A Comparative Study of the Stochastic Averaging Method and the Path Integration Method for Nonlinear Ship Roll Motion in Random Beam Seas

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Abstract: In this paper, the energy-based stochastic averaging method and the path integration (PI) method are applied to study the stochastic response of the nonlinear roll motion in random beam seas. Specifically, the Markov diffusion theory is applied to describe the random roll motion such that the probabilistic properties of the ship roll motion are governed by the Fokker-Planck (FP) equation. The stochastic averaging method focuses on the roll energy envelope process and reduces the difficulty in calculating the stochastic response via a dimensional reduction of the original system. In contrast, the path integration method is based on the Markov property of the original dynamical system and provides approximate solutions to the FP equation by linking the explicitly known local solutions. Although the stochastic averaging technique is well established in the study of the stochastic responses of various dynamical systems, its accuracy for determining the high-level roll response has not yet been a focus of study. This paper aims to provide a comparative study for the performance of the two abovementioned methods, additionally, the advantages and shortcomings of using each method for studying the nonlinear roll motion in beam seas are demonstrated by practical calculations.

Keywords: Stochastic averaging method; Path integration method; Nonlinear roll motion; Stochastic response.

1. Introduction

The danger of severe roll motions to vessels sailing in the open seas has been known to seafarers for millennia. Although scientific descriptions of this classical problem have been dated to Euler's time in the 18th century [1], this problem has received attention in the past decades from the fields of ship stability [2, 3], nonlinear dynamics [4, 5], stochastic dynamics [6], etc. For ships rolling in random seas, challenges have been reported in determining the stochastic responses of such nonlinear systems [7].

A methodology based on a Markov model for the nonlinear roll motion in random beam seas has popular for analyzing the corresponding stochastic response [6, 8-10], mainly because the probabilistic properties and the stochastic evolutions of Markov dynamical systems are governed by the Fokker-Planck (FP) equation. In this work, the roll motion in random beam seas is decoupled from the other motion modes and governed by a single-degree-of-freedom (SDOF) model, where the associated wave excitation moment is assumed to be a non-white Gaussian process described by a specific spectrum. Subsequently, the stationary random wave excitation process is approximated as filtered (colored) white noise by introducing a second-order linear filter [11]. Therefore, the original SDOF model, also a second order differential equation, is extended into a four-dimensional (4D) Markov dynamical system.

Generally, for such nonlinear systems, numerical methods based on a direct solution of the FP equation are hardly feasible. Therefore, several alternative techniques have been developed to provide approximation solutions of the FP equation, such as the statistical linearization method [9], the cumulant-neglect closure method [12, 13], the stochastic averaging method [14-16] and the path integration (PI) method [17, 18]. Since our focus is on the nonlinear stochastic response of the roll motion, the former two methods are not considered as they cannot provide accurate information of the high-level responses. The principles of the stochastic averaging method and the PI method are described as follows.

The stochastic averaging method has been extensively used in the field of random vibration and serves as a convenient tool to obtain the (approximate) stationary or non-stationary responses of stochastic dynamical systems. This method simplifies the problem by introducing a dimensional reduction of the original system, but the essential behavior of the system is retained. Then, a numerical solution or even analytical solution of the low-dimensional FP equation can be obtained. Roberts was the pioneer and introduced the stochastic averaging technique to study the stochastic roll response [19, 20]. Instead of basing the stochastic averaging method on generalized harmonic functions with the assumption that the amplitude and phase of the roll response are slowly varying functions with respect to time [21], in his studies, the roll energy envelope process was considered in an energy-based stochastic averaging method. The drift and diffusion coefficients of the subsequent one-dimensional FP equation were approximated by a Fourier series, then, the governing equation was solved numerically. In recent years, Dostal et al. [14, 22] applied the energy-based stochastic averaging method to ship roll dynamics in order to find the closed-form solutions as well as the numerical solutions of the averaged roll energy process. By applying this procedure, they have obtained analytical results for the roll motion behavior in random seas.

The path integration (PI) method is an efficient approximation for solving the FP equation of the original system and providing the stationary or non-stationary response statistics of the dynamical system. This method is based on the Markov property of the original system and the global solution, i.e., the evolution of the response statistics is calculated by linking the explicitly known local solutions by a step-by-step solution technique [18] according to the basic Chapman-Kolmogorov equation. The PI method has been applied to a variety of problems in the discipline of engineering [17, 18, 23-25]. Recently, a 4D PI procedure has been formulated in order to study the stochastic roll response of a vessel in random beam seas [7]. The rationality for using the 4D extended system to describe the roll motion in random beam seas and the efficiency and accuracy of the 4D PI procedure were reported in Ref. [11].

In this work, both the energy-based stochastic averaging method and the PI method are applied in order to calculate the stochastic roll response, in particular, the high-level response in the tail region, since this response is important for the safety of the vessel. Although the energy-based stochastic averaging method has been applied to the random roll motion in beam seas [6, 10] and also for high-dimensional dynamical systems [14], its accuracy and performance for the high-level response of the roll motion in random beam seas have not yet been studied. This work provides a comparative study for the performance of the two abovementioned methods that are based on different principles. Furthermore, a straightforward Monte Carlo simulation (MCS) is introduced as a verification tool for the comparative study.

2. Mathematical Model of Roll Motion

Under the dead ship condition, i.e., a ship with zero speed (or low speed) under unidirectional beam seas, the roll motion can be represented by the following SDOF equation [6, 26]:

$$\ddot{\theta}(t) + b_{44}\dot{\theta}(t) + b_{44q}\dot{\theta}(t)\left|\dot{\theta}(t)\right| + c_1\theta(t) - c_3\theta^3(t) = m(t) \quad (1)$$

where $\theta(t)$ and $\dot{\theta}(t)$ are the roll angle and roll velocity, respectively. The parameters b_{44} and b_{44q} are the linear and quadratic damping coefficients, c_1 and c_3 are the linear and cubic roll restoring coefficients, and $m(t)$ is the relative roll excitation moment.

The roll motion has a softening stiffness characteristic since the cubic restoring term in Eq. (1) is negative. For the softening cases, the response trajectories eventually deviate from the region of stability in the phase plane, i.e., ship capsizing would occur when the roll angle exceeds a certain limit (such as the angle of vanishing stability) and the roll response is non-stationary by nature [21]. Nevertheless, the roll response can be treated as a stationary diffusion process when the mean time to capsize is very long and the rationality of this approximation has been noted in Ref. [6]. On the other hand, many ships exhibit hardening behavior up to critical angles (e.g., 30-40 degrees and more), and this would produce different features in the roll motion, such as bifurcations and jumps, especially for the narrow-band excitation. Moreover, it should be noted that the SDOF model in Eq. (1) and the associated linear-plus-cubic restoring term are valid for qualitative studies and for understanding the nonlinear behavior of the roll motion under random excitation.

The random wave excitation moment $m(t)$ is a stationary Gaussian process and described by the spectrum $S_{mm}(\omega)$, which is expressed as:

$$S_{mm}(\omega) = |F_{roll}(\omega)|^2 S_{\xi\xi}(\omega) \big/ (I_{44} + A_{44})^2 \tag{2}$$

in which I_{44} denotes the moment of inertia in the roll and A_{44} is the added mass moment term. $|F_{roll}(\omega)|$ represents the roll moment amplitude per unit wave height at frequency ω. $S_{\xi\xi}(\omega)$ is the wave spectrum and is given as a modified Pierson-Moskowitz spectrum with a significant wave height H_s and peak period T_p [27].

Subsequently, the linear filter technique is introduced in order to approximate the driving process $m(t)$ in Eq. (1) as a filtered white noise, then, the original SDOF model will be extended into a Markov dynamical system. In this work, the target spectrum, $S_{mm}(\omega)$ is approximated by a second-order linear filter, which is presented as follows:

$$\begin{cases} dx_3 = (x_4 - \beta x_3)dt + \gamma dW \\ dx_4 = -\alpha x_3 dt \end{cases} \tag{3}$$

where x_3 and x_4 are the state variables in the filter equation with x_3 representing the output $m(t)$. $dW(t)=W(t+dt)-W(t)$ represents an infinitesimal increment of a standard Wiener process with $E\{dW(t)\}=0$ and $E\{dW(t)dW(s)\}=0$ for $t \neq s$ and $E\{dW(t)^2\}=dt$. The spectrum generated by the second-order linear filter (3) is denoted as $S_{Filter}(\omega)$ and given as flowws:

$$S_{Filter}(\omega)=\frac{1}{2\pi}\frac{\gamma^2\omega^2}{(\alpha-\omega^2)^2+(\beta\omega)^2} \tag{4}$$

where α, β and γ are the parameters of the linear filter.

By combining Eq. (1) with Eq. (3), the nonlinear roll motion in random beam seas is described by the following 4D state-space equation:

$$\begin{cases} dx_1 = x_2 dt \\ dx_2 = (-b_{44}x_2 - b_{44q}x_2 |x_2| - c_1 x_1 + c_3 x_1^3 + x_3)dt \\ dx_3 = (x_4 - \beta x_3)dt + \gamma dW \\ dx_4 = -\alpha x_3 dt \end{cases} \tag{5}$$

where $x_1 = \theta(t)$, $x_2 = \dot{\theta}(t)$ and $x_3 = m(t)$.

Furthermore, Eq. (5) can be constructed as an Itô stochastic differential equation (SDE), which is given as:

$$d\mathbf{x} = \mathbf{a}(\mathbf{x},t)dt + \mathbf{b}(t)d\mathbf{W}(t) \tag{6}$$

where $\mathbf{x}(t)=(x_1(t),\ldots, x_4(t))^T$ is a 4D state-space vector process, the vector $\mathbf{a}(\mathbf{x},t)$ is the drift term and $\mathbf{b}(t)d\mathrm{W}(t)$ represents the diffusion term. The vector $d\mathbf{W}(t)=\mathbf{W}(t+dt)-\mathbf{W}(t)$ denotes the independent increments of a standard Wiener process. For the Markov process $\mathbf{x}(t)$, its transition probability density function (PDF), $p(\mathbf{x},t \mid \mathbf{x}',t')$, satisfies the FP equation that is expressed as [28]:

$$\frac{\partial}{\partial t} p(\mathbf{x},t\,|\,\mathbf{x}',t') = -\sum_{i=1}^{4} \frac{\partial\left[a_i(\mathbf{x},t)\right]}{\partial x_i} p(\mathbf{x},t\,|\,\mathbf{x}',t') + \frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4} \frac{\partial^2\left[(b(t)\cdot b^T(t))_{ij}\right]}{\partial x_i \partial x_j} p(\mathbf{x},t\,|\,\mathbf{x}',t') \tag{7}$$

where $\mathbf{x}'$ denotes the state-space vector at time t' and $t' < t$.

The response statistics of the nonlinear roll motion in random beam seas can be obtained by solving the governing FP equation, Eq. (7) and the principles of the two numerical methods for providing approximate solutions of the FP equation are presented in the following sections.

3. Energy based Stochastic Averaging Method

The stochastic averaging method has been well established as a powerful approximate technique for predicting of the response statistics of nonlinear dynamical systems. Instead of solving the FP equation of the original system, this method is focused on the energy envelope process of the roll dynamics, and a one-dimensional SDE is established to describe the distribution of the roll energy process [19]. Therefore, the problem is simplified as a calculation of the relevant PDFs for the low-dimensional cases.

Generally, the damping coefficients of the roll motion are smaller than the restoring coefficients and a perturbation parameter $\varepsilon \ll 1$ is introduced in the SDOF model, Eq. (1), in order to provide a scaled system for the stochastic averaging procedure. The scaled system with $t_\varepsilon = \varepsilon t$ is given as follows [14]:

$$\begin{cases} \dfrac{d}{dt_\varepsilon} x = y \\ \dfrac{d}{dt_\varepsilon} y = -\alpha_1 x + \alpha_3 x^3 - \varepsilon(\beta_1 y + \beta_2 y|y|) + \sqrt{\varepsilon}\upsilon m_s(t_\varepsilon) \end{cases} \tag{8}$$

where $x = x_1$, $y = \varepsilon^{-1} x_2, \alpha_1 = \varepsilon^{-2} c_1, \alpha_3 = \varepsilon^{-2} c_3, \beta_1 = \varepsilon^{-2} b_{44}$, $\beta_3 = \varepsilon^{-1} b_{44q}, \upsilon = \varepsilon^{-\frac{5}{2}}$ and $m_s(t_\varepsilon) = m(\varepsilon t)$.

Since we are interested in the total energy of the roll dynamics, the Hamilton function $H(x, y)$ is introduced:

$$H(x,y) = \frac{y^2}{2} + \alpha_1 \frac{x^2}{2} - \alpha_3 \frac{x^4}{4} \tag{9}$$

The contour lines of the Hamilton function, $H(x, y)$ are shown in Fig. 1. The fixed points of the system in Eq. (8) without dissipation and random perturbation (i.e., ε=0) are as follows: $P_1=(\sqrt{\alpha_1/\alpha_3},0)$, $P_2=(-\sqrt{\alpha_1/\alpha_3},0)$ and $S=(0, 0)$. Furthermore, the change in the total energy is the time derivate of the Hamilton function:

$$\frac{d}{dt_\varepsilon}H = \varepsilon y^2(-\beta_1 - \beta_2|y|) + \sqrt{\varepsilon}y\upsilon m_s(t_\varepsilon) \tag{10}$$

From the Hamiltonian, we introduce the following relationship:

$$Q(x,H) = y^2 = 2H - \alpha_1 x^2 + \alpha_3 \frac{x^4}{2} \tag{11}$$

Then, combing the first equation of Eq. (8) with Eq. (10), the following system is obtained

$$\begin{cases} \frac{d}{dt_\varepsilon}x = \sqrt{Q(x,H)} \\ \frac{d}{dt_\varepsilon}H = \varepsilon Q(x,H)(-\beta_1 - \beta_2\sqrt{Q(x,H)}) + \sqrt{\varepsilon}\sqrt{Q(x,H)}\upsilon m_s(t_\varepsilon) \end{cases} \tag{12}$$

in which the variable y was eliminated by defining the function $Q(x,H)$.

The scaling (perturbation) parameter in Eq. (8) helps to indicate the relative order of magnitude of the roll damping term and the external excitation term. An important property of the reformulated system in Eq. (13) is that the energy level H changes slowly compared to the oscillations of the variable x. This enables the application of the stochastic averaging method to this system since the fast oscillatory dynamics of the roll motion can be averaged over the roll period. For the multiple scale model in Eq. (12), the period $T(H)$ at the energy level H ($0 \leq H \leq H_c$) of one oscillation of the fast variable x in the absence of noise and damping (i.e., $\varepsilon=0$), is given as [20]:

$$T(H) = \int_0^{T(H)} dt = 2\int_{-b(H)}^{b(H)} \frac{dx}{\sqrt{Q(x,H)}} = \frac{4}{q}K(k) \tag{13}$$

where H_c is the energy level corresponding to the roll angle of vanishing stability, which is

$$H_c = \alpha_1^2 / 4\alpha_3 \tag{14}$$

with

$$q = a\sqrt{\frac{\alpha_3}{2}},\ a = \sqrt{\frac{4H}{b^2\alpha_3}} \tag{15}$$

and the function $K(k)$ is the complete elliptic integral of the first kind with elliptic modulus k given by $k=b/a$ [29]. The limits of integration of $\pm b(H)$ are the points where $y = \sqrt{Q(x,H)} = 0$, and the periodic orbit interacts with the x-axis. In other words, $b(H)$ is the maximum value, i.e., the roll amplitude, of x for each energy level H and is given by the following equation:

$$b = \sqrt{-\frac{-\alpha_1 + \sqrt{\alpha_1^2 - 4\alpha_3 H}}{\alpha_3}} \tag{16}$$

It should be noted that the roll amplitude b for each level of H is independent of the scaling parameter ε.

The process H converges weakly to a diffusion Markov process as the perturbation parameter $\varepsilon \to 0$, then, the corresponding one-dimensional Itô equation for the Markov process is expressed as follows:

$$dH = m(H)dt_\varepsilon + \sigma(H)dW(t_\varepsilon) \tag{17}$$

where the drift and diffusion coefficient are as follows [14, 30]:

$$m(H) = \frac{4}{Tq}\int_{-\infty}^{0} \upsilon^2 R_{m_s m_s}(\tau)\int_0^{K(k)} \frac{cn_{t_\varepsilon+\tau} dn_{t_\varepsilon+\tau}}{cn_{t_\varepsilon} dn_{t_\varepsilon}} du d\tau + \frac{1}{T}\int_0^T Q(x(t_\varepsilon), H)(-\beta_1 - \beta_2\sqrt{Q(x(t_\varepsilon), H)})dt_\varepsilon \tag{18}$$

$$\sigma^2(H) = \frac{4b^2 q}{T}\int_{-\infty}^{\infty} \upsilon^2 R_{m_s m_s}(\tau)\int_0^{K(k)} cn_{t_\varepsilon} dn_{t_\varepsilon} cn_{t_\varepsilon+\tau} dn_{t_\varepsilon+\tau} du d\tau \tag{19}$$

where $R_{m_s m_s}$ is the autocorrelation function of the stochastic process $m_s(t_\varepsilon)$ and $cn(\cdot, k)$ and $dn(\cdot, k)$ are the Jacobian elliptic functions and the following abbreviations are used:

$$cn := cn(qt_\varepsilon, k); dn := dn(qt_\varepsilon, k) \text{ and } u := qt_\varepsilon \tag{20}$$

in which, if the substitute τ or $t_\varepsilon + \tau$ is used, we refer to the argument $q\tau$ or $q(t_\varepsilon + \tau)$, respectively.

As mentioned in Section 2, the roll response can be assumed to be an approximately stationary diffusion process when the mean time to capsize is very long. Therefore, a stationary solution of the one-dimensional FP equation that governs the SDE (17) can be obtained and expressed as [8, 30]:

$$f(H) = \frac{C}{\sigma^2(H)}\exp\left(2\int_0^H \frac{m(h)}{\sigma^2(h)}dh\right) \tag{21}$$

where C is a non-dimensional parameter that can be obtained by a normality condition.

In addition, the autocorrelation function $R_{m_s m_s}$, which is important for calculating the drift and diffusion coefficients for Eq. (17) is given as:

$$\begin{aligned} R_{m_s m_s}(\tau) &= R_{mm}(\tau) = -\sum_{i=1}^{2} e^{\mu_i|\tau|}\frac{\gamma^2\mu_i^2}{(2\mu_i+\beta)(\mu_i^2-\beta\mu_i+\alpha)} \\ \mu_{1,2} &= -\beta/2 \pm \sqrt{(\beta/2)^2-\alpha} \end{aligned} \tag{22}$$

Based on Eq. (16), the stationary PDF of the maximum roll amplitude $b(H)$ corresponding to different levels of H is expressed as:

$$p_{st}(b) = f(H)\sqrt{\alpha_1^2 - 4\alpha_3 H}\sqrt{-\frac{-\alpha_1+\sqrt{\alpha_1^2-4\alpha_3 H}}{\alpha_3}} \tag{23}$$

Furthermore, the stationary joint PDF of x, y in Eq. (8) can be obtained from $f(H)$ as follows:

$$f_{XY}(x,y) = \left.\frac{f(H)}{T(H)}\right|_{H(x,y)=\frac{y^2}{2}+\alpha_1\frac{x^2}{2}-\alpha_3\frac{x^4}{4}} \tag{24}$$

and then the joint PDF of the roll angle and roll velocity can be calculated by applying the following relationship :

$x = x_1 = \theta$, $y = \varepsilon^{-1}x_2 = \varepsilon^{-1}\dot{\theta}$. The relevant derivations for Eq. 24 are presented in Appendix A.

4. PI Method

The PI method is based on the Markov property of the response process $\mathbf{x}(t)$ and provides approximate solutions to the FP equation, Eq. (7), of the original system, Eq. (5), by applying an iterative scheme which follows the basic Chapman-Kolmogorov equation:

$$p(\mathbf{x},t) = \int_{R^4} p(\mathbf{x},t \mid \mathbf{x}',t')p(\mathbf{x}',t')d\mathbf{x}' \tag{25}$$

where $d\mathbf{x}' = \prod_{i=1}^{4} dx_i'$.

For the numerical solution of the time-continuous process $\mathbf{x}(t)$, a time-discrete approximation should be introduced. In this regard, Naess and Moe [17] proposed a fourth-order Runge-Kutta-Maruyama approximation:

$$\mathbf{x}(t) = \mathbf{x}(t') + \mathbf{r}(\mathbf{x}(t'),t',\Delta t) + \mathbf{b}(t')\Delta\mathbf{W}(t') \tag{26}$$

where $\Delta t = t-t'$ is the time increment and the vector $\mathbf{r}(\mathbf{x}(t'), t', \Delta t)$ denotes the explicit fourth-order Runge-Kutta increment. Since $\mathbf{W}(t)$ is a Wiener process, the independent increment $\Delta\mathbf{W}(t')= \mathbf{W}(t)-\mathbf{W}(t')$ is a Gaussian variable for every t' when the time increment Δt is sufficiently small. Time sequence $\{\mathbf{x}(i\cdot\Delta t)\}_{i=0}^{\infty}$ is a Markov chain and can be used to approximate the time-continuous Markov process $\mathbf{x}(t)$ when Δt is sufficiently small.

Furthermore, the transition PDF, $p(\mathbf{x},t \mid \mathbf{x}',t')$, follows a (degenerate) Gaussian distribution, which is written as:

$$p(\mathbf{x},t \mid \mathbf{x}',t') = \delta(x_1 - x_1' - r_1(\mathbf{x}',t',\Delta t))\cdot \delta(x_2 - x_2' - r_2(\mathbf{x}',t',\Delta t))\cdot \tilde{p}(x_3,t \mid x_3',t')\cdot \delta(x_4 - x_4' - r_4(\mathbf{x}',t',\Delta t)) \tag{27}$$

where $\tilde{p}(x_3,t \mid x_3',t')$ is given as:

$$\tilde{p}(x_3,t \mid x_3',t') = \frac{1}{\sqrt{2\pi\gamma^2\Delta t}}\cdot\exp\left\{-\frac{(x_3 - x_3' - r_3(\mathbf{x}',t',\Delta t))^2}{2\gamma^2\Delta t}\right\} \tag{28}$$

in which $\mathbf{x}'= \mathbf{x}(t')$ and $r_i(\mathbf{x}', t', \Delta t) = r_i(\mathbf{x}(t'), t', \Delta t)$, where i=1,2,3,4 are the Runge-Kutta increments for the state space variables.

The time evolution of the PDF of $\mathbf{x}(t)$ can be determined by the following iterative scheme when the initial PDF, $p(\mathbf{x}^{(0)}, t_0)$ is given:

$$p(\mathbf{x},t) = \int_{R^4}\cdots\int_{R^4}\prod_{i=1}^{n} p(\mathbf{x}^{(s)},t_s \mid \mathbf{x}^{(s-1)},t_{s-1})\cdot p(\mathbf{x}^{(0)},t_0)d\mathbf{x}^{(0)}\ldots d\mathbf{x}^{(n-1)} \tag{29}$$

where $\mathbf{x}= \mathbf{x}^{(n)} = \mathbf{x}(t_n)$, $t = t_n = t_0+n\cdot\Delta t$, $\mathbf{x}^{(s)} = \mathbf{x}(t_s)$ and $t_s= t_0+s\cdot\Delta t$.

The numerical iterative algorithm, i.e., Eq. (29) describes the mathematical principle of the PI method and the associated numerical implementations have been systematically described in Refs. [7, 18]. The PDF of the energy level H can be obtained by applying the transformation (30).

$$f(H)=4\int_0^{b(H)}\frac{1}{\sqrt{Q(x,H)}}f_{XY}(x,\sqrt{Q(x,H)})dx=4\int_0^{\sqrt{-\frac{-\alpha_1+\sqrt{\alpha_1^2-4\alpha_3 H}}{\alpha_3}}}\frac{1}{\sqrt{Q(x,H)}}f_{XY}(x,\sqrt{Q(x,H)})dx \tag{30}$$

in which $f_{XY}(x,y)$ is provided by the joint PDF of the roll angle and roll velocity. The relevant derivations for Eq. (30) are given in Appendix B. Subsequently, the distribution of the maximum roll amplitude $b(H)$ can be obtained by using Eq. (23).

5. Numerical Results

In this work, a fishing research vessel is selected to study the rolling behavior in random beam seas. The main parameters of the ship model are given in Table 1, and it is seen that the damping coefficients are small when compared with the restoring coefficients. In order to conduct a detailed study of the performance for the stochastic averaging method and the PI method, Gaussian white noise and filtered white noise are applied as the driving processes for the SDOF model, Eq. (1).

5.1 System excited by Gaussian white noise

Under the excitation of pure Gaussian white noise, the 4D state-space equation, Eq. (5), is simplified as:

$$\begin{cases} dx_1 = x_2 dt \\ dx_2 = (-b_{44}x_2 - b_{44q}x_2\,|\,x_2\,| - c_1 x_1 + c_3 x_1^3)dt + \sigma_0 dW \end{cases} \tag{31}$$

where σ_0 is the noise level.

For the two-dimensional (2D) system in Eq. (31), the corresponding numerical implementation for the PI method and the derivations of the drift and diffusion coefficients for the one-dimensional Itô equation, Eq. (17) in the stochastic averaging method have been described in Chai et al. [11] and Dostal et al. [14], respectively. In this study, the noise level σ_0 is 0.067. For the stochastic averaging method, the drift coefficients and diffusion coefficients for different levels of roll amplitude b are plotted in Figs. 2 and 3, respectively. Subsequently, the distribution for the roll amplitude b can be obtained by applying Eqs. (22) and (23) and the result is presented in Fig. 4. A simple MCS based on the fourth-order Runge-Kutta-Maruyama method [7] is applied to check the performance of the stochastic averaging method and the result is also plotted in Fig. 4.

Furthermore, the joint PDF of the roll angle process and the roll velocity process obtained by applying the stochastic averaging method is shown in Fig. 5. Correspondingly, Fig. 6 presents the joint PDF of the roll response for the 2D system in Eq. (31) calculated by the PI method. It is observed in Figs. 4-6 that the stochastic averaging method provides satisfactory results of the response statistics for the 2D system when compared with the response statistics calculated by the PI method and MCS. In order to study the accuracy and performance of the stochastic averaging method and the PI method for the high-level roll response, a logarithmic scale is applied to the vertical axis of Fig. 4. Subsequently, the PDFs of the roll amplitude b with a logarithmic scale, provided by the stochastic averaging method, the PI method and the MCS are presented in Fig. 7. It is seen in Fig. 7 that both the stochastic averaging method and the PI method can provide satisfactory and reliable results of the response statistics, even in the tail region. The straightforward MCS can present stable results to only approximately to the 10^{-6} level. Nevertheless, the MCS can provide a reasonable evaluation for the performance of the stochastic averaging method and the PI method.

5.2 System excited by filtered white noise

In this section, the performance of the stochastic averaging method and the PI method for the 4D system in Eq. (5) is studied. For the random wave excitation, the information for the roll moment amplitude per unit wave height at frequency ω is given in Ref. [31]. The sea state with H_s=4.0 m and T_p=11.0 s is selected for this study. The parameters α, β and γ in the linear filter Eq. (3) are obtained by the least square error method.

For the 4D system in Eq. (5), the stochastic averaging method is applied in order to provide the PDF of the roll amplitude b. The drift and diffusion coefficients for different levels are shown in Figs. 8 and 9. The PDF of b obtained by the stochastic averaging method is presented in Fig. 10 and the distribution of the roll amplitude b, calculated by the MCS, is also plotted. In addition, the joint PDF of the roll angle process and the roll velocity process, provided by the stochastic averaging method and the PI method, are presented in Figs. 11 and 12, respectively. Fig. 10 shows that the stochastic averaging method slightly overestimates the peak value of the PDF for the roll amplitude b and underestimates the response statistics for the region with $b > 15$ degrees. Moreover, in the central regions of Figs. 11 and 12, it is observed that the values of the joint PDF of the roll response calculated by the stochastic averaging method are slightly higher than the joint PDF obtained by the PI method.

Furthermore, the vertical axis of Fig. 13 is given a logarithmic scale in order to study the performance of the stochastic averaging method and the PI method regarding the response statistics in the tail region. Fig. 13 shows that when compared with the result provided by MCS, the PI method is able to provide accurate and reliable result in the tail region with very low probability levels. However, the stochastic averaging method underestimates the response statistics in the tail region for the current system that corresponds to the sea state with H_s=4.0 m and T_p=11.0 s.

Based on the above studies, differences have been observed in the performance of the stochastic averaging method applied for calculating the response statistics of the systems excited by Gaussian white noises and random wave excitation (approximated as filtered white noise). The basic assumption of the stochastic averaging method lies in the fact that the ship has light roll damping and then the spectrum of the roll response is narrow-banded and peaked near the natural roll frequency. In particular, for the sea state with H_s=4.0 m and T_p=11.0 s, the filtered spectrum, the equivalent white noise with the spectrum S_0 and the spectrum of the roll response due to the random wave excitation are shown in Fig. 14. It is seen that the input spectrum is much broader than the output spectrum due to the light roll damping. In addition, for the sea state with H_s=4.0 m and T_p=11.0 s, the corresponding noise level used in the 2D system, Eq. (31), for the equivalent white noise is 0.067 [32].

For the current sea state with H_s=4.0 m and T_p=11.0 s, the stochastic averaging method could not provide reliable and satisfactory results for the high-level roll response in the tail region. Therefore, it would be interesting to find an index or the relevant requirements under which the stochastic averaging method would provide satisfactory results, especially in the tail region. Based on the fact that the stochastic averaging method performs accurately for the system with light damping and broad-banded excitation, such as the roll motion excited by Gaussian white noise, Roberts et al. [20] suggested that the ratio between the bandwidth of the excitation process and the bandwidth of the response process could serve as an index for the performance of the stochastic averaging method. However, it is still very difficult to determine, theoretically, a definite range of ratios for which the stochastic averaging method will provide acceptable results.

On the other hand, according to the fundamental work of Stratonovich [30], a requirement for a good approximation by the stochastic averaging scheme is that the correlation time τ_{cor} of the random excitation process $m(t)$ should be much smaller than the relaxation time τ_0 of the dynamical system, i.e., $\tau_{cor} \ll \tau_0$. The correlation time of the stationary excitation process is defined with its autocorrelation function $R_{mm}(\tau)$ as follows:

$$\tau_{cor} = \frac{\int_0^\infty \tau \left| R_{mm}(\tau) \right| d\tau}{\int_0^\infty \left| R_{mm}(\tau) \right| d\tau} \tag{32}$$

Since the relaxation time τ_0 is defined for only a linear system, the stochastic linearization technique [33] is introduced in order to obtain the corresponding linearized damping term. The relaxation time is approximated as:

$$\tau_0 = 2/\beta_{lin} \tag{33}$$

where β_{lin} is the linearized damping coefficient provided by the stochastic linearization method.

For the dynamical system excited by Gaussian white noise, the correlation time is zero (i.e., there is no correlation in time space), and the requirement $\tau_{cor} << \tau_0$ is easy to meet. From Eqs. (32) and (33), when the dynamical system has smaller damping coefficients or a smaller correlation time τ_{cor}, the stochastic averaging method is expected to offer more accurate approximations of the response statistics. Furthermore, for the effectiveness of the correlation index τ_{cor}/τ_0 applied as an index for the performance of the stochastic averaging method can be investigated by evaluating more numerical cases corresponding to different sea states.

5.3 Comparisons and discussions

The principles of the stochastic averaging method and the PI method are quite different: the former method is based on the fact that the fast oscillatory roll motion can be averaged over the roll period due to the light roll damping and wide-banded external excitation terms, however, the latter is based on the Markov property of the original dynamical system and the response statistics are obtained by solving the governing FP equation of the original system. Based on the abovementioned numerical results and comparisons, the advantages and shortcomings of these two methods are summarized in this section.

First, both the stochastic averaging method and the PI method can provide sufficient response statistics. In addition to the joint PDF of the roll response, the distribution of the roll amplitude b shown in the above examples, the mean upcrossing rate, and the exceedance probability, as well as some characteristic extreme values that are used for the reliability evaluation of the system can also be obtained. Therefore, the accuracy and computational cost of the two numerical methods would be the most important aspects with which to evaluate for their applications.

Regarding the accuracy of these methods, it is seen in Sections 5.1 and 5.2 that the PI method can provide satisfactory results of the response statistics, both for the system excited by Gaussian white noise and for the system driven by filtered white noises. On the other hand, the accuracy of the stochastic averaging method depends on the system and the excitation terms. Specifically, the stochastic averaging method exhibits a reliable performance for the system in Eq. (38) excited by Gaussian white noise, however, for the 4D system with filtered white noise corresponding to the sea state with H_s=4.0 m and T_p=11.0 s, its performance for the high-level roll response is not satisfactory.

Furthermore, the PI method suffers the dimensionality problem. The PI method solves the FP equation of the original system, and the computational cost increases dramatically with the dimension of the system and the grid number in each dimension. For the 2D system in Eq. (31), there are 128 grid nodes in each direction, and the computational time for one simulation is less than 1 minute, while for the 4D system in Eq. (5), the computational time of the PI procedure with 64×64×32×32 nodes (i.e., the grid number is 64 for the first two dimensions and 32 for the third and fourth dimensions) for each simulation is approximately 1.5 hours on a laptop. On the other hand, the MCS method does not critically suffer from the dimensionality problem since the statistics of the response are obtained directly from the realizations. For the 2D and 4D systems, 3000 realizations with a predetermined 3-hour simulation time (approximately 1840 natural roll periods) for each realization are included in the direct MCS. Approximately 15-20 minutes are required to obtain the response statistics shown in Figs. 7 and 13. Therefore, regarding the computational cost, the MCS method has a large advantage over the 4D PI method. Nevertheless, the straightforward MCS can provide stable response statistics to the level only approximately equal or lower than 10^{-6}. For a stable result at a level lower than 10^{-7}, the computational cost for the conventional MCS is nearly formidable.

The stochastic averaging method reduces the original 2D and 4D systems into one-dimensional systems, and the results are then given by analytical formulas, rather than numerical calculations. For the computational efficiency of the stochastic averaging method, the drift coefficient and diffusion coefficient of the one-dimensional Markov process in Eq. (17), i.e., Eqs. (18) and (19), are obtained analytically [16]. The computational times to obtain the response statistics of the dynamical system excited by Gaussian white noise and filtered white noise are 20 seconds and 7 minutes for the stochastic averaging method, respectively. In addition, the drift and diffusion coefficients can also be calculated by direct numerical integration, which takes a longer time than the analytical solutions.

6. Conclusions

In this work, two approaches with different principles, the stochastic averaging method and the PI method, were applied in order to calculate the response statistics of the roll motion excited by Gaussian white noise and filtered white noise. Comparative studies for the accuracy and computational cost of the two methods have been performed with the assistance of the conventional MCS.

The advantages and shortcomings of the two abovementioned methods have been demonstrated via practical calculations and comparisons in Section 5. Based on the comparisons and discussions above, it is seen that the PI method can provide more accurate estimations of the roll response statistics than those of the stochastic averaging method, especially for the high-level responses with low probability levels. Since the high-level response is directly related to the safety of the vessel for the nonlinear roll motion in random seas, the PI method is recommended since its accuracy in the tail region is always reliable. However, the stochastic averaging method has the advantage of lesser computational time since this method simplifies the problem of solving the FP equation of the original system by applying a dimension-reduction technique. On the other hand, the PI method does suffer from the dimensionality problem. Therefore the computational cost would increase dramatically with the dimensionality of the system.

Furthermore, the performance of the stochastic averaging method depends on the excitation term and the damping terms of the dynamical system. In order to obtain satisfactory results by the time-saving stochastic averaging method, a reliable index for the performance of the stochastic averaging method could be introduced. Verification and application of such an index could be a valuable area of work after current study.

Acknowledgements

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Appendix A

Firstly, the following functional relation is introduced:

$$\begin{cases} X = X = g_1(X,Y) \\ H = \dfrac{Y^2}{2} + \alpha_1 \dfrac{X^2}{2} - \alpha_3 \dfrac{X^4}{4} = g_2(X,Y) \end{cases} \tag{A.1}$$

where X and Y are random variables corresponds to x, y defined by Eq. (8), respectively. Following Stratonovich [30], the PDF of the displacement x given an energy level value H is inversely proportional to the velocity y and thus

$$f_{X|H}(x\,|\,H) = \frac{1}{T(H)} \frac{1}{\sqrt{Q(x,H)}} \tag{A.2}$$

Using $f_{XH}(x, H) = f_{X|H}(x|H) f(H)$, the joint PDF of x and y is obtained as:

$$f_{XY}(x,y) = f_{XH}(x,H) \begin{vmatrix} \dfrac{\partial g_1}{\partial x} & \dfrac{\partial g_1}{\partial y} \\ \dfrac{\partial g_2}{\partial x} & \dfrac{\partial g_2}{\partial y} \end{vmatrix} = f_{XH}(x,H)\left|y\right| = \frac{1}{T(H)} \frac{1}{\sqrt{Q(x,H)}} \cdot f(H) \cdot \sqrt{Q(x,H)} = \frac{f(H)}{T(H)} \tag{A.3}$$

where the Hamilton function $H(x, y)$ is given in Eq. (9) and the period function $T(H)$ is obtained from Eq. (13).

Appendix B

In order to get the expression of $f(H)$ (Eq. 30) and the corresponding PDF of roll amplitude b (Eq. 23) by using the joint PDF of the roll response obtained by the PI method, the following transformation will be applied.

Firstly, according to Eq. (A.1), we have the function:

$$\begin{cases} X = g_1^{-1}(X,H) = X \\ Y = g_2^{-1}(X,H) = \pm\sqrt{Q(X,H)} \end{cases} \tag{B.1}$$

and the function $Q(X,H)$ is presented in Eq. (11). By introducing the Jacobian determinant, the following distributions can be obtained.

$$f_{XH}(x,H) = f_{XY}(x = g_1^{-1}(x,H), y = g_2^{-1}(x,H)) \cdot |J| \tag{B.2}$$

and the Jacobian determinant is given by:

$$J = \begin{vmatrix} \frac{\partial g_1^{-1}}{\partial x} & \frac{\partial g_1^{-1}}{\partial H} \\ \frac{\partial g_2^{-1}}{\partial x} & \frac{\partial g_2^{-1}}{\partial H} \end{vmatrix} = \frac{\partial g_2^{-1}}{\partial H} \tag{B.3}$$

Thereby, the distribution of the energy level H is expressed as:

$$f(H) = \int f_{XH}(x,H)dx = \int \frac{1}{|y|} f_{XY}(x,y)dx \tag{B.4}$$

In addition, from Eq. (11), the following relationship is obtained:

$$|y| = \sqrt{Q(x,H)} \tag{B.5}$$

According to the Fig. 1, which presents the contour lines of the Hamilton function $H(x, y)$, Eq. (B.5) can be simplified by the following expression considering the symmetry in the integration with respect to the x-axis for each H level:

$$\begin{aligned} f(H) &= 4\int_0^{b(H)} \frac{1}{\sqrt{Q(x,H)}} f_{XY}(x,\sqrt{Q(x,H)})dx \\ &= 4\int_0^{\sqrt{\frac{-\alpha_1+\sqrt{\alpha_1^2-4\alpha_3 H}}{\alpha_3}}} \frac{1}{\sqrt{Q(x,H)}} f_{XY}(x,\sqrt{Q(x,H)})dx \end{aligned} \tag{B.6}$$

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Tables:

Table 1 List of ship parameters

Parameters	Dimensional value
$I_{44}+A_{44}$	$5.540\times10^{7}\ kg\cdot m^{2}$
Δ	$2.017\times10^{7}\ N$
b_{44}	$0.095 s^{-1}$
b_{44q}	0.0519
c_{1}	$1.153\ s^{-2}$
c_{3}	$0.915\ s^{-2}$

Figures

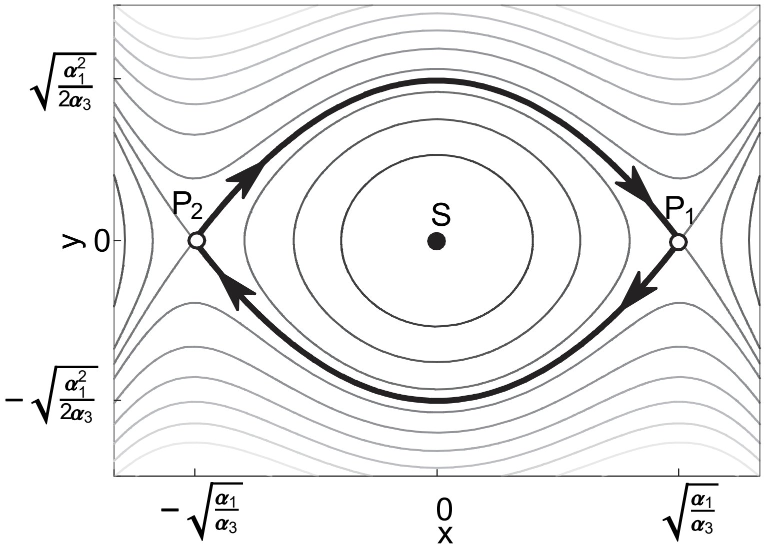

Fig. 1 Contour lines of $H(x,y)$

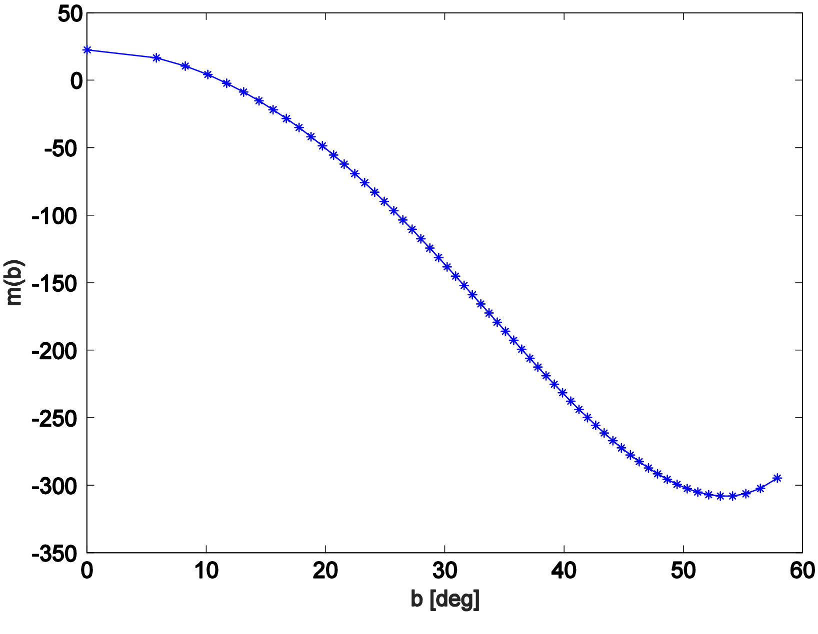

Fig. 2 Drift coefficients for the 2D system in Eq. (31) with σ_0=0.067

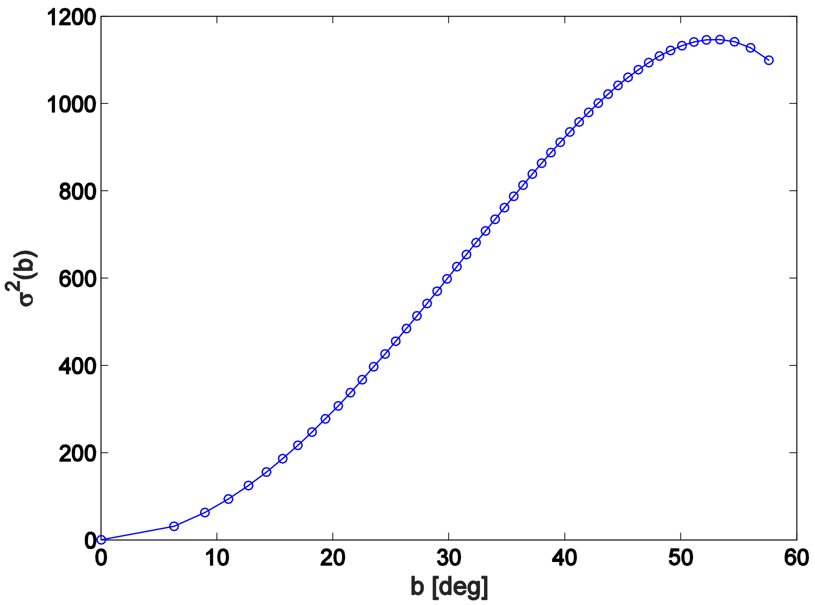

Fig. 3 Diffusion coefficients for the 2D system in Eq. (31) with σ_0=0.067

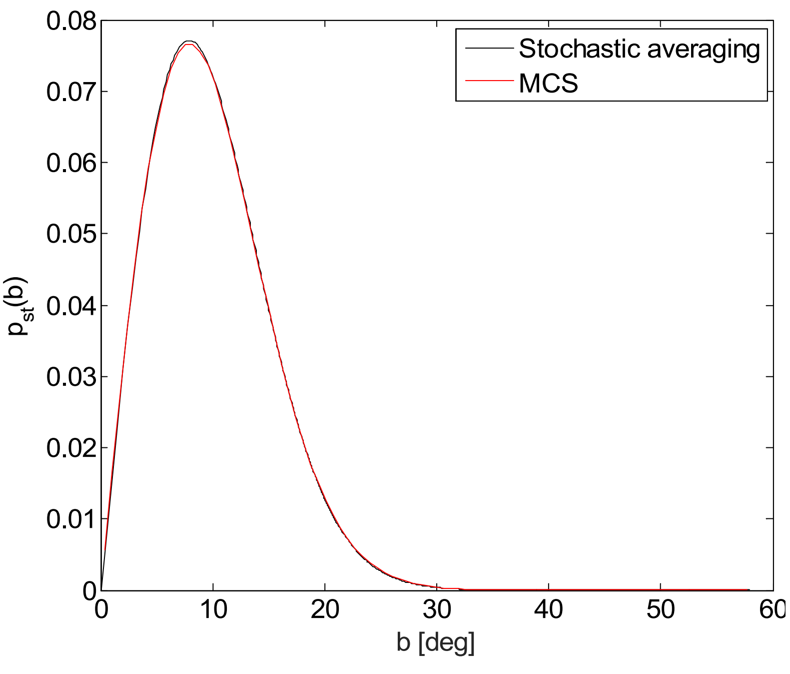

Fig. 4 PDFs for the maximum roll amplitude *b* for the 2D system with σ_0=0.067, provided by the stochastic averaging method and MCS

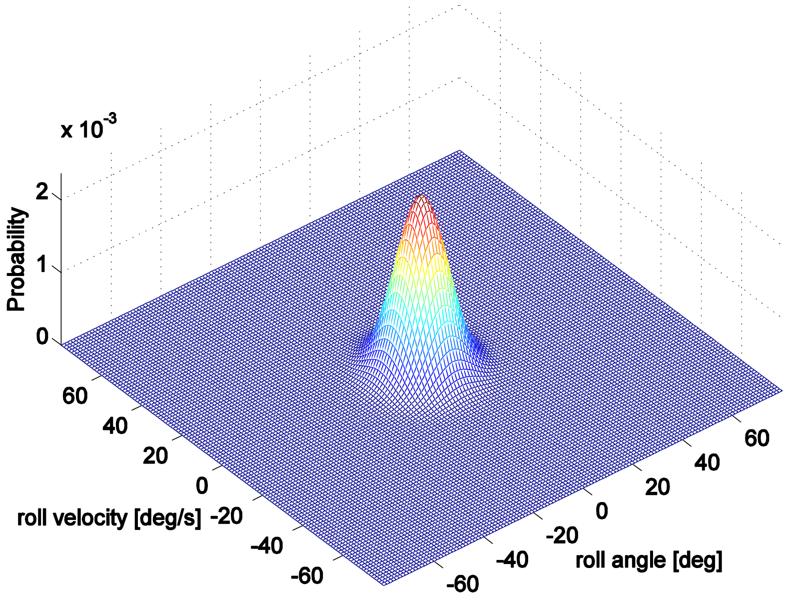

Fig. 5 Joint PDF of the roll angle and roll velocity for the 2D system in Eq. (31) with σ_0=0.067, obtained by the stochastic averaging method

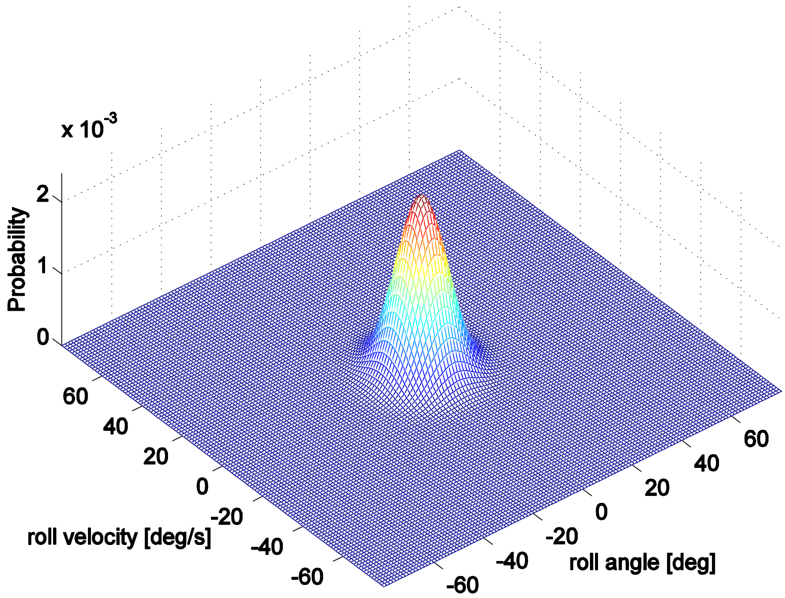

Fig. 6 Joint PDF of the roll angle and roll velocity for the 2D system in Eq. (31) with σ_0=0.067, calculated by the PI method

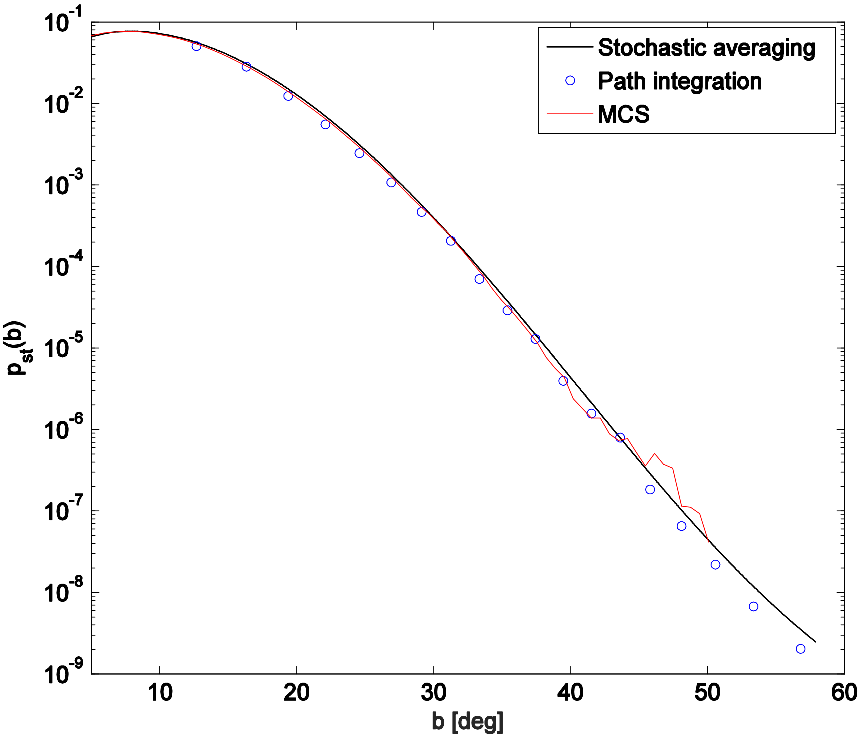

Fig. 7 PDFs for the maximum roll amplitude b for the 2D system with σ_0=0.067, plotted with a logarithmic scale

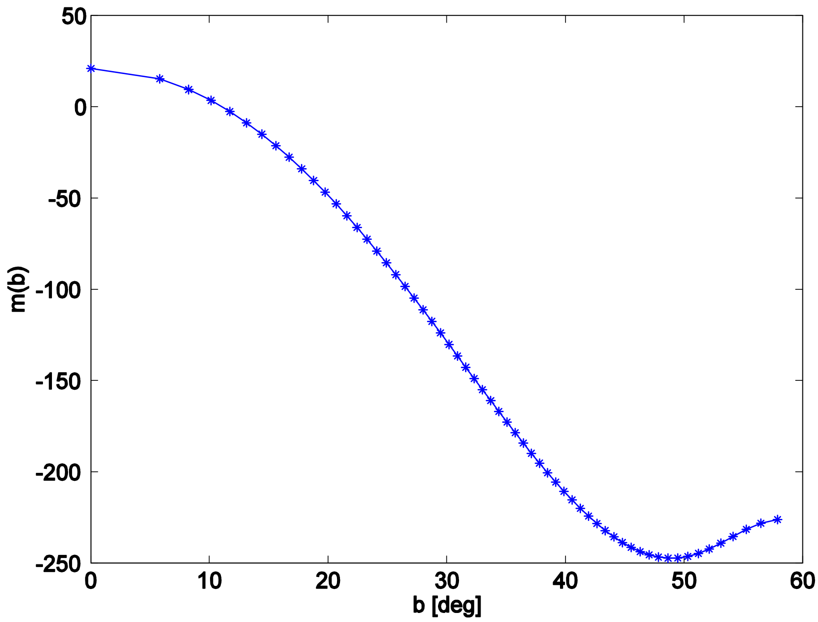

Fig. 8 Drift coefficients for the 4D system in Eq. (5) with the sea state H_s=4.0 *m* and T_p=11.0 *s*

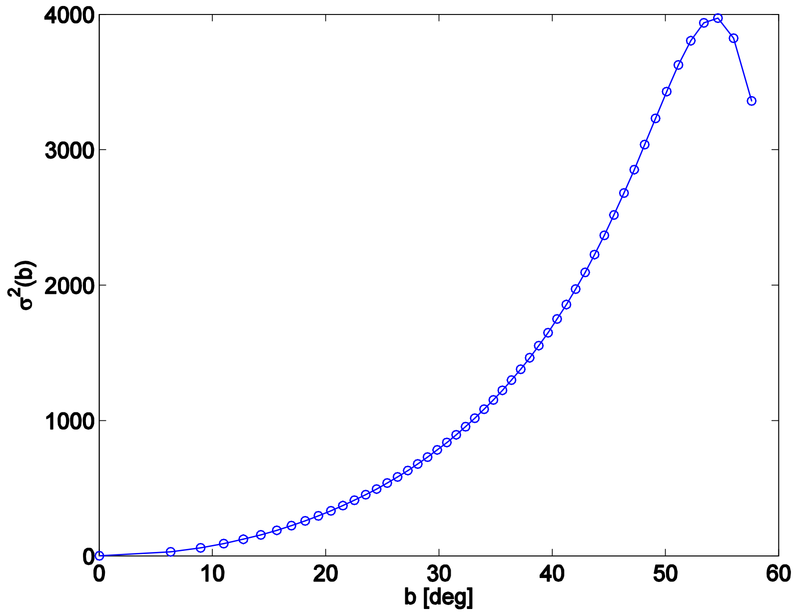

Fig. 9 Diffusion coefficients for the 4D system in Eq. (5) with the sea state H_s=4.0 m and T_p=11.0 s

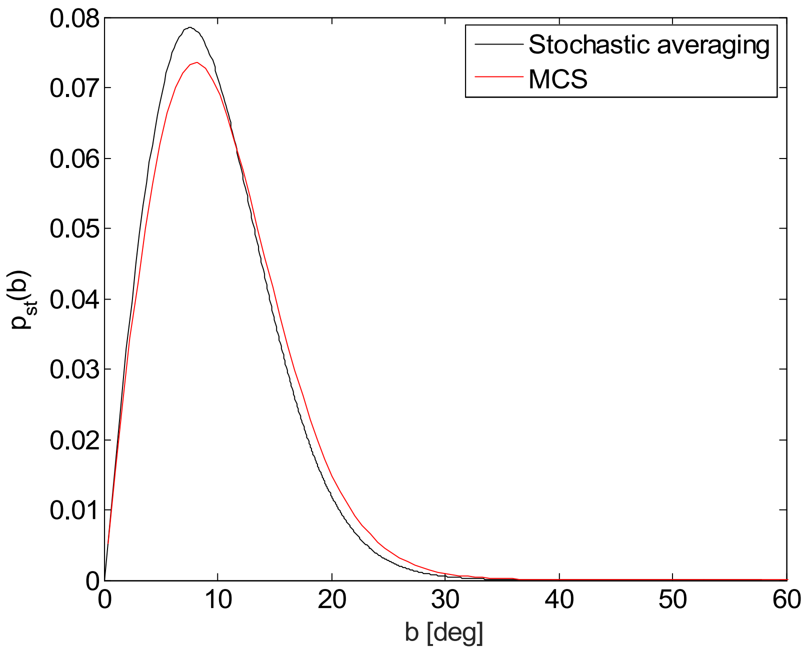

Fig. 10 Stationary PDF for the maximum roll amplitude b for the 4D dynamical system in Eq. (5) with the sea state H_s=4.0 m and T_p=11.0 s

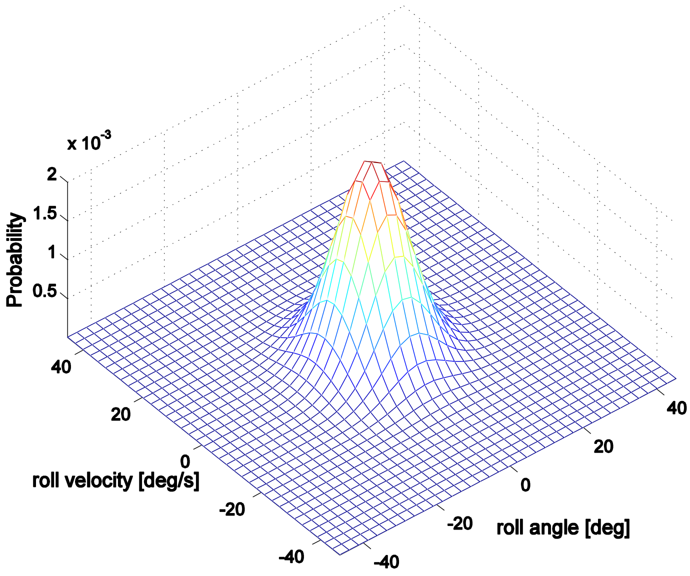

Fig. 11 Joint PDF of the roll angle and roll velocity for the 4D system in Eq. (5) with the sea state with H_s=4.0 *m* and T_p=11.0 *s*, obtained by the stochastic averaging method

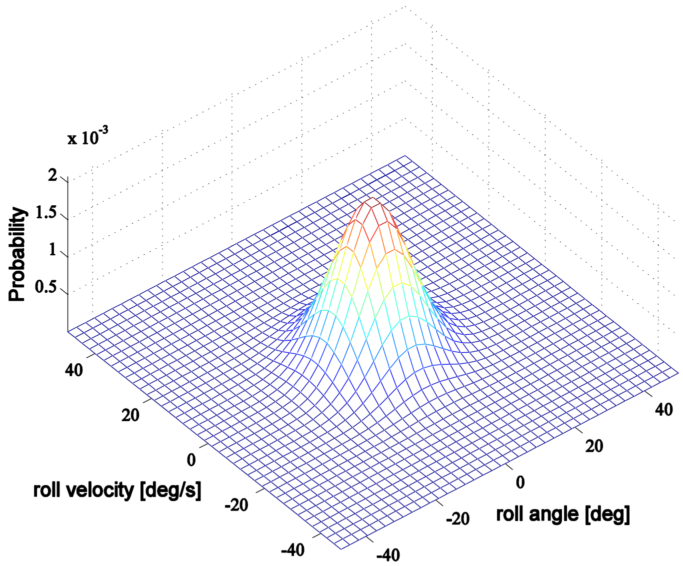

Fig. 12 Joint PDF of the roll angle and roll velocity for the 4D system in Eq. (5) with the sea state with H_s=4.0 *m*, T_p=11.0 *s*, obtained by the PI method

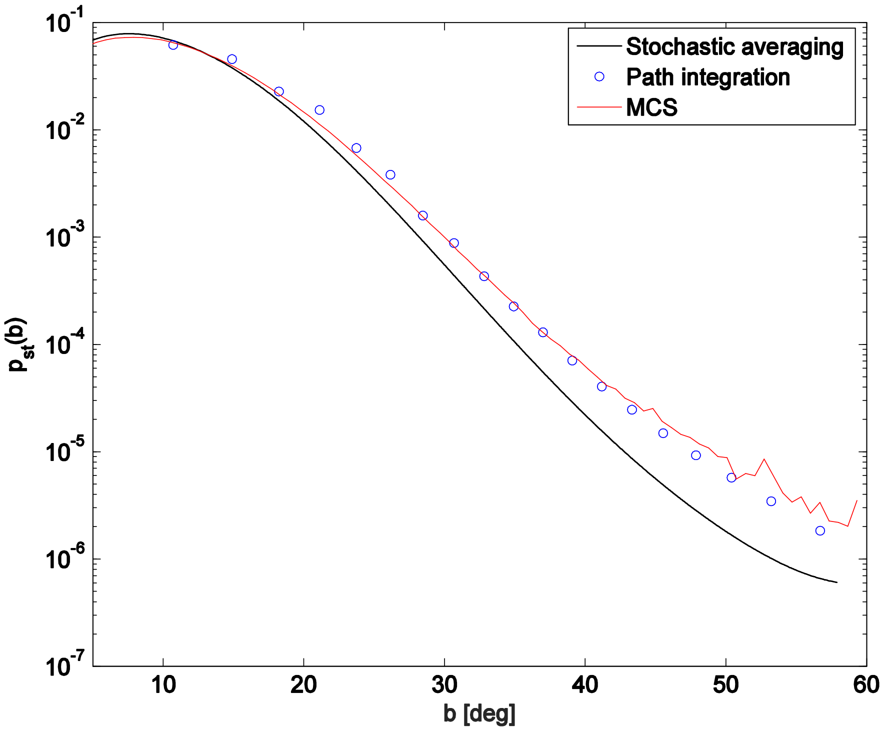

Fig. 13 Logarithmic scale of the stationary PDF for the maximum roll amplitude b for the 4D system in Eq. (5) with the sea state H_s=4.0 m and T_p=11.0 s

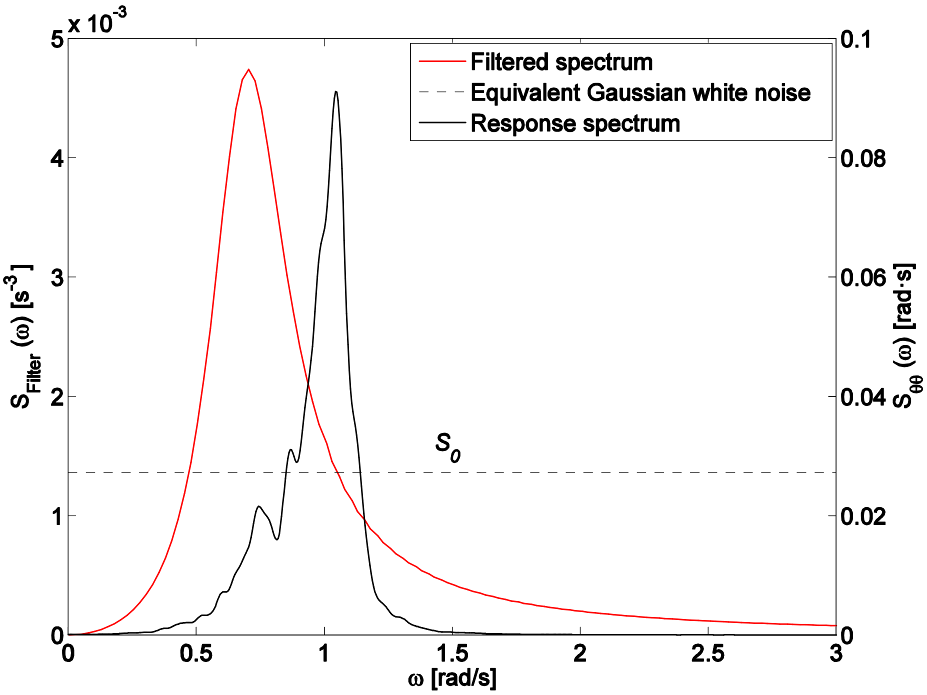

Fig. 14 Filtered spectrum for the input process $m(t)$, spectrum of the equivalent Gaussian white noise and spectrum of the output process $\theta(t)$ for the sea state with H_s=4.0 m and T_p=11.0 s