

A Statistical Property of Wireless Channel Capacity: Theory and Application

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ABSTRACT

This paper presents a set of new results on wireless channel capacity by exploring its special characteristics. An appealing discovery is that the instantaneous and cumulative capacity distributions of typical fading channels are light-tailed. An implication of this property is that these distributions and subsequently the distributions of delay and backlog for constant arrivals can be upper-bounded by some exponential functions, which is often assumed but not justified in the literature of wireless network performance analysis. In addition, three representative dependence structures of the capacity process are studied, namely comonotonicity, independence, and Markovian, and bounds are derived for the cumulative capacity distribution and delay-constrained capacity. To help gain insights in the performance of a wireless channel whose capacity process may be too complex or detailed dependence information is lacking, stochastic orders are introduced to the capacity process, based on which, comparison results of delay and delay-constrained capacity are obtained. Moreover, the impact of self-interference in communication, which is an open problem in stochastic network calculus (SNC), is investigated and original results are derived. These results complement the SNC literature, easing its application to wireless networks and its extension towards a calculus for wireless networks.

Keywords

Wireless Channel Capacity; Stochastic Process

1. INTRODUCTION

The wireless communication system is entering a new generation, namely 5G. 5G is transformative, since it will advance mobile communication technology from largely a set of technologies, connecting people to people and people to information, to a unified connectivity fabric connecting people to everything [24], i.e., 5G will thrust mobile technology into the exclusive realm of general purpose technologies, e.g., electricity and automobile. The profound effects arising from these innovations range widely from the positive impacts on human and machine productivity to ultimately elevating the living standards of people around the world [24]. On the other hand, there will be a continuing wireless data explosion and an increasing requirement of higher data rate

and less latency. It has been depicted that the amount of IP data handled by wireless networks will exceed 500 exabytes by 2020, the aggregate data rate and edge rate will increase respectively by $1000\times$ and $100\times$ from 4G to 5G, and the round-trip latency needs to be less than 1ms in 5G [1]. The capacity demand and supply is a paradox, and the potential digital traffic jams threaten to throttle the information-technology revolution [22]. Evidently, it becomes more and more crucial to explore the ultimate capacity that a wireless channel can provide under stringent delay constraints and to analyze what delay limit may be achieved in specific wireless channel situations.

In this paper, we ask and answer three questions on the statistical properties of wireless channel capacity that is treated as a stochastic process, and the obtained results are supposed to provide some insights to cope with the above challenges.

1. What is the fundamental property of this stochastic process?

We discover that the tail distribution of wireless channel capacity is light-tailed. A simple explanation is that the capacity is a logarithm function of some random variables, so long as these random variables are as light as fat tails, the capacity is light-tailed. Though intuitive, it has been taken for granted without being taken fully advantage of. Moreover, this property is fundamental as it holds for all typical wireless channel models. This property has been extended from flat-fading to frequency-selective fading, from instantaneous to cumulative time regime, and from single-hop to multiple-hop scenarios.

2. What is the hidden resource to be utilized in wireless channels?

As a stochastic process, the wireless channel capacity is dependent over time, we classify the dependence structure into three categories, i.e., positive dependence, independence, and negative dependence, and we show that the negative dependence greatly improves the channel performance even with a smaller capacity mean with respect to independence, while the positive dependence has an opposite effect. It is worth noting that the dependence control can be implemented in practice, e.g., the negative dependence in power control will bring its impact into capacity. On the other hand, the negative dependence in environment can be taken advantage of, e.g., fading.

3. What is the impact of self-interference on ad hoc network scalability?

It is well known that the self-interference has a huge impact on the end-to-end throughput, and we prove it mathematically that the impact of self-interference can be localized. Specifically, when a common channel is shared among a group of nodes, the end-to-end network can be seen as a single-hop system. This result can be used to reduce the complexity of network topology in analysis. In addition, the wireless channel with self-interference is a feedback system, which is difficult for analysis and is regarded as an open problem in stochastic network calculus, a methodology suitable for end-to-end network performance analysis in feedforward networks. The solution here indicates the potential of extending stochastic network calculus to non-feedforward networks.

To date, wireless channel capacity has mostly been analyzed for its average rate in the asymptotic regime, i.e., ergodic capacity, or at one time instant/short time slot, i.e., instantaneous capacity. For instance, the first and second order statistical properties of instantaneous capacity have been extensively investigated, e.g., [38, 42]. However, the previous works focus on deriving explicit expressions of considered statistics in specific channel models without exploiting the general capacity property of different channels, e.g., [38, 23, 41, 39], or rely on the assumption that the distribution of the service process is exponential without mathematical justification, e.g., [31]. In [20], the capacity of ad hoc networks is studied based on constant transmission rate, while we focus on the stochastic process of wireless channel capacity. In [32], the self-interference in ad hoc networks is investigated in the protocol layer and it is shown that the average distance between source and destination must be small for network scalability, in contrast, we focus on the physical layer capacity and provide a mathematical proof of the localization property.

The remainder of this paper is structured as follows. Sec. 2 focuses on the basic concepts and fundamental property of wireless channel capacity. First, concepts of ergodic capacity, instantaneous capacity, cumulative capacity, and transient capacity are introduced; second, the fundamental property that the tail distribution of the capacity process is light-tailed irrelevant to temporal dependence, is proved; third, it is shown how specific dependence structures can be taken advantage of for result improvement. Sec. 3 is dedicated to applications of the light-tail property of wireless channel capacity. First, the wireless channel is modeled as a queueing system following a general queuing principle, delay-constrained capacity is defined as a complementary to the classical capacity concepts with a focus on delay performance, and Lundberg's inequality is invoked for explicit results in view of the light-tailed distribution; second, for dependence scenarios where explicit results are not tractable, the influence of dependence is manifested by stochastic ordering, again, the results are based on the light-tail property; last, the performance analysis is extended from feedforward to non-feedforward systems and from single-hop to multi-hop systems, with application to self-interference modeling in and scalability investigation of ad hoc networks. Finally, the paper is concluded in Sec. 4.

2. THEORY

2.1 Concepts

Basic concepts of wireless channel capacity, including ergodic capacity, instantaneous capacity, cumulative capacity, and transient capacity, are introduced in this part.

The maximum mutual information over input distribution at t , denoted as $C(t)$ throughout this paper, is defined as *instantaneous capacity* [9]:

$$C(t) = \max_{p(x)} I(X; Y|h(t)), \quad (1)$$

where $h(t)$ is a stochastic process describing wireless channel fading, X and Y are input and output random variables with alphabets \mathcal{X} and \mathcal{Y} , and the maximum is taken over all possible input distributions $p(x) = P\{X = x\}$, $x \in \mathcal{X}$.

Consider a discrete-time flat fading channel with input $x(t)$, output $y(t)$, and stationary fading process $h(t)$, the complex baseband representation is as follows,

$$y(t) = h(t)x(t) + n(t), \quad (2)$$

where $n(t)$ is an additive white Gaussian noise (AWGN) process $\mathcal{CN}(0, N_0)$. Conditional on a realization of $h(t)$, the mutual information is expressed as [47]

$$I(X; Y|h(t)) = \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} P(x, y|h_t) \log_2 \frac{P(x, y|h_t)}{P(x|h_t)P(y|h_t)}. \quad (3)$$

Particularly, for a single input single output channel, if the channel side information is only known at the receiver, solving the right hand side of (1) with (3), the instantaneous capacity is obtained [48], i.e.,

$$C(t) = W \log_2 (1 + \gamma |h(t)|^2), \quad (4)$$

where $|h(t)|$ denotes the envelope of $h(t)$, $\gamma = P/N_0W$ denotes the average received SNR per complex degree of freedom, P denotes the average transmission power per complex symbol, $N_0/2$ denotes the power spectral density of AWGN, and W denotes the channel bandwidth. For multiple input and multiple output channels, a generalized form of (4) is available in [47, 17].

Averaging the instantaneous capacity over the probability space of channel gain, the mean is defined as *ergodic capacity* [47]:

$$\bar{C} = E[C(t)]. \quad (5)$$

The definition implies that the ergodic capacity is a constant and is a concept for infinite code length in infinite time regime, i.e., it defines the maximum transmission rate of the channel with asymptotically small error probability for the code with sufficiently long length that the received codewords is affected by all fading states [19].

To account for finite time regimes, the sum of instantaneous capacity over a time period $(s, t]$, denoted as $S(s, t)$, is defined as *cumulative capacity*:

$$S(s, t) = \sum_{i=s+1}^t C(i). \quad (6)$$

For $S(0, t)$, we use $S(t)$ as simplification. The time average of the cumulative capacity through $(0, t]$ is defined as *transient capacity* [48]:

$$\bar{C}(t) = \frac{S(t)}{t}. \quad (7)$$

Note that the transient capacity is random, which essentially defines the achievable capacity for a code with finite length that the received codewords only experience partial fading states. The probabilistic average of the transient capacity in a stationary process is expressed as

$$E[\bar{C}(t)] = \bar{C}, \quad (8)$$

where \bar{C} is the ergodic capacity of the channel. According to the law of large numbers, the transient capacity converges to the ergodic capacity when time goes to infinity, i.e.,

$$\lim_{t \rightarrow \infty} \bar{C}(t) \rightarrow \bar{C}, \quad (9)$$

for independent and identically distributed instantaneous capacity. However, the dependence in capacity may be unknown, and a more general result for the transient capacity on finite time horizon is expressed by the Chebyshev inequality [37],

$$P\{|\bar{C}(t) - \bar{C}| \geq x\} \leq \frac{\text{Var}[\bar{C}(t)]}{x^2}, \quad (10)$$

which is a basic result of concentration [5]. It indicates that, in view of temporal behavior, statistical properties of the cumulative process should be taken into account besides the instantaneous capacity.

2.2 Light-tail Behavior

A distribution is said to be light-tailed, if the tail $\bar{F}(x) = 1 - F(x)$ is exponentially bounded, i.e.,

$$\bar{F}(x) = O(e^{-\theta x}), \quad (11)$$

for some $\theta > 0$; equivalently, it means the moment generating function $\hat{F}[\theta]$ is finite for some $\theta > 0$. Otherwise, the distribution is said to be heavy-tailed [3, 43], specifically, if $\bar{F}(x) = O(x^{-\theta})$, it is said to be fat-tailed.

The following theorem gives the condition for the wireless channel capacity distribution to be light-tailed.

THEOREM 1. *For flat fading, the instantaneous capacity is expressed as the logarithm transform of the instantaneous channel gain, i.e., $C(t) = W \log_2(1 + \gamma h(t)^2)$, $\forall t$. If the distribution of the fading process is not heavier than fat tail, the distribution of the instantaneous capacity is light-tailed.*

PROOF. For convenience, we omit the time index t and write $C = W \log_2(1 + \gamma h^2)$. Correspondingly, the tail of the instantaneous capacity is a function of the tail of the channel gain, i.e.,

$$\bar{F}_C(x) = \bar{F}_h\left(\sqrt{\frac{2^{\frac{x}{W}} - 1}{\gamma}}\right). \quad (12)$$

Let $r = \sqrt{\frac{2^{\frac{x}{W}} - 1}{\gamma}}$, for some $\theta > 0$, $\bar{F}_C(x) = O(e^{-\theta x})$ entails

$$\bar{F}_h(r) = O(r^{-\theta}), \quad (13)$$

which completes the proof. \square

The following corollary shows that the capacity distributions of the typical wireless fading channels are light-tailed.

COROLLARY 1. *If a wireless channel is Rayleigh, Rice, Nakagami- m , Weibull, or lognormal fading channel, its instantaneous capacity distribution is light-tailed.*

PROOF. For Weibull fading channel, the tail of fading is expressed as

$$\bar{F}_h(r) = e^{-cr^k}, \quad (14)$$

where $c > 0$ and $k > 0$ are constants. Applying Taylor's theorem to expand e^{-cr^k} , it is easily shown that, for some θ satisfying $k > \theta > 0$

$$\lim_{r \rightarrow \infty} \frac{e^{-cr^k}}{r^{-\theta}} = \lim_{r \rightarrow \infty} \frac{r^\theta}{1 + cr^k + \dots} = 0. \quad (15)$$

This limit shows that though the Weibull distribution is heavy-tailed for $0 < k < 1$, it is lighter than the fat tail. Hence from Theorem 1, the instantaneous capacity under Weibull fading is light-tailed.

Rayleigh fading is a special case of Weibull fading with $k = 2$. The distribution of its instantaneous capacity is expressed as [23]

$$F(x) = 1 - e^{\frac{1-2^{\frac{x}{W}}}{\gamma}}. \quad (16)$$

It is trivial to show that the tail is exponentially bounded

$$\bar{F}(x) \leq e^{\frac{1}{\gamma}} e^{-\theta x}, \quad (17)$$

for $0 < \theta \leq \frac{1}{W\gamma} 2^{\frac{1}{\log 2}} \log 2$. Hence, the instantaneous capacity under Rayleigh fading is light-tailed.

For Rice fading channel, the tail of the instantaneous capacity is expressed as [41]

$$\bar{F}(x) = Q_1\left(\frac{s}{\sigma_0}, \frac{\sqrt{2^{x/W} - 1/\gamma_s s}}{\sigma_0^2}\right), \quad (18)$$

where W is the bandwidth, s the amplitude of the LOS (light of sight) component, σ_0^2 the variance of the underlying Gaussian process, and γ_s the average SNR. According to the exponential bound of the Marcum Q-function [46],

$$\alpha_F = \limsup_{x \rightarrow \infty} \frac{-\log \bar{F}(x)}{x} \quad (19)$$

$$\geq \limsup_{x \rightarrow \infty} \frac{1}{2x} \left(\frac{\sqrt{2^{x/W} - 1/\gamma_s s}}{\sigma_0^2} - \frac{s}{\sigma_0} \right)^2 \quad (20)$$

$$= \infty, \quad (21)$$

which means that the instantaneous capacity of a Rice fading channel is light-tailed [43].

For Nakagami- m fading channel [39], since the square of the Nakagami- m random variable follows a gamma distribution, which is light-tailed [3], the distribution of its instantaneous capacity is thus light-tailed.

For lognormal fading channel [40], since the lognormal distribution has all the moments, which means that it has a lighter tail than the fat-tailed distribution [21], the distribution of its instantaneous capacity is light-tailed. \square

The rest of this subsection shows that the light-tailed property is extended from flat-fading to frequency-selective fading, from instantaneous to cumulative time regime, and from single-hop to multiple-hop scenarios.

COROLLARY 2. *For frequency-selective fading modeled by L parallel independent channels with the instantaneous capacity $C = \sum_{\ell=1}^L W_\ell \log_2(1 + \gamma h_\ell^2)$, if the distribution of the instantaneous capacity of each sub-channel $C_\ell = W_\ell \log_2(1 + \gamma h_\ell^2)$ is light-tailed, so is the instantaneous capacity distribution of the frequency-selective fading channel.*

PROOF. For this frequency-selective fading channel, its instantaneous capacity is by definition related to the instantaneous capacity of each sub-channel as

$$C = \sum_{\ell=1}^L C_\ell. \quad (22)$$

The tail of the distribution of the instantaneous capacity can then be expressed by [26]

$$\bar{F}_C(x) = 1 - F_{C_1} \circledast \dots \circledast F_{C_L}(x) \quad (23)$$

$$\leq \bar{F}_{C_1} \oplus \dots \oplus \bar{F}_{C_L}(x), \quad (24)$$

where $f \circledast g(x) = \int_{-\infty}^{\infty} f(x-y)dg(y)$ is the Stieltjes convolution and $f \oplus g(t) = \inf_{0 \leq s \leq t} \{f(s) + g(t-s)\}$ is the univariate min-plus convolution [4] or infimal convolution [44]. The first step results from sum of independent random variables, and the second step results from that the distribution of sum of independent random variables is upper bounded by the distribution of such a sum without dependence consideration [26]. As is illustrated in the proof of the next theorem, the latter is light-tailed. \square

COROLLARY 3. Consider a wireless channel, if the distribution of its instantaneous capacity at any time is light-tailed, the distribution of the cumulative capacity is light-tailed, and the distribution of the cumulative capacity of a concatenation of such wireless channels is light-tailed.

PROOF. Without considering any dependence constraint, the tail of the cumulative capacity, $S(t) = C(1) + \dots + C(t)$, is bounded by [26]

$$\bar{F}_{S(t)}(x) \leq \bar{F}_{C(1)} \oplus \dots \oplus \bar{F}_{C(t)}(x), \quad (25)$$

which is exactly the infimal convolution of the Fréchet upper bound [44]. If the instantaneous capacity is light tailed, i.e.,

$$\bar{F}_C(x) \leq ae^{-bx}, \quad (26)$$

applying a distribution bound for the sum of exponentially bounded random variables [26], the tail of the cumulative capacity is exponentially bounded, i.e.,

$$\bar{F}_{S(t)}(x) \leq \prod_{k=1}^t (a_k b_k w)^{\frac{1}{b_k w}} \cdot e^{-\frac{x}{w}}, \quad (27)$$

where $w = \sum_{k=1}^t \frac{1}{b_k}$.

For a concatenation of wireless channels, each with a cumulative capacity $S_i(s, t)$, the cumulative capacity process is essentially the service process of the channel, and the cumulative capacity of the concatenation channel is expressed as [26, 16]

$$S(s, t) = S_1 \otimes \dots \otimes S_N(s, t), \quad (28)$$

where $f \otimes g(x) = \inf_{0 \leq y \leq x} \{f(y) + g(y, x)\}$ is the bivariate min-plus convolution [6]. Then, the tail is expressed as

$$\bar{F}_{S(t)}(x) = P\{S_1 \otimes \dots \otimes S_N(t) \geq x\} \quad (29)$$

$$= P\left\{\inf_{\mathbf{u} \in \mathcal{U}(x)} \sum_{i=1}^N S_i(u_{i-1}, u_i) \geq x\right\} \quad (30)$$

$$\leq \inf_{\mathbf{u} \in \mathcal{U}(x)} P\left\{\sum_{i=1}^N S_i(u_{i-1}, u_i) \geq x\right\} \quad (31)$$

$$\leq \inf_{\mathbf{u} \in \mathcal{U}(x)} E\left[e^{\theta \sum_{i=1}^N S_i(u_{i-1}, u_i)}\right] \cdot e^{-\theta x}, \quad (32)$$

where $\mathcal{U}(x) = \{\mathbf{u} = (u_0, u_1, \dots, u_N) : u_0 = 0, u_N = t, 0 \leq u_1 \leq \dots \leq u_{N-1} \leq t\}$, for some $\theta > 0$. \square

2.3 Dependence Refinement

In general, the capacity is dependent over time, which results from the temporal dependence in the environment or in the controllable parameters of the system. Specifically for the cumulative capacity, the influence of stochastic dependence is characterized by the Fréchet bounds [44]

$$\check{F}_{S(t)}(x) \leq F_{S(t)}(x) \leq \hat{F}_{S(t)}(x), \quad (33)$$

where

$$\check{F}_{S(t)}(x) = \left[\sup_{\mathbf{u} \in \mathcal{U}(x)} \sum_{i=1}^t F_{C(i)}(u_i) - (t-1) \right]^+, \quad (34)$$

$$\hat{F}_{S(t)}(x) = \left[\inf_{\mathbf{u} \in \mathcal{U}(x)} \sum_{i=1}^t F_{C(i)}(u_i) \right]_1, \quad (35)$$

with $\mathcal{U}(x) = \{\mathbf{u} = (u_1, \dots, u_t) : \sum_{i=1}^t u_i = x\}$. The Fréchet bounds hold in general, making use of specific dependence information among $C(1), C(2), \dots$, the bounds can be improved. To this aim, three representative capacity processes are investigated in this subsection, which are comonotonic process, additive process, and Markov additive process.

2.3.1 Comonotonic Process

The upper Fréchet bound expresses the extremal positive dependence indicating the largest sum with respect to convex order, and the dependence structure is represented by the comonotonic copula [12, 13, 14], i.e.,

$$F(c_1, \dots, c_t) = \min_{1 \leq i \leq t} F_{C(i)}(c_i); \quad (36)$$

equivalently [12], for $U \sim U(0, 1)$,

$$(C(1), \dots, C(t)) \stackrel{d}{=} (F_{C(1)}^{-1}(U), \dots, F_{C(t)}^{-1}(U)), \quad (37)$$

which implicates that comonotonic random variables are increasing functions of a common random variable [13].

If the increment of the cumulative capacity has comonotonicity, the cumulative capacity is defined as a comonotonic process in this paper (which is different from a similar concept regarding the comonotonicity between different processes [27]). The distribution results of cumulative capacity and transient capacity are as follows.

THEOREM 2. For a stationary capacity process, the distributions of the cumulative capacity and transient capacity with comonotonicity are expressed as

$$F_{S(t)}(x) = F_C\left(\frac{x}{t}\right), \quad (38)$$

$$F_{\bar{C}(t)}(x) = F_C(x). \quad (39)$$

PROOF. In the special case that all marginal distribution functions are identical $F_{C(i)} \sim F_C$, comonotonicity of $C(i)$ is equivalent to saying that $C(1) = C(2), \dots, = C(t)$ holds almost surely [12]. In other words, the sample function of the capacity process is stationary and depends only on the initial value of the capacity in each realization. \square

2.3.2 Additive Process

The independence structure of an additive process is expressed by a product copula

$$F(c_1, \dots, c_t) = \prod_{i=1}^t F_{C(i)}(c_i), \quad (40)$$

and the distribution of the cumulative capacity is expressed via Stieltjes convolution as

$$F_{S(t)}(x) = F_{C(1)} \circledast \dots \circledast F_{C(t)}(x). \quad (41)$$

The cumulative capacity with independent increment is modeled as an additive process [25]. The distribution bounds of cumulative capacity and transient capacity are as follows.

THEOREM 3. *For a stationary capacity process, the distribution of the cumulative capacity with independence is expressed as, for some $\theta > 0$,*

$$1 - e^{t\kappa(\theta) - \theta x} \leq F_{S(t)}(x) \leq e^{t\kappa(-\theta) + \theta x}, \quad (42)$$

where $\kappa(\theta) = \log E[e^{\theta C(i)}]$ is the cumulant generating function of the instantaneous capacity, and the distribution of the transient capacity is expressed as

$$1 - e^{-y_l} \leq P\{\bar{C}(t) \leq c^*\} \leq e^{-y_u}, \quad (43)$$

where $c^* = \frac{t\kappa(\theta^*) + y^*}{\theta^* t}$, with $y^* = y_u$ for $\theta^* < 0$ for the upper bound, and $y^* = y_l$ for $\theta^* > 0$ for the lower bound.

PROOF. In the special case that all marginal distribution functions are identical $F_{C(i)} \sim F_C$, a likelihood ratio process of the cumulative capacity is formulated and expressed as [2]

$$L(t) = e^{\theta S(t) - t\kappa(\theta)}, \quad (44)$$

where $L(t)$ is a mean-one martingale and $\kappa(\theta)$ is the cumulant generating function, i.e.,

$$\kappa(\theta) = \log E[e^{\theta C(i)}] = \log \int e^{\theta x} F(dx), \quad (45)$$

where $\theta \in \Theta = \{\theta \in \mathbb{R} : \kappa(\theta) < \infty\}$.

According to Markov inequality, for any $\mu > 0$,

$$P\{L(t) \geq \mu\} \leq \frac{1}{\mu} E[L(t)] = \frac{1}{\mu}. \quad (46)$$

Letting $\mu = e^{-t\kappa(\theta) + \theta x}$, for $\theta \leq 0$, the cumulative distribution function is bounded by

$$P\{S(t) \leq x\} \leq e^{t\kappa(\theta) - \theta x}, \quad (47)$$

while for $\theta > 0$, the complementary cumulative distribution function is expressed as

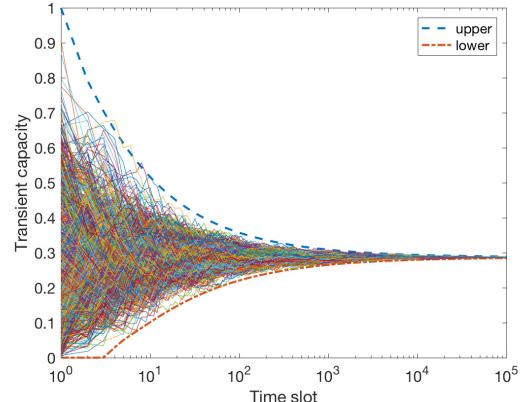
$$P\{S(t) \geq x\} \leq e^{t\kappa(\theta) - \theta x}, \quad (48)$$

which shows that the distribution has a light tail. Letting $-y^* = t\kappa(\theta) - \theta x \leq 0$, the distribution of the transient capacity is bounded by

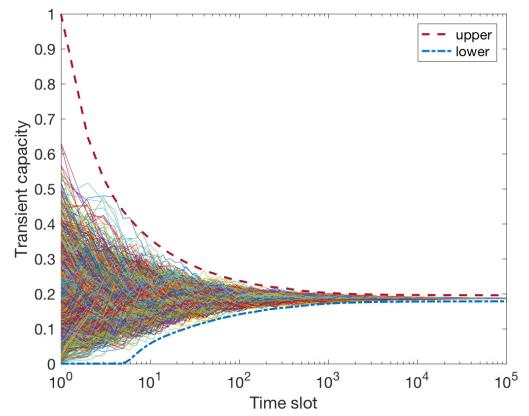
$$1 - e^{-y_l} \leq P\{\bar{C}(t) \leq c^*\} \leq e^{-y_u}, \quad (49)$$

where $c^* = \frac{t\kappa(\theta) + y^*}{\theta t}$, with $y^* = y_u$ for $\theta < 0$ for the upper bound, and $y^* = y_l$ for $\theta > 0$ for the lower bound. \square

REMARK 1. *The upper and lower bound of $F_{S(t)}(x)$ do not hold simultaneously, the upper bound is useful for $x < \bar{S}(t)$, the lower bound is useful for $x > \bar{S}(t)$, and both bounds are worthless for $x = \bar{S}(t)$ [18]. Considering $\bar{F}_{S(t)}(x) = 1 - F_{S(t)}(t)$, which means that the upper and lower bound can not decrease or increase simultaneously, this property holds in general. An indication of this property is that, for a fixed violation probability, the obtained bounds on $S(t)$ or $\bar{C}(t)$ based on the upper and lower distribution bounds are lower and upper bounds of $S(t)$ or $\bar{C}(t)$ with respect to their mean. It is illustrated in Fig. 1.*



(a) Additive process.



(b) Markov additive process.

Figure 1: Transient capacity of additive and Markov additive Rayleigh channel. According to the strong law of large numbers for the additive process and extended to the Markov additive process, the transient capacity converges to the mean as time goes to infinity, i.e., the convergence of sample paths. The large deviation results are upper bound and lower bound with respect to the mean. Results are normalized, with violation probability $\epsilon = 10^{-3}$, $W = 20\text{kHz}$, $SNR = e^{0.5}$ for the additive process, $\mathbf{SNR} = [e^{0.5} \ 0.9e^{0.5}; 0.8e^{0.5} \ 0.7e^{0.5}]$ and $\mathbf{P} = [0.3 \ 0.7; 0.1 \ 0.9]$ for the Markov additive process with initial distribution $\boldsymbol{\pi} = [0.5 \ 0.5]$, and 1000 sample paths.

2.3.3 Markov Additive Process

The Markov property is solely a dependence property that can be modeled exclusively in terms of copulas. As a consequence, starting with a Markov process, a multitude of other Markov processes can be constructed by just modifying the marginal distributions [10, 30, 36]. It is worth noting that the Markov property indicates both positive and negative dependence, which is determined by the underlying copula. For a Markov chain, the selection of the copula and the marginal distribution is coupled [10], the transition matrix can be expressed in terms of the copula and marginal distribution and vice versa. Particularly for an idempotent copula, the process is conditionally independently and identically distributed given the initial state [30].

Specifically, if the dependence in capacity follows a Markov process and the instantaneous capacity has a corresponding distribution with respect to a state transition, then the cumulative capacity is a Markov additive process, which is a bivariate process with strong Markov property and the increment process is conditionally independent given a realization of the underlying Markov process. A formal definition of Markov additive process is in Appendix.

THEOREM 4. *For a Markov additive process, conditional on initial state J_0 , the distribution of the cumulative capacity is expressed as, for some $\theta > 0$,*

$$1 - \frac{h^{(\theta)}(J_0)e^{t\kappa(\theta)-\theta x}}{\min_{j \in E}(h^{(\theta)}(J_j))} \leq F_{S(t)}(x) \leq \frac{h^{(-\theta)}(J_0)e^{t\kappa(-\theta)+\theta x}}{\min_{j \in E}(h^{(-\theta)}(J_j))}, \quad (50)$$

and the distribution of the transient capacity is expressed as

$$1 - \frac{h^{(\theta)}(J_0)e^{-y_l}}{\min_{j \in E}(h^{(\theta)}(J_j))} \leq P\{\bar{C}(t) \leq c^*\} \leq \frac{h^{(-\theta)}(J_0)e^{-y_u}}{\min_{j \in E}(h^{(-\theta)}(J_j))}, \quad (51)$$

where $c^* = \frac{t\kappa(\theta^*)+y^*}{\theta^* t}$, with $y^* = y_u$ for $\theta^* < 0$ for the upper bound, and $y^* = y_l$ for $\theta^* > 0$ for the lower bound.

PROOF. Like the independent case, a likelihood ratio process is formulated with an exponential change of measure [2],

$$L(t) = \frac{h^{(\theta)}(J_t)}{h^{(\theta)}(J_0)} e^{\theta S(t)-t\kappa(\theta)}, \quad (52)$$

which is a mean-one martingale. $\kappa(\theta)$ and $h^{(\theta)}$ are respectively the logarithm of the Perron-Frobenius eigenvalue and the corresponding right eigenvector of the kernel for the Markov additive process $C(t)$, i.e., $\hat{\mathbf{F}}[\theta]$. In order to provide exponential upper bound for the distribution of the cumulative capacity, define [18]

$$\underline{L}(t) = \frac{\min_{j \in E}(h^{(\theta)}(J_j))}{h^{(\theta)}(J_0)} e^{\theta S(t)-t\kappa(\theta)}, \quad (53)$$

where $\underline{L}(t) \leq L(t)$, i.e., $E[\underline{L}(t)] \leq 1$. Apply Markov inequality to $\underline{L}(t)$ and get, for any $\mu > 0$,

$$P\{\underline{L}(t) \geq \mu\} \leq \frac{1}{\mu} E[\underline{L}(t)] \leq \frac{1}{\mu}. \quad (54)$$

Choose $\mu = e^{-t\kappa(\theta)+\theta x} \frac{\min_{j \in E}(h^{(\theta)}(J_j))}{h^{(\theta)}(J_0)}$, for $\theta \leq 0$,

$$P\{S(t) \leq x\} \leq \frac{h^{(\theta)}(J_0)}{\min_{j \in E}(h^{(\theta)}(J_j))} e^{t\kappa(\theta)-\theta x}, \quad (55)$$

while for $\theta > 0$,

$$P\{S(t) \geq x\} \leq \frac{h^{(\theta)}(J_0)}{\min_{j \in E}(h^{(\theta)}(J_j))} e^{t\kappa(\theta)-\theta x}, \quad (56)$$

which indicates that the distribution has a light tail. Letting $-y^* = t\kappa(\theta) - \theta x \leq 0$, the distribution of the transient capacity is bounded by

$$1 - \frac{h^{(\theta)}(J_0)e^{-y_l}}{\min_{j \in E}(h^{(\theta)}(J_j))} \leq P\{\bar{C}(t) \leq c^*\} \leq \frac{h^{(\theta)}(J_0)e^{-y_u}}{\min_{j \in E}(h^{(\theta)}(J_j))}, \quad (57)$$

where $c^* = \frac{t\kappa(\theta)+y^*}{\theta t}$, with $y^* = y_u$ for $\theta < 0$ for the upper bound, and $y^* = y_l$ for $\theta > 0$ for the lower bound. \square

REMARK 2. *The Markov additive process can be seen as a non-stationary additive process defined on a Markov process. If the Markov process has only one state, then it reduces to a stationary additive process [7]. In addition, the strong law of large numbers applies to the Markov additive process [35], and the mean of transient capacity exists [3], i.e.,*

$$\lim_{t \rightarrow \infty} \frac{E_{J_0 \in E}[S(t)]}{t} = \kappa'(0). \quad (58)$$

It is demonstrated in Fig. 1.

3. APPLICATION

3.1 Performance Guarantee

In the regime of information theory, the focus is on the asymptotic limit of the tradeoff between accuracy and rate of communication ignoring the role of delay that may affect this tradeoff [15]. Instead, we use queueing analysis and focus on two performance metrics, i.e., delay and delay-constrained capacity.

3.1.1 Queueing Principle

The wireless channel is essentially a queueing system with cumulative service process $S(t)$ and cumulative arrival process $A(0, t) = \sum_{s=0}^t a(s)$, where $a(t)$ denotes the traffic input to the channel at time t , and the temporal increment in the system is expressed as

$$X(t) = a(t) - C(t). \quad (59)$$

The queueing principle of the wireless channel is expressed through the backlog in the system, which is a reflected process of the temporal increment $X(t)$ [2], i.e.,

$$B(t+1) = [B(t) + X(t)]^+. \quad (60)$$

Throughout this paper, $B(0) = 0$ is assumed, and the backlog function is expressed as

$$B(t) = \sup_{0 \leq s \leq t} (A(s, t) - S(s, t)). \quad (61)$$

For a lossless system, the output is the difference between the input and backlog, $A^*(t) = A(t) - B(t)$, which is further represented by

$$A^*(t) = A \otimes S(t), \quad (62)$$

where $f \otimes g(s, t) = \inf_{s \leq \tau \leq t} \{f(s, \tau) + g(\tau, t)\}$ is the bivariate min-plus convolution [4, 6], and the delay is defined via the input-output relationship [8], i.e.,

$$D(t) = \inf \{d \geq 0 : A(t-d) \leq A^*(t)\}, \quad (63)$$

which is the virtual delay that a hypothetical arrival has experienced on departure.

The *delay-constrained capacity or throughput* is defined as the maximum rate of traffic with delay requirement that the system can support without dropping [49], i.e.,

$$\bar{C}(d, \epsilon) = \sup_{P(D(t) > d) \leq \epsilon, \forall t} E \left[\frac{A(t)}{t} \right]. \quad (64)$$

To avoid nontrivial considerations, we assume that the input is a constant fluid process, i.e.,

$$A(t) = \lambda t. \quad (65)$$

Then, the delay-constrained capacity is expressed as

$$\bar{C}(d, \epsilon) = \sup_{P(D(t) > d) \leq \epsilon, \forall t} \lambda. \quad (66)$$

It is a folk law that the regularity of arrival or service processes results in better performance measures, and it has been proved that for some involved system the queue length of a constant fluid input is the shortest for all types of inputs that have the same average traffic rate [34], thus the minimal delay and maximal delay-constrained capacity.

For the constant fluid arrival, it is trivial to show that the tail distributions of backlog and delay are respectively expressed as,

$$P(B(t) > x) = P \left\{ \sup_{0 \leq s \leq t} \{\lambda(t-s) - S(s, t)\} > x \right\}, \quad (67)$$

and

$$P(D(t) > d) = P \left\{ \sup_{0 \leq s \leq t} \{\lambda(t-s) - S(s, t)\} > \lambda d \right\}, \quad (68)$$

in addition, their relationship is easily verified,

$$P(B(t) > x) = P \left(D(t) > \frac{x}{\lambda} \right). \quad (69)$$

The requirement of the delay-constrained capacity on performance analysis indicates that the cumulative process of wireless channel capacity should be considered, in contrast to the asymptotic or instantaneous behavior in Shannon capacity.

3.1.2 Metric Analysis

Performance of wireless channels with three representative capacity processes are analyzed in this part.

THEOREM 5 (COMONOTONIC PROCESS). Consider a constant arrival process $A(t) = \lambda t$, the delay on finite time horizon is expressed as

$$P(D(t) > d) = P \left\{ C(1) < \lambda - \frac{\lambda d}{t} \right\}, \quad (70)$$

while the delay on infinite time horizon is expressed as

$$P(D > d; \forall d > 0) = P\{C(1) < \lambda\}. \quad (71)$$

PROOF. For a constant arrival process $A(t) = \lambda t$, the delay is expressed as

$$P(D(t) > d) = P \left\{ \sup_{0 \leq s \leq t} (A(s) - S(s)) > \lambda d \right\} \quad (72)$$

$$= P \left\{ C(1) < \lambda - \frac{\lambda d}{t} \right\}. \quad (73)$$

Letting time go to infinity gives

$$P(D > d; \forall d > 0) = P\{C(1) < \lambda\}. \quad (74)$$

This completes the proof. \square

It indicates that, for a comonotonic capacity process, a delay bound makes sense only on the finite time horizon, and on the infinite time horizon, whenever there is deep fade, there will be infinite delay, which is relevant to the outage probability for slow fading [48].

THEOREM 6 (ADDITIVE PROCESS). Consider a constant arrival process $A(t) = \lambda t$, the delay at the wireless channel is bounded by

$$C_- e^{-\theta \lambda d} \leq P(D \geq d) \leq C_+ e^{-\theta \lambda d}. \quad (75)$$

Letting $P(D \geq d) = \epsilon$, the delay-constrained capacity is expressed by

$$\frac{-\log \frac{\epsilon}{C_-}}{\theta d} \leq \lambda \leq \frac{-\log \frac{\epsilon}{C_+}}{\theta d}, \quad (76)$$

where

$$C_- = \inf_{x \in [0, x_0]} \frac{\bar{B}(x)}{\int_x^\infty e^{\theta(y-x)} B(dy)}, \quad (77)$$

$$C_+ = \sup_{x \in [0, x_0]} \frac{\bar{B}(x)}{\int_x^\infty e^{\theta(y-x)} B(dy)}, \quad (78)$$

and B is the distribution of $\lambda - C$ and $x_0 = \sup\{x : B(x) < 1\}$.

PROOF. For a constant arrival process $A(t) = \lambda t$, the delay is bounded by

$$P(D \geq d) = P \left\{ \sup_{t \geq 0} (A(t) - S(t)) \geq \lambda d \right\} \quad (79)$$

$$\leq e^{-\theta \lambda d}, \quad (80)$$

where the last inequality follows the Lundberg's inequality [43, 3], if $\theta (> 0)$ satisfies the Lundberg equation $\kappa(\theta) = 0$, where

$$\kappa(\theta) = \log \int e^{\theta(\lambda - C(t))} F(dx). \quad (81)$$

The approach to obtain the lower bound and to improve the prefactors is available in [43, 3]. \square

THEOREM 7 (MARKOV ADDITIVE PROCESS). Consider a constant arrival process $A(t) = \lambda t$, the delay conditional on the initial state $J_0 = i$ is bounded by

$$\frac{h^{(\theta)}(J_i) e^{-\theta \lambda d}}{\max_{j \in E} h^{(\theta)}(J_j)} \leq P_i(D \geq d) \leq \frac{h^{(\theta)}(J_i) e^{-\theta \lambda d}}{\min_{j \in E} h^{(\theta)}(J_j)}, \quad (82)$$

and, given the initial state distribution π , the stationary delay is thus bounded by

$$P(D \geq d) = \sum_{i \in E} \pi_i P_i(D \geq d). \quad (83)$$

Letting $P(D \geq d) = \epsilon$, the delay-constrained capacity is expressed as

$$\frac{-1}{\theta d} \log \frac{\epsilon \cdot \max_{j \in E} h^{(\theta)}(J_j)}{\sum_{i \in E} \pi_i h^{(\theta)}(J_i)} \leq \lambda \leq \frac{-1}{\theta d} \log \frac{\epsilon \cdot \min_{j \in E} h^{(\theta)}(J_j)}{\sum_{i \in E} \pi_i h^{(\theta)}(J_i)}. \quad (84)$$

PROOF. For a constant arrival process $A(t) = \lambda t$, the delay conditional on initial state $J_0 = i$ is bounded by [50]

$$P_i(D \geq d) = P_i \left\{ \sup_{t \geq 0} (\lambda(t) - S(t)) \geq \lambda d \right\} \quad (85)$$

$$\leq \frac{h^{(\theta)}(J_i)}{\min_{j \in E} h^{(\theta)}(J_j)} e^{-\theta \lambda d}, \quad (86)$$

where the last inequality follows the Lundberg's inequality, if $\theta (> 0)$ satisfies the Lundberg equation $\kappa(-\theta) = 0$. $\kappa(\theta)$ and $h^{(\theta)}$ are respectively the logarithm of the Perron-Frobenius eigenvalue and the corresponding right eigenvector of the kernel for the Markov additive process $S(t) - \lambda t$, i.e., $\hat{F}[\theta]$. The lower delay bound is available in [50]. \square

Specifically, if F_{ij} is independent of j , the prefactor in the Lundberg inequality can be improved and the doubly-sided bound is expressed as

$$C_- h^{(\theta)}(J_i) e^{-\theta \lambda d} \leq P_i(D \geq d) \leq C_+ h^{(\theta)}(J_i) e^{-\theta \lambda d}, \quad (87)$$

where

$$C_- = \min_{j \in E} \frac{1}{h_j^{(\theta)}} \cdot \inf_{x \geq 0} \frac{\bar{B}_j(x)}{\int_x^\infty e^{\theta(y-x)} B_j(dy)}, \quad (88)$$

$$C_+ = \max_{j \in E} \frac{1}{h_j^{(\theta)}} \cdot \sup_{x \geq 0} \frac{\bar{B}_j(x)}{\int_x^\infty e^{\theta(y-x)} B_j(dy)}, \quad (89)$$

and B_j is the distribution of the instantaneous capacity C_j [3].

3.2 Channel Comparison

For more involved dependence scenarios, explicit results of performance measures are hard to derive or no more tractable. As an alternative, we investigate the influence trend of different dependence structures, by first defining stochastic orders on cumulative capacity and then studying their impact on delay. For convenience, we omit the time index in this subsection.

3.2.1 Stochastic Ordering

The cumulative capacity S_X is said to be smaller than S_Y in stochastic order, i.e.,

$$S_X \leq_{st} S_Y, \quad (90)$$

if the distribution functions F_{S_X} and F_{S_Y} are comparable in the sense that $P(S_X \leq x) \geq P(S_Y \leq x)$, $\forall x$. In particular, the pointwise comparison of $S_X \leq S_Y$ implies the stochastic ordering $S_X \leq_{st} S_Y$. An equivalent condition for stochastic ordering is that the expectation of all increasing functions \mathcal{F} is larger for S_Y than for S_X , i.e., $E[f(S_X)] \leq E[f(S_Y)]$, $\forall f \in \mathcal{F}$. Considering the convexity of the functions, two other stochastic orders are defined. The cumulative capacity S_X is said to be smaller than S_Y in convex order (respectively increasing convex order), written as

$$S_X \leq_{cx} S_Y, \quad (91)$$

(respectively $S_X \leq_{icx} S_Y$), if for all convex functions \mathcal{F}_{cx} (respectively all increasing convex functions \mathcal{F}_{icx}), $E[f(S_X)] \leq E[f(S_Y)]$, $\forall f \in \mathcal{F}_{cx}$ (respectively $\forall f \in \mathcal{F}_{icx}$).

Intuitively, positive dependence implies that large or small values of random variables tend to occur together, while negative dependence implies that large values of one variable tend to occur together with small values of others [11]. By comparing to the probability measure of independence, positive dependence and negative dependence are defined under stochastic orders. In particular, the cumulative capacity S is said to have a positive dependence structure in the sense of increasing convex order, if

$$S_\perp \leq_{icx} S_P, \quad (92)$$

or a negative dependence structure in the sense of increasing convex order, if

$$S_N \leq_{icx} S_\perp, \quad (93)$$

where S_\perp has an independence structure. Since the cumulative capacity is an additive function, the relationship between convex order and increasing convex order is expressed in the following Lemma.

LEMMA 1. *For cumulative capacities S_N , S_\perp and S_P which respectively have negative dependence, independence, and positive dependence structures, if their marginal distributions are identical for all t , their convex ordering is equivalent to their increasing convex ordering, i.e.,*

$$S_N \leq_{icx} S_\perp \leq_{icx} S_P \iff S_N \leq_{cx} S_\perp \leq_{cx} S_P. \quad (94)$$

PROOF. Since the mean of sum of random variables equals the sum of means of individual random variables, i.e.,

$$E[S_N] = E[S_\perp] = E[S_P], \quad (95)$$

the proof follows directly from that the increasing convex order is identical to the convex order under equal expectations [28]. \square

3.2.2 Ordering of Delay

The Chernoff bound provides a general way to calculate the exponential bound of delay, i.e., for some $\theta > 0$,

$$P(D \geq d) \leq \sum_{t=0}^{\infty} P\{A(t) - S(t) \geq \lambda d\} \quad (96)$$

$$\leq \sum_{t=0}^{\infty} E\left[e^{\theta(\lambda t - S(t))}\right] \cdot e^{-\theta \lambda d}. \quad (97)$$

As the distribution of wireless channel capacity is light-tailed, the asymptotic behavior of the bounding function is still exponential for weak forms of dependence, while it becomes heavy-tailed for stronger dependence [3]. Specifically, the decay rate of the tail distribution is reflected by the adjustment coefficient, which gives a crude comparison of the exponential bounds.

THEOREM 8. *Consider two wireless channel capacity processes, if the cumulative capacities are convex ordered, then the adjustment coefficients for the delay bounds are correspondingly ordered, i.e.,*

$$S(t) \leq_{cx} \tilde{S}(t), \quad \forall t \in \mathbb{N} \implies \tilde{\theta} \leq \theta. \quad (98)$$

PROOF. Consider the negative increment process, i.e.,

$$-X(t) = C(t) - a(t). \quad (99)$$

If it is light-tailed, then the delay violation probability has an exponential bound with adjustment coefficient $\theta > 0$ defined by $\kappa(\theta) = 0$, where [3, 33]

$$\kappa(\theta) = \lim_{t \rightarrow \infty} \frac{1}{t} E\left[e^{\theta \sum_{i=1}^t X(i)}\right]. \quad (100)$$

By exploring the ordering of the cumulative increment process,

$$\sum_{i=1}^n -X(i) \leq_{cx} \sum_{i=1}^n -\tilde{X}(i), \quad \forall n \in \mathbb{N}, \quad (101)$$

the adjustment coefficients are ordered as follows [3, 33]

$$\tilde{\theta} \leq \theta. \quad (102)$$

Specifically, for constant arrival, the ordering of the cumulative capacity results in the ordering of the cumulative negative increment process. \square

The ordering of the adjustment coefficients gives an ordering of the asymptotic delay tail distribution, with some restrictions, the result can be applied to the delay-constrained capacity.

COROLLARY 4. For delay bounding functions with the same prefactor or are bounded by a same prefactor before the exponential term, the ordering of the cumulative capacity $S_N \leq_{cx} S_\perp \leq_{cx} S_P$ indicates the ordering of the delay, i.e.,

$$P(D_N \geq x) \leq P(D_\perp \geq x) \leq P(D_P \geq x), \quad (103)$$

and the ordering of the delay-constrained capacity for the same prefactor, i.e.,

$$\lambda_N \geq \lambda_\perp \geq \lambda_P. \quad (104)$$

Since every multi-dimensional distribution functions $\mathbf{y} \mapsto I(\mathbf{y} \leq \mathbf{x})$ and multi-dimensional survival functions $\mathbf{y} \mapsto I(\mathbf{y} > \mathbf{x})$ are both supermodular functions [45], i.e., $f(\mathbf{x}) + f(\mathbf{y}) \leq f(\mathbf{x} \wedge \mathbf{y}) + f(\mathbf{x} \vee \mathbf{y})$, the supermodular ordering of the instantaneous increment, i.e.,

$$-\mathbf{X} \leq_{sm} -\tilde{\mathbf{X}}, \quad (105)$$

indicates that the marginal distributions of the instantaneous increments are identical, which can be used for comparison between scenarios of instantaneous increment with identical marginal distributions and different dependence structures. Specifically, if $-\mathbf{X} \leq_{sm} -\tilde{\mathbf{X}}$, then $\sum_{i=1}^n -X(i) \leq_{cx} \sum_{i=1}^n -\tilde{X}(i)$.

REMARK 3. The ordering results indicates that we can use an alternative system model for tractable analysis, if the dependence structure of the new model has a monotonic relationship with the original one that is mathematically complex. For instance, the results under independence assumption can be treated as a conservative approximation for negative dependence cases. Particularly, the impact of negative and positive dependence on delay in Markov additive capacity process, and comparison with the additive capacity process, are shown in Fig. 2.

3.3 Ad hoc Scalability

In an ad hoc network, nodes may communicate in a multi-hop way and the output from the previous hop is exactly the input to the next hop [20]. An instinct feature of such systems is self-interference, i.e., the total input to the channel consists of both the output from the previous hop and the output of itself [32]. A wireless channel with self-interference is a queueing system with feedback, which is extremely difficult for end-to-end stochastic performance analysis and regarded as an open problem in stochastic network calculus [26]. In this subsection, we give a solution to this problem. Since the delay lower bound of systems without feedback holds in general, we focus on the delay upper bound here.

3.3.1 Single-hop Case

Consider a wireless channel with capacity $S(t)$, input $A(t)$, and output $A^*(t)$, where the output $A^*(t)$ is directly fed back into the wireless channel, the total input $\tilde{A}(t)$ to the channel is

$$\tilde{A}(t) = A(t) + A^*(t). \quad (106)$$

In the more general case, if the output $A^*(t)$ passes through a server with capacity process $\hat{S}(t)$ on the path of feedback, the overall input to the channel becomes

$$\tilde{A}(t) = A(t) + A^* \otimes \hat{S}(t). \quad (107)$$

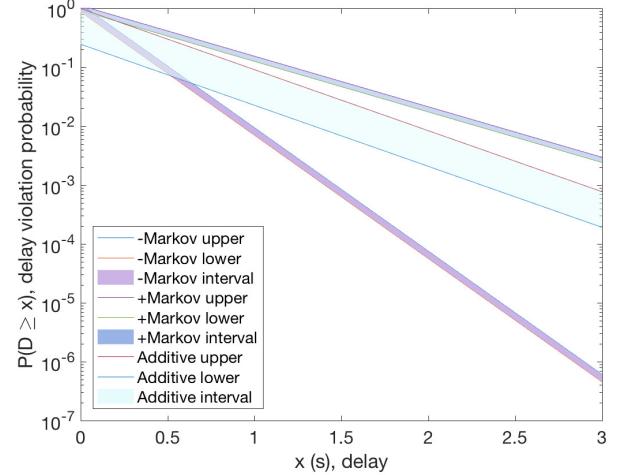


Figure 2: Delay tail distribution of Rayleigh channel. “-” and “+” depict respectively negative and positive dependence in capacity, the lines depict the double-sided bounds with the intervals depicted as the shaded areas. $\lambda = 10\text{kbits}$, $W = 20\text{kHz}$, $SNR = e^{0.5}$ for the additive capacity process, $\mathbf{SNR} = [e^{0.5} \ 0.9e^{0.5}; 0.8e^{0.5} \ 0.7e^{0.5}]$, and $\mathbf{P} = [0.4125 \ 0.5875; 0.2518 \ 0.7482]$ for $\lambda - C(t)$ indicating negative dependence in capacity and $\mathbf{P} = [0.2875 \ 0.7125; 0.3054 \ 0.6946]$ for $\lambda - C(t)$ indicating positive dependence in capacity, for the Markov additive capacity process with initial distribution $\pi = [0.5 \ 0.5]$. In case F_{ij} is independent of j , similar phenomena appears.

For such a feedback system, we can treat it as a blackbox providing service $\tilde{S}(t)$ only to the input $A(t)$, i.e.,

$$A^*(t) = A \otimes \tilde{S}(t). \quad (108)$$

The following theorem establishes a relation between $\tilde{S}(t)$, $S(t)$ and $A(t)$.

THEOREM 9. The service process $\tilde{S}(t)$ for the input $A(t)$ is lower bounded by

$$\tilde{S}(t) \geq S(t) - A(t), \quad (109)$$

correspondingly, the delay is upper bounded by

$$P(D \geq d) \leq P \left\{ \sup_{t \geq 0} (A(t) + A(t) - S(t)) \geq A(d) \right\}. \quad (110)$$

PROOF. The service for the input $A(t)$ is bounded by

$$\tilde{S}(t) \geq S(t) - A^* \otimes \hat{S}(t) \quad (111)$$

$$\geq S(t) - A^*(t) \quad (112)$$

$$\geq S(t) - A(t), \quad (113)$$

where the first inequality follows the leftover service under blind scheduling [26], the second inequality follows the monotonicity of bivariate min-plus convolution [6], i.e., $\forall t$, $f \otimes g \leq g$ if $f(t, t) = 0$ or $f \otimes g \leq f$ if $g(t, t) = 0$, and the last inequality takes advantage of system causality, i.e., $A(t) \geq A^*(t)$. By definition (63), the delay is bounded by

$$P(D \geq d) \leq P \left\{ \sup_{t \geq 0} (A(t) - (S(t) - A(t))) \geq A(d) \right\}, \quad (114)$$

where time reversibility is assumed. \square

For additive and Markov additive capacity processes with constant arrivals, explicit delay results directly follow.

COROLLARY 5 (ADDITIVE CASE). *For the constant arrival process $A(t) = \lambda t$, the delay is bounded by*

$$P(D \geq d) \leq P \left\{ \sup_{t \geq 0} (2\lambda(t) - S(t)) \geq d\lambda \right\} \quad (115)$$

$$\leq e^{-\theta d\lambda}, \quad (116)$$

where the last inequality follows Lundberg's inequality, if $\theta > 0$ satisfies the Lundberg equation $\kappa(\theta) = 0$, where $\kappa(\theta) = \log \int e^{\theta(2\lambda - C(t))} F(dx)$.

COROLLARY 6 (MARKOV ADDITIVE CASE). *For the constant arrival process $A(t) = \lambda t$, the delay conditional on initial state $J_0 = i$ is bounded by*

$$P_i(D \geq d) \leq P_i \left\{ \sup_{t \geq 0} (2\lambda(t) - S(t)) \geq d\lambda \right\} \quad (117)$$

$$\leq \frac{h^{(\theta)}(J_i)}{\min_{j \in E} h^{(\theta)}(J_j)} e^{-\theta d\lambda}, \quad (118)$$

where the last inequality follows Lundberg's inequality, if $\theta > 0$ satisfies the Lundberg equation $\kappa(-\theta) = 0$. $\kappa(\theta)$ and $\mathbf{h}^{(\theta)}$ are respectively the logarithmic Perron-Frobenius eigenvalue and the corresponding right eigenvector of the kernel for the Markov additive process $S(t) - 2\lambda t$, i.e., $\hat{\mathbf{F}}[\theta]$. Then the delay is bounded by $P(D \geq d) \leq \sum_{i \in E} \pi_i P_i(D \geq d)$.

3.3.2 Multiple-hop Case

A simple example of self-interference is neighbor interference, i.e., interference only exists in adjacent hops, the end-to-end capacity is expressed as

$$(S_1 - A_1^*) \otimes \dots \otimes (S_N - A_N^*)(t) \quad (119)$$

$$\geq (S_1 - A_1) \otimes \dots \otimes (S_N - A_1)(t) \quad (120)$$

$$= \inf_{\mathbf{u} \in \mathcal{U}(x)} \sum_{i=1}^N (S_i - A_1)(u_{i-1}, u_i), \quad (121)$$

where $\mathcal{U}(x) = \{\mathbf{u} = (u_0, u_1, \dots, u_N) : u_0 = 0, u_N = t, 0 \leq u_1 \leq \dots \leq u_{N-1} \leq t\}$, $A_1(t) \geq A_1^*(t) \geq \dots \geq A_N^*(t)$, and the inequality holds because of the monotonicity of the bivariate min-plus convolution [6], i.e., $f \otimes g(s, t) \leq \tilde{f} \otimes \tilde{g}$, $\forall f \leq \tilde{f}$ and $g \leq \tilde{g}$.

The neighbor interference is the extremal scenario where only output interference should be considered. For the generic K hop interference, where K is independent of the network size N in principle, both output and input interference should be taken into account and the most severe interference contains K output interference and $K-1$ input interference. In contrast to output interference towards previous hops, input interference is the interference to the next hops. Under the same assumption for neighbor interference and with the same approach for analysis, the service at each hop is lower bounded by

$$S_i(t) - K^* A_1(t), \quad (122)$$

where $K^* = \min(2K-1, N)$. It is worth noting that the interference of the input is absolute while the interference of the output is relative in that it exists only when the output is fed back into the wireless channel.

Based on the above insight, the delay result is summarized in the following theorem.

THEOREM 10. *Consider a concatenation of wireless channels with cumulative capacity process $S_i(t)$, $1 \leq i \leq N$. Then, for constant arrival $A_1(t) = \lambda t$, the end to end delay is expressed as*

$$P(D \geq d) \leq \sum_{t=0}^{\infty} \sum_{\mathbf{u} \in \mathcal{U}(x)} E \left[e^{-\theta \sum_{i=1}^N S_i^*(u_{i-1}, u_i)} \right] \cdot e^{\theta \lambda(t-d)}, \quad (123)$$

where $S_i^*(u_{i-1}, u_i) = (S_i - K^* A_1)(u_{i-1}, u_i)$, $K^* = \min(2K-1, N)$, and $\mathcal{U}(x) = \{\mathbf{u} = (u_0, u_1, \dots, u_N) : u_0 = 0, u_N = t, 0 \leq u_1 \leq \dots \leq u_{N-1} \leq t\}$.

PROOF. Recall that the distribution function of the cumulative capacity of a concatenation of wireless channels is bounded by

$$F_{S_t}(x) = P \{S_1 \otimes \dots \otimes S_N(t) \leq x\} \quad (124)$$

$$\leq \sum_{0 \leq u_1 \leq \dots \leq u_{N-1} \leq t} E \left[e^{-\theta \sum_{i=1}^N S_i(u_{i-1}, u_i)} \right] \cdot e^{\theta x}, \quad (125)$$

where $u_0 = 0$ and $u_N = t$, for some $\theta > 0$.

Specifically, the network capacity with interference is bounded by

$$S(t) = (S_1 - A_1^*) \otimes \dots \otimes (S_N - A_N^*)(t) \quad (126)$$

$$\geq (S_1 - K^* A_1) \otimes \dots \otimes (S_N - K^* A_1)(t), \quad (127)$$

where $K^* = \min(2K-1, N)$. Thus the end to end delay is bounded by

$$P(D \geq d) \leq \sum_{t=0}^{\infty} P \{S(t) \leq \lambda(t-d)\} \quad (128)$$

$$\leq \sum_{t=0}^{\infty} \sum_{\mathbf{u} \in \mathcal{U}(x)} E \left[e^{-\theta \sum_{i=1}^N S_i^*(u_{i-1}, u_i)} \right] \cdot e^{\theta \lambda(t-d)}, \quad (129)$$

where $S_i^*(u_{i-1}, u_i) = (S_i - K^* A_1)(u_{i-1}, u_i)$ and $\mathcal{U}(x) = \{\mathbf{u} = (u_0, u_1, \dots, u_N) : u_0 = 0, u_N = t, 0 \leq u_1 \leq \dots \leq u_{N-1} \leq t\}$, if the summation converges for some $\theta > 0$. \square

In the special case, where a common service process $S(t)$ is shared among each hop, which means that the service process at each hop is interfered by all the output processes synchronously, the end-to-end capacity is expressed as

$$\left(S - \sum_{i=1}^N A_i^* \right) \otimes \dots \otimes \left(S - \sum_{i=1, i \neq N-1}^N A_i^* - A_1 \right) (t) \quad (130)$$

$$\geq (S - NA_1) \otimes \dots \otimes (S - NA_1)(t) \quad (131)$$

$$= \inf_{\mathbf{u} \in \mathcal{U}(x)} \sum_{i=1}^N (S - NA_1)(u_{i-1}, u_i) \quad (132)$$

$$\geq (S - NA_1)(t), \quad (133)$$

where the second inequality holds under the assumption that $S(t) - NA_1(t)$ is a subadditive process [29], e.g., a stationary additive process. In addition, a corresponding upper bound is available as $(S - NA_N^*)(t)$. The insight is summarized in the following corollary.

COROLLARY 7. *Consider a concatenation of wireless channels with self-interference and all hops share a common wireless channel $S(t)$. If the transient network capacity $\bar{C}_1^N(t)$ converges, it is asymptotically expressed as*

$$\lim_{t \rightarrow \infty} \bar{C}_1^N(t) = \lim_{t \rightarrow \infty} \frac{(S - NA_1)(t)}{t}, \quad (134)$$

which results from $\lim_{t \rightarrow \infty} A_1(t)/t = \lim_{t \rightarrow \infty} A_N^*(t)/t$.

This result indicates that, if a common channel is shared among a group of nodes, a multi-hop network can be modeled as a single-hop system, and the impact of multiple hops on the time-average network capacity is equivalent to the impact of multiple identical inputs. On the other hand, for different groups of nodes far apart sufficiently for channel reuse, the impact of routing hops is localized in each group, and the group with the most severe interference is the bottleneck of the end-to-end routing. This localization property provides a diversity to the network structure, which increases the scalability of the ad hoc networks.

4. CONCLUSION

Future wireless communication calls for exploration of more efficient use of wireless channel capacity to meet the increasing demand on higher data rate and less latency. This motivates the analysis to maximally take into account the special characteristics of the wireless channel capacity process, which include the tail behavior, stochastic dependence, and self-interference in wireless communication. To this aim, a set of new results directly exploring these characteristics have been presented in this paper. Among them, an appealing finding is that, for typical fading channels, their instantaneous capacity and cumulative capacity are both light-tailed. It immediately implicates that the cumulative capacity and subsequently the delay and backlog performance can be upper-bounded by some exponential distributions, and provides evident justification for the exponential distribution assumption used in the literature. Specifically, various bounds have been derived for distributions of the cumulative capacity and the delay-constrained capacity, considering three representative dependence structures in the capacity process, namely comonotonicity, independence, and Markovian. To help gain insights in the performance of a general wireless channel, stochastic orders are introduced to the cumulative capacity process, based on which, results to compare the delay and delay-constrained capacity performance have been obtained. Moreover, the open SNC problem of performance analysis of self-interference in wireless communication is tackled through a novel approach that models the wireless channel as a feedback system, taking advantage of system causality, original results have been derived. In all, the set of results obtained in this paper provide fundamental contributions to linking the SNC theory to wireless networks and hence contribute significantly to its extension towards a calculus for wireless networks.

APPENDIX

Markov Additive Process

A Markov additive process is defined as a bivariate Markov process $\{X_t\} = \{(J_t, S(t))\}$ where $\{J_t\}$ is a Markov process with state space E and the increments of $\{S(t)\}$ are governed by $\{J_t\}$ in the sense that [3]

$$E[f(S(t+s) - S(t))g(J_{t+s})|\mathcal{F}_t] = E_{J_t,0}[f(S(s))g(J_s)]. \quad (135)$$

For finite state space and discrete time, a Markov additive process is specified by the measure-valued matrix (kernel) $\mathbf{F}(dx)$ whose ij th element is the defective probability distribution

$$F_{ij}(dx) = P_{i,0}(J_1 = j, Y_1 \in dx), \quad (136)$$

where $Y_t = S(t) - S(t-1)$. An alternative description is in terms of the transition matrix $\mathbf{P} = (p_{ij})_{i,j \in E}$ (here $p_{ij} = P_i(J_1 = j)$) and the probability measures

$$H_{ij}(dx) = P(Y_1 \in dx | J_0 = i, J_1 = j) = \frac{F_{ij}(dx)}{p_{ij}}. \quad (137)$$

Consider the matrix $\widehat{\mathbf{F}}_t[\theta] = (E_i[e^{\theta S(t)}; J_t = j])_{i,j \in E}$, it is proved that [2]

$$\widehat{\mathbf{F}}_t[\theta] = \widehat{\mathbf{F}}[\theta]^t, \quad (138)$$

where $\widehat{\mathbf{F}}[\theta] = \widehat{\mathbf{F}}_1[\theta]$ is a $E \times E$ matrix with ij th element $\widehat{F}^{(ij)}[\theta] = p_{ij} \int e^{\theta x} F^{(ij)}(dx)$, and $\theta \in \Theta = \{\theta \in \mathbb{R} : \int e^{\theta x} F^{(ij)}(dx) < \infty\}$. By Perron-Frobenius theory, $e^{\kappa(\theta)}$ and $\mathbf{h}^{(\theta)} = (h_i^{(\theta)})_{i \in E}$ are respectively the positive real eigenvalue with maximal absolute value and the corresponding right eigenvector of $\widehat{\mathbf{F}}[\theta]$, i.e., $\widehat{\mathbf{F}}[\theta]\mathbf{h}^{(\theta)} = e^{\kappa(\theta)}\mathbf{h}^{(\theta)}$. In addition, for the left eigenvector $\mathbf{v}^{(\theta)}$, $\mathbf{v}^{(\theta)}\mathbf{h}^{(\theta)} = 1$ and $\boldsymbol{\varpi}\mathbf{h}^{(\theta)} = 1$, where $\boldsymbol{\varpi} = \mathbf{v}^{(0)}$ is the stationary distribution and $\mathbf{h}^{(0)} = \mathbf{e}$.

5. REFERENCES

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