

On Heat Transfer In Bödewadt Flow

Muhammad Rahman^{a,b}, Helge I. Andersson^a

^aDepartment of Energy and Process Engineering,
Norwegian University of Science and Technology, Trondheim, Norway

^bDepartment of Mathematics, Mirpur University of Science and Technology, Mirpur, Pakistan

Abstract

Heat transfer in revolving Bödewadt flow above a planar surface has been considered. We have shown that a similarity solution of the thermal energy problem does not exist as long as the surface is impermeable. The failure of the existence of physically realistic similarity solutions for the thermal field is ascribed to the fact that the axial flow component is directed away from the surface. If the planar surface is porous and allows for suction, the direction of the axial flow can be reversed. Similarity solutions have been obtained for some different values of the dimensionless suction velocity A and the Prandtl number Pr . The thermal boundary layer became gradually thinner with increasing suction A and for higher Pr , thereby also increasing the heat transfer rate through the planar surface.

Keywords: Revolving flow; Bödewadt flow; Heat transfer; Similarity solutions.

1. Introduction

The steadily revolving flow of a viscous fluid above a solid surface was first studied by Bödewadt [1] who transformed the governing partial differential equations into a set of ordinary differential equations by means of the same similarity transformation as originally used by Von Karman [2] in his classical study of the swirling flow driven by a constantly rotating disk. The Bödewadt flow can be considered as a reversed Von Karman flow with the axial velocity component directed away from the planar surface rather than towards the rotating disk. However, the three velocity components in the Bödewadt flow exhibit a more complex variation than in the Von Karman flow and the Bödewadt boundary layer is substantially thicker than the corresponding Von Karman boundary layer.

In this paper we consider the heat transfer in steadily revolving Bödewadt flow. The heat transfer in this prototype flow seems to have received only negligible attention in comparison with the heat transfer in the Von Karman flow. Heat transfer in flow above a rotating disk was first studied by Millsaps & Pohlhausen [3] and Sparrow & Gregg [4], followed by many others. See, e.g., the book by Shevchuk [5]. Shevchuk and Buschmann [6], for instance, found self-similar solutions for the flow and heat transfer in a fluid co-rotating with a rotating disk with a radially varying disk temperature. One may speculate whether the lack of studies of heat transfer in Bödewadt flow is due the relatively higher complexity of the three-dimensional flow field.

The only earlier studies that we are aware of are the recent papers by Sahoo [7], Sahoo et al. [8], and Turkyilmazoglu [9]. Sahoo [7] included heat transfer analysis in his study of Bödewadt

flow of an electrically conducting fluid with partial slip. Some temperature profiles were presented, but not for pure Bödewadt flow of a Newtonian fluid with no-slip at the solid surface. Sahoo et al. [8] also focussed on non-Newtonian fluid properties and the majority of their results were concerned with the flow field, but two figures showing heat transfer results also for Newtonian fluids were included. As we will see later, these results might be questionable. Even more recently, Turkyilmazoglu [9] studied the heat transfer in Bödewadt flow over a stretching but non-rotating disk. In the absence of stretching, however, his results suggested a constant temperature all across the viscous boundary layer and therefore failed to satisfy the outer boundary condition for the thermal field. In this paper heat transfer in Bödewadt flow will be revisited with the view to clarify the contradictory findings of Sahoo et al. [8] and Turkyilmazoglu [9].

2. Mathematical Model Equations

Let us consider the steadily revolving flow of a viscous fluid above a planar surface. In cylindrical polar coordinates (r, θ, z) the governing mass conservation, momentum and thermal energy equations become:

$$\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw) = 0, \quad (1)$$

$$u \frac{\partial u}{\partial r} - \frac{v^2}{r} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left[\frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r} \left(\frac{u}{r} \right) + \frac{\partial^2 u}{\partial z^2} \right], \quad (2)$$

$$u \frac{\partial v}{\partial r} + \frac{uv}{r} + w \frac{\partial v}{\partial z} = \nu \left[\frac{\partial^2 v}{\partial r^2} + \frac{\partial}{\partial r} \left(\frac{v}{r} \right) + \frac{\partial^2 v}{\partial z^2} \right], \quad (3)$$

$$u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} \right], \quad (4)$$

*Corresponding author

Email address: muhammad.rahman@ntnu.no (Muhammad Rahman)

$$\rho C_p \left(u \frac{\partial T}{\partial r} + w \frac{\partial T}{\partial z} \right) = k \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \left(\frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial z^2} \right), \quad (5)$$

where (u, v, w) are the velocity components of the fluid in the radial, circumferential and axial directions, respectively, and T is the temperature. Here, we have assumed rotational symmetry about the vertical z -axis, *i.e.* $\partial/\partial\theta = 0$. The kinematic viscosity of the fluid is ν and C_p is the specific heat at constant pressure of the fluid. k is the thermal conductivity of the fluid. The boundary conditions are

$$\left. \begin{aligned} u = 0, \quad v = 0, \quad w = -\sqrt{\nu\Omega}A, \quad T = T_w, \quad \text{at } z = 0, \\ u = 0, \quad v = r\Omega, \quad p = \frac{1}{2}r^2\Omega^2, \quad T = T_\infty, \quad \text{as } z \rightarrow \infty, \end{aligned} \right\} \quad (6)$$

Here, Ω and T_∞ are the angular velocity and temperature of the revolving fluid high above the surface, while A and T_w are the dimensionless suction velocity and temperature at the solid surface at $z = 0$. In the present study the temperature T_w of the stationary disk is constant. Shevchuk and Buschmann [6], however, allowed for a radial power-law variation of the surface temperature of a rotating disk.

3. Similarities Transformations

Following Bödewadt we express the fluid velocity components and pressure as

$$\left. \begin{aligned} u(r, z) &= r\Omega F(\eta), \\ v(r, z) &= r\Omega G(\eta), \\ w(r, z) &= \sqrt{\nu\Omega}H(\eta), \\ p(r, z) &= \rho(-\nu\Omega P(\eta) + \frac{1}{2}r^2\Omega^2), \\ T(r, z) &= T_\infty + (T_w - T_\infty)\Theta(\eta), \end{aligned} \right\} \quad (7)$$

where η is a dimensionless variable defined by

$$\eta = z\sqrt{\Omega/\nu}. \quad (8)$$

In terms of the non-dimensional variables defined by (7)-(8) the governing equations (1)-(5) become:

$$2F + H' = 0, \quad (9)$$

$$F'' - HF' - F^2 + G^2 = 1, \quad (10)$$

$$G'' - HG' - 2FG = 0, \quad (11)$$

$$P' + 2FH - 2F' = 0, \quad (12)$$

$$\Theta'' - PrH\Theta' = 0, \quad (13)$$

where Pr is the Prandtl number, $Pr = \frac{C_p\mu_k}{k}$. The corresponding boundary conditions specified in (6) transfer to :

$$\left. \begin{aligned} F(\eta) = 0, \quad G(\eta) = 0, \quad H(\eta) = -A, \quad \Theta(\eta) = 1 \quad \text{at } \eta = 0, \\ F(\eta) = 0, \quad G(\eta) = 1, \quad P(\eta) = 0, \quad \Theta(\eta) = 0 \quad \text{as } \eta \rightarrow \infty. \end{aligned} \right\} \quad (14)$$

4. Exact Analytical Solution

The ODE for the thermal field (13) can be integrated twice to give the solution for the temperature profile:

$$\Theta(\eta) = 1 - \frac{I(\eta)}{I(\infty)}, \quad (15)$$

where

$$I(\eta) = \int_0^\eta \exp \left[Pr \int_0^\eta H ds \right] d\eta, \quad (16)$$

and $H(\eta)$ is the axial velocity component obtained from the solution of the accompanying flow problem.

Now, $I'(\eta) = \exp \left[Pr \int_0^\eta H ds \right]$ and therefore $I'(0) = 1$ so that the temperature gradient at the surface becomes:

$$\Theta'(0) = -\frac{I'(0)}{I(\infty)} = -\frac{1}{I(\infty)}. \quad (17)$$

If we for simplicity assume that the axial velocity component is constant, *i.e.* $H(\eta) = H_0$, the integration can be performed analytically as:

$$\begin{aligned} I(\eta) &= \int_0^\eta \exp \left[Pr \int_0^\eta H_0 ds \right] d\eta = \int_0^\eta \exp [PrH_0\eta] d\eta \\ &= \frac{1}{PrH_0} \exp [PrH_0\eta]_0^\eta, \end{aligned} \quad (18)$$

and finally, $I(\eta) = \frac{1}{PrH_0} [\exp (PrH_0\eta) - 1]$ which gives $I(\infty) = -\frac{1}{PrH_0}$ provided that $H_0 < 0$. This gives the temperature gradient $\Theta'(0) = -\frac{1}{I(\infty)} = PrH_0 < 0$. The assumption of a constant axial velocity H_0 is made here to be able to demonstrate that the sign of temperature gradient at the surface is determined by the sign of H_0 . In reality, however, the axial velocity H varies with η . Nevertheless, it was shown by Turkyilmazoglu [10] that the axial velocity component of the von Karman flow becomes constant in presence of strong suction. The same tendency is likely to appear also in the Bödewadt flow.

5. Numerical Approach

We solved the two-point boundary value problem consisting of the coupled set of ordinary differential equations (9)-(13) subjected to the boundary conditions (14). For this purpose we have used the bvp4c MATLAB solver, which gives very good results for the non-linear ODEs with multipoint BVPs. This finite-difference code utilizes the 3-stage Lobatto IIIa formula, that is a collocation formula and the collocation polynomial provides a C1-continuous solution that is fourth-order accurate uniformly in [a,b]. For multipoint BVPs, the solution is C1-continuous within each region, but continuity is not automatically imposed at the interfaces. Mesh selection and error control are based on the residual of the continuous solution. Analytical condensation is used when the system of algebraic equations is formed ; see e.g. Shampine et al.[11]. Numerical solutions of the three-dimensional flow problem were provided by Nath and Venkatachala [12] for three different values of the

suction parameter A . The comparisons in Table 1 show that the results of the present computations compare very well with their data. Moreover, the entries for $A = 0$ match exactly with the corresponding data tabulated by Turkyilmazoglu [9].

Table 1: Effect of suction parameter A on the shear stress characteristics $F'(0)$ and $G'(0)$. Comparisons with results computed by Nath and Venkatachala [12] in their Table 1.

A	$-F'(0)$		$G'(0)$	
	present	N & V	present	N & V
0	0.9420	0.9420	0.7729	0.7729
0.5	0.9033	NA	1.0652	NA
1.0	0.8351	0.8350	1.3885	1.3863
2.0	0.6468	0.6462	2.1560	2.1550
3.0	0.4800	NA	3.0611	NA

6. Numerical Results

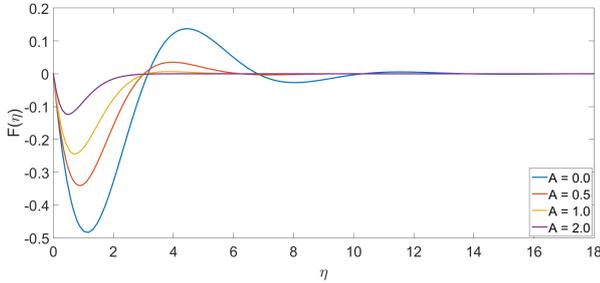


Figure 1: Radial velocity component $F(\eta)$ for some different values of the suction parameter A .

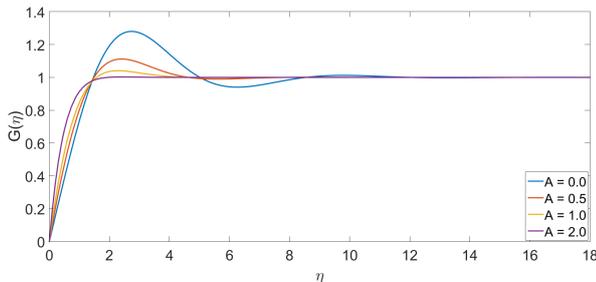


Figure 2: Circumferential velocity component $G(\eta)$ for some different values of the suction parameter A .

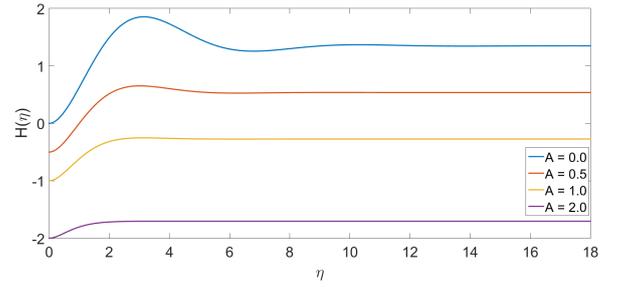


Figure 3: Axial velocity component $H(\eta)$ for some different values of the suction parameter A .

6.1. Velocity Field

Computed results for the three velocity components are shown in Figures 1- 3. In absence of suction, the fluid motion well above the surface is characterized by a uniform angular velocity G , which is reduced through a viscous boundary layer in order for the fluid to adhere to the no-slip condition $G = 0$ at the solid surface. The reduction of the circumferential velocity component in the vicinity of the surface reduces the radial directed centripetal acceleration (or centrifugal force) such that the prevailing radial pressure gradient induces an inward fluid motion F . In order to assure mass conservation, this inward fluid motion gives in turn rise to an axial outward flow $H > 0$. Such a spiralling flow exists near the planar surface, although more complex variations of the velocity field are seen further away, but yet before the uniformly rotating flow conditions are reached for $\eta > 12$. This oscillatory nature of the three velocity components was reported already by Bödewadt [1] and makes the Bödewadt flow qualitatively different from the Von Karman flow. However, it is interesting to notice that these oscillations are damped and even suppressed in presence of a magnetic field (King and Lewellen [13]), partial slip (Sahoo et al.[14]) or if the disk is stretched (Turkyilmazoglu [9]). Of particular relevance for the present study is the effect of surface suction. Nath and Venkatachala [12] showed that sufficient suction through the planar surface suppressed the oscillatory nature of the three-dimensional flow field and, moreover, made the axial flow be directed in the inward direction rather than outward, as is the case in the classical Bödewadt flow.

Nath and Venkatachala [12] showed results for two different values of the suction parameter A (1.0 and 2.0) and compared these results with the pure Bödewadt case $A = 0$. Here, we have revisited the same cases and also included results with modest suction ($A = 0.5$) and stronger suction ($A = 3$). The trends observed by Nath and Venkatachala [12] are reproduced here. The radial inflow F in Figure 1 is monotonically reduced with increasing suction and this is accompanied with a reduction of the axial flow H in Figure 3. The direction of the axial flow is inverted when the suction rate $A \geq 1$, such that the fluid instead moves towards the solid surface. With modest suction, however, the fluid in the vicinity of the surface moves away from the surface, whereas the fluid further away is directed towards it such that H changes sign at about $\eta = 1$ for $A = 0.5$. The radial and tangential shear stresses are proportional with the slopes of

the respective velocity profiles $F'(0)$ and $G'(0)$ given in Table 1 and compared with corresponding results reported by Nath and Venkatachala [12]. Here, we can see that while the tangential shear stress $G'(0)$ increases with A , the magnitude of the radial shear stress $-F'(0)$ decreases with increasing suction.

6.2. Heat Transfer

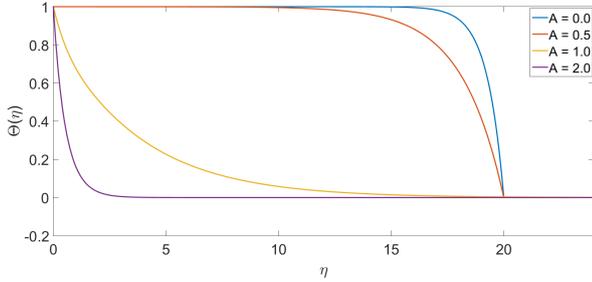


Figure 4: Variation of temperature $\Theta(\eta)$ for some different values of suction parameter A and $Pr = 1$.

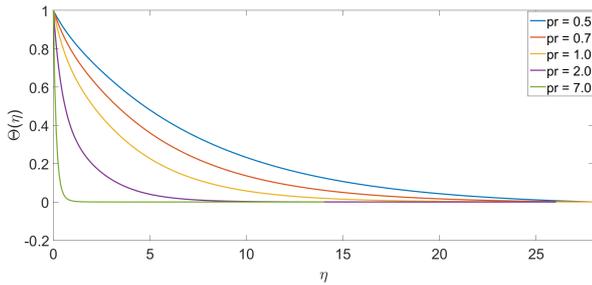


Figure 5: Variation of temperature $\Theta(\eta)$ for some different values of Pr and $A = 1.0$.

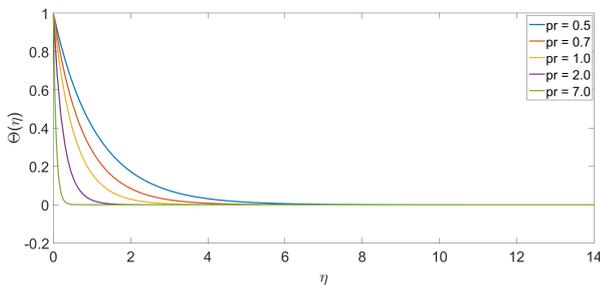


Figure 6: Variation of temperature $\Theta(\eta)$ for some different values of Pr and $A = 2.0$.

While the three-dimensional flow field is a one-parameter problem determined by the suction parameter A , the accompanying thermal problem is a two-parameter problem in A and Pr . Computed temperature profiles for various values of A and Prandtl number $Pr = 1$ are shown in Figure 4. These results

show a remarkable effect of the suction parameter depending on whether $A \geq 1$ or not. The temperature profiles for $A = 0$ and $A = 0.5$ show that the temperature is constant from the surface and far beyond $\eta = 12$ before the temperature drops to zero in order to satisfy the outer boundary condition. These profiles resemble $\Theta(\eta) = 1$ seen in Figure 3 in Turkyilmazoglu [9] for zero stretching, i.e. pure Bödewadt flow. Although all the results in Figure 4 are solutions of the ODE which satisfy the two thermal boundary conditions, only results for $A \geq 1$ are physically plausible in the sense that the heat flux $-\Theta'$ far away from the surface asymptotes to zero. The auxiliary condition that $\Theta' \rightarrow 0$ as $\eta \rightarrow \infty$ is commonly overlooked in analysis of thermal boundary layers, similarly as the auxiliary conditions $F' \rightarrow 0$ and $G' \rightarrow 0$ as $\eta \rightarrow \infty$ for the momentum boundary layer; see Andersson [15] and Pantokratoras [16]. Realistic solutions are therefore obtained only in presence of sufficient suction.

From Figure 3 we recall that the axial velocity component H changes sign from positive to negative as the suction parameter A is gradually increased. Since the only effect of the fluid flow on the thermal energy problem is through the axial velocity component $H(\eta)$, the presence of suction is likely to have a major impact on the temperature distribution. Indeed, we showed in Section 4 that if $H = H_0$ the temperature gradient $\Theta'(0) = PrH_0$ becomes negative only provided that $H_0 < 0$. This explains why we only obtain plausible solutions when sufficiently strong suction is applied, i.e. when the axial velocity turned negative everywhere. It is therefore remarkable that Shaoo et al. [8] showed temperature profiles for $Pr = 1$ in their Figure 9 which resemble those for $A \geq 1$ in our Figure 4. We believe that these results were obtained due to an unnoticed sign error in their thermal energy equation.

The slope Θ' of the four temperature profiles in Figure 4 is negative. However, the magnitude of the slope of the physically plausible temperature variations decreases with η , whereas the unphysical profiles exhibit a gradually increasing $|\Theta'|$. A critical value A_{crit} of the dimensionless suction parameter can therefore be defined as the A -value that leads to an inflection point in the temperature profile, i.e. $\Theta'' = 0$. The second derivative of the temperature distribution can readily be obtained from the exact analytical solution given by equations (15)-(16) as

$$\Theta''(\eta) = \Theta'(0) \cdot Pr \cdot H(\eta) \cdot \exp \left[Pr \int_0^\eta H d\eta \right]. \quad (19)$$

The critical A -value accordingly corresponds to the amount of suction $H(0) = -A$ which is just sufficient to make the axial velocity component H negative for all values of η . By means of systematic integrations of the three-dimensional flow problem we found that $A_{crit} = 0.85$.

The realistic solutions in Figure 4 show that the adaption of the dimensionless temperature from $\Theta = 1$ at the surface to $\Theta = 0$ far above the surface occurs over a gradually shorter distance with increasing A . Stronger suction accordingly tends to make the thermal boundary layer thinner for a given value of the Prandtl number. Figures 5 and 6 show temperature profiles for a range of Prandtl numbers for prescribed suction $A = 1$ and

$A = 2$, respectively. For a given flow field, the thermal boundary layer becomes gradually thinner as Pr increases from 0.5 to 7.0. This trend is consistent with the general knowledge that the importance of thermal conduction, relative to viscous diffusion, diminishes with increasing Pr . The excess surface temperature is therefore felt only in the near vicinity of the surface for $Pr \gg 1$. In-depth discussions on high-Prandtl-number effects on the thermal boundary layer thickness and the surface heat transfer can be found in Shevchuk [5, 17].

The Nusselt number Nu is a convenient non-dimensional measure of the local heat transfer rate at the surface $\eta = 0$. The Nusselt number is usually defined as $Nu = \frac{q(0)L}{k(T_w - T_\infty)} = \frac{-\partial T / \partial z|_{z=0}L}{T_w - T_\infty}$. Shevchuk & Buschmann [6] and Shevchuk [5, 17] took $L = r$, which gives $Nu = -\Theta'(0)\sqrt{\Omega r^2/\nu}$. Here, however, the length scale L is taken as $\sqrt{\nu/\Omega}$ to give $Nu = -\Theta'(0)$. Nu is proportional with $-\Theta'(0)$ tabulated in Table 2 for all values of A and Pr considered herein. Here, we observe that the heat transfer increases with increasing suction and for higher Prandtl numbers.

Table 2: Heat transfer rate $-\Theta'(0)$ at the surface.

$Pr \setminus A$	0	0.5	1.0	2.0	3.0
0.5			0.1934	0.9239	1.4736
0.7			0.2984	1.3154	2.0713
1.0			0.4943	1.9114	2.9705
2.0			1.4013	3.9215	5.9743
7.0			6.7857	13.9633	20.9879

7. Concluding Remarks

We have shown that realistic similarity solutions of the thermal energy problem do not exist for Bödewadt flow in absence of suction ($A = 0$). The only earlier results for pure Bödewadt flow by Sahoo et al. [8] and Turkyilmazoglu [9] are therefore not physically realistic. This explains why heat transfer in Bödewadt flow seems to have escaped the attention of the research community. If sufficient suction is applied, however, plausible solutions do exist. This phenomenon is explained by the reversal of the axial velocity component when suction is imposed. This finding is consistent with the fact that similarity solutions of the heat transfer problem associated with Von Karman flow do exist. The thermal energy equation in that case is exactly the same as in the Bödewadt flow, but the sign of the axial velocity component H is opposite. One can therefore conjecture that solutions of the thermal problem of the Von Karman flow ceases to exist if sufficient blowing through the rotating disk is applied, so that the axial flow direction is reversed.

It should be noted that our conclusions are valid only for the actual heat and fluid flow problem studied herein. The presence of either a magnetic field [12], partial slip [7] or stretching of the surface [9] will alter the flow field and thereby also the possible existence of similarity solutions of the thermal field. Moreover, if, for instance, viscous dissipation or ohmic heating is included in the thermal energy equation, see e.g. Sahoo [7],

the situation is different and similarity solutions may exist even in absence of suction.

Similarity solutions of the heat transfer problem associated with Bödewadt flow subjected to significant suction ($A \geq 1$) have been provided here for the first time. Increasing suction tends to make the thermal boundary layer gradually thinner and thereby increases the heat transfer rate through the solid surface. Likewise, as the relative importance of thermal diffusivity reduces for higher Prandtl numbers, the thermal boundary layer becomes gradually thinner.

8. References

- [1] Bödewadt, U. T. "Die Drehströmung Über festem grunde." Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik 20. (1940): 241-253.
- [2] Kármán, Th V. "Über laminare und turbulente Reibung." Journal of Applied Mathematics and Mechanics/Zeitschrift für Angewandte Mathematik und Mechanik 1 (1921): 233-252.
- [3] Millsaps, K., and K. Pohlhausen. "Heat transfer by laminar flow from a rotating plate." Journal of the Aeronautical Sciences 19 (1952): 120-126.
- [4] E. M. Sparrow and J. L. Gregg. "Heat transfer from a rotating disk to fluids of any Prandtl number." Journal of Heat Transfer, 81 (1959): 249-251.
- [5] Shevchuk, I. V. "Convective heat and mass transfer in rotating disk systems." Springer Science & Business Media, Vol. 45 (2009).
- [6] Shevchuk, I.V., and M.H.Buschmann. "Rotating disk heat transfer in a fluid swirling as a forced vortex." Heat and Mass Transfer 41 (2005): 1112-1121.
- [7] Sahoo, B. "Effects of slip on steady Bödewadt flow and heat transfer of an electrically conducting non-Newtonian fluid." Communications in Nonlinear Science and Numerical Simulation 16 (2011): 4284-4295.
- [8] Sahoo, B, R. A. Van Gorder, and H. I. Andersson. "Steady revolving flow and heat transfer of a non-Newtonian Reiner-Rivlin fluid." International Communications in Heat and Mass Transfer 39 (2012): 336-342.
- [9] Turkyilmazoglu, M. "Bödewadt flow and heat transfer over a stretching stationary disk." International Journal of Mechanical Sciences 90 (2015): 246-250.
- [10] Turkyilmazoglu, Mustafa. "An asymptotic investigation into the stabilization effect of suction in the rotating-disk boundary layer flow." Computers & Mathematics with Applications 53 (2007): 750-759.
- [11] Shampine, L.F., J.Kierzenka, and M.W. Reichelt. "Solving boundary value problems for ordinary differential equations in MATLAB with bvp4c." Tutorial notes (2000): 437-448.
- [12] Nath, G., and B. J. Venkatchala. "The effect of suction on boundary layer for rotating flows with or without magnetic field." Proceedings of the Indian Academy of Sciences-Section A. Vol. 85. No. 5. Springer India, (1977): 332-337.
- [13] King, W. S., and W.S.Lewellen. "Boundary-layer similarity solutions for rotating flows with and without magnetic interaction." Physics of Fluids 7 (1964): 1674-1680.
- [14] Sahoo, B., S., Abbasbandy, S. Poncet. "A brief note on the computation of the Bödewadt flow with Navier slip boundary conditions." Computers Fluids 90 (2014): 133-137.
- [15] Andersson, H.I. "Comments on BVPs in non-Newtonian boundary-layer flow." Applied Mathematics and Computation 182 (2006): 1714-1716.
- [16] Pantokratoras, A. "A common error made in investigation of boundary layer flows." Applied Mathematical Modelling 33 (2009): 413-422.
- [17] Shevchuk, I.V. "Turbulent heat and mass transfer over a rotating disk for the Prandtl or Schmidt numbers much larger than unity: an integral method." Heat and Mass Transfer 45 (2009): 1313-1321.