ALMOST SQUARE AND OCTAHEDRAL NORMS IN TENSOR PRODUCTS OF BANACH SPACES

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Abstract. The aim of this note is to study some geometrical properties like diameter two properties, octahedrality and almost squareness in the setting of (symmetric) tensor product spaces. In particular, we show that the injective tensor product of two octahedral Banach spaces is always octahedral, the injective tensor product of an almost square Banach space with any Banach space is almost square, and the injective symmetric tensor product of an octahedral Banach space is octahedral.

1. Introduction

Let X be a (real) Banach space with closed unit ball B_X and unit sphere S_X. Following [4], we will say that

(i) X has the *local diameter two property* (LD2P) whenever each slice of B_X has diameter two.

(ii) X has the *diameter two property* (D2P) whenever each nonempty relatively weakly open subset of B_X has diameter two.

(iii) X has the *strong diameter two property* (SD2P) whenever each finite convex combination of slices of B_X has diameter two.

Similarly, for dual spaces one can define the w^*-LD2P, the w^*-D2P and the w^*-SD2P by replacing slices and weakly open subsets with weak* slices and weak* open subsets in the above definitions.

The starting point for the study of diameter two properties was probably [24] and [26]. Recent years have seen a lot of activity in the study of diameter two properties, see [2, 4, 10, 19] and the references therein.

In order to characterize the dual of spaces with diameter two properties Haller, Langemets and Põldvere studied octahedral norms in [19]. There they proved the following characterizations of octahedral norms that we will take as our definitions. A Banach space X is

(i) *locally octahedral* (LOH) if for every $x \in S_X$ and $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x \pm y\| > 2 - \varepsilon$.

(ii) *weakly octahedral* (WOH) if for every $x_1, \ldots, x_n \in S_X$, $x^* \in B_{X^*}$ and $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x_i \pm ty\| \geq (1-\varepsilon)(|x^*(x_i)| + t)$ for all $i \in \{1, \ldots, n\}$ and $t > 0$.

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(iii) *octahedral* (OH) if for every $x_1, \ldots, x_n \in S_X$ and $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x_i + y\| > 2 - \varepsilon$ for all $i \in \{1, \ldots, n\}$.

It is known that a Banach space X has the SD2P (respectively D2P, LD2P) if, and only if, X^* is OH (respectively WOH, LOH) (cf. e.g. [19]).

A family of geometrical properties closely related to diameter two properties and octahedrality is the following which was introduced in [3]. A Banach space X is

(i) *locally almost square* (LASQ) if for every $x \in S_X$ there exists a sequence $\{y_n\}$ in B_X such that $\|x \pm y_n\| \to 1$ and $\|y_n\| \to 1$.
(ii) *weakly almost square* (WASQ) if for every $x \in S_X$ there exists a sequence $\{y_n\}$ in B_X such that $\|x \pm y_n\| \to 1$, $\|y_n\| \to 1$ and $y_n \to 0$ weakly.
(iii) *almost square* (ASQ) if for every $x_1, \ldots, x_k \in S_X$ there exists a sequence $\{y_n\}$ in B_X such that $\|y_n\| \to 1$ and $\|x_i \pm y_n\| \to 1$ for every $i \in \{1, \ldots, k\}$.

It is known that the sequence involved in the definition of ASQ can be chosen to be weakly-null [3, Theorem 2.8], so ASQ implies WASQ which in turn implies LASQ. Moreover, ASQ implies the SD2P, WASQ implies the D2P, and LASQ implies the LD2P (see [3], but note that the latter two statements were proved in [22]).

Diameter two properties of (symmetric) tensor products, both injective and projective, have attracted the attention of many researchers. In [4, Question (b)], it was explicitly posed as an open question how diameter two properties are preserved by tensor products. Partial answers, which strongly rely on infinite-dimensional centralizers, appeared in [5] and [8]. For instance, it is known that the projective tensor product of a $C(K)$ space with any non-zero Banach space has the D2P [8, Theorem 4.1] and that the symmetric projective tensor product of any $L_1(\mu)$ space has the D2P [5, Theorem 3.3]. However, the assumption of having infinite-dimensional centralizer have been shown to be far from necessary. In the symmetric case it has been recently proved that the symmetric projective tensor product of any ASQ space has the SD2P [13, Theorem 3.3]. Furthermore, in the non-symmetric case there are even stability results for some diameter two properties, e.g. both the LD2P and the SD2P are stable by taking projective tensor products (see [4, Theorem 2.7] and [11, Corollary 3.6]). In spite of the previous nice results, the interplay between diameter two properties, octahedrality, and almost squareness with respect to tensor products is currently not well understood. Thus, the aim of this note is to go further and study octahedrality and almost squareness and their relations to (symmetric) tensor products equipped with the injective or projective norm.

In Section 2 we start by showing that octahedrality is stable by taking injective tensor products, see Theorem 2.2 and Corollary 2.4. Our results on octahedrality of injective tensor products are refinements of the results of [11], where octahedrality of spaces of operators was deeply studied. Next we turn to almost squareness. A consequence of Theorem 2.6 is that the injective tensor product of an ASQ space with any non trivial Banach space is ASQ. The projective norm is much more difficult to work with, but we are able to show that the projective tensor product of c_0 with any non-zero Banach space is LASQ in Proposition 2.10.

In Section 3 we study the symmetric tensor products. Our main result is Theorem 3.1 which is a stability result of octahedrality for injective symmetric tensor products. It will allow us to significantly improve some results from [6, Section 4] (see Corollary 3.2). Combining WASQ and the Dunford-Pettis property we get LD2P of the projective symmetric tensor product in Proposition 3.6. We will also give an inheritance result for WASQ spaces which, combined with the above result, will result in a criterion for the symmetric projective tensor product of a subspace of $L_1(\mu)$ to have the LD2P.

We close the paper with some open questions in Section 4.

We use standard Banach space notation, as can be found in e.g. [9]. Given Banach spaces X and Y, $L(X,Y)$ (respectively $K(X,Y)$) will denote the space of bounded linear operators (respectively compact operators) from X to Y. We will consider only non-zero Banach spaces.

2. Tensor product spaces

Recall that given two Banach spaces X and Y, the *injective tensor product* of X and Y, denoted by $X\widehat{\otimes}_\varepsilon Y$, is the completion of $X\otimes Y$ under the norm given by

$$\|u\| := \sup\left\{\sum_{i=1}^{n} |x^*(x_i)y^*(y_i)| : x^*\in S_{X^*}, y^*\in S_{Y^*}\right\},$$

where $u := \sum_{i=1}^{n} x_i\otimes y_i$. Every $u\in X\widehat{\otimes}_\varepsilon Y$ can be viewed as an operator $T_u : X^*\to Y$ which is weak*-to-weakly continuous. In particular, $y^*\circ T_u$ belongs to X for all $y^*\in Y^*$ (see [25] for background).

Quite a lot is known about octahedrality of spaces of operators. It is essentially known that the injective tensor product $X\widehat{\otimes}_\varepsilon Y$ of two Banach spaces is LOH if one of the spaces is LOH (see the comment following Lemma 2.3 in [11] or [23, Theorem 3.39]). Hence it is natural to wonder when an injective tensor product is OH. Before we can state our results about octahedrality of $X\widehat{\otimes}_\varepsilon Y$ we will need to introduce a bit of notation. Given a Banach space X and a norm one element u in X, define

$$D(X,u) := \{f\in B_{X^*} : f(u) = 1\}.$$

Define $n(X,u)$ as the largest non-negative real number k satisfying

$$k\|x\| \leq \sup\{|f(x)| : f \in D(X,u)\}$$

for all $x \in X$ (see the discussion following Theorem 3.6 in [8] for examples of spaces which have such unitaries). Note that $n(X,u) = 1$ if, and only if, $D(X,u)$ is a norming subset for X. We will also need the following geometrical characterization of octahedrality that is proved in [23].

Theorem 2.1 ([23, Theorem 3.21])**.** *Let X be a Banach space. X is OH if, and only if, whenever E is a finite-dimensional subspace of X, $x_1^*, \ldots, x_n^* \in B_{X^*}$, $\varepsilon > 0$ and $0 < \varepsilon_0 < \varepsilon$ there exists $y \in S_X$ such that, whenever $|\gamma_i| \leq 1 + \varepsilon_0$, there exists $y_i^* \in X^*$ satisfying $\|y_i^*\| \leq 1 + \varepsilon$ for all $i \in \{1, \ldots, n\}$, $y_i^*|_E = x_i^*|_E$, and $y_i^*(y) = \gamma_i$.*

Our first theorem is a version of [11, Theorem 3.5] and [11, Theorem 3.1] stated in the context of weak*-to-weakly continuous operators. We include a proof that uses Theorem 2.1 instead of working with the w^*-SD2P of the dual space.

Theorem 2.2. *Let X and Y be Banach spaces and let $H \subseteq L(X^*, Y)$ be a closed subspace such that $X \otimes Y \subseteq H$. Assume that each $T \in H$ is weak*-to-weakly continuous.*

(i) If X and Y are OH, then H is OH.
(ii) If X is OH and there exists $y \in S_Y$ such that $n(Y,y) = 1$, then H is OH.

Proof. (i). Let $T_1, \ldots, T_k \in S_H$ and $\varepsilon > 0$. For each i find $y_i^* \in S_{Y^*}$ and $x_i^* \in S_{X^*}$ such that

$$y_i^*(T_i x_i^*) = x_i^*(T_i^* y_i^*) > 1 - \varepsilon.$$

Let $E = \operatorname{span}\{T_i^* y_i^* : i \in \{1, \ldots, k\}\} \subset X$. By Theorem 2.1 we find $w \in S_X$ and $w_i^* \in X^*$, $i \in \{1, \ldots, k\}$, such that $w_i^*(T_i^* y_i^*) = x_i^*(T_i^* y_i^*)$, $w_i^*(w) = 1$ and $\|w_i^*\| \leq 1 + \varepsilon$.

Let $F = \operatorname{span}\{T_i w_i^* : i \in \{1, \ldots, k\}\} \subset Y$. Using Theorem 2.1 again we find $z \in S_Y$ and $z_i^* \in Y^*$, $i \in \{1, \ldots, k\}$, such that $z_i^*(T_i w_i^*) = y_i^*(T_i w_i^*)$, $z_i^*(z) = 1$ and $\|z_i^*\| \leq 1 + \varepsilon$.

Define $S = w \otimes z \in X \otimes Y$. We have $S \in S_H$ and

$$\begin{aligned}\|T_i + S\| &\geq \frac{1}{(1+\varepsilon)^2} z_i^*(T_i w_i^* + S w_i^*) = \frac{1}{(1+\varepsilon)^2}(y_i^*(T_i w_i^*) + w_i^*(w) z_i^*(z)) \\ &= \frac{1}{(1+\varepsilon)^2}(y_i^*(T_i x_i^*) + 1) > \frac{2-\varepsilon}{(1+\varepsilon)^2}.\end{aligned}$$

Consequently H is OH.

(ii). Let $T_1, \ldots, T_k \in S_H$ and $\varepsilon > 0$. For each i find $y_i^* \in S_{Y^*}$ and $x_i^* \in S_{X^*}$ such that

$$y_i^*(T_i x_i^*) = x_i^*(T_i^* y_i^*) > 1 - \varepsilon.$$

By assumption $D(Y,y) = \{y^* \in S_{Y^*} : y^*(y) = 1\}$ is norming for Y so we may assume that $y_i^*(y) = 1$ for each $i \in \{1, \dots, n\}$. We now proceed as in (i) by finding $w \in S_X$ and $w_i^* \in X^*$. We can then use $z_i^* = y_i^*$ and $z = y$ to conclude the proof as above. □

Remark 2.3. The assumption that every operator in H is weak*-to-weakly continuous is necessary in the second statement of Theorem 2.2. Indeed, $Z := L(\mathcal{C}[0,1]^*, \ell_\infty) = L(\ell_1, \mathcal{C}[0,1]^{**})$ is not OH from [11, Corollary 3.12].

Since $H = X \widehat{\otimes}_\varepsilon Y$ is a subspace of $L(X^*, Y)$, Theorem 2.2 says the following about octahedrality of injective tensor products.

Corollary 2.4. *Let X and Y be Banach spaces.*

(i) If both X and Y are OH, then $X \widehat{\otimes}_\varepsilon Y$ is OH.

(ii) If X is OH and there exists $y \in S_Y$ such that $n(Y, y) = 1$, then $X \widehat{\otimes}_\varepsilon Y$ is OH.

Let us also note the following necessary conditions for injective tensor product spaces to be OH, WOH or LOH. Proof of the below proposition follows the ideas in [11, Proposition 3.9] and [23, Theorem 3.46] and will not be included. A Banach space has a *non-rough norm* if the dual unit ball has weak* slices of arbitrarily small diameter (see [15, Proposition II.1.11]).

Proposition 2.5. *Let X and Y be Banach spaces. Assume that Y has a non-rough norm.*

(i) If $X \widehat{\otimes}_\varepsilon Y$ is LOH, then X is LOH.

(ii) If $X \widehat{\otimes}_\varepsilon Y$ is WOH, then X is WOH.

(iii) If $X \widehat{\otimes}_\varepsilon Y$ is OH, then X is OH.

From octahedrality we now turn to almost squareness and diameter two properties. Acosta, Becerra Guerrero and Rodríguez-Palacios have shown that if Y is a Banach space, then $X \widehat{\otimes}_\varepsilon Y$ has the D2P whenever X is a Banach space such that the supremum of the dimension of the centralizer of all the even duals of X is unbounded [8, Theorem 5.3]. Now, we prove a stability result for ASQ, which will provide a wide class of injective tensor product spaces which have the SD2P.

Theorem 2.6. *Let X and Y be Banach spaces. Assume that X is ASQ.*

(i) If $H \subseteq K(Y, X)$ is a closed subspace with $Y^* \otimes X \subset H$, then H is ASQ.

(ii) If $H \subseteq K(Y^*, X)$ is a closed subspace with $Y \otimes X \subset H$, then H is ASQ.

Proof. We will prove only (ii). The proof of (i) is similar.

Let $T_1, \dots, T_k \in S_H$ and $\varepsilon > 0$. The set

$$K = \cup_{i=1}^k T_i(B_{Y^*})$$

is a relatively compact subset of B_X hence there exists $x_1, \dots, x_n \in B_X$ such that $K \subset \cup_{i=1}^n B(x_i, \varepsilon)$. Define

$$E := \operatorname{span}\{x_i : i \in \{1, \dots, n\}\}.$$

As E is a finite-dimensional subspace of X and X is ASQ there exists $z \in S_X$ such that

$$\|e + \lambda z\| \leq (1 + \varepsilon) \max\{\|e\|, |\lambda|\}$$

for all $e \in E$ and $\lambda \in \mathbb{R}$ by [3, Theorem 2.4]. Pick $y \in S_Y$, and define $S := z \otimes y \in S_H$. For each T_i and $y^* \in B_{Y^*}$ there exists j such that $\|T_i y^* - x_j\| < \varepsilon$, so

$$\begin{aligned}\|(T_i + S)y^*\| &\leq \|T_i y^* - x_j\| + \|x_j + y^*(y)z\| \\ &\leq \varepsilon + (1 + \varepsilon) \max\{\|x_j\|, |y^*(y)|\} \leq 1 + 2\varepsilon.\end{aligned}$$

Taking supremum over all such y^* we get $\|T_i + S\| \leq 1 + 2\varepsilon$, which means that H is ASQ [3, Proposition 2.1]. □

Example. The converse of Theorem 2.6 does not hold. Indeed, given $1 < p \leq q < \infty$ then $K(\ell_p, \ell_q)$ is an M-ideal in its bidual $L(\ell_p, \ell_q)$ [20, Example VI.4.1] and, consequently, it is ASQ [3, Corollary 4.3]. Thus, $K(Y, X)$ can be ASQ even if both X and Y are uniformly convex.

Corollary 2.7. *Let X and Y be Banach spaces. If X is ASQ, then $X \widehat{\otimes}_\varepsilon Y$ is ASQ. In particular, $X \widehat{\otimes}_\varepsilon Y$ has the SD2P.*

Using the above corollary we can characterize the $\mathcal{C}(K, X)$ spaces which are ASQ.

Corollary 2.8. *Let K be compact Hausdorff space and X be a Banach space. The following assertions are equivalent:*

(i) $\mathcal{C}(K, X)$ is ASQ.
(ii) X is ASQ.

Proof. (i) $\Rightarrow$ (ii). Pick $x_1, \dots, x_n \in S_X$ and $\varepsilon > 0$, and let us find $x \in S_X$ such that $\|x_i \pm x\| \leq 1 + \varepsilon$ for all $i \in \{1, \dots, n\}$. For each $i \in \{1, \dots, n\}$ define

$$f_i(t) = x_i \text{ for all } t \in K,$$

which is an element of $S_{\mathcal{C}(K,X)}$. Since $\mathcal{C}(K, X)$ is ASQ there exists $f \in S_{\mathcal{C}(K,X)}$ such that $\|f_i \pm f\| \leq 1 + \varepsilon$ for all $i \in \{1, \dots, n\}$. Pick $t \in K$ such that $f(t) \in S_X$, and define $x := f(t)$. Now

$$\|x_i \pm x\| = \|f_i(t) \pm f(t)\| \leq \|f_i \pm f\| \leq 1 + \varepsilon \text{ for all } i \in \{1, \dots, n\}.$$

So X is ASQ.

(ii) $\Rightarrow$ (i). As $\mathcal{C}(K, X) = \mathcal{C}(K) \widehat{\otimes}_\varepsilon X$, we get the desired result from Corollary 2.7. □

Next we turn to projective tensor products. Given two Banach spaces X and Y, recall that the *projective tensor product* of X and Y, denoted by $X\widehat{\otimes}_\pi Y$, is the completion of $X \otimes Y$ under the norm given by

$$\|u\| := \inf\left\{\sum_{i=1}^n \|x_i\|\|y_i\| : u = \sum_{i=1}^n x_i \otimes y_i\right\}.$$

It is known that $B_{X\widehat{\otimes}_\pi Y} = \overline{\operatorname{conv}}(B_X \otimes B_Y) = \overline{\operatorname{conv}}(S_X \otimes S_Y)$ [25, Proposition 2.2]. Moreover, given Banach spaces X and Y, it is well known that $(X\widehat{\otimes}_\pi Y)^* = L(X, Y^*)$ (see [25] for background).

From Corollary 2.7 we get examples of projective tensor products that are octahedral.

Corollary 2.9. *Let X and Y be Banach spaces. Assume that X is ASQ and Y is Asplund. If either X^* or Y^* has the approximation property, then $X^*\widehat{\otimes}_\pi Y^*$ is OH.*

Proof. By assumption, we have $X^*\widehat{\otimes}_\pi Y^* = (X\widehat{\otimes}_\varepsilon Y)^*$ (cf. e.g. [25, Theorem 5.33]). The result follows from Corollary 2.7. □

As noted in the Introduction, if a Banach space is LASQ, then it has the LD2P. Given two Banach spaces X and Y it is known that $X\widehat{\otimes}_\pi Y$ has the LD2P whenever X has the LD2P (see for example [4, Theorem 2.7]). Furthermore, $X\widehat{\otimes}_\pi Y$ has the SD2P whenever both X and Y have the SD2P [11, Corollary 3.6]. The next proposition gives us examples of projective tensor product spaces which are LASQ.

Proposition 2.10. *If X is a Banach space, then $c_0\widehat{\otimes}_\pi X$ is LASQ. Moreover, $c_0\widehat{\otimes}_\pi X$ has the D2P.*

Proof. Let $Y := c_0\widehat{\otimes}_\pi X$. Let $u \in S_Y$ and $\varepsilon > 0$. Since $\operatorname{conv}(S_{c_0} \otimes S_X)$ is dense in B_Y we can find $v = \sum_{i=1}^n \lambda_i z_i \otimes x_i$ such that $\|u - v\| \leq \varepsilon$, where $z_i \in S_{c_0}$, $x_i \in S_X$ and $\lambda_i \geq 0$ for each $i \in \{1, \dots, n\}$ with $\sum_{i=1}^n \lambda_i = 1$. In particular, $\|v\| \geq 1 - \varepsilon$. We may assume that $z_1, \dots, z_n$ have finite support, so we can find $m \in \mathbb{N}$ such that $n \geq m$ implies $z_i(n) = 0$. Define

$$\tilde{z}_i := \sum_{j=1}^{m-1} z_i(j) e_{m+j} \quad \text{and} \quad w := \sum_{i=1}^n \lambda_i \tilde{z}_i \otimes x_i.$$

We have

$$\|v \pm w\| \leq \sum_{i=1}^n \lambda_i \|z_i \pm \tilde{z}_i\| \|x_i\| \leq 1$$

because $\|z_i \pm \tilde{z}_i\| = \max\{\|z_i\|, \|\tilde{z}_i\|\} = 1$ since z_i and $\tilde{z}_i$ have disjoint support for each $i \in \{1, \dots, n\}$. Let θ be a permutation of $\mathbb{N}$ that swaps k and $m + k$ for $k \in \{1, \dots, m - 1\}$. Let Φ_θ be the isometry on c_0 defined by θ. We have $\Phi_\theta(v) = w$ and if $T \in L(X, \ell_1) = Y^*$, then

$$\langle \Phi_\theta^* T, v\rangle = \langle T, \Phi_\theta v\rangle = \langle T, w\rangle.$$

It follows that $\|w\| = \|v\|$ and $\left\| u \pm \frac{w}{\|w\|} \right\| \leq 1 + 2\varepsilon$, hence Y is LASQ [3, Proposition 2.1].

Finally, let us prove that Y has the D2P. Let $u_0 \in B_Y$, $T_1, \ldots, T_k \in Y^*$ and $\alpha > 0$. Consider the relatively weakly open neighborhood

$$\mathcal{U} = \{y \in B_Y : |\langle y - u_0, T_j \rangle| < \alpha,\ j \in \{1, \ldots, k\}\}.$$

Then $\mathcal{U} \cap S_Y \neq \emptyset$. Let u from the first part of the proof be in $\mathcal{U}$. With ε small enough we may assume that v is also in $\mathcal{U}$. Now $\{T_j x_i\}$, $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, k\}$, is a finite set of elements in ℓ_1 so, replacing e_{m+j} with e_{N+j} for N big enough in the definition of $\tilde{z}_i$, we can make

$$\langle w, T_j \rangle = \sum_{i=1}^{n} \lambda_i \langle \tilde{z}_i, T_j x_i \rangle$$

as small as we wish. Consequently, we may assume that $v \pm w \in \mathcal{U}$. Since

$$\|v + w - (v - w)\| = \|2w\| \geq 2 - 2\varepsilon,$$

we conclude that Y has the D2P. $\square$

Proposition 2.11. *Let X and Y be Banach spaces. Assume that there exists $f \in S_{Y^*}$ such that $n(Y^*, f) = 1$. If X is ASQ, then $X \widehat{\otimes}_\pi Y$ is LASQ.*

Proof. Denote by $Z := X \widehat{\otimes}_\pi Y$. Let $u \in S_Z$ and $\varepsilon > 0$. Since $\operatorname{conv}(S_X \otimes S_Y)$ is dense in S_Z we can find $v = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i$ such that $\|u - v\| \leq \varepsilon$, where $x_i \in S_X, y_i \in S_Y$ and $\lambda_i \geq 0$ for each $i \in \{1, \ldots, n\}$ with $\sum_{i=1}^{n} \lambda_i = 1$. Since $n(Y^*, f) = 1$ we may assume (see [8, Corollary 3.5]) that $f(y_i) = 1$ for each $i \in \{1, \ldots, n\}$, and hence $\|\sum_{i=1}^{n} \lambda_i y_i\| = 1$.

As X is ASQ we can find $x \in S_X$ such that $\|x_i \pm x\| \leq 1 + \varepsilon$ for all $i \in \{1, \ldots, n\}$. Define $z := x \otimes \sum_{i=1}^{n} \lambda_i y_i$, which is a norm one element. Now

$$\|u \pm z\| \leq \|u - v\| + \|v \pm z\| \leq \varepsilon + \sum_{i=1}^{n} \lambda_i \|x_i \pm x\| \|y_i\| < 1 + 2\varepsilon.$$

We conclude that Z is LASQ [3, Proposition 2.1]. $\square$

Remark 2.12. From [11, Theorem 3.1] it is immediate that $X \widehat{\otimes}_\pi Y$ has the SD2P in the above proposition. However, we can not conclude that $X \widehat{\otimes}_\pi Y$ is ASQ. Indeed, let X be ASQ and $Y = \ell_1^n$. Then $X \widehat{\otimes}_\pi Y = \ell_1^n(X)$ is not ASQ [3, Lemma 5.5]. Given a Banach space X it is not even clear whether $X \widehat{\otimes}_\pi Y$ can be ASQ for any Banach space Y with $\dim(Y) \geq 2$.

3. Symmetric tensor product spaces

Given $N \in \mathbb{N}$, the *(N-fold) symmetric tensor product*, $\otimes^{s,N} X$, of a Banach space X is the linear span of the tensors $x^N := x \otimes \cdots \otimes x$ in the N-fold tensor product $\otimes^N X$. The *(N-fold) injective symmetric* *tensor product* of X, denoted by $\widehat{\otimes}_{\varepsilon,s,N}X$, is the completion of the space $\otimes^{s,N}X$ under the norm

$$\|u\| := \sup\left\{\left|\sum_{i=1}^{n} x^*(x_i)^N\right| : x^* \in B_{X^*}\right\},$$

where $u := \sum_{i=1}^{n} x_i^N$ (cf. e.g. [18]). Note that $(\widehat{\otimes}_{\varepsilon,s,N}X)^* = \mathcal{P}_I(^N X)$, the Banach space of N-homogeneous integral polynomials on X.

We have seen in Theorem 2.2 (i) that octahedrality is stable for non-symmetric injective tensor products. Now we shall establish the analogous result for the symmetric case.

Theorem 3.1. *Let X be a Banach space and $N \in \mathbb{N}$. If X is OH, then $\widehat{\otimes}_{\varepsilon,s,N}X$ is OH.*

Proof. Denote by $Y := \widehat{\otimes}_{\varepsilon,s,N}X$. Pick $u_1, \ldots, u_k \in S_Y$ and $\varepsilon > 0$. Assume, with no loss of generality, that $u_i := \sum_{j=1}^{n_i} x_{ij}^N$ for each $i \in \{1, \ldots, k\}$. For each $i \in \{1, \ldots, k\}$ pick $x_i^* \in S_{X^*}$ such that $\sum_{j=1}^{n_i} x_i^*(x_{ij})^N > 1 - \varepsilon$. As X is OH we can ensure from Theorem 2.1 the existence of $y \in S_X$ and $y_1^*, \ldots, y_k^* \in X^*$ such that

$$y_i^*(x_{ij}) = x_i^*(x_{ij}) \text{ for all } j \in \{1, \ldots, n_i\},$$
$$y_i^*(y) = 1 \text{ and } \|y_i^*\| \leq 1 + \varepsilon \text{ for all } i \in \{1, \ldots, k\}.$$

Define $u := y^N \in S_Y$. Now, given $i \in \{1, \ldots, k\}$, one has

$$\|u_i + u\| \geq \frac{\sum_{j=1}^{n_i} y_i^*(x_{ij})^N + y_i^*(y)^N}{1+\varepsilon} = \frac{\sum_{j=1}^{n_i} x_i^*(x_{ij})^N + 1}{1+\varepsilon} > \frac{2-\varepsilon}{1+\varepsilon}.$$

As $\varepsilon > 0$ was arbitrary we conclude that Y is OH, as desired. □

Given a Banach space X, denote by $X^{(\infty}$ the completion of the linear space given by the union of all even duals of X. A linear projection P on X is an *L-projection* if

$$\|x\| = \|Px\| + \|x - Px\|$$

for all $x \in X$. The closed linear span of the set of L-projections on X is called the *Cunningham algebra* of X, denoted by $\mathrm{Cun}(X)$ (cf. e.g. [20, p. 46]). The above theorem allows us to improve [6, Theorem 4.2], where Acosta and Becerra Guerrero obtained w^*-D2P of spaces of homogeneous integral polynomials when the Cunningham algebra $\mathrm{Cun}(X^{(\infty})$ of $X^{(\infty}$ is infinite-dimensional. As $\dim \mathrm{Cun}(X^{(\infty}) = \infty$ implies that X (and even X^{**}) is OH (this follows from [7, Theorem 3.4]), we get the following result from Theorem 3.1.

Corollary 3.2. *Let X be a Banach space. If $\mathrm{Cun}(X^{(\infty})$ is infinite-dimensional, then $\mathcal{P}_I(^N X)$ has the w^*-SD2P for each $N \in \mathbb{N}$.*

Next we will weaken the assumptions of Theorem 3.1 and get examples of Banach spaces whose injective symmetric tensor products are locally octahedral. The proof relies on the following geometrical characterization of weak octahedrality, whose proof can be found in [19].

Theorem 3.3 ([19, Theorem 2.6]). *Let X be a Banach space. Then X is WOH if, and only if, for every finite-dimensional subspace E of X, $x^* \in B_{X^*}$, and $\varepsilon > 0$, there are $y \in S_X$ and $x_1^*, x_2^* \in X^*$ with $\|x_1^*\|, \|x_2^*\| \leq 1+\varepsilon$ satisfying $x_1^*|_E = x_2^*|_E = x^*|_E$ and $x_1^*(y) - x_2^*(y) > 2-\varepsilon$.*

Proposition 3.4. *Let X be a Banach space and let N be an odd number. If X is WOH, then $\widehat{\otimes}_{\varepsilon,s,N}X$ is LOH. In particular, $\mathcal{P}_I(^N X)$ has the w^*-LD2P.*

Proof. Denote by $Y := \widehat{\otimes}_{\varepsilon,s,N}X$. Consider $u := \sum_{i=1}^k x_i^N \in S_Y$ and $\varepsilon > 0$. It is enough to prove the existence of $v \in S_Y$ such that $\|u \pm v\| > \frac{1-\varepsilon+(1-\varepsilon)^N}{1+\varepsilon}$. To this aim consider $x^* \in S_{X^*}$ such that $1-\varepsilon < x^*(u) = \sum_{i=1}^n x^*(x_i)^N$.

Using Theorem 3.3 we can find $x_1^*, x_2^* \in X^*$ and $y \in S_X$ such that

$$\begin{aligned}
&x_1^*(x_i) = x_2^*(x_i) = x^*(x_i) \text{ for all } i \in \{1, \dots, k\},\\
&x_1^*(y) - x_2^*(y) > 2-\varepsilon, \text{ and}\\
&\|x_1^*\| \leq 1+\varepsilon, \|x_2^*\| \leq 1+\varepsilon.
\end{aligned}$$

Now define $v := y^N \in S_Y$. From $x_1^*(y) - x_2^*(y) > 2-\varepsilon$ we conclude that $x_1^*(y) > 1-\varepsilon$ and $-x_2^*(y) > 1-\varepsilon$. Consequently

$$\begin{aligned}
\|u+v\| &\geq \frac{\sum_{i=1}^k x_1^*(x_i)^N + x_1^*(y)^N}{1+\varepsilon} > \frac{\sum_{i=1}^k x^*(x_i)^N + (1-\varepsilon)^N}{1+\varepsilon}\\
&> \frac{1-\varepsilon+(1-\varepsilon)^N}{1+\varepsilon}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\|u-v\| &\geq \frac{\sum_{i=1}^k x_2^*(x_i)^N - x_2^*(y)^N}{1+\varepsilon} > \frac{\sum_{i=1}^k x^*(x_i)^N + (1-\varepsilon)^N}{1+\varepsilon}\\
&> \frac{1-\varepsilon+(1-\varepsilon)^N}{1+\varepsilon}.
\end{aligned}$$

Hence we have $\|u \pm v\| > \frac{1-\varepsilon+(1-\varepsilon)^N}{1+\varepsilon}$, so Y is LOH ☐

Remark 3.5. We do not know whether one can get WOH in the above proposition. Neither do we know whether the above proposition holds for an even number N.

We pass now to projective symmetric tensor products. Given a Banach space X, we define the *(N-fold) projective symmetric tensor product* of X, denoted by $\widehat{\otimes}_{\pi,s,N}X$, as the completion of the space $\otimes^{s,N}X$ under the norm

$$\|u\| := \inf\left\{\sum_{i=1}^n |\lambda_i|\|x_i\|^N : u := \sum_{i=1}^n \lambda_i x_i^N, n \in \mathbb{N}, x_i \in X\right\}.$$

The dual, $(\widehat{\otimes}_{\pi,s,N}X)^* = \mathcal{P}(^N X)$, is the Banach space of N-homogeneous continuous polynomials on X (see [18] for background).

It is known that $\widehat{\otimes}_{\pi,s,N}X$ has the SD2P provided the Banach space X is ASQ [13, Theorem 3.3]. The proof of this result relies heavily on the fact that the sequences involved in the definition of ASQ can be chosen to be c_0-sequences (see [3, Lemma 2.6]). If one tries to copy the idea of the ASQ proof in the WASQ setting it does not work because WASQ Banach spaces do not have to contain any isomorphic copy of c_0 [3, Proposition 3.5]. Consequently, in order to connect WASQ spaces with diameter two properties in projective symmetric tensor products, we shall need an extra assumption.

A Banach space X has the *Dunford-Pettis property* if every weakly compact operator $T : X \longrightarrow Y$ is weak-to-norm sequentially continuous, i.e., whenever $x_n \to x$ in X weakly then $Tx_n \to Tx$ in norm in Y (cf. e.g. [9, Proposition 5.4.2]). It is known that every continuous polynomial on a Banach space having the Dunford-Pettis property is weakly sequentially continuous [16, Proposition 2.34].

Proposition 3.6. *Let X be a Banach space with the Dunford-Pettis property. If X is WASQ, then $\widehat{\otimes}_{\pi,s,N}X$ has the LD2P for each $N \in \mathbb{N}$.*

Proof. Let $N \in \mathbb{N}$. It is enough to prove that $\left(\widehat{\otimes}_{\pi,s,N}X\right)^* = \mathcal{P}(^N X)$ is LOH. To this aim pick $P \in S_{\mathcal{P}(^N X)}$ and $\varepsilon > 0$. Pick $x \in S_X$ such that $P(x) > 1 - \varepsilon$.

Since X is WASQ we can find a sequence $\{y_n\}$ in B_X such that $y_n \to 0$ weakly, $\|y_n\| \to 1$ and $\|x \pm y_n\| \to 1$. As $\{y_n\}$ is a weakly-null sequence and X has the Dunford-Pettis property we conclude that $P(y_n) \to 0$ [16, Proposition 2.34]. Consequently $P(x \pm y_n) \to P(x) > 1 - \varepsilon$ [17, Lemma 1.1]. So we can find $n \in \mathbb{N}$ big enough to ensure

$$P(x \pm y_n) > 1 - \varepsilon,$$

$\|x \pm y_n\| \leq 1 + \varepsilon$ and $\|y_n\| > 1 - \varepsilon$. Choose $f \in S_{X^*}$ such that $f(y_n) > 1 - \varepsilon$. Now

$$1 + \varepsilon \geq \|y_n \pm x\| \geq f(y_n \pm x) \geq 1 - \varepsilon \pm f(x),$$

which implies that $|f(x)| < 2\varepsilon$.

Now we shall argue by cases: If N is odd, define $Q(x) := f(x)^N$ for each $x \in X$, which is a norm one N-homogeneous polynomial. Now

$$\begin{aligned}
\|P \pm Q\| \geq (P \pm Q)\left(\frac{x \pm y_n}{\|x \pm y_n\|}\right) &= \frac{P(x \pm y_n) \pm Q(x \pm y_n)}{\|x \pm y_n\|^N} \\
> \frac{1 - \varepsilon \pm (f(x) \pm f(y))^N}{(1+\varepsilon)^N} &= \frac{1 - \varepsilon + (f(y) \pm f(x))^N}{(1+\varepsilon)^N} \\
> \frac{1 - \varepsilon - (1 - 3\varepsilon)^N}{(1+\varepsilon)^N}.&
\end{aligned}$$

If N is even, pick $g \in S_{X^*}$ such that $g(x) = 1$, which implies that $|g(y_n)| < \varepsilon$. Define $Q(z) = g(z)f(z)^{N-1}$ for all $z \in X$, which is a norm one polynomial. Then

$$\begin{aligned}\|P \pm Q\| &\geq (P \pm Q)\left(\frac{x \pm y_n}{\|x \pm y_n\|}\right) = \frac{P(x \pm y_n) \pm Q(x \pm y_n)}{\|x \pm y_n\|^N} \\ &> \frac{1 - \varepsilon \pm (g(x) \pm g(y))(f(x) \pm f(y))^{N-1}}{(1+\varepsilon)^N}.\end{aligned}$$

Similar estimates to the ones above allow us to conclude that

$$\frac{1 - \varepsilon \pm (g(x) \pm g(y))(f(x) \pm f(y))^{N-1}}{(1+\varepsilon)^N} > \frac{1 - \varepsilon + (1-\varepsilon)(1-3\varepsilon)^{N-1}}{(1+\varepsilon)^N}.$$

In any case, as $\varepsilon > 0$ was arbitrary, we conclude that $\mathcal{P}(^N X)$ is LOH. $\square$

In order to exhibit examples of Banach spaces where the above proposition applies we shall prove the following result about inheritance of WASQ to subspaces. Note that it extends the result for the D2P from [14, Theorem 2.2], where it is proved that D2P is inherited by finite codimensional subspaces. In addition, the theorem below seems to be the only known result about inheritance of WASQ by subspaces.

Theorem 3.7. *Let X be WASQ and let $Y \subseteq X$ be a closed subspace. If X/Y has the Schur property, then Y is WASQ.*

Proof. Pick $y \in S_Y \subseteq S_X$. As X is WASQ we can find $\{x_n\}$ a sequence in B_X such that $\|y \pm x_n\| \to 1$, $\|x_n\| \to 1$, and $x_n \to 0$ weakly.

Consider the quotient map $\pi : X \longrightarrow X/Y$, which is a weak-to-weakly continuous map. We get $\pi(x_n) \to 0$ weakly and, by the Schur property of X/Y, $\pi(x_n) \to 0$ even in norm. Hence, for each $n \in \mathbb{N}$, we can find $y_n \in B_Y$ such that

$$\|y_n - x_n\| < \|\pi(x_n)\| + \frac{1}{n}.$$

We shall prove that $\{y_n\} \subseteq B_Y$ satisfies our requirements. On the one hand

$$1 \geq \|y_n\| \geq \|x_n\| - \|y_n - x_n\| \text{ for all } n \in \mathbb{N},$$

so $\|y_n\| \to 1$. On the other hand

$$\|y \pm x_n\| - \|x_n - y_n\| \leq \|y \pm y_n\| \leq \|y \pm x_n\| + \|x_n - y_n\|$$

holds for each $n \in \mathbb{N}$, hence $\|y \pm y_n\| \to 1$. Finally, as $x_n \to 0$ weakly and $x_n - y_n \to 0$ weakly (even in norm) we conclude that $y_n \to 0$ weakly. By definition, Y is WASQ, as desired. $\square$

Remark 3.8. It is clear that the above proof also shows that given a Banach space X which is ASQ and $Y \subseteq X$ a closed subspace such that X/Y has the Schur property, then Y is ASQ. However, this can be deduced from a more general result [1, Theorem 3.6], where it is proved that ASQ is inherited by closed subspaces whose quotient space does not contain any isomorphic copy of c_0.

In [5] it was shown that the N-fold projective symmetric tensor product of $L_1(\mu)$, μ a σ-finite measure, has the LD2P for each $N \in \mathbb{N}$. Now we can combine Proposition 3.6 and Theorem 3.7 to extend Theorem 3.1 of that paper, by considering some closed subspaces of $L_1(\mu)$. The proof relies on the fact that $L_1(\mu)$ is WASQ, a fact that can be proved using the ideas in [22, Lemma 3.3].

Corollary 3.9. *Let X be a complemented subspace of $L_1(\mu)$, where (Ω, Σ, μ) is a measure space and μ is a σ-finite measure. If $L_1(\mu)/X$ is isomorphic to ℓ_1 then $\widehat{\otimes}_{\pi,s,N}X$ has the LD2P for each $N \in \mathbb{N}$.*

Proof. As $L_1(\mu)/X$ is isomorphic to ℓ_1 then $L_1(\mu)/X$ has the Schur property. Consequently, X is WASQ by Theorem 3.7. In addition, X has the Dunford-Pettis property because $L_1(\mu)$ has the Dunford-Pettis property [9, Theorem 5.4.5] and X is complemented in $L_1(\mu)$. So the result holds as an immediate application of Proposition 3.6. □

Remark 3.10. Note that given $L_1(\mu)$ as in the above corollary and given $X \subseteq L_1(\mu)$ an infinite-dimensional and complemented subspace, there are conditions on $L_1(\mu)/X$ which guarantee that it is isomorphic to ℓ_1 such as having an unconditional basis or the Radon-Nikodým property (see [9, p. 122]). Moreover, it is conjectured that each infinite-dimensional complemented subspace of $L_1(\mu)$ is isomorphic either to $L_1(\mu)$ or to ℓ_1 (see e.g. [9, Conjecture 5.6.7]). If this conjecture were correct, the above corollary would have a wide range of applications.

4. Some remarks and open questions

In this section we will pose some open questions related to our main results. In light of Theorem 2.2 the following question arises.

Question 4.1. *Let X and Y be Banach space and $H \subseteq L(X^*, Y)$ a closed subspace such that $X \otimes Y \subseteq H$ and that each element of H is weak*-to-weakly continuous. Is H OH whenever X is OH?*

Theorem 2.2 (ii) provides a partial positive answer. On the other hand, in [21, Theorem 4.2] an example is given of a complex two-dimensional Banach space E such that $L_1^{\mathbb{C}}([0,1])\widehat{\otimes}_\varepsilon E$ fails to have the Daugavet property and, consequently, is a natural candidate for a negative answer to the above question.

In Section 2 we saw conditions on Banach spaces X and Y which ensure that certain subspaces of $L(X^*, Y)$ are LOH or OH. So it is natural to wonder

Question 4.2. *Let X and Y be Banach spaces and $H \subseteq L(X^*, Y)$ be a closed subspace such that $X \otimes Y \subseteq H$. When is H WOH?*

We note that from Proposition 2.10 we get that $L(X, \ell_1) = (c_0\widehat{\otimes}_\pi X)^*$ is WOH for any Banach space X. Similarly, for any K infinite compact Hausdorff space, $L(X, C(K)^*) = (C(K)\widehat{\otimes}_\pi X)^*$ is WOH for any Banach space X by [8, Theorem 4.1].

Corollary 2.7 says that the injective tensor product is ASQ provided one of the factors is ASQ. For the symmetric case we ask:

Question 4.3. *Let X be a Banach space. Is $\widehat{\otimes}_{\varepsilon,s,N} X$ ASQ for each $N \in \mathbb{N}$ whenever X is ASQ?*

Concerning octahedrality in projective tensor products we have only seen one result, Corollary 2.9. However, there is a wider class of examples of OH projective tensor spaces.

Example. Let X and Y be Banach spaces.

(i) We saw in the example following Corollary 2.7 that given $1 < p \leq q < \infty$ then $K(\ell_p, \ell_q)$ is ASQ. By [3, Proposition 2.5] the dual space $\ell_p\widehat{\otimes}_\pi \ell_{q^*}$ is OH, where $\frac{1}{q} + \frac{1}{q^*} = 1$.

(ii) If $X = L_1(\mu)$ then $X\widehat{\otimes}_\pi Y = L_1(\mu, Y)$ is OH by a straightforward computation.

(iii) If $X = \ell_1(I)$ for a suitable infinite set I, then $X\widehat{\otimes}_\pi Y$ is OH. This follows from the identification $\ell_1(I)\widehat{\otimes}_\pi Y = \ell_1(I, Y)$ [25, Example 2.6] and the fact that infinite ℓ_1-sums of Banach spaces are always OH.

(iv) If $\mathrm{Cun}(X)$ is infinite-dimensional, then $X\widehat{\otimes}_\pi Y$ is OH. This follows from the fact that $(X\widehat{\otimes}_\pi Y)^* = L(Y, X^*)$ has an infinite-dimensional centralizer [20, Lemma VI.1.1] and thus it has the SD2P [7, Proposition 3.3].

(v) If H is a Hilbert space, then $\mathcal{F}(H)\widehat{\otimes}_\pi H$ is OH, where $\mathcal{F}(H)$ is the Lipschitz-free space over H. This follows from [12, Theorem 2.4] and from the fact that the space of Lipschitz functions on H which vanish at 0 under the Lipschitz norm can identified with $L(\mathcal{F}(H), H)$.

In view of (ii),(iii), (iv) and (v) of the above example we wonder

Question 4.4. *Let X and Y be Banach spaces. Is $X\widehat{\otimes}_\pi Y$ OH whenever X is OH?*

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