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Classifying Subcategories in Quotients of Exact Categories

Classifying Triangulated and Thick
Triangulated Subcategories of an Algebraic
Triangulated Category

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Master of Science in Mathematical Sciences
Submission date: December 2017
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Abstract

The goal of this thesis is to prove a one-to-one correspondence between (thick) triangulated subcategories of the stable category associated with a Frobenius category and certain subcategories of the Frobenius category. The result is also generalized in the setting of an exact category and a quotient category. This thesis starts with an introduction to exact categories, quotient categories, Frobenius categories and the stable category of a Frobenius category. After this, the main results are explained and proved. Finally, a few applications are given.

Sammendrag

Målet med denne masteroppgaven er å konstruere en en-til-en korrespondanse mellom (tykke) triangulerte underkategorier av den stabile kategorien tilhørende en Frobenius-kategori og visse underkategorier av Frobenius-kategorien. Resultatet er videre generalisert slik at det holder for en eksakt kategori og en kvotientkategori. Oppgaven begynner med en introduksjon av eksakte kategorier, kvotekategorier, Frobenius-kategorier og den stabile kategorien av en Frobenius-kategori. Deretter blir hovedresultatene forklart og bevist. Avslutningsvis gis det noen anvendelser.

Acknowledgments

I would like to thank my supervisor Professor Petter Bergh for all his help and support. I visited Professor Ryo Takahashi at the University of Nagoya in Nagoya, Japan for 3 weeks during my work, and I would like to thank him and others at the university for their help and hospitality during my stay.

Contents

Notation

In this thesis we normally denote general categories by $\mathscr{C}$, (pre)additive and abelian categories by $\mathscr{A}$, exact categories by $(\mathscr{E}, \mathcal{S})$, Frobenius categories by $(\mathscr{F}, \mathcal{S})$ and triangulated categories by $(\mathscr{T}, T, \Delta)$, where T is the autoequivalence and Δ is the collection of distinguished triangles. We assume the reader to be familiar with additive, abelian and triangulated categories. However, a short introduction of the latter is given in Appendix B. We do not assume any prior knowledge about exact or Frobenius categories.

Preface

This thesis was written in 2017 under the supervision of Professor Petter Bergh at the Norwegian University of Science and Technology. It treats exact categories, quotient categories, Frobenius categories and the stable category of a Frobenius category. The reader needs no prior knowledge about these subjects to understand this thesis. However, it is assumed that the reader is familiar with additive, abelian and triangulated categories. Nevertheless, a short introduction of the latter is given in Appendix B. This includes the definition and some results, but no proofs.

The stable category of a Frobenius category is triangulated, and several authors have studied the (thick) triangulated subcategories of such a category. In [7] Takahashi considers the category of maximal Cohen-Macaulay modules over a Gorenstein ring R, which is Frobenius, and the associated stable category. He classifies the thick triangulated subcategories of the stable category. The motivation of this thesis was to prove a similar classification theorem for arbitrary Frobenius categories. This has been accomplished, and generalized further in the case of an exact category.

Chapter 1 introduces exact categories and quotient categories. Section 1.1 defines exact categories and gives some results. Our definition of an exact category is equivalent with the one given by Quillen in [9], and the prof of this is given. Section 1.2 treats quotient categories, especially what we mean with $\mathscr{A}/\mathscr{N}$ for a preadditive category $\mathscr{A}$ and a subcategory $\mathscr{N}$. A Frobenius category is an exact category satisfying some extra assumptions, and the stable category of a Frobenius category is a quotient category. The purpose of this chapter is therefore to prepare for Chapter 2 on Frobenius categories, and Chapter 3 containing the main results of this thesis.

Chapter 2 introduces Frobenius categories and their stable categories. The definitions are given in section 2.1, while section 2.2 is concerned with the triangulated structure of the stable category. Section 2.3 explains how the short exact sequences in a Frobenius category are closely related to the distinguished triangles in the stable category.

Chapter 3 contains the main results of this thesis. Section 3.1 treats the most general case in the setting of an exact category. First, the definitions of complete and thick subcategories of an exact category $\mathscr{E}$ and of a quotient category $\mathscr{E}/\mathscr{N}$ are given. Then we provide the necessary assumptions to construct a one-to-one correspondence between complete/thick subcategories of $\mathscr{E}$ and complete/thick subcategories of $\mathscr{E}/\mathscr{N}$. In section 3.2 we treat the special case when the exact category is Frobenius and the quotient category is the associated stable category. This results in a one-to-one correspondence between complete/thick subcategories of the Frobenius category and triangulated/thick triangulated subcategories of the stable category.

Chapter 4 gives some examples of Frobenius categories and applies the main results from Chapter 3 to these. Section 4.1 defines Gorenstein projective objects in an abelian category $\mathscr{A}$ and proves that the full subcategory consisting of these objects, $\operatorname{Gproj}\mathscr{A}$, is Frobenius. Section 4.2 treats the special case where the abelian category is $\operatorname{mod} R$, the category of finitely generated modules over a commutative ring R. In this case the Gorenstein projective objects are precisely the totally reflexive R-modules. Furthermore, we consider the case where the ring R is a Gorenstein local ring. In this case the Gorenstein projective objects are the maximal Cohen-Macaulay modules over R. As mentioned, the classification of the thick triangulated subcategories of the stable category associated with this category is already given by Takahashi in [7]. While working on this thesis I visited Professor Ryo Takahashi at the University of Nagoya, Japan. Chapter 4 is a result of the work done under his guidance.

1. Exact categories and quotient categories

1.1. Exact categories

The content of this section is taken from [9], [5] and [2]. We normally denote an additive category by $\mathscr{A}$ and an exact category by $\mathscr{E}$. However, we will use the notation $\mathscr{E}$ leading up to the definition of an exact category.

Definition 1.1. *Let $\mathscr{E}$ be an additive category and $A \xrightarrow{f} B \xrightarrow{g} C$ a sequence in $\mathscr{E}$. We call (f, g) a* **kernel-cokernel pair** *if f is a kernel of g and g is a cokernel of f. Let $\mathcal{S}$ be a family of kernel-cokernel pairs in $\mathscr{E}$. If $(f, g) \in \mathcal{S}$, then we call f an* **admissible monomorphism** *and g an* **admissible epimorphism**.

We use the notations $A \overset{f}{\rightarrowtail} B$ and $B \overset{g}{\twoheadrightarrow} C$ to specify that f is an admissible monomorphism and g is an admissible epimorphism, respectively.

Definition 1.2. *Let $\mathscr{E}$ be an additive category and $\mathcal{S}$ a family of kernel-cokernel pairs in $\mathscr{E}$. Assume that $\mathcal{S}$ is closed under isomorphisms and satisfies the following:*

Ex0 *For all $A \in \mathscr{E}$ the identity morphism 1_A is an admissible monomorphism.*

Ex0^{op} *For all $A \in \mathscr{E}$ the identity morphism 1_A is an admissible epimorphism.*

Ex1 *The class of admissible monomorphisms is closed under composition.*

Ex1^{op} *The class of admissible epimorphisms is closed under composition.*

Ex2 *Admissible monomorphisms are stable under pushout along arbitrary morphisms:*

$$\begin{array}{ccc} A & \overset{f}{\rightarrowtail} & B \\ \downarrow{\scriptstyle h} & {\scriptstyle PO} & \downarrow{\scriptstyle h'} \\ A' & \overset{f'}{\rightarrowtail} & B' \end{array}$$

Ex2^{op} *Admissible epimorphisms are stable under pullback along arbitrary morphisms:*

$$\begin{array}{ccc} B' & \overset{g'}{\twoheadrightarrow} & C' \\ \downarrow{\scriptstyle h'} & {\scriptstyle PB} & \downarrow{\scriptstyle h} \\ B & \overset{g}{\twoheadrightarrow} & C \end{array}$$

In this case $\mathcal{S}$ is an **exact structure** *on $\mathscr{E}$ and $(\mathscr{E}, \mathcal{S})$ is an* **exact category**. *The elements of $\mathcal{S}$ are called* **short exact sequences**.

Remark 1.3. **(1)** We will soon see that the axioms can be weakened and that they are equivalent to the classical definition given by Quillen in [9].

(2) By the duality of the axioms, $\mathcal{S}$ is an exact structure on $\mathscr{E}$ if and only if $\mathcal{S}^{op}$ is an exact structure on $\mathscr{E}^{op}$.

(3) Isomorphisms are both admissible monomorphisms and admissible epimorphisms. As the diagram below illustrates, this follows from the fact that $\mathcal{S}$ is closed under isomorphisms and axiom Ex0 and Ex0^{op}, respectively.

$$\begin{array}{ccccc} A & \xrightarrow[\cong]{f} & B & \longrightarrow & 0 \\ {\scriptstyle\cong}\downarrow{\scriptstyle 1_A} & & {\scriptstyle\cong}\downarrow{\scriptstyle f^{-1}} & & {\scriptstyle\cong}\downarrow \\ A & \overset{1_A}{\rightarrowtail} & A & \twoheadrightarrow & 0 \end{array}$$

(4) Assume that we have $A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C \in \mathcal{S}$ and $h : A \to D$. Then by axiom Ex2 there exists a pushout given by the left square of the following diagram

$$\begin{array}{ccccc} A & \overset{f}{\rightarrowtail} & B & \overset{g}{\twoheadrightarrow} & C \\ \downarrow{\scriptstyle h} & PO & \downarrow{\scriptstyle h'} & & \| \\ D & \overset{f'}{\rightarrowtail} & P & \overset{g'}{\dashrightarrow} & C \end{array}$$

By Lemma A.1 there exists a morphism $g' : P \to C$ such that the diagram commutes and $(f', g') \in \mathcal{S}$. Dually, given $i : D \to C$, then we have a pullback

$$\begin{array}{ccccc} A & \overset{f'}{\dashrightarrow} & P & \overset{g'}{\twoheadrightarrow} & D \\ \| & & \downarrow{\scriptstyle i'} & PB & \downarrow{\scriptstyle i} \\ A & \overset{f}{\rightarrowtail} & B & \overset{g}{\twoheadrightarrow} & C \end{array}$$

and a morphism $f' : A \to P$ such that $(f', g') \in \mathcal{S}$.

(5) An admissible epimorphism is always an epimorphism since it is a cokernel. Indeed, let $A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C \in \mathcal{S}$ and assume that $ag = bg$ for morphisms $a, b : C \to D$. Then $agf = 0$, so the cokernel property gives a unique morphism $c : C \to D$ such that $cg = ag$. However, both a and b satisfies this, hence $a = b$ and g is an epimorphism.

$$\begin{array}{ccccc} A & \overset{f}{\rightarrowtail} & B & \overset{g}{\twoheadrightarrow} & C \\ & & & \searrow^{ag} & \downarrow \\ & & & & D \end{array}$$

Dually, an admissible monomorphism is always a monomorphism since it is a kernel.

Example 1.4. Let $\mathscr{A}$ be an abelian category and define

$$\mathcal{S} := \left\{ X \overset{f}{\to} Y \overset{g}{\to} Z \;\middle|\; 0 \to X \overset{f}{\to} Y \overset{g}{\to} Z \to 0 \text{ exact} \right\}.$$

Then $\mathcal{S}$ is an exact structure on $\mathscr{A}$ and we call it the **standard exact structure** on $\mathscr{A}$. Another exact structure on $\mathscr{A}$ is given by

$$\mathcal{S}' := \left\{ X \xrightarrow{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]} X \oplus Y \xrightarrow{\left[\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right]} Y \;\middle|\; X, Y \in \mathscr{A} \right\}.$$

Let axiom Ex0' be that 1_0 is an admissible epimorphism, where 0 denotes the zero object. In [5, Appendix A] Keller defines $\mathcal{S}$ to be an exact structure on $\mathscr{E}$ if it $\mathcal{S}$ is closed under isomorphisms and satisfies our axioms Ex0', Ex1^{op}, Ex2 and Ex2^{op}. In our terminology, Quillen defines in [9] that $(\mathscr{E}, \mathcal{S})$ is an exact category if $\mathcal{S}$ is a family of kernel-cokernel pairs in $\mathscr{E}$ which is closed under isomorphisms and satisfies the following:

a) For all A, B in $\mathscr{E}$, $A \overset{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}{\rightarrowtail} A \oplus B \overset{\left[\begin{smallmatrix}0 & 1\end{smallmatrix}\right]}{\twoheadrightarrow} B$ is short exact.

b) The axioms Ex1, Ex1^{op}, Ex2 and Ex2^{op} hold.

c) If $g : B \to C$ has a kernel in $\mathscr{E}$ and if there exists a morphism $h : D \to B$ such that $gh : D \twoheadrightarrow C$ is an admissible epimorphism, then g is an admissible epimorphism.

c)^{op} If $f : A \to B$ has a cokernel in $\mathscr{E}$ and if there exists a morphism $i : B \to E$ such that $if : A \rightarrowtail E$ is an admissible monomorphism, then f is an admissible monomorphism.

Proposition 1.5. *Our definition of an exact category as given in Definition 1.2, Keller's definition and Quillen's definition are all equivalent.*

Before we prove the theorem we need the following lemma.

Lemma 1.6. *Assume that $\mathcal{S}$ is a family of kernel-cokernel pairs that satisfies Ex2 and Ex2^{op} and which is closed under isomorphisms. Then in the setting of Ex2 and respectively Ex2^{op}, i.e.*

$$\begin{array}{ccc} A & \overset{f}{\rightarrowtail} & B \\ \downarrow h & PO & \downarrow h' \\ A' & \overset{f'}{\rightarrowtail} & B' \end{array} \qquad resp. \qquad \begin{array}{ccc} B' & \overset{g'}{\twoheadrightarrow} & C' \\ \downarrow h' & PB & \downarrow h \\ B & \overset{g}{\twoheadrightarrow} & C \end{array}$$

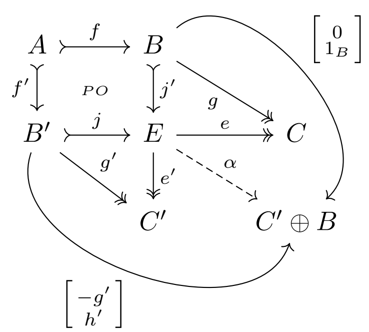
the sequence $A \overset{\left[\begin{smallmatrix}-f\\h\end{smallmatrix}\right]}{\rightarrowtail} B \oplus A' \overset{\left[\begin{smallmatrix}h' & f'\end{smallmatrix}\right]}{\twoheadrightarrow} B'$, *respectively* $B' \overset{\left[\begin{smallmatrix}-g'\\h'\end{smallmatrix}\right]}{\rightarrowtail} C' \oplus B \overset{\left[\begin{smallmatrix}h & g\end{smallmatrix}\right]}{\twoheadrightarrow} C$ *is in $\mathcal{S}$. Moreover, Lemma A.3 gives that the squares are both pushouts and pullbacks.*

Proof. We prove it in the case of Ex2^{op}. Let $(f, g) \in \mathcal{S}$. From the dual part of Remark 1.3 (4) we have a commutative diagram

$$\begin{array}{ccccc} A & \overset{f'}{\rightarrowtail} & B' & \overset{g'}{\twoheadrightarrow} & C' \\ \| & & \downarrow h' & PB & \downarrow h \\ A & \overset{f}{\rightarrowtail} & B & \overset{g}{\twoheadrightarrow} & C \end{array}$$

with $(f', g') \in \mathcal{S}$. Using Remark 1.3 (4) again, we get the following commutative diagram, to the left, with rows and columns in $\mathcal{S}$.

$$\begin{array}{ccccc} A & \overset{f}{\rightarrowtail} & B & \overset{g}{\twoheadrightarrow} & C \\ f' \downarrow & PO & \downarrow j' & & \| \\ B' & \overset{j}{\rightarrowtail} & E & \overset{e}{\twoheadrightarrow} & C \\ g' \downarrow & & \downarrow e' & & \\ C' & = & C' & & \end{array}$$



Since $\begin{bmatrix} -g' \\ h' \end{bmatrix} f' = \begin{bmatrix} 0 \\ 1_B \end{bmatrix} f$, the pushout property gives a morphism $\alpha : E \to C' \oplus B$ with

$$\alpha j' = \begin{bmatrix} 0 \\ 1_B \end{bmatrix}, \quad \alpha j = \begin{bmatrix} -g' \\ h' \end{bmatrix}.$$

Moreover, since $(j'h' - j)f' = 0$ and g' is the cokernel of f', there exists a morphism $\gamma : C' \to E$ such that

$$\gamma g' = j'h' - j.$$

Furthermore, α is an isomorphism with inverse $\beta = \begin{bmatrix} \gamma & j' \end{bmatrix}$. To see this, first note that we have $\alpha\gamma g' = \begin{bmatrix} 1_C & 0 \end{bmatrix}^t g'$. Since g' is an epimorphism by Remark 1.3 (5), this gives that $\alpha\gamma = \begin{bmatrix} 1_C \\ 0 \end{bmatrix}$. Thus $\alpha\beta = \begin{bmatrix} \alpha\gamma & \alpha j' \end{bmatrix} = 1_{C'\oplus B}$. To get the second part, note that

$$\beta\alpha j' = \begin{bmatrix} \gamma & j' \end{bmatrix} \begin{bmatrix} 0 \\ 1_B \end{bmatrix} = j' = 1_E \circ j',$$

$$\beta\alpha j = \begin{bmatrix} \gamma & j' \end{bmatrix} \begin{bmatrix} -g' \\ h' \end{bmatrix} = j = 1_E \circ j.$$

The pushout property then gives that $\beta\alpha = 1_E$.

Moreover, $e\gamma = h$ since $e\gamma g' = hg'$ and g' is an epimorphism. Hence we therefore have that $e\alpha^{-1} = \begin{bmatrix} e\gamma & ej' \end{bmatrix} = \begin{bmatrix} h & g \end{bmatrix}$. Because $\mathcal{S}$ is closed under isomorphisms and $(j, e) \in \mathcal{S}$, we get $(\alpha j, e\alpha^{-1}) \in \mathcal{S}$, which equals

$$B' \xrightarrow{\begin{bmatrix} -g' \\ h' \end{bmatrix}} C' \oplus B \xrightarrow{\begin{bmatrix} h & g \end{bmatrix}} C. \qquad \square$$

Proof of Proposition 1.5. Quillen's definition gives our definition: axiom a) gives axiom Ex0 and Ex0^{op} by letting B and A be the zero object, respectively. Our definition clearly gives Keller's definition. So the only thing remaining to prove is that Keller's definition gives Quillen's definition. We follow the proof given by Keller in [5, Appendix A].

1st step: axiom a) holds. First note that 1_A is an admissible epimorphism by axiom Ex0' and Ex2^{op} because of the pullback

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ \downarrow & PB & \downarrow \\ 0 & \xrightarrow{1_0} & 0 \end{array}$$

The fact that axiom a) holds follows now from Lemma 1.6 since

$$\begin{array}{ccc} A & \xrightarrow{-1_A} & A \\ \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} \\ B & \xrightarrow{1_B} & B \end{array}$$

is a pullback, giving that $A \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} A \oplus B \xrightarrow{\begin{bmatrix} 0 & 1 \end{bmatrix}} B$ is short exact.

2nd step: axiom c) and c)^{op} hold. Let $D \xrightarrow{h} B \xrightarrow{g} C$ be as in c) and let f be the kernel of g. By Lemma 1.6 $\begin{bmatrix} g & gh \end{bmatrix} : B \oplus D \to C$ is an admissible epimorphism and so is

$$\begin{bmatrix} g & 0 \end{bmatrix} = \begin{bmatrix} g & gh \end{bmatrix} \begin{bmatrix} 1_B & -h \\ 0 & 1_D \end{bmatrix}$$

since $\mathcal{S}$ is closed under isomorphisms. The kernel of $\begin{bmatrix} g & 0 \end{bmatrix}$ is $f \oplus 1_D$, so $f \oplus 1_D$ is therefore an admissible monomorphism. Thus, because of the pushout

$$\begin{array}{ccc} A \oplus D & \overset{f\oplus 1}{\rightarrowtail} & B \oplus D \\ {\scriptstyle \begin{bmatrix} 1 & 0 \end{bmatrix}}\downarrow & PO & \downarrow{\scriptstyle \begin{bmatrix} 1 & 0 \end{bmatrix}} \\ A & \overset{f}{\rightarrowtail} & B \end{array}$$

we can conclude that f is also an admissible monomorphism. Since $\begin{bmatrix} g & 0 \end{bmatrix}$ is the cokernel of $f \oplus 1_D$, g is a cokernel of f. Hence $(f, g) \in \mathcal{S}$. The proof of c)^{op} is done dually.

3rd step: axiom b) holds. Proving Ex1 is the only thing remaining. Let $A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C$ be short exact and let $B \overset{h}{\rightarrowtail} B'$ be an admissible monomorphism. From Ex2 we get the following pushout

$$\begin{array}{ccc} B & \overset{h}{\rightarrowtail} & B' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ C & \overset{k}{\rightarrowtail} & C' \end{array} \tag{1.1}$$

By Lemma 1.6 this is also a pullback and $\begin{bmatrix} g' & k \end{bmatrix}$ is an admissible epimorphism. The following diagram is a pullback

$$\begin{array}{ccc} B' \oplus B & \overset{1\oplus g}{\twoheadrightarrow} & B' \oplus C \\ {\scriptstyle \begin{bmatrix} 0 & 1 \end{bmatrix}}\downarrow & PB & \downarrow{\scriptstyle \begin{bmatrix} 0 & 1 \end{bmatrix}} \\ B & \overset{g}{\twoheadrightarrow} & C \end{array}$$

Hence by axiom Ex2^{op}, $1_{B'} \oplus g$ is an admissible epimorphism as well. So by Ex1^{op}

$$\begin{bmatrix} g' & k \end{bmatrix} \begin{bmatrix} 1_{B'} & 0 \\ 0 & g \end{bmatrix} = \begin{bmatrix} g' & kg \end{bmatrix}$$

is also an admissible epimorphism. Since $\begin{bmatrix} g' & kg \end{bmatrix} = g' \begin{bmatrix} 1_{B'} & h \end{bmatrix}$, the second step gives that g' is an admissible epimorphism if it has a kernel. Moreover, hf is the kernel of g', following from the fact that f is the kernel of g and the pullback property of (1.1). Hence $(hf, g') \in \mathcal{S}$, so hf is an admissible monomorphism. □

Corollary 1.7. *Let $A \overset{f}{\rightarrowtail} B$ be an admissible monomorphism and $A \overset{g}{\twoheadrightarrow} C$ be an admissible epimorphism and consider the pushout*

$$\begin{array}{ccc} A & \overset{f}{\rightarrowtail} & B \\ \downarrow{\scriptstyle g} & PO & \downarrow{\scriptstyle g'} \\ C & \overset{f'}{\rightarrowtail} & D \end{array}$$

Then g' is an admissible epimorphism. Moreover, if g is an isomorphism, so is g'.

Proof. The fact that g' is admissible epimorphism follows directly from the 3rd step in the proof of Proposition 1.5. To get the moreover part, assume that g is an isomorphism with inverse $h : C \to A$. Note that $fhg = f \circ 1_A = 1_B \circ f$, so the pushout property gives a unique morphism $h' : D \to B$ such that $h'f' = fh$ and $h'g' = 1_B$. We get $g'h'g' = g' \circ 1_B = 1_D \circ g'$, which implies that $g'h' = 1_D$ since g' is an (admissible) epimorphism. Hence h' is the inverse of g'. □

Proposition 1.8. *Let*

$$K:\quad \begin{array}{ccc} A & \overset{i}{\rightarrowtail} & B \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ A' & \overset{i'}{\rightarrowtail} & B' \end{array}$$

be commutative with i and i' admissible monomorphisms. Then the following are equivalent:

(i) The square K is a pushout.

(ii) The sequence $A \overset{\left[\begin{smallmatrix}-i\\ f\end{smallmatrix}\right]}{\rightarrowtail} B \oplus A' \overset{\left[\begin{smallmatrix}f' & i'\end{smallmatrix}\right]}{\twoheadrightarrow} B'$ *is short exact.*

(iii) The square K is both a pushout and a pullback.

(iv) The square K is part of a commutative diagram of the form

$$\begin{array}{ccccc} A & \overset{i}{\rightarrowtail} & B & \overset{p}{\twoheadrightarrow} & C \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} & & \| \\ A' & \overset{i'}{\rightarrowtail} & B' & \overset{p'}{\twoheadrightarrow} & C \end{array}$$

with short exact rows.

Proof. (i) ⇒ (ii) follows from Lemma 1.6, (ii) ⇒ (iii) follows from Lemma A.3 and (iii) ⇒ (i) is obvious.

(i) ⇒ (iv) follows from Lemma A.1: Since i is an admissible monomorphism it is the kernel of a morphism $p : B \to C$ with $(i, p) \in \mathcal{S}$. Hence by the lemma, there exists $p' : B' \to C$ such that the diagram commutes and with p' the cokernel of i'. Moreover, i' is an admissible monomorphism by assumption, hence $(i', p') \in \mathcal{S}$.

(iv) ⇒ (ii) Since p, p' are admissible epimorphisms there exists a pullback as in the diagram below.

$$\begin{array}{ccccc} & & A & = & A \\ & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i} \\ A' & \overset{j'}{\rightarrowtail} & P & \overset{q'}{\twoheadrightarrow} & B \\ \| & & \downarrow{\scriptstyle q} & PB & \downarrow{\scriptstyle p} \\ A' & \overset{i'}{\rightarrowtail} & B' & \overset{p'}{\twoheadrightarrow} & C \end{array}$$

From the dual of the implication (i) ⇒ (iv), we get the rest of the commutative diagram above with short exact rows and columns. Our goal is to prove that $A \overset{j}{\rightarrowtail} P \overset{q}{\twoheadrightarrow} B' \in \mathcal{S}$ is isomorphic to the sequence

$$A \xrightarrow{\left[\begin{smallmatrix}-i\\ f\end{smallmatrix}\right]} B \oplus A' \xrightarrow{\left[\begin{smallmatrix}f' & i'\end{smallmatrix}\right]} B'.$$

Since $p = p'f' : B \to C$, the pullback property gives a unique morphism $k : B \to P$ such that

$$q'k = 1_B \text{ and } qk = f'.$$

Now $q'(1_P - kq') = 0$, so since j' is the kernel of q', there exists a unique $l : P \to A'$ with

$$j'l = 1_P - kq'.$$

By Remark 1.3 (5) j' is a monomorphism, hence

$$j'lk = (1_P - kq')k = 0 \implies lk = 0,$$
$$j'lj' = (1_P - kq')j' = j' \implies lj' = 1_{A'}.$$

Similarly, since i' is a monomorphism, we get

$$i'lj = (qj')lj = q(1_P - kq')j = qj - (qk)(q'j) = -f'i = -i'f \implies lj = -f.$$

The morphisms

$$\begin{bmatrix} k & j' \end{bmatrix} : B \oplus A' \to P \qquad \text{and} \qquad \begin{bmatrix} q' \\ l \end{bmatrix} : P \to B \oplus A'$$

are inverses of each other:

$$\begin{bmatrix} k & j' \end{bmatrix} \begin{bmatrix} q' \\ l \end{bmatrix} = kq' + j'l = 1_P \qquad \text{and} \qquad \begin{bmatrix} q' \\ l \end{bmatrix} \begin{bmatrix} k & j' \end{bmatrix} = \begin{bmatrix} q'k & q'j' \\ lk & lj' \end{bmatrix} = \begin{bmatrix} 1_B & 0 \\ 0 & 1_{A'} \end{bmatrix}.$$

Note that

$$\begin{bmatrix} f' & i' \end{bmatrix} = q \begin{bmatrix} k & j' \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} i & -f \end{bmatrix} = \begin{bmatrix} q' & l \end{bmatrix} j.$$

Hence we get an isomorphism

$$\begin{array}{ccccc} A & \overset{j}{\rightarrowtail} & P & \overset{q}{\twoheadrightarrow} & B' \\ \downarrow{\scriptstyle -1_A} & & \downarrow{\scriptstyle \begin{bmatrix} q' \\ l \end{bmatrix}} & & \| \\ A & \xrightarrow{\begin{bmatrix} -i \\ f \end{bmatrix}} & B \oplus A' & \xrightarrow{\begin{bmatrix} f' & i' \end{bmatrix}} & B' \end{array}$$

Thus $A \xrightarrow{\begin{bmatrix} -i \\ f \end{bmatrix}} B \oplus A' \xrightarrow{\begin{bmatrix} f' & i' \end{bmatrix}} B' \in \mathcal{S}$. □

Definition 1.9. *Let $\mathscr{E}'$ be a subcategory of an exact category $(\mathscr{E}, \mathcal{S})$. Then $\mathscr{E}'$ is* **extension closed** *if whenever $X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z$ is in $\mathcal{S}$ with $X, Z \in \mathscr{E}'$, then $Y \in \mathscr{E}'$.*

Proposition 1.10. *Let $\mathscr{E}'$ be a full subcategory of an exact category $(\mathscr{E}, \mathcal{S})$ with $0 \in \mathscr{E}'$. Assume that $\mathscr{E}'$ is extension closed and define*

$$\mathcal{S}' := \left\{ X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z \in \mathcal{S} \;\middle|\; X, Y, Z \in \mathscr{E}' \right\}.$$

Then $(\mathscr{E}', \mathcal{S}')$ is an exact category and $\mathscr{E}'$ closed under isomorphisms. We say that the exact structure on $\mathscr{E}'$ is **induced by the exact structure on** $\mathscr{E}$.

Proof. $\mathscr{E}'$ is additive: Since $\mathscr{E}'$ is a full subcategory of an an additive category and contains the zero object, $\mathscr{E}'$ is preadditive. For any objects X, Y in $\mathscr{E}'$, $X \rightarrowtail X \oplus Y \twoheadrightarrow Y$ is short exact. Hence $X \oplus Y \in \mathscr{E}'$ since $\mathscr{E}'$ is extension closed. Thus $\mathscr{E}'$ is additive.

By Proposition 1.5 it is enough to show that the axioms Ex0', Ex1^{op}, Ex2 and Ex2^{op} are satisfied. Axiom Ex0' follows immediately from $0 \in \mathscr{E}'$ and the definition of $\mathcal{S}$.

Ex2: Let $A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C$ be in $\mathcal{S}'$ and let $h : A \to D$ be in $\mathscr{E}'$. Since $\mathscr{E}$ is exact, the pushout along f and h exists, thus we get the following commutative diagram in $\mathscr{E}$:

$$\begin{array}{ccccc} A & \overset{f}{\rightarrowtail} & B & \overset{g}{\twoheadrightarrow} & C \\ \downarrow{\scriptstyle h} & PO & \downarrow{\scriptstyle h'} & & \| \\ D & \overset{f'}{\rightarrowtail} & P & \overset{g'}{\twoheadrightarrow} & C \end{array}$$

By Proposition 1.8 the second row is in $\mathcal{S}$. Since $\mathscr{E}'$ is extension closed and D, C are in $\mathscr{E}'$, so is P. Thus the pushout along f and h lies in $\mathscr{E}'$. Ex2^{op} is done dually.

Ex1^{op}: Let $A \overset{f}{\rightarrowtail} B \overset{g}{\twoheadrightarrow} C$ and $P \overset{p}{\rightarrowtail} B' \overset{h}{\twoheadrightarrow} B$ be in $\mathcal{S}'$. Note that gh is an admissible epimorphism in $(\mathscr{E}, \mathcal{S})$ since the category is exact. Hence gh is an admissible epimorphism in $(\mathscr{E}', \mathcal{S}')$ as well if it has a kernel which lies in $\mathscr{E}'$. By Ex2^{op} there exists a pullback along h and f with objects in $\mathscr{E}'$:

$$\begin{array}{ccccc} A' & \xrightarrow{f'} & B' & \xrightarrow{gh} & C \\ \downarrow{\scriptstyle h'} & {\scriptstyle PB} & \downarrow{\scriptstyle h} & & \| \\ A & \overset{f}{\rightarrowtail} & B & \overset{g}{\twoheadrightarrow} & C \end{array}$$

By the dual of Lemma A.2 f' is the kernel of gh, so $(f', gh) \in \mathcal{S}'$.

Closed under isomorphisms: Let $f : X \to Y$ be an isomorphism and assume that $X \in \mathscr{E}'$. Then $X \overset{f}{\rightarrowtail} Y \twoheadrightarrow 0$ is in $\mathcal{S}$, hence $Y \in \mathscr{E}'$ since $\mathscr{E}'$ is extension closed. □

The next proposition consider the special case when the exact category is an abelian category.

Proposition 1.11. *Let $\mathscr{E}$ be a full additive subcategory of an abelian category $\mathscr{A}$. Suppose that $\mathscr{E}$ is extension closed and define*

$$\mathcal{S} := \left\{ X \xrightarrow{f} Y \xrightarrow{g} Z \text{ in } \mathscr{E} \;\middle|\; 0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0 \text{ exact in } \mathscr{A} \right\}.$$

Then $(\mathscr{E}, \mathcal{S})$ is an exact category.

Conversely, if $(\mathscr{E}, \mathcal{S})$ is an essentially small exact category, then there exists an abelian category $\mathscr{A}$ such that $\mathscr{E}$ is an extension closed, full additive subcategory of $\mathscr{A}$ and with $\mathcal{S}$ as above, i.e. consisting of the sequences in $\mathscr{E}$ that are exact in $\mathscr{A}$.

Proof. Part I: This follows immediately from Proposition 1.10 where $\mathscr{A}$ has the standard exact structure as defined in Example 1.4.

Part II: Let $\mathscr{A}$ be the category of left-exact contravariant functors from $\mathscr{E}$ into the category of abelian groups. Then $\mathscr{A}$ is abelian, and $\mathscr{E}$ becomes a subcategory of $\mathscr{A}$ via the Yoneda embedding. It can be shown that $\mathscr{E}$ is extension closed, and that a sequence $X \to Y \to Z$ in $\mathscr{E}$ is in $\mathcal{S}$ if and only if $0 \to X \to Y \to Z \to 0$ is exact in $\mathscr{A}$. The detailed proof of this can be found in both [5, Appendix A] and [2, Appendix A]. □

1.2. Quotient categories

The content of this section is mostly taken from [4].

Definition 1.12. *Let $\mathscr{A}$ be a preadditive category. A collection of morphisms $\mathcal{I}$ is a (two-sided)* **ideal** *of $\mathscr{A}$ if*

(i) $\mathcal{I}(X, Y) := \mathcal{I} \cap \operatorname{Hom}(X, Y)$ *is a subgroup of the abelian group* $\operatorname{Hom}(X, Y)$*, and*

(ii) whenever $f \in \operatorname{Hom}(X, Y)$, $g \in \mathcal{I}(Y, Z)$, $h \in \operatorname{Hom}(Z, W)$, *then* $hgf \in \mathcal{I}(X, W)$.

Remark 1.13. Given $f, g \in \operatorname{Hom}(X, Y)$, we say that f is related to g if $f - g \in \mathcal{I}(X, Y)$. It is clear that this is an equivalence relation on $\operatorname{Hom}(X, Y)$.

Definition 1.14. *Let $\mathscr{A}$ be a preadditive category and $\mathcal{I}$ an ideal of $\mathscr{A}$. We define the* **quotient category** *$\mathscr{A}/\mathcal{I}$ by*

$$\operatorname{obj}\mathscr{A}/\mathcal{I} := \operatorname{obj}\mathscr{A}$$

$$\operatorname{Hom}_{\mathscr{A}/\mathcal{I}}(X,Y) := \operatorname{Hom}_{\mathscr{A}}(X,Y)/\mathcal{I}(X,Y).$$

This means that $\mathscr{A}/\mathcal{I}$ has the same objects as $\mathscr{A}$ and that the morphisms in $\mathscr{A}/\mathcal{I}$ are the equivalence classes of the morphisms in $\mathscr{A}$. The equivalence class of a morphism $f : X \to Y$ in $\mathscr{A}$ will be denoted by $\underline{f}$ in $\mathscr{A}/\mathcal{I}$. When it is clear from the context, the notation $\underline{\operatorname{Hom}}_{\mathscr{A}}(X,Y)$, or simply $\underline{\operatorname{Hom}}(X,Y)$, will be used instead of $\operatorname{Hom}_{\mathscr{A}/\mathcal{I}}(X,Y)$.

Note that composition in the quotient category $\mathscr{A}/\mathcal{I}$ is in fact well defined. Indeed, assume that $f - f' \in \mathcal{I}(X,Y)$ and $g - g' \in \mathcal{I}(Y,Z)$. Then

$$gf - g'f' = gf - gf' + gf' - g'f' = g(f - f') + (g - g')f' \in \mathcal{I}(X,Z),$$

which implies that $\underline{gf} = \underline{g'f'}$. Associativity of composition in the quotient category $\mathscr{A}/\mathcal{I}$ follows directly from the associativity in $\mathscr{A}$. Thus $\mathscr{A}/\mathcal{I}$ is indeed a category.

Proposition 1.15. *Let $\mathscr{A}$ be a (pre)additive category and $\mathcal{I}$ an ideal of $\mathscr{A}$. Then the quotient category $\mathscr{A}/\mathcal{I}$ is (pre)additive.*

Proof. Since $\mathcal{I}(X,Y)$ is a subgroup of the abelian group $\operatorname{Hom}(X,Y)$, the factor group

$$\underline{\operatorname{Hom}}(X,Y) = \operatorname{Hom}_{\mathscr{A}}(X,Y)/\mathcal{I}(X,Y)$$

is also an abelian group. The bilinearity of composition in $\mathscr{A}/\mathcal{I}$ follows directly from the bilinearity of composition in $\mathscr{A}$. If 0 is a zero object in $\mathscr{A}$, then it is a zero object in $\mathscr{A}/\mathcal{I}$ as well since $\underline{\operatorname{Hom}}(0,X)$ and $\underline{\operatorname{Hom}}(X,0)$ will contain only one morphism. If a biproduct of X, Y in $\mathscr{A}$ is given by

$$X \underset{p_1}{\overset{i_1}{\rightleftarrows}} X \oplus Y \underset{p_2}{\overset{i_2}{\leftrightarrows}} Y$$

then it is trivial to see that a biproduct of X, Y in $\mathscr{A}/\mathcal{I}$ is given by

$$X \underset{\underline{p_1}}{\overset{\underline{i_1}}{\rightleftarrows}} X \oplus Y \underset{\underline{p_2}}{\overset{\underline{i_2}}{\leftrightarrows}} Y.$$

□

Definition 1.16. *We define $\Sigma : \mathscr{A} \to \mathscr{A}/\mathrm{I}$ to be the functor given by*

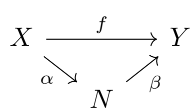
$$X \mapsto X$$

$$(f : X \to Y) \mapsto (\underline{f} : X \to Y)$$

We may refer to $\underline{f}$ as the image of f.

It is trivial to see that Σ is in fact an additive functor.

Example 1.17. We say that a morphism $f : X \to Y$ **factors through** the object N if there exist morphisms such that the diagram below commutes.



Let $\mathscr{A}$ be a preadditive category and $\mathscr{N}$ a (full) subcategory which is closed under finite direct sums. Define $\mathcal{I}(X,Y) \subseteq \operatorname{Hom}(X,Y)$ to be the collection of all morphisms which factor through some object in $\mathscr{N}$, and denote by $\mathcal{I}$ the union of all $\mathcal{I}(X,Y)$. Then $\mathcal{I}$ is an ideal of $\mathscr{A}$. To see this, first assume that $f_1, f_2 \in \mathcal{I}(X,Y)$ factors through N_1, N_2, respectively, with $f_i = \beta_i \alpha_i$ for $i = 1, 2$. Then the diagram

$$\begin{array}{ccccc} X & & \xrightarrow{\quad f_1 - f_2 \quad} & & Y \\ & {\scriptstyle \left[\begin{smallmatrix} \alpha_1 \\ -\alpha_2 \end{smallmatrix}\right]} \searrow & & \nearrow {\scriptstyle \left[\begin{smallmatrix} \beta_1 & \beta_2 \end{smallmatrix}\right]} & \\ & & N_1 \oplus N_2 & & \end{array}$$

commutes. Since $\mathscr{N}$ is closed under finite direct sums, this means that $f_1 - f_2 \in \mathcal{I}(X,Y)$. Thus $\mathcal{I}(X,Y)$ is a subgroup of $\operatorname{Hom}(X,Y)$. Now assume that we are in the setting of axiom (ii). Then we have

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & & \xrightarrow{\quad g \quad} & & Z & \xrightarrow{h} & W \\ & & & {\scriptstyle \alpha} \searrow & & \nearrow {\scriptstyle \beta} & & & \\ & & & & N & & & & \end{array}$$

Hence $hgf \in \mathcal{I}(X,W)$ since it factors through N via $hgf = (h\beta)(\alpha f)$.

Definition 1.18. *Let $\mathscr{A}$ be a preadditive category. Assume that $\mathscr{N}$ is a (full) subcategory which is closed under finite direct sums and let $\mathcal{I}$ be as in the example above. Then we define $\mathscr{A}/\mathscr{N}$ to be the quotient category $\mathscr{A}/\mathcal{I}$.*

2. Frobenius categories and the associated stable categories

The main sources of this chapter are [3] and [4].

2.1. Definition

In this section, we define what it means for an exact category to be a Frobenius category. We normally denote a general exact category by $(\mathscr{E}, \mathcal{S})$ and a Frobenius category by $(\mathscr{F}, \mathcal{S})$. However, we use the notation $(\mathscr{F}, \mathcal{S})$ in the next definitions leading up to the definition of a Frobenius category.

Definition 2.1. *Let $\mathcal{S}$ be a family of kernel-cokernel pairs in an additive category $\mathscr{F}$. An object P in $\mathscr{F}$ is* $\mathcal{S}$**-projective** *if for all admissible epimorphisms $g : Y \twoheadrightarrow Z$ and for all morphisms $a : P \to Z$ there exists a (not necessarily unique) morphism $b : P \to Y$ such that $a = gb$:*

$$\begin{array}{ccccc} & & P & & \\ & & {\scriptstyle \exists b}\,\downarrow & \searrow^{a} & \\ X & \overset{f}{\rightarrowtail} & Y & \overset{g}{\twoheadrightarrow} & Z \end{array} \qquad \in \mathcal{S}$$

Dually, an object I in $\mathscr{F}$ is $\mathcal{S}$**-injective** *if for all admissible monomorphisms $f : X \rightarrowtail Y$ and for all morphisms $a : X \to I$ there exists a morphism $b : Y \to I$ such that $a = bf$:*

$$\begin{array}{ccccc} X & \overset{f}{\rightarrowtail} & Y & \overset{g}{\twoheadrightarrow} & Z \\ & {\scriptstyle a}\searrow & \downarrow{\scriptstyle \exists b} & & \\ & & I & & \end{array} \qquad \in \mathcal{S}$$

Example 2.2. Any initial object is $\mathcal{S}$-projective and any terminal object is $\mathcal{S}$-injective. In particular, the zero object is both $\mathcal{S}$-projective and $\mathcal{S}$-injective.

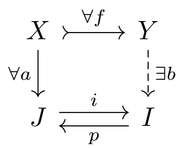
Definition 2.3. *Let $\mathcal{S}$ be a family of kernel-cokernel pairs in an additive category $\mathscr{F}$. We define* $\operatorname{proj} \mathscr{F}$*, resp.* $\operatorname{inj} \mathscr{F}$*, to be the full subcategory of $\mathscr{F}$ consisting of all $\mathcal{S}$-projective objects, resp. $\mathcal{S}$-injective objects.*

Definition 2.4. *An exact category $(\mathscr{F}, \mathcal{S})$ has* **enough** $\mathcal{S}$**-projectives** *if for all objects X in $\mathscr{F}$ there exists an admissible epimorphism $g : P \twoheadrightarrow X$ with P an $\mathcal{S}$-projective object in $\mathscr{F}$, and* **enough** $\mathcal{S}$**-injectives** *if for all objects X in $\mathscr{F}$ there exists an admissible monomorphism $f : X \rightarrowtail I$ with I an $\mathcal{S}$-injective object in $\mathscr{F}$.*

Definition 2.5. *An object A is a* **direct summand** *of X if there exist morphisms $A \xrightarrow{f} X \xrightarrow{g} A$ such that $gf = 1_A$.*

Lemma 2.6. *The direct sum of two $\mathcal{S}$-injective objects is $\mathcal{S}$-injective and a direct summand of an $\mathcal{S}$-injective object is $\mathcal{S}$-injective. Dually, the direct sum of two $\mathcal{S}$-projective objects is $\mathcal{S}$-projective and a direct summand of an $\mathcal{S}$-projective object is $\mathcal{S}$-projective.*

Proof. The first part about direct sum is trivial, so we only prove the second part about direct summand. Let J be a direct summand of an $\mathcal{S}$-injective object I. Then there exist morphisms $J \underset{p}{\overset{i}{\rightleftarrows}} I$ with $pi = 1_J$.



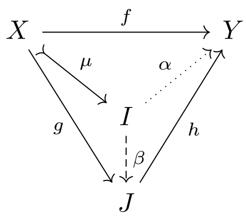
Since I is $\mathcal{S}$-injective we have that for all admissible monomorphisms $f : X \rightarrowtail Y$ and morphisms $a : X \to J$ there exists a morphism $b : Y \to I$ such that $ia = bf$. Let $c : Y \to J$ with $c = pb$. Then $cf = pbf = pia = a$. Hence J is an $\mathcal{S}$-injective object. □

Remark 2.7. A sequence $X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z \in \mathcal{S}$ is right split whenever Z is $\mathcal{S}$-projective and left split whenever X is $\mathcal{S}$-injective. To see this, simply let a in the definition of $\mathcal{S}$-projective and $\mathcal{S}$-injective objects be the identity morphism.

Definition 2.8. *An exact category $(\mathscr{F}, \mathcal{S})$ is a* **Frobenius category** *if it has enough $\mathcal{S}$-projectives, enough $\mathcal{S}$-injectives and if* $\operatorname{proj} \mathscr{F} = \operatorname{inj} \mathscr{F}$.

Let $(\mathscr{F}, \mathcal{S})$ be a Frobenius category unless otherwise stated. For a pair of objects X, Y in $\mathscr{F}$ let $\mathcal{I}(X, Y)$ denote the additive subgroup of $\operatorname{Hom}(X, Y)$ consisting of the morphisms which factor through an $\mathcal{S}$-injective object. The **stable category** $\underline{\mathscr{F}}$ associated with $\mathscr{F}$ is the category whose objects are the objects of $\mathscr{F}$ and the set of morphisms from X to Y is $\operatorname{Hom}_{\mathscr{F}}(X, Y)/\mathcal{I}(X, Y)$. In other words, $\underline{\mathscr{F}}$ is the quotient category $\mathscr{F}/\operatorname{inj}\mathscr{F}$ as defined in Definition 1.18. We use the notation $\underline{\operatorname{Hom}}(X, Y)$ instead of $\operatorname{Hom}_{\underline{\mathscr{F}}}(X, Y)$ to denote the set of morphisms from X to Y in $\underline{\mathscr{F}}$. The equivalence class of a morphism $f : X \to Y$ in $\mathscr{F}$ is denoted by $\underline{f}$ in $\underline{\mathscr{F}}$.

Remark 2.9. Assume that $f : X \to Y$ factors through some $\mathcal{S}$-injective object J as in the diagram below. Let $\mu : X \rightarrowtail I$ be an admissible monomorphism, I an $\mathcal{S}$-injective. Then there exists $\alpha : I \to Y$ such that $\alpha\mu = f$.



Namely, since J is an $\mathcal{S}$-injective object and μ is an admissible monomorphism, there exists $\beta : I \to J$ such that $\beta\mu = g$. Let $\alpha := h\beta$. Then $\alpha\mu = h\beta\mu = hg = f$.

2.2. Triangulation of the stable category

The stable category of a Frobenius category is triangulated in a very natural way, as we will see in this section. The triangulated categories that arise in this way are called **algebraic**, and the ones that appear naturally in homological algebra (homotopy categories of complexes, derived categories) are all of this form. Given a stable category $\underline{\mathscr{F}}$, we construct an autoequivalence $T : \underline{\mathscr{F}} \to \underline{\mathscr{F}}$, a collection of triangles Δ, and prove that this give a triangulated structure on $\underline{\mathscr{F}}$. Appendix B gives a short introduction to triangulated categories. It is assumed that the reader is familiar with the content of that appendix. We will therefore refer to the axioms (TR1) to (TR4) without explaining their content.

Lemma 2.10. *Let $X \overset{\mu}{\rightarrowtail} I \overset{\pi}{\twoheadrightarrow} X'$ and $Y \overset{\mu'}{\rightarrowtail} I' \overset{\pi'}{\twoheadrightarrow} Y'$ be exact in $(\mathscr{F}, \mathcal{S})$ with I' (but not necessarily I) an $\mathcal{S}$-injective object. Given any morphism $f : X \to Y$ there exist morphisms such that the following diagram commutes.*

$$\begin{array}{ccccc} X & \overset{\mu}{\rightarrowtail} & I & \overset{\pi}{\twoheadrightarrow} & X' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle I(f)} & & \downarrow{\scriptstyle f'} \\ Y & \overset{\mu'}{\rightarrowtail} & I' & \overset{\pi'}{\twoheadrightarrow} & Y' \end{array}$$

Proof. We have that μ is an admissible monomorphism and I' is $\mathcal{S}$-injective, so there exists a morphism $I(f) : I \to I'$ such that $I(f)\mu = \mu' f$. Since $0 = \pi'\mu' f = \pi' I(f)\mu$ and π is the cokernel of μ, there exists a morphism $f' : X' \to Y'$ such that $f'\pi = \pi' I(f)$. □

Lemma 2.11. *Let $X \overset{\mu}{\rightarrowtail} I \overset{\pi}{\twoheadrightarrow} X'$ and $X \overset{\mu'}{\rightarrowtail} I' \overset{\pi'}{\twoheadrightarrow} X''$ be in $\mathcal{S}$, where I and I' are $\mathcal{S}$-injectives. Then X' and X'' are isomorphic in $\underline{\mathscr{F}}$.*

Proof. From Lemma 2.10 we get morphisms such that the following diagram commutes

$$\begin{array}{ccccc} X & \overset{\mu}{\rightarrowtail} & I & \overset{\pi}{\twoheadrightarrow} & X' \\ \| & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ X & \overset{\mu'}{\rightarrowtail} & I' & \overset{\pi'}{\twoheadrightarrow} & X'' \\ \| & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle g'} \\ X & \overset{\mu}{\rightarrowtail} & I & \overset{\pi}{\twoheadrightarrow} & X' \end{array}$$

Since $0 = (f'f - 1_I)\mu$ and π is the cokernel of μ, there exists $h : X' \to I$ such that $h\pi = f'f - 1_I$. Thus $\pi h\pi = \pi(f'f - 1_I) = \pi f'f - \pi = g'g\pi - \pi = (g'g - 1_{X'})\pi$, and since π is an epimorphism it follows that $\pi h = g'g - 1_{X'}$. This means that $g'g - 1_{X'}$ factors through the $\mathcal{S}$-injective object I, hence $\underline{g'g} = \underline{1}_{X'}$. Similarly, $\underline{gg'} = \underline{1}_{X''}$. □

For an object X in $\mathscr{F}$, let $[X]$ denote the isomorphism class of X in $\underline{\mathscr{F}}$. Let $X \rightarrowtail I \twoheadrightarrow X'$ be in $\mathcal{S}$ with I an $\mathcal{S}$-injective object. Then the lemma states that $[X']$ is independent of the choice of $X \rightarrowtail I \twoheadrightarrow X'$. For each object X in $\mathscr{F}$ choose a sequence $X \overset{\mu(X)}{\rightarrowtail} I(X) \overset{\pi(X)}{\twoheadrightarrow} TX \in \mathcal{S}$ with $I(X)$ an $\mathcal{S}$-injective object, and define $T(X) := TX$. Given $f : X \to Y$ we get from Lemma 2.10 a commutative diagram

$$\begin{array}{ccccc} X & \overset{\mu(X)}{\rightarrowtail} & I(X) & \overset{\pi(X)}{\twoheadrightarrow} & TX \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle I(f)} & & \downarrow{\scriptstyle T(f)} \\ Y & \overset{\mu(Y)}{\rightarrowtail} & I(Y) & \overset{\pi(Y)}{\twoheadrightarrow} & TY \end{array}$$

Moreover, the next lemma shows that the equivalence class of $T(f)$ is independent of the choice of $I(f)$.

Lemma 2.12. *Given a commutative diagram*

$$\begin{array}{ccccc} X & \overset{\mu(X)}{\rightarrowtail} & I(X) & \overset{\pi(X)}{\twoheadrightarrow} & TX \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle I_i(f)} & & \downarrow{\scriptstyle T_i(f)} \\ Y & \overset{\mu(Y)}{\rightarrowtail} & I(Y) & \overset{\pi(Y)}{\twoheadrightarrow} & TY \end{array}$$

for $i = 1, 2$*, then* $\underline{T_1(f)} = \underline{T_2(f)}$ *in* $\underline{\mathscr{F}}$*.*

Proof. We have $[I_1(f) - I_2(f)]\mu(X) = \mu(Y)f - \mu(Y)f = 0$, so the cokernel property of $\pi(X)$ gives that there exists a unique $g : T(X) \to I(Y)$ such that $g\pi(X) = I_1(f) - I_2(f)$. Hence

$$\pi(Y) \circ g \circ \pi(X) = \pi(Y) \circ [I_1(f) - I_2(f)] = [T_1(f) - T_2(f)] \circ \pi(X).$$

Since $\pi(X)$ is an epimorphism, this implies that $\pi(Y)g = T_1(f) - T_2(f)$. Thus $T_1(f) - T_2(f)$ factors through the $\mathcal{S}$-injective object $I(Y)$, so $\underline{T_1(f)} = \underline{T_2(f)}$ in $\underline{\mathscr{F}}$. □

As a consequence of this, $T : \underline{\mathscr{F}} \to \underline{\mathscr{F}}$ is a well-defined functor, and the next theorem shows that T is an autoequivalence. Furthermore, T is an automorphism if we are able to choose TX such that $T : [X] \to [X']$ is a bijection for all X. Readers who only want to consider triangulated categories where the functor is an automorphism would have to make this assumption throughout this thesis.

Theorem 2.13. $T : \underline{\mathscr{F}} \to \underline{\mathscr{F}}$ *is an autoequivalence. Moreover, if* $T : [X] \to [X']$ *is a bijection for all* X*, then* T *is an automorphism.*

Proof. *T is dense:* Let $Y \in \underline{\mathscr{F}}$. Since $\mathscr{F}$ has enough $\mathcal{S}$-projectives there exists $X \overset{\mu}{\rightarrowtail} P \overset{\pi}{\twoheadrightarrow} Y$ in $\mathcal{S}$ with P an $\mathcal{S}$-projective object. The $\mathcal{S}$-projectives coincide with the $\mathcal{S}$-injectives, so P is also injective. Hence $T(X)$ and Y are isomorphic in $\underline{\mathscr{F}}$ by Lemma 2.11. Moreover, if $T : [X] \to [Y]$ is a bijection, then there exists a unique $X' \in [X]$ with $T(X') = Y$.

T is full: Assume that we have $g : T(X) \to T(Y)$. We want to construct $f : X \to Y$ such that $T(\underline{f}) = \underline{g}$.

$$\begin{array}{ccccc} X & \overset{\mu(X)}{\rightarrowtail} & I(X) & \overset{\pi(X)}{\twoheadrightarrow} & TX \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ Y & \overset{\mu(Y)}{\rightarrowtail} & I(Y) & \overset{\pi(Y)}{\twoheadrightarrow} & TY \end{array}$$

Since $I(X)$ is $\mathcal{S}$-projective there exists $g' : I(X) \to I(Y)$ such that $\pi(Y)g' = g\pi(X)$. Hence $\pi(Y)g'\mu(X) = g\pi(X)\mu(X) = 0$, so by the kernel property of $\mu(Y)$, there exists $f : X \to Y$ such that $g'\mu(X) = \mu(Y)f$. Thus by Lemma 2.12, $T(\underline{f}) = \underline{g}$ in $\underline{\mathscr{F}}$ and T is full.

T is faithful: Assume that $T(f_1) = T(f_2)$. We want to prove that $\underline{f_1} = \underline{f_2}$.

$$\begin{array}{ccccc} X & \overset{\mu(X)}{\rightarrowtail} & I(X) & \overset{\pi(X)}{\twoheadrightarrow} & TX \\ \downarrow{\scriptstyle f_1 - f_2} & \overset{h}{\swarrow} & \downarrow{\scriptstyle I(f_1) - I(f_2)} & & \downarrow{\scriptstyle T(f_1) - T(f_2) = 0} \\ Y & \underset{\mu(Y)}{\rightarrowtail} & I(Y) & \underset{\pi(Y)}{\twoheadrightarrow} & TY \end{array}$$

Note that $\pi(Y)[I(f_1) - I(f_2)] = [T(f_1) - T(f_2)]\pi(X) = 0$, so the kernel property of $\mu(Y)$ gives a unique morphism $h : I(X) \to Y$ such that $I(f_1) - I(f_2) = \mu(Y)h$. Furthermore, $\pi(Y)[I(f_1) - I(f_2)]\mu(X) = 0$ as well, so there exists a unique morphism $\alpha : X \to Y$ such that $[I(f_1) - I(f_2)]\mu(X) = \mu(Y)\alpha$. Both $f_1 - f_2$ and $h\mu(X)$ satisfies the condition of α, hence $f_1 - f_2 = h\mu(X)$, giving $\underline{f_1} = \underline{f_2}$ in $\underline{\mathscr{F}}$. □

Remark 2.14. **(1)** The construction of the functor T involves a choice for each object X. If the functors T_1 and T_2 are obtained from different choices, then it is possible to show that T_1 and T_2 are naturally isomorphic functors. See for example Happel's proof in [3, Section 2.2] for details.

(2) To simplify notation we will use $X \overset{x}{\rightarrowtail} I(X) \overset{\overline{x}}{\twoheadrightarrow} TX$ instead of $X \overset{\mu(X)}{\rightarrowtail} I(X) \overset{\pi(X)}{\twoheadrightarrow} TX$.

Take a morphism $u : X \to Y$ in $\mathscr{F}$ and let $X \overset{\mu}{\rightarrowtail} I \overset{\pi}{\twoheadrightarrow} X' \in \mathcal{S}$ with I an $\mathcal{S}$-injective object. Consider the solid part of the following diagram in $\mathscr{F}$

$$\begin{array}{ccccccc}
X & \overset{\mu}{\rightarrowtail} & I & \overset{\pi}{\twoheadrightarrow} & X' & & \\
\downarrow{\scriptstyle u} & {\scriptstyle PO} & \downarrow{\scriptstyle \overline{u}} & & \| & & \\
Y & \overset{v}{\rightarrowtail} & C_u & \overset{w'}{\twoheadrightarrow} & X' & \overset{g'}{\dashrightarrow} & TX
\end{array}
\qquad w := g'w' : C_u \dashrightarrow TX$$

where C_u is the pushout along μ and u, and w' is the cokernel of v as given in Lemma A.1. Note that $Y \overset{v}{\rightarrowtail} C_u \overset{w'}{\twoheadrightarrow} X' \in \mathcal{S}$ by Proposition 1.8. Lemma 2.11 provides a morphism $g' : X' \to TX$ such that $\underline{g'}$ is an isomorphism in $\underline{\mathscr{F}}$. Define $w := g'w' : C_u \to TX$.

Definition 2.15. *Using the notation as above, we call the triangle $X \overset{u}{\to} Y \overset{v}{\to} C_u \overset{w}{\to} TX$ and its image in $\underline{\mathscr{F}}$ a* **standard triangle**. *We define Δ to be the collection of all triangles in $\underline{\mathscr{F}}$ which are isomorphic to a standard triangle, and we call such triangles* **distinguished triangles**.

Remark 2.16. Some authors have a different definition of a standard triangle, which we call strictly standard: A triangle $X \overset{u}{\to} Y \overset{v}{\to} C_u \overset{w}{\to} TX$ is **strictly standard** if it is obtained from the pushout

$$\begin{array}{ccccc}
X & \overset{x}{\rightarrowtail} & I(X) & \overset{\overline{x}}{\twoheadrightarrow} & TX \\
\downarrow{\scriptstyle u} & {\scriptstyle PO} & \downarrow{\scriptstyle \overline{u}} & & \| \\
Y & \overset{v}{\rightarrowtail} & C_u & \overset{w}{\twoheadrightarrow} & TX
\end{array}$$

We will see in Corollary 2.18 that every standard triangle is isomorphic to the strictly standard triangle constructed from the same morphism $u : X \to Y$. Hence the collection of all triangles which are isomorphic in $\underline{\mathscr{F}}$ to a *strictly* standard triangle equals Δ. Thus both definitions of a standard triangle will give the same triangulation on $\underline{\mathscr{F}}$.

We shall now prove that $\underline{\mathscr{F}}$ is a triangulated category. However, we will need the Triangulated Five Lemma (Lemma B.4) in order to prove (TR4). Therefore, we first prove that $\underline{\mathscr{F}}$ is pretriangulated.

Theorem 2.17. *The triple $(\underline{\mathscr{F}}, T, \Delta)$ is a pretriangulated category.*

Proof. (TR1). It is clear from the definition that Δ is closed under isomorphisms and that every morphism is part of a triangle. Consider the following commutative diagram of triangles:

$$\begin{array}{ccccccc}
X & \overset{1}{\longrightarrow} & X & \overset{x}{\longrightarrow} & I(X) & \overset{\overline{x}}{\longrightarrow} & TX \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & {\scriptstyle PO} & \downarrow{\scriptstyle \overline{1}} & & \downarrow{\scriptstyle 1} \\
X & \overset{1}{\longrightarrow} & X & \overset{v}{\longrightarrow} & C_{1_X} & \overset{w}{\longrightarrow} & TX
\end{array}$$

Since $1_X : X \to X$ is an isomorphism, so is $\overline{1_X}$ by Corollary 1.7. Hence we have in fact an isomorphism of triangles. The bottom row is a standard triangle, thus the image in $\underline{\mathscr{F}}$ of the top row is distinguished. It is clear that $X \overset{1}{\to} X \overset{x}{\to} I(X) \overset{\overline{x}}{\to} TX \in \Delta$ and $X \overset{1}{\to} X \to 0 \to TX$ are isomorphic in $\underline{\mathscr{F}}$, making the latter triangle distinguished.

(TR2'). Note that (TR2') is half of (TR2), i.e. rotating only one way: see Appendix B. We only need to prove the axiom for standard triangles. Let $X \overset{u}{\to} Y \overset{v}{\to} C_u \overset{w}{\to} TX$ be a standard triangle given by the following diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{\mu} & I & \xrightarrow{\pi} & X' & & \\
\downarrow{\scriptstyle u} & PO & \downarrow{\scriptstyle \overline{u}} & & \| & & \\
Y & \xrightarrow{v} & C_u & \xrightarrow{w'} & X' & \xrightarrow{g'} & TX
\end{array}
\qquad w = g'w'$$

From Lemma 2.10 we get $u' : I \to I(Y)$ and $u'' : X' \to TY$ such that the following diagram commutes

$$\begin{array}{ccccc}
X & \overset{\mu}{\rightarrowtail} & I & \overset{\pi}{\twoheadrightarrow} & X' \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle u''} \\
Y & \overset{y}{\rightarrowtail} & I(Y) & \overset{\overline{y}}{\twoheadrightarrow} & TY
\end{array}$$

Since $yu = u'\mu$, the pushout property of C_u gives a unique morphism $\theta : C_u \to I(Y)$ such that the following diagram commutes

$$\begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow{\scriptstyle \mu} & PO & \downarrow{\scriptstyle v} \\
I & \xrightarrow{\overline{u}} & C_u
\end{array}
\qquad \theta : C_u \dashrightarrow I(Y),\quad \theta v = y,\quad \theta\overline{u} = u'$$

Note that $\overline{y}\theta v = \overline{y}y = 0 = u''w'v$ and $\overline{y}\theta\overline{u} = \overline{y}u' = u''\pi = u''w'\overline{u}$. Hence $\overline{y}\theta = u''w'$ by the pushout property of C_u. We get the following commutative diagram

$$\begin{array}{ccccc}
Y & \overset{v}{\rightarrowtail} & C_u & \overset{w'}{\twoheadrightarrow} & X' \\
\downarrow{\scriptstyle y} & & \downarrow{\scriptstyle \begin{bmatrix}\theta \\ w'\end{bmatrix}} & & \| \\
I(Y) & \overset{\begin{bmatrix}1 \\ 0\end{bmatrix}}{\rightarrowtail} & I(Y) \oplus X' & \overset{\begin{bmatrix}0 & 1\end{bmatrix}}{\twoheadrightarrow} & X' \\
\downarrow{\scriptstyle \overline{y}} & & \downarrow{\scriptstyle \begin{bmatrix}\overline{y} & -u''\end{bmatrix}} & & \\
TY & = & TY & &
\end{array} \tag{2.1}$$

As mentioned before Definition 2.15, the first row is exact. It follows from Quillen's axiom a) that the second row is exact. Hence from Proposition 1.8 the upper left square is a pushout. Since

$$\begin{bmatrix}\overline{y} & -u''\end{bmatrix}\begin{bmatrix}\theta \\ w'\end{bmatrix} = \overline{y}\theta - u''w' = 0,$$

Diagram 2.1 yields that

$$Y \xrightarrow{v} C_u \xrightarrow{\begin{bmatrix}\theta \\ w'\end{bmatrix}} I(Y) \oplus X' \xrightarrow{\begin{bmatrix}\overline{y} & -u''\end{bmatrix}} TY$$

is a standard triangle.

Now consider the following diagram:

$$\begin{array}{ccccccc}
Y & \xrightarrow{v} & C_u & \xrightarrow{\begin{bmatrix}\theta \\ w'\end{bmatrix}} & I(Y)\oplus X' & \xrightarrow{\begin{bmatrix}\overline{y} & -u''\end{bmatrix}} & TY \\
\Vert & & \Vert & & \big\downarrow{\scriptstyle \begin{bmatrix}0 & g'\end{bmatrix}} & & \Vert \\
Y & \xrightarrow{v} & C_u & \xrightarrow{w} & TX & \xrightarrow{-Tu} & TY
\end{array}$$

We have $\begin{bmatrix}0 & g'\end{bmatrix}\begin{bmatrix}\theta \\ w'\end{bmatrix} = g'w' = w$, making the middle square commutative. It is possible to use Lemma 2.12 to prove that $\underline{(Tu)g'} = \underline{u''}$ in $\underline{\mathscr{F}}$. Hence the square to the right commutes in $\underline{\mathscr{F}}$ since $-\underline{Tu}\begin{bmatrix}0 & \underline{g'}\end{bmatrix} = \begin{bmatrix}0 & -\underline{(Tu)g'}\end{bmatrix} = \begin{bmatrix}\underline{y} & -\underline{u''}\end{bmatrix}$. Moreover, we know that $\underline{g'}$ is an isomorphism in $\underline{\mathscr{F}}$, hence the diagram above is an isomorphism of triangles, thus $Y \xrightarrow{v} C_u \xrightarrow{w} TX \xrightarrow{-Tu} TY$ is distinguished.

(TR3). Consider two standard triangles

$$\begin{array}{ccccccc}
X & \xrightarrow{\mu} & I & \xrightarrow{\pi} & X' & & \\
\big\downarrow{\scriptstyle u} & & \big\downarrow{\scriptstyle \overline{u}} & & \Vert & & \\
Y & \xrightarrow{v} & Z & \xrightarrow{s} & X' & \xrightarrow{\tau} & TX
\end{array}
\qquad \textit{and} \qquad
\begin{array}{ccccccc}
A & \xrightarrow{\mu'} & I' & \xrightarrow{\pi'} & A' & & \\
\big\downarrow{\scriptstyle u'} & & \big\downarrow{\scriptstyle \overline{u'}} & & \Vert & & \\
B & \xrightarrow{v'} & C & \xrightarrow{s'} & A' & \xrightarrow{\lambda} & TA
\end{array}$$

with $\underline{\tau}, \underline{\lambda}$ isomorphisms. Let $w := \tau s$ and $w' := \lambda s'$. Assume that we have the following commutative diagram in $\underline{\mathscr{F}}$

$$\begin{array}{ccccccc}
X & \xrightarrow{\underline{u}} & Y & \xrightarrow{\underline{v}} & Z & \xrightarrow{\underline{w}} & TX \\
\big\downarrow{\scriptstyle \underline{\varphi}} & & \big\downarrow{\scriptstyle \underline{\psi}} & & & & \big\downarrow{\scriptstyle \underline{T\varphi}} \\
A & \xrightarrow{\underline{u'}} & B & \xrightarrow{\underline{v'}} & C & \xrightarrow{\underline{w'}} & TA
\end{array}$$

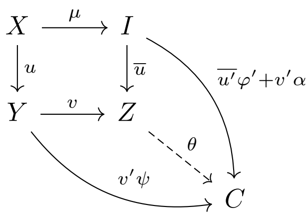
From Lemma 2.10 and 2.11 we know that we can construct the following commutative diagram with $\underline{\nu} : TX \to X'$ the inverse of $\underline{\tau} : X' \to TX$

$$\begin{array}{ccccc}
X & \xrightarrow{\mu(X)} & I(X) & \xrightarrow{\pi(X)} & TX \\
\Vert & & \big\downarrow{\scriptstyle \nu'} & & \big\downarrow{\scriptstyle \nu} \\
X & \xrightarrow{\mu} & I & \xrightarrow{\pi} & X' \\
\big\downarrow{\scriptstyle \varphi} & & \big\downarrow{\scriptstyle \varphi'} & & \big\downarrow{\scriptstyle \varphi''} \\
A & \xrightarrow{\mu'} & I' & \xrightarrow{\pi'} & A' \\
\Vert & & \big\downarrow{\scriptstyle \lambda'} & & \big\downarrow{\scriptstyle \lambda} \\
A & \xrightarrow{\mu(A)} & I(A) & \xrightarrow{\pi(A)} & TA
\end{array}$$

We get that $T\underline{\varphi} = \underline{\lambda\varphi''\nu}$ from Lemma 2.12. By assumption $\underline{\psi u} = \underline{u'\varphi}$, hence there is a morphism $\alpha : I \to B$ such that $\psi u = u'\varphi + \alpha\mu$. Now

$$v'\psi u = v'(u'\varphi + \alpha\mu) = v'u'\varphi + v'\alpha\mu = \overline{u'}\mu'\varphi + v'\alpha\mu = \overline{u'}\varphi'\mu + v'\alpha\mu = (\overline{u'}\varphi' + v'\alpha)\mu.$$

Since Z is a pushout we get a morphism $\theta : Z \to C$ such that $\theta\overline{u} = \overline{u'}\varphi' + v'\alpha,\ \theta v = v'\psi$ as in the diagram below.



The pushout property of Z gives that $s'\theta = \varphi'' s$ since

$$(s'\theta - \varphi'' s)\overline{u} = s'(\overline{u'}\varphi' + v'\alpha) - \varphi''\pi = \pi'\varphi' - \varphi''\pi = 0,$$

$$(s'\theta - \varphi'' s)v = s'\theta v - \varphi'' s v = s'v'\psi - \varphi'' s v = 0 - 0 = 0.$$

Hence we get a commutative diagram

$$\begin{CD} X @>{u}>> Y @>{v}>> Z @>{s}>> X' \\ @V{\underline{\varphi}}VV @V{\underline{\psi}}VV @V{\underline{\theta}}VV @V{\underline{\varphi''}}VV \\ A @>{\underline{u'}}>> B @>{\underline{v'}}>> C @>{\underline{s'}}>> A' \end{CD}$$

giving us the following morphism of triangles

$$\begin{CD} X @>{u}>> Y @>{v}>> Z @>{\underline{w}=\underline{\tau s}}>> TX \\ @| @| @| @V{\underline{\nu}}VV \\ X @>{u}>> Y @>{v}>> Z @>{\underline{s}}>> X' \\ @V{\underline{\varphi}}VV @V{\underline{\psi}}VV @V{\underline{\theta}}VV @V{\underline{\varphi''}}VV \\ A @>{\underline{u'}}>> B @>{\underline{v'}}>> C @>{\underline{s'}}>> A' \\ @| @| @| @V{\underline{\lambda}}VV \\ A @>{\underline{u'}}>> B @>{\underline{v'}}>> C @>{\underline{w'}=\underline{\lambda s'}}>> TA \end{CD} \qquad T\underline{\varphi} = \underline{\lambda}\underline{\varphi''}\underline{\nu}$$

Thus (TR3) holds for standard triangles. $\square$

Corollary 2.18. *Let $u : X \to Y$ be a morphism in $\mathscr{F}$. The two standard triangles given by*

$$\begin{CD} X @>{\mu}>> I @>{\pi}>> X' \\ @V{u}VV PO @V{\overline{u}}VV @| \\ Y @>{v}>> Z @>{s}>> X' @>{\tau}>> TX \end{CD} \qquad \textit{and} \qquad \begin{CD} X @>{\mu'}>> I' @>{\pi'}>> X'' \\ @V{u}VV PO @V{\overline{u}'}VV @| \\ Y @>{v'}>> Z' @>{s'}>> X'' @>{\tau'}>> TX \end{CD}$$

are isomorphic in $\underline{\mathscr{F}}$. Moreover, every standard triangle is isomorphic to the strictly standard triangle constructed from the same morphism $u : X \to Y$.

Proof. This follows immediately from (TR3) and the Triangulated Five Lemma (Lemma B.4) applied to the following commutative diagram:

$$\begin{CD} X @>{\underline{u}}>> Y @>{\underline{v}}>> Z @>{\underline{w}}>> TX \\ @| @| @. @| \\ X @>{\underline{u}}>> Y @>{\underline{v'}}>> Z' @>{\underline{w'}}>> TX \end{CD}$$

$\square$

Theorem 2.19. *The triple $(\underline{\mathscr{F}}, T, \Delta)$ is a triangulated category.*

Proof. (TR4). We only need to consider the case of standard triangles. By Corollary 2.18 we may take the triangles to be strictly standard, as defined in Remark 2.16. Assume that we have three standard triangles given by

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ \big\downarrow{\scriptstyle x} & & \big\downarrow{\scriptstyle i} \\ I(X) & \xrightarrow{\overline{u}} & Z' \\ \big\downarrow{\scriptstyle \overline{x}} & & \big\downarrow{\scriptstyle i'} \\ TX & = & TX \end{array}, \qquad \begin{array}{ccc} Y & \xrightarrow{v} & Z \\ \big\downarrow{\scriptstyle y} & & \big\downarrow{\scriptstyle j} \\ I(Y) & \xrightarrow{\overline{v}} & X' \\ \big\downarrow{\scriptstyle \overline{y}} & & \big\downarrow{\scriptstyle j'} \\ TY & = & TY \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \xrightarrow{w:=vu} & Z \\ \big\downarrow{\scriptstyle x} & & \big\downarrow{\scriptstyle k} \\ I(X) & \xrightarrow{\overline{w}} & Y' \\ \big\downarrow{\scriptstyle \overline{x}} & & \big\downarrow{\scriptstyle k'} \\ TX & = & TX \end{array}$$

Our goal is to prove that there exist morphisms such that the diagram below commutes in $\underline{\mathscr{F}}$ with $Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{(T\underline{i})j'} TZ' \in \Delta$.

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{i} & Z' & \xrightarrow{i'} & TX \\ \| & & \big\downarrow{\scriptstyle v} & & \big\downarrow{\scriptstyle f} & & \| \\ X & \xrightarrow{w=vu} & Z & \xrightarrow{k} & Y' & \xrightarrow{k'} & TX \\ & & \big\downarrow{\scriptstyle j} & & \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle Tu} \\ & & X' & = & X' & \xrightarrow{j'} & TY \\ & & \big\downarrow{\scriptstyle j'} & & \big\downarrow{\scriptstyle (Ti)j'} & & \\ & & TY & \xrightarrow{Ti} & TZ' & & \end{array} \tag{2.2}$$

Let $Z' \overset{l}{\rightarrowtail} I(Z') \overset{\overline{l}}{\twoheadrightarrow} TZ' \in \mathcal{S}$ with $I(Z')$ $\mathcal{S}$-injective. Since i and l are admissible monomorphisms, so is li, thus there exists $p : I(Z') \to M$ such that $(li, p) \in \mathcal{S}$. Now, by Corollary 2.18, we may take $Y \overset{li}{\rightarrowtail} I(Z') \overset{p}{\twoheadrightarrow} M$ instead of $Y \overset{y}{\rightarrowtail} I(Y) \overset{\overline{y}}{\twoheadrightarrow} TY$. Hence we may assume that $I(Y) = I(Z')$ and $y = li$. Then $yu = liu = l\overline{u}x$. Define $I_u := l\overline{u} : I(X) \to I(Y)$ as in the diagram below. Then there exists $Tu : TX \to TY$ with $Tu\overline{x} = \overline{y}I_u = \overline{y}l\overline{u}$. Let $1 : I(Y) \to I(Z')$, giving a morphism $Ti : TY \to TZ'$ with $Ti\overline{y} = \overline{l}$. In other words the following diagrams commute:

$$\begin{array}{ccccc} X & \overset{x}{\rightarrowtail} & I(X) & \overset{\overline{x}}{\twoheadrightarrow} & TX \\ \big\downarrow{\scriptstyle u} & & \big\downarrow{\scriptstyle I_u = l\overline{u}} & & \big\downarrow{\scriptstyle Tu} \\ Y & \overset{y}{\rightarrowtail} & I(Y) & \overset{\overline{y}}{\twoheadrightarrow} & TY \end{array} \qquad \begin{array}{ccccc} Y & \overset{y}{\rightarrowtail} & I(Y) & \overset{\overline{y}}{\twoheadrightarrow} & TY \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle 1} & & \big\downarrow{\scriptstyle Ti} \\ Z' & \overset{l}{\rightarrowtail} & I(Z') & \overset{\overline{l}}{\twoheadrightarrow} & TZ' \end{array}$$

Since $\overline{w}x = kw = kvu$ and $jw = jvu = \overline{v}yu = \overline{v}liu = \overline{v}l\overline{u}x$, we get the following pushouts:

$$\begin{array}{ccccc} X & \xrightarrow{u} & Y & & \\ \big\downarrow{\scriptstyle x} & & \big\downarrow{\scriptstyle i} & \searrow{\scriptstyle kv} & \\ I(X) & \xrightarrow{\overline{u}} & Z' & & \\ & \searrow{\scriptstyle \overline{w}} & & \searrow{\scriptstyle f} & \\ & & & & Y' \end{array} \qquad \begin{array}{ccccc} X & \xrightarrow{w} & Z & & \\ \big\downarrow{\scriptstyle x} & & \big\downarrow{\scriptstyle k} & \searrow{\scriptstyle j} & \\ I(X) & \xrightarrow{\overline{w}} & Y' & & \\ & \searrow{\scriptstyle \overline{v}l\overline{u}} & & \searrow{\scriptstyle g} & \\ & & & & X' \end{array}$$

By the pushout property of Z' we get that $gf = \overline{v}l$, since $gfi = gkv = jv = \overline{v}y = \overline{v}li$ and $gf\overline{u} = g\overline{w} = \overline{v}l\overline{u}$.

We now want to show that with f and g as defined above, diagram 2.2 commutes. We note that $fi = kv$ and $gk = j$ by construction. Hence we have to prove that $i' = k'f$ and $(Tu)k' = j'g$. Now $k'f = i'$ follows from the pushout property of Z' since

$$k'fi = k'kv = 0 = i'i,$$

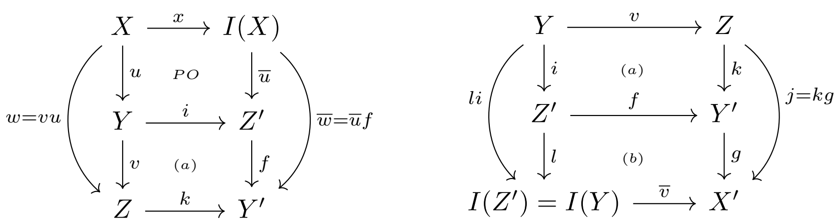
$$k'f\overline{u} = k'\overline{w} = \overline{x} = i'\overline{u}.$$

Similarly $(Tu)k' = j'g$ follows from the pushout property of Y' since

$$(Tu)k'k = 0 = j'j = j'gk,$$

$$(Tu)k'\overline{w} = (Tu)\overline{x} = \overline{y}l\overline{u} = j'\overline{v}l\overline{u} = j'gf\overline{u} = j'g\overline{w}.$$

The only thing left to prove is that $Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{(Ti)j'} TZ'$ is a standard triangle. Look at the following two commutative diagrams



In the diagram to the left, the outer square and the top square are pushouts. Hence by Lemma A.4, so is the bottom square, square (a). Furthermore, since the outer square of the diagram to the right is a pushout we can now use Lemma A.4 again and conclude that (b) is a pushout. Recall that $\bar{l} = (Ti)\overline{y} = (Ti)j'\overline{v}$. Hence we get the following commutative diagram

$$\begin{array}{ccc} Z' & \xrightarrow{f} & Y' \\ \downarrow{\scriptstyle l} & PO & \downarrow{\scriptstyle g} \\ I(Z') & \xrightarrow{\overline{v}} & X' \\ \downarrow{\scriptstyle \bar{l}} & & \downarrow{\scriptstyle (Ti)j'} \\ TZ' & = & TZ' \end{array}$$

which proves that $Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{(Ti)j'} TZ'$ is a standard triangle. □

2.3. Correspondence between short exact sequences and distinguished triangles

Short exact sequences in $\mathscr{F}$ are closely related to the distinguished triangles in $\underline{\mathscr{F}}$, as the following two theorems describe.

Theorem 2.20. *Any short exact sequence is part of a distinguished triangle in the following sense: Let $X \overset{u}{\rightarrowtail} Y \overset{u'}{\twoheadrightarrow} Z \in \mathcal{S}$. Then there exists a morphism $u'' : Z \to TX$ such that $X \xrightarrow{u} Y \xrightarrow{u'} Z \xrightarrow{-u''} TX \in \Delta$.*

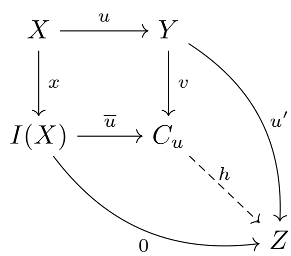
More precisely, we have by Lemma 2.10 the following commutative diagram with exact rows, which gives us the morphism u''.

$$\begin{CD} X @>{u}>> Y @>{u'}>> Z \\ @| @VV{i}V @VV{u''}V \\ X @>{x}>> I(X) @>{\overline{x}}>> TX \end{CD}$$

Proof. Let $X \xrightarrow{u} Y \xrightarrow{v} C_u \xrightarrow{w} TX$ be the strictly standard triangle given by the following commutative diagram

$$\begin{CD} X @>{u}>> Y \\ @VV{x}V @VV{v}V \\ I(X) @>{\overline{u}}>> C_u \\ @VV{\overline{x}}V @VV{w}V \\ TX @= TX \end{CD}$$

Our goal is to prove that $X \xrightarrow{u} Y \xrightarrow{u'} Z \xrightarrow{-u''} TX$ and $X \xrightarrow{u} Y \xrightarrow{v} C_u \xrightarrow{w} TX$ are isomorphic triangles in $\underline{\mathscr{F}}$. Consider the pushout



Since $u'u = 0 = 0 \circ x$ the pushout property of C_u gives a unique $h : C_u \to Z$ such that $hv = u'$ and $h\overline{u} = 0$. We claim that $(\underline{1}_X, \underline{1}_Y, \underline{h})$ is an isomorphism of triangles. First we will prove that $\underline{h}$ is an isomorphism in $\underline{\mathscr{F}}$.

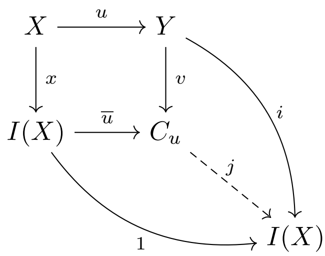
Since $(v - \overline{u}i)u = vu - \overline{u}iu = \overline{u}x - \overline{u}x = 0$, the cokernel property of u' gives a unique morphism $g : Z \to C_u$ such that $gu' = v - \overline{u}i$:

$$\begin{array}{ccccc} X & \overset{u}{\rightarrowtail} & Y & \overset{u'}{\twoheadrightarrow} & Z \\ & & & \searrow^{v-\overline{u}i} & \downarrow g \\ & & & & C_u \end{array}$$

Moreover, we get that

$$hgu' = h(v - \overline{u}i) = hv - h\overline{u}i = u' - 0i = u' = 1_Z u'.$$

Note that u' is an epimorphism by Remark 1.3 (5), hence $hg = 1_Z$. We shall now see that $gh - 1_{C_u}$ factors through an $\mathcal{S}$-injective object, giving $\underline{gh} = \underline{1_{C_u}}$ in $\underline{\mathscr{F}}$. This will prove that $\underline{h}$ is an isomorphism. Consider the pushout



Let $j : C_u \to I(X)$ be the unique morphism such that $jv = i$ and $j\overline{u} = 1$. If $ghv = (1_{C_u} - \overline{u}j)v$ and $gh\overline{u} = (1_{C_u} - \overline{u}j)\overline{u}$, then $gh = 1_{C_u} - \overline{u}j$ since C_u is a pushout. We have

$$(1_{C_u} - \overline{u}j)v = v - \overline{u}jv = v - \overline{u}i = gu' = ghv,$$

$$(1_{C_u} - \overline{u}j)\overline{u} = \overline{u} - \overline{u}j\overline{u} = \overline{u} - \overline{u} \circ 1 = 0 = g0 = gh\overline{u}.$$

Thus $gh = 1_{C_u} - \overline{u}j$, which gives $\underline{gh} = \underline{1}_{C_u}$ in $\underline{\mathscr{F}}$.

Now consider the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & C_u & \xrightarrow{w} & TX \\ \| & & \| & & \downarrow{\scriptstyle h} & & \| \\ X & \xrightarrow{u} & Y & \xrightarrow{u'} & Z & \xrightarrow{-u''} & TX \end{array}$$

This is an isomorphism of triangles if the diagram commutes. We have $hv = u'$ by construction, hence we only have to prove that $\underline{w} = -\underline{u''h}$ in $\underline{\mathscr{F}}$. Let $\beta = w - (-u''h) = w + u''h$ and let j be as above. The pushout property of C_u gives that $\beta = \overline{x}j$ if $\beta v = \overline{x}jv$ and $\beta\overline{u} = \overline{x}j\overline{u}$. We have

$$\beta v = (u''h + w)v = u''hv + wv = u''u' + 0 = \overline{x}i = \overline{x}jv,$$

$$\beta\overline{u} = (u''h + w)\overline{u} = u''h\overline{u} + w\overline{u} = u''0 + \overline{x} = \overline{x} = \overline{x}j\overline{u}.$$

Hence $X \xrightarrow{u} Y \xrightarrow{u'} Z \xrightarrow{-u''} TX$ is a distinguished triangle. □

The next theorem is the converse of Theorem 2.20.

Theorem 2.21. *For any distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ in $\underline{\mathscr{F}}$ there exist in $\mathscr{F}$ a short exact sequence $X' \overset{u'}{\rightarrowtail} Y' \overset{v'}{\twoheadrightarrow} Z'$ and a morphism $w' : Z' \to TX'$ such that $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{v} TX$ and $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} TX'$ are isomorphic as triangles in $\underline{\mathscr{F}}$.*

Proof. Assume that the triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX$ is distinguished, then so is the rotated triangle $T^{-1}Z \xrightarrow{T^{-1}w} X \xrightarrow{u} Y \xrightarrow{v} Z$ by (TR2). By the definition of Δ and Corollary 2.18 we have that $T^{-1}Z \xrightarrow{T^{-1}w} X \xrightarrow{u} Y \xrightarrow{v} Z \in \Delta$ is isomorphic to a strictly standard triangle $Z'' \xrightarrow{w''} X' \xrightarrow{u'} Y' \xrightarrow{v'} TZ''$, arriving from the following diagram

$$\begin{array}{ccccc} Z'' & \overset{\mu}{\rightarrowtail} & I(Z'') & \overset{\pi}{\twoheadrightarrow} & TZ'' \\ {\scriptstyle w''}\downarrow & PO & \downarrow & & \| \\ X' & \overset{u'}{\rightarrowtail} & Y' & \overset{v'}{\twoheadrightarrow} & TZ'' \end{array}$$

Note that $X' \overset{u'}{\rightarrowtail} Y' \overset{v'}{\twoheadrightarrow} TZ''$ is short exact in $\mathscr{F}$ by Proposition 1.8. Define $Z' := TZ''$ and $w' := T(w'') : Z' \to TX'$. Then the distinguished triangles $X \overset{u}{\to} Y \overset{v}{\to} Z \overset{v}{\to} TX$ and $X' \overset{u'}{\longrightarrow} Y' \overset{v'}{\longrightarrow} Z' \overset{w'}{\longrightarrow} TX'$ are isomorphic as triangles in $\underline{\mathscr{F}}$ with $X' \overset{u'}{\rightarrowtail} Y' \overset{v'}{\twoheadrightarrow} Z' \in \mathcal{S}$. $\square$

3. Classifying subcategories

3.1. Subcategories of an exact category and a quotient category

Let $(\mathscr{E}, \mathcal{S})$ be an exact category and $\mathscr{N}$ a full subcategory which is closed under finite direct sums, and let $\mathscr{E}/\mathscr{N}$ be the quotient category as defined in Definition 1.18. We denote morphisms in $\mathscr{E}/\mathscr{N}$ by $\underline{f}$. Denote by $\mathcal{S}_{\mathscr{N}}$ the collection of all sequences $X \xrightarrow{\underline{f}} Y \xrightarrow{\underline{g}} Z$ in $\mathscr{E}/\mathscr{N}$ for which there exists an isomorphism of sequences

$$\begin{array}{ccccc} X & \xrightarrow{\underline{f}} & Y & \xrightarrow{\underline{g}} & Z \\ \downarrow\cong & \circlearrowleft & \downarrow\cong & \circlearrowleft & \downarrow\cong \\ X' & \xrightarrow{\underline{f'}} & Y' & \xrightarrow{\underline{g'}} & Z' \end{array}$$

with $X' \overset{f'}{\rightarrowtail} Y' \overset{g'}{\twoheadrightarrow} Z' \in \mathcal{S}$.

We shall establish a bijection between certain subcategories of the exact category $\mathscr{E}$ and the quotient category $\mathscr{E}/\mathscr{N}$.

Definition 3.1. *A nonempty subcategory $\mathscr{E}'$ of an exact category $(\mathscr{E}, \mathcal{S})$ is called a* **complete subcategory** *if the following hold*

(i) $\mathscr{E}'$ is a full subcategory,

(ii) 2 out of 3: If $X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z \in \mathcal{S}$ and two of X, Y, Z are in $\mathscr{E}'$, then so is the third.

Moreover, $\mathscr{E}'$ is a **thick subcategory** *if in addition the following holds*

(iii) $\mathscr{E}'$ is closed under direct summands, i.e. if A is a direct summand of X, then $X \in \mathscr{E}'$ implies $A \in \mathscr{E}'$.

Remark 3.2. Let $\mathscr{E}'$ be a complete subcategory of $\mathscr{E}$. Take $X \in \mathscr{E}'$: such an object exists since $\mathscr{E}'$ is nonempty. Then $0 \in \mathscr{E}'$ since $X \overset{1_X}{\rightarrowtail} X \twoheadrightarrow 0$ is short exact. Hence by Proposition 1.10 $\mathscr{E}'$ is additive, closed under isomorphisms and admits an exact structure induced by the exact structure on $\mathscr{E}$.

By Proposition 1.5 all sequences of the form $A \overset{\left[\begin{smallmatrix}1 & 0\end{smallmatrix}\right]^t}{\rightarrowtail} A \oplus B \overset{\left[\begin{smallmatrix}0 & 1\end{smallmatrix}\right]}{\twoheadrightarrow} B$ are short exact. Hence if two of A, B and $A \oplus B$ are in $\mathscr{E}'$, then so is the third by axiom (ii). We will use this fact repeatedly without referring to Proposition 1.5 every time.

Definition 3.3. *A nonempty subcategory $\mathscr{E}'_{\mathscr{N}}$ of the quotient category $\mathscr{E}/\mathscr{N}$ is called a* **complete subcategory** *if the following hold*

(i) $\mathscr{E}'_{\mathscr{N}}$ is a full subcategory,

(ii) 2 out of 3: If $X \xrightarrow{\underline{f}} Y \xrightarrow{\underline{g}} Z \in \mathcal{S}_{\mathscr{N}}$ and two of X, Y, Z are in $\mathscr{E}'_{\mathscr{N}}$, then so is the third.

Moreover, $\mathscr{E}'_{\mathscr{N}}$ is a **thick subcategory** *if in addition the following holds*

(iii) $\mathscr{E}'_{\mathscr{N}}$ is closed under direct summands.

Our goal is to establish a one-to-one correspondence between the complete/thick subcategories of $\mathscr{E}$ containing $\mathscr{N}$ and the complete/thick subcategories of $\mathscr{E}/\mathscr{N}$, under the right assumptions. First we need to prove that complete and thick subcategories of $\mathscr{E}$ containing $\mathscr{N}$ are closed under isomorphisms when we pass to $\mathscr{E}/\mathscr{N}$. This is always true in the case of a thick subcategory, as the corollary of the next proposition shows. However, we need some extra assumptions in the case of a complete subcategory.

Proposition 3.4. *Let $\mathscr{E}'$ be a thick subcategory of $\mathscr{E}$ containing $\mathscr{N}$. If Y is a direct summand of X in the quotient category $\mathscr{E}/\mathscr{N}$, then $X \in \mathscr{E}'$ implies $Y \in \mathscr{E}'$.*

Proof. Assume that we have $Y \xrightarrow{g} X \xrightarrow{f} Y$ with $\underline{fg} = \underline{1}_Y$ in $\mathscr{E}/\mathscr{N}$. Then there exists a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{fg-1_Y} & Y \\ & {\scriptstyle\alpha}\searrow \quad \nearrow{\scriptstyle\beta} & \\ & N & \end{array}$$

with $N \in \mathscr{N}$. Now define morphisms i, p by

$$Y \xrightarrow{i := \begin{bmatrix} g \\ \alpha \end{bmatrix}} X \oplus N \xrightarrow{p := \begin{bmatrix} f & -\beta \end{bmatrix}} Y$$

Then $pi = \begin{bmatrix} f & -\beta \end{bmatrix} \begin{bmatrix} g \\ \alpha \end{bmatrix} = fg - \beta\alpha = 1_Y$, so Y is a direct summand of $X \oplus N$. Moreover, since $X, N \in \mathscr{E}'$ the 2 out of 3 property gives that $X \oplus N \in \mathscr{E}'$. Hence Y is a direct summand of an object in $\mathscr{E}'$, implying $Y \in \mathscr{E}'$. □

Corollary 3.5. *Let $\mathscr{E}'$ be a thick subcategory of $\mathscr{E}$ containing $\mathscr{N}$. Then $\mathscr{E}'$ is closed under isomorphisms in the quotient category $\mathscr{E}/\mathscr{N}$: if X and Y are isomorphic in $\mathscr{E}/\mathscr{N}$, then $X \in \mathscr{E}'$ implies $Y \in \mathscr{E}'$.*

We now state the additional assumptions we need to prove the above for complete (not necessarily thick) subcategories containing $\mathscr{N}$.

Definition 3.6. *A (full) subcategory $\mathscr{N}$ of $\mathscr{E}$ is* **factorization admissible** *if it is closed under finite direct sums, direct summands, and if whenever a morphism $f : X \to Y$ factors through an object in $\mathscr{N}$, then there exists a factorization*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle\alpha}\searrow \quad \nearrow{\scriptstyle\beta} & \\ & N & \end{array}$$

with $N \in \mathscr{N}$ such that either α is an admissible monomorphism or β is an admissible epimorphism.

As the following example shows, such subcategories appear naturally.

Example 3.7. If $\mathscr{E}$ has enough $\mathcal{S}$-injective objects, then $\operatorname{inj} \mathscr{E}$ is a factorization admissible subcategory. Indeed, we may take α to be an admissible monomorphism $\mu : X \rightarrowtail I$ as described in Remark 2.9. Moreover, $\operatorname{inj} \mathscr{E}$ is closed under finite direct sums and direct summands. Similarly, if $\mathscr{E}$ has enough $\mathcal{S}$-projective objects, then $\operatorname{proj} \mathscr{E}$ is a factorization admissible subcategory of $\mathscr{E}$.

The other technical condition we need when we deal with complete subcategories is an adapted version of the Five Lemma.

Definition 3.8. *We say that the quotient category $\mathscr{E}/\mathscr{N}$ satisfies the* **Weak Five Lemma** *for $\mathcal{S}_{\mathscr{N}}$ if whenever we have a morphism of sequences in $\mathcal{S}_{\mathscr{N}}$*

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \qquad & \in \mathcal{S}_{\mathscr{N}} \\ \downarrow{\scriptstyle \underline{f}} & & \downarrow{\scriptstyle \underline{g}} & & \downarrow{\scriptstyle \underline{h}} & & \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \qquad & \in \mathcal{S}_{\mathscr{N}} \end{array}$$

then the following hold:

(i) If $\underline{f}$ and $\underline{g}$ are isomorphisms and $Z' = 0$, then $\underline{h}$ is an isomorphism (i.e. $Z \cong 0$).

(ii) Dually, if $\underline{g}$ and $\underline{h}$ are isomorphisms and $X = 0$, then $\underline{f}$ is an isomorphism.

Lemma 3.9. *If $X \cong 0$ in $\mathscr{E}/\mathscr{N}$, then X is a direct summand of an object in $\mathscr{N}$.*

Proof. We have $\underline{1}_X = \underline{0}$ in $\mathscr{E}/\mathscr{N}$, so there is a factorization

$$\begin{array}{ccc} X & \xrightarrow{1_X - 0} & X \\ & {\scriptstyle \alpha}\searrow \quad \nearrow{\scriptstyle \beta} & \\ & N & \end{array}$$

in $\mathscr{E}$ with $N \in \mathscr{N}$. Hence $\beta\alpha = 1_X$, so X is a direct summand of N. □

Proposition 3.10. *Assume that $\mathscr{N}$ is a factorization admissible subcategory of $\mathscr{E}$ and that $\mathscr{E}/\mathscr{N}$ satisfies the Weak Five Lemma for $\mathcal{S}_{\mathscr{N}}$. If $\mathscr{E}'$ is a complete subcategory of $\mathscr{E}$ containing $\mathscr{N}$, then $\mathscr{E}'$ is closed under isomorphisms in the quotient category $\mathscr{E}/\mathscr{N}$.*

Proof. Assume that $X \in \mathscr{E}'$ is isomorphic to Y in $\mathscr{E}/\mathscr{N}$, and that $X \underset{\underline{g}}{\overset{\underline{f}}{\rightleftarrows}} Y$ are inverse isomorphisms. Then $fg - 1_Y$ factors through some object $N \in \mathscr{N}$ as in the diagram below.

$$\begin{array}{ccc} Y & \xrightarrow{fg - 1_Y} & Y \\ & {\scriptstyle \alpha}\searrow \quad \nearrow{\scriptstyle \beta} & \\ & N & \end{array}$$

We may assume that either α is an admissible monomorphism or that β is an admissible epimorphism. If α is an admissible monomorphism, consider the pushout

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ {\scriptstyle \alpha}\downarrow & PO & \downarrow{\scriptstyle a} \\ N & \xrightarrow{g'} & C \end{array}$$

By Lemma 1.6 we get that

$$Y \xrightarrow{\begin{bmatrix} g \\ \alpha \end{bmatrix}} X \oplus N \xrightarrow{\begin{bmatrix} a & -g' \end{bmatrix}} C$$

lies in $\mathcal{S}$. Hence we get the following morphism of sequences in $\mathcal{S}_{\mathscr{N}}$

$$\begin{array}{ccccccc}
Y & \xrightarrow{\underline{\left[\begin{smallmatrix} g \\ \alpha \end{smallmatrix}\right]}} & X \oplus N & \xrightarrow{\underline{\left[\begin{smallmatrix} a & -g' \end{smallmatrix}\right]}} & C & & \in \mathcal{S}_{\mathscr{N}} \\
\big\downarrow \underline{g} & \circlearrowleft & \big\downarrow \underline{\left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right]} & \circlearrowleft & \big\downarrow & & \\
X & \xrightarrow{\underline{1}_X} & X & \longrightarrow & 0 & & \in \mathcal{S}_{\mathscr{N}}
\end{array}$$

Since $\underline{g}$ and $\underline{\left[\begin{smallmatrix} 1 & 0 \end{smallmatrix}\right]}$ are isomorphisms in $\mathscr{E}/\mathscr{N}$, so is the map $C \to 0$ by the Weak Five Lemma for $\mathcal{S}_{\mathscr{N}}$. By Lemma 3.9, $C \cong 0$ implies that $C \in \mathscr{N}$ since $\mathscr{N}$ is closed under direct summands. Furthermore, $\mathscr{E}'$ contains $\mathscr{N}$, so this gives that $C \in \mathscr{E}'$. We have $X \oplus N \in \mathscr{E}'$ by the 2 out of 3 property, since $X, N \in \mathscr{E}'$. Hence the 2 out of 3 property applied to $Y \rightarrowtail X \oplus N \twoheadrightarrow C$ implies that $Y \in \mathscr{E}'$.

If instead β is an admissible epimorphism consider the pullback

$$\begin{array}{ccc}
C & \xrightarrow{b} & X \\
f' \big\downarrow & {\scriptstyle PB} & \big\downarrow f \\
N & \overset{\beta}{\twoheadrightarrow} & Y
\end{array}$$

We get $C \overset{\left[\begin{smallmatrix} b \\ f' \end{smallmatrix}\right]}{\rightarrowtail} X \oplus N \overset{\left[\begin{smallmatrix} f & -\beta \end{smallmatrix}\right]}{\twoheadrightarrow} Y \in \mathcal{S}$ and a morphism of sequences in $\mathcal{S}_{\mathscr{N}}$

$$\begin{array}{ccccccc}
0 & \longrightarrow & X & \xrightarrow{\underline{1}_X} & X & & \in \mathcal{S}_{\mathscr{N}} \\
\big\downarrow & & \big\downarrow \underline{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]} & & \big\downarrow \underline{f} & & \\
C & \xrightarrow{\underline{\left[\begin{smallmatrix} b \\ f' \end{smallmatrix}\right]}} & X \oplus N & \xrightarrow{\underline{\left[\begin{smallmatrix} f & -\beta \end{smallmatrix}\right]}} & Y & & \in \mathcal{S}_{\mathscr{N}}
\end{array}$$

As before, the fact that $\underline{f}$ and $\underline{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]}$ are isomorphisms in $\mathscr{E}/\mathscr{N}$ implies that $C \in \mathscr{N} \subseteq \mathscr{E}'$, which again implies that $Y \in \mathscr{E}'$. $\square$

Recall that $\mathscr{N}$ is a full subcategory of the exact category $\mathscr{E}$, and that this subcategory is closed under finite direct sums. We assume that $\mathscr{N}$ is factorization admissible only when stated. We now construct the maps

$$\{\text{complete subcategories of } \mathscr{E} \text{ containing } \mathscr{N}\} \xrightarrow{F} \{\text{subcategories of } \mathscr{E}/\mathscr{N}\}$$

$$\{\text{subcategories of } \mathscr{E} \text{ containing } \mathscr{N}\} \xleftarrow{G} \{\text{complete subcategories of } \mathscr{E}/\mathscr{N}\}$$

that will induce inverse bijections in several cases.

Definition 3.11. **(a)** *For a complete subcategory $\mathscr{E}'$ of $\mathscr{E}$ containing $\mathscr{N}$, define $F\mathscr{E}'$ to be the full subcategory of $\mathscr{E}/\mathscr{N}$ whose objects are the objects of $\mathscr{E}'$.*

(b) *For a complete subcategory $\mathscr{E}'_{\mathscr{N}}$ of $\mathscr{E}/\mathscr{N}$, define $G\mathscr{E}'_{\mathscr{N}}$ to be the full subcategory of $\mathscr{E}$ whose objects are the objects in $\mathscr{E}'_{\mathscr{N}}$ (including $\mathscr{N}$).*

We first prove that F and G are maps between the two collections of complete subcategories, under the right assumptions.

Theorem 3.12. *If $\mathscr{E}'_{\mathscr{N}}$ is a complete subcategory of $\mathscr{E}/\mathscr{N}$, then $G\mathscr{E}'_{\mathscr{N}}$ is a complete subcategory of $\mathscr{E}$ containing $\mathscr{N}$. Moreover, if $\mathscr{E}'_{\mathscr{N}}$ is thick, then so is $G\mathscr{E}'_{\mathscr{N}}$.*

Proof. The subcategory $G\mathscr{E}'_{\mathscr{N}}$ is full by definition. Now assume that $X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z$ is in $\mathcal{S}$. Then $X \overset{f}{\longrightarrow} Y \overset{g}{\longrightarrow} Z$ is in $\mathcal{S}_{\mathscr{N}}$. Hence the 2 out of 3 property of $G\mathscr{E}'_{\mathscr{N}}$ follows from 2 out of 3 property of $\mathscr{E}'_{\mathscr{N}}$ since $\operatorname{obj} G\mathscr{E}'_{\mathscr{N}} = \operatorname{obj} \mathscr{E}'_{\mathscr{N}}$. Furthermore, if A is a direct summand of X in $\mathscr{E}$, then A is also a direct summand of X in $\mathscr{E}/\mathscr{N}$. Hence $G\mathscr{E}'_{\mathscr{N}}$ is closed under direct summands in $\mathscr{E}$ since $\mathscr{E}'_{\mathscr{N}}$ is closed under direct summands in $\mathscr{E}/\mathscr{N}$. Note that all objects in $\mathscr{N}$ are isomorphic to the zero object in $\mathscr{E}/\mathscr{N}$. Since $\mathscr{E}'_{\mathscr{N}}$ is closed under isomorphisms and $0 \in \mathscr{E}'_{\mathscr{N}}$, this implies that $\mathscr{N}$ is contained in $\mathscr{E}'_{\mathscr{N}}$, and therefore also in $G\mathscr{E}'_{\mathscr{N}}$. □

Lemma 3.13. *Let $\mathscr{E}'$ be a complete subcategory of $\mathscr{E}$ (containing $\mathscr{N}$) and assume that $\mathscr{E}'$ is closed under isomorphisms in $\mathscr{E}/\mathscr{N}$. Then $F\mathscr{E}'$ is a complete subcategory of $\mathscr{E}/\mathscr{N}$.*

Proof. The subcategory $F\mathscr{E}'$ is full by definition. Let the sequence $X \overset{f}{\to} Y \overset{g}{\to} Z$ be in $\mathcal{S}_{\mathscr{N}}$. By the definition of $\mathcal{S}_{\mathscr{N}}$ there exists an isomorphism

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' \end{array}$$

of sequences in $\mathscr{E}/\mathscr{N}$ with $X' \overset{f'}{\rightarrowtail} Y' \overset{g'}{\twoheadrightarrow} Z'$ short exact in $\mathscr{E}$. By assumption, $\mathscr{E}'$ is closed under isomorphisms in $\mathscr{E}/\mathscr{N}$. Thus the 2 out of 3 property of $F\mathscr{E}'$ follows directly from the 2 out of 3 property of $\mathscr{E}'$. Indeed, assume for example that $X, Y \in \operatorname{obj} F\mathscr{E}' = \operatorname{obj} \mathscr{E}'$. Then we have $X', Y' \in \operatorname{obj} \mathscr{E}'$ since $\mathscr{E}'$ is closed under isomorphisms in $\mathscr{E}/\mathscr{N}$. Furthermore, since $X' \overset{f'}{\rightarrowtail} Y' \overset{g'}{\twoheadrightarrow} Z'$ short exact in $\mathscr{E}$, the 2 out of 3 property of $\mathscr{E}'$ implies that $Z' \in \operatorname{obj} \mathscr{E}'$. Again, $\mathscr{E}'$ is closed under isomorphisms in $\mathscr{E}/\mathscr{N}$, so this implies that $Z \in \operatorname{obj} \mathscr{E}' = \operatorname{obj} F\mathscr{E}'$. □

Theorem 3.14. *(1) Assume that $\mathscr{N}$ is a factorization admissible subcategory of $\mathscr{E}$ and that $\mathscr{E}/\mathscr{N}$ satisfies the Weak Five Lemma for $\mathcal{S}_{\mathscr{N}}$. If $\mathscr{E}'$ is a complete subcategory of $\mathscr{E}$ containing $\mathscr{N}$, then $F\mathscr{E}'$ is a complete subcategory of $\mathscr{E}/\mathscr{N}$.*

(2) If $\mathscr{E}'$ contains $\mathscr{N}$ and is thick, then so is $F\mathscr{E}'$.

Proof. The subcategory $\mathscr{E}'$ is closed under isomorphisms in $\mathscr{E}/\mathscr{N}$ in both cases by Proposition 3.10 and Corollary 3.5, respectively. Thus by Lemma 3.13 $F\mathscr{E}'$ is a complete subcategory of $\mathscr{E}/\mathscr{N}$. Moreover, if $\mathscr{E}'$ is closed under direct summands in $\mathscr{E}$, then $F\mathscr{E}'$ is closed under direct summands in $\mathscr{E}/\mathscr{N}$ by Proposition 3.4. □

We now have the following main result, which follows from the above.

Theorem 3.15. *Let $\mathscr{E}$ be an exact category. Assume that $\mathscr{N}$ is a factorization admissible subcategory of $\mathscr{E}$ and that $\mathscr{E}/\mathscr{N}$ satisfies the Weak Five Lemma for $\mathcal{S}_{\mathscr{N}}$. Then there is a one-to-one correspondence between complete subcategories of $\mathscr{E}$ containing $\mathscr{N}$ and complete subcategories of the quotient category $\mathscr{E}/\mathscr{N}$ given by*

$$\{\textit{complete subcategories of } \mathscr{E} \textit{ containing } \mathscr{N}\} \underset{G}{\overset{F}{\rightleftarrows}} \{\textit{complete subcategories of } \mathscr{E}/\mathscr{N}\}.$$

When we consider thick subcategories, we can drop the technical assumptions.

Theorem 3.16. *Let $\mathscr{E}$ be an exact category. Then there is a one-to-one correspondence between thick subcategories of $\mathscr{E}$ containing $\mathscr{N}$ and thick subcategories of the quotient category $\mathscr{E}/\mathscr{N}$ given by*

$$\{\textit{thick subcategories of } \mathscr{E} \textit{ containing } \mathscr{N}\} \underset{G}{\overset{F}{\rightleftarrows}} \{\textit{thick subcategories of } \mathscr{E}/\mathscr{N}\}.$$

3.2. Subcategories of a Frobenius category and the stable category

In this section we look at the special case where $(\mathscr{F}, \mathcal{S})$ is Frobenius and $\mathscr{N} = \operatorname{inj}\mathscr{F}$, i.e. the quotient category $\mathscr{F}/\mathscr{N}$ is the stable category $\underline{\mathscr{F}}$. Note that $\operatorname{inj}\mathscr{F}$ is a factorization admissible subcategory of $\mathscr{F}$ as described in Example 3.7.

Lemma 3.17. *A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is in $\mathcal{S}_{\mathscr{N}}$ if and only if there exists a morphism $\underline{h} : Z \to TX$ such that $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ is in Δ.*

Proof. Assume that $X \xrightarrow{f} Y \xrightarrow{g} Z \in \mathcal{S}_{\mathscr{N}}$. Then there exists an isomorphism of sequences given by the solid part of the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \overset{T\varphi_1^{-1}h'\varphi_3}{\dashrightarrow} & TX \\ \underline{\varphi_1}\downarrow\cong & & \underline{\varphi_2}\downarrow\cong & & \underline{\varphi_3}\downarrow\cong & & T\underline{\varphi_1}\downarrow\cong \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \overset{\underline{h'}}{\dashrightarrow} & TX' \end{array}$$

with $X' \overset{f'}{\rightarrowtail} Y' \overset{g'}{\twoheadrightarrow} Z' \in \mathcal{S}$. By Theorem 2.20 there exists a morphism $h' : Z' \to TX'$ such that $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} TX' \in \Delta$. Define $\underline{h} := (T\underline{\varphi_1})^{-1} \circ \underline{h'} \circ \underline{\varphi_3}$. Then $(\varphi_1, \varphi_2, \varphi_3)$ is an isomorphism between the triangles $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} TX' \in \Delta$ and $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$. Hence the latter triangle is also in Δ.

Now assume that $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX \in \Delta$. Then by Theorem 2.21 there exists a short exact sequence $X' \overset{f'}{\rightarrowtail} Y' \overset{g'}{\twoheadrightarrow} Z'$ and a morphism $h' : Z' \to TX'$ such that $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} TX'$ are isomorphic as triangles. This isomorphism of triangles clearly gives an isomorphism of the sequences $X \xrightarrow{f} Y \xrightarrow{g} Z$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \in \mathcal{S}_{\mathscr{N}}$, hence $X \xrightarrow{f} Y \xrightarrow{g} Z \in \mathcal{S}_{\mathscr{N}}$. ☐

Recall that a nonempty subcategory $\mathscr{T}'$ of a triangulated category $(\mathscr{T}, T, \Delta)$ is a **triangulated subcategory** if the following hold

(i) it is a full subcategory,

(ii) 2 out of 3: If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ is a distinguished triangle in $\mathscr{T}$ and two out of three of X, Y, Z are in $\mathscr{T}'$, then so is the third.

Moreover, $\mathscr{T}'$ is a **thick triangulated subcategory** if in addition the following holds

(iii) it is closed under direct summands.

The stable category $\underline{\mathscr{F}}$ of a Frobenius category $\mathscr{F}$ is triangulated by Theorem 2.19. The following proposition gives an equivalent definition of (thick) triangulated subcategories of $\underline{\mathscr{F}}$.

Proposition 3.18. *A subcategory $\underline{\mathscr{F}}'$ of $\underline{\mathscr{F}}$ is a triangulated subcategory if and only if it is a complete subcategory. Furthermore, $\underline{\mathscr{F}}'$ is a thick triangulated subcategory if and only if it is a thick subcategory as in the sense of Definition 3.1.*

Proof. It follows directly from Lemma 3.17 that the following are equivalent

(a) If $X \overset{f}{\to} Y \overset{g}{\to} Z \overset{h}{\to} TX \in \Delta$ and two of X, Y, Z are in $\underline{\mathscr{F}}'$, then so is the third.

(b) If $X \overset{f}{\to} Y \overset{g}{\to} Z \in \mathcal{S}_{\mathscr{N}}$ and two of X, Y, Z are in $\underline{\mathscr{F}}'$, then so is the third. □

Remark 3.19. The Triangulated Five Lemma (Lemma B.4) implies that $\underline{\mathscr{F}}$ satisfies the Weak Five Lemma for $\mathcal{S}_{\mathscr{N}}$.

We are now ready to state the two main Theorems 3.15 and 3.16 in the case of a Frobenius category and the associated stable category. The results follow from above.

Theorem 3.20. *Let $\mathscr{F}$ be a Frobenius category. Then there is a one-to-one correspondence between complete subcategories of $\mathscr{F}$ containing* inj $\mathscr{F}$ *and triangulated subcategories of the associated stable category $\underline{\mathscr{F}}$ given by*

$$\{\textit{complete subcategories of } \mathscr{F} \textit{ containing } \operatorname{inj} \mathscr{F}\} \underset{G}{\overset{F}{\rightleftarrows}} \{\textit{triangulated subcategories of } \underline{\mathscr{F}}\}.$$

Theorem 3.21. *Let $\mathscr{F}$ be a Frobenius category. Then there is a one-to-one correspondence between thick subcategories of $\mathscr{F}$ containing* inj $\mathscr{F}$ *and thick triangulated subcategories of the associated stable category $\underline{\mathscr{F}}$ given by*

$$\{\textit{thick subcategories of } \mathscr{F} \textit{ containing } \operatorname{inj} \mathscr{F}\} \underset{G}{\overset{F}{\rightleftarrows}} \{\textit{thick triangulated subcategories of } \underline{\mathscr{F}}\}.$$

4. Applications

In this chapter we consider certain Frobenius categories and apply the main theorems from last chapter, Theorem 3.20 and Theorem 3.21, to these. In Section 4.1 we regard Gorenstein projective objects in an abelian category and prove that this gives a Frobenius category. In Section 4.2 we take the abelian category to be the category of all finitely generated modules over a commutative Noetherian ring. In this case, the Gorenstein projective objects are the totally reflexive modules. Specializing further, we let the ring be a Gorenstein local ring. Then the Gorenstein projective objects are the maximal Cohen-Macaulay modules.

4.1. Gorenstein projective objects in abelian categories

Recall from Example 1.4 that an abelian category $\mathscr{A}$ is equipped with the standard exact structure if a sequence $X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z$ is short exact if and only if $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is exact. We call an object projective if it is projective with respect to this exact structure. In this section, let $\mathscr{A}$ be an abelian category with the standard exact structure, and assume that $\mathscr{A}$ has enough projective objects.

For a subcategory $\mathscr{A}'$ of $\mathscr{A}$, define

$$\mathcal{S}' := \{X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z \text{ short exact in } \mathscr{A} \mid X, Y, Z \in \mathscr{A}'\}.$$

Define by $\operatorname{proj} \mathscr{A}'$, resp. $\operatorname{inj} \mathscr{A}'$', the full subcategory of $\mathscr{A}'$ consisting of all $\mathcal{S}'$-projective, resp. $\mathcal{S}'$-injective, objects. Note that $(\mathscr{A}', \mathcal{S}')$ need not to be an exact category.

Let X be an object in $\mathscr{A}$. Then a sequence

$$\mathbf{P} = \quad (\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to 0)$$

is a **projective resolution** of X if P_n is projective for all n and the sequence is exact, except for in position 0, where $\operatorname{Cok} d_1 = X$. Note that every object has a projective resolution since $\mathscr{A}$ has enough projectives.

Definition 4.1. *An object $X \in \mathscr{A}$ is a* **Gorenstein projective object** *if there exists an exact sequence*

$$\mathbf{P} = \quad (\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{f} P_{-1} \to P_{-2} \to \dots)$$

of projective objects in $\mathscr{A}$, such that $X = \operatorname{Im} f$ and

$$\operatorname{Hom}_{\mathscr{A}}(\mathbf{P}, Q) = \quad (\cdots \to \operatorname{Hom}_{\mathscr{A}}(P_{-1}, Q) \to \operatorname{Hom}_{\mathscr{A}}(P_0, Q) \to \operatorname{Hom}_{\mathscr{A}}(P_1, Q) \to \dots)$$

is exact for all $Q \in \operatorname{proj} \mathscr{A}$. In this case $\mathbf{P}$ is called a **complete projective resolution** *of X.*

We denote by $\operatorname{Gproj} \mathscr{A}$ the full subcategory of $\mathscr{A}$ consisting of all Gorenstein projective objects in $\mathscr{A}$.

Lemma 4.2. *All projective objects in $\mathscr{A}$ are Gorenstein projective objects, i.e.*

$$\operatorname{proj} \mathscr{A} \subseteq \operatorname{Gproj} \mathscr{A}.$$

Proof. Let $P \in \operatorname{proj} \mathscr{A}$, then it is easily seen that

$$\mathbf{P} = \quad (\cdots \to 0 \to 0 \to P \xrightarrow{1_P} P \to 0 \to 0 \to \dots)$$

is a complete projective resolution of P. □

Remark 4.3. If $X \in \operatorname{Gproj} \mathscr{A}$ and $Q \in \operatorname{proj} \mathscr{A}$, then $\operatorname{Ext}^i(X, Q) = 0$ for all $i > 0$. This can be proved in the following way. Let $\mathbf{P}$ be a complete projective resolution of X. Then

$$\operatorname{Hom}_{\mathscr{A}}(\mathbf{P}, Q) = \quad (\cdots \to \operatorname{Hom}_{\mathscr{A}}(P_{-1}, Q) \to \operatorname{Hom}_{\mathscr{A}}(P_0, Q) \to \operatorname{Hom}_{\mathscr{A}}(P_1, Q) \to \dots)$$

is exact. Hence

$$\operatorname{Ext}^i(X, P) = H^i\big(0 \to \operatorname{Hom}_{\mathscr{A}}(Q_0, P) \to \operatorname{Hom}_{\mathscr{A}}(Q_1, P) \to \dots\big) = 0$$

for all $i > 0$.

Lemma 4.4. *For any short exact sequence* $0 \to X \to Y \to Z \to 0$ *in* $\mathscr{A}$ *with* $Z \in \operatorname{Gproj} \mathscr{A}$*, the sequence*

$$0 \to \operatorname{Hom}(Z, P) \to \operatorname{Hom}(Y, P) \to \operatorname{Hom}(X, P) \to 0$$

is exact whenever $P \in \operatorname{proj} \mathscr{A}$.

Proof. It follows from Remark 4.3 that $\operatorname{Ext}^1(Z, P) = 0$. By applying $\operatorname{Hom}(-, P)$ to the exact sequence $0 \to X \to Y \to Z \to 0$ we get the following exact sequence:

$$0 \to \operatorname{Hom}(Z, P) \to \operatorname{Hom}(Y, P) \to \operatorname{Hom}(X, P) \to \operatorname{Ext}^1(Z, P) = 0.$$ □

Corollary 4.5. *All projective objects in* $\mathscr{A}$ *are injective objects in* $\operatorname{Gproj} \mathscr{A}$*, i.e.*

$$\operatorname{proj} \mathscr{A} \subseteq \operatorname{inj}(\operatorname{Gproj} \mathscr{A}).$$

Proof. This follows immediately from Lemma 4.2 and Lemma 4.4. □

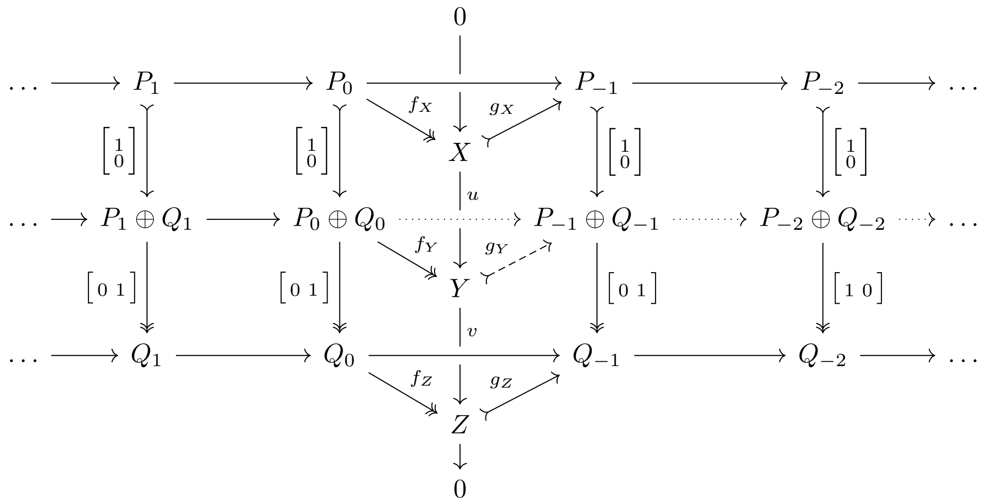
Recall that a subcategory $\mathscr{E}'$ of an exact category $(\mathscr{E}, \mathcal{S})$ is **extension closed** if whenever $X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z$ is in $\mathcal{S}$ with $X, Z \in \mathscr{E}'$, then $Y \in \mathscr{E}'$.

Proposition 4.6. *The subcategory* $\operatorname{Gproj} \mathscr{A}$ *is extension closed in* $\mathscr{A}$.

Proof. Assume that $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ is exact in $\mathscr{A}$ with $X, Z \in \operatorname{Gproj} \mathscr{A}$. Let $\mathbf{P}$ and $\mathbf{Q}$ be complete projective resolutions of X and Z, respectively. By the Horseshoe Lemma (Lemma A.6), there exists a projective resolution

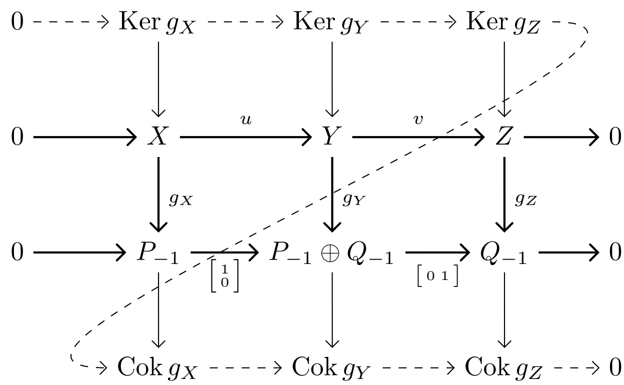
$$\mathbf{R}^{\geq \mathbf{0}} = (\cdots \to P_2 \oplus Q_2 \to P_1 \oplus Q_1 \to P_0 \oplus Q_0 \to 0)$$

of Y such that the solid part of the following diagram commutes.



Since $Z \in \operatorname{Gproj} \mathscr{A}$ and $P_{-1} \in \operatorname{proj} \mathscr{A}$, Lemma 4.4 gives that there exists $a : Y \to P_{-1}$ such that $au = g_X$. Let $b := g_Z \circ v : Y \to Q_{-1}$ and define $g_Y := \begin{bmatrix} a \\ b \end{bmatrix} : Y \to P_{-1} \oplus Q_{-1}$. Then this morphism makes the diagram commute.

Now consider the following diagram.



The Snake Lemma (Lemma A.5) applied to the solid part of the diagram gives that the dashed morphisms exist and give an exact sequence. Since g_X and g_Z are (admissible) monomorphisms, this implies that $\operatorname{Ker} g_Y = 0$. That is g_Y is an (admissible) monomorphism as well. It follows that $Y = \operatorname{Im}(g_Y \circ f_Y)$ and that

$$P_1 \oplus Q_1 \to P_0 \oplus Q_0 \xrightarrow{g_Y f_Y} P_{-1} \oplus Q_{-1}$$

is exact. Furthermore, $0 = \operatorname{Ker} g_Z \to \operatorname{Cok} g_X \to \operatorname{Cok} g_Y \to \operatorname{Cok} g_Z \to 0$ is exact. Thus we get by induction that there exists a morphism $P_i \oplus Q_i \to P_{i-1} \oplus Q_{i-1}$ for all $i \leq 0$ such that

$$\mathbf{R} = \quad (\cdots \to P_1 \oplus Q_1 \to P_0 \oplus Q_0 \to P_{-1} \oplus Q_{-1} \to P_{-2} \oplus Q_{-2} \to \dots)$$

is an exact sequence. This is a complete projective resolution of Y if $\operatorname{Hom}_{\mathscr{A}}(\mathbf{R}, S)$ is exact for all $S \in \operatorname{proj} \mathscr{A}$. Note that since Q_i is projective, $\operatorname{Ext}(Q_i, S) = 0$ for all i. Thus

$$0 \to \operatorname{Hom}(Q_i, S) \to \operatorname{Hom}(P_i \oplus Q_i, S) \to \operatorname{Hom}(P_i, S) \to \operatorname{Ext}(Q_i, S) = 0$$

is exact. Therefore we get an exact sequence of chain complexes

$$0 \to \operatorname{Hom}(\mathbf{Q}, S) \to \operatorname{Hom}(\mathbf{R}, S) \to \operatorname{Hom}(\mathbf{P}, S) \to 0$$

of abelian groups. The resulting long exact sequence in homology implies that $\operatorname{Hom}(\mathbf{R}, S)$ is exact since $\operatorname{Hom}(\mathbf{Q}, S)$ and $\operatorname{Hom}(\mathbf{P}, S)$ are exact. □

Definition 4.7. *Define $\mathcal{S}$ be the collection of all short exact sequences in $\mathscr{A}$ with objects in* $\operatorname{Gproj} \mathscr{A}$*, i.e.*

$$\mathcal{S} = \{0 \to X \to Y \to Z \to 0 \textit{ exact in } \mathscr{A} \mid X, Y, Z \in \operatorname{Gproj} \mathscr{A}\}.$$

Corollary 4.8. *The collection $\mathcal{S}$ is an exact structure on* $\operatorname{Gproj} \mathscr{A}$*.*

Proof. Proposition 1.11 states that $(\operatorname{Gproj} \mathscr{A}, \mathcal{S})$ is exact if $\operatorname{Gproj} \mathscr{A}$ is extension closed in $\mathscr{A}$ and $0 \in \operatorname{Gproj} \mathscr{A}$. The first part holds by Proposition 4.6 and the second part is trivial. □

Definition 4.9. *Given $X \in \mathscr{A}$ and a projective resolution*

$$\mathbf{P} = \quad (\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to 0)$$

*of X, we define the n-**th syzygy** of X with respect to $\mathbf{P}$ to be $\Omega^n_{\mathbf{P}} X := \operatorname{Im} d_n$ for all $n \geq 1$ and $\Omega^0_{\mathbf{P}} X := X$.*

If X is in $\operatorname{Gproj} \mathscr{A}$ *and*

$$\mathbf{P} = \quad (\cdots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_{-1} \xrightarrow{d_{-1}} P_{-2} \to \dots)$$

*is a complete projective resolution of X, then we extend the definition to negative integers as well. That is, we define the n-**th syzygy** of X with respect to $\mathbf{P}$ to be $\Omega^n_{\mathbf{P}} X := \operatorname{Im} d_n$ for all $n \in \mathbb{Z}$.*

Remark 4.10. **(1)** Note that $\Omega^0_{\mathbf{P}} X = X$ when $\mathbf{P}$ is a complete projective resolution of X.

(2) We use the notation $\Omega_{\mathbf{P}} X$ for $\Omega^1_{\mathbf{P}} X$.

(3) The n-th syzygy $\Omega^n_{\mathbf{P}} X$ depends on the (complete) projective resolution $\mathbf{P}$. If we do not want to specify the (complete) projective resolution we use the notation $\Omega^n X$. However, $\Omega^n X$ is not uniquely determined in this case.

(4) Let $X \in \operatorname{Gproj} \mathscr{A}$. The complete projective resolution $\mathbf{P}$ of X shifted n times to the right (or $-n$ times to the left) is a complete projective resolution of $\Omega^n_{\mathbf{P}} X$, so $\Omega^n_{\mathbf{P}} X \in \operatorname{Gproj} \mathscr{A}$.

(5) Let $X \in \operatorname{Gproj} \mathscr{A}$. We have $\Omega^n_{\mathbf{P}} X = \operatorname{Cok} d_{n+1} = \operatorname{Im} d_n = \operatorname{Ker} d_{n-1}$ since the complete projective resolution $\mathbf{P}$ is exact. Thus it breaks down into short exact sequences

$$\begin{array}{ccccccccccc} \mathbf{P} = & \dots \longrightarrow & P_1 & & \xrightarrow{\quad d_1 \quad} & & P_0 & \xrightarrow{\quad d_0 \quad} & P_{-1} & \xrightarrow{\quad d_{-1} \quad} & P_{-2} \longrightarrow \dots \\ & & & \searrow & & \nearrow & & \searrow \quad \nearrow & & \searrow \quad \nearrow & \\ & & & & \Omega_{\mathbf{P}} X & & & X & & \Omega^{-1}_{\mathbf{P}} X & \end{array}$$

Hence the sequence $0 \to \Omega^{n+1}_{\mathbf{P}} X \to P_n \to \Omega^n_{\mathbf{P}} X \to 0$ lies in $\mathcal{S}$ for all $n \in \mathbb{Z}$.

Proposition 4.11. *The $\mathcal{S}$-projective objects of* $\operatorname{Gproj} \mathscr{A}$ *coincide with the projective objects of $\mathscr{A}$, that is* $\operatorname{proj}(\operatorname{Gproj} \mathscr{A}) = \operatorname{proj} \mathscr{A}$*.*

Proof. It is trivial to see that if $P \in \operatorname{proj} \mathscr{A}$ lies in a subcategory $\mathscr{C} \subseteq \mathscr{A}$, then $P \in \operatorname{proj} \mathscr{C}$. Hence $\operatorname{proj} \mathscr{A} \subseteq \operatorname{proj}(\operatorname{Gproj} \mathscr{A})$ follows from the inclusion $\operatorname{proj} \mathscr{A} \subseteq \operatorname{Gproj} \mathscr{A}$ given by Lemma 4.2.

Now assume that $X \in \operatorname{proj}(\operatorname{Gproj} \mathscr{A})$. By Remark 4.10 (5) there exists a short exact sequence $0 \to \Omega X \to P \to X \to 0 \in \mathcal{S}$. Since X is an $\mathcal{S}$-projective object this implies that the sequence is right split, as shown in Remark 2.7. Thus X is a direct summand of $P \in \operatorname{proj} \mathscr{A}$, which gives $X \in \operatorname{proj} \mathscr{A}$ by Lemma 2.6. Hence $\operatorname{proj}(\operatorname{Gproj} \mathscr{A}) \subseteq \operatorname{proj} \mathscr{A}$. □

Proposition 4.12. *The $\mathcal{S}$-injective objects of* $\operatorname{Gproj} \mathscr{A}$ *coincide with the projective objects of* $\mathscr{A}$*, that is* $\operatorname{inj}(\operatorname{Gproj} \mathscr{A}) = \operatorname{proj} \mathscr{A}$.

Proof. Assume that $X \in \operatorname{inj}(\operatorname{Gproj} \mathscr{A})$. Just as in the proof of Proposition 4.11 there exists a sequence $0 \to X \to P \to \Omega^{-1}X \to 0 \in \mathcal{S}$, which is left split since X is $\mathcal{S}$-injective. Hence X is a direct summand of $P \in \operatorname{proj} \mathscr{A}$, which implies $X \in \operatorname{proj} \mathscr{A}$. Thus $\operatorname{inj}(\operatorname{Gproj} \mathscr{A}) \subseteq \operatorname{proj} \mathscr{A}$. Furthermore, $\operatorname{proj} \mathscr{A} \subseteq \operatorname{inj}(\operatorname{Gproj} \mathscr{A})$ by Corollary 4.5. □

Corollary 4.13. *(a) The $\mathcal{S}$-projective and $\mathcal{S}$-injective objects of* $\operatorname{Gproj} \mathscr{A}$ *coincide, and equals the projective objects of* $\mathscr{A}$*. That is,* $\operatorname{proj}(\operatorname{Gproj} \mathscr{A}) = \operatorname{proj} \mathscr{A} = \operatorname{inj}(\operatorname{Gproj} \mathscr{A})$.

(b) The exact category $(\operatorname{Gproj} \mathscr{A}, \mathcal{S})$ *has enough $\mathcal{S}$-projective objects and enough $\mathcal{S}$-injective objects.*

(c) The exact category $(\operatorname{Gproj} \mathscr{A}, \mathcal{S})$ *is Frobenius.*

Proof. (a) This is Proposition 4.11 and Proposition 4.12 combined.

(b) This follows immediately from Remark 4.10 (5) and (a).

(c) By Corollary 4.8 $(\operatorname{Gproj} \mathscr{A}, \mathcal{S})$ is exact, hence it is Frobenius by (a) and (b). □

From Theorem 3.20, Theorem 3.21 and the above we deduce the following result.

Theorem 4.14. *Let $\mathscr{A}$ be an abelian category,* $\operatorname{Gproj} \mathscr{A}$ *the Frobenius category of Gorenstein projective objects, and* $\underline{\operatorname{Gproj}} \mathscr{A}$ *the corresponding stable category. Then we have the following one-to-one correspondences*

$$\{\textit{complete subcategories of } \operatorname{Gproj} \mathscr{A} \textit{ containing } \operatorname{proj} \mathscr{A}\}$$
$$\updownarrow {\scriptstyle 1-1}$$
$$\{\textit{triangulated subcategories of } \underline{\operatorname{Gproj}} \mathscr{A}\}$$

$$\{\textit{thick subcategories of } \operatorname{Gproj} \mathscr{A} \textit{ containing } \operatorname{proj} \mathscr{A}\}$$
$$\updownarrow {\scriptstyle 1-1}$$
$$\{\textit{thick triangulated subcategories of } \underline{\operatorname{Gproj}} \mathscr{A}\}$$

4.2. Gorenstein projective objects over $\operatorname{mod} R$

Let R be a commutative Noetherian ring and denote by $\operatorname{mod} R$ the abelian category of all finitely generated R-modules. It is well known that $\operatorname{mod} R$ has enough projective objects. We define $\operatorname{Gproj} R := \operatorname{Gproj} \operatorname{mod} R$ and $\operatorname{proj} R := \operatorname{proj} \operatorname{mod} R$ to simplify notation. The dual of a module $M \in \operatorname{mod} R$ is defined to be $M^* := \operatorname{Hom}(M, R)$. Note that $P \in \operatorname{proj} R$ implies $P^* \in \operatorname{proj} R$, and that the natural homomorphism $P \to P^{**}$ is an isomorphism. In this section we show that the Gorenstein projective objects in $\operatorname{mod} R$ are the totally reflexive modules, as defined below.

Definition 4.15. *A module $M \in \operatorname{mod} R$ is* **totally reflexive** *if the following hold*

(i) the natural homomorphism $M \to M^{**}$ is an isomorphism,

(ii) $\operatorname{Ext}_R^{>0}(M, R) = 0$,

(iii) $\operatorname{Ext}_R^{>0}(M^*, R) = 0$.

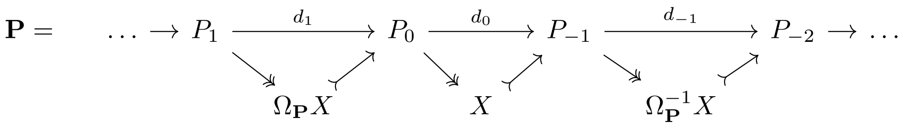
We define $\mathcal{G}(R)$ to be the full subcategory of $\operatorname{mod} R$ *consisting of all totally reflexive modules.*

Remark 4.16. **(1)** A module $P \in \operatorname{mod} R$ is projective if and only if there exist $Q \in \operatorname{mod} R$ and $n \geq 0$ such that $P \oplus Q \cong R^n$. Thus since Ext_R commutes with finite direct sums, we have that $\operatorname{Ext}_R^{>0}(M, R) = 0$ implies $\operatorname{Ext}_R^{>0}(M, P) = 0$ for all $P \in \operatorname{proj} R$.

(2) Assume that

$$\mathbf{P} = \quad (\cdots \to P_2 \to P_1 \to P_0 \to P_{-1} \to P_{-2} \to \dots)$$

is an exact sequence with projective terms, and that it breaks down into short exact sequences in the following way

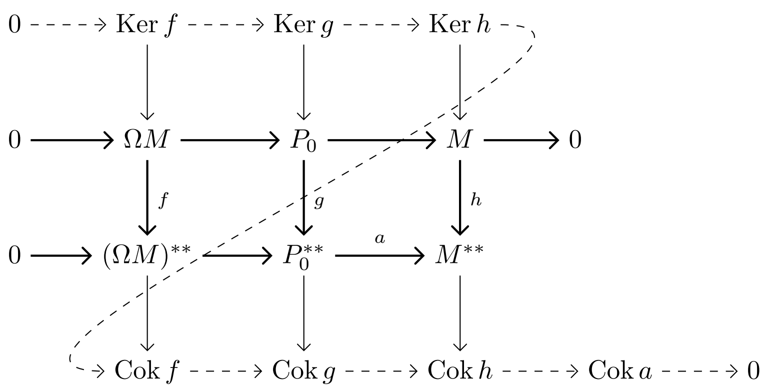


If for an object $Q \in \operatorname{mod} R$ we have that $\operatorname{Ext}_R(\Omega^n_{\mathbf{P}} X, Q) = 0$ for all $n \in \mathbb{Z}$, then $\operatorname{Hom}(\mathbf{P}, Q)$ is exact as well.

Proposition 4.17. *The Gorenstein projective objects of* $\operatorname{mod} R$ *are precisely the totally reflexive modules, i.e.* $\mathcal{G}(R) = \operatorname{Gproj} R$*. Thus* $\mathcal{G}(R)$ *is a Frobenius category.*

Proof. Assume that M is a Gorenstein projective object with a complete projective resolution $\mathbf{P}$, and let $0 \to \Omega_{\mathbf{P}} M \to P_0 \to M \to 0$ be exact. Then $0 \to M^* \to P_0^* \to (\Omega_{\mathbf{P}} M)^* \to 0$ is exact by Lemma 4.4. Thus $0 \to (\Omega_{\mathbf{P}} M)^{**} \to P_0^{**} \to M^{**} \to \operatorname{Ext}((\Omega_{\mathbf{P}} M)^*, R) \to \operatorname{Ext}(P_0^*, R) = 0$ is exact as well.

Let f, g, h in the diagram below be the natural homomorphisms. The Snake Lemma (Lemma A.5) applied to the solid part of the diagram below gives that the dashed arrows exist and form an exact sequence.



Since P_0 is projective, the map $g : P_0 \to P_0^{**}$ is an isomorphism. Moreover, $\operatorname{Ker} g = 0 = \operatorname{Cok} g$ implies that $\operatorname{Ker} f = 0$. By using the same argument on the short exact sequence $0 \to M \to P_{-1} \to \Omega_{\mathbf{P}}^{-1} M \to 0$ we get that $\operatorname{Ker} h = 0$. Furthermore, this implies that $\operatorname{Cok} f = 0$ since $0 = \operatorname{Ker} h \to \operatorname{Cok} f \to 0$ is exact. Again, the same argument applied to $0 \to M \to P_{-1} \to \Omega_{\mathbf{P}}^{-1} M \to 0$ gives that $\operatorname{Cok} h = 0$. Thus $h : M \to M^{**}$ is an isomorphism.

Moreover, $\operatorname{Cok}(P_0^{**} \to M^{**}) = \operatorname{Cok} a = 0$, since $\operatorname{Cok} h = 0$. Note that this implies that $\operatorname{Ext}((\Omega_{\mathbf{P}} M)^*, R) = 0$ since

$$0 \to (\Omega_{\mathbf{P}} M)^{**} \to P_0^{**} \xrightarrow{a} M^{**} \to \operatorname{Ext}((\Omega_{\mathbf{P}} M)^*, R) \to 0$$

is exact. Similarly, $\operatorname{Ext}((\Omega^n_{\mathbf{P}} M)^*, R) = 0$ for all n. Thus it follows that $\operatorname{Ext}^{>0}(M^*, R) = 0$ since $0 = \operatorname{Ext}((\Omega^n_{\mathbf{P}} M)^*, R) = \operatorname{Ext}^n(M^*, R)$ for all $n \geq 0$. Moreover, $\operatorname{Ext}^n(M, R) = 0$ by Remark 4.3. Hence M is a totally reflexive module.

Now assume that $M \in \mathcal{G}(R)$. Let

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0 \tag{4.1}$$

$$\cdots \to Q_2 \to Q_1 \to Q_0 \to M^* \to 0 \tag{4.2}$$

be projective resolutions of M and M^* respectively. By applying $(-)^* = \operatorname{Hom}(-, R)$ to the sequence 4.2, we get the sequence

$$0 \to M^{**} \to Q_0^* \to Q_1^* \to Q_2^* \to \dots \tag{4.3}$$

which is exact since $\operatorname{Ext}_R^{>0}(M^*, R)$. Since $M \cong M^{**}$, it follows that we can combine the sequences 4.1 and 4.3 into the following sequence

$$\mathbf{P} = \quad \begin{array}{ccccccccc} \dots \to P_1 & & \longrightarrow & & P_0 & \longrightarrow & Q_0^* & \longrightarrow & Q_{-1}^* \to \dots \\ & \searrow & & \nearrow & & \searrow \nearrow & & \searrow \nearrow & \\ & & \Omega M & & & M \cong M^{**} & & (\Omega M^*)^* & \end{array}$$

Moreover, $\mathbf{P}$ is exact: Note that since M is totally reflexive we get that the following sequence is exact:

$$0 \to (\Omega^n M^*)^* \to Q_n^* \to (\Omega^{n+1} M^*)^* \to \operatorname{Ext}_R(\Omega^n M^*, R) = \operatorname{Ext}^n_R(M^*, R) = 0.$$

Thus $\mathbf{P}$ is exact with $M = \operatorname{Im}(P_0 \to Q_0^*)$. Furthermore, it is fairly easy to show that $\mathcal{G}(R)$ is closed under taking duals and syzygies, thus $(\Omega^n M^*)^*, \Omega^n M \in \mathcal{G}(R)$. Hence by Remark 4.16 (1) and (2), $\operatorname{Hom}(\mathbf{P}, S)$ is exact for all $S \in \operatorname{proj} R$. Thus $\mathbf{P}$ is a complete projective resolution of M. □

Note that if a subcategory $\mathscr{C}$ of $\mathcal{G}(R)$ is closed under finite direct sums and direct summands, then it contains $\operatorname{proj} R$ if and only if it contains the object R. Thus the condition that a thick subcategory must contain $\operatorname{proj} R$ can be replaced by the condition that it must contain the object R. Consequently, in this setting, Theorem 4.14 takes the following form.

Theorem 4.18. *Let R be a commutative Noetherian ring, $\mathcal{G}(R)$ the Frobenius category of totally reflexive modules, and $\underline{\mathcal{G}}(R)$ the corresponding stable category. Then we have the following one-to-one correspondences*

$$\{\textit{complete subcategories of } \mathcal{G}(R) \textit{ containing } \operatorname{proj} R\}$$
$$\updownarrow \scriptstyle{1-1}$$
$$\{\textit{triangulated subcategories of } \underline{\mathcal{G}}(R)\}$$

$$\{\textit{thick subcategories of } \mathcal{G}(R) \textit{ containing } R\}$$
$$\updownarrow \scriptstyle{1-1}$$
$$\{\textit{thick triangulated subcategories of } \underline{\mathcal{G}}(R)\}$$

Specializing further, we now consider at a special type of rings, called Gorenstein local rings.

Definition 4.19. *Let* $(R, \mathfrak{m}, k)$ *be a commutative Noetherian local ring, where* $\mathfrak{m}$ *is the maximal ideal and* $k := R/\mathfrak{m}$ *is the residue field.*

(1) The **(Krull) dimension** *of R is*

$$\dim R := \sup\{n \geq 0 \mid \exists \textit{ chain of prime ideals } \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n\}.$$

(2) The **depth** *of* $M \in \operatorname{mod} R$ *is*

$$\operatorname{depth} M = \inf\{n \geq 0 \mid \operatorname{Ext}^n_R(k, M) \neq 0\}.$$

(3) $M \in \operatorname{mod} R$ *is a* **maximal Cohen-Macaulay module** *if* $M = 0$ *or* $\operatorname{depth} M = \dim R$.

(4) R *is a* **Gorenstein local ring** *if it has finite injective dimension as a module over itself.*

(5) $\mathrm{CM}(R)$ *is the full subcategory of* $\operatorname{mod} R$ *consisting of all maximal Cohen-Macaulay modules over* R.

The following characterization of maximal Cohen-Macaulay modules over a Gorenstein local ring is well known, and the proof can be found in [8, Theorem 4.8] or [1].

Proposition 4.20. *For a Gorenstein local ring a module* $M \in \operatorname{mod} R$ *is maximal Cohen-Macaulay if and only if it is totally reflexive. Thus* $\mathrm{CM}(R) = \mathcal{G}(R) = \operatorname{Gproj} R$ *and* $\mathrm{CM}(R)$ *is a Frobenius category.*

By the above, Theorem 4.18 takes the following form. The "thick" part of this result is known, c.f. [7].

Theorem 4.21. *Let* R *be a Gorenstein local ring,* $\mathrm{CM}(R)$ *the Frobenius category of maximal Cohen-Macaulay modules, and* $\underline{\mathrm{CM}}(R)$ *the corresponding stable category. Then we have the following one-to-one correspondences*

$$\{\textit{complete subcategories of } \mathrm{CM}(R) \textit{ containing } \operatorname{proj} R\}$$
$$\updownarrow 1-1$$
$$\{\textit{triangulated subcategories of } \underline{\mathrm{CM}}(R)\}$$

$$\{\textit{thick subcategories of } \mathrm{CM}(R) \textit{ containing } R\}$$
$$\updownarrow 1-1$$
$$\{\textit{thick triangulated subcategories of } \underline{\mathrm{CM}}(R)\}$$

A. Basic results

This section contains several basic results, which we use throughout this thesis. Unless otherwise stated we work in a general category $\mathscr{C}$.

Lemma A.1. *Assume that g is a cokernel of f and that the right-hand square of the diagram below is a pullback along g and h.*

$$\begin{array}{ccccc} A & \overset{f'}{\dashrightarrow} & P & \overset{g'}{\longrightarrow} & D \\ \Vert & & \downarrow{\scriptstyle h'} & {\scriptstyle PB} & \downarrow{\scriptstyle h} \\ A & \overset{f}{\longrightarrow} & B & \overset{g}{\longrightarrow} & C \end{array}$$

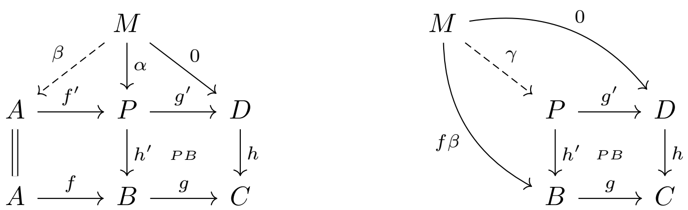
Then there exists a morphism $A \xrightarrow{f'} P$ such that the diagram commutes and f' is the kernel of g'.

Dually, assume that f is a kernel of g and that the left-hand square of the diagram below is a pushout along f and h.

$$\begin{array}{ccccc} A & \overset{f}{\longrightarrow} & B & \overset{g}{\longrightarrow} & C \\ \downarrow{\scriptstyle h} & {\scriptstyle PO} & \downarrow{\scriptstyle h'} & & \Vert \\ D & \overset{f'}{\longrightarrow} & P & \overset{g'}{\dashrightarrow} & C \end{array}$$

Then there exists a morphism $P \xrightarrow{g'} C$ such that the diagram commutes and g' is the cokernel of f'.

Proof. We prove the first part. Let f' be the unique morphism such that $h'f' = f$ and $g'f' = 0$, which exists since P is a pullback and $gf = 0 = h \circ 0$. Assume that we have a morphism $\alpha : M \to P$ with $g'\alpha = 0$. Then f' is the kernel of g' if there exists a unique $\beta : M \to A$ such that $f'\beta = \alpha$.



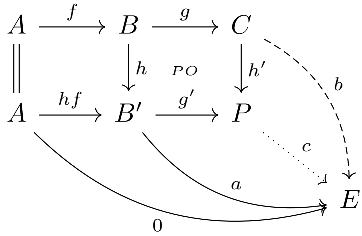
Since g is the cokernel of f and $gh'\alpha = hg'\alpha = 0$, there exists a unique $\beta : M \to A$ such that $h'\alpha = f\beta = h'f'\beta$. Moreover, the pullback property gives a unique $\gamma : M \to P$ satisfying $g'\gamma = 0$ and $h'\gamma = f\beta$. However, both α and $f'\beta$ satisfies this, so $f'\beta = \alpha$. □

Lemma A.2. *Assume that g is a cokernel of f and that the right-hand square of the diagram below is a pushout along g and h.*

$$\begin{array}{ccccc} A & \overset{f}{\longrightarrow} & B & \overset{g}{\longrightarrow} & C \\ \Vert & & \downarrow{\scriptstyle h} & {\scriptstyle PO} & \downarrow{\scriptstyle h'} \\ A & \overset{hf}{\longrightarrow} & B' & \overset{g'}{\longrightarrow} & P \end{array}$$

Then g' is the cokernel of hf.

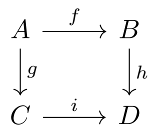
Proof. Assume that $a : B' \to E$ with $ahf = 0$. Since g is the cokernel of f, there exists a unique $b : C \to E$ with $bg = ah$. Hence the pushout property gives that there exists a unique $c : P \to E$ with $cg' = a$ and $ch' = b$.



For g' to be the cokernel of hf we need a unique $c' : P \to E$ with $c'g' = a$, but not necessarily $c'h' = b$. Assume therefore that $c'g' = a$ and define $b' := c'h'$. Then $b'g = c'h'g = c'g'h = ah$. Since b is unique with the property that $bg = ah$, this implies that $b' = b$. Thus $c'g' = a$ and $b = c'h'$, hence by the uniqueness of c, $c = c'$. □

Lemma A.3. *In an additive category* $\mathscr{A}$, *consider* $A \xrightarrow{\begin{bmatrix}-f\\ g\end{bmatrix}} B \oplus C \xrightarrow{\begin{bmatrix}h & i\end{bmatrix}} D$ *and*

K :

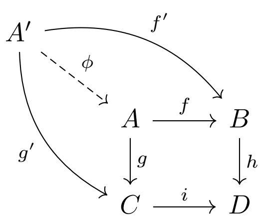


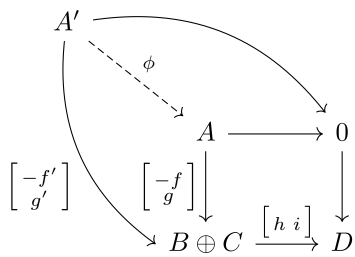
Assume that K is a commutative square, i.e. that $\begin{bmatrix}h & i\end{bmatrix}\begin{bmatrix}-f\\ g\end{bmatrix} = 0$. *Then*

(i) K is a pullback if and only if $\begin{bmatrix}-f\\ g\end{bmatrix}$ *is a kernel of* $\begin{bmatrix}h & i\end{bmatrix}$.

(ii) K is a pushout if and only if $\begin{bmatrix}h & i\end{bmatrix}$ *is a cokernel of* $\begin{bmatrix}-f\\ g\end{bmatrix}$.

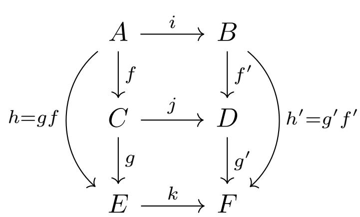
Proof. It is trivial to see that (i) holds by regarding the following two diagrams and recalling the definition of pullback and kernel:





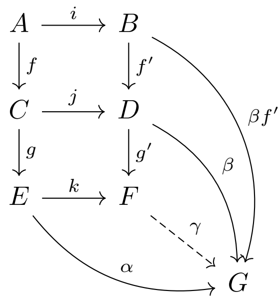
Part (ii) holds by duality. □

Lemma A.4. *Assume that we have the following commutative diagram*



with the outer square and the top square being pushouts, i.e. F is a pushout along i and $h = gf$ and D is a pushout along i and f. Then the bottom square is a pushout too, i.e. F is a pushout along j and g.

Proof. Assume that we have $\alpha : E \to G$ and $\beta : D \to G$ with $\alpha g = \beta j$. Then $\alpha g f = \beta j f = \beta f' i$, so by the pushout property of the outer square, there exists a unique $\gamma : F \to G$ such that $\gamma k = \alpha$ and $\gamma h' = \gamma g' f' = \beta f'$:



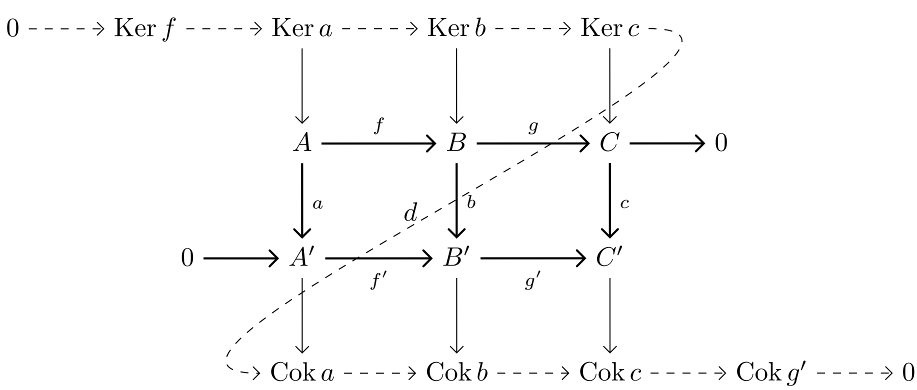
By the pushout property of the top square, $\gamma g' = \beta$ if $\gamma g' f' = \beta f'$ and $\gamma g' j = \beta j$. The first equality holds by assumption, hence we only need to prove that $\gamma g' j = \beta j$. However, this holds since $\gamma g' j = \gamma k g = \alpha g = \beta j$.

Now assume that $\gamma_i k = \alpha$ and $\gamma_i g' = \beta$ for $i = 1, 2$. Then $\gamma_i h' = \gamma_i g' f' = \beta f'$ and $\gamma_i k = \alpha$, so by the uniqueness of γ, $\gamma = \gamma_1 = \gamma_2$. □

Some well-known results

We will only state the remaining well-known results.

Lemma A.5 (Snake Lemma)**.** *In an abelian category $\mathscr{A}$, assume that we are given the solid part of the diagram below. Then the dotted arrows exist and form an exact sequence.*



Lemma A.6 (Horseshoe Lemma)**.** *Let $\mathscr{A}$ be an abelian category. Assume that $X \rightarrowtail Y \twoheadrightarrow Z$ is a short exact sequence in $\mathscr{A}$ and that $\mathbf{P^X}$ and $\mathbf{P^Z}$ are projective resolutions for X and Z, respectively. Then there exists a projective resolution $\mathbf{P^Y}$ of Y with $P_i^Y = P_i^X \oplus P_i^Z$ such that* *the following diagram commutes*

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_2^X & \longrightarrow & P_1^X & \longrightarrow & P_0^X & \longrightarrow & X \\
& & \big\downarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} & & \big\downarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} & & \big\downarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} & & \big\downarrow \\
\cdots & \longrightarrow & P_2^X \oplus P_2^Z & \longrightarrow & P_1^X \oplus P_1^Z & \longrightarrow & P_0^X \oplus P_0^Z & \longrightarrow & Y \\
& & \big\downarrow \begin{bmatrix} 0 & 1 \end{bmatrix} & & \big\downarrow \begin{bmatrix} 0 & 1 \end{bmatrix} & & \big\downarrow \begin{bmatrix} 0 & 1 \end{bmatrix} & & \big\downarrow \\
\cdots & \longrightarrow & P_2^Z & \longrightarrow & P_1^Z & \longrightarrow & P_0^Z & \longrightarrow & Z
\end{array}$$

Lemma A.7 (Five Lemma)**.** *Let $\mathscr{A}$ be an abelian category and assume that we have the following commutative diagram with exact rows.*

$$\begin{array}{ccccccccc}
A_0 & \xrightarrow{\alpha_0} & A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 \\
\big\downarrow f_0 & & \big\downarrow f_1 & & \big\downarrow f_2 & & \big\downarrow f_3 & & \big\downarrow f_4 \\
B_0 & \xrightarrow{\beta_0} & B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & B_4
\end{array}$$

Then the following holds

(i) If f_0 is an epimorphism and f_1, f_3 are monomorphisms, then f_2 is a monomorphism.

(ii) If f_4 is a monomorphism and f_1, f_3 are epimorphism, then f_2 is an epimorphism.

Combining these two results, we get that if f_0, f_1, f_3, f_4 are isomorphisms, then so is f_2.

B. Triangulated categories

In this Appendix we give a short introduction to triangulated categories. This includes the definition and some results, but no proofs.

Definition

Definition B.1. *Let $\mathscr{T}$ be an additive category and $T : \mathscr{T} \to \mathscr{T}$ an additive autoequivalence. A* **triangle** *is a sequence of the form*

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX.$$

A **morphism of triangles** *ϕ is a triple $\phi = (\phi_X, \phi_Y, \phi_Z)$ of morphisms such that the diagram*

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & TX \\ \downarrow{\scriptstyle \phi_X} & & \downarrow{\scriptstyle \phi_Y} & & \downarrow{\scriptstyle \phi_Z} & & \downarrow{\scriptstyle T\phi_X} \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & TX' \end{array}$$

commutes. Furthermore, ϕ is an **isomorphism of triangles** *if ϕ_X, ϕ_Y and ϕ_Z are isomorphisms in $\mathscr{T}$.*

Definition B.2. *A* **pretriangulated category** *is an additive category $\mathscr{T}$ together with an additive autoequivalence T and a collection of triangles Δ, called distinguished triangles, which satisfies the following axioms.*

(TR1) (a) *Δ is closed under isomorphisms of triangles.*

(b) *For every object $X \in \mathscr{T}$, we have $X \xrightarrow{1_X} X \longrightarrow 0 \longrightarrow TX \in \Delta$.*

(c) *For every morphism $f : X \to Y$ there exists a distinguished triangle of the form $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow TX$.*

(TR2) Δ is closed under rotations, meaning that if $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ is a distinguished triangle, then so is $Y \xrightarrow{g} Z \xrightarrow{h} TX \xrightarrow{-Tf} TY$ and $T^{-1}Z \xrightarrow{-T^{-1}h} X \xrightarrow{f} Y \xrightarrow{g} Z$.

(TR3) Given a commutative diagram

$$\begin{array}{cccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & TX & \quad \in \Delta \\ \downarrow{\scriptstyle \phi_X} & & \downarrow{\scriptstyle \phi_Y} & & & & \downarrow{\scriptstyle T\phi_X} & \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & TX' & \quad \in \Delta \end{array}$$

then there exists a (not necessarily unique) $\phi_Z : Z \to Z'$ such that (ϕ_X, ϕ_Y, ϕ_Z) is a morphism of triangles.

The category $\mathscr{T}$ is **triangulated** *if in addition the following axiom holds.*

(TR4) *Assume that we have distinguished triangles*

$$X \xrightarrow{f} Y \longrightarrow Z' \longrightarrow TX,$$

$$Y \xrightarrow{g} Z \longrightarrow X' \xrightarrow{h} TY \text{ and}$$

$$X \xrightarrow{gf} Z \longrightarrow Y' \longrightarrow TX.$$

Then there exists a distinguished triangle $Z' \longrightarrow Y' \longrightarrow X' \longrightarrow TZ'$ such that the following diagram commutes

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & Z' & \longrightarrow & TX \\
\downarrow{\scriptstyle 1_X} & & \downarrow{\scriptstyle g} & & \vdots & & \downarrow{\scriptstyle 1_{TX}} \\
X & \xrightarrow{gf} & Z & \longrightarrow & Y' & \longrightarrow & TX \\
 & & \downarrow & & \vdots & & \downarrow{\scriptstyle Tf} \\
 & & X' & \xrightarrow{1_{X'}} & X' & \xrightarrow{h} & TY \\
 & & \downarrow{\scriptstyle h} & & \downarrow & & \\
 & & TY & \longrightarrow & TZ' & &
\end{array}$$

We say that T and Δ give a triangulated structure on $\mathscr{T}$. An additive category may have several triangulated structures. However, when there is no fear of confusion, we will often refer to the triangulated category as $\mathscr{T}$ instead of $(\mathscr{T}, T, \Delta)$.

Note that many authors require the additive functor T to be an automorphism, not only an autoequivalence. The triangulated categories we consider in this thesis are the stable categories of Frobenius categories. In this case, the functor T is always an autoequivalence. However, an extra assumption is needed for T to be an automorphism. The details are given in Chapter 2, and readers who prefer T to be an automorphism are free to make the necessary assumption.

Some results for pretriangulated categories

In this section, we list some well-known results about pretriangulated categories without proof. Therefore, let $(\mathscr{T}, T, \Delta)$ be a pretriangulated category unless otherwise stated.

Proposition B.3. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ be a distinguished triangle. Then $g \circ f = 0$ and $h \circ g = 0$.*

Lemma B.4 (Triangulated Five lemma)**.** *Assume that we have a morphism of distinguished triangles*

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & TX \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle T\alpha} \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & TX'
\end{array}$$

Then if two of the morphisms α, β and γ are isomorphisms, then so is the third.

Corollary B.5. *Given a morphism $X \xrightarrow{f} Y$, then there exists exactly one distinguished triangle of the form $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ up to isomorphism.*

Definition B.6. *The* **direct sum** *of two triangles* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ *and* $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} TA$ *is the triangle*

$$X \oplus A \xrightarrow{\begin{bmatrix} f & 0 \\ 0 & \alpha \end{bmatrix}} Y \oplus B \xrightarrow{\begin{bmatrix} g & 0 \\ 0 & \beta \end{bmatrix}} Z \oplus C \xrightarrow{\begin{bmatrix} h & 0 \\ 0 & \gamma \end{bmatrix}} TX \oplus TA.$$

Proposition B.7. *The direct sum of two distinguished triangles is distinguished.*

Definition B.8. *The triangle* $A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} TA$ *is a* **direct summand** *of the triangle* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ *if there exists two morphisms of triangles* $(\psi_1, \phi_1, \theta_1)$ *and* $(\psi_2, \phi_2, \theta_2)$ *such that the diagram below commutes and such that* $\psi_2 \circ \psi_1 = 1_A$, $\phi_2 \circ \phi_1 = 1_B$ *and* $\theta_2 \circ \theta_1 = 1_C$.

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & TA \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle T\psi_1} \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & TX \\ \downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \theta_2} & & \downarrow{\scriptstyle T\psi_2} \\ A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & TA \end{array}$$

Proposition B.9. *A direct summand of a distinguished triangle is distinguished.*

Weakening of the axioms

The axioms for a (pre)triangulated category can be weakened. In fact, it can be shown that it suffices with half of (TR2), and that (TR3) is redundant. The latter was first proven by J.P. May in [6].

Proposition B.10. *Axiom (TR3) follows from axiom (TR1) and (TR4).*

Axiom (*TR2'*)**.** *If* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ *is a distinguished triangle, then so is the triangle* $Y \xrightarrow{g} Z \xrightarrow{h} TX \xrightarrow{-Tf} TY$.

Lemma B.11. *A category satisfying the axioms (TR1), (TR2') and (TR3) is pretriangulated.*

Theorem B.12. *A category satisfying the axioms (TR1), (TR2') and (TR4) is triangulated.*

Triangulated subcategories

Definition B.13. *A nonempty subcategory* $\mathscr{T}'$ *of a triangulated category* $\mathscr{T}$ *is a* **triangulated subcategory** *if the following hold*

(i) $\mathscr{T}'$ *is a full subcategory,*

(ii) 2 out of 3: If $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} TX$ *is a distinguished triangle in* $\mathscr{T}$ *and two out of three of* X, Y, Z *are in* $\mathscr{T}'$*, then so is the third.*

Moreover, $\mathscr{T}'$ *is a* **thick triangulated subcategory** *if in addition the following holds*

(iii) $\mathscr{T}'$ *is closed under direct summands, meaning that if* A *is a direct summand of* X *and* $X \in \mathscr{T}'$*, then* $A \in \mathscr{T}'$ *as well.*

Remark B.14. **(1)** $\mathscr{T}'$ is additive. Let $X \in \mathscr{T}'$: such an object exists since $\mathscr{T}'$ is nonempty. Then $X \to X \to 0 \to TX \in \Delta$ by (TR1) (b). Hence by (ii), $0 \in \mathscr{T}'$. Note that by (TR1) (b), (TR2) and Proposition B.7, $X \to X \oplus Y \to Y \to TX$ is distinguished. Thus by (ii), $X, Y \in \mathscr{T}'$ implies that $X \oplus Y \in \mathscr{T}'$. The additivity of $\mathscr{T}'$ follows now immediately from the additivity of $\mathscr{T}$.

(2) $\mathscr{T}'$ is closed under isomorphisms. Assume that $f : X \to Y$ is an isomorphism. Then it follows from (TR1) (a) and (b) that $X \xrightarrow{f} Y \to 0 \to TX \in \Delta$. Hence we have that $Y \in \mathscr{T}'$ if $X \in \mathscr{T}'$ by (ii), since $0 \in \mathscr{T}'$.

(3) $\mathscr{T}'$ is closed under T and T^{-1}. This follows directly from (TR2) and (ii) applied to $X \to X \to 0 \to TX \in \Delta$.

(4) $\mathscr{T}$ induces a triangulated structure on $\mathscr{T}'$. Let T' be the functor T restricted to $\mathscr{T}'$. It can easily be shown that this is in fact an autoequivalence on $\mathscr{T}'$. Let Δ' be the collection of all distinguished triangles with entries in $\mathscr{T}'$. Then $(\mathscr{T}', T', \Delta')$ is triangulated.

(5) Some authors requires a thick triangulated subcategory to be closed under direct summands of distinguished triangles instead of being closed under direct summands of objects. However, it is easy to show that these two definitions are equivalent.

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