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Optimal time decay of the Boltzmann equation with frictional force

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ABSTRACT

In this paper, we use the combination of energy method and Fourier analysis to obtain the optimal time decay of the Boltzmann equation with frictional force towards equilibrium. Precisely speaking, we decompose the equation into macroscopic and microscopic partitions and perform the energy estimation. Then, we construct a special solution operator to a linearized equation without source term and use Fourier analysis to obtain the optimal decay rate to this solution operator. Finally, combining the decay rate with the energy estimation for nonlinear terms, the optimal decay rate to the Boltzmann equation with frictional force is established.

1. Introduction

In this paper we consider the Boltzmann equation with frictional force:

$$\partial_t f + \xi \cdot \nabla_x f - \alpha u \cdot \nabla_\xi f = Q(f, f), \tag{1.1}$$

with the initial data

$$f(0, x, \xi) = f_0(x, \xi). \tag{1.2}$$

Here, $f = f(x, t, \xi) \in \mathbb{R}$ represents the probability (mass, number) density of gas particles around position $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ with velocity $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ at time $t \in \mathbb{R}^+$. Q is the nonlinear collision operator for hard-sphere model which is defined by

$$Q(f, g) = \frac{1}{2} \int_{\mathbb{R}^3} \int_{S_+^2} \left(f(\xi') g(\xi'_*) + f(\xi'_*) g(\xi') - f(\xi) g(\xi_*) - f(\xi_*) g(\xi) \right) \left| (\xi - \xi_*) \cdot \Omega \right| d\xi_* \, d\Omega. \tag{1.3}$$

Here $S_+^2 := \{\Omega \in S^2 \mid (\xi - \xi_*) \cdot \Omega \geqslant 0\}$ and

$$\xi' = \xi - \left[(\xi - \xi_*) \cdot \Omega \right] \Omega, \tag{1.4}$$

$$\xi'_* = \xi_* + \left[(\xi - \xi_*) \cdot \Omega \right] \Omega, \tag{1.5}$$

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ξ, ξ_* are the velocities before collision while ξ', ξ'_* are the velocities after collision. In the damping term $-\alpha u\cdot\nabla_\xi f$, the frictional force $-\alpha u$ ($\alpha>0$) is proportional to the macroscopic velocity $u=u(x,t)=\frac{\int_{\mathbb{R}^3}\xi f\,d\xi}{\int_{\mathbb{R}^3} f\,d\xi}$. Without loss of generality, we take $\alpha=1$ throughout this paper.

It is well known that $\{1,\xi_1,\xi_2,\xi_3,\frac{1}{2}|\xi|^2\}$ are the five *collision invariants* satisfying

$$\int_{\mathbb{R}^3}\phi(\xi)Q(f,h)\,d\xi=0. \tag{1.6}$$

Clearly, the following global *Maxwellian* M is a stationary solution to (1.1),

$$M=\frac{1}{(2\pi)^{3/2}}\exp\bigl(-|\xi|^2/2\bigr). \tag{1.7}$$

We decompose f as

$$f=M+\sqrt{M}g. \tag{1.8}$$

Then the Boltzmann equation (1.1) can be reformulated into

$$\partial_t g+\xi\cdot\nabla_x g-u\cdot\nabla_\xi g+u\cdot\xi\sqrt{M}+\frac{1}{2}u\cdot\xi g=Lg+\Gamma(g,g), \tag{1.9}$$

where L is the *linearized collision operator* and Γ is the corresponding nonlinear collision operator, given by

$$Lg=\frac{1}{\sqrt{M}}\bigl(Q(M,\sqrt{M}g)+Q(\sqrt{M}g,M)\bigr),$$
$$\Gamma(g,g)=\frac{1}{\sqrt{M}}Q(\sqrt{M}g,\sqrt{M}g). \tag{1.10}$$

Now we consider the Cauchy problem of (1.9) with the corresponding initial data

$$g(0,x,\xi)=g_0(x,\xi)=\frac{1}{\sqrt{M}}\bigl(f_0(x,\xi)-M\bigr),\quad (x,\xi)\in\mathbb{R}^3\times\mathbb{R}^3. \tag{1.11}$$

It can be mathematically derived that, from (1.1) we can obtain a fluid-type system for the macroscopic components whose leading term is the Euler system with frictional force which models the compressible flow through porous media. Since Euler system with frictional force has been extensively studied in [13], etc., we thought it would be interesting to consider our model.

Before starting our own discussion, let's mention some of the recent work related to our work.

In the fundamental work [10] and [11,12], the energy method for the Boltzmann equation in the whole space was independently developed. Later in [8] a new energy method for the Boltzmann equation was introduced. Based on a refined energy method, in [2] the author proved the global existence and uniform-in-time stability of the solution in the space $L^2_\xi(H^N_x)$ to the Cauchy problem for the Boltzmann equation around a global Maxwellian, in that paper the author introduced a free energy functional to control the macroscopic part of the solution, which was of the same spirit with Kawashima's compensation function in the Fourier space [9].

As for the decay rates problem, the first breakthrough was made by Ukai in [14], where the author used the spectral analysis to obtain a exponential rates for the Boltzmann equation with hard potentials on torus. In [15] the method was improved to cope with the existence of time-periodic states with time-periodic sources. In [16] convergence rate to stationary solutions for Boltzmann equation with external force was given by combining the dissipation from the viscosity and heat conductivity on the fluid components and the dissipation on the non-fluid component through the celebrated H-theorem. Recently in [6], a new approach is introduced by combining the energy method and spectral analysis to study the optimal convergence rates of several gas motions. Later in [1] the author combine the energy method and Fourier analysis to obtain the optimal time decay of the Vlasov–Poisson–Boltzmann system in $\mathbb{R}^3$. In addition, the optimal convergence rates of the Navier–Stokes equation is studied in [3,4] and references therein.

As for the Boltzmann equation with frictional force (1.1), it is firstly investigated in [7] where the existence near the global Maxwellian is proved by using the nonlinear energy method developed in [11,12]. In this paper we use the different energy method given in [10] to prove the global existence and the uniform stability. Then, combining with the Fourier analysis we obtain the convergence rate of the solution to (1.1) towards the stationary solution M.

Before giving our main results, we list some notations that will be used in our paper. $\langle\cdot,\cdot\rangle$ denotes the inner product in the Hilbert space $L^2(\mathbb{R}^3\times\mathbb{R}^3)$ or $L^2(\mathbb{R}^3)$ without any ambiguity. And in the following we use $\|\cdot\|$ to denote the corresponding L^2 norm. When the norms need to be distinguished from each other, we write $\|\cdot\|_{L^2(\mathbb{R}^3_\xi)}$, $\|\cdot\|_{L^2(\mathbb{R}^3_x)}$ and $\|\cdot\|_{L^2(\mathbb{R}^3_x\times\mathbb{R}^3_\xi)}$ respectively. Set

$$\langle g,h\rangle_\nu\equiv\bigl\langle\nu(\xi)g,h\bigr\rangle,$$

for any functions $g = g(x,\xi)$ and $h = h(x,\xi)$ to be the weighted inner product in $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, and use $\|\cdot\|_\nu$ for the corresponding weighted L^2 norm. We use H^N to denote the Sobolev space $H^N(\mathbb{R}^3 \times \mathbb{R}^3)$, and H^N_ν to denote the space $H^N(\mathbb{R}^3 \times \mathbb{R}^3; \nu(\xi)\,dx\,d\xi)$. For the multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, we denote

$$\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}, \quad \text{and} \quad |\alpha| = \sum_{i=1}^{3} \alpha_i.$$

For simplicity, we use ∂_i to denote ∂_{x_i} for each $i = 1, 2, 3$.

In order to state our main results later, we need to define the energy functional:

$$\big[\big[g(t)\big]\big]^2 \equiv \sum_{|\alpha|+|\beta|\leqslant N} \big\|\partial_x^\alpha \partial_\xi^\beta g(t)\big\|^2, \tag{1.12}$$

and the dissipation rate

$$\big[\big[g(t)\big]\big]_\nu^2 \equiv \sum_{|\alpha|+|\beta|\leqslant N} \big\|\partial_x^\alpha \partial_\xi^\beta g_2(t)\big\|_\nu^2 + \sum_{0<|\alpha|\leqslant N} \big\|\partial_x^\alpha g_1(t)\big\|_\nu^2 + \|b\|^2, \tag{1.13}$$

where $N \geqslant 4$ is an integer. Functionals g_1, g_2 and b will be defined in the next section. In addition, C denotes a generic positive (generally large) constant and λ denotes a generic positive (generally small) constant. They may be different from line to line.

We also use $\hat{g}$ to denote the Fourier transform of g, $(a|b) = a \cdot \bar{b}$ denotes the complex inner product of complex vectors a and b.

The main results of this paper can be stated as follows:

Theorem 1.1. *There exist $\delta_0 > 0$, $\lambda_0 > 0$ and $C_0 > 0$ such that if $g_0(x,\xi)$ satisfies $[[g_0]] \leqslant \delta_0$, then there exists a unique global solution $g(x,t,\xi)$ to the Cauchy problem* (1.9) *and* (1.11) *such that*

$$\big[\big[g(t)\big]\big]^2 + \lambda_0 \int_0^t \big[\big[g(s)\big]\big]_\nu^2\,ds \leqslant C_0[[g_0]]^2, \quad \forall t \geqslant 0.$$

Theorem 1.2. *Suppose all the conditions in Theorem* 1.1 *hold. Let $g = g(x,t,\xi)$ and $h = h(x,t,\xi)$ be two solutions to the Boltzmann equation* (1.9) *corresponding to given initial data $g_0(x,\xi)$ and $h_0(x,\xi)$, then there exist δ_1 $(0 < \delta_1 < \delta_0)$, $\lambda_1 > 0$ and $C_1 > 0$ such that, if*

$$\max\big\{\big[\big[g(0)\big]\big], \big[\big[h(0)\big]\big]\big\} \leqslant \delta_1$$

then $g(x,t,\xi)$, $h(x,t,\xi)$ satisfy

$$\big[\big[g(t)-h(t)\big]\big]^2 + \lambda_1 \int_0^t \big[\big[g(s)-h(s)\big]\big]_\nu^2\,ds \leqslant C_1\big[\big[g(0)-h(0)\big]\big]^2, \quad \forall t \geqslant 0.$$

Theorem 1.3. *Suppose $g_0(x,\xi)$ satisfies $\|g_0\|_{H^N}$ and $\|g_0\|_\nu$ small enough, then, g enjoys the estimate with algebraic decay rate in time*:

$$\big\|g(t)\big\|_{H^N} \leqslant C\|g_0\|_{H^N\cap Z_1}(1+t)^{-\frac{3}{4}}.$$

Here $Z_q = L^2_\xi(L^q_x)$, $\|g\|_{Z_q} = (\int(\int |g|^q\,dx)^{\frac{2}{q}}\,d\xi)^{\frac{1}{2}}$.

Remark 1. The proof of the global existence is based on the energy method by combining the local existence and the closure of the a priori estimate. The uniform-in-time stability of the global solutions is proved in the completely same way. Here we only need to close the a priori estimate. That is, under the a priori assumption that $[[g(t)]]$ is very small, say,

$$\big[\big[g(t)\big]\big] \leqslant \delta, \tag{1.14}$$

where δ is a sufficiently small positive constant, we want to prove that there exist functionals $\mathcal{E}(g(t))$ and $\mathcal{D}(g(t))$ which are equivalent to $[[g(t)]]^2$ and $[[g(t)]]_\nu^2$ respectively, such that

$$\frac{d}{dt}\mathcal{E}\big(g(t)\big) + \lambda\mathcal{D}\big(g(t)\big) \leqslant 0, \tag{1.15}$$

The rest of the paper is organized as follows. In the next section, we review some basic properties of the linearized operator L and nonlinear operator Γ, and give the macro–micro decomposition in Section 2. We perform the energy estimates and prove Theorem 1.1 and 1.2 in Section 3. The optimal convergence rate in Theorem 1.3 is given in the last section.

2. Some basic properties and macro–micro decomposition

We know that the properties of L and Γ are very crucial in energy estimation for the Boltzmann equation. We list them here for later use.

(1) $L = -\nu(\xi) + K$, where $\nu(\xi)$ is a nonnegative measurable function called the *collision frequency*, while K is a self-adjoint compact operator on $L^2(\mathbb{R}^3)$ with a real symmetric integral kernel.
(2) There exists a constant $\nu_0 > 0$ such that for $\forall\xi$, $\nu_0^{-1}(1+|\xi|) \leqslant \nu(\xi) \leqslant \nu_0(1+|\xi|)$.
(3) The null space of the operator L is the 5-dimensional space of collision invariants:

$$\mathcal{N} = \operatorname{Ker} L = \operatorname{span}\left\{\sqrt{M};\ \xi_i\sqrt{M},\ i=1,2,3;\ |\xi|^2\sqrt{M}\right\}.$$

(4) Following from the Boltzmann's H-theorem, L is self-adjoint and non-positive in $L^2(\mathbb{R}^3)$. Furthermore, there exists a constant $\lambda > 0$ such that:

$$-\int_{\mathbb{R}^3} gLg\,d\xi \geqslant \lambda \int_{\mathbb{R}^3} \nu(\xi)\big(\{\mathbf{I}-\mathbf{P}\}g\big)^2\,d\xi, \quad \forall g \in D(L), \tag{2.1}$$

where $\mathbf{P}$ denotes the projection operator from $L^2(\mathbb{R}^3)$ to $\mathcal{N}$ and $D(L)$ is the domain of L given by $D(L) = \{g \in L^2(\mathbb{R}^3) \mid \nu(\xi)g \in L^2(\mathbb{R}^3)\}$.

(5)

$$\begin{aligned} &\big|\langle\Gamma(g,h),w\rangle\big| \leqslant C\left\{\int_{\mathbb{R}^3}\big\|\nu^{1/2}g\big\|_{L^2_\xi}\|h\|_{L^2_\xi}\big\|\nu^{1/2}w\big\|_{L^2_\xi}\,dx + \int_{\mathbb{R}^3}\big\|\nu^{1/2}h\big\|_{L^2_\xi}\|g\|_{L^2_\xi}\big\|\nu^{1/2}w\big\|_{L^2_\xi}\,dx\right\}, \\ &\big\|\langle\Gamma(g,h),w\rangle\big\|_{L^2_x} + \big\|\langle\Gamma(h,g),w\rangle\big\|_{L^2_x} \leqslant C\big\|\nu^3 w\big\|_{L^\infty_{x,\xi}}\|g\|_{L^\infty_x(L^2_\xi)}\|h\|. \end{aligned} \tag{2.2}$$

Now we give the macro–micro decomposition to prepare for the later energy estimates

$$\begin{cases} g(t,x,\xi) = g_1 + g_2, \\ g_1 = \mathbf{P}g \in \mathcal{N}, \\ g_1 = \left\{a(t,x) + \displaystyle\sum_{i=1}^3 b_i(t,x)\xi_i + c(t,x)|\xi|^2\right\}\sqrt{M}, \\ g_2 = g - g_1 = (\mathbf{I}-\mathbf{P})g \in \mathcal{N}^\perp. \end{cases} \tag{2.3}$$

Then Eq. (1.9) can be rewritten as

$$\partial_t g_1 + \xi\cdot\nabla_x g_1 - u\cdot\nabla_\xi g_1 + u\cdot\xi\sqrt{M} + \frac{1}{2}u\cdot\xi g_1 = r + l + h, \tag{2.4}$$

with

$$\begin{aligned} &r = -\partial_t g_2, \\ &l = -\xi\cdot\nabla_x g_2 + u\cdot\nabla_\xi g_2 - \frac{1}{2}u\cdot\xi g_2 + Lg_2, \\ &h = \Gamma(g,g). \end{aligned}$$

Next we derive the evolution equations for (a,b,c). In fact, by putting $(2.3)_3$ into (2.4) and collecting the coefficients w.r.t. the basis $\{e_k,\ k=13\}$ consisting of

$$\sqrt{M}, \quad (\xi_i\sqrt{M})_{1\leqslant i\leqslant 3}, \quad \big(|\xi_i|^2\sqrt{M}\big)_{1\leqslant i\leqslant 3}, \quad (\xi_i\xi_j\sqrt{M})_{1\leqslant i<j\leqslant 3}, \quad \big(|\xi|^2\xi_i\sqrt{M}\big)_{1\leqslant i\leqslant 3}, \tag{2.5}$$

we have the following macroscopic equations on the coefficients (a,b,c) of g_1:

$$\begin{aligned} &\partial_t a - u\cdot b = \gamma^{(0)}, \\ &\partial_t b_i + \partial_i a + u_i a - 2u_i c + u_i = \gamma_i^{(1)}, \\ &\partial_t c + \partial_i b_i + u_i b_i = \gamma_i^{(2)}, \\ &\partial_i b_j + \partial_j b_i + u_i b_j + u_j b_i = \gamma_{ij}^{(2)}, \\ &\partial_i c + c u_i = \gamma_i^{(3)}, \\ &\quad \text{where } i \neq j,\ \text{with } u = \frac{b}{1+a+3c}. \end{aligned} \tag{2.6}$$

All terms on the right are the coefficients of $r + l + h$ w.r.t. the corresponding elements in the basis $\{e_{13}\}$ are given by:

$$\begin{aligned}
&\gamma^{(0)} = -\partial_t \bar{r}^{(0)} + l^{(0)} + h^{(0)},\\
&\gamma_i^{(1)} = -\partial_t \bar{r}_i^{(1)} + l_i^{(1)} + h_i^{(1)},\\
&\gamma_i^{(2)} = -\partial_t \bar{r}_i^{(2)} + l_i^{(2)} + h_i^{(2)},\\
&\gamma_{ij}^{(2)} = -\partial_t \bar{r}_{ij}^{(2)} + l_{ij}^{(2)} + h_{ij}^{(2)},\\
&\gamma_i^{(3)} = -\partial_t \bar{r}_i^{(3)} + l_i^{(3)} + h_i^{(3)},\\
&\quad \text{where } i \neq j, \text{ with } r = -\partial_t \bar{r}.
\end{aligned} \tag{2.7}$$

We cite the following lemma for later use, it says that the coefficients of the separated part $\bar{r}$, the linear part l and the nonlinear part h can be bounded by the microscopic dissipation rate.

Lemma 2.1. *(See [5].) It holds that*

$$\sum_{|\alpha|\leqslant N-1} \sum_{ij} \left\| \partial_x^\alpha \left[\bar{r}^{(0)}, \bar{r}_i^{(1)}, \bar{r}_i^{(2)}, \bar{r}_{ij}^{(2)}, \bar{r}_i^{(3)} \right] \right\|^2 \leqslant C \sum_{|\alpha|\leqslant N-1} \left\| \partial_x^\alpha g_2 \right\|^2,$$
$$\sum_{|\alpha|\leqslant N-1} \sum_{ij} \left\| \partial_x^\alpha \left[l^{(0)}, l_i^{(1)}, l_i^{(2)}, l_{ij}^{(2)}, l_i^{(3)} \right] \right\|^2 \leqslant C \sum_{|\alpha|\leqslant N} \left\| \partial_x^\alpha g_2 \right\|^2,$$
$$\sum_{|\alpha|\leqslant N} \sum_{ij} \left\| \partial_x^\alpha \left[h^{(0)}, h_i^{(1)}, h_i^{(2)}, h_{ij}^{(2)}, h_i^{(3)} \right] \right\|^2 \leqslant C \left[\left[g(t) \right]\right]^2 \left[\left[g(t) \right]\right]_\nu^2.$$

Based on $(2.6)_3$ and $(2.6)_4$, the macroscopic component $b = (b_1, b_2, b_3)$ satisfies an elliptic-type equation

$$-\Delta_x b_j - \partial_j \partial_j b_j = -\sum_{i\neq j} \partial_i \left(\gamma_{ij}^{(2)} - (u_i b_j + u_j b_i) \right) + \sum_{i\neq j} \partial_j \left(\gamma_i^{(2)} - u_i b_i \right) - 2\partial_j \left(\gamma_j^{(2)} - u_j b_j \right). \tag{2.8}$$

Also, (a, b, c) satisfies the local macroscopic balance laws. In fact, multiplying Eq. (1.1) by the collision invariants $1, \xi, \frac{1}{2}|\xi|^2$ and integrating the products on ξ over $\mathbb{R}^3$ we have

$$\begin{aligned}
&\partial_t \int f\, d\xi + \nabla_x \cdot \int \xi f\, d\xi = 0,\\
&\partial_t \int \xi f\, d\xi + \nabla_x \cdot \int \xi \otimes \xi f\, d\xi = -u \int f\, d\xi,\\
&\partial_t \int \frac{1}{2}|\xi|^2 f\, d\xi + \nabla_x \cdot \int \frac{1}{2}|\xi|^2 \xi f\, d\xi = -u \cdot \int \xi f\, d\xi.
\end{aligned}$$

By using the perturbation (1.8) and the decomposition (2.3), direct calculation gives the macroscopic balance laws on the coefficients (a, b, c) of g_1:

$$\begin{aligned}
&\partial_t (a + 3c) + \nabla_x \cdot b = 0,\\
&\partial_t b + \nabla_x (a + 5c) + \nabla_x \cdot \int \xi \otimes \xi \sqrt{M} g_2\, d\xi = -b,\\
&\partial_t (3a + 15c) + 5\nabla_x \cdot b + \nabla_x \cdot \int |\xi|^2 \xi \sqrt{M} g_2\, d\xi = -2u \cdot b.
\end{aligned}$$

These can be easily rewritten as

$$\partial_t a - \nabla_x \cdot \int \frac{1}{2}|\xi|^2 \xi \sqrt{M} g_2\, d\xi = u \cdot b, \tag{2.9}$$

$$\partial_t b_i + \partial_i (a + 5c) + \nabla_x \cdot \int \xi \xi_i \sqrt{M} g_2\, d\xi = -b_i, \tag{2.10}$$

$$\partial_t c + \frac{1}{3} \nabla_x \cdot b + \frac{1}{3} \nabla_x \cdot \int \frac{1}{2}|\xi|^2 \xi \sqrt{M} g_2\, d\xi = -\frac{1}{3} u \cdot b. \tag{2.11}$$

It is straightforward to calculate that, those terms containing the microscopic part g_2 can be bounded by the microscopic dissipation rate:

Lemma 2.2. *(See [5].) It holds that*

$$\sum_{|\alpha|\leqslant N-1}\big\|\partial_x^\alpha\nabla_x\cdot\big[\big\langle|\xi|^2\xi\sqrt{M},g_2\big\rangle,\langle\xi\otimes\xi\sqrt{M},g_2\rangle\big]\big\|^2\leqslant C\sum_{0<|\alpha|\leqslant N}\big\|\partial_x^\alpha g_2\big\|^2.$$

3. Energy estimates

Part 1. First we estimate the microscopic component g_2 by proving the following lemma.

Lemma 3.1.

(1) *Estimates on the zero-th order*:

$$\frac{1}{2}\frac{d}{dt}\big\|g(t)\big\|^2+C\|b\|^2+\lambda\big\|g_2(t)\big\|_\nu^2\leqslant C\big[\big[g(t)\big]\big]\big[\big[g(t)\big]\big]_\nu^2. \tag{3.1}$$

(2) *Estimates on the spatial derivatives*:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{1\leqslant|\alpha|\leqslant N}\big\|\partial_x^\alpha g\big\|^2+C\sum_{1\leqslant|\alpha|\leqslant N}\big\|\partial_x^\alpha b\big\|^2+\lambda\sum_{1\leqslant|\alpha|\leqslant N}\big\|\partial_x^\alpha g_2\big\|_\nu^2\\&\quad\leqslant C\big[\big[g(t)\big]\big]\big[\big[g(t)\big]\big]_\nu^2+C\delta\sum_{1\leqslant|\alpha|\leqslant N}\big\|\partial_x^\alpha(a,b,c)\big\|^2+C\delta\sum_{1\leqslant|\alpha|\leqslant N-1}\big\|\partial_x^\alpha\nabla_\xi g_2\big\|^2.\end{aligned} \tag{3.2}$$

(3) *Estimates on the mixed derivatives*:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{\substack{|\beta|=k\\|\alpha|+|\beta|\leqslant N}}\big\|\partial_x^\alpha\partial_\xi^\beta g_2\big\|^2+\lambda\sum_{\substack{|\beta|=k\\|\alpha|+|\beta|\leqslant N}}\big\|\partial_x^\alpha\partial_\xi^\beta g_2\big\|_\nu^2\\&\quad\leqslant C\big[\big[g(t)\big]\big]\big[\big[g(t)\big]\big]_\nu^2+C\sum_{|\alpha|\leqslant N-k+1}\big\|\partial_x^\alpha g_2\big\|_\nu^2+C_{\chi_{\{k\geqslant2\}}}\sum_{\substack{1\leqslant|\beta|\leqslant k-1\\|\alpha|+|\beta|\leqslant N}}\big\|\partial_x^\alpha\partial_\xi^\beta g_2\big\|_\nu^2\\&\qquad+C\sum_{|\alpha|\leqslant N-k}\big\|\partial_x^\alpha\nabla_x(a,b,c)\big\|^2+C\sum_{0\leqslant|\alpha|\leqslant N-k}\big\|\partial_x^\alpha b\big\|^2,\end{aligned} \tag{3.3}$$

where the integer $1\leqslant k\leqslant N$ and $\chi_{\{k\geqslant2\}}$ is the characteristic function of the set $\{k\geqslant2\}$.

Proof. In the proof we use the a priori assumption (1.14) and the equivalent relationship between u and b

$$u=\frac{b}{1+a+3c}. \tag{3.4}$$

Also, the following Sobolev inequality is frequently used in performing our energy estimates

$$\|g\|_{L^\infty}\leqslant C\|\nabla g\|^{\frac{1}{2}}\|\nabla^2 g\|^{\frac{1}{2}}\leqslant C\|\nabla g\|_{H^1}.$$

(1) To prove (3.1), we multiply (1.9) by g and take the integration over $\mathbb{R}^3\times\mathbb{R}^3$:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\iint g^2\,d\xi\,dx+\frac{1}{2}\iint\xi\cdot\nabla_x g^2\,d\xi\,dx-\frac{1}{2}\iint u\cdot\nabla_\xi g^2\,d\xi\,dx+\iint u\cdot\xi\sqrt{M}g\,d\xi\,dx+\frac{1}{2}\iint u\cdot\xi g^2\,d\xi\,dx\\&\quad=\iint gLg\,d\xi\,dx+\iint g\Gamma(g,g)\,d\xi\,dx.\end{aligned} \tag{3.5}$$

First we notice $\iint\xi\cdot\nabla_x g^2\,d\xi\,dx=\iint u\cdot\nabla_\xi g^2\,d\xi\,dx=0$. Then we use (2.1) and (2.2) to obtain the estimates involving Lg and $\Gamma(g,g)$. Terms involving the frictional force u are estimated as follows:

$$\begin{aligned}&\iint u\cdot\xi\sqrt{M}g\,d\xi\,dx=\int u\cdot b\,dx=\int\frac{|b|^2}{1+a+3c}\,dx\sim C\|b\|^2,\\&\frac{1}{2}\iint u\cdot\xi g^2\,d\xi\,dx=\frac{1}{2}\iint u\cdot\xi g_1^2\,d\xi\,dx+\iint u\cdot\xi g_1g_2\,d\xi\,dx+\frac{1}{2}\iint u\cdot\xi g_2^2\,d\xi\,dx,\end{aligned}$$

using (1.14) and (3.4), direct calculation gives

$$\frac{1}{2}\iint u\cdot\xi g_1^2\,d\xi\,dx=\frac{1}{2}\int(a+5c)u\cdot b\,dx=\frac{1}{2}\int\frac{a+5c}{1+a+3c}|b|^2\,dx\leqslant C\delta\|b\|^2,$$

using the Cauchy inequality and again the a priori assumption (1.14) together with (3.4) we have

$$\iint u\cdot\xi g_1 g_2\,d\xi\,dx\leqslant C\delta\|g_2\|_\nu^2+C\delta\|b\|^2,$$

at last use (3.4) to obtain

$$\frac{1}{2}\iint u\cdot\xi g_2^2\,d\xi\,dx\leqslant C\delta\|g_2\|_\nu^2.$$

Combining estimates of all terms we obtain (3.1).

(2) To prove (3.2), we take derivatives ∂_x^α $(1\leqslant|\alpha|\leqslant N)$ of (1.9) and multiply the resulting equation by $\partial_x^\alpha g$, then integrate over $\mathbb{R}^3\times\mathbb{R}^3$:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\partial_x^\alpha g\right\|^2+\iint\partial_x^\alpha g\xi\cdot\nabla_x\left(\partial_x^\alpha g\right)dx\,d\xi-\iint\partial_x^\alpha g\partial_x^\alpha(u\cdot\nabla_\xi g)\,dx\,d\xi\\&\quad+\iint\partial_x^\alpha g\partial_x^\alpha u\cdot\xi\sqrt{M}\,dx\,d\xi+\frac{1}{2}\iint\partial_x^\alpha g\partial_x^\alpha(u\cdot\xi g)\,dx\,d\xi\\&=\iint\partial_x^\alpha gL\partial_x^\alpha g\,dx\,d\xi+\iint\partial_x^\alpha g\sum_{|\beta|\leqslant|\alpha|}C_\alpha^\beta\Gamma\left(\partial_x^\beta g,\partial_x^{\alpha-\beta}g\right)dx\,d\xi.\end{aligned}\tag{3.6}$$

The right-hand side can be estimated using (2.1) and (2.2) easily as before. The second term on the left-hand side equals to 0. The rest terms we give the detailed estimation as follows:

(i)

$$\begin{aligned}\iint\partial_x^\alpha g\partial_x^\alpha(u\cdot\nabla_\xi g)\,dx\,d\xi&=\iint\partial_x^\alpha gu\cdot\nabla_\xi\left(\partial_x^\alpha g\right)dx\,d\xi+\iint\partial_x^\alpha g\sum_{1\leqslant|\beta|\leqslant|\alpha|}C_\alpha^\beta\partial_x^\beta u\cdot\nabla_\xi\left(\partial_x^{\alpha-\beta}g\right)dx\,d\xi\\&:=\sum_{i=1}^4 I_i.\end{aligned}$$

Using the explicit expression of g_1 in $(2.3)_3$ we have $\nabla_\xi g_1=(b+2c\xi)\sqrt{M}-\frac{1}{2}\xi g_1$, this together with the a priori assumption (1.14), (3.4) and the Cauchy inequality, we do the calculation as follows:

$$\begin{aligned}I_1:\quad&\iint\partial_x^\alpha g_1\sum_{1\leqslant|\beta|\leqslant|\alpha|}C_\alpha^\beta\partial_x^\beta u\cdot\nabla_\xi\left(\partial_x^{\alpha-\beta}g_1\right)dx\,d\xi\\&=\int\sum_{1\leqslant|\beta|\leqslant|\alpha|}C_\alpha^\beta\partial_x^\beta u\cdot\partial_x^{\alpha-\beta}b\partial_x^\alpha(a+3c)\,dx+2\int\sum_{1\leqslant|\beta|\leqslant|\alpha|}C_\alpha^\beta\partial_x^\beta u\cdot\partial_x^\alpha b\partial_x^{\alpha-\beta}c\,dx\\&\quad-\frac{1}{2}\int\sum_{1\leqslant|\beta|\leqslant|\alpha|}C_\alpha^\beta\partial_x^\beta u\cdot\left[\partial_x^{\alpha-\beta}b\partial_x^\alpha(a+5c)+\partial_x^\alpha b\partial_x^{\alpha-\beta}(a+5c)\right]dx\\&\leqslant C\delta\sum_{1\leqslant|\beta|\leqslant|\alpha|}\left\|\partial_x^\beta(a,b,c)\right\|^2+C\delta\sum_{1\leqslant|\beta|\leqslant|\alpha|}\left\|\partial_x^\beta b\right\|^2,\end{aligned}$$

$$\begin{aligned}I_2:\quad&\iint\partial_x^\alpha g_1\sum_{1\leqslant|\beta|\leqslant|\alpha|}C_\alpha^\beta\partial_x^\beta u\cdot\nabla_\xi\left(\partial_x^{\alpha-\beta}g_2\right)dx\,d\xi\\&=\iint\partial_x^\alpha g_1\partial_x^\alpha u\cdot\nabla_\xi g_2\,dx\,d\xi+\iint\partial_x^\alpha g_1\sum_{1\leqslant|\beta|\leqslant|\alpha|-1}C_\alpha^\beta\partial_x^\beta u\cdot\nabla_\xi\partial_x^{\alpha-\beta}g_2\,dx\,d\xi\\&\leqslant C\delta\sum_{1\leqslant|\beta|\leqslant|\alpha|}\left\|\partial_x^\beta(a,b,c)\right\|^2+C\delta\sum_{1\leqslant|\beta|\leqslant|\alpha|-1}\left\|\partial_x^\beta\nabla_\xi g_2\right\|^2,\end{aligned}$$

$$\begin{aligned}I_3:\quad&\iint\partial_x^\alpha g_2\sum_{1\leqslant|\beta|\leqslant|\alpha|}C_\alpha^\beta\partial_x^\beta u\cdot\nabla_\xi\left(\partial_x^{\alpha-\beta}g_1\right)dx\,d\xi\\&=\iint\partial_x^\alpha g_2\sum_{1\leqslant|\beta|\leqslant|\alpha|}C_\alpha^\beta\partial_x^\beta u\cdot\partial_x^{\alpha-\beta}\left((b+2c\xi)\sqrt{M}-\frac{1}{2}\xi g_1\right)dx\,d\xi\\&\leqslant C\delta\sum_{1\leqslant|\beta|\leqslant|\alpha|}\left\|\partial_x^\beta b\right\|^2+C\delta\left\|\partial_x^\alpha g_2\right\|_\nu^2+C\delta\sum_{1\leqslant|\beta|\leqslant|\alpha|}\left\|\partial_x^\beta(a,b,c)\right\|^2,\end{aligned}$$

$$I_4:\quad \iint \partial_x^\alpha g_2 \sum_{1\leqslant|\beta|\leqslant|\alpha|} C_\alpha^\beta \partial_x^\beta u\cdot\nabla_\xi\big(\partial_x^{\alpha-\beta} g_2\big)\,dx\,d\xi$$
$$= \iint \partial_x^\alpha g_2 \partial_x^\alpha u\cdot\nabla_\xi g_2\,dx\,d\xi + \iint \partial_x^\alpha g_2 \sum_{1\leqslant|\beta|\leqslant|\alpha|-1} C_\alpha^\beta \partial_x^\beta u\cdot\partial_x^{\alpha-\beta}\nabla_\xi g_2\,dx\,d\xi$$
$$\leqslant C\delta\big\|\partial_x^\alpha b\big\|^2 + C\delta\big\|\partial_x^\alpha g_2\big\|_\nu^2 + C\delta \sum_{1\leqslant|\beta|\leqslant|\alpha|-1}\big\|\partial_x^\beta\nabla_\xi g_2\big\|^2 + C\delta\sum_{1\leqslant|\beta|\leqslant|\alpha|}\big\|\partial_x^\beta(a,b,c)\big\|^2.$$

Combining I_i, $i=1,2,3,4$ we have

$$\iint \partial_x^\alpha g\partial_x^\alpha(u\cdot\nabla_\xi g)\,dx\,d\xi \leqslant C\delta\big\|\partial_x^\alpha g_2\big\|_\nu^2 + C\delta\sum_{1\leqslant|\beta|\leqslant|\alpha|-1}\big\|\partial_x^\beta\nabla_\xi g_2\big\|^2 + C\delta\sum_{1\leqslant|\beta|\leqslant|\alpha|}\big\|\partial_x^\beta(a,b,c)\big\|^2. \tag{3.7}$$

(ii)

$$\iint \partial_x^\alpha g\partial_x^\alpha u\cdot\xi\sqrt{M}\,dx\,d\xi = \iint \partial_x^\alpha g_1\partial_x^\alpha u\cdot\xi\sqrt{M}\,dx\,d\xi$$
$$= \int \frac{\partial_x^\alpha b\cdot\partial_x^\alpha b}{1+a+3c}\,dx + \int\bigg[\partial_x^\alpha u - \frac{\partial_x^\alpha b}{1+a+3c}\bigg]\cdot\partial_x^\alpha b\,dx$$
$$= I_5 + I_6, \tag{3.8}$$

$$I_5 \sim \big\|\partial_x^\alpha b\big\|^2,\qquad |I_6|\leqslant C\delta\sum_{1\leqslant|\beta|\leqslant|\alpha|}\big\|\partial_x^\beta(a,b,c)\big\|^2.$$

(iii)

$$\frac12\iint \partial_x^\alpha g\partial_x^\alpha(u\cdot\xi g)\,dx\,d\xi = \frac12\iint u\cdot\xi\big(\partial_x^\alpha g\big)^2\,dx\,d\xi + \frac12\iint \partial_x^\alpha g\xi\cdot\sum_{1\leqslant|\beta|\leqslant|\alpha|} C_\alpha^\beta\partial_x^\beta u\partial_x^{\alpha-\beta} g\,dx\,d\xi$$
$$\leqslant C\delta\sum_{1\leqslant|\beta|\leqslant|\alpha|}\big\|\partial_x^\beta(a,b,c)\big\|^2 + C\delta\sum_{1\leqslant|\beta|\leqslant|\alpha|}\big\|\partial_x^\alpha g_2\big\|_\nu^2. \tag{3.9}$$

In fact, in the above inequality we have used

$$\frac12\iint u\cdot\xi\big(\partial_x^\alpha g\big)^2\,dx\,d\xi \leqslant \frac12\iint |u\cdot\xi|\big(\partial_x^\alpha g_1\big)^2\,dx\,d\xi + \frac12\iint |u\cdot\xi|\big(\partial_x^\alpha g_2\big)^2\,dx\,d\xi$$
$$\leqslant C\delta\big\|\partial_x^\alpha(a,b,c)\big\|^2 + C\delta\big\|\partial_x^\alpha g_2\big\|_\nu^2.$$

The estimates of the other partitions are similar to those on $I_i, i=1,2,3,4$, only much simpler.
Put (3.7), (3.8), (3.9) into (3.6) and take the summation w.r.t. $|\alpha|$ from 1 to N thus we have (3.2) proved.

(3) To prove (3.3), first we rewrite (1.9) into

$$\partial_t g_2 + \xi\cdot\nabla_x g_2 - u\cdot\nabla_\xi g_2 + u\cdot\xi\sqrt{M} + \frac12 u\cdot\xi g_2$$
$$= Lg_2 + \Gamma(g,g) - \partial_t g_1 - \xi\cdot\nabla_x g_1 + u\cdot\nabla_\xi g_1 - \frac12 u\cdot\xi g_1, \tag{3.10}$$

to (3.10), we take ∂_ξ^β with $|\beta|=k$ $(1\leqslant k\leqslant N)$ and then ∂_x^α, multiply the resulting equation by $\partial_x^\alpha\partial_\xi^\beta g_2$ and take integrations over $\mathbb{R}^3\times\mathbb{R}^3$. Following the same argument as the proof to (3.1) and (3.2), that is, using the a priori assumption (1.14) and (3.4), together with the Cauchy inequality, (3.3) can be proved easily. But one thing we would like to point out is that, we need to use the macroscopic balance laws (2.9), (2.10) and (2.11), then with the help of Lemma 2.2, we could remove the time derivative $\partial_t g_1$. This completes the proof of Lemma 3.1. □

Now the dissipation rate of the microscopic component can be obtained by a suitable linear combination of (3.1), (3.2) and (3.3). In fact, the summation of (3.1) and (3.2) gives

$$\frac{1}{2}\frac{d}{dt}\sum_{|\alpha|\leqslant N}\left\|\partial_x^\alpha g\right\|^2 + C\sum_{|\alpha|\leqslant N}\left\|\partial_x^\alpha b\right\|^2 + \lambda\sum_{|\alpha|\leqslant N}\left\|\partial_x^\alpha g_2\right\|_\nu^2$$
$$\leqslant C\big[\big[g(t)\big]\big]\big[\big[g(t)\big]\big]_\nu^2 + C\delta\sum_{1\leqslant|\alpha|\leqslant N}\left\|\partial_x^\alpha(a,b,c)\right\|^2 + C\delta\sum_{1\leqslant|\alpha|\leqslant N-1}\left\|\partial_x^\alpha\nabla_\xi g_2\right\|^2. \tag{3.11}$$

The linear combination of (3.3) with k taking values from 1 to N gives

$$\frac{1}{2}\frac{d}{dt}\sum_{\substack{|\alpha|+|\beta|\leqslant N\\|\beta|\geqslant 1}}\left\|\partial_x^\alpha\partial_\xi^\beta g_2\right\|^2 + \lambda\sum_{\substack{|\alpha|+|\beta|\leqslant N\\|\beta|\geqslant 1}}\left\|\partial_x^\alpha\partial_\xi^\beta g_2\right\|_\nu^2$$
$$\leqslant C\big[\big[g(t)\big]\big]\big[\big[g(t)\big]\big]_\nu^2 + C\sum_{|\alpha|\leqslant N}\left\|\partial_x^\alpha g_2\right\|_\nu^2 + C\sum_{|\alpha|\leqslant N-1}\left\|\partial_x^\alpha\nabla_x(a,b,c)\right\|^2 + C\sum_{|\alpha|\leqslant N-1}\left\|\partial_x^\alpha b\right\|^2, \tag{3.12}$$

where the energy functional is actually

$$\sum_{k=1}^N C_{N,k}\sum_{\substack{|\beta|=k\\|\alpha|+|\beta|\leqslant N}}\left\|\partial_x^\alpha\partial_\xi^\beta g_2\right\|^2,$$

for some properly chosen $C_{N,k}>0$, in order to eliminate the term

$$C\chi_{\{k\geqslant 2\}}\sum_{\substack{1\leqslant|\beta|\leqslant k-1\\|\alpha|+|\beta|\leqslant N}}\left\|\partial_x^\alpha\partial_\xi^\beta g_2\right\|_\nu^2$$

on the right-hand side of (3.3).

At last, the linear combination of (3.11) and (3.12) gives

$$\frac{1}{2}\frac{d}{dt}\Bigg(\sum_{|\alpha|\leqslant N}\left\|\partial_x^\alpha g\right\|^2 + \sum_{\substack{|\alpha|+|\beta|\leqslant N\\|\beta|\geqslant 1}}\left\|\partial_x^\alpha\partial_\xi^\beta g_2\right\|^2\Bigg) + \lambda\sum_{|\alpha|+|\beta|\leqslant N}\left\|\partial_x^\alpha\partial_\xi^\beta g_2\right\|_\nu^2 + C\sum_{|\alpha|\leqslant N}\left\|\partial_x^\alpha b\right\|^2$$
$$\leqslant C\big[\big[g(t)\big]\big]\big[\big[g(t)\big]\big]_\nu^2 + C\sum_{|\alpha|\leqslant N-1}\left\|\partial_x^\alpha\nabla_x(a,b,c)\right\|^2. \tag{3.13}$$

Part 2. Now we turn to the estimation on the macroscopic dissipation rate, the main result of this part is the following lemma:

Lemma 3.2.

$$\frac{d}{dt}\mathcal{G}\big(g(t)\big) + \sum_{|\alpha|\leqslant N-1}\left\|\partial_x^\alpha\nabla_x(a,b,c)\right\|^2 \leqslant C\Bigg\{\sum_{|\alpha|\leqslant N}\left\|\partial_x^\alpha g_2\right\|^2 + \big[\big[g(t)\big]\big]^2\big[\big[g(t)\big]\big]_\nu^2 + \sum_{|\alpha|\leqslant N-1}\left\|\partial_x^\alpha b\right\|^2\Bigg\}. \tag{3.14}$$

Here,

$$\mathcal{G}\big(g(t)\big) := \sum_{|\alpha|\leqslant N-1}\sum_{i=1}^3\big[\mathcal{G}_{\alpha,i}^a\big(g(t)\big) + \mathcal{G}_{\alpha,i}^b\big(g(t)\big) + \mathcal{G}_{\alpha,i}^c\big(g(t)\big) + \mathcal{G}_{\alpha,i}^{ab}\big(g(t)\big)\big],$$

with

$$\mathcal{G}_{\alpha,i}^a\big(g(t)\big) = \big\langle\partial_x^\alpha\bar{r}_i^{(1)},\partial_i\partial_x^\alpha a\big\rangle,$$
$$\mathcal{G}_{\alpha,i}^b\big(g(t)\big) = -\sum_{j\neq i}\big\langle\partial_x^\alpha\bar{r}_j^{(2)},\partial_i\partial_x^\alpha b_i\big\rangle + \sum_{j\neq i}\big\langle\partial_x^\alpha\bar{r}_{ji}^{(2)},\partial_j\partial_x^\alpha b_i\big\rangle + 2\big\langle\partial_x^\alpha\bar{r}_i^{(2)},\partial_i\partial_x^\alpha b_i\big\rangle,$$
$$\mathcal{G}_{\alpha,i}^c\big(g(t)\big) = \big\langle\partial_x^\alpha\bar{r}_i^{(3)},\partial_i\partial_x^\alpha c\big\rangle,$$
$$\mathcal{G}_{\alpha,i}^{ab}\big(g(t)\big) = \big\langle\partial_x^\alpha b_i,\partial_i\partial_x^\alpha a\big\rangle.$$

$\mathcal{G}_{\alpha,i}^a(g(t))$, $\mathcal{G}_{\alpha,i}^b(g(t))$, and $\mathcal{G}_{\alpha,i}^c(g(t))$ stand for the interactive energy functionals between the microscopic part g_2 with the macroscopic part a, b and c separately, while $\mathcal{G}_{\alpha,i}^{ab}(g(t))$ stands for the interactive energy functional between a and b.

Proof. *Estimates on b.* Applying ∂_x^α with $|\alpha| \leqslant N-1$ to (2.8), multiplying it by $\partial_x^\alpha b_j$ and then integrating it over $\mathbb{R}^3$, simply integrating by parts we can have

$$\begin{aligned}&\left\|\nabla_x\partial_x^\alpha b_j\right\|^2+\left\|\partial_j\partial_x^\alpha b_j\right\|^2+\frac{d}{dt}\mathcal{G}^b_{\alpha,j}\big(g(t)\big)\\&\quad\leqslant C\varepsilon\left\{\left\|\partial_j\partial_x^\alpha(a,b,c)\right\|^2+C\delta\sum_{1\leqslant|\alpha_1|\leqslant|\alpha|}\left\|\partial_x^{\alpha_1}(a,b,c)\right\|^2+\|b\|^2+\left\|\partial_x^\alpha\nabla_x\cdot\langle\xi_j\xi\sqrt{M},g_2\rangle\right\|^2\right\}\\&\qquad+C\left\{\sum_{ij}\left\|\nabla_x\partial_x^\alpha\big[\bar r_i^{(2)},\bar r_{ij}^{(2)},\bar r_j^{(2)}\big]\right\|^2+\sum_{ij}\left\|\partial_x^\alpha\big[l_i^{(2)},l_{ij}^{(2)},h_i^{(2)},h_{ij}^{(2)}\big]\right\|^2\right\},\end{aligned}\tag{3.15}$$

then we take the summation of (3.15) w.r.t. α over $|\alpha| \leqslant N-1$ and j over $\{1,2,3\}$, and using Lemma 2.1 and Lemma 2.2 we have

$$\begin{aligned}&\frac{d}{dt}\sum_{|\alpha|\leqslant N-1}\sum_{j=1}^3\mathcal{G}^b_{\alpha,j}\big(g(t)\big)+\sum_{|\alpha|\leqslant N-1}\left\|\nabla_x\partial_x^\alpha b\right\|^2\\&\quad\leqslant C\varepsilon\sum_{|\alpha|\leqslant N-1}\left\|\nabla_x\partial_x^\alpha(a,b,c)\right\|^2+C\varepsilon\|b\|^2+C\left\{\sum_{|\alpha|\leqslant N}\left\|\partial_x^\alpha g_2\right\|^2+\big[\big[g(t)\big]\big]^2\big[\big[g(t)\big]\big]_\nu^2\right\}.\end{aligned}\tag{3.16}$$

Estimates on c. We use the macroscopic equations about c, $(2.6)_5$ and (2.11), through direct calculation can obtain:

$$\begin{aligned}&\frac{d}{dt}\sum_{|\alpha|\leqslant N-1}\sum_{j=1}^3\mathcal{G}^c_{\alpha,j}\big(g(t)\big)+\sum_{|\alpha|\leqslant N-1}\left\|\nabla_x\partial_x^\alpha c\right\|^2\\&\quad\leqslant C\varepsilon\sum_{|\alpha|\leqslant N-1}\left\|\nabla_x\partial_x^\alpha(a,b,c)\right\|^2+C\delta\sum_{|\alpha|\leqslant N-1}\left\|\partial_x^\alpha b\right\|^2+C\left\{\sum_{|\alpha|\leqslant N}\left\|\partial_x^\alpha g_2\right\|^2+\big[\big[g(t)\big]\big]^2\big[\big[g(t)\big]\big]_\nu^2\right\}.\end{aligned}\tag{3.17}$$

Estimates on a. Similar to the estimation on c, using the macroscopic equations about a, $(2.6)_2$ and (2.9), we can obtain

$$\begin{aligned}&\frac{d}{dt}\sum_{|\alpha|\leqslant N-1}\sum_{j=1}^3\big[\mathcal{G}^a_{\alpha,j}\big(g(t)\big)+\mathcal{G}^{ab}_{\alpha,j}\big(g(t)\big)\big]+\sum_{|\alpha|\leqslant N-1}\left\|\nabla_x\partial_x^\alpha a\right\|^2\\&\quad\leqslant C\varepsilon\sum_{|\alpha|\leqslant N-1}\left\|\nabla_x\partial_x^\alpha(a,b,c)\right\|^2+C\sum_{|\alpha|\leqslant N-1}\left\|\partial_x^\alpha b\right\|^2+C\left\{\sum_{|\alpha|\leqslant N}\left\|\partial_x^\alpha g_2\right\|^2+\big[\big[g(t)\big]\big]^2\big[\big[g(t)\big]\big]_\nu^2\right\}.\end{aligned}\tag{3.18}$$

The lemma is proved by combining (3.16), (3.17) and (3.18). □

Now we are fully prepared to prove our first two main theorems.

Proof of Theorem 1.1. We only need to close the a priori estimate (1.15). The linear combination of (3.11) and (3.14) gives

$$\begin{aligned}&\frac{d}{dt}\left[\frac{A}{2}\sum_{|\alpha|\leqslant N}\left\|\partial_x^\alpha g(t)\right\|^2+\mathcal{G}\big(g(t)\big)\right]+CA\sum_{|\alpha|\leqslant N}\left\|\partial_x^\alpha b\right\|^2+\lambda\left(\sum_{|\alpha|\leqslant N}\left\|\partial_x^\alpha g_2\right\|_\nu^2+\sum_{|\alpha|\leqslant N-1}\left\|\nabla_x\partial_x^\alpha(a,b,c)\right\|^2\right)\\&\quad\leqslant C\big(\big[\big[g(t)\big]\big]+\big[\big[g(t)\big]\big]^2\big)\big[\big[g(t)\big]\big]_\nu^2+C\delta\sum_{1\leqslant|\alpha|\leqslant N-1}\left\|\partial_x^\alpha\nabla_\xi g_2\right\|^2,\end{aligned}\tag{3.19}$$

where $A>0$ is large enough so that the energy functional in (3.19) is equivalent to its first part $\sum_{|\alpha|\leqslant N}\|\partial_x^\alpha g(t)\|^2$. Finally, recall the definitions of the norms $[[g(t)]]$ and $[[g(t)]]_\nu$, the linear combination of (3.13) and (3.19) gives the Lyapunov inequality:

$$\frac{d}{dt}\mathcal{E}\big(g(t)\big)+\lambda\mathcal{D}\big(g(t)\big)\leqslant C\sqrt{\mathcal{E}\big(g(t)\big)}\mathcal{D}\big(g(t)\big),\tag{3.20}$$

where the energy functions satisfy

$$\begin{aligned}\mathcal{E}\big(g(t)\big)&\sim\sum_{|\alpha|\leqslant N}\left\|\partial_x^\alpha g\right\|^2+\sum_{\substack{|\alpha|+|\beta|\leqslant N\\|\beta|\geqslant1}}\left\|\partial_x^\alpha\partial_\xi^\beta g_2\right\|^2+\frac{A}{2}\sum_{|\alpha|\leqslant N}\left\|\partial_x^\alpha g(t)\right\|^2+\mathcal{G}\big(g(t)\big)\\&\sim\big[\big[g(t)\big]\big]^2,\end{aligned}$$

$$\mathcal{D}\big(g(t)\big) \sim \sum_{|\alpha|\leqslant N} \big\|\partial_x^\alpha g_2\big\|_\nu^2 + \sum_{|\alpha|\leqslant N-1} \big\|\nabla_x \partial_x^\alpha (a,b,c)\big\|^2 + CA \sum_{|\alpha|\leqslant N} \big\|\partial_x^\alpha b\big\|^2 + \sum_{|\alpha|+|\beta|\leqslant N} \big\|\partial_x^\alpha \partial_\xi^\beta g_2\big\|_\nu^2$$
$$\sim \big[\big[g(t)\big]\big]_\nu^2.$$

Thus the a priori estimate (1.15) is closed. Take integration over $[0,t]$ to (1.15) gives

$$\mathcal{E}\big(g(t)\big) + \lambda \int_0^t \mathcal{D}\big(g(s)\big)\,ds \leqslant \mathcal{E}\big(g(0)\big). \tag{3.21}$$

We now complete the proof of Theorem 1.1. □

Proof of Theorem 1.2. To prove Theorem 1.2, we set

$$w(x,t,\xi) = g(x,t,\xi) - h(x,t,\xi).$$

Then $w = w(x,t,\xi)$ satisfies

$$\partial_t w + \xi \cdot \nabla_x w - u \cdot \nabla_\xi w + \frac{1}{2} u \cdot \xi w = Lw + \Gamma(w,g) + \Gamma(h,w). \tag{3.22}$$

Perform the same procedure as in the proof of Theorem 1.1, we can easily obtain the similar Lyapunov-type inequality of w:

$$\frac{d}{dt}\mathcal{E}\big(w(t)\big) + \lambda \mathcal{D}\big(w(t)\big) \leqslant C\big\{\big[\big[g(t)\big]\big]_\nu^2 + \big[\big[h(t)\big]\big]_\nu^2\big\}\mathcal{E}\big(w(t)\big),$$

use the result of Theorem 1.1 to g and h, then Gronwall's inequality implies that

$$\mathcal{E}\big(w(t)\big) + \lambda \int_0^t \mathcal{D}\big(w(s)\big)\,ds \leqslant C\mathcal{E}\big(w(0)\big).$$

Thus Theorem 1.2 is proved. □

4. Optimal convergence rate

In order to obtain the convergence rate, we define e^{tB} as the solution operator to the Cauchy problem (4.1) with $h \equiv 0$. Now, we consider the Cauchy problem

$$\begin{cases} \partial_t g + \xi \cdot \nabla_x g + b \cdot \xi \sqrt{M} = Lg + h, \\ g(x,0,\xi) = g_0(x,\xi), \end{cases} \tag{4.1}$$

with the additional assumption $\mathbf{P}h = 0$.

Similar procedure as in Section 2 we can rewrite (4.1) into the form:

$$\partial_t (a + 3c) + \nabla_x \cdot b = 0, \tag{4.2}$$
$$\partial_t b_j + \partial_j (a + 3c) + 2\partial_j c + \sum_m \partial_m A_{jm}\big(\{\mathbf{I} - \mathbf{P}\}g\big) + b_j = 0, \tag{4.3}$$
$$\partial_t c + \frac{1}{3}\nabla_x \cdot b + \frac{1}{6}\sum_j \partial_j B_j\big(\{\mathbf{I} - \mathbf{P}\}g\big) = 0, \tag{4.4}$$

and

$$\partial_t\big[A_{jj}\big(\{\mathbf{I} - \mathbf{P}\}g\big) + 2c\big] + 2\partial_j b_j = A_{jj}(R + h), \tag{4.5}$$
$$\partial_t\big[A_{jm}\big(\{\mathbf{I} - \mathbf{P}\}g\big)\big] + \partial_j b_m + \partial_m b_j = A_{jm}(R + h), \quad j \neq m \tag{4.6}$$
$$\partial_t\big[B_j\big(\{\mathbf{I} - \mathbf{P}\}g\big)\big] + 10\partial_j c = B_j(R + h), \tag{4.7}$$

here $1 \leqslant j, m \leqslant 3$,

$$A_{jm}(g) := \big\langle (\xi_j \xi_m - 1)\sqrt{M}, g\big\rangle,$$
$$B_j(g) := \big\langle \big(|\xi|^2 - 5\big)\xi_j \sqrt{M}, g\big\rangle,$$

and

$$R = -\xi \cdot \nabla_x \{\mathbf{I} - \mathbf{P}\} g + L\{\mathbf{I} - \mathbf{P}\} g.$$

It is initially observed in [10] and later in [1] that, with direct calculation we can obtain

$$\begin{aligned}
&-\partial_t\Bigg[\sum_j \partial_j A_{jm}\big(\{\mathbf{I} - \mathbf{P}\} g\big) + \frac{1}{2}\partial_m A_{mm}\big(\{\mathbf{I} - \mathbf{P}\} g\big)\Bigg] - \Delta_x b_m - \partial_m \partial_m b_m \\
&\quad = \frac{1}{2}\sum_{j \neq m} \partial_m A_{jj}(R + h) - \sum_j \partial_j A_{jm}(R + h), \qquad (4.8)
\end{aligned}$$

for some fixed m.

First let's prove the following two lemmas, Lemma 4.1 and Lemma 4.2, they will later be used in the proof of Lemma 4.3, which is crucial in obtaining the decay rate in Theorem 1.3.

Lemma 4.1. *If $E(\hat{g})$ takes the form*

$$\begin{aligned}
E(\hat{g}) &= \kappa_1 \sum_m \Bigg(\sum_j \frac{ik_j}{1 + |k|^2} A_{jm}\big(\{\mathbf{I} - \mathbf{P}\}\hat{g}\big) + \frac{1}{2}\frac{ik_m}{1 + |k|^2} A_{mm}\big(\{\mathbf{I} - \mathbf{P}\}\hat{g}\big)\Big| - \hat{b}_m \Bigg) \\
&\quad + \kappa_1 \sum_j \Bigg(B_j\big(\{\mathbf{I} - \mathbf{P}\}\hat{g}\big)\Big|\frac{ik_j}{1 + |k|^2}\hat{c}\Bigg) + \sum_m \Bigg(\hat{b}_m \Big| \frac{ik_m}{1 + |k|^2}(\hat{a} + 3\hat{c})\Bigg), \qquad (4.9)
\end{aligned}$$

for some $\kappa_1 > 0$, then for any $t \geqslant 0$ and $k \in \mathbb{R}^3_k$, we obtain

$$\partial_t \operatorname{Re} E(\hat{g}) + \lambda \frac{|k|^2}{1 + |k|^2}\big(|\hat{b}|^2 + |\hat{c}|^2 + |\hat{a} + 3\hat{c}|^2\big) \leqslant C\big(\big\|\{\mathbf{I} - \mathbf{P}\}\hat{g}\big\|^2_{L^2_\xi} + \big\|\nu^{-\frac{1}{2}}\{\mathbf{I} - \mathbf{P}\}\hat{h}\big\|^2_{L^2_\xi}\big) + C|\hat{b}|^2. \qquad (4.10)$$

Proof. *Estimate on $\hat{b}$*, for $0 < \delta_1 < 1$ we claim

$$\begin{aligned}
&\partial_t \operatorname{Re}\Bigg[\sum_m \Bigg(\sum_j ik_j A_{jm}\big(\{\mathbf{I} - \mathbf{P}\}\hat{g}\big) + \frac{1}{2} ik_m A_{mm}\big(\{\mathbf{I} - \mathbf{P}\}\hat{g}\big)| - \hat{b}_m\Bigg)\Bigg] + (1 - \delta_1)|k|^2|\hat{b}|^2 \\
&\quad \leqslant \frac{C}{\delta_1}\big(1 + |k|^2\big)\big(\big\|\{\mathbf{I} - \mathbf{P}\}\hat{g}\big\|^2_{L^2_\xi} + \big\|\nu^{-\frac{1}{2}}\{\mathbf{I} - \mathbf{P}\}\hat{h}\big\|^2_{L^2_\xi}\big) + C\delta_1|k|^2\big(|\hat{a} + 3\hat{c}|^2 + |\hat{c}|^2\big). \qquad (4.11)
\end{aligned}$$

In fact, perform the Fourier transform to (4.8) and take the complex inner product with $\hat{b}_m$, we obtain

$$\begin{aligned}
&\Bigg(-\partial_t\Bigg[\sum_j ik_j A_{jm}\big(\{\mathbf{I} - \mathbf{P}\}\hat{g}\big) + \frac{1}{2} ik_m A_{mm}\big(\{\mathbf{I} - \mathbf{P}\}\hat{g}\big)\Bigg] + |k|^2\hat{b}_m + k_m^2\hat{b}_m\Big|\hat{b}_m\Bigg) \\
&\quad = \Bigg(\frac{1}{2}\sum_{j \neq m} ik_m A_{jj}(\hat{R} + \hat{h}) - \sum_j ik_j A_{jm}(\hat{R} + \hat{h})\Big|\hat{b}_m\Bigg),
\end{aligned}$$

then rearranging the equation we have

$$\begin{aligned}
&\partial_t\Bigg(\Bigg[\sum_j ik_j A_{jm}\big(\{\mathbf{I} - \mathbf{P}\}\hat{g}\big) + \frac{1}{2} ik_m A_{mm}\big(\{\mathbf{I} - \mathbf{P}\}\hat{g}\big)\Bigg]\Big| - \hat{b}_m\Bigg) + |k|^2\hat{b}_m^2 + k_m^2\hat{b}_m^2 \\
&\quad = \Bigg(\frac{1}{2}\sum_{j \neq m} ik_m A_{jj}(\hat{R} + \hat{h}) - \sum_j ik_j A_{jm}(\hat{R} + \hat{h})\Big|\hat{b}_m\Bigg) + \Bigg(\Bigg[\sum_j ik_j A_{jm}\big(\{\mathbf{I} - \mathbf{P}\}\hat{g}\big) + \frac{1}{2} ik_m A_{mm}\big(\{\mathbf{I} - \mathbf{P}\}\hat{g}\big)\Bigg]\Big| - \partial_t\hat{b}_m\Bigg).
\end{aligned}$$

We use the Fourier transform of (4.3) to eliminate the time derivative $\partial_t\hat{b}_m$, and notice the following estimates on A_{jm}:

$$\begin{aligned}
&\big|A_{jm}\big(\{\mathbf{I} - \mathbf{P}\}\hat{g}\big)\big|^2 \leqslant C\big\|\{\mathbf{I} - \mathbf{P}\}\hat{g}\big\|^2_{L^2_\xi}, \\
&\big|A_{jm}(\hat{h})\big|^2 \leqslant C\big\|\nu^{-\frac{1}{2}}\{\mathbf{I} - \mathbf{P}\}\hat{h}\big\|^2_{L^2_\xi}, \\
&\big|A_{jm}(\hat{R})\big|^2 \leqslant C\big(1 + |k|^2\big)\big\|\{\mathbf{I} - \mathbf{P}\}\hat{g}\big\|^2_{L^2_\xi},
\end{aligned}$$

then through the Cauchy inequality we complete the estimation on $\hat{b}$.

Estimate on $\hat{c}$, for $0 < \delta_2 < 1$ we obtain

$$\partial_t \operatorname{Re}\left[\sum_j \big(B_j\big(\{\mathbf{I}-\mathbf{P}\}\hat{g}\big)\Big|ik_j\hat{c}\big)\right] + (1-\delta_2)|k|^2|\hat{c}|^2$$
$$\leqslant \frac{C}{\delta_2}\big(1+|k|^2\big)\big(\big\|\{\mathbf{I}-\mathbf{P}\}\hat{g}\big\|^2_{L^2_\xi} + \big\|\nu^{-\frac{1}{2}}\{\mathbf{I}-\mathbf{P}\}\hat{h}\big\|^2_{L^2_\xi}\big) + \delta_2|k|^2|\hat{b}|^2. \tag{4.12}$$

In fact, Fourier transformation of (4.7) gives

$$\partial_t\big[B_j\big(\{\mathbf{I}-\mathbf{P}\}\hat{g}\big)\big] + 10ik_j\hat{c} = B_j(\hat{R}+\hat{h}),$$

then, taking complex inner product with $ik_j\hat{c}$ we obtain

$$\partial_t\big(B_j\big(\{\mathbf{I}-\mathbf{P}\}\hat{g}\big)\big|ik_j\hat{c}\big) + 10k_j^2|\hat{c}|^2 = \big(B_j(\hat{R}+\hat{h})\big|ik_j\hat{c}\big) + \big(B_j\big(\{\mathbf{I}-\mathbf{P}\}\hat{g}\big)\big|ik_j\partial_t\hat{c}\big).$$

Using the Fourier transformation of (4.4) to eliminate $\partial_t\hat{c}$, taking summation over $1 \leqslant j \leqslant 3$ and using Cauchy inequality we can obtain (4.12). Here we used the following estimates on B_j:

$$\big|B_j\big(\{\mathbf{I}-\mathbf{P}\}\hat{g}\big)\big|^2 \leqslant C\big\|\{\mathbf{I}-\mathbf{P}\}\hat{g}\big\|^2_{L^2_\xi},$$
$$\big|B_j(\hat{h})\big|^2 \leqslant C\big\|\nu^{-\frac{1}{2}}\{\mathbf{I}-\mathbf{P}\}\hat{h}\big\|^2_{L^2_\xi},$$
$$\big|B_j(\hat{R})\big|^2 \leqslant C\big(1+|k|^2\big)\big\|\{\mathbf{I}-\mathbf{P}\}\hat{g}\big\|^2_{L^2_\xi}.$$

Estimate on $\hat{a}+3\hat{c}$, similarly, for $0 < \delta_3 < 1$ we have

$$\partial_t \operatorname{Re}\sum_m \big(\hat{b}_m\big|ik_m(\hat{a}+3\hat{c})\big) + (1-\delta_3)|k|^2|\hat{a}+3\hat{c}|^2 \leqslant \frac{C}{\delta_3}|k|^2\big\|\{\mathbf{I}-\mathbf{P}\}\hat{g}\big\|^2_{L^2_\xi} + \frac{C}{\delta_3}|k|^2|\hat{c}|^2 + \frac{C}{\delta_3}|\hat{b}|^2 + |k|^2|\hat{b}|^2. \tag{4.13}$$

We complete the proof by combining (4.11), (4.12) and (4.13). □

Lemma 4.2.

$$\partial_t\|\hat{g}\|^2_{L^2_\xi} + \lambda\big\|\nu^{\frac{1}{2}}\{\mathbf{I}-\mathbf{P}\}\hat{g}\big\|^2_{L^2_\xi} + |\hat{b}|^2 \leqslant C\big\|\nu^{-\frac{1}{2}}\{\mathbf{I}-\mathbf{P}\}\hat{h}\big\|^2_{L^2_\xi}, \tag{4.14}$$

for any $t \geqslant 0$ *and* $k \in \mathbb{R}^3_k$.

Proof. The Fourier transform of (4.1) gives

$$\partial_t\hat{g} + i\xi\cdot k\hat{g} + \hat{b}\cdot\xi\sqrt{M} = L\hat{g} + \hat{h}.$$

Taking the complex inner product with $\hat{g}$ and taking the real part we obtain

$$\frac{1}{2}\partial_t\|\hat{g}\|^2_{L^2_\xi} + \operatorname{Re}\int_{\mathbb{R}^3}(\hat{b}\cdot\xi\sqrt{M}|\hat{g})\,d\xi = \operatorname{Re}\int_{\mathbb{R}^3}(L\hat{g}|\hat{g})\,d\xi + \operatorname{Re}\int_{\mathbb{R}^3}(\hat{h}|\hat{g})\,d\xi.$$

Then we can deduce the result immediately. □

Lemma 4.3. *Let* $1 \leqslant q \leqslant 2$.

(i) *For any* α, α' *with* $\alpha' \leqslant \alpha$, *and for any* g_0 *satisfying* $\partial_x^\alpha g_0 \in L^2$ *and* $\partial_x^{\alpha'} g_0 \in Z_q$, *one has*

$$\big\|\partial_x^\alpha e^{tB}g_0\big\| \leqslant C(1+t)^{-\sigma[q,m]}\big(\big\|\partial_x^{\alpha'}g_0\big\|_{Z_q} + \big\|\partial_x^\alpha g_0\big\|\big). \tag{4.15}$$

(ii) *Similarly for any* α, α' *with* $\alpha' \leqslant \alpha$, *and for any* h *satisfying* $\nu(\xi)^{-\frac{1}{2}}\partial_x^\alpha h \in L^2$ *and* $\nu(\xi)^{-\frac{1}{2}}\partial_x^{\alpha'} h \in Z_q$ *for* $t \geqslant 0$, *one has*

$$\left\|\partial_x^\alpha\int_0^t e^{(t-s)B}\{\mathbf{I}-\mathbf{P}\}h(s)\,ds\right\|^2$$
$$\leqslant C\int_0^t (1+t-s)^{-2\sigma[q,m]}\big(\big\|\nu(\xi)^{-\frac{1}{2}}\partial_x^{\alpha'}\{\mathbf{I}-\mathbf{P}\}h\big\|^2_{Z_q} + \big\|\nu(\xi)^{-\frac{1}{2}}\partial_x^\alpha\{\mathbf{I}-\mathbf{P}\}h\big\|^2\big). \tag{4.16}$$

Here $m = |\alpha-\alpha'|$, $\sigma[q,m] = \frac{3}{2}(\frac{1}{q}-\frac{1}{2}) + \frac{m}{2}$, $Z_q = L^2_\xi(L^q_x)$.

Proof. Set

$$\tilde{E}(\hat{g}) := \|\hat{g}\|^2_{L^2_\xi} + \kappa_2 \operatorname{Re} E(\hat{g}), \tag{4.17}$$

here κ_2 is a suitably small positive constant.

Combining the results of Lemmas 4.2 and 4.1 for some suitable $\lambda > 0$ we obtain

$$\partial_t \tilde{E}(\hat{g}) + \lambda \frac{|k|^2}{1+|k|^2}\big(|\hat{b}|^2 + |\hat{c}|^2 + |\hat{a}+3\hat{c}|^2\big) + \lambda \big\|\nu^{-\frac{1}{2}}\{\mathbf{I}-\mathbf{P}\}\hat{g}\big\|^2_{L^2_\xi} + \lambda|\hat{b}|^2 \leqslant C\big\|\nu^{-\frac{1}{2}}\{\mathbf{I}-\mathbf{P}\}\hat{h}\big\|^2_{L^2_\xi}.$$

Notice that

$$\begin{aligned}
&\lambda \frac{|k|^2}{1+|k|^2}\big(|\hat{b}|^2 + |\hat{c}|^2 + |\hat{a}+3\hat{c}|^2\big) + \lambda \big\|\nu^{-\frac{1}{2}}\{\mathbf{I}-\mathbf{P}\}\hat{g}\big\|^2_{L^2_\xi} + \lambda|\hat{b}|^2 \\
&\quad\geqslant \lambda \frac{|k|^2}{1+|k|^2}\big(|\hat{b}|^2 + |\hat{c}|^2 + |\hat{a}+3\hat{c}|^2\big) + \lambda \big\|\nu^{-\frac{1}{2}}\{\mathbf{I}-\mathbf{P}\}\hat{g}\big\|^2_{L^2_\xi} \\
&\quad\geqslant \lambda \frac{|k|^2}{1+|k|^2}\big[\big\|\{\mathbf{I}-\mathbf{P}\}\hat{g}\big\|^2_{L^2_\xi} + |\hat{b}|^2 + |\hat{c}|^2 + |\hat{a}+3\hat{c}|^2\big] \\
&\quad\geqslant \lambda \frac{|k|^2}{1+|k|^2}\tilde{E}(\hat{g}).
\end{aligned}$$

Then, we obtain

$$\partial_t \tilde{E}(\hat{g}) + \lambda \frac{|k|^2}{1+|k|^2}\tilde{E}(\hat{g}) \leqslant C\big\|\nu^{-\frac{1}{2}}\{\mathbf{I}-\mathbf{P}\}\hat{h}\big\|^2_{L^2_\xi},$$

which implies

$$\tilde{E}\big(\hat{g}(t,k)\big) \leqslant \tilde{E}\big(\hat{g}(0,k)\big)e^{-\frac{\lambda|k|^2}{1+|k|^2}t} + C\int_0^t e^{-\frac{\lambda|k|^2}{1+|k|^2}(t-s)}\big\|\nu^{-\frac{1}{2}}\{\mathbf{I}-\mathbf{P}\}\hat{h}\big\|^2_{L^2_\xi}(s)\,ds.$$

We now prove (4.15), let $h=0$, so $g(t)=e^{Bt}g_0$ is the solution to (4.1), then we have

$$\begin{aligned}
\big\|\partial_x^\alpha e^{Bt}g_0\big\|^2 &= \int_{\mathbb{R}^3_k} \big|k^{2\alpha}\big|\big|\tilde{E}\big(\hat{g}(t,k)\big)\big|\,dk \\
&\leqslant \int_{\mathbb{R}^3_k} \big|k^{2\alpha}\big|e^{-\frac{\lambda|k|^2}{1+|k|^2}t}\big\|\hat{g}_0(k)\big\|^2_{L^2_\xi}\,dk \\
&\leqslant \int_{|k|^2\leqslant 1} \big|k^{2(\alpha-\alpha')}\big|e^{-\frac{\lambda|k|^2}{1+|k|^2}t}\big|k^{2\alpha'}\big|\big\|\hat{g}_0(k)\big\|^2_{L^2_\xi}\,dk + \int_{|k|^2\geqslant 1} e^{-\frac{\lambda}{2}t}\big|k^{2\alpha}\big|\big\|\hat{g}_0(k)\big\|^2_{L^2_\xi}\,dk \\
&\leqslant C(1+t)^{-\frac{3}{q}+\frac{3-2|\alpha-\alpha'|}{2}}\big\|\partial_x^{\alpha'}g_0\big\|^2_{Z_q} + Ce^{-\frac{\lambda}{2}t}\big\|\partial_x^\alpha g_0\big\|^2.
\end{aligned} \tag{4.18}$$

The proof of (4.16) is similar to that of (4.15), we set $g_0=0$, so

$$g(t) = \int_0^t e^{B(t-s)}\{\mathbf{I}-\mathbf{P}\}h(s)\,ds$$

is the solution to (4.1), then, we obtain

$$\bigg\|\partial_x^\alpha \int_0^t e^{B(t-s)}\{\mathbf{I}-\mathbf{P}\}h(s)\,ds\bigg\|^2 \leqslant C\int_0^t\int_{\mathbb{R}^3_k} \big|k^{2\alpha}\big|e^{-\frac{\lambda|k|^2}{1+|k|^2}(t-s)}\big\|\nu^{\frac{1}{2}}\{\mathbf{I}-\mathbf{P}\}\hat{h}(s)\big\|^2_{L^2_\xi}\,dk\,ds. \tag{4.19}$$

Now, we complete the proof of the lemma. □

Lemma 4.4. *g, g_2 are as defined in Section 2, then we have*

$$\frac{1}{2}\frac{d}{dt}\big\|\nu^{\frac{1}{2}}g_2\big\|^2+\lambda\|\nu g_2\|^2\leqslant C\big[1+\sqrt{\mathcal{E}\big(g(t)\big)}\big]\mathcal{D}\big(g(t)\big). \tag{4.20}$$

Proof. From (1.9) we have

$$\begin{aligned}&\partial_t g_2+\xi\cdot\nabla_x g_2-u\cdot\nabla_\xi g_2-\frac{1}{2}u\cdot\xi g_2\\&\quad=Lg_2+\Gamma(g,g)-\left[\partial_t g_1+\xi\cdot\nabla_x g_1-u\cdot\nabla_\xi g_1-\frac{1}{2}u\cdot\xi g_1+u\cdot\xi\sqrt{M}\right].\end{aligned}$$

Multiplying the last equation by νg_2 and taking integration over $\mathbb{R}^3\times\mathbb{R}^3$ gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big\|\nu^{\frac{1}{2}}g_2\big\|^2+\int_{\mathbb{R}^3\times\mathbb{R}^3}\left(\xi\cdot\nabla_x g_2-u\cdot\nabla_\xi g_2+\frac{1}{2}u\cdot\xi g_2\right)\nu g_2\,dx\,d\xi\\&\quad=-\|\nu g_2\|^2+\int_{\mathbb{R}^3\times\mathbb{R}^3}\big[Kg_2+\Gamma(g,g)\big]\nu g_2\,dx\,d\xi\\&\qquad-\int_{\mathbb{R}^3\times\mathbb{R}^3}\left[\partial_t g_1+\xi\cdot\nabla_x g_1-u\cdot\nabla_\xi g_1-\frac{1}{2}u\cdot\xi g_1+u\cdot\xi\sqrt{M}\right]\nu g_2\,dx\,d\xi.\end{aligned}$$

Then we obtain the result of the lemma. In fact,

$$\begin{aligned}&\int_{\mathbb{R}^3\times\mathbb{R}^3}u\cdot\nabla_\xi g_2(\nu g_2)\,dx\,d\xi\leqslant\sqrt{\mathcal{E}\big(g(t)\big)}\mathcal{D}\big(g(t)\big),\\&\int_{\mathbb{R}^3\times\mathbb{R}^3}[\partial_t g_1+\xi\cdot\nabla_x g_1]\nu g_2\,dx\,d\xi\leqslant C\mathcal{D}\big(g(t)\big).\qquad\Box\end{aligned}$$

Now we are ready to prove the last main theorem on the decay rate.

Proof of Theorem 1.3. 1. Let g be the solution to the Cauchy problem (1.9) and (1.11). Denote δ_0, K_0 and $\delta_{0\nu}$ by

$$\delta_0=\big[[g_0]\big],\qquad K_0=\|g_0\|_{Z_1},\qquad \delta_{0\nu}=\big\|\nu^{\frac{1}{2}}g_0\big\|.$$

2. From the earlier energy estimates one can obtain

$$\frac{d}{dt}\mathcal{E}\big(g(t)\big)+\lambda\mathcal{D}\big(g(t)\big)\leqslant 0.$$

Direct calculation gives

$$\frac{d}{dt}\mathcal{E}\big(g(t)\big)+\lambda\mathcal{E}\big(g(t)\big)\leqslant C\|\mathbf{P}g\|^2.$$

Integrating it over $[0,t]$ we have

$$\mathcal{E}\big(g(t)\big)\leqslant e^{-\lambda t}\mathcal{E}(g_0)+C\int_0^t e^{-\lambda(t-s)}\|\mathbf{P}g\|^2(s)\,ds. \tag{4.21}$$

3. We rewrite (1.9) as

$$\partial_t g+\xi\cdot\nabla_x g+b\cdot\xi\sqrt{M}=Lg+S(g)\quad\Rightarrow\quad\frac{d}{dt}g=Bg+S(g),$$

where

$$S(g)=\Gamma(g,g)+u\cdot\nabla_\xi g-\frac{1}{2}u\cdot\xi g+(b-u)\cdot\xi\sqrt{M},$$

then the solution to the Cauchy problem (1.9) and (1.11) can be written as

$$g(t)=e^{tB}g_0+\int_0^t e^{(t-s)B}S\big(g(s)\big)\,ds.$$

4. We rewrite the solution g as

$$g=e^{tB}g_0+\int_0^t e^{(t-s)B}\{\mathbf{I}-\mathbf{P}\}\big[S_1\big(g(s)\big)+S_2\big(g(s)\big)\big]\,ds+\int_0^t e^{(t-s)B}\mathbf{P}S_2\big(g(s)\big)\,ds,$$

here $S_1(g(s))=\Gamma(g,g)$, $S_2(g(s))=u\cdot\nabla_\xi g-\frac{1}{2}u\cdot\xi g+(b-u)\cdot\xi\sqrt{M}$.

Define

$$\mathcal{E}_\infty(t)=\sup_{0\leqslant s\leqslant t}(1+s)^{\frac{3}{2}}\mathcal{E}\big(g(s)\big).$$

Using Lemmas 4.3 and 4.4 we have

$$\begin{aligned}
\|\mathbf{P}g\|^2(t)&\leqslant C(1+t)^{-\frac{3}{2}}\big[\|g_0\|_{Z_1}^2+\|g_0\|^2\big]\\
&\quad+C\sum_{j=1}^{2}\int_0^t(1+t-s)^{-\frac{3}{2}}\Big(\big\|\nu^{-\frac{1}{2}}\{\mathbf{I}-\mathbf{P}\}S_j\big(g(s)\big)\big\|_{Z_1}^2+\big\|\nu^{-\frac{1}{2}}\{\mathbf{I}-\mathbf{P}\}S_j\big(g(s)\big)\big\|^2\Big)\,ds\\
&\quad+C\left(\int_0^t(1+t-s)^{-\frac{3}{4}}\Big(\big\|\mathbf{P}S_2\big(g(s)\big)\big\|_{Z_1}+\big\|\mathbf{P}S_2\big(g(s)\big)\big\|\Big)\,ds\right)^2\\
&\leqslant C(1+t)^{-\frac{3}{2}}\big(\delta_0^2+K_0^2\big)+C\big(\delta_{0\nu}^2+\delta_0^2\big)\int_0^t(1+t-s)^{-\frac{3}{2}}\mathcal{E}\big(g(s)\big)\,ds\\
&\quad+C\left(\int_0^t(1+t-s)^{-\frac{3}{4}}\mathcal{E}\big(g(s)\big)\,ds\right)^2\\
&\leqslant C(1+t)^{-\frac{3}{2}}\big[\delta_0^2+K_0^2+\big(\delta_{0\nu}^2+\delta_0^2\big)\mathcal{E}_\infty(t)+\mathcal{E}_\infty^2(t)\big],
\end{aligned}$$

in which we have used the following estimations:

$$\begin{aligned}
&\big\|\nu^{-\frac{1}{2}}\{\mathbf{I}-\mathbf{P}\}S_1\big(g(s)\big)\big\|_{Z_1}^2+\big\|\nu^{-\frac{1}{2}}\{\mathbf{I}-\mathbf{P}\}S_1\big(g(s)\big)\big\|^2\leqslant C\big\|\nu^{\frac{1}{2}}g\big\|^2\mathcal{E}(g),\\
&\big\|\nu^{-\frac{1}{2}}\{\mathbf{I}-\mathbf{P}\}S_2'\big(g(s)\big)\big\|_{Z_1}^2+\big\|\nu^{-\frac{1}{2}}\{\mathbf{I}-\mathbf{P}\}S_2'\big(g(s)\big)\big\|^2\leqslant C\big(\big\|\nu^{\frac{1}{2}}g_0\big\|^2+\mathcal{E}(g_0)\big)\mathcal{E}(g),\\
&\big\|\nu^{-\frac{1}{2}}\{\mathbf{P}\}S_2\big(g(s)\big)\big\|_{Z_1}^2+\big\|\nu^{-\frac{1}{2}}\{\mathbf{P}\}S_2\big(g(s)\big)\big\|^2\leqslant C\|u\|_{L_x^2\cap L_x^\infty}\|g\|,\\
&\big\|(b-u)\cdot\xi\sqrt{M}\,\big\|_{Z_1}+\big\|(b-u)\cdot\xi\sqrt{M}\,\big\|\leqslant C\|u\|_{L_x^2\cap L_x^\infty}\|a+3c\|,\\
&\int_0^t(1+t-s)^{-\frac{3}{2}}(1+s)^{-\frac{3}{2}}\,ds\leqslant C(1+t)^{-\frac{3}{2}},\\
&\int_0^t(1+t-s)^{-\frac{3}{4}}(1+s)^{-\frac{3}{2}}\,ds\leqslant C(1+t)^{-\frac{3}{4}}.
\end{aligned}$$

Here $S_2'(g(s))=u\cdot\nabla_\xi g-\frac{1}{2}u\cdot\xi g$.

Now from (4.21) we have

$$\mathcal{E}\big(g(t)\big)\leqslant C(1+t)^{-\frac{3}{2}}\big[\delta_0^2+K_0^2+\big(\delta_{0\nu}^2+\delta_0^2\big)\mathcal{E}_\infty(t)+\mathcal{E}_\infty^2(t)\big].$$

This implies

$$\mathcal{E}_\infty(t)\leqslant C\big[\delta_0^2+K_0^2+\big(\delta_{0\nu}^2+\delta_0^2\big)\mathcal{E}_\infty(t)+\mathcal{E}_\infty^2(t)\big]. \tag{4.22}$$

So we get

$$\|g(t)\|_{H^N} \leqslant C\|g_0\|_{H^N \cap Z_1}(1+t)^{-\frac{3}{4}}. \tag{4.23}$$

Till now the proof of Theorem 1.3 ends. □

Remark 2. Using (4.22) to obtain our result (4.23) is a standard argument which can be proved rigorously. In fact, this inequality is to say that if $\mathcal{E}_\infty(t)$ is priorily small uniformly in time, then it will be smaller than what is expected. Now, as long as it is initially small, it must be uniformly bounded in all time due to the continuity argument.

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