalities $(X_s^{(1)} - X_s^{(0)}) \geq 0$ and $X_s^{(1)} \geq 0$.

Consider iteration (13) with $k = 0$ and $i = s$. Following representation (4) we have

$$\begin{aligned}
& -{A^{(0)}}^T(X_s^{(1)} - X_s^{(0)}) - (X_s^{(1)} - X_s^{(0)})\, A^{(0)} \\
= \; & -\mathcal{R}_s(\mathbf{X}^{(0)}) + \sum_{j<s} [X_i^{(0)} S_j (X_j^{(0)} - X_j^{(1)}) + (X_j^{(0)} - X_j^{(1)}) S_j X_s^{(0)}] \\
& + \textstyle\sum_{j<s} \left[X_j^{(0)} S_{sj}\, (X_j^{(1)} - X_j^{(0)}) + (X_j^{(1)} - X_j^{(0)}) S_{sj} X_j^{(0)}\right]\,.
\end{aligned}$$

Since $X_j^{(1)} - X_j^{(0)}$ is nonnegative for $j < s$ and $S_j \leq 0$, we conclude that the right-hand side of the above equation is a nonnegative matrix. Thus, the inequalities $(X_s^{(1)} - X_s^{(0)}) \geq 0$ and $X_s^{(1)} \geq 0$ hold. We deduce that $X_i^{(1)}$ is nonnegative for $i = 1, \ldots, N$ and $\mathbf{X}^{(1)} \geq \mathbf{X}^{(0)}$.

We construct the matrix sequence $\mathbf{X}^{(s)}$ with iteration (13). We prove items (i) and (ii) by induction. Assume that $\mathcal{R}_i(\mathbf{X}^{(k-1)}) \;\leq\; 0$, $A - \sum_{j=1}^N S_j\, X_j^{(k-1)}$ is asymptotically stable and $\hat{\mathbf{X}} \geq \mathbf{X}^{(k)} \geq \mathbf{X}^{(k-1)}$. We first prove that $\mathcal{R}_i(\mathbf{X}^{(k)}) \;\leq\; 0$ and $A - \sum_{j=1}^N S_j\, X_j^{(k)}$ is asymptotically stable. Secondly, we will compute $\mathbf{X}^{(k+1)}$ as a unique solution of (13). Third we will prove that $\hat{\mathbf{X}} \geq \mathbf{X}^{(k+1)} \geq \mathbf{X}^{(k)}$.

We begin with the inequality $\mathcal{R}_i(\mathbf{X}^{(k)}) \leq 0,\ i = 1, \ldots, N$. We apply identity (5) with $\mathbf{X} = \mathbf{X}^{(k)}$ and $\mathbf{Z} = \mathbf{X}^{(k-1)}$ for $\mathcal{R}_i(\mathbf{X}^{(k)}) = \mathcal{R}_i(\mathbf{X}^{(k-1)}, \mathbf{X}^{(k)})$ and combine it with iteration (13) for the computation of $X_i^{(k)}$.

We execute some matrix manipulations and derive

$$
\begin{aligned}
&\mathcal{R}_i(\mathbf{X}^{(k)}) = \ +(X_i^{(k)} - X_i^{(k-1)})S_i(X_i^{(k)} - X_i^{(k-1)}) \,, \\
&+\sum_{j<i} [(X_i^{(k-1)} - X_i^{(k)})\, S_j(X_j^{(k-1)} - X_j^{(k)}) + (X_j^{(k-1)} - X_j^{(k)})S_j(X_i^{(k-1)} - X_i^{(k)})] \\
&-\sum_{j<i} \left[(X_j^{(k)} - X_j^{(k-1)})S_{ij}\,(X_j^{(k)} - X_j^{(k-1)})\right] \\
&-\sum_{j>i}[X_j^{(k-1)}S_{ij}(X_j^{(k)} - X_j^{(k-1)}) + (X_j^{(k)} - X_j^{(k-1)})\, S_{ij}\, X_j^{(k)}] \\
&- \sum_{j>i} (X_j^{(k-1)} - X_j^{(k)})\, S_j\, X_i^{(k)} + X_i^{(k)}\, S_j(X_j^{(k-1)} - X_j^{(k)})\,, \qquad i = 1, \ldots, N\,.
\end{aligned}
$$

Note that $\mathbf{X}^{(k)} \geq \mathbf{X}^{(k-1)} \geq 0$ and $S_i \leq 0, S_{ij} \geq 0, j \neq i$. Then $X_j^{(k-1)}S_{ij}(X_j^{(k)} - X_j^{(k-1)}) \geq 0$ and $(X_j^{(k)} - X_j^{(k-1)})\, S_{ij}\, X_j^{(k)} \geq 0$ for $j = 1, \ldots, N$. Moreover, $\mathcal{R}_i(\mathbf{X}^{(k)}) \ \leq\ 0\ , i = 1, \ldots, N\,.$

We will prove that $A^{(k)} = A - \sum_{j=1}^{N} S_j\, X_j^{(k)}$ is asymptotically stable. For this purpose we consider $M^{(k)} = diag[-(I_n \otimes A^{(k)}), \ldots, -(I_n \otimes A^{(k)})]$.

From the difference $\mathcal{R}_i(\mathbf{X}^{(k)}) - \mathcal{R}_i(\hat{\mathbf{X}})$ we express $(i = 1, \ldots, N)$:

$$
-{A^{(k)}}^T\, (X_i^{(k)} - \hat{X}_i) - (X_i^{(k)} - \hat{X}_i)\, A^{(k)} = T_i^{(k)},
$$

where

$$
\begin{aligned}
T_i^{(k)} \ &= \ \mathcal{R}_i(\mathbf{X}^{(k)}) + (\hat{X}_i - X_i^{(k)})S_i(\hat{X}_i - X_i^{(k)}) + (\hat{X}_i - X_i^{(k)})S_i(\hat{X}_i - X_i^{(k)}) \\
&-\sum_{j\neq i}^{N} [\hat{X}_j\, S_{ij}\, (\hat{X}_j - X_j^{(k)}) + (\hat{X}_j - X_j^{(k)})\, S_{ij}\, X_j^{(k)}] + \sum_{j\neq i} [(\hat{X}_j - X_j^{(k)})\, S_j\, \hat{X}_i + \hat{X}_i\, S_j(\hat{X}_j - X_j^{(k)})]\,.
\end{aligned}
$$

The inequalities $\hat{X}_j\, S_{ij}\, (\hat{X}_j - X_j^{(k)}) \geq 0,\ (\hat{X}_j - X_j^{(k)})\, S_{ij}\, X_j^{(k)} \geq 0$ hold true because $\hat{\mathbf{X}} \geq \mathbf{X}^{(k)} \geq 0$ and $S_{ij} \geq 0$ for $i, j = 1, \ldots, N,\ j \neq i$. Analogously, we conclude that $(\hat{X}_j - X_j^{(k)})\, S_j\, \hat{X}_i \leq 0$ and $\hat{X}_i\, S_j(\hat{X}_j - X_j^{(k)}) \leq 0, j \neq i$ because $S_j \leq 0, j = 1, \ldots, N$. Thus $T_i^{(k)} \leq 0, i = 1, \ldots, N$ and therefore $M^{(k)} = diag[-(I_n \otimes A^{(k)}), \ldots, -(I_n \otimes A^{(k)})]$ is an M-matrix.

Now, we could compute the matrix $\mathbf{X}^{(k+1)}$ via (13). We have to prove the inequalities $\hat{\mathbf{X}} \geq \mathbf{X}^{(k+1)} \geq \mathbf{X}^{(k)}$.

We begin with the proof of the inequality $\mathbf{X}^{(k+1)} \geq \mathbf{X}^{(k)}$. Combining iteration (13) for computing $X_i^{(k+1)}$ and identity $\mathcal{R}_i(\mathbf{X}^{(k)}) = \mathcal{R}_i(\mathbf{X}^{(k)}, \mathbf{X}^{(k)})$ we deduce:

$$
\begin{aligned}
&-{A^{(k)}}^T\, (X_i^{(k)} - X_i^{(k+1)}) - (X_i^{(k)} - X_i^{(k+1)})\, A^{(k)} \\
= \ &\mathcal{R}_i(\mathbf{X}^{(k)}) - \sum_{j<i} [X_i^{(k)} S_j (X_j^{(k)} - X_j^{(k+1)}) + (X_j^{(k)} - X_j^{(k+1)}) S_j X_i^{(k)}] \\
&-\sum_{j<i} \left[X_j^{(k)} S_{ij}\, (X_j^{(k+1)} - X_j^{(k)}) + (X_j^{(k+1)} - X_j^{(k)})\, S_{ij}\, X_j^{(k)}\right]\,, \quad i = 1, \ldots, N\,.
\end{aligned}
$$

Assume $i=1$. We have for $(X_1^{(k)} - X_1^{(k+1)})$:

$$-A^{(k)^T}(X_1^{(k)} - X_1^{(k+1)}) - (X_1^{(k)} - X_1^{(k+1)})A^{(k)} = \mathcal{R}_1(\mathbf{X}^{(k)}).$$

Thus, $(X_1^{(k)} - X_1^{(k+1)}) \le 0$. Assume $i > 1$ and $(X_s^{(k)} - X_s^{(k+1)}) \le 0$ for $s = 1, \ldots, i-1$. We have

$$\begin{aligned} & -A^{(k)^T}(X_i^{(k)} - X_i^{(k+1)}) - (X_i^{(k)} - X_i^{(k+1)})A^{(k)} \\ = \ & \mathcal{R}_i(\mathbf{X}^{(k)}) - \sum_{j<i}[X_i^{(k)}S_j(X_j^{(k)} - X_j^{(k+1)}) + (X_j^{(k)} - X_j^{(k+1)})S_jX_i^{(k)}] \\ & - \sum_{j<i}\left[X_j^{(k)}S_{ij}(X_j^{(k+1)} - X_j^{(k)}) + (X_j^{(k+1)} - X_j^{(k)})S_{ij}X_j^{(k)}\right], \quad i = 1, \ldots, N. \end{aligned}$$

Under the assumption $(X_j^{(k)} - X_j^{(k+1)}) \le 0$ for $j < i$ we find that the right-hand side of the above identity is nonpositive. Thus, $(X_i^{(k)} - X_i^{(k+1)}) \le 0$. Hence, the inequality $\mathbf{X}^{(k+1)} \ge \mathbf{X}^{(k)}$ is proved.

Further on, we consider the difference $\left(-A^{(k)^T}(X_i^{(k+1)} - \hat{X}_i) - (X_i^{(k+1)} - \hat{X}_i)A^{(k)}\right)$.

We signify the term $-A^{(k)^T}\hat{X}_i - \hat{X}_iA^{(k)}$ from the identity $\mathcal{R}_i(\hat{\mathbf{X}}) = \mathcal{R}_i(\mathbf{X}^{(k)}, \hat{\mathbf{X}})$. Combining with iteration (13) we derive

$$\begin{aligned} & -A^{(k)^T}(X_i^{(k+1)} - \hat{X}_i) - (X_i^{(k+1)} - \hat{X}_i)A^{(k)} = -\mathcal{R}_i(\hat{\mathbf{X}}) + (\hat{X}_i - X_i^{(k)})S_i(\hat{X}_i - X_i^{(k)}) \\ & + \sum_{j<i}(\hat{X}_j - X_j^{(k)})S_j(\hat{X}_i - X_i^{(k)}) + (\hat{X}_i - X_i^{(k)})S_j(\hat{X}_j - X_j^{(k)}), \\ & + \sum_{j<i}[+X_i^{(k)}S_j(\hat{X}_j - X_j^{(k+1)}) + (\hat{X}_j - X_j^{(k+1)})S_jX_i^{(k)}] \\ & + \sum_{j>i}(\hat{X}_j - X_j^{(k)})S_j\hat{X}_i + \hat{X}_iS_j(\hat{X}_j - X_j^{(k)}) - [X_j^{(k)}S_{ij}(\hat{X}_j - X_j^{(k)}) + (\hat{X}_j - X_j^{(k)})S_{ij}\hat{X}_j], \\ & - \sum_{j<i}(X_j^{(k)} - X_j^{(k+1)})S_{ij}(X_j^{(k)} - X_j^{(k+1)}) \\ & - \sum_{j<i}[+\hat{X}_jS_{ij}(\hat{X}_j - X_j^{(k+1)}) + (\hat{X}_j - X_j^{(k+1)})S_{ij}X_j^{(k+1)}], \quad i = 1, \ldots, N. \end{aligned}$$

We fix $i = 1$. Then

$$\begin{aligned} & -A^{(k)^T}(X_1^{(k+1)} - \hat{X}_1) - (X_i^{(k+1)} - \hat{X}_1)A^{(k)} \\ = \ & -\mathcal{R}_1(\hat{\mathbf{X}}) + (\hat{X}_1 - X_1^{(k)})S_1(\hat{X}_1 - X_i^{(1)}) + \sum_{j>1}(\hat{X}_j - X_j^{(k)})S_j\hat{X}_1 + \hat{X}_1S_j(\hat{X}_j - X_j^{(k)}), \\ & - \sum_{j>1}[X_j^{(k)}S_{1j}(\hat{X}_j - X_j^{(k)}) + (\hat{X}_j - X_j^{(k)})S_{1j}\hat{X}_j]. \end{aligned}$$

The right-hand side of the last equality is nonpositive, moreover $X_1^{(k+1)} - \hat{X}_1 \le 0$. Assume that $X_j^{(k+1)} - \hat{X}_j \le 0$ for $j < i \le N$. We would prove now the inequality $X_i^{(k+1)} - \hat{X}_i \le 0$. For the right-hand side of the last identity we have

$$X_i^{(k)}S_j(\hat{X}_j - X_j^{(k+1)}) + (\hat{X}_j - X_j^{(k+1)})S_jX_i^{(k)} \le 0, \quad j < i,$$

because $\hat{X}_j - X_j^{(k+1)} \geq 0$, $X_i^{(k)} \geq 0$ and $S_j \leq 0$. For the same reason we could conclude that

$$(X_j^{(k)} - X_j^{(k+1)}) S_{ij} (X_j^{(k)} - X_j^{(k+1)}) \geq 0 \,, \quad \text{and} \quad \hat{X}_j \, S_{ij} \, (\hat{X}_j - X_j^{(k+1)}) + (\hat{X}_j - X_j^{(k+1)}) S_{ij} \, X_j^{(k+1)} \geq 0 \,.$$

Thus $-A^{(k)^T}(X_i^{(k+1)} - \hat{X}_i) - (X_i^{(k+1)} - \hat{X}_i)\, A^{(k)} \leq 0$ and moreover $X_i^{(k+1)} - \hat{X}_i \leq 0$ for $i = 1, \ldots, N$.

The induction process for the proof of (i) and (ii) is now complete.

The matrix sequence of nonnegative matrices $\{\mathbf{X}^{(k)}\}_{k=0}^{\infty}$ converges to the nonnegative solution $\tilde{\tilde{\mathbf{X}}} = (\tilde{\tilde{X}}_1, \ldots, \tilde{\tilde{X}}_N)$ to the set of Riccati equations (3). Moreover, the "limit" matrix has the properties $\tilde{\tilde{\mathbf{X}}} \leq \hat{\mathbf{X}}$ and the matrix $-A + \sum_{j=1}^{N} S_j \, \tilde{\tilde{X}}_j$ is an M-matrix. Thus the matrix $A - \sum_{j=1}^{N} S_j \, \tilde{\tilde{X}}_j$ is asymptotically stable.

The proof is complete. □

3. Numerical simulations

We carry out some numerical experiments for computing the stabilizing solution to the set of generalized Riccati equations (3). The Newton method (8) and the accelerated Newton method (13) are applied.

Example. We consider a three players game ($N = 3$) where the matrix coefficients: A, B_i, Q_i and R_{ij} for $i, j = 1, \ldots, N$ are the following. We define them using the Matlab description.

A=abs(randn(n,n))/10; A=A-5*eye(n,n);
B_1= zeros(n,1); B_1(1,1)=5; B_1(3,1)=2; B_1(n,1)=4;
B_2=full(abs(sprandn(n,4,0.8))/10);
B_3=full(abs(sprandn(n,3,0.8))/10);,
Q_1=4.5*eye(n,n); Q_1(1,n)=3.5; Q_1(n,1)=3.5;
Q_2=3.75*eye(n,n); Q_2(1,n)=4.5; Q_2(n,1)=4.5;
Q_3=2.85*eye(n,n); Q_3(1,n)=1.5; Q_3(n,1)=1.5;
R_{11} = -90; R_{21} = R_{31} = 200;
R_{22} = [-400 0 0 -10; 0 -100 0 0; 0 0 -200 0; -10 0 0 -400];
R_{33} = [-80 0 0; 0 -90 -5; 0 -5 -60]*10;
R_{12} = [40 0 0 0; 0 200 0 0; 0 0 500 0;0 0 0 30];
R_{13} = [120 0 0; 0 75 0; 0 0 140];
R_{23} = [220 0 0; 0 180 0; 0 0 190];
R_{32} = [100 0 0 0; 0 250 0 0; 0 0 240 0; 0 0 0 300];

We execute this example for different values of n and 100 runs for each values of n. We take $X_1^{(0)} = X_2^{(0)} = X_3^{(0)} = 0$ and thus $\mathcal{R}_i(\mathbf{X}^{(0)}) = -Q_i \leq 0$ (i.e. the matrix is nonpositive). In addition, we take $\hat{X}_1 = 1.25\, e\, e^T, \hat{X}_2 = \hat{X}_3 = 2.5\, e\, e^T$, with $e^T = (1, 1, \ldots, 1)$. For the above choice the conditions of theorems 2.3 and 2.5 are fulfilled, i.e. $\mathbf{X}^{(0)} \leq \hat{\mathbf{X}}$, $\mathcal{R}_i(\mathbf{X}^{(0)}) \leq 0$ and $\mathcal{R}_i(\hat{\mathbf{X}}) \geq 0$, $i = 1, 2, 3$. The computed solution $\tilde{\mathbf{X}}$ satisfies the inequality $\tilde{\mathbf{X}} \leq \hat{\mathbf{X}}$.

We summarize the results from experiments for $n = 10$. The Newton iteration (8) needs 4 iteration steps to find the stabilizing nonnegative and positive definite solution $\tilde{\mathbf{X}}_N$. The accelerated Newton iteration (13) needs 6 iteration steps to find the stabilizing nonnegative and positive definite solution $\tilde{\mathbf{X}}_{AC}$. The CPU time is 1.4s for executing the Newton iteration with 100 runs and 0.5s for executing the accelerated Newton iteration with 100 runs.

We would comment the results from experiments for $n = 15$. The Newton iteration (8) needs 4 iteration steps to find the stabilizing nonnegative and positive definite solution $\tilde{\mathbf{X}}_N$. The accelerated Newton iteration (13) needs 6 iteration steps to find the stabilizing nonnegative and positive definite solution $\tilde{\mathbf{X}}_{AC}$. The CPU time is 11.4s for executing the Newton iteration with 100 runs and 1.2s for executing the accelerated Newton iteration with 100 runs.

4. Conclusion

We have studied two iterative processes for finding the stabilizing solution to a set of generalized Riccati equations (3). The convergence properties of both methods have been derived and formulated as sufficient conditions for the existence of a stabilizing solution to (3). The accelerated Newton method is a new method, which uses the specific type of the Newton iteration. We have made numerical experiments for the computation of this solution and we have compared the numerical results. First of all, our numerical experiments confirm the effectiveness of the proposed new iterative methods (8) and (13). Furthermore, the second method, based on the solution the Lyapunov equations at each iteration step, is faster than the Newton method.

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