Introduction to the papers of TWG03: Algebraic thinking

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In CERME-10, the Thematic Working Group 3 "Algebraic thinking" continued the work carried out in previous CERME conferences. There were a total of 16 papers and 5 posters with a total of 29 group participants representing countries from Europe and other continents: Canada, Finland, Germany, Greece, Ireland, Norway, Portugal, Spain, Sweden, Tunisia, Turkey, UK, and USA.

Recurring issues

While the importance of algebra education is universally acknowledged, the problem of teaching it successfully to most students is not yet solved. Thus, there is a need to go back to basics over and over again and a lot of issues occur repeatedly in the history of CERME working groups on algebra. A broader overview is given in Hodgen, Oldenburg and Strømskag (2017). The discussion during CERME 10 brought up the following fundamental issues:

- What is algebra? There is still no uniform definition of what is the particularity of this field and what are the relations to other mathematical fields like combinatorics or geometry (that use letters as well). Moreover, many notions are not fully defined.
- How can it be empirically determined what works? We still have no universal measures of algebraic competence. Hence, many ad hoc tests are used.
- What should be taught? Too little is known about how knowledge builds up in the long term. For instance, it may be that certain concepts and metaphors that work well in some grades will give raise to obstacles later on.

Regarding the first point in this list, the group discussed the question of whether it would be sensible to rename the group's title just to "Algebra", because the word "thinking" gives the cognitive a higher weight than it might deserve. But this was resolved by the shared understanding that "algebraic thinking" is interpreted to include language, affect and possibly further factors.

The second point was taken up in a series of discussions about the quality of research and communication. Conceptual validity is seen often to be a problem. To rely just on Cronbach's alpha to ensure internal consistency seems not adequate. Perhaps the community should ensure that whole tests, measurement instruments and data are made accessible for other participants? Still, it will be difficult to ensure a common understanding of notions, given the plurality of theories and terminology used.

Despite these questions, there are substantial areas where a consensus has been reached: It is accepted that early algebra "works", in the sense that it is possible to develop algebraic thinking using, or just beginning to use, formal symbolic notation. Furthermore, most researchers see structure as a guiding principle in algebra and especially the structure of equations and the role of the equal sign is identified as central. The context/environment of each research event is relevant and especially the tasks and its implementation by the teacher are crucial together with the role of the researcher. Regarding ideas for the curriculum, we agreed that equation solving should not start with too simple equations. Filloy and Rojano's (1989) distinction between arithmetical and algebraic equations is important to exemplify the domain in which algebraic methods can show there power to students.

Some comments on issues dealt with in the papers

Functions have been identified my many colleagues as central issue in algebraic thinking and hence we have seen several papers (Isler et al., Pinto & Cañadas, Postelnicu, and Weber) that deepen the understanding of functions.

Isler et al. report results from a quantitative study in the US of Grade 6 students' written work on a functional thinking assessment item. The results show that students who experienced an early algebra intervention during Grades 3-5 were more likely to successfully represent a function rule in words and variables than students who did not. Also, both comparison and intervention groups of students were found to be more successful representing a function rule in variables than in words. The results underscore the impact of early algebra on students' later success in algebra, and challenge the view that the concept of variable should not be introduced until secondary school.

Pinto and Cañadas report from a study of 24 Spanish Grade 3 students' functional thinking during engagement with a contextualised linear problem (placing tiles). Two types of functional relationships were identified—*correspondence* and *covariation*—and the ability to generalise was observed in some of the students. The study was part of a broader teaching experiment, and the data were collected through a task-based questionnaire.

Postelnicu conducted a study of 58 US high school students' difficulties with writing equations of parallel and perpendicular lines (in the context of Algebra 1). Chevallard's theory of didactic transposition was employed to account for the relativity of the mathematical knowledge with respect to the institutions where the knowledge was created. The analysis shows that the mathe-matical knowledge (through the didactic transposition) lost its essential feature—the *proof*—with serious consequences for the curriculum. What remained to be learned was how to execute tasks.

Weber presents a theoretical paper, where vom Hofe's construct of 'Grundvorstellungen' and Sfard's distinction between operational and structural conceptions are used to analyse structural and operational models of logarithmic functions. Weber claims that logarithmic functions should not be introduced structurally, as inverse exponential functions. Instead, several operational models of the logarithmic concept are proposed, and their explanatory power for graphing is expounded.

Zindel presents a model for conceptualizing the core of the function concept, which is made up of those facets that are equally important for all types of functions and common to all representations. The so-called *facet model* enables the identification of potential obstacles and a detailed description of students' learning processes when connecting representations (e.g., verbal and symbolic representations when solving word problems). In total, 19 design experiments with overall 96 learners (mainly Grades 9-10) were conducted and qualitatively analyzed.

A focus on the thinking in algebraic thinking has been laid by four papers: Palatnik and Koichu; Twohill; Soneira, González-Calero and Arnau; and, Proulx.

Palatnik and Koichu took a detailed view on how students make sense of formula they found on various ways. The authors found that the process of sense making is consists of formulating and justifying claims, making generalizations, finding mechanisms and established coherence among the explored objects.

Twohill investigated number sequences from geometric patterns and the path of students to general terms. It turned out that between figural and numerical aspects of the patterns there is a whole continuum of ways that students think about these sequences. It is not easily said what aspects students should look at to be successful in finding a proper generalization.

Soneira et al. investigated in details the well-known error that students might produce expressions in which different occurrings of the same variable have different (but often related) reference. They explain this by idiosyncratic semiotic systems used by the students. The process of translation between algebra and natural language is highly complex.

Proulx investigated how teachers and students solve algebraic problems mentally. Forcing them not to use paper and pencil or other techniques allows to get close to their thinking. This revealed a wide variety of approaches and students and teachers differed in these. In the end, a sense for the diversity should be developed especially by the teachers.

Röj-Lindburg et al. considered the transition from informal to formal methods of equations solving in Grade 6 (12 years old) in Finland. The approaches taken by three teachers were analysed. One teacher used the image of a balance scale; another used uncomplicated 'real-world' situations; and the third had an emphasis on formal methods, in particular the need to 'do the same thing on both sides'. The third teacher's lesson was analysed and concluded that the discussion focused strongly on memorizing the procedure and did not develop an algebraic understanding of equality. In fact, it was concluded that in none of the teachers' lessons was there a need for students to adopt an algebraic way of thinking about equality.

Steinweg brought out the fact that the mathematics teaching units in Germany primary education lack explicit algebra learning environments. She offered ways in which key algebraic ideas can be used as guiding principles to rethink 'arithmetic' topics in six German primary school classes so that they can be used as learning environments for algebraic thinking. She focused on work from a pupil who was working on a task to decompose the area of a given rectangle and who appeared to show an awareness of the inherent distributive structures. Pre- and post-tests showed an increase in the percentage of children giving answers deemed to be algebraic in nature.

Papadopoulos and Patsiala studied the use of a particular learning environment called "Father Woodland" with seventy Grade 3 students (8-9 year olds) from two different primary schools in Greece. The approaches taken by the students were categorized into four types and it was noted that over the course of eight tasks, there was increased use of approaches which were classified as either 'combining words and symbolic language' or 'using symbolic language to express relationships". An argument was made that the environment helped develop the students' algebraic thinking.

As mentioned above, several researchers were concerned with the issues of "what should be taught" and what constitutes proficiency from the learners' points of view. Pinkernell, Düsi and Vogel proposed a way to construct validity for the concept of proficiency in elementary algebra, and presented the methodology of constructing a "model" of proficiency, together with the resulting product – the “revised model.” Wladis and colleagues described an instructor-generated “concept framework” for elementary algebra in the tertiary context.

Chimoni and Pitta-Pantazi addressed the issue of determining empirically “what works” for teaching algebra. They conducted a study with 96 early algebra students and compared two intervention courses. The first intervention course included real life scenarios and semi-structured tasks, while the second intervention course involved mathematical investigations and structured tasks. The results showed that the first course had better learning outcomes.

Two papers reported on structural aspects of algebra, at the elementary and university level, respectively. Strømskag and Valenta addressed the issue of justifying the commutative property of multiplication of natural numbers for Grade 6 students. At the heart of the study were the limitations of the visual representation used by the observed student teacher to help her students justify the property of commutativity of multiplication. Mutambara and Bansilal investigated the understanding of the concept of vector subspace. Participating students were 84 in-service teachers enrolled in a mathematics course at a Zimbabwean university. The action, process, object schema (APOS) theory, based on Piaget’s genetic epistemology, was proposed for the analysis of two tasks. The results highlighted the teachers’ difficulties with the concepts of sets, matrices, and vector subspace.

Outlook

The synopsis of papers given above shows the wide variety of theories, topics and methods used in this group. Such a pluralistic situation is highly welcomed as it allows to test the validity of research results from multiple perspectives. Thus, the consensus described above, can be viewed as solidly grounded and form the base for further research that can and should address questions that are not yet understood well enough. One such area is the domain of high school algebra. Weber’s paper has shown the potential of better understanding such concepts. Another point to be developed further is the perspective of teachers and teacher education.

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