Decay estimation in nonlinear hyperbolic system of conservation laws

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ABSTRACT

In this paper we study some decay estimates in nonlinear hyperbolic system of conservation laws. This research is not only interesting in itself but also crucial in studying the large time behavior problem. By introducing a proper Glimm functional, we obtain some useful decay estimates which are proved helpful in obtaining decay rates of the admissible solutions to nonlinear hyperbolic conservation laws as $t \to \infty$.

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1. Introduction

We consider the Cauchy problem

$$\begin{cases} u_t + f(u)_x = 0, \\ u(x, 0) = u_0(x), \end{cases} \tag{1.1}$$

with the additional assumptions:

(A) The total variation of $u_0(x)$ is small.
(B) For each characteristic field, $\nabla \lambda_i(u) \cdot r_i(u)$ vanishes at a single manifold of codimension one, which is transversal to the characteristic vector $r_i(u)$. We denote the linear degeneracy manifold by $LD_i \equiv \{u: \nabla \lambda_i(u) \cdot r_i(u) = 0, \ u \in \mathbb{R}^n\}$. Assume $(\nabla_{uu} \lambda_i(u) \cdot r_i(u)) \cdot r_i(u) \neq 0$ and $0 \in LD_i$.

The initial data of (1.1) satisfies $u_0(x) = 0$ for $|x| \geqslant N$. Here $u \in \mathbb{R}^n$, $f : \Omega \mapsto \mathbb{R}^n$ is a smooth vector function with $\Omega \subset \mathbb{R}^n$ being an open set. Denote $A(u)$ as $Df(u)$ the $n \times n$ Jacobian matrix of the flux function f. The system (1.1) is assumed to be strictly hyperbolic, that is, for every $u \in \Omega$, the matrix $A(u)$ has n real distinct eigenvalues, denoted by

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u).$$

Corresponding to these eigenvalues, there are n linearly independent right eigenvectors $r_i(u)$ and left eigenvectors $l_i(u)$. We normalize $r_i(u)$ by $\|r_i(u)\| = 1$ and $l_i(u) \cdot r_i(u) = 1$.

For the same problem with the assumptions (A) and

(C) each characteristic field is genuinely nonlinear,

we can use a similar argument to obtain similar conclusion, see Remarks 2.1 and 4.4.

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Let us introduce some basic concepts of weak solutions to the hyperbolic conservation laws, which will be used in later discussion. The weak solution considered is defined as follows:

Definition 1.1. A function $u : [0, T] \times \mathbb{R} \mapsto \mathbb{R}$ is a weak solution of the problem (1.1), if u is a bounded measurable function and

$$\iint\limits_{t \geqslant 0} \left[u\phi_t + f(u)\phi_x \right] dx \, dt + \int\limits_{t=0} u_0(x)\phi(x, 0) \, dx = 0, \tag{1.2}$$

holds for any smooth function ϕ with compact support in $\{(x, t) \mid (x, t) \in \mathbb{R}^2\}$.

Under the assumptions (A) and (B), the global existence of weak solutions to (1.1) is demonstrated by introducing the Glimm scheme or wavefront tracking algorithm and using the solutions to the Riemann problems solved by Lax as building blocks, cf. [1,3,6,15,16,18–22] and references therein. There are two main elements to consider in proving the existence theory of the Cauchy problem. One is to prove the decreasing of Glimm functional and the other one is to approximate the initial data by piecewise constant functions and solve the Riemann problems locally in space and time. Here, the Riemann problem is (1.1) with the initial data given by

$$u_0(x) = \begin{cases} u^- & \text{if } x < 0, \\ u^+ & \text{if } x > 0, \end{cases} \tag{1.3}$$

where $u^{\pm}$ are constants. In the Riemann problem it is well known that corresponding to the n characteristic fields of the system, there are n Hugoniot curves. Since the weak solution is not unique we need the following Liu's entropy condition to choose the physical shock.

Definition 1.2. (See [19].) A discontinuity (u_-, u_+) is admissible if

$$\sigma_i(u_-, u_+) \leqslant \sigma_i(u_-, u), \tag{1.4}$$

for any state u on the Hugoniot curve $S_i(u_-)$ between u_- and u_+, where $S_i(u_-) \equiv \{u: \sigma_i(u_-, u)(u_- - u) = f(u_-) - f(u)\}$.

Any state u on the i-th Hugoniot curve $S_i(u_0)$ is connected to u_0 by an i-th shock wave, if the above entropy condition is satisfied. We denote $S_i(\alpha)(u_0)$ as the state which can be connected to u_0 by an i-th shock wave of strength α. Note that the shock wave described here includes the case of contact discontinuity.

Another basic wave pattern used to solve Riemann problem is called the rarefaction wave. The state $R_i(\alpha)(u_0)$ $(i = 1, 2, \ldots, n)$ is connected to u_0 by an i-th rarefaction wave of strength α, if

$$\begin{cases} \dfrac{d}{d\alpha} R_i(\alpha)(u_0) = r_i\big(R_i(\alpha)(u_0)\big), & \text{for } \lambda_i\big(R_i(\alpha)(u_0)\big) > \lambda_i(u_0), \\ R_i(0)(u_0) = u_0. \end{cases}$$

By implicit function theorem, the Riemann problem for general systems is solved by piecing together waves in different families. Moreover, with Liu's entropy condition, each wave in the i-th family, called an i-wave, may be the composition of several i-th admissible shocks and rarefaction waves.

In [20] the authors construct the wave curve $W_i(\alpha)(u_0)$ as the curve consisting of all the end states that can be connected to u_0 by admissible shocks, rarefaction waves or their combination of the i-th family. Here α is a non-degenerate parameter along the curve. Up to a linear transformation, this parameter can be chosen as the i-th component of u, i.e. u^i. The wave curve has the following regularity result:

Lemma 1.1. (See [2].) With the assumptions (A) and (B), the admissible i-th curve $W_i(s)(u_0)$ has Lipschitz continuous first order derivatives.

Let us introduce the notations to be used later. The backward and forward generalized characteristics are defined in [14]. We shall take the point of view that a left characteristic does not terminate at a contact shock, but crosses the shock curve and continues with altered speed and value as a right characteristic. We shall call the resulting piecewise linear, convex curve a backward generalized characteristic, noting that we regain the property that any point is connected by a backward generalized characteristic to the initial axis, see Fig. 4. We define a forward generalized characteristic to follow a characteristic line until it terminates at a shock, and to follow the shock curve thereafter. Define

$$N^+(t) \doteq \inf\{y: u(x, t) = 0, \ \forall x > y\}, \qquad N^+(0) = N, \tag{1.5}$$

$$N^-(t) \doteq \sup\{y: u(x, t) = 0, \ \forall x < y\}, \qquad N^-(0) = -N. \tag{1.6}$$

It is well known that

$$\left|N^{\pm}(t)\right| \leqslant C_1 t + N, \tag{1.7}$$

where C_1 is a constant independent of t. As in [17] for any given t_0 large enough, we denote by $Y_L^i(t)$, $Y_R^i(t)$, $i = 1, \ldots, n$, the i-th forward characteristics issuing from the point $(N^+(t_0), t_0)$ and $(N^-(t_0), t_0)$, respectively. It follows from the strict hyperbolicity of (1.1) that there exists t_1 such that $Y_L^i(t)$ intersects $Y_R^j(t)$, $i, j = 1, \ldots, n$, $i > j$, before time t_1. Similarly from t_1 we can obtain t_2. Therefore, if we choose t_0 large enough we can obtain

$$t_2 \leqslant \mathcal{K} t_0. \tag{1.8}$$

Here $\mathcal{K}$ is a constant independent of t. Without any ambiguity we use $Y_L^i(t)$, $Y_R^i(t)$, $i = 1, \ldots, n$, to denote the forward i-th characteristics issuing from any t_0. $\xi_i(t; t_0)$ denotes the interval $[Y_L^i(t), Y_R^i(t)]$, while $\varsigma_i(t; t_0)$ denotes the horizontal interval outside $Y_L^i(t)$ and $Y_R^i(t)$ issuing from t_0.

We set $X_i(t) \doteq T.V.\{\lambda_i(u(x,t)): x \in R\}$, $X(t) \doteq \sum_i X_i(t)$. Let $X_i^+(t)$ and $X_i^-(t)$ denote the increasing and decreasing variation of λ_i respectively at the fixed time t. It is clear that $X_i(t) = X_i^+(t) + X_i^-(t)$. In this paper we use $X_i^{\pm}(\xi_i(t; \tau))$ to denote the increasing and decreasing variation of λ_i in $\xi_i(t; \tau)$, using $X_i^{\pm}(\varsigma_i(t; \tau))$ to denote the increasing and decreasing variation of λ_i outside $\xi_i(t; \tau)$. We use the symbols α, β, etc. to represent both waves (in symbolic sense) and their strengths (in numerical sense) in this paper.

The main result of this paper can be stated as follows.

Theorem 1.1. *For any time $t \geqslant t_2$*

$$X_i^{\pm}\left(\varsigma_i(t; t_1)\right) \leqslant O(1) Q^2(t_0), \tag{1.9}$$

$$X_i^+(t) \leqslant \frac{|\xi_i(t; t_0)|}{t - t'} + \left[\frac{1}{4} + O(1) V(0)\right] X_i^+\left(\xi_i(t'; t_0)\right) + O(1)\left[Q^2(t_0) + Q(t')\right], \tag{1.10}$$

$$Q(t_2) \leqslant V(t_2) X^{\frac{3}{2}}(t_2) + O(1) Q(t_0) V(t_2), \tag{1.11}$$

provided the total variation of the initial data $u(x, 0)$ is small enough. Here $t_1 < t' < t$, $|A|$ denotes the Lebesgue measure of set A, $Q(t)$ and $V(t)$ will be defined in the next section.

The remainder of the paper will be organized as follows. In the next section, we review some basic properties of the system (1.1) under assumptions (A) and (B), and the new wave potential in Glimm functional is introduced, together with some preliminaries of the wave interaction estimates in the wave tracing argument. In Section 3 we give some useful decay estimates and the proof of the main Theorem 1.1. In the last section we give an application of the estimation in Section 3.

2. Basic properties and Glimm functional

Without ambiguity we will use $W_i(u_0)$ to denote the composite wave curve through the state u_0. As in [11,13], in the following discussion, we will assume that a rarefaction wave is divided into several small rarefaction shocks with strength as a pre-chosen small constant in the wave front tracking method. In this way, the shock waves and rarefaction waves can be treated in the same way and the error thus caused tends to zero as the small constant approaches zero.

Definition 2.1. (See [20].) Let $u_r \in W_i(u_l)$ so that u_l is connected to u_r by i-discontinuities (u_{j-1}, u_j), and i-rarefaction waves (u_j, u_{j+1}), j odd, $1 \leqslant j \leqslant m - 1$, $u_0 = u_l$ and $u_m = u_r$. A set of vectors $\{v_0, v_1, \ldots, v_p\}$ is a partition of (u_l, u_r) if

(i) $v_0 = u_l$, $v_p = u_r$, $v_{k-1}^i \leqslant v_k^i$, $k = 1, 2, \ldots, p$,
(ii) $\{u_0, u_1, \ldots, u_m\} \subset \{v_0, v_1, \ldots, v_p\}$,
(iii) $v_k \in R_i(u_j)$, j odd, if $u_j^i < v_k^i < u_{j+1}^i$,
(iv) $v_k \in D_i(u_{j-1}, u_j)$, j odd, if $u_{j-1}^i < v_k^i < u_j^i$. Here

$$D_i(u_l, u_r) \equiv \left\{u: (u - u_l)\sigma(u_l, u_r) - \left(f(u) - f(u_l)\right) = c(u)r_i(u) \text{ for some scalar } c(u)\right\}.$$

Then set

(1) $y_k \equiv v_k - v_{k-1}$,
(2) $\lambda_{i,k} \equiv \lambda_i(v_{k-1})$ and $[\lambda_i]_k \equiv [\lambda_i](v_{k-1}, v_k) \equiv \lambda_i(v_k) - \lambda_i(v_{k-1}) > 0$ if (iii) holds,
(3) $\lambda_{i,k} \equiv \sigma(u_{j-1}, u_j)$ and $[\lambda_i]_k \equiv [\lambda_i](v_{k-1}, v_k) \equiv 0$ if (iv) holds.

The partition is stable under the perturbation in the following sense.

Lemma 2.1. *(See [20].) Suppose that $u_r \in W_i(u_l)$, $\bar{u}_r \in W_i(\bar{u}_l)$, with $u_r^i - u_l^i = \bar{u}_r^i - \bar{u}_l^i \equiv \alpha > 0$, and $|u_l - \bar{u}_l| \equiv \beta$. Then there exist partitions $\{v_0, v_1, \ldots, v_p\}$ and $\{\bar{v}_0, \bar{v}_1, \ldots, \bar{v}_p\}$ for the i-waves (u_l, u_r) and $(\bar{u}_l, \bar{u}_r)$ respectively such that $\bar{v}_k^i - \bar{v}_0^i = v_k^i - v_0^i$, $k = 1, 2, \ldots, p$, and the following hold:*

(i) $\sum_{k=1}^p |y_k - \bar{y}_k| = O(1)\alpha\beta$.
(ii) $|\lambda_{i,k} - \bar{\lambda}_{i,k}| = O(1)\beta$, $k = 1, 2, \ldots, p$.
(iii) *Let $\Theta^+(u_l, u_r)$ represent the value of λ_i at the right state u_r minus the wave speed of the right-most i-wave in (u_l, u_r). Similar definition holds for $\Theta^-(u_l, u_r)$.*
  Then

$$\left|\Theta^-(u_l, u_r) - \Theta^-(\bar{u}_l, \bar{u}_r)\right| + \left|\Theta^+(u_l, u_r) - \Theta^+(\bar{u}_l, \bar{u}_r)\right| = O(1)\alpha\beta.$$

  Moreover, the index set $\{1, 2, \ldots, p\}$ can be written as a disjointed union of subsets I, II and III such that
(iv) *for $k \in$ I corresponding to rarefaction waves, both v_k and $\bar{v}_k$ are of type (iii) of Definition 2.1 and*

$$\sum_{k \in I} \left|[\lambda_i]_k - [\bar{\lambda}_i]_k\right| = O(1)\alpha\beta,$$

(v) *for $k \in$ II corresponding to discontinuities, both v_k and $\bar{v}_k$ are of the type (iv) of Definition 2.1,*
(vi) *for $k \in$ III corresponding to mixed types, v_k and $\bar{v}_k$ are of different types and*

$$\sum_{k \in III} \left|[\lambda_i]_k + [\bar{\lambda}_i]_k\right| = O(1)\alpha\beta.$$

Defining a proper Glimm functional is crucial in obtaining some decay estimates which are useful in studying the large time behavior problem and N-waves, cf. [4,5,7–11,13,14,17], etc. For instance, in paper [17] the author gives the decay rates of total variation of the wave speed as $t^{-\frac{1}{2}}$. The scalar case of this problem in genuinely nonlinear conservation laws dates back to 1975 [8] and 1965 [9].

In the problem (1.1), with initial data compactly supported, for t large enough, we know the main interaction comes from the same family, so the proper definition of the same family wave potential Q_s is crucial in the study of decay estimation. We define the following Glimm type functional:

$$\mathcal{F}(t) \equiv V(t) + M_0 Q(t).$$

In the above definition,

$$V(t) = \sum\{|\alpha|: \alpha \text{ is any wave in } u(t,x)\}, \qquad Q(t) = Q_d(t) + Q_s(t),$$

$$Q_d(t) = \sum\{|\alpha||\beta|: \alpha \text{ and } \beta \text{ are strengths of } i\text{-th and } j\text{-th waves of } u(t,x), \text{ respectively, } i > j,$$

$$\text{and } \alpha \text{ lies to the right of } \beta\},$$

$$Q_s(t) = \sum_{i=1}^n Q_s^i,$$

$$Q_s^i = \sum\left\{X^{\frac{1}{2}}(t)\left(|\alpha|\left|\Delta\lambda_i(\beta)\right| + |\beta|\left|\Delta\lambda_i(\alpha)\right|\right): \alpha \text{ and } \beta \text{ are two } i\text{-waves in } u(t,x)\right\}. \tag{2.1}$$

Here M_0 is a suitably large constant and $|\Delta\lambda_i(\alpha)|$ is the total variation of λ_i across the wave α.

Remark 2.1. In the genuinely nonlinear case, Q_s^i is defined as

$$Q_s^i = \sum\{|\alpha||\beta|V(t): \alpha \text{ and } \beta \text{ are two } i\text{-waves in } u(t,x)\}. \tag{2.2}$$

Theorem 2.2. *$\mathcal{F}(t)$ is non-increasing at the times of interaction provided that M_0 is chosen large enough and the total variation of $u(x,0)$ is suitably small.*

The theorem will be proved in next section.

3. Some decay estimates

The purpose of this section is to formulate our main estimation. We show that $X^+(t)$ and $Q(t)$ can be estimated step by step. Without loss of generality we assume $(\nabla_{uu}\lambda_i(u) \cdot r_i(u)) \cdot r_i(u) < 0$ for $u \in LD_i$.

We review some known results for later use. The following lemma estimates the speed variations.

Lemma 3.1. *(See [12].) There exists a constant $\eta_0 > 0$ such that any i-wave $\alpha = (u_l, u_r)$ with $|\alpha| < \eta_0$ satisfies*

$$\left|\Delta\lambda_i(\alpha)\right| \approx \left(\left|u_m^i - u_*^i\right| + |\alpha|\right)|\alpha|,$$

where u_m is any state on $W_i(u_l)$ and lies between u_l and u_r.

Lemma 3.2. *(See [12].) $u_* \in LD_i \cap W_i(u_l)$, we write $u_* = u_*(u_l)$. Then for any $u \in R_i^+(u_l) \cup S_i^-(u_l)$ and any $u_r \in S_i(u_l)$, we have*

$$\lambda_i(u) - \lambda_i(u_*) = \frac{c(u_*)}{2}\left(u^i - u_*^i\right)^2 + O\left[\left(\left|u^i - u_*^i\right|^2 + \left|u_l^i - u_*^i\right|^2\right)\left|u^i - u_*^i\right|\right], \tag{3.1}$$

$$\sigma_i(u_r, u_l) - \lambda_i(u_*) = \frac{c(u_*)}{6}\left(\left(u_r^i - u_*^i\right)^2 + \left(u_r^i - u_*^i\right)\left(u_l^i - u_*^i\right) + \left(u_l^i - u_*^i\right)^2\right) + O\left(\left|u_l^i - u_*^i\right|^3 + \left|u_r^i - u_l^i\right|^3\right), \tag{3.2}$$

where $c(u_*) = (\nabla_{uu}\lambda_i(u_*) \cdot r_i(u_*)) \cdot r_i(u_*) \neq 0$.

Lemma 3.3. *(See [13].) Suppose that (u_l, u_r) is a left contact i-shock, and $u_* \in LD_i \cap S_i(u_l)$, then $u_*^i - u_l^i \to 0$ as $u_r^i - u_*^i \to 0$.*

Lemma 3.4. *(See [13].) Suppose that (u_l, u_r) is a composite i-wave. Furthermore, u_l is related to u_r by an i-th rarefaction (u_l, u_m) and a left contact i-shock (u_m, u_r). We set $u_r = W_i(\alpha)(u_l) = S_i(\alpha - \alpha_l) \circ R_i(\alpha_l)(u_l)$, where $\alpha_l = u_m^i - u_l^i$, $\alpha_r = u_r^i - u_m^i$, $\alpha = u_r^i - u_l^i$. Take $u_* \in LD_i \cap W_i(u_l)$. Then $\frac{d\alpha_l}{d\alpha}|_{W_i(\alpha)=u_*} = -\frac{1}{2}$.*

Lemma 3.5. *With the same assumption as in Lemma 3.4 we can obtain*

$$\left.\frac{d\lambda_i(R_i(\alpha_l))}{d\alpha}\right|_{W_i(\alpha)=u_*} = \frac{1}{4}\left.\frac{d\lambda_i(W_i(\alpha))}{d\alpha}\right|_{W_i(\alpha)=u_*}. \tag{3.3}$$

Proof. In the following, for simplicity of notation, we omit the dependency of S_i, R_i, W_i on u_l. The left contact shock satisfies the following equality:

$$\lambda_i(u_m)(u_m - u_r) - \left(f(u_m) - f(u_r)\right) = 0. \tag{3.4}$$

Simple calculation shows that

$$\nabla f\left(W_i(\alpha)\right)\dot{W}_i(\alpha) - \frac{d\alpha_l}{d\alpha}\nabla f\left(R_i(\alpha_l)\right)\dot{R}_i(\alpha_l)$$
$$= \nabla\lambda_i \cdot r_i\left(R_i(\alpha_l)\right)\frac{d\alpha_l}{d\alpha}\left(W_i(\alpha) - R_i(\alpha_l)\right) + \lambda_i\left(R_i(\alpha_l)\right)\left(\dot{W}_i(\alpha) - \frac{d\alpha_l}{d\alpha}\dot{R}_i(\alpha_l)\right).$$

Using the above equality we can deduce

$$\frac{d\lambda_i(R_i(\alpha_l))}{d\alpha} = \frac{[\lambda_i(W_i(\alpha)) - \lambda_i(R_i(\alpha_l))]l_i(W_i(\alpha)) \cdot \dot{W}_i(\alpha)}{\nabla_u\lambda_i(W_i(\alpha)) \cdot \dot{W}_i(\alpha)l_i(W_i(\alpha)) \cdot (W_i(\alpha) - R_i(\alpha_l))}\frac{d\lambda_i(W_i(\alpha))}{d\alpha}. \tag{3.5}$$

Using a similar calculation as that of [13] and Lemma 3.3 we obtain

$$\left.\frac{[\lambda_i(W_i(\alpha)) - \lambda_i(R_i(\alpha_l))]l_i(W_i(\alpha)) \cdot \dot{W}_i(\alpha)}{\nabla_u\lambda_i(W_i(\alpha)) \cdot \dot{W}_i(\alpha)l_i(W_i(\alpha)) \cdot (W_i(\alpha) - R_i(\alpha_l))}\right|_{W_i(\alpha)=u_*} = \frac{1}{4}. \quad \square$$

Now we are ready to give the proof of Theorem 2.2.

Proof of Theorem 2.2. As we know the difference between our definition of Glimm functional and the original definition in [12] is Q_s. We only need to estimate the change of $\mathcal{F}(t)$ at times of the interaction of the same family.

Suppose that there are i-waves α, β, ε_i and k-waves ε_k. α interacts with β at time t yielding i-wave γ and k-waves δ_k, $k \neq i$, as shown in Fig. 1:
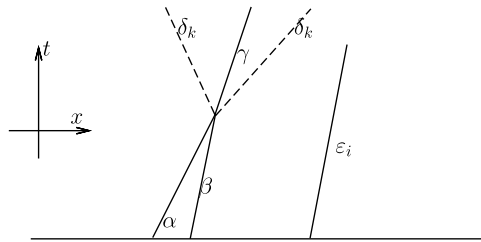
$$\alpha + \beta \to \gamma + \sum_{k \neq i}\delta_k.$$

Fig. 1. Wave interaction of the same family.

Case 1: $\alpha, \beta > 0$ or $\alpha, \beta < 0$, that is, α, β are in the same direction. From estimates of wave interaction in [12], we know that

$$|\gamma - \alpha - \beta| + \sum_{k \neq i} |\delta_k| = O(1)|\alpha||\beta|\left(\left|\Delta\lambda_i(\alpha)\right| + \left|\Delta\lambda_i(\beta)\right|\right) \tag{3.6}$$

and

$$\left|\Delta\lambda_i(\gamma)\right| - \left|\Delta\lambda_i(\alpha)\right| - \left|\Delta\lambda_i(\beta)\right| = O(1)|\alpha||\beta|\left(\left|\Delta\lambda_i(\alpha)\right| + \left|\Delta\lambda_i(\beta)\right|\right). \tag{3.7}$$

Then,

$$\begin{aligned}
\Delta Q(t) &= Q(t+) - Q(t-) \\
&= -X^{\frac{1}{2}}(t-)\left(|\alpha|\left|\Delta\lambda_i(\beta)\right| + |\beta|\left|\Delta\lambda_i(\alpha)\right|\right) + \sum_{\varepsilon_k} |\varepsilon_k|\left[|\gamma| - |\alpha| - |\beta|\right] + \sum_{\varepsilon_i} |\varepsilon_i|\left[\left(\left|\Delta\lambda_i(\gamma)\right|X^{\frac{1}{2}}(t+)\right.\right. \\
&\quad \left.\left. - \left(\left|\Delta\lambda_i(\alpha)\right| + \left|\Delta\lambda_i(\beta)\right|\right)X^{\frac{1}{2}}(t-) + \left|\Delta\lambda_i(\varepsilon_i)\right|\left(|\gamma|X^{\frac{1}{2}}(t+) - \left(|\alpha| + |\beta|\right)X^{\frac{1}{2}}(t-)\right)\right] \\
&\leqslant -\frac{1}{2}X^{\frac{1}{2}}(t-)\left(|\alpha|\left|\Delta\lambda_i(\beta)\right| + |\beta|\left|\Delta\lambda_i(\alpha)\right|\right).
\end{aligned}$$

Case 2: $\alpha > 0$, $\beta < 0$ and $\gamma = \sum_{k=1}^{m-1} \gamma_{ik} + \gamma_{im} > 0$, here $\sum_{k=1}^{m-1} \gamma_{ik} \doteq \gamma^r$, $\gamma_{im} \doteq \gamma^s$.

$$\begin{aligned}
\Delta Q(t) &= Q(t+) - Q(t-) \\
&\leqslant -X^{\frac{1}{2}}(t-)\left(|\alpha|\left|\Delta\lambda_i(\beta)\right| + |\beta|\left|\Delta\lambda_i(\alpha)\right|\right) + X^{\frac{1}{2}}(t+) \sum_{k,j=1}^{m-1}\left[|\gamma_{ik}|\left|\Delta\lambda_i(\gamma_{ij})\right| + |\gamma_{ij}|\left|\Delta\lambda_i(\gamma_{ik})\right|\right] \\
&\quad + X^{\frac{1}{2}}(t+) \sum_{k=1}^{m-1}\left[|\gamma_{ik}|\left|\Delta\lambda_i(\gamma_{im})\right| + |\gamma_{im}|\left|\Delta\lambda_i(\gamma_{ik})\right|\right] \\
&\leqslant -X^{\frac{1}{2}}(t-)\left(|\alpha|\left|\Delta\lambda_i(\beta)\right| + |\beta|\left|\Delta\lambda_i(\alpha)\right|\right) + \frac{3}{4}X^{\frac{1}{2}}(t+)|\beta|\left|\Delta\lambda_i(\alpha)\right| + \frac{1}{2}X^{\frac{1}{2}}(t+)|\alpha|\left|\Delta\lambda_i(\beta)\right| \\
&\leqslant 0.
\end{aligned}$$

In the above estimation we have used the results in Lemmas 3.4 and 3.5.

Finally, if M_0 is chosen large enough and the total variation of u_0 is suitably small, we obtain

$$\Delta\mathcal{F}(t) = \mathcal{F}(t+) - \mathcal{F}(t-) \leqslant 0.$$

This ends the proof. □

Remark 3.1. From the proof of Theorem 2.2, it is clear that $Q(t)$ is strictly decreasing with respect to t.

With the help of the Glimm functional and the above lemmas, we can obtain the following lemma, which will be used in the proof of Theorem 1.1.

Lemma 3.6. For any $t \geqslant t_1$, we have

$$X_i^{\pm}\left(\varsigma_i(t; t_0)\right) \leqslant O(1)Q^2(t_0), \tag{3.8}$$

$$X_i^+(t) \leqslant X_i^+\left(\xi_i(t'; t_0)\right) + O(1)O(1)\left[Q^2(t_0) + Q(t')\right], \tag{3.9}$$

$$Q(t_1) \leqslant Q_s(t_1) + O(1)Q(t_0)V(t_1). \tag{3.10}$$
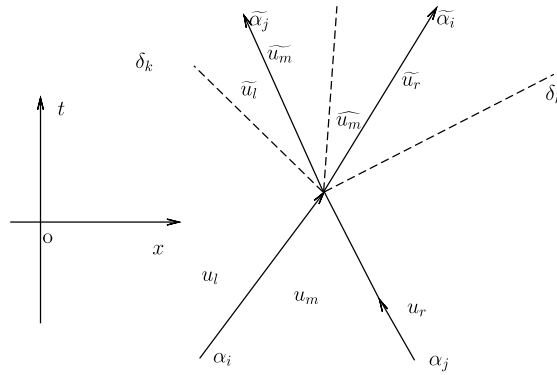
Here $t_0 < t' < t$.

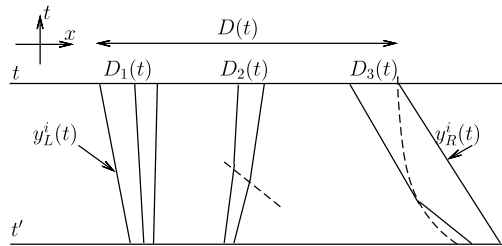**Fig. 2.** Wave interaction of different families.

Fig. 3. Estimate for rarefaction waves.

Proof. Clearly for $t \geqslant t_1$ the total amount of i-waves in $\varsigma_i(t; t_0)$ is $Q(t_0)$. With the help of Lemma 3.1 we can easily obtain (3.8). From the definition of $Q(t)$ and (3.8), we can deduce (3.10) directly. Now we demonstrate the proof of (3.9). From (3.8) we have

$$X_i^+(t) \leqslant X_i^+\big(\xi_i(t; t_0)\big) + O(1)Q^2(t_0). \tag{3.11}$$

Lemma 3.5 shows that $X_i^+(\xi_i(t; t_0)) \leqslant X_i^+(\xi_i(t'; t_0))$ provided that there is no wave interaction of different families. Otherwise, as in paper [13], assume that $\alpha_i = (u_l, u_m)$ is the wave of the i-th family, on the left of $\alpha_j = (u_m, u_r)$ $(i > j)$ and the interaction

$$\alpha_i + \alpha_j \to \widetilde{\alpha}_i + \widetilde{\alpha}_j + \sum_{k \neq i, j} \delta_k$$

takes place at time t with the location x, see Fig. 2. Use the notation $\widetilde{\alpha}_j \doteq (\widetilde{u}_l, \widetilde{u}_m)$, $\widetilde{\alpha}_i \doteq (\widehat{u}_m, \widetilde{u}_r)$.

Take $\widehat{u}_r \in W_i(\widehat{u}_m)$ such that $\alpha_i = \widehat{u}_r^i - \widehat{u}_m^i$. We know that $|\widehat{u}_r - \widetilde{u}_r| = O(1)|\alpha_i||\alpha_j|$. Then using Lemma 2.1 to (u_l, u_m) and $(\widehat{u}_m, \widehat{u}_r)$ we obtain

$$X_i^+\big(\xi_i(t; t_0)\big) \leqslant X_i^+\big(\xi_i(t'; t_0)\big) + O(1)Q(t') \leqslant X_i^+(t') + O(1)Q(t'). \tag{3.12}$$

Hence, the combination of (3.11) and (3.12) yields (3.10). □

Proof of Theorem 1.1. With the help of strict decreasing of $Q(t)$ the proof of (1.9) and (1.11) is similar to Lemma 3.6. We will not give it here. Now we prove the estimation (1.10). Let $D(t)$ be the distance between any backward generalized characteristic $y_L^i(t)$ and $y_R^i(t)$ at time t. Divide $D(t)$ into intervals $D_i(t)$ where λ_i is increasing.

For every D_j, if the characteristics extend from t' to t without crossing any waves, refer to D_1 in Fig. 3. Clearly

$$X_i^+\big(D_j(t)\big) \leqslant \frac{|D_j(t)|}{t - t'}. \tag{3.13}$$

If the backward characteristics do encounter waves of other families, refer to D_2 in Fig. 3. Lemma 2.1 shows that

$$X_i^+\big(D_j(t)\big) \leqslant \frac{|D_j(t)|}{t - t'} + O(1)Q(t'). \tag{3.14}$$

For the case of an i-rarefaction crossing an i-shock, refer to $D_3(t)$ in Fig. 3. From the result of Lemma 3.5 we know that

$$X_i^+\big(D_3(t)\big) \leqslant \left[\frac{1}{4} + O(1)V(0)\right] X_i^+\big(D_3(t')\big). \tag{3.15}$$

Combining (3.13), (3.14), and (3.15) and given that rarefaction waves arising from interaction can be bounded by $O(1)Q(t')$, the proof of the lemma is completed. □

Remark 3.2. In scalar conservation laws, for any $t' < t$, we have

$$X^+(t) = X^+\big(\xi(t)\big) \leqslant \frac{|\xi(t)|}{t - t'} + \left[\frac{1}{4} + O(1)V(0)\right] X^+\big(\xi(t')\big). \tag{3.16}$$

Here $\xi(t) = [Y_L(t), Y_R(t)]$, $Y_L(t)$, $Y_R(t)$ are the forward characteristics issuing from $(-N, 0)$ and $(N, 0)$ respectively.

4. Application

Applying the estimation in Section 3 and the following lemma we obtain the decay rates of $X(t)$ and $Q(t)$. The rates of decay can be stated as follows.

Theorem 4.1. *For total variation of the initial data $u(x, 0)$ sufficiently small, and $t > 0$, we have*

$$X(t) = O(1)t^{-1+\varrho}, \tag{4.1}$$

$$Q(t) = O(1)t^{-\frac{3}{2}+\frac{3}{2}\varrho}. \tag{4.2}$$

Here $\varrho = \frac{4+3\theta-\sqrt{9\theta^2-24\theta+4}}{6}$, $\theta > 0$ is a small fixed constant.

In order to obtain Theorem 4.1 we first give the following lemma.

Lemma 4.1. *Suppose that for all $t \leqslant T$ and T large enough, we have*

$$\max\big\{\big|\lambda_i\big(u(\cdot, t)\big) - \lambda_i(0)\big|\big\} \leqslant Ct^{-1+\varrho}, \tag{4.3}$$

$$X^+(t) \leqslant HCt^{-1+\varrho}, \tag{4.4}$$

$$Q(t) \leqslant (HC)^{\frac{3}{2}}t^{-\frac{3}{2}+\frac{3}{2}\varrho}, \tag{4.5}$$

where $C < 1$ and H is sufficiently large. Then for $t \leqslant 2T$ we have

$$\max\big\{\big|\lambda_i\big(u(\cdot, t)\big) - \lambda_i(0)\big|\big\} \leqslant C't^{-1+\varrho}, \tag{4.6}$$

$$X^+(t) \leqslant HC't^{-1+\varrho}, \tag{4.7}$$

$$Q(t) \leqslant (HC')^{\frac{3}{2}}t^{-\frac{3}{2}+\frac{3}{2}\varrho}, \tag{4.8}$$

for $C' = (1 + H_2T^{-\theta})C + H_2T^{-\theta}$, where $\varrho = \frac{4+3\theta-\sqrt{9\theta^2-24\theta+4}}{6}$, $\theta > 0$ is a small constant.

Proof. We prove the lemma by three steps.

Step 1. Since $u(x, t)$ is of compact support,

$$X^-(T) = X^+(T) + O(1)Q(\phi T), \tag{4.9}$$

so

$$X(T) = 2X^+(T) + O(1)Q(\phi T). \tag{4.10}$$

Here and in the following $\phi = \frac{1}{\mathcal{K}}$ is independent of t. Using Theorem 1.1 and the definition of $Q(t)$, for T large enough, we have

$$Q(T) \leqslant O(1)V(T)\big[X^{\frac{3}{2}}(T) + Q(\phi T)\big]. \tag{4.11}$$

From Remark 3.1, for $T \leqslant t \leqslant 2T$, we obtain

$$Q(t) \leqslant Q(T) \leqslant (HC)^{\frac{3}{2}}(2T)^{-\frac{3}{2}+\frac{3}{2}\varrho}\big[O(1)V(0)2^{\frac{3(1-\varrho)}{2}}\big(1 + \phi^{-\frac{3}{2}+\frac{3}{2}\varrho}\big)\big].$$

Choose $V(0)$ small enough, such that $O(1)V(0)2^{\frac{3(1-\varrho)}{2}}(1 + \phi^{-\frac{3}{2}+\frac{3}{2}\varrho}) \leqslant 1$. Now we complete the proof of (4.8).

Step 2. Let $T_* = T^\varrho \cdot T^{-\theta}$, $Y_L^i = (u_l^L, u_r^L)$, $Y_R^i = (u_l^R, u_r^R)$, and for $t \in [T_*, T]$.
By direct calculation, it is clear that

$$\frac{dY_R^i}{dt} - \lambda_i(0) \leqslant O(1)Q^2(\phi T_*). \tag{4.12}$$

Using (1.9), Lemmas 3.1 and 3.2 we have

$$\frac{dY_L^i}{dt} - \lambda_i(0) \geqslant \frac{1}{3}\left(\lambda_i(u_r^L) - \lambda_i(0)\right) - O(1)X^{\frac{3}{2}}(t) - O(1)Q^2(\phi T_*). \tag{4.13}$$

Thus we obtain

$$Y_R^i(T) - Y_L^i(T) = \int_{T_*}^{T}\left[\frac{dY_R^i}{dt} - \frac{dY_L^i}{dt}\right]dt + Y_R^i(T_*) - Y_L^i(T_*) \tag{4.14}$$

$$\leqslant \frac{1}{3\varrho}CT^\varrho + KT^\varrho T^{-\theta} + O(1)(HC)^3 T^\varrho T^{-\theta}.$$

From (1.10) for $T' \leqslant T$

$$X_i^+(T) \leqslant \frac{Y_R^i(T) - Y_L^i(T)}{T - T'} + \frac{1}{4}\left(1 + O(1)V(0)\right)X_i^+\left(\xi_i(T'; \phi T_*)\right) + O(1)\left[Q^2(\phi T_*) + Q(T')\right]. \tag{4.15}$$

Taking $T' = \xi T$ in (4.15) we obtain

$$X_i^+(T) \leqslant \frac{1}{3\varrho}C(1-\xi)^{-1}T^{-1+\varrho} + K(1-\xi)^{-1}T^{-1+\varrho}T^{-\theta}$$

$$+ \left(\frac{1}{4} + O(1)V(0)\right)HC(\xi T)^{-1+\varrho} + O(1)(HC)^3 T^{-1+\varrho}(1-\xi)^{-1}T^{-\theta}$$

$$= C(2T)^{-1+\varrho}\left[2^{1-\varrho}\left(\frac{1}{3\varrho}(1-\xi)^{-1} + T^{-\theta}\frac{K}{C}(1-\xi)^{-1} + O(1)H^3(1-\xi)^{-1}T^{-\theta}\right)\right.$$

$$\left. + \left(\frac{1}{4} + O(1)V(0)\right)H\left(\frac{2}{\xi}\right)^{1-\varrho}\right].$$

We choose T large enough, $V(0)$ small enough and ξ suitably close to 1 such that $\left(\frac{1}{4} + O(1)V(0)\right)\left(\frac{2}{\xi}\right)^{1-\varrho} < \frac{2}{3}$, then, there exists H satisfying $36\left[\frac{1}{3\varrho}(1-\xi)^{-1} + T^{-\theta}\frac{K}{C}(1-\xi)^{-1}\right] \leqslant H \leqslant \sqrt{\frac{(1-\xi)T^\theta}{36O(1)}}$, so $H \geqslant 18\left[\frac{1}{3\varrho}(1-\xi)^{-1} + T^{-\theta}\frac{K}{C}(1-\xi)^{-1} + O(1)H^3(1-\xi)^{-1}T^{-\theta}\right]$.

From the above calculation for $t \in [T, 2T]$ we can deduce

$$X_i^+(t) \leqslant X_i^+(T) + O(1)Q(T) \leqslant HC(2T)^{-1+\varrho}. \tag{4.16}$$

So far, we may take $C' = C$.

Step 3. Let $T_0 = T_*^{\frac{1}{\varrho}} = T \cdot T^{-\frac{\theta}{\varrho}}$. With the help of Lemmas 3.1 and 3.2, by direct calculation we have

$$\frac{dY_L^i(t)}{dt} - \lambda_i(0) \geqslant \frac{1}{3}\left[\lambda_i\left(u\left(Y_L^i(t)+, t\right)\right) - \lambda_i(0)\right] - O(1)\left(X^{\frac{3}{2}}(t) + Q^2(\phi T_*)\right)$$

$$\geqslant \frac{1}{3}\left[\lambda_i\left(u\left(Y_L^i(t)+, t\right)\right) - \lambda_i(0)\right] - O(1)\left[(HC)^3 T^{-1+\varrho-\theta} + (HC)^{\frac{3}{2}}t^{-\frac{3}{2}+\frac{3}{2}\varrho}\right].$$

Denote $Y(t)$ as $Y_L^i(t) - Y_R^i(T_0) - \lambda_i(0)(t - T_0)$. Then, from the above calculation we have

$$\frac{dY(t)}{dt} \geqslant \frac{1}{3}\frac{Y(t)}{t - T_0} - O(1)\left[(HC)^3 T^{-1+\varrho-\theta} + (HC)^{\frac{3}{2}}t^{-\frac{3}{2}+\frac{3}{2}\varrho}\right]. \tag{4.17}$$

Solving, we obtain for $t \geqslant T$

$$Y(t) \geqslant Y(T)\left(\frac{t - T_0}{T - T_0}\right)^{\frac{1}{3}} + O(1)T^{-1+\varrho-\theta}(t - T_0). \tag{4.18}$$
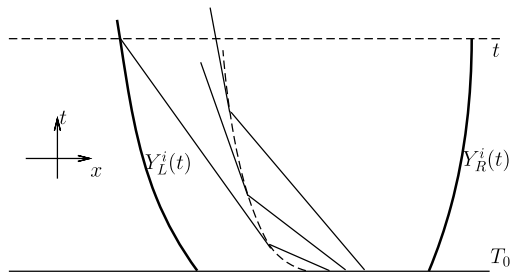
Fig. 4. $Y_L^i(t)$.

From Fig. 4 and combining (4.12), (4.13), (4.14), (4.18) we have for $T \leqslant t \leqslant 2T$

$$
\begin{aligned}
\lambda_i\big(u(x,t)\big) - \lambda_i(0) &\geqslant \frac{Y_L^i(t) - Y_R^i(T_0) - \lambda_i(0)(t - T_0)}{t - T_0} - O(1)Q^2(\phi T_*) \\
&\geqslant -C\big(1 + O(1)T^{-\theta}\big)t^{-1+\varrho} + O(1)T^{-1+\varrho-\theta} \\
&\geqslant -t^{-1+\varrho}\big[1 + H_2 T^{-\theta} C + H_2 T^{-\theta}\big].
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\lambda_i\big(u(x,t)\big) - \lambda_i(0) &\leqslant \frac{Y_R^i(t) - Y_L^i(T_0) - \lambda_i(0)(t - T_0)}{t - T_0} + O(1)Q(\phi T_*) \\
&\leqslant t^{-1+\varrho}\big[H_2 T^{-\theta} C + H_2 T^{-\theta}\big].
\end{aligned}
$$

For T and H_2 large enough, let $C' = (1 + H_2 T^{-\theta})C + H_2 T^{-\theta}$, we complete the proof of (4.6). □

Remark 4.2. In the proof of Lemma 4.1 we omit some lower order items as they are absorbed into higher order ones.

Proof of Theorem 4.1. The proof of this theorem is the same to the paper [14], we only give an outline of the proof here.
Use Lemma 4.1, and let $T_n = 2^n T$, $C_0 = C$. We need to prove $C_{n+1} = (1 + H_2 T_n^{-\theta})C_n + H_2 T_n^{-\theta} < 1$. In fact,

$$
C_{n+1} + 1 = \big(1 + H_2 T_n^{-\theta}\big)(C_n + 1). \tag{4.19}
$$

The above equality gives $C_n \leqslant C_0 + O(1)(C_0 + 1)T^{-\theta}$. Choosing T large enough and $C_0 < 1$, we obtain $C_n < 1$ for all n. □

Remark 4.3. From the proof of Lemma 4.1 and Theorem 4.1 we can see for scalar cubic nonlinear conservation laws the decay rates can be $X(t) = O(1)t^{-\frac{1}{3}}$, which is consistent with the result in [14].

Remark 4.4. For the genuinely nonlinear system of conservation laws, using the Q defined in Remark 2.1 and similar argument as in the main theorems we can deduce

$$
X_i^{\pm}\big(\varsigma_i(t; t_1)\big) \leqslant O(1)Q(t_1) \leqslant O(1)Q(t_0), \tag{4.20}
$$

$$
X_i^+(t) \leqslant \frac{|\xi_i(t; t_0)|}{t - t'} + O(1)Q(t_0), \tag{4.21}
$$

$$
Q(t_2) \leqslant X^3(t_2) + O(1)Q(t_0)X(t_2), \tag{4.22}
$$

for any $t \geqslant t_2$ and sufficiently small $V(0)$. Here $t_1 < t' < t$.
Using the above estimation and a similar argument as in Theorem 4.1 the following decay rates can be established.
For $t > 0$, we have the following estimations

$$
X(t) = O(1)t^{-1+\epsilon}, \tag{4.23}
$$

$$
Q(t) = O(1)t^{-3+3\epsilon}, \tag{4.24}
$$

provided the total variation of the initial data $u(x, 0)$ is small enough. Here $\epsilon = \frac{1}{2}$, which is consistent with the result in [17].

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