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Automated Advice: A Portfolio Management Perspective on Robo-Advisors

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Abstract

In this paper we investigate the predominant robo-advisor model, uncovering that however novel this solution might be, it also relies religiously on imperative contributions to modern portfolio theory that have been made in the past half a century. Despite conforming by and large to passive investment, we find that the slight variations in the methodologies used by robo-advisors introduce significant variability in risk-adjusted returns across the robo-advisor spectrum. Nonetheless, our performance estimations show that three out of the four robo-advisors considered in this paper produce higher risk-adjusted return than the benchmark. In testing the robo-advisor model on the Norwegian market, we also find that a robo-advisor strategy based on a multifactor approach, outperforms the benchmark for the investment horizons considered.

Preface

This thesis is written in pursuance of our Master of Science degrees at the Norwegian University of Science and Technology (NTNU), with specialization in Financial Engineering. The purpose of this paper is to evaluate the methodology applied by so-called robo-advisors - the automated alternative to traditional investment managers that has emerged in the years following the recent Great Recession - as well as estimating the performance it yields. While no robo-advisor is currently operating in Norway, we apply the robo-advisor methodology to the Norwegian market and assess its suitability.

Our interest in this topic has largely sprung from the interdisciplinary nature of our studies at NTNU, combining courses in computer science, finance and investment analysis. We have thoroughly enjoyed studying one of the latest and most applauded additions to the world of finance, emerging from its cross-section with computer science. We were also intrigued by the potential of robo-advisors to provide the wider public with an opportunity to wisely and effortlessly partake in financial markets.

We would like to express our gratitude to our supervisor Alexei Gaivoronski at the Department of Industrial Economics and Technology Management, for his guidance and support throughout the project.

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1 INTRODUCTION

Automated investment managers, popularly referred to as 'robo-advisors', have become one of the fastest growing areas within the investment landscape, challenging traditional solutions to the age-old question: How should I invest my money? In essence, a robo advisor is an online financial advisor that uses algorithms to automatically construct, optimize and manage the investment portfolio of a client. Riding a wave of entrepreneurial effort and venture capital backing, robo-advisors are bringing change to the investment management industry, rethinking business models and expanding the wealth management client base. In 2014, venture capitalists poured nearly 300 million dollars into various robo-advisors (Demos, 2015). By 2020, the assets under management of robo-advisors in the US, the leading market, is forecast to reach an estimated two trillion dollars, which implies a formidable compound annual growth rate of 68 percent over a five-year period (Epperson et al., 2015).

Robo-advisors have emerged from the entwinement of two branches of history; investment theory and computer science. Up until the middle of the 20th century, investing was mostly considered a form of gambling for the wealthy, the goal of which was to acquire attractive assets at the best possible price – a practice fuelled by the slow speed at which information travelled and thus affected prices. Economist John Burr Williams (1938) pioneered the use of a company's fundamentals in investment decisions, and articulated the present value model in his paper "The Theory of Investment Value", which became predominant in the understanding of stock prices. However, Williams was under the impression that risk could be completely diversified away, and therefore largely ignorant to the effects risk has on valuation.

A paradigm shift was underway in 1952, when Harry Markowitz introduced mean-variance optimization and thereby sparked the era of modern portfolio theory. Markowitz (1952) was the first to mathematically formulate the role of risk and diversification, concluding that it is a security's covariance with the given portfolio that determines incremental risk, and while diversification offers a "free lunch" in the sense that risk can be reduced without sacrificing expected portfolio return, exposure to market risk will remain. The work of Markowitz also marked a starting point for the use of sophisticated computer science in finance. His techniques for solving the portfolio selection problem required more computational power than was available at the time, which led Markowitz to go on developing algorithms for approximate solutions in his later work (Berk & DeMarzo, 2014; Markowitz, 1991).

Building on Markowitz' efficient frontier of portfolios offering the highest return for the given level of risk, Tobin (1958) identified the unique efficient portfolio of risky securities to be combined with a risk-free investment, weighted according to risk preference. The Capital Asset Pricing Model (CAPM), introduced in the 1960s, identified this efficient portfolio as the market portfolio, containing all tradable securities in proportion to its market capitalization (Sharpe, 1964). CAPM relied on the same assumptions regarding investor behavior as the efficient market hypothesis (EMH) of Eugene Fama (1970) did, who argued that the high degree to which prices reflect available information makes outperforming the overall market a matter of luck. To arrive at this conclusion, Fama depended on sophisticated computer processing used to analyze vast amounts of financial data (Fox, 2009).

With the contributions of such as Markowitz, Tobin, Sharpe and Fama, all of whom became Nobel laureates, the road had been paved for passive management – attempting to copy rather than beat market performance – to enter the realm of investment. By then, market indices had already existed for a long time, but these were not originally designed as investment vehicles (Hebner, 2006). Mutual funds attempting to replicate the performance of market indices, by holding the same stocks in the same proportions, were introduced in the 1970s. With his book A Random Walk Down Wall Street, Princeton economist Burton Malkiel (1973) established himself as one of the first proponents of these index funds. Malkiel's views were rooted in the random walk hypothesis, closely related to the EMH and stating that the past movement of an asset price cannot be used to predict its future movement, but that over time, prices maintain an upward trend.

Exchange traded funds (ETFs), the next market replication instrument in the evolution of passive investing, were introduced in the late 1980s. As index funds, ETFs represent ownership in a fully diversified portfolio that typically track a market index, but, unlike index funds, trades directly on an exchange. ETFs have shown rapid adoption in the past decade, providing commoners access to well-diversified portfolios that meet a variety of investment needs, including harder-to-access asset classes such as commodities and bonds, or stocks in specific countries, size categories, or industries (Berk & DeMarzo, 2014).

By the beginning of the 1990s, computer-aided quantitative practices, many of which stemming from fields of engineering, had become well-established in finance. Digital investment tools were used by financial professionals to develop investor profiles, allocate assets or recommend specific portfolios or securities to investors. At the turn of the millennium, the number of digital analytical tools available directly to investors began to grow, fuelled by regulation and the commercialization of the internet in the decade that had past (FINRA, 2016).

The landscape expanded further following the 2008 financial crisis and the resulting loss of clients' trust in established financial services institutions. A number of new entrants began offering a range of client-facing digital financial tools. While traditional wealth management firms were busy meeting new regulatory requirements and the complexities of crisis-driven consolidation (Nanayakkara et al., 2015), these firms saw an opportunity to leverage their technological competence to build out simpler and cheaper methods of delivering investment advice, offering functionality previously only available to financial professionals (FINRA, 2016).

The robo-advice space was pioneered by companies such as Betterment and Wealthfront, the latter of which has efficient market hypothesis-proponent Burton Malkiel serving as chief investment officer. Robo-advisor Personal Capital, which we will not cover in this paper due to the difficulty of replicating technicalities in its investment methodology, can boast of having modern portfolio theory originator Harry Markowitz at its board of advisors. Industry incumbents have responded by acquiring or developing client-facing digital tools of their own, or by employing their professional-facing counterparts in order to enhance efficiency and quality of service. The most notable instance is perhaps Charles Schwab, the wealth-management giant that in 2015 rolled-out its own automated wealth service, targeting a far less affluent segment than its traditional clients. The world's largest asset manager BlackRock entered the scene later that year by acquiring FutureAdvisor, a digital investment manager founded in 2010 by former Microsoft engineers.

The predominant robo-advisor model today is founded on modern portfolio theory's technical way of constructing an optimal portfolio given the investor's risk preference, combined with the rationale for passive investing provided by the efficient market hypothesis. The platforms work by assessing the investor's risk preference and investment objectives through a questionnaire, before applying modern portfolio techniques for building a balanced portfolio, typically consisting of around a dozen ETFs. Thus, robo-advisors can largely be considered a natural next step in the evolution of passive investing, standing on the shoulders of giants both in terms of portfolio theory and computer science. However, as we shall see, robo-advisors also depart from the theoretical definition of a true passive strategy, in ways that however subtle, introduces significant variability in risk-adjusted returns across the robo-advisor spectrum.

Notwithstanding the formidable growth and praise robo-advisors have enjoyed in the past few years, the question of whether these automated investment managers will end up a complete game-changer or a niche in the periphery of the affluent and well-guarded financial system, remains to be answered. In this paper, we investigate the inner workings of the predominant robo advisor model, as well as evaluate its performance. Thus, we contribute a view on robo-advisors' benefits and limitations, providing a basis for better understanding its future potential.

While the robo-advisor model has caught the attention of investors and entrepreneurs around the world, with there currently being more than 350 robo-advisors operating globally (Kocianski, 2016), none has entered the Norwegian market as of yet. To assess how well the robo-advisor model is suited for the Norwegian market, we estimate the performance of hypothetical robo-advisor portfolios, and discuss the compatibility of key robo-advisor features with Norwegian market conditions.

The remainder of this paper is structured as follows: in section 2 we provide an overview of the investment management industry, before laying the groundwork for our evaluation of the robo-advisor model in section 3, in which we review the literature it is based upon. Section 4 contains details on the methodologies of the robo-advisors covered in this paper, including elaborate discussions on two of the most valuable robo-advisor features: automated rebalancing and tax-loss harvesting. In section 5 we estimate the performances of major American robo-advisors, before testing the robo-advisor model on the Norwegian market in section 6. Finally, we discuss our findings and provide concluding remarks in section 7.

2 The Investment Management Industry

The financial system composes mechanisms that allow strangers to strike agreements that let them to move money through time, hedge risks, and exchange assets. Investors move money from the present to the future when they save, expecting an appropriate rate of return for bearing risk through time. Borrowers move money from the future to the present to finance current projects; hedgers trade to reduce their exposure to undesirable risks; and speculative traders attempt to identify under- and overvalued securities (McMillan et al., 2011).

The investment management industry functions as a type of financial intermediary, whose activities transform cash flows and risk from one form to another, and services allow buyers and sellers to connect through instruments fulfilling their needs. Specifically, the industry serves as intermediary between issuers of financial products (governments, corporations or financial institutions), and private and institutional investors looking to allocate their funds optimally. Most of the industry's income stems from personal wealth, worth hundreds of trillions of US dollars on a global basis (Sironi, 2016).

Three main parties contribute to the supply-demand chain of the industry: issuers in search of the cheapest funding, intermediaries maximizing their profits through margins, and final investors looking for the highest risk-adjusted return. Securities, such as bonds and stocks, are first sold in primary markets by their issuers, then traded in secondary markets ensuring liquidity (McMillan et al., 2011). Intermediaries, the topic of interest in this paper, are professional players advising final investors on suitable products or portfolios.

From the 1950s onward, technological advancement has left its clear marks on the investment management industry, particularly in the sense of automating back-office processes and trading. Nonetheless, and perhaps somewhat perplexing, the critical decision-making stage has largely been left untouched. The conventional model, in which investors are assisted by human advisors or brokers, is predominant to this day (Sironi, 2016). It was only recently that investment management was almost exclusively conducted through human advisors and bundled with other services. With the advent of robo-advisors, consumers have gained direct access to portfolio management tools (Huang, 2014).

2.1 Traditional Investment Managers

A traditional investment manager is a professional who makes investments on behalf of a third-party, with the aim of generating the highest possible return given the investor's specific time- and risk-related preferences. Investment managers offer their clients financial advice and manage clients' portfolios by trading component securities, with a frequency that depends on the type of investment management.

Investment managers' revenues are generated by charging fees from clients. These fees cover custody fees and administration costs, such as record keeping, accounting services and trading, as well as costs related to the ongoing management of the portfolio, such as due diligence, monitoring and account rebalancing (LCP, 2015). The most common fee structure involves the advisor or investment management firm charging the client a percentage of the assets being managed. In 2016, the industry average for traditional investment managers was 1.35% of assets under management. Expenses for any fund within the portfolio comes in addition, ranging from 0.12% of annual fund value for index mutual funds, to 0.65% for actively managed mutual funds (AdvisoryHQ, 2016).

Traditional investment managers are typically employed by financial institutions such as banks. Thus, in the case of financial conglomerates offering services to a variety of parties, including both issuers and investors, conflicts of interest may arise (Sironi, 2016).

2.2 Robo-advisors

Robo-advisors first appeared between 2008 and 2010, as part of a larger phenomenon coined financial technology (fintech), in which the disruptive forces of technology are finally beginning to erode the resilient walls of the financial industry. A set of concomitant factors helped robo-advisors rise to prominence towards 2013, including a tightening of international regulations in favor of investor protection; the tremendous market penetration of smartphones; and increasing popularity not only among target low-margin customers, but also the high-net-worth regulars of traditional investment firms (Haffenden & Melone, 2016; Sironi, 2016).

Robo-advisors are automated investment solutions offering initiation and management of clients' portfolios using proprietary algorithms. In contrast to traditional investment managers, robo-advisors interact with their customers through online platforms featuring advanced customer experience, offering little or no human intervention (Lieber, 2014). Customers are guided through an online questionnaire in which risk tolerance and investment objectives are assessed, which in turn is determinative of the portfolio selected. In addition to asset allocation, robo-advisors institutionalize automatic reinvestment of dividends and portfolio rebalancing, as well as regular account maintenance and other services (Nasdaq, 2014). Several robo-advisors also offer tax optimization.

In general, robo-advisors have much lower minimum investment requirements than their traditional counterparts, attracting investors looking to invest smaller funds (The Economist, 2015). The average minimum investment amount with a traditional investment manager is $50,000, substantially higher than what required by the robo-advisors presented in table 2.1 (AdvisoryHQ, 2016). Thus far, robo-advisors are nowhere near the assets under management of the largest traditional investment managers, ranging from 2.3 to 5.1 trillion dollars (IPE, 2016).

Robo-Advisor	**Year Founded**	**Minimum Investment ($)**	**Assets Under Management ($M)**	**Investors Served**
Betterment	2008	0	8,000	210,000
Wealthfront	2011	500	5,500	74,000
Schwab	2015	5,000	17,000	N/A*
FutureAdvisor	2010	0	600	6,300

Table 2.1: Sources: Betterment, 2017a; Wealthfront, 2017a; Schwab, 2017a; FutureAdvisor, 2017a; Credio 2017

* Information not available. However, Schwab manages 140,000 accounts.

As we show next, robo-advisors are considered a low-cost alternative to traditional investment management. Annual advisory fees of the largest robo-advisors are concentrated around 0.15% to 0.25% of assets under management, and increases with the extent of human involvement. As with traditional investment firms, investors additionally pay the expenses of underlying funds.

Fee Structure and Costs

The fees charged to those investing with a robo-advisor usually include an annual advisory fee for the robo-advisor services, in addition to the expense ratios of the constituent ETFs of the portfolio. The latter are costs associated with managing and operating the exchange-traded funds, usually charged by lowering the dividends payed by the ETF. The advisory fees and expense ratios are most often calculated as a percentage of assets under management, and while there is general agreement among robo-advisors to keep these low, there are some variations among them.

Betterment (2017a) charges their customers a flat fee of 0.25% of assets managed. However, no fee is charged on assets exceeding $2 million. The expense ratios of the ETFs that Betterment invest in range from 0.03% to 0.40%; the average expense ratio being 0.12%. Wealthfront (2017a) also charges an advisory fee of 0.25%, but with the exception of the first $10,000 invested, which are managed for free. With ETF expense ratios ranging from 0.03% to 0.25%, the average expense ratio is 0.14%.

Schwab Intelligent Portfolios (2017a) stands out in the group by omitting the advisory fee altogether. Customers pay only the operating expense ratios of the ETFs in their portfolio, which range from

0.03% to 0.65%. For conservative, moderate and aggressive portfolios the average cost is 0.07%, 0.16% and 0.21%, respectively, of assets under management. The zero advisory fee is made possible by the fact that Schwab Intelligent Portfolios mainly invests in funds issued by the Charles Schwab Corporation, thus making money off of the underlyind operating expense ratios. This is a noteworthy observation in the sense that lower fees are traded off against bias in the investment vehicle selection caused by the advisor's own financial incentives. Schwab Intelligent Portfolios is also the only robo-advisor recommending substantial allocations to cash, which are as high as 20%. The cash positions represent another source of revenue for Schwab; given that these positions are held in Schwab's own cash vehicles, it collects the spread between the earnings on reinvesting the money and what is paid out to customers.

FutureAdvisor (2017a) charges a flat advisory fee of 0.5%. Unlike the other three robo-advisors, who do not charge any transaction costs, trading commissions come in addition to this advisory fee. Trading commissions and expense ratios are nevertheless minimized when constructing and managing the customer portfolios, and the average total fee when investing with FutureAdvisor is 0.65% of assets managed per year.

To illustrate the differences in costs for a moderate investor, tables 2.2-2.6 show the costs associated with investing $5,000, $50,000, $500,000, $1,500,000 and $4,000,000. The effective advisory fee is the annual advisory fee one would pay as a percentage of the funds invested, calculated as the combination of fee ratios for each of the underlying investment intervals.

Investment: $5,000

	Betterment	Wealthfront	Schwab	FutureAdvisor
Effective Advisory Fee	0.25%	0%	0%	0.50%
Average Expense Ratio	0.12%	0.14%	0.16%*	0.15%
Total Cost (%)	0.37%	0.14%	0.16%	0.65%
Total Cost ($)	$18.50	$7.00	$8.00	$32.50

Table 2.2: Costs associated with a $5,000 investment.

*Average expense ratio for a moderate investor.

Investment: $50,000

	Betterment	Wealthfront	Schwab	FutureAdvisor
Effective Advisory Fee	0.25%	0.20%	0%	0.50%
Average Expense Ratio	0.12%	0.14%	0.16%	0.15%
Total Cost (%)	0.37%	0.34%	0.16%	0.65%
Total Cost ($)	$185.00	$170.00	$80.00	$325.00

Table 2.3: Costs associated with a $50,000 investment.

Investment: $500,000

	Betterment	Wealthfront	Schwab	FutureAdvisor
Effective Advisory Fee	0.25%	0.25%	0%	0.50%
Average Expense Ratio	0.12%	0.14%	0.16%	0.15%
Total Cost (%)	0.37%	0.39%	0.16%	0.65%
Total Cost ($)	$1,850.00	$1,925.00	$800.00	$3,250.00

Table 2.4: Costs associated with a $500,000 investment.

Investment: $1,500,000

	Betterment	Wealthfront	Schwab	FutureAdvisor
Effective Advisory Fee	0.25%	0.25%	0%	0.50%
Average Expense Ratio	0.12%	0.14%	0.16%	0.15%
Total Cost (%)	0.37%	0.39%	0.16%	0.65%
Total Cost ($)	$5,550.00	$5,825.00	$2,400.00	$9,750.00

Table 2.5: Costs associated with a $1,500,000 investment.

Investment: $4,000,000

	Betterment	Wealthfront	Schwab	FutureAdvisor
Effective Advisory Fee	0.13%	0.25%	0%	0.50%
Average Expense Ratio	0.12%	0.14%	0.16%	0.15%
Total Cost (%)	0.25%	0.39%	0.16%	0.65%
Total Cost ($)	$9,800.00	$15,575.00	$6,400.00	$26,000.00

Table 2.6: Costs associated with a $4,000,000 investment.

Tables 2.2-2.6 show that Wealthfront has the lowest fees for investments less than $10,000. Whenever the investment exceeds this amount, however, Schwab Intelligent Portfolios offers significantly lower fees than the other robo-advisors examined.

Any assessment of investment management is incomplete without evaluating the investment philosophy executed. Robo-advisors have widely adopted a passive investment philosophy, incorporating a strong focus on obtaining diversification at the lowest possible cost. Next, we turn to the theoretical and empirical foundation for this choice.

3 Literature Review

3.1 Market Efficiency

Investment advisors, whether human professionals or those designing the algorithms that run robo-advisors, typically adhere to differing investment philosophies and standpoints on best practice, which in turn impacts the results provided to clients. A key pillar on which the choice of investment strategy rests, is the degree to which one believes the market is efficient. The efficient-market hypothesis was introduced by Eugene Fama in 1970, and has played a critical role in the understanding of financial markets in the decades thereafter, though also being subject to much debate. The hypothesis is rooted in the work of Maurice Kendall (1953), who found that share and commodity prices follow an unpredictable "random walk". The fundamental assumption is that market participants are rational and profit-maximizing, thus accounting for all relevant information when pricing an asset. According to Fama (1970), this will ultimately lead to all securities being traded at their fair value, making it impossible to consistently outperform the overall market.

The efficient-market hypothesis was later revised to state that financial markets are nearly efficient most of the time, and that market participants who aim to exploit anomalies are a necessity in order to eliminate mispricing. Hence, paradoxically, it is those actively searching for market inefficiencies that are bringing the market towards efficiency (Grossman & Striglitz, 1980; Sharpe, 2002). Equivalently, should all investors be passive and valuations be deemed unprofitable, a gap between companies' intrinsic value and their share price will eventually arise.

Since there are costs associated with obtaining and analyzing information, the rational and profit-maximizing market participant who takes on such tasks should be compensated in the form of higher return. Thus, the modern efficient-market hypothesis allows for excess return, but only to the extent that it covers the costs of obtaining new information (Fama, 1991). Others will mimic successful investment managers, effectively driving profit towards zero. As prices reflect an increasing amount of information, outperformance will slow as marginal cost of obtaining information increases, eventually reaching an equilibrium (Jensen, 1978).

If there are winners in the market, there are also losers. Modern efficient market theory claims that value is transferred from investors with little information and high costs, to investors with good information and low costs. Black (1986) labels these two types of market participants informed traders and noise traders. Noise traders may act on noise misinterpreted as information, or may have specific motives such as an immediate need for liquidity, translating into willingness to pay a premium. Informed traders are able to seek out the mispricing created by the noise traders, and capitalize on it, thus absorbing the market anomalies. Black maintains that both types of traders are necessary for financial markets to function. If everyone had the same beliefs, no one would act. Different perceptions create inefficiency which in turn constitutes the basis for trading.

3.2 Portfolio Management

As a natural first step in the process of investing, the investor decides upon an investment policy – often referred to as the strategic allocation – in which asset classes and normal weights are selected in order to meet the investor's objectives (Brinson et al., 1986). Given that each asset class will have an associated risk and return, the allocation plan should primarily be determined on the basis of the investors' risk appetite, as well as the time horizon for the investment (Cochrane, 1999).

In the case of robo-advisors, this input is provided by the online questionnaire, and fed to an algorithm which will then automatically determine what asset classes to invest in, and in what proportions (Moyer, 2015b). The algorithms employed in the predominant robo-advisor model performs asset allocation using mean-variance analysis, a crucial component of modern portfolio theory, which we turn to shortly.

Once the desired asset allocation plan is set, the investor can choose to actively manage selected asset classes (in part or in total), in attempt to maintain diversification while partaking in opportunities for excess returns in desired parts of the market.

Passive Management

Passive investors believe they have more to gain from reducing investment costs than trying to beat the average, and will therefore attempt to replicate the market rather than outperforming it. The portfolio of the passive investor is composed such that each security is represented in the same proportion as in the corresponding market (Sharpe, 1991). In practice, the passive investor will typically invest in indices and exchange-traded funds (ETFs), and hold these positions for a long time, which is precisely what robo-advisors encourage. With passive management, considerably lower management costs incur relative to active management, transaction costs are avoided through infrequent trading, and broad diversification can be enjoyed - all of which are contributing factors to robo-advisors' choice of passive management.

Active Management

Active management is based on the desire to achieve risk-adjusted excess return from exploiting market inefficiencies using some information advantage, effectively claiming that price predictions can be made. Countless active investment strategies exist, including momentum-, value-, growth- and contrarian investment. Several of these are based on security analyses that are typically divided into two broad methodologies: fundamental and technical analysis. Fundamental analysis maintains that markets may misprice a security in the short run. Consequently, its proponents make use of financial information such as key performance indicators, to carry out their own valuations with the objective of detecting and profiting from mispricing. Technical analysis is meant to forecast the direction of prices through the study of trends and past market data (Murphy, 1999).

The potential excess return achieved through active management derives from two components: market timing and security selection. Timing, also referred to as active asset allocation, is the strategic under or overweighting of an asset class, or subclass, relative to its normal weight in an attempt to capture excess returns from short-term fluctuations in asset class prices (Brinson et al., 1986). For example, the investor who believes the stock market is undervalued, may decide to increase the weight of stocks relative to the other portfolio asset classes, such as bonds. Other examples include switching between sectors or regions within a market. Security selection, on the other hand, is the active selection of investments within an asset class, that are believed to be mispriced.

In the strictest sense, an active investor is any investor who does not hold every security from the market, with each represented in the same percentage as in the market. Given this definition of passive and active management, Sharpe (1991) observed that mathematically it must be the case that:

1. Before costs, the return on the average actively managed dollar will equal the return on the average passively managed dollar and

2. After costs, the return on the average actively managed dollar will be less than the return on the average passively managed dollar

This serves to emphasize a point we have already made; if there are winners, there must be losers. Active management is indeed a zero-sum game.

3.3 Modern Portfolio Theory

Mean-variance analysis is, as noted, the primary asset allocation framework employed by robo-advisors. It was introduced by Markowitz (1952) for the pursuit of portfolio construction, and has become interchangeable and inseparable from modern portfolio theory (MPT). James Tobin's (1958) extension of the framework is commonly known as mean-variance optimization (MVO), and Sharpe (1964) contributed further with his theory on asset pricing. Today, the umbrella term modern portfolio theory refers to a mathematical framework for constructing portfolios of securities, where the objective is to maximize expected return for a specified level of risk or, equivalently, minimizing portfolio risk for a specified level of return. In the following section, we present modern portfolio theory along with its advantages and limitations.

Diversification

Although the notion of diversification has existed for centuries, exemplified with the proverb "do not put all eggs in one basket", it was not mathematically formulated prior to modern portfolio theory. Diversification is the process through which exposure to any one particular risk is reduced, commonly achieved through allocating capital in a variety of assets that are imperfectly correlated. That way, risk reduction can be achieved without compromising portfolio returns. Diversification is risk-reducing because different assets and asset classes respond differently to market events, and can thus be combined in order to offset divergent effects.

Markowitz (1952) found that portfolio risk cannot be entirely eliminated through diversification. Asset risk is divided into two component risk types: firm-specific and systematic risk. *Firm-specific* (idiosyncratic) risk refers to risk associated with a specific company. Fluctuations in the return of an asset caused by firm-specific news are independent, and can be averaged out in a large portfolio (Berk & De Marzo, 2014). *Systematic* risk, however, refers to return fluctuations subject to market-wide news, such as interest rate changes and economic cycles. Systematic risk affects all firms, albeit to a varying degree, and will therefore be present in any portfolio.

Markowitz demonstrated that by combining stocks into a portfolio, risk is reduced depending on the extent to which the stocks face common risks and has prices moving identically. Specifically, each security contributes to the volatility of the portfolio according to its total risk, scaled by its correlation with the portfolio. The lower the correlation between a stock and the remaining portfolio, the lower the portfolio volatility – reflecting firm-specific risk being averaged out. Thus, for a portfolio of long positions, unless all of the stocks are perfectly correlated, the risk of the portfolio will be *lower* than the weighted average volatility (total risk) of the individual stocks. In contrast, the expected return of the portfolio will be *equal* to the weighted average expected return (Berk & DeMarzo, 2014). This is the very deed of diversification: a diversified portfolio can yield higher return yet have less volatility than the least volatile of its constituents (Sullivan & Sheffrin, 2003).

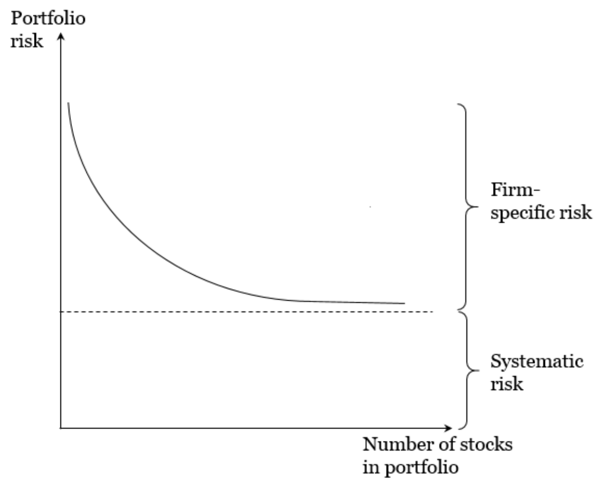


Figure 3.1: The portfolio risk decreases as the number of stocks increases

Portfolio risk is dependent on the relative proportion of asset classes in the portfolio, their associated risk and correlations with the other asset classes. Given that all of these factors are non-static over time, simply investing in two assets with initially offsetting risks will typically be insufficient to obtain long-term diversification. Multiple studies have shown that increasing the number of securities in a portfolio will significantly reduce firm-specific risk, as illustrated in Figure 3.1, yet the exact appropriate number has been a topic of debate.

In their empirical studies, Evan and Archer (1968) found that the dispersion of risk as a function of the number of securities in the portfolio has an asymptotic form. Furthermore, the asymptotic limit of this function was shown to represent systematic risk, and their results indicated that firm-specific risk could be diversified by constructing portfolios of merely ten securities. Later studies have shown that portfolios should be larger in order to obtain satisfying diversification. Elton and Gruber (1977) showed that while portfolio risk decreases at a diminishing rate the more securities that are added, a portfolio of 15 stocks has 32% higher risk than a portfolio of 100 stocks. Bird and Tippett (1986) reached the same conclusion using a more mathematical approach, while the most recent literature has shown that a well-diversified portfolio should include 30-40 stocks (Statman, 1987).

Time Diversification

The time horizon of an investment describes the period of time an investment is held before the investor is intended to make use of its returns, liquidating the investment. The question of what effect investment time horizon has on portfolio risk has been widely discussed in financial literature. Paul Samuelson's (1963) *myth of time diversification*, based on variance risk and utility theory, states that investors' risk tolerance should be unaffected by time horizon so long as the following conditions hold:

1. Investors have constant relative risk aversion, in the sense that the risk exposure is independent of changes in wealth

2. Investment returns are independent and identically distributed, following a random walk

3. Future wealth depends solely on investment results, not on any other forms of income

Samuelson's conclusions was later subject to critique by theorists maintaining that time does indeed have diversifying effect. Merrill and Thorley (1996) argue that conclusions drawn upon utility theory stand or fall by the assumed form of the utility function. By instead evaluating risk through financial option pricing, which is independent of investors' risk attitudes, Merrill and Thorley found that risk is reduced when investments have larger time horizons.

In the article "Beware of Dogma - The truth about time diversification" Kritzman and Rich (1998) describe time diversification from different perspectives. On one hand, they maintain that an investor's willingness to take on more risk over a longer time horizon depends on the investor's utility function. Thus, the investor could be indifferent to a risky and a risk-free investment with the same level of utility, regardless of investment horizon. On the other hand, the authors characterize time diversification as dependent on the investor's definition of risk. They argue that an investor measuring risk as variance will experience increasing risk with an increased time horizon, due to the linear property of the measure in the sense that variance grows proportionally with time. An investor measuring risk as volatility will however experience decreasing risk, as volatility grows with the square root of time. Consequently, depending on whether risk is defined in terms of the total size of the potential loss or the variability of returns, risk is increasing or decreasing with time horizon.

MEAN-VARIANCE OPTIMIZATION

Mean-variance analysis is a framework introduced by Markowitz (1952) for assembling optimal portfolios. We consider the problem of optimally investing capital in m risky assets $i = 1, 2, \ldots, m$, for a single period, with respective returns given by the following multivariate random vector:

$$\mathbf{R} = [R_1, R_2, ..., R_m]' \tag{3.1}$$

The following vector of returns and covariance matrix represent the mean and covariance of these asset returns, respectively:

$$E[\mathbf{R}] = \boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix} \tag{3.2}$$

$$\mathrm{Cov}[\mathbf{R}] = {}^\circ = \begin{bmatrix} {}^\circ{}_{1,1} & \cdots & {}^\circ{}_{1,m} \\ \vdots & \ddots & \vdots \\ {}^\circ{}_{m,1} & \cdots & {}^\circ{}_{m,m} \end{bmatrix} \tag{3.3}$$

The expected portfolio return is thus given by the linear combination of the underlying expectations:

$$\alpha_{\mathbf{w}} = E[R_{\mathbf{w}}] = \mathbf{w}'\boldsymbol{\alpha} \tag{3.4}$$

Similarly, the variance of the portfolio is given by the variance of the weighted average of the individual returns:

$$\sigma^2_{\mathbf{w}} = \text{var}[R_{\mathbf{w}}] = \mathbf{w}'^{\circ}\mathbf{w} \tag{3.5}$$

Given preferences for higher expected returns and lower variance, Markowitz posed the evaluation of different portfolios' **w** as a quadratic programming problem, in which the objective is to maximize the expected return subject to a target return variance σ_0^2:

$$\begin{aligned} \text{Maximize:} \quad & E[R_{\mathbf{w}}] = \mathbf{w}'\boldsymbol{\alpha} \\ \text{Subject to:} \quad & \mathbf{w}'^{\circ}\mathbf{w} = \sigma_0^2 \\ & \mathbf{w}'\mathbf{1}_m = 1 \end{aligned} \tag{3.6}$$

Solving the maximization problem for every possible target variance, or the equivalent minimization problem for every possible target expected return α_0, yields the *efficient frontier*:

$$\{(\alpha_0, \sigma_0^2) = (E(R_{\mathbf{w}_0}), \text{var}(R_{\mathbf{w}_0})) \mid \mathbf{w}_0 \text{ optimal}\}$$

To arrive at these efficient portfolios, diversification plays a key role. Specifically, adding new investment opportunities allows for greater diversification and improves the efficient frontier. Thus, every available investment opportunity should be represented in order to obtain the best possible set of risk and return opportunities (Berk & DeMarzo, 2014).

Markowitz' framework was extended by Tobin (1958), who recognized the possibility of constructing portfolios that combine risky securities with a risk-free investment. Tobin identified the *unique* portfolio of risky investments to be optimally combined with borrowing or lending at the risk-free rate. This *tangent portfolio*, so named because it represents the tangency point on the efficient frontier of risky investments, is the portfolio with the highest *Sharpe ratio* of any portfolio in the economy. The Sharpe ratio is a measure of a portfolio's risk-adjusted return, presented in equation 3.7. Here, r_p represents the portfolio risk; r_f the risk free rate; and σ_p the portfolio volatility:

$$S = \frac{r_p - r_f}{\sigma_p} \tag{3.7}$$

Given that the tangent portfolio has the highest Sharpe ratio, it provides the largest reward per unit of volatility of any portfolio available. The astonishing implication is that *all* investors should hold the tangent portfolio, weighted relative to the risk-free investment in accordance to the investor's ideal exposure to risk.

Tobin proved what is known as the *separation theorem*, stating that optimal portfolio choice is separated into two stages. First, the optimal portfolio of risky assets is identified. Second, the appropriate ratio of investments in the tangent portfolio to risk-free assets is determined. Thus, all investors should have portfolios placed on the straight line representing the *efficient frontier including risk-free investment*, as illustrated in Figure 3.2.

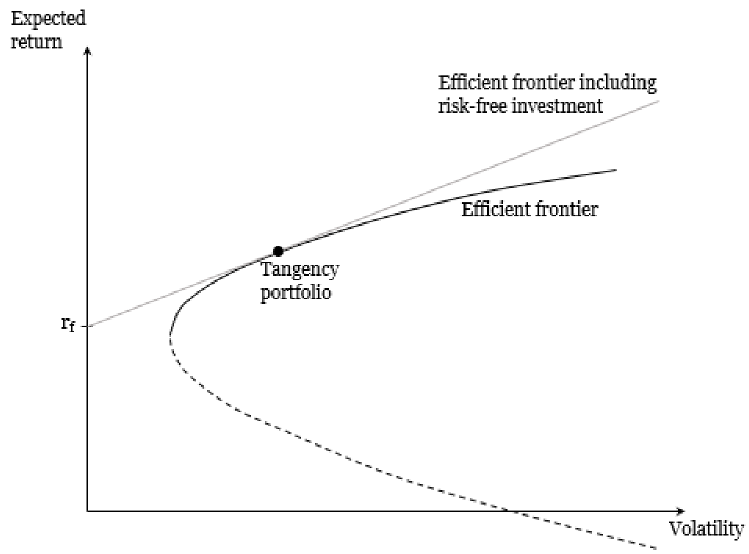


Figure 3.2: Efficient frontier

THE CAPITAL ASSET PRICING MODEL

Based on Tobin's definition of the efficient (tangent) portfolio, Sharpe (1964) were among the theorists who in the 1960s developed the Capital Asset Pricing Model (CAPM) by applying many of the same assumptions as Fama did in his development of the efficient market hypothesis. The main assumptions are as follows:

1. Investors can trade all securities at competitive market prices, without incurring taxes or transactions costs, and can borrow and lend at the risk-free interest rate.

2. Investors are rational and hold only efficient portfolios of traded securities.

3. Investors have homogeneous expectations regarding the volatilities, correlations, and expected returns of securities.

The CAPM is a theory of equilibrium; rational investors with homogeneous expectations will identify and demand the same efficient portfolio, and since supply must equal demand, the efficient portfolio must be the market portfolio. Under these conditions, the efficient frontier including risk-free rate from Figure 3.2 is known as the *capital market line*, and the tangency portfolio equals the market portfolio. CAPM shows that the expected return of an asset, as given by equation 3.8, depends solely on its beta:

$$E(r_i) = r_f + \beta_i(E(r_m) - r_f) \tag{3.8}$$

Here $E(r_i)$ represents the expected return of asset i; r_f the risk free return; r_m the market return; and β_i the asset beta. The beta of the asset measures its volatility due to market risk relative to the market as a whole, thus capturing the security's sensitivity to market risk (Berk & DeMarzo, 2014). Intuitively, given that idiosyncratic risk is diversifiable, the expected return on an asset should be determined by the market risk premium, scaled by beta.

CAPM can be described as a one-factor model, as the asset return depends on a single risk factor, namely beta. When real returns differ from returns estimated by CAPM, alpha is used to describe this abnormality, and the asset alpha is defined as the excess return of the beta-adjusted market return (Bodie et al., 2009).

3.4 Limitations of Mean-Variance Optimization

Notwithstanding being a widely deployed and renowned framework, mean-variance optimization also has its limitations. In this section, we present the limitations of MVO related to approximation and estimation errors, static input, and the impact of time horizon, along with techniques to overcome some of these. Models developed in response to the limitations of MVO regarding approximation and estimation errors are presented in section 3.5.

Approximation Error

To arrive at optimal portfolios for each given level of risk, MVO relies on assumptions about the means, variances, and covariances of the different assets, while neglecting information on the assets' specific periodic returns and other features of their distributions, such as skewness or kurtosis. This approach is adequate for maximizing expected utility if at least one of two conditions holds (Cremers et al., 2003). The first is the assumption that returns are normally distributed. However, a significant limitation to the normality assumption is that it insufficiently accounts for extreme market moves (Swensen, 2009). Empirical studies have shown that real-world returns possess non-normal characteristics; two separate studies performed by economist William Nordhaus (2011) and investment research firm Morningstar (2011) found that asset class returns were more extreme than what could be predicted by a normal distribution. The financial crisis of 2007-2008 provided additional evidence of non-normal asset returns, with returns in this period being significantly skewed to the left. Variance, not only a crucial input to mean-variance optimization but also a symmetric measure, fails to identify such skewed distributions. Asset returns with a positive (negative) skew will thus appear riskier (less risky) to investors than they really are, leading to under-allocation (over-allocation) of the asset.

The alternative assumption that mean-variance optimization may rely upon is that investors have quadratic utility, as shown in Figure 3.3a. With μ being the expected return of the portfolio; λ the risk aversion; and σ the portfolio variance, quadratic utility is defined as:

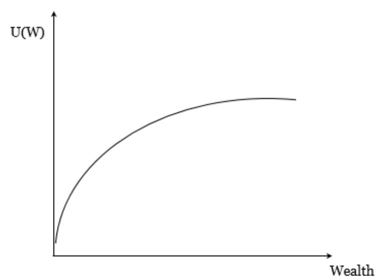
$$E(U) = \mu - \lambda\sigma^2 \tag{3.9}$$

Quadratic utility implies that risks are symmetrical; that investors are indifferent to upside deviations and downside deviations (Adler & Kritzman, 2007). However, this assumption stands in contrast to the findings of behavioral economists such as Kahnemann and Tversky (1979), suggesting that investors are risk averse in the domain of gains, but *risk seeking* in the domain of losses. Investors' tendency to strongly prefer avoiding losses to acquiring gains is a phenomenon known as loss aversion. This behavior is captured by an S-shaped value function, which Kahnemann and Tversky model as follows:

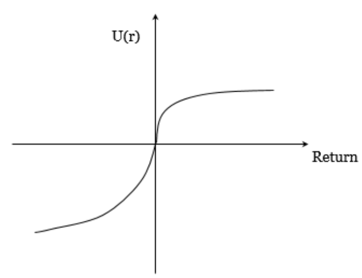
$$U(x) = \{ \begin{matrix} -A(\theta - x)^{\gamma_1}, \text{for } x \leq \theta \\ +B(x-\theta)^{\gamma_2}, \text{for } x > \theta \end{matrix} \tag{3.10}$$

Subject to:
$A, B > 0$
$0 < \gamma_1, \gamma_2 \leq 1$

Here, the portfolio's return is represented by x, and A and B are parameters that control the degree of loss aversion and the curvature of the function for outcomes above and below the return threshold, γ. Figure 3.3b shows an S-shaped value function with a threshold of 0 return.



(a) Quadratic utility function.



(b) S-shaped utility function.

Figure 3.3: Utility functions.

STATIC INPUT

Another limitation of MVO is its use of static input. In reality, asset correlations are time-varying (Swensen, 2009), which in times of market stress induce correlations that can deviate substantially from long-term correlation levels. Specifically, it has been shown that correlations among assets tend to increase during financial crises (Sandoval & Franca, 2012), implying that assets become increasingly correlated with the market, which in turn represents a raise in systematic risk. Given that MVO does not distinguish between firm-specific and systematic risk, the beta exposure of the portfolio will increase regardless of the diversification strategies implemented. Fortunately, in contrast to the short term deviations of asset correlations, long term correlations are significantly lower (Siegel, 2014). Thus, investors could overcome the static input limitation by extending their time horizon.

Estimation Error

In the process of selecting an optimal portfolio, some estimation error is likely to arise given the use of expected values as input parameters, rather than true values. Several studies have found that even small errors in input parameters may result in significant manifestations in portfolio weights (Lummer et al., 1994). Broadie (1993) distinguishes between the *estimated* frontier, the *actual* frontier and the *true* efficient frontier. The estimated frontier is obtained using estimated input parameters, as intended, while the actual frontier is obtained using the same weights as in the estimated frontier, but with actual returns and variances. The true efficient frontier is obtained using true, but unknown, input parameters. Investors relying on MVO to calculate the efficient frontier will thus find the actual frontier, which is always placed below the true efficient frontier due to estimation errors, thus consisting of inefficient portfolios.

The expected value of input parameters plays a pivotal role in the construction of optimal portfolios. Particularly, studies suggests that the expected asset class returns have the most significant impact on portfolio weights, followed by asset correlations and covariances (Chopra, 1993). Critics of MVO have argued that the framework essentially maximizes estimation error, by overweighting securities with large expected returns, negative correlations and small variances, and underweighting securities with small expected returns, positive correlations and large variances (Michaud, 1989). Simulations of estimation error has supported this view, showing that the estimated frontier increasingly overstates the performance of the actual frontier as the number of asset classes grows (Broadie, 1993). This phenomenon of overestimating values of measure is commonly known as the optimizer's curse, implying that actual outcomes on average are worse compared to the original estimates (Smith & Winkler, 2006).

To overcome the limitations of estimation error there are several applicable solutions. Imposing reasonable constraints could prevent highly concentrated portfolios, and ensure the inclusion of asset classes with desirable diversification qualities. Swensen (2009) proposes a lower limit of five percent and an upper limit of 25 to 30 percent for each asset class weight. Another option is to perform sensitivity analysis, varying input parameters to observe consequent changes in the efficient frontier, with the aim of selecting a portfolio that is close to efficient under several plausible sets of parameters (Lummer et al., 1994). By incorporating input uncertainty into the model, a portfolio that performs well in a number of different scenarios can be identified - a practice referred to as robust optimization (Fabozzi et al., 2007). Yet another solution is offered by the Black-Litterman model, which we cover in section 3.5.

Time Horizon

The MVO framework is a single-period model, most commonly employed using a one-year time horizon. Investors, however, may have objectives for their investments spanning several time horizons. This discrepancy may lead to suboptimal investment decisions made on the basis of MVO (Swensen, 2009).

Despite the extensive treatment of asset returns as completely independent of past returns, there is evidence suggesting the contrary. Stock returns have been shown to exhibit *mean-reverting* behavior, in which periods of poor performance relative to long-term returns have been followed by periods of overperformance, and vice versa (Poterba & Summers, 1988; De Bondt & Thaler, 1989). Contrarily, bonds have been found to exhibit *mean-averting* behavior; periods of deviation from the long-term return tend to be followed by periods in which the same type of deviation persists. Thus, the relative risk exposure inherent in different asset classes is dependent on the holding period. As a result, when the time horizon increases, the risk of stocks is declining relative to bonds, consequently affecting the efficient frontier (Siegel, 2014).

Goetzmann and Edwards (1994) simulated long-term returns incorporating auto-correlation between returns, showing that different time horizons lead to different efficient frontiers. Thus, investors' time horizons should be clearly defined in order to correspond with the time-horizon used in the mean-variance optimization.

3.5 Extensions and Alternatives to Mean-Variance Optimization

Full-Scale Optimization

In response to the approximation error associated with mean-variance optimization, full-scale optimization has been utilized as an alternative. Contrary to MVO, full-scale optimization can accommodate any set of return distributions and any description of investor preferences. Therefore, to the extent that the search algorithm used in full-scale optimization is effective, it yields the true optimal portfolio in sample, whereas MVO yields an approximation. The flip-side is that full-scale optimization is computationally intensive, and has thus, until recently, been infeasible to run due to lacking computational efficiency. To identify the portfolio weights that yield the highest expected utility, based on a plausible utility function in the presence of non-normal return distributions, expected utility for every period in the sample and for every possible combination of assets must be computed.

For instance, if the investor has an S-shaped value function, the utility for each period in the sample is computed as $B(x-\theta)\gamma^2$ if the weighted portfolio return is greater than the threshold θ, and as $-A(\theta-x)\gamma^1$ if the weighted portfolio return is less than or equal to θ, where x equals the weighted portfolio return. The threshold θ is where the investor's pain of losses becomes greater than the joy of an equal sized gain. As such, full-scale optimization can incorporate an investor's preference of loss aversion. A numerical search algorithm is used to search among the combinations of portfolio weights and evaluate each according to its utility value, to find the one that yields the maximum expected utility. To decrease the running time of the algorithm, heuristic functions are typically used to avoid searching through combinations of weights that are unlikely to be attractive.

This approach implicitly considers all of the features of the empirical sample, including skewness, kurtosis, and any other peculiarity in the distribution. However, like mean-variance optimization, full-scale optimization still suffers from estimation error. To the extent any of the features of the sampling distribution do not prevail out of sample, the full-scale solution will be suboptimal (Adler & Kritzman, 2007).

The Black-Litterman Model

Black and Litterman (1992) observed that quantitative optimization models such as MVO have an unfortunate tendency to produce unreasonable portfolio weights, particularly assigning large weights to a few assets while assigning zero weights to the majority. To their understanding, this predisposition is mainly caused by an extreme sensitivity to asset return assumptions, and driven by the difficulty of estimating asset returns. While these imbalanced weights might be optimal from a static viewpoint, they are unlikely to offer sufficient diversification benefits in the long run.

To overcome this problem, Black and Litterman introduced the idea of equilibrium returns. Their model is an extension to MVO in the sense that it derives expected asset returns – a critical input to MVO – from a combination of equilibrium returns and investor views. Given historical asset covariances, the Black-Litterman model calculates what each asset's expected return has to be in order for MVO to generate a portfolio in which its weight equals its observable market capitalization; i.e. its weight in the market portfolio. This process is known as *reverse optimization*, and the outputs produced are the equilibrium returns.

If the investor has assumptions about the expected returns of the various assets that differ from those of the market, the expected returns are adjusted accordingly. The investor's degree of confidence in each of these personal assumptions can also be incorporated into the model. In absence of investor views about the market, the expected returns will equal the equilibrium returns. The expected return estimates, whether tilted in favor of the investor's views or not, can finally be used as input to MVO in order to obtain the efficient frontier.

While the Black-Litterman model introduces new obstacles, particularly related to the difficulty of identifying the market portfolio, its primary benefit is that its use of equilibrium asset returns leads to intuitive and reasonable portfolio weights, without the need to add constraints to the optimization process. The model also provides a flexible framework for incorporating qualitative views about the market into the asset allocation.

3.6 Empirical Studies and Multi-Factor Models

Robo-advisors have generally adopted a strategy of passive management, the merits of which we will now review. Numerous empirical studies of market efficiency have been conducted, some of which making strong cases for passive management, while others have identified trading strategies generating return in excess of risk compensation. However, if a given investment strategy seemingly creates risk-adjusted excess return, this may be due to market inefficiency, but one can never rule out the possibility of the pricing model being inaccurately specified in its evaluation of risk and return, making any resolute refutation of the efficient market hypothesis practically impossible (Campbell et al., 1997).

According to Brinson et al. (1986; 1991), the total return on a portfolio is a net result of the strategic allocation of asset classes, market timing and security selection. Their studies found that the strategic allocation – the investment policy – provided not only the larger portion of return, but also explained more than 90% of the total variation in portfolio returns. Two main conclusions can be drawn from the majority of the studies in this field: Actively managed funds have on average not generated excess return net of costs, although studies isolating the skill level of managers show that managers have on average succeeded in selecting stocks that outperform the market.

Some studies even observe that managers are able to obtain risk-adjusted excess return net of costs (Berk & Green, 2004). However, Fama and French (2009) use simulations to show that good funds achieving risk-adjusted excess return net of costs are undistinguishable from the lucky bad funds. If outperforming the market is a result of skill, one would expect the outperformance to persist over multiple time periods. The literature has thus far provided contradicting conclusions on persistence

(Hendricks et al., 1993; Brown & Goetzmann, 1995; Carhart, 1997). Bessler (2008) finds that funds' performance persistence depends on the individual managers, and that employment turnover weakens fund level persistence.

Over the past three decades, several empirical studies have documented how certain investment strategies would generate returns that cannot be explained using traditional pricing models such as the CAPM. Consequently, these are called anomalies, of which the following three have been studied in most detail:

> *Size*: Historically, stocks of firms with smaller market capitalizations have on average outperformed stocks of firms with larger market capitalizations, even after adjusting for the differences in beta (Banz, 1981).
>
> *Value*: Historically, so-called value stocks, which are stocks of companies that have a high book value relative to market value, have on average had higher beta-adjusted return than stocks of companies with low book-to-market ratio (Stattman, 1980).
>
> *Momentum*: Jegadeesh and Titman (1993) found that short term trends in stock prices has a tendency to persist over some period of time.

These anomalies have broadly been interpreted in two different ways. Behavioral theorists consider them deviations from the assumed rational investor behavior underlying the efficient market hypothesis, typically under- or overreaction to news and an inclination to believe that trends will persist (Shiller, 1981; Lakonishok et al., 1994). DeBondt and Thaler (1985) find that stocks with past extreme bad returns outperform stocks with past extreme good returns, which in effect suggests that the value effect is a result of overreaction among investors.

Proponents of the efficient market hypothesis, on the other hand, argue that anomalies such as these represent compensation for risk not captured by the pricing model used when detecting them. Among those advocating the risk interpretation is Fama and French (1993; 1996), who went on to design the Fama-French three-factor model, and later the five-factor model, to replace CAPM as a pricing model for risky securities. They claim that the excess return generated as a result of the size and value factor represents premium for the increased risk of default that companies with small market capitalization or high book-to-market ratio typically carry (Fama & French, 1996). Carhart's (1997) four-factor model is an extension of the Fama-French three-factor model that also includes the momentum effect, although proponents of the efficient market hypothesis have thus far been unsuccessful in providing sufficient explanation to the momentum effect.

The Carhart Four-Factor Model

The Carhart four-factor model recognize the size, value and momentum effects as compensation for risk not captured by CAPM. In order to account for market capitalization, book-to-market ratio and past returns in expected return calculations, the four-factor model, described by equation 3.11, considers the returns of three self-financing portfolios. These are commonly referred to as the small-minus-big portfolio (SMB), the high-minus-low portfolio (HML), and the momentum portfolio (PR1YR).

$$E(r_i) = r_f + \beta_i(E(r_m) - r_f) + \beta_{\text{i},SMB}E(r_{\text{SMB}}) + \beta_{\text{i},HML}E(r_{\text{HML}}) + \beta_{\text{i},PR1YR}E(r_{\text{PR1}YR}) \tag{3.11}$$

Here, $E(r_{\text{S}MB})$ is the expected return on SMB; a factor portfolio made by a long position in an equal weight portfolio consisting of the stocks with market capitalization below the market median, financed by a short position in an corresponding portfolio of stocks with market capitalizartion above the median.

$E(r_{\text{H}ML})$ is the expected the return on the HML factor portfolio, which is equally weighted, long the stocks with book-to-market ratios greater than the 70th percentile on the market, and short the stocks with book-to-market ratios less than the 30th percentile.

Finally, $E(r_{\text{P}R1YR})$ is the expected return on the momentum portfolio, which is created by ranking all stocks in terms of their return over the past year. The momentum portfolio consists of a long position in the top 30% of the stocks with the highest returns and short the bottom 30%. The PR1YR portfolio must be updated every year to account for changes in relative returns.

Each of the factor betas, β_i, $\beta_{\text{i},SMB}$, $\beta_{\text{i},HML}$ and $\beta_{\text{i},PR1YR}$, is the expected percentage change in the return of a security for a one percent change in the return of the corresponding factor portfolio.

The Fama-French Five-Factor Model

In response to studies suggesting that much of the variation in average returns related to profitability and investment is left unexplained by the three-factor model, Fama and French (2015) added these two factors to form the five-factor model. The profitability factor (RMW) is the difference between the returns of firms with robust (high) and weak (low) operating profitability; and the investment factor (CMA) is the difference between the returns of firms that invest conservatively and firms that invest aggressively.

The expected return on an asset, as given by equation 3.12, is thus influenced by its exposure to these additional risk factors, $\beta_{\text{i},RMW}$ and $\beta_{\text{i},CMA}$:

$$E(r_i) = r_f + \beta_i(E(r_m) - r_f) + \beta_{\text{i},SMB}E(r_{\text{SMB}}) + \beta_{\text{i},HML}E(r_{\text{HML}}) + \beta_{\text{i},RMW}E(r_{\text{RMW}}) + \beta_{\text{i},CMA}E(r_{\text{C}MA}) \tag{3.12}$$

To identify the robust-minus-weak portfolio, RMW, all stocks are ranked according to their operating profitability. The RMW portfolio consists of a long position in stocks that have profitability placing them above the 70th percentile, and a short position in stocks with sufficiently low profitability to be placed in the 30th percentile. The investment factor is accounted for by the conservative-minus-aggressive portfolio (CMA), obtained by ranking stocks according to their total asset growth. The CMA portfolio has a long position in the 30% stocks with the lowest total asset growth, and a short position in the 30% stocks with the highest total asset growth.

Whilst Fama and French did not include the momentum factor in the model, given that it failed to to produce statistically significant changes in model performance in their tests, others have made the case for its inclusion.

4 Robo-Advisor Methodology

In the following section, we provide details on how the robo-advisors discussed in this paper - Betterment, Wealthfront, FutureAdvisor and Schwab Intelligent Portfolios - implement modern portfolio theory and passive investment as cornerstones of their methodologies.

4.1 Investment Philosophy

Considering robo-advisors in terms of the framework of Brinson et al. (1986; 1991), we observe that their portfolio returns derives solely from the component that is strategic allocation of asset class weights; disclaiming market timing and security selection. Betterment (2017) explicitly states that no tactical allocations are made, and Wealthfront's (2017) guidelines inhibit market timing. In order to maintain the initial investment policy, portfolio rebalancing, which we cover in section 4.6, is triggered by significant deviations in asset class weights.

Robo-advisors maintain investment discretion with respect to customers' accounts, which compromises the authority to make trades on customers' behalf. The robo-advisor model, grounded in a passive investment philosophy, assumes that events such as market swings can trigger irrational rebalancing of portfolios on the part of investors, violating the investment policy. For that reason, robo-advisors offer few opportunities for customers to make adjustments to their portfolios. The customer is usually only allowed to select specific trades that are proposed by the robo-advisor. Schwab Intelligent Portfolios (2017) is even more restrictive, allowing customer intervention only at the initial stage, when the strategic allocation is determined.

The robo-advisor methodology employs five steps:

1. *Asset class selection*: Identify an ideal set of asset classes to invest in
2. *Investment vehicle selection*: Select ideal investment vehicles to represent each asset class
3. *Constructing optimal portfolios*: Apply Modern Portfolio Theory to generate efficient portfolios of the chosen asset classes
4. *Risk tolerance assessment*: Determine the investor's risk tolerance in order to select an efficient portfolio with the appropriate level of risk
5. *Ongoing portfolio management*: Monitoring, rebalancing and tax-loss harvesting

In the rest of this section, we will elaborate on the techniques used by robo-advisors in each of these steps, and discuss the justification for their choices.

4.2 Asset Class Selection

The first step in the robo-advisors' investment methodology is to select asset classes that has desirable risk and return characteristics. According to Modern Portfolio Theory, it is necessary to choose asset classes with low correlation in order to increase the portfolios' diversification benefits. The traditional approach to asset allocation has typically yielded a blend of domestic stocks and bonds, and potentially a cash position. In response to the changing economic conditions in recent years, robo-advisors' asset allocation models have evolved from merely extrapolating from stocks and bonds' long-term historical results. With MPT as foundation, adjustments are made to reflect the new market realities, as well as the continuing expansion to non-traditional asset classes, such as gold and other commodities, and sub-asset classes. Stocks are divided into large and small, domestic and international, and developed and emerging markets. Bonds include Treasuries, agencies, investment grade corporate bonds and high-yield bonds.

One of these recent developments triggering the need for adjustment is that asset classes have become more highly correlated. This is a clear trend since the late 1990s, caused by the greater interconnectivity between global markets, and is especially true for equity assets (Schwab Intelligent Portfolios, 2017c; Wealthfront, 2017c). Global interconnectivity is further fuelled by the amounts of information available, and the speed with which it travels. The tendency to act quickly on breaking news is a contributing factor to market volatility, particularly in times of unease. In 2008, correlations increased due to the global financial crisis. Investors were surprised to find that the core equity portion of their traditional portfolios of 60% stocks and 40% bonds accounted for 99% of the total risk. To accommodate investors' loss aversion in recognition of these developments, robo-advisors seek to reduce equity risk and achieve a better balance of risk-taking (Schwab Intelligent Portfolios, 2017c).

While US government bonds' correlation with equities remain low, thereby offering significant diversification benefit in a portfolio weighted towards stocks (Wealthfront, 2017c), their income-producing appeal has lessened in recent years. Long-term interest rates, as measured by the yield on the 10-year Treasury bond, have declined considerably since the middle of the 1980s. Given that interest income is at historically low levels, particularly affecting investors that are in or near retirement and with a traditional asset allocation, robo-advisors feature other fixed income investments, including high-yield bonds, developed market bonds and emerging-market bonds. Since the expected stock return is the sum of the risk-free rate and the stock's risk premium, low Treasury yields also lowers the expected return on stocks.

Risk and return are essential factors when considering the inclusion of an asset class in an investment portfolio. The evaluation is however multifaceted. To identify an ideal set of asset classes for the current investment environment, robo-advisors consider each asset class' long-term historical behavior in different economic scenarios, risk-return relationship according to asset pricing theories, and expected future behavior based on long-term market trends and the macroeconomic environment. Asset class correlation is minimized to achieve greater diversification benefits. Each asset class is also evaluated in terms of its potential for capital growth and income generation, inflation protection, implementation cost and tax efficiency (Wealthfront, 2017c). For Betterment, Wealthfront, Schwab Intelligent Portfolios and FutureAdvisor, asset classes must be accessible through a sufficient number of securities in order to support tax-loss harvesting, which we cover in section 4.7.

Asset classes can broadly be divided into three main categories: equities, bonds and inflation assets. These serve several different purposes in a portfolio:

> *Growth*: Equity asset classes (US stock market, developed markets and emerging markets) have historically given the highest exposure to economic growth. While having a high volatility, equities offer the opportunity for long-term capital gains.
>
> *Income*: The most income-producing asset classes are fixed income investments, such as corporate bonds, international emerging market bonds, preferred stocks, bank loans and other

floating-rate notes. Dividend-paying stocks offer potential for both high yield and high return (income and growth).

Inflation protection: With Treasury Inflation-Protected Securities (TIPS), the principal, and thereby also the coupon (given by the constant coupon rate), increases with inflation and thus provide protection from it. Also real estate (REITs) and natural resources provide long-term protection from inflation in moderate and high inflation environments, given that their prices tend to be highly correlated with inflation. Stocks provide some degree of long-run inflation protection, however less than what REITs do.

Defensive assets: These are asset classes that generally have low or negative correlations with equities, and therefore provide a cushion for stock-heavy portfolios during economic turmoil. Examples include US Treasuries, developed markets government bonds, and gold.

Tax efficiency: Equities are relatively tax efficient due to the favorable tax treatment on long-term capital gains and stock dividends. Most bonds are tax-inefficient due to bond interest income being taxed at ordinary income tax rates, except for tax-exempt Municipal Bonds.

International Asset Classes

In accordance with their passive investment philosophy, robo-advisors include international asset classes to globally diversify their portfolios. For instance, a Betterment portfolio may be invested in up to 102 countries and more than 5,000 publicly traded companies across the world—along with exposure to bonds issued by governments, corporations and supranational institutions, as well as securitized debt. The rationale for investing in international asset classes is, as before, given by modern portfolio theory and the efficient market hypothesis. First, the global market offers additional diversification opportunities, which yields more efficient portfolios. Second, the argument of active management being a zero-sum game can be extended to world markets. The *average* active manager who holds a subset of the world market, such as a particular national stock market, will never outperform an investor holding the world portfolio weighted by market capitalization (Malkiel, 2015).

Home country bias is a well-documented behavioral bias among investors leading them to underrepresent foreign market investments in their portfolios; many even invest solely in domestic securities. Robo-advisor allocations, on the other hand, are close to the relative size of these markets, but does however vary with the level of investor risk tolerance. In particular, investing in emerging market equities usually increases the risk level of the portfolio. The upside is a high growth potential; The World Bank (2017) forecasts that in 2017, growth in developing economies will be twice as high as in advanced economies. Thus, emerging market equities should boost expected returns for investors with long enough time horizons to ride out the inevitable market fluctuations (Betterment, 2017c; Malkiel, 2015). On a related note, dollar-denominated emerging market bonds have high volatility, but also high coupon rates. In fact, given the increased correlations between equity asset classes in a world market that is tied closer together, as described above, growth potential is the primary reason for including foreign stock market asset classes in robo-advisor portfolios (Wealthfront, 2017c).

Tax Efficiency

Robo-advisors offer both taxable and tax-deferred accounts. The latter include Individual Retirement Accounts (IRAs) and 401(k)s. These account types reflect different investment goals. Tax-deferred accounts are designed for retirement savings and provide tax advantages; contributions are usually tax-deductible and all transactions and earnings within the tax-deferred account have no tax impact. As such, taxes are deferred until retirement, at which withdrawals are taxed as income. Generally, early withdrawal prior to retirement age is subject to being included in gross income plus a 10 percent additional tax penalty (Internal Revenue Service, 2012). Taxable accounts do not have the tax-advantages of a tax-deferred account, but does however offer more flexibility, as funds can be withdrawn at any time without incurring income taxes or penalties.

Robo-advisors attempt to minimize taxes by forecasting the taxes likely to be generated by any given asset class, and allocate different asset classes in taxable and tax-deferred accounts accordingly (Wealthfront, 2017c). Asset class selection with respect to account type can have a large impact on investment returns. Research indicates that tax-efficient asset placement can increase portfolio returns by 10 to 20 basis points per year when compared to simpler asset placement strategies (Daryanani & Cordaro, 2005; FutureAdvisor, 2017c). Robo-advisors start by placing the least tax-efficient funds into tax-deferred accounts, whenever doing so complies with the investor's goals and time horizon. For taxable accounts, Wealthfront (2017c) found that seven asset classes were sufficiently tax efficient to be deployed – TIPS, municipal bonds, dividend growth stocks, US stocks, foreign developed stocks, emerging market stocks and natural resources.

Given that income is generally taxed at a higher rate than capital gains, income-generating asset classes, such as bonds, are held in tax-deferred accounts. This allows income to compound and grow over time, deferred from taxation until withdrawal. The exception is federally tax-exempt municipal bonds, which due to their tax advantage is held in taxable accounts (Betterment, 2017a). Also REITs, which by law must return the majority – typically 90% – of their net rental income to investors every year, are held in tax-deferred accounts. In 2015, the yield on REITs were more than double the taxable income of US stocks (FutureAdvisor, 2017c). For taxable accounts, robo-advisors offer tax-loss harvesting.

4.3 Investment Vehicle Selection

The next step in the investment methodology is to identify the most appropriate investment vehicles to represent each selected asset class. All robo-advisors discussed in this paper invest in ETFs rather than in individual securities. ETFs are securities that typically track broad market indices for different asset classes, thus representing portfolios of securities. Unlike mutal funds, ETFs are traded on listed exchanges, like stocks, and are bought and sold during all open market hours.

Combining ETFs is a simple and cost-efficient diversification strategy, considering that each ETF has some degree of diversification on its own (Adjei, 2009). For instance, Vanguard Total Stock Market Index Fund ETF has 3,598 stock holdings, making it highly diversified according to Statman (1987), who found that a well-diversified stock portfolio should be composed of 30-40 stocks. Furthermore, due to the infrequent changes in the constituents of the underlying benchmarks, the expense ratios of ETFs tend to be much lower than the ones of mutual funds.

Other advantages of using ETFs as investment vehicles include transparency and clear goals and mandates. ETFs passively track benchmarks, and publicly disclose constituent assets and their weightings. Furthermore, over the last decade, the global ETF market has seen remarkable growth, resulting in a robust market of liquid ETFs, which also makes them easily substitutable (Betterment, 2017). Combined with the possibility of intraday transactions, ETFs offer investors a great deal of flexibility.

The very first stage of the ETF selection phase is to eliminate all ETFs that conflict with the investment philosophy, such as ETFs that are inverse, leveraged, actively managed, or have a narrow strategic focus (niche, category, geography, etc.) In general, robo-advisors select the ETFs that are most representative of the chosen asset classes, while offering market liquidity at the lowest possible fees and expenses. Betterment (2017) defines this as minimizing *frictions*, which is the set of factors causing the performance of the ETF to deviate from that of its benchmark. Betterment's measure of these frictions is summarized as the total annual cost of ownership, which is the sum of the costs associated with the aggregate trading activities of the average customer, as part of the ongoing management of the portfolio, and the costs associated with owning the fund.

$$\begin{gathered} Total\ annual\ cost\ of\ ownership = \\ \underbrace{bidask\ spread + low\ liquidity}_{\text{Cost-to-Trade}} + \underbrace{expense\ ratio + tracking\ difference}_{\text{Cost-to-Hold}} \end{gathered} \tag{4.1}$$

Since Betterment transactions are commission-free, the only component of transaction costs included in the Cost-to-Trade is the bid-ask spread. The second factor affecting Cost-to-Trade is the liquidity; the degree to which the ETF can be quickly bought or sold in the market without impacting its price. This is generally a function of the number of buyers and sellers in the market, which in turn is indicated by the average daily trading volume for the ETF. The more shares of an ETF Betterment needs to buy on behalf of its customers; the more volume is needed to complete the trades without affecting market prices. Therefore, average market volume is measured as a percentage of Betterment's regular trading activity.

Additional considerations are taken to select ETFs whose natural market efficiencies will not be disturbed by Betterment trading. Specifically, ETFs are assessed with respect to Betterment's potential relative share (RS) of assets under management and average daily trading volume, selecting only those that satisfy a maximum amount.

$$RS_{\mathrm{AUM}} = \frac{AUM_{\mathrm{Robo}-Advisor}}{AUM_{\mathrm{ETF}}} \leq Threshold_{\mathrm{AUM}} \tag{4.2}$$

$$RS_{\mathrm{Vol}} = \frac{Vol_{\mathrm{Robo}-Advisor}}{Vol_{\mathrm{ETF}}} \leq Threshold_{\mathrm{Vol}} \tag{4.3}$$

The principal component of the Cost-to-Hold, and of frictions as a whole, is the fund's expense ratio. The higher the expenses imposed by an ETF administrator, the lower the return net-of-fees that the investor is left with. Finally, the Cost-to-Hold also includes the benchmark tracking error; the under- or outperformance of an ETF relative to its benchmark index, for reasons such as weight deviations and trades with respect to the fund's holdings. The higher the tracking error, the less appropriate an ETF is to represent its asset class. ETF issuers can generally reduce the tracking error by improving their operational systems, but this adds expense that is ultimately passed on to the investor. In other words, robo-advisors must typically strike a trade-off between tracking error and expense. Tracking error may also stem from rebates from lending securities. Many ETF issuers generate income from lending out underlying securities to hedge funds to enable short sales. The more prevalent the lending, the higher the risk to the ETF buyer, which is particularly undesirable whenever the rebates are not shared among the ETF investors in the form of lower management fees.

Like Betterment, Wealthfront (2017) also searches for ETFs that minimize cost and tracking error - including the lending of underlying securities - and offer sufficient market liquidity. Newly issued ETFs are usually considered inappropriate for recommendation. Wealthfront emphasizes that each ETF is not evaluated in isolation, but rather in terms of its potential impact on the portfolio's overall risk-adjusted, after-tax return net-of-fees. For instance, while the other robo-advisors use three to four ETFs to represent the US stock market, Wealthfront has chosen to use only the Vanguard's Total Stock Market ETF (VTI). According to Wealthfront, this choice is justified by the broad market exposure provided by VTI while ultimately yielding a higher risk-adjusted, after-tax return net-of-fees. Similarly, ETFs with higher management fees may be chosen if its superior anticorrelation with the other asset classes results in a higher return.

Schwab Intelligent Portfolios (2017) does also consider bid-ask spread, liquidity, expense ratio and tracking error when selecting ETFs. Similar to Wealthfront, Schwab Intelligent Portfolios also rule out any ETF that has less than three months' of history. In addition, Schwab excludes all ETFs without sufficient assets under management, since these are at greater risk of closing, which may in turn cause tax complexities. Given that Schwab Intelligent Portfolios does not support fractional shares, ETFs that are below share price thresholds are selected in order to allow even low balance accounts to include all ETFs. In addition to tracking error, Schwab considers several factors when assessing whether an ETF reflects an asset class, such as geographic exposure, sector concentration, market capitalization and legal structure.

In addition to the ETF criteria common to the robo-advisors – bid-ask spread, liquidity, expense ratio and tracking error – FutureAdvisor also considers fund manager reputation, assets under management, and commissions (FutureAdvisor Support, personal communication, May 1, 2017).

4.4 Portfolio Allocation

In determining optimal combinations of selected asset classes, all four robo-advisors take a modern portfolio theory approach, combined with methods selected to overcome limitations of mean-variance optimization.

Forecasting Asset Class Behavior

Mean-variance optimization requires, as inputs, estimates for each asset class' expected return, standard deviation, and correlations with other asset classes. Robo-advisors base the latter two on short-term and long-term historical values. While long-term historical values benefit from larger sample sizes, short-term values more accurately capture current conditions stemming from market evolution. Forward-looking volatility implied by option prices is also included in the estimation of expected volatility (Wealthfront, 2017c).

There are some differences in methodology across the robo-advisor spectrum regarding estimation of expected asset class returns. Common to all is the use of the Black-Litterman model to modify the initial expected return estimates. Wealthfront (2017c) use CAPM as the baseline estimate, and make further adjustments using the Black-Litterman model and the Gordon growth model. The Gordon growth model is used to compare the market price of an asset against its predicted value, which is based on a future series of dividends that grow at a constant rate. The difference in the two values is viewed as an indication that the stock may be over- or undervalued by the market (Gordon &

Shapiro, 1956). Using the Black-Litterman model, Wealthfront incorporate their views on long-term return expectations for each asset class based on interest rates, credit spreads, dividend yields, GDP growth and other macroeconomic variables. Finally, ETF expenses and estimated tax liability due on each asset class' return is subtracted to derive a net-of-fee, after-tax expected return used as input to the MVO model.

In recognition of the outperformance tendency of small-capitalization stocks and value stocks, Betterment use the Fama-French three-factor model to estimate the expected return of each asset class. The Black-Litterman model is once again applied to adjust these estimates, however without ascribing their own market views (Egan, 2017). As such, the Black-Litterman model is used to produce diversified and intuitive portfolios by largely mitigating the optimizer's sensitivity problem and benefitting from the collective knowledge of the markets.

FutureAdvisor (2017) goes one step further by applying the Fama-French five-factor model in conjunction with the Black-Litterman model, expanding on the three-factor model with factors for profitability and investment. The assumption is, as previously described, that investors can reap persistently higher returns without any additional long-term risk by increasing their exposure to firms with high operating profitability, and to firms with low total asset growth. Schwab Intelligent Portfolios does not disclose their model of choice for estimating expected asset class returns.

Portfolio Optimization

The robo-advisors discussed in this paper all use mean-variance optimization to solve the efficient frontier of portfolios with maximum expected return for every level of risk. In extension of the previous discussion on asset class selection, robo-advisors also use MVO as an important tool for evaluating the number of asset classes to use in a portfolio, given that adding an asset class might raise the efficient frontier. While Wealthfront and FutureAdvisor rely by and large on MVO as a stand-alone model, Betterment and Schwab Intelligent Portfolios combine it with other models to adjust for its inability to capture investors' loss aversion.

To complement MVO, which is used to calculate the efficient frontier for *expected* outcomes, Betterment use a downside-risk optimization model to calculate the efficient frontier for *worse than expected* outcomes. This modeling for worst-case scenarios is used in an attempt to minimize downside risk by generating and evaluating a full range of future outcomes. The model is also used to stress-test their allocations through negative market scenarios in order to understand the potential severity of a drawdown, as well as its duration.

Schwab Intelligent Portfolios takes a somewhat different approach, whereby optimized portfolios are constructed by taking the average of the weights produced by mean-variance optimization and by full-scale optimization, as described in section 3.5. The full-scale optimization return threshold is set to zero. Consistent with the findings by Kahneman and Tversky (1979), the slope below the threshold is set to two, meaning that he undesirability of a loss is twice as high as the desirability of a similar sized gain. By averaging the two optimization methods, Schwab Intelligent Portfolios attempts to strike a balance between seeking the portfolio with the highest risk-reward relationship, and a portfolio that has preferences for loss aversion.

Weight Constraints and Final Adjustments

As a final precaution against estimation errors and to ensure proper portfolio diversification, robo-advisors enforce minimum and maximum allocation constraints on the mean-variance optimization, or make adjustments to the weights generated. Wealthfront (2017c) use 5% as a minimum allocation for all asset classes except for TIPS, which are inefficient for investors with moderate to high risk tolerance. The maximum allocation of any asset class is set to 35%. While Schwab Intelligent Portfolios (2017c) states that "investors should have at least some long-term exposure to all of the major equity markets", specific asset class weight constraints are not disclosed.

Both Betterment and Schwab Intelligent Portfolios (2017c) “de-risk” their portfolios by adjusting certain asset class weights through risk allocation. Higher weights are assigned to asset classes that contribute less risk relative to asset classes that contribute more risk. Schwab Intelligent Portfolios finishes by applying some qualitative judgment to ensure that the portfolios are in alignment with the investors’ intuition, goals and preferences.

4.5 Determining Customer Risk

In contrast to traditional portfolio managers, who assess customers’ risk tolerance through conversations, robo-advisors evaluate customers’ risk profile using an online questionnaire (Moyer, 2015b). Robo-advisors strive to ask as few questions as possible in order to keep the process straightforward and fast. To that end, Wealthfront (2017) uses behavioral economics in attempt to simplify the questionnaire. Combining subjective and objective questions, and placing more weight on whichever component that reflects the most risk aversion, the customer’s risk tolerance is determined.

Customers can later re-evaluate and adjust their own risk profile, allowing them to take on greater risk in periods where they have more funds to invest, and to decrease the risk in other periods. Wealthfront (2017) allows customers to adjust their risk profile every 30 days. The restriction on frequency is used as a measure against customers attempting to speculate in the form of market timing. Moreover, Wealthfront gradually adjusts customers’ portfolios in order to decrease volatility as the customer approaches retirement.

4.6 Rebalancing

The automated nature of robo-advisors makes them particularly well suited for performing portfolio rebalancing. Rebalancing is a key component of passive management, and refers to the process of periodically buying or selling assets that are over- or underrepresented in a portfolio, relative to the strategic allocation. Thus, given unchanged risk preferences and beliefs about the future, rebalancing allows the investor to revert to the optimal portfolio while also profiting from short-term gains. As rebalancing involves decreasing holdings of stocks that have increased in value while reinvesting in stocks that have decreased, it is effectively a way of buying low and selling high. The resulting gains are often referred to as the rebalancing bonus. Rebalancing may also occur as a consequence of changes in risk preferences, for instance due to the investor getting closer to retirement.

Robo-advisors hold an advantage over their human counterparts – be it individual investors or professional managers – when it comes to rebalancing. The rebalancing process should be carried out in a strictly disciplined manner, without the cognitive biases that humans are vulnerable to. Firstly, it is counterintuitive for humans to commit more capital to underperforming asset classes, particularly in times of market distress. However, history has shown that average annualized returns for equities bounce back to exceed the returns for fixed income securities for time horizons of 5 years or more following significant stock market downturns (Jaconetti et al., 2010). Investors who rebalanced during trying times were therefore rewarded; they maintained their target asset allocation while enjoying the subsequent equity returns.

Humans are evolutionary wired for herd mentality – following what others are doing – and recency bias – privileging information recently retained when making predictions about the future (Pompian, 2011). While both biases serve several basic purposes, they tend to create overconfidence in financial markets among investors, and have in turn played a significant role in causing market crashes. Investors are reluctant to sell high-performing assets, which according to theory are most likely to be the riskiest ones, notably equities. Without rebalancing, the portfolio weight on equities will over time approach 100%. Such portfolio has a higher expected return, but is exposed to risks that may be inappropriate for the investor.

A study by David Swensen, Chief Investment Officer at Yale University, found that rebalanced portfolios earned an average of 0.4% more per year, with less risk, over the ten-year period from 1992 to 2002, than portfolios that were not rebalanced (Swensen, 2005). Kaissar (2017) compared an annually rebalanced portfolio to a never-rebalanced portfolio over the time period of 1926 to 2016, which is as far back as stock market data exists. Both portfolios start out with a target asset allocation of 60% US stocks and 40% US bonds, represented by the S&P 500 and five-year Treasuries respectively. Consistent with modern portfolio theory, the non-rebalanced portfolio's stock allocation gradually drifts upward, to a maximum of approximately 99% stocks and 1% bonds in 2016. As its equity exposure increases, the portfolio displays higher return; its average annual return of 9.4% exceeds the rebalanced portfolio's return, 8.6% annually, by 0.8% per year. The results are nearly identical over shorter periods; 9.7% average annual return over rolling 10-year periods for the neglected portfolio, and 8.8% for the portfolio rebalanced yearly.

The catch is that the non-rebalanced portfolio is far riskier; it has a standard deviation of 16.4%, compared to 12.1% for the rebalanced portfolio. While the non-rebalanced portfolio produced the highest absolute return, the rebalanced portfolio produced the highest risk-adjusted return (Sharpe ratio). According to Kaissar (2017), the reason why investors taking on extra risk often end up with lower risk-adjusted returns is that they have trouble sticking to their guns through market ups and downs. Many make poor timing decisions, such as selling off investments during a market decline. Consequently, long-term returns suffer. For instance, over the past fifteen years, the average investor in the Vanguard 500 Index Fund, tracking the S&P 500, managed to capture just 68% of the fund's return. In comparison, the average investor in the Vanguard Intermediate-Term Treasury Fund captured 85% of the fund's return.

Thus, there is clear theoretical and empirical evidence in favor of rebalancing, and, given their deliberate nature, computers are ideal executors of the rebalancing process. This is however not the only advantage of robo-advisors over human investment managers with regard to rebalancing. Whereas investment managers incur time and labor costs when carrying out rebalancing (Jaconetti et al., 2010), these costs are close to zero for robo-advisors. They do however share other costs that come with rebalancing, including:

1. *Taxes*: If rebalancing within taxable registrations, capital gains taxes may be due upon the sale of appreciated assets.

2. *Transaction costs*: For ETFs and individual securities, transaction costs are likely to include brokerage commissions and bid-ask spreads. Thus, finding the ideal rebalancing strategy means optimizing the trade-off between risk control and cost minimization.

At the highest level, we distinguish between rebalancing strategies in terms of how frequently the portfolio is monitored for rebalancing purposes. Calendar rebalancing is the simplest and most widespread rebalancing strategy among individual investors and wealth managers (Daryanani, 2008; Fuscaldo, 2015). The strategy involves monitoring and, if necessary, rebalancing, at predetermined time intervals, such as monthly, quarterly, annually and so forth. Intermediate fluctuations not apparent in the drift at the time of rebalancing are thus unaccounted for. The other major rebalancing strategy is market-based rebalancing, also referred to as opportunistic or threshold-based rebalancing. Portfolio drift is monitored continuously, and rebalancing is triggered whenever the drift exceeds a predetermined percentage. Both calendar- and market-based rebalancing may vary in terms of the level of drift triggering rebalancing, and to what extent the portfolio is rebalanced to target. Resulting variations are sometimes referred to as hybrid rebalancing strategies.

Unsurprisingly, given computers' excellent ability to monitor continuously, all robo-advisors discussed in this paper perform market-based rebalancing. The portfolios are inspected for drift by an algorithm on a daily basis, and automatically rebalanced in either of these three cases:

1. *Sell/buy rebalancing*: Whenever the allocation of one or more asset classes deviates by more than the specified amount.

2. *Cash flow rebalancing*: Whenever there is a cash deposit or withdrawal, or if dividends from the portfolio accrues. These proceeds are used to invest in asset classes that are underrepresented, or, in the case of cash withdrawal, come from selling asset classes that are overrepresented.

3. *Allocation change rebalancing*: Whenever there is a change in the strategic allocation due to the customer changing the investment goals, time horizon or risk preferences.

The rebalancing triggers used by different robo-advisors vary. Schwab tolerates a drift of 2% in any asset class weight before rebalancing (Schwab Customer Service, personal communication, May 10, 2017). Betterment (2016) notifies customers whenever the asset class drift exceeds 2%; informing them on what deposit is required to reduce the drift to zero. Rebalancing is triggered once the drift reaches 3%. Wealthfront allows for a larger drift before rebalancing; for tax-deferred accounts, a drift of 4-6% is tolerated, and as much as 6-10% is tolerated for taxable accounts (Wealthfront, 2011; Wealthfront Client Services, personal communication, May 9, 2017). FutureAdvisor does not disclose their rebalancing triggers specifically, but employs a multi-level method. For instance, their rebalancing algorithm is twice as sensitive to deviations in the split between stocks, bonds and cash in the portfolio, as it is to deviations in subclasses such as domestic and international stocks (Simpson, 2017; FutureAdvisor Support, personal communication, May 10, 2017). Like Betterment, FutureAdvisor also notifies their customers of the need to rebalance.

Betterment (2016) is particularly sensitive to changes in the strategic allocation; even a change of 1% triggers the portfolio to be rebalanced entirely to match the new desired allocation, regardless of tax consequences. Wealthfront (2017) rebalance tax-sheltered accounts following a change in the strategic allocation within one business day, but may wait to rebalance taxable accounts for an extended period of time should it be deemed tax-inefficient to rebalance. [Schwab? FutureAdvisor?]

All robo-advisors employ algorithms that attempt to optimize rebalancing subject to tax and trading expense effects. As a first step, portfolio inflows and outflows that are taking place regardless of rebalancing are channeled to lessen the drift. Not only does this lower taxes, but unnecessary trading costs are also avoided. Using inflows such as deposits and dividends to buy underrepresented assets reduce the need to sell assets that have appreciated disproportionately, thus avoiding capital gains tax. Whenever the investor wants to withdraw cash from the portfolio, these funds are made available through the sale of overrepresented assets.

In the absence of sufficient cash flows to maintain tolerable portfolio drift, robo-advisors will reduce the drift by selling overweight asset classes, and use the proceeds to buy into the underweight asset classes. Hence, while there may be only one asset class differing enough from its target weight to trigger the rebalancing, other asset classes may also experience rebalancing trades (Koenig, 2017). FutureAdvisor state that they will rebalance in a tax-sheltered account whenever possible, and factor in capital gains when not (Simpson, 2017). Betterment (2016) will in principle reduce the drift to zero, but with the exception of cases when rebalancing demands sales that would realize short-term capital gains. Since short-term capital gains are taxed at a higher rate than long-term capital gains, Betterment will wait for these assets to become long-term before rebalancing. Meanwhile, the portfolio drift may stay above the rebalancing trigger of 3%. Such instances are usually a result of the portfolio being less than a year old.

Given that portfolios managed by Schwab, Wealthfront and FutureAdvisor consist of whole shares of ETFs, the rebalancing of these portfolios involves buying or selling at least one ETF share (Ludwig, 2016; Schwab, 2017a). Since both these firms place low limits on the minimum account size, this may represent a challenge for small accounts in the sense that obtaining the optimal asset class weight might require investing in only a fraction of an ETF. As a result, asset class weights may deviate from their targets. A feature unique to Betterment is that all funds are invested, even when it implies investing in fractional shares, thus avoiding this challenge faced by the other robo-advisors (Ludwig, 2016).

As the defining characteristic of market-based rebalancing, the actual number of rebalancing trades will vary depending on the market conditions. For instance, about twenty rebalancing trades would hypothetically have been triggered by Schwab's rebalancing algorithm during the financial crisis of 2008-2009; substantially higher than the median of three rebalancing trades per two years over the period 2002-2015 (Schwab, 2017a). As described above, the number of rebalancing trades will also depend on the number of cash flows entering or exiting the portfolio. FutureAdvisor report that their customers' portfolios are on average rebalanced 4-6 times per year (Simpson, 2017). While the robo-advisors share a reliance on market-based rebalancing to keep portfolio risk levels as consistent as possible without incurring unnecessary costs, they evidently disagree on the implementation details, which may cause significant variations.

4.7 Tax-Loss Harvesting

Selling appreciated assets for the sake of rebalancing can generate capital gains taxes, which tax-loss harvesting trades can help offset. Under US Law, capital losses can be used to offset capital gains, and thereby be utilized for tax purposes. A capital loss occurs whenever an asset is sold for less than its original purchase price. Should capital losses exceed capital gains in any tax year, they generally may be used to also offset up to $3,000 of ordinary income a year (Schwab, 2017a). Remaining losses can be carried forward to be used against future capital gains. Tax-loss harvesting is the systematic attempt to benefit from such laws, and to do so requires selling the depreciated asset in order to realize the capital loss. In order to retain a similar market exposure, the robo-advisors discussed in this paper follow a tax-loss harvesting strategy in which the proceeds from the sale are used to purchase a substitute security.

US law does however not allow for tax-loss harvesting in cases in which a *substantially identical* security is acquired within 30 calendar days before or after the sale. This is known as the *wash sale rule*, and it also applies to purchases made in a separate account belonging to the investor, in a spouse's account or by a company acting on behalf of the investor. The wash sale rule also covers contracts and options to acquire the security. Whenever a wash sale occurs, the loss resulting from the initial sale is disallowed for tax purposes, and instead added to the cost basis of the substitute security. The holding period for the substitute security will include the holding period of the original security. This adjustment postpones the loss deduction until the eventual sale of the substitute security (Internal Revenue Service, 2017).

A common way of avoiding a wash sale is to simply refrain from purchasing any securities for the 30 days after the tax-loss harvest, keeping the proceeds in cash. It is not just humans who resort to this strategy in order to ease the potentially complex task of tax-loss harvesting; even basic tax-loss harvesting algorithms may opt for holding all new deposits and dividends in cash during the 30 days following a harvest (Betterment, 2017b). However, a *cash drag* not only leaves a part of the portfolio unexposed to the market, thereby lowering the expected long-term return, but it also represents a deviation from the investor's optimal allocation. Therefore, the robo-advisors in this paper reinvest harvest proceeds into closely correlated assets, as well as investing unforeseen cash flows from the investor; attempting to maximize harvesting opportunities without sacrificing the investor's asset allocation.

Selecting an appropriate substitute asset can however be a bit of a challenge. US Congress has not provided any clear definition of *substantially identical*, thereby giving the Internal Revenue Service a lot of flexibility to rule on existing and future security types that could be used in transactions without substance (Michaels & Tilkin, 2012). For robo-advisor portfolios consisting of passive ETFs or index funds, two securities that track different indices are typically necessary to avoid violating the wash sale rule. Juggling two index funds from different issuers (e.g., Vanguard and Schwab) that track the *same* index is normally deemed substantially identical (Betterment, 2017b; Wealthfront, 2017b).

For these reasons, the robo-advisors discussed in this paper use a set of secondary ETFs whenever substituting for the primary ETFs becomes necessary (Betterment, 2017b; FutureAdvisor, 2017b; Schwab Customer Service, personal communication, April 18, 2017; Wealthfront, 2017b). Secondary ETFs track a different but highly correlated index from the primary ETFs. Each set of primary and secondary ETFs used by the respective robo-advisors are listed in Appendices A-D, along with correlations between the ETFs, and their expense ratios.

Betterment refers to this practice as *parallel position management*; allowing each asset class to be comprised of two closely related securities. A preference for the primary security is built into their algorithm, but always subject to tax considerations. Secondary ETFs are the first to be sold in the event of withdrawals or sales from rebalancing, but not at the expense of triggering short-term capital gains – in that case, lots of the primary ETF may be sold first. Similarly, primary ETFs are the first to be bought in the event of deposits, dividends and buys from rebalancing, unless doing so incurs larger wash sale costs than buying the secondary ETF. The direction of a harvest event – selling one

ETF to buy the other – depends purely on what has the greatest expected value. Dual representation of asset classes is not only a proven way of avoiding wash sales, but also sufficiently robust to allow the investor the flexibility of making frequent and haphazard transactions, without precluding the tax-loss harvesting process. Betterment (2017b) also states that their algorithm may even allocate deposits to secondary assets in anticipation of harvesting.

While the robo-advisors discussed in this paper share the tax-loss harvesting strategy of utilizing dual-security asset classes, there are some differences as to how long the secondary ETF is held before being switched back to the primary ETF, all other things being equal. Wealthfront switches back to the primary ETF after waiting out the 30-day wash-sale window, unless doing so would generate a short-term capital gain. Hence, if markets are flat in the time following the tax-loss harvest, Wealthfront will switch back to the primary ETF (Wealthfront Client Services, personal communication, April 18, 2017). The switchback condition of avoiding short-term capital gains is highly significant; given that all asset classes have positive expected returns, omitting this condition would likely mean a reduction in the tax-loss harvesting benefit caused by realizing capital gains from the sale of the secondary ETF. In cases where such capital gains exceed the capital losses on the initial harvest sale, rigid switchback strategies will even leave the investor worse off.

The other three robo-advisors – Betterment, Schwab and FutureAdvisor – will contrarily keep the secondary ETF indefinitely, only switching back to the primary ETF in the event of rebalancing, cash flows or sufficient losses triggering further tax-loss harvesting, as explained above (Betterment, 2017b; FutureAdvisor Support, personal communication, April 18, 2017; Schwab Customer Service, personal communication, April 19, 2017). While this strategy is more prone to the slight cost differences between the primary and secondary securities; the amount of trading, and thereby also transaction cost, is limited. The trade-off might be worthwhile; the difference in average expense ratio between a moderate risk portfolio at Betterment consisting of only primary securities, and one consisting of an equal split between the primary and the secondary, is less than two basis points (Betterment, 2017b). Other advantages of this strategy include an increased likelihood of reaching long-term holding periods for the assets in the portfolio, thereby being subject to a lower tax rate.

However, holding both the primary and secondary ETF indefinitely, and potentially harvesting losses of off both, increases the chance of triggering a wash sale though an (unrelated) inflow to a tax-deferred account. To avoid such instances, both Betterment and FutureAdvisor have a third correlated security (tracking a third index) for each harvestable asset class. These are only utilized to hold deposits in tax-deferred accounts. The tertiary ticker is immediately sold following the wash sale period, and replaced with the primary ticker (Betterment Support, personal communication, April 20, 2017). The tax-loss harvesting services of the robo-advisors discussed in this paper are summarized in Table 4.1.

Robo-advisor	**Account minimum**	**Tax-loss harvesting strategy**	**Secondary ETF holding period**	**TLH-monitoring**
Betterment	$0	Tertiary ticker system	Indefinitely	Daily
Wealthfront	$500	Conditional switchback (dual tickers)	31 days, subject to STCG	Daily
Schwab	$50,000	Dual ticker system	Indefinitely	Daily
FutureAdvisor	$20,000	Tertiary ticker system	Indefinitely	Daily

Table 4.1: Robo-advisor tax-loss harvesting services

The robo-advisors use a cost-benefit analysis framework to evaluate potential harvesting opportunities for each ETF lot currently trading below its cost basis [1]. A tax-loss harvest is executed if the benefit minus the cost exceeds a threshold. The depreciated ETF is then sold to recognize the loss, and the same dollar amount of the similar ETF is purchased. The benefit equals the potential capital loss multiplied by the long- or short-term capital gains tax rate, τ, depending on the holding period of the ETF being more or less than one year, respectively. The cost is the trading cost of selling one ETF and buying the other. The threshold is an estimate of the expected *future* harvesting benefit, modeled by assessing the likelihood of potentially capturing an even larger decline by waiting to harvest the loss. Expected return and volatility of each asset class are used to calculate this estimate (Wealthfront, 2017b).

$$Capital\,Loss_{t=1} \times \tau_{t=1} - Trading\,Cost \geq E(TLH\,Benefit_{\mathrm{t>1}}) \tag{4.4}$$

Another common feature of these robo-advisors is the *daily* scanning for harvesting opportunities. Although this has clear advantages that we will turn to shortly, increasing the tax-loss harvesting frequency also increases the risk of a wash sale occurring – particularly if the ongoing portfolio management also supports regular deposits, dividends and rebalancing. The complexity increases further in the case of multiple accounts – whether taxable, tax-deferred or belonging to a spouse. Given the manual effort thus required to coordinate the tax-loss harvesting process successfully, it has traditionally been conducted with less frequency and offered only to investors with high tax brackets and accounts of $5 million or more (Traff, 2016; Wealthfront, 2017b).

The tedious and continuous work of tax-loss harvesting is however ideally suited for computers. While there are now software alternatives available to assist investment managers with the tax-loss harvesting process, these solutions are expensive, and implementation is still not automated. Thus, not only is the customer likely to incur the higher trading costs associated with traditional investment managers, but also costs related to the time and labor necessary to execute the tax-loss harvesting trades. These additional costs cut down on the number of opportunities to successfully harvest a tax-loss (Traff, 2016).

Many investment managers still resort to the traditional strategy of only looking to harvest losses at the end of each fiscal year. Horan and Adler (2009) surveyed 322 wealth managers (mostly CFA charterholders), and found that the largest proportion of them harvest losses at specific periodic intervals, such as once a year. Infrequent harvesters are however likely to miss numerous loss-harvesting opportunities during the course of a year. Berkin and Ye (2003) quantified the benefits of *monthly* tax-loss harvesting for index portfolios under normal market conditions, by running a series of Monte Carlo simulations to generate a performance distribution over a 25-year period. Relative to the standard buy-and-hold strategy, the median cumulative after-tax benefit of monthly harvesting, net of all liquidation taxes, was a substantial 58 percentage point spread in alpha.

The authors also found that the number of tax-loss harvests are high at first, and then diminishes over time. Generally, the opportunity for loss realization reduces with the holding period, given that markets have positive expected return in the long run. Moreover, relative to the *periodic* tax-loss harvester, the *continuous* harvester will be able to harvest additional losses off of reinvested

[1] The original value, usually the purchase price, adjusted for reinvested dividends, capital gains, commissions and transaction fees.

proceeds from intermediate tax-loss sales. These can once again be reinvested, and a fraction of which turn into further additional harvest opportunities. The result is an exponentially decreasing function of tax-loss harvesting opportunities. This pattern implies that a diligent effort to harvest substantial losses whenever they occur is highly rewarding, as opposed to the infrequent harvester leaving many loss-harvesting opportunities on the table (Berkin & Ye, 2003).

Tax-loss harvesting is designed to take advantage of market volatility, and the consensus among financial empiricists is that returns are generally more volatile over shorter time horizons. Using over two centuries of U.S. equity returns, Siegel (2014) found that variances realized over investment horizons of several decades are substantially lower than short-horizon variances on a per-year basis. Under the random walk hypothesis, the standard deviation of each asset class' average real annual returns will fall by the square root of the holding period due to the Central Limit Theorem. With mean reversion, as shown by Siegel (2014), the standard deviation of these returns falls even faster.

Continuous tax-loss harvesting makes it possible to capitalize on short-term periods of market turbulence. By analyzing S&P 500 returns from 1950 to the present, FutureAdvisor (2017b) found that the broader the interval over which potential losses are evaluated, the more likely the underlying index is to have risen in price, and hence the fewer the tax-loss harvesting opportunities. While there were only 15 tax-loss harvesting opportunities on a yearly basis during this whole period, there were 6,586 opportunities on a daily basis. On a smaller scale, an interesting instance is Britain's vote to leave the European Union at the end of June 2016, which led to a short burst of volatility. The S&P 500 fell about 5.5% during the two trading days following the vote, but recovered quickly and went on to set new all-time highs within weeks, before entering a period of low volatility. Schwab (2017b) reports that this brief market turmoil led to a surge in tax-loss harvesting trades. Their algorithm triggered a total of 1200 tax-loss harvesting sell trades in June 2016; substantially higher than the median of 70 monthly tax-loss harvesting sell trades from April to September that year.

Research conducted by Wealthfront (2017b) indicates that daily tax-loss harvesting offers more than double the benefit than the year-end version. Their analysis suggests that the benefit of their tax-loss harvesting is an additional 1.08% in annual after-tax return for the average customer, assuming full liquidation after the holding period. Backtesting conducted by Betterment arrives at a more conservative estimate; tax-loss harvesting would have added 0.77% in after-tax return to the average customer over the past 13 years. The value of tax-loss harvesting can primarily be attributed to the compounding of reinvested tax savings, and pushing income and capital gains into the long-term capital gains tax rate, which is significantly lower than the tax rate for short-term capital gains. Tax-loss harvesting is a strategy to *defer* taxes, not avoid them. Specifically, realizing losses and purchasing the same dollar amount of a similar security will lower the cost basis of the asset class, which in turn is used to determine the capital gain upon liquidation.

For instance, assume a portfolio includes an investment of $100,000 in VEA as the primary security representing the Developed Markets asset class. The market declines to the point where VEA is worth $90,000. All shares of VEA are sold to harvest the loss, and replaced with a $90,000 position in SCHF as the secondary security for Developed Markets, realizing a loss of $10,000. The asset class recovers and the SCHF position closes the year at $100,000. Liquidating the portfolio at that time would trigger a short-term capital gain of $10,000, given that the cost basis for the Developed Markets asset class is now only $90,000. In this case, the harvesting generated no value.

However, assume instead liquidation is postponed, and that the harvested $10,000 capital loss is used to offset income and other unavoidable gains in the portfolio. The tax savings generated from the tax-loss harvest is then reinvested. Suppose the portfolio is liquidated several years later, at which time tax is due on the $10,000 appreciation of the Developed Markets asset class. However, this gain is then considered a long-term capital gain, and therefore taxed at a lower rate. Moreover, the money saved from deferring taxes initially is similar to an interest-free loan and is likely to have compounded over time. Consequently, tax-loss harvesting is most valuable for long-term investments. In special cases where the appreciated assets are never liquidated, but rather donated to charity or passed on to an heir, capital gains taxes are avoided permanently.

It is important to note that tax-loss harvesting is not necessarily universally beneficial. First, the harvesting of tax losses resets the holding period of the asset class, which is used to distinguish between long-term and short-term capital gains. This is unlikely to become an issue for long-term investors, but could represent an increased tax burden for investors with sporadic imminent withdrawal needs. Second, should all of the additional gains due to tax-loss harvesting be realized over the course of a single year, this might push the taxpayer into a higher tax bracket. Third, tax deferral is undesirable for tax bracket climbers, such as investors with high income growth potential. To the extent that the disadvantage of a higher future tax bracket outweighs the time value of potential reinvested tax-savings, it may even make sense to harvest gains, not losses. Conversely, tax-loss harvesting is particularly beneficial for high income earners; investors who incur substantial gains every year; and steady savers. By adding new price points to a portfolio of gradually decreasing cost basis, regular deposits create fresh potential tax-loss harvesting opportunities (Berkin & Ye, 2003; Betterment, 2017b).

5 Estimating Robo-Advisor Performance

In the following section we present our performance evaluation of the four robo-advisors covered in this paper: Betterment, Wealthfront, Schwab Intelligent Portfolios and FutureAdvisor. The assessment is divided into four parts. First, we backtest the robo-advisor asset allocations for an investor with moderate risk tolerance over investment horizons of eight and ten years. Second, we calculate the true efficient frontier based on empirical data, and compare the estimated performances of moderate, conservative and aggressive robo-advisor portfolios to those of efficient portfolios. Third, using the above-mentioned gross returns, we estimate robo-advisor returns net of fees and tax-loss harvesting effects. Finally, we approximate actual gross returns on each of the exact robo-advisor allocations, for every risk tolerance and over the time periods in which the robo-advisors have been in operation.

5.1 Methodology

Recreating Robo-Advisor Asset Allocations

In considering the publicly available robo-advisor asset allocations we observe that these clearly vary across the robo-advisor spectrum, as is evident from each robo-advisor's proposed portfolio for a moderate investor, which is shown in Table 5.1 (Betterment, 2017a; Wealthfront, 2017a; Schwab, 2017a; FutureAdvisor, 2017a). The specific investment vehicles chosen to represent each asset class are also publicized, and provided in Appendices A-D. The differences in asset class weightings, and to a lesser extent selected ETFs, are particularly noteworthy given that these robo-advisors all employ a version of mean-variance optimization and allegedly swear by passive management. Had all of them adhered to true passive management by holding the market portfolio, we would not expect to see such discrepancies in their portfolio weights for a given risk profile. The slight variances in their asset allocation frameworks are clearly causing notable differences in the resulting portfolios.

Asset Class	**Betterment**	**Wealthfront**	**Schwab**	**FutureAdvisor**
US stocks	33.3%	41.0%	30.0%	28.9%
International stocks	35.7%	31.0%	26.0%	43.4%
US bonds	19.6%	23.0%	15.5%	10.9%
International bonds	11.4%	0.0%	8.0%	4.1%
Other	0.0%	5.0%	20.5%	12.8%

Table 5.1: Portfolio composition for a moderate investor

Given that many of the ETFs robo-advisors invest in have existed for only a limited period of time, they cannot be used in the longer-term performance assessment of robo-advisors' investment strategies. Therefore, ETFs and research asset data with similar properties have been identified to construct portfolios with the same characteristics as the ones robo-advisors hold.

Investment research firm Morningstar (2017a) was used to analyze the asset compositions of the ETFs that robo-advisors invest in. For the equity part of the funds, Morningstar uses a 3x3 matrix to decompose the ETF into different stock categories (holding styles) along two dimensions: book-to-market ratio and market capitalization. As such, stocks are categorized as value stocks (high book-to-market ratio), blend stocks or growth stocks (low book-to-market ratio), as well as large cap, mid cap or small cap. Finally, we used data on the percentages of US and non-US stocks, as well as the division into developed and emerging market stocks, to obtain a full decomposition of each ETF.

Portfolio Visualizer (2017), an online portfolio analysis tool, was used for backtesting and for providing the data input necessary to calculate the efficient frontier. Portfolio Visualizer tracks returns for 37 asset classes, sourced from various ETFs, indices and research data sets. These well-established asset classes were weighted to reconstruct the robo-advisor portfolios by first analyzing and decomposing the asset-classes using Morningstar data, and later mapping this detailed data against the decomposition of robo-advisors' component ETFs. The same approach was repeated for the non-equity part of the portfolios.

Specifically, each ETF used by each robo-advisor was decomposed into the nine stock categories defined by Morningstar. Thereafter, each category was further divided into two subcategories with respect to emerging market stocks and developed market stocks - for a total of 18 categories. Having decomposed each individual ETF this way, the decompositions were then aggregated and normalized to yield the characteristics of the robo-advisor portfolio as a whole, resulting in a target portfolio which we aimed to replicate using some linear combination of the asset classes. The weights of each asset class were found solving the resulting linear equation system. To this end, we used Matlab solver tools, and non-negativity constraints were imposed on the weights, thus only allowing long positions. The resulting asset allocations for an investor with moderate risk preference are shown in Table 5.2.

Asset Class	Betterment	Wealthfront	Schwab	FutureAdvisor
US Large Cap Value	8.9%	9.3%	6.4%	8.9%
US Large Cap Blend	7.7%	11.3%	5.1%	7.7%
US Large Cap Growth	3.9%	9.4%	3.8%	3.8%
US Mid Cap Value	4.6%	2.4%	2.4%	2.4%
US Mid Cap Blend	3.2%	2.5%	2.0%	1.7%
US Mid Cap Growth	1.6%	2.7%	1.9%	1.6%
US Small Cap Value	1.6%	1.1%	3.5%	0.9%
US Small Cap Blend	1.3%	1.2%	3.2%	0.9%
US Small Cap Growth	0.6%	1.1%	1.7%	0.9%
International Stock Market	–	–	11.4%	–
Emerging Markets	6.2%	14.0%	4.5%	17.0%
International Value Stocks	–	–	4.6%	–
International Small Cap	–	–	5.5%	–
International Developed Markets	29.5%	17.0%	–	26.4%
Total bond	17.8%	23.0%	7.5	4.1%
Global bond	11.4%	–	8.0%	4.1%
Cash/money market	–	–	10.5%	–
Corporate bonds	1.8%	–	8.0%	–
Gold	–	–	5.0%	–
Commodities	–	5.0%	–	–
Domestic REIT	–	–	5.0%	6.4%
International REIT	–	–	–	6.4%
TIPS	–	–	–	6.8%

Table 5.2: Asset class weights for a moderate investor

BACKTESTING

The backtesting of the portfolios was performed using Portfolio Visualizer, for a selected time period of 8 years, from January 2009 to December 2016. The investment horizon considered was limited to the number of years for which data on the benchmark ETFs is available. However, by also backtesting the robo-advisor portfolios over a ten-year investment horizon, we are able to provide annual portfolio returns for each year since 2007, thereby including an assessment of robo-advisors' hypothetical performance during the financial crisis.

The backtesting shows the effect of investing $10,000 in each portfolio at the start of the period. In accordance with robo-advisors' rebalancing strategies, we perform market-based rebalancing with tolerance bands given by the minimum of an absolute 5% weight change and a 25% weight change relative to the target allocation. Thus, the deviation in any asset class weight never exceeds 5%. Note that this type of long-term performance estimation is possible given that only negligible changes has been made to robo-advisor allocations thus far. Similarly, most ETFs (whether those robo-advisors invest in or those representing asset classes) attempt to limit deviations from their target allocations.

Efficient Frontier

All 37 asset classes represented in Portfolio Visualizer, including US equities, international equities, fixed income, gold, cash and commodities, were included in the mean-variance optimization to obtain the true efficient frontier. As such, we used data on asset class correlations, annual returns and standard deviations provided by Portfolio Visualizer for the time period between January 2009 and December 2016. Equations 3.4 and 3.5 were used to calculate portfolio returns and standard deviatons.

Mean-variance optimization was performed in Excel using the Solver add-in, and the efficient portfolios were identified by solving for the minimum amount of volatility for each level of return. The optimization was carried out twice to generate two different efficient frontiers: one subject to only non-negativity constraints disallowing short sales, and one subject to an additional constraint limiting the maximum weight of any asset class to 30%. The latter is consistent with Swensen's (2009) recommendation for avoiding estimation error using MVO.

Benchmarks

Appropriate benchmarks were constructed as standards to compare the performances of the robo-advisors against. Given that robo-advisor portfolios are globally diversified, the Vanguard Total World Stock ETF (VT) was chosen as the measurement of "holding the stock market". Its return approximates the return of all stocks in the world - large and small, developed countries and emerging markets, in the exact proportions that these are valued in the global market. The traditional choice of the S&P 500 as a market proxy was discarded due to the fact that it merely represents Large-Cap U.S. stocks - about half of the stocks asset class as a whole.

Given that robo-advisors invest in bonds as well as equities, the Vanguard Total Bond Market ETF was combined with the Vanguard Total World Stock ETF to form the benchmark, in order to reflect the amount of risk carried by the portfolios. The different levels of risk inherent in conservative, moderate and aggressive robo-advisor portfolios are accounted for by considering three respective benchmark portfolios, each with a bond-to-equity ratio given by the the American Association of Individual Investors (2017), and shown in table 5.3.

Risk Level	**Equity (VT)**	**Bond (BND)**
Conservative	50%	50%
Moderate	70%	30%
Aggressive	90%	10%

Table 5.3: Bond and equity allocation used for benchmark construction.

5.2 Results

In this section, we present the results obtained from backtesting the robo-advisor portfolios and constructing the efficient frontier. We conclude the section by outlining the effects tax-loss harvesting, fees and other costs may have on the net returns of moderate robo-advisor portfolios. In section 5.3, we also provide the robo-advisors' actual returns for the relatively short time span they have been operating.

BACKTESTING

Backtesting of the portfolios show that all four robo-advisors outperformed the benchmark in terms of cumulative return over the eight-year post-financial crisis investment horizon. Cumulative returns from January 2009 onward are graphed in Figure 5.1. Table 5.4 shows the final balance at the end of the investment period for each of the portfolios, and also includes selected performance measures such as each portfolio's annualized volatility, Sharpe ratio, and correlation and beta with the US stock market.

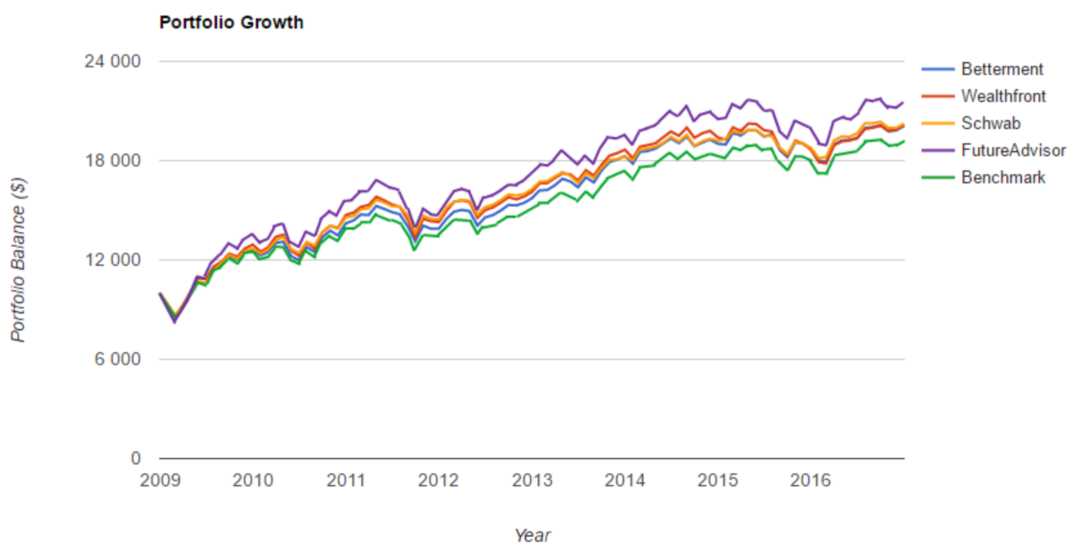


Figure 5.1: The growth of moderate robo-advisor portfolios and benchmark between January 2009 and December 2016

Portfolio	Final Balance	Standard Deviation	Sharpe Ratio	US Market Correlation	Portfolio Beta
FutureAdvisor	$21,411	14.20%	0.74	0.93	0.92
Schwab	$20,244	10.63%	0.87	0.93	0.69
Wealthfront	$20,150	11.91%	0.79	0.96	0.79
Betterment	$20,104	11.44%	0.81	0.95	0.75
Benchmark	$19,122	11.37%	0.76	0.95	0.74

Table 5.4: Selected performance measures for moderate robo-advisor portfolios and benchmark between January 2009 and December 2016, sorted by final portfolio balance. The portfolio beta equals the beta against the US stock market.

As shown in Table 5.4, FutureAdvisor's portfolio yields the highest growth over the investment period. The amount of risk it takes on is however sufficiently large for it to ultimately have the lowest risk-adjusted return. With the exception of FutureAdvisor, all robo-advisors have higher reward per unit of risk than the benchmark. We also observe that all but one portfolio follow a well-known pattern of higher risk yielding higher return, for which there is solid theoretical explanation. The exception is Schwab, whose high return for the relatively low level of risk taken on produce the highest risk-adjusted return out of all the portfolios.

To have a closer look at what might be driving the discrepancies in the performances of robo-advisors, which all claim to adhere to passive investing, we decomposed the equity part of their moderate portfolios and scaled it up to represent a full portfolio on its own. We then compared their allocations to stock categories of particular interest to the market capitalization of those stock categories, represented by their weight in the Vanguard Total World Stock ETF. The results are shown in Table 5.5.

	Betterment	**Wealthfront**	**Schwab**	**FutureAdvisor**	**Benchmark (VT)**
Large Cap Stocks	70.92%	75.63%	59.34%	72.43%	77.00%
Small Cap Stocks	6.69%	6.31%	18.38%	7.92%	6.00%
Value Stocks	40.37%	33.13%	42.27%	41.89%	35.00%
Emerging Markets	7.53%	15.89%	9.74%	20.21%	6.95%
Developed Markets ex US	48.08%	56.41%	53.53%	39.77%	52.33%
US Stocks	44.39%	27.70%	36.73%	40.01%	40.72%

Table 5.5: Decomposition of equity part of robo-advisor portfolios and benchmark, in terms of size, style and geography.

FutureAdvisor has a clear tilt in favor of emerging markets, which explains its higher volatility and higher return, which is a distinctive feature of emerging markets. In absolute terms, Wealthfront and Betterment have the least overall deviations from the market composition. With the Black-Litterman model as a starting point, used without applying personal views, and then tilting the portfolios in accordance with the Fama-French three-factor model, Betterment deliver market-like exposure with a value orientation. Its small cap tilt is however negligible. Given that Betterment makes value allocations with US equities only, as shown in Appendix A, and that US large cap value funds emphasize giant caps (Morningstar, 2017a), Betterment's value tilt increases its portfolio-weighted average market capitalization.

While Schwab does not disclose their model of choice for estimating expected asset class returns, the fact that Schwab places the largest weight on small cap and value stocks out of all the robo-advisors, might indicate its use of the Fama-French three-factor model. Schwab also invests in so-called *fundamental* funds, which are weighted based on multiple fundamental factors, including book value, dividends and cash flow, rather than just one, such as market capitalization. As such, fundamental funds represent another sort of value tilt. The rationale for using fundamental funds is that regular value-weighted funds are prone to place too much weight in over-priced stocks, and too little weight in under-priced stocks, thereby having higher exposure to market bubbles and crashes.

The positive beta values of the robo-advisor portfolios, as displayed in Table 5.4, imply that their returns move in the same direction as the US stock market return, which is unsurprising given their significant equity proportions. With beta values ranging from 0.75 to 0.92, robo-advisors will according to CAPM outperform the US market whenever the latter experiences a downturn, but perform worse in bull markets characterized by growth. We can observe this tendency in Figure 5.2, showing annual portfolio returns from January 2007 to December 2016. While the robo-advisors would have performed slightly better than the S&P 500 during the financial crisis of 2008, the S&P 500 has generally outperformed the hypothetical robo-advisor portfolios following the market upswing in 2009. Once again we can observe the risk inherent in the FutureAdvisor portfolio, which has the highest beta with respect to the US market of all the robo-advisors, and experiences large fluctuations during the period of market turmoil.

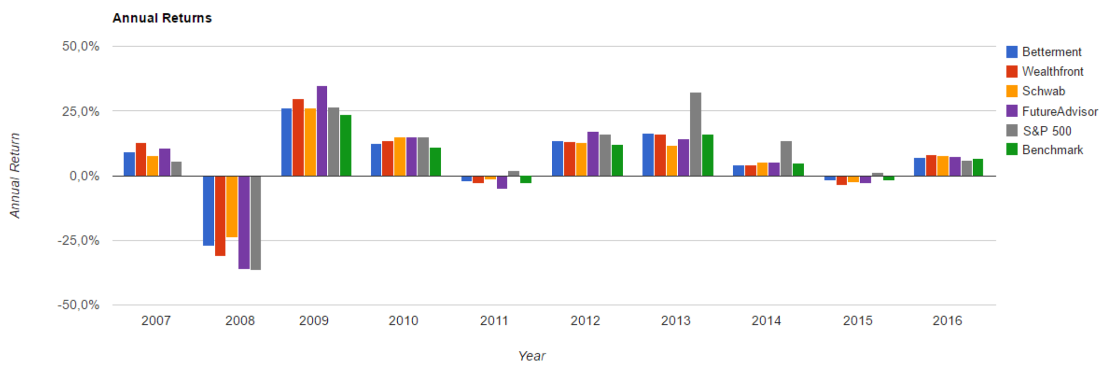


Figure 5.2: Annual returns of moderate robo-advisor portfolios compared to the S&P 500 and the moderate benchmark from January 2007 to December 2016. Note that the benchmark does not come into existence before 2009.

Efficient Frontier

Figure 5.3 shows the true efficient frontier, based on asset class returns, volatilities and correlations for the time period between January 2009 and December 2016. The light blue data points represent the risk and return of each asset class included in the optimization.

While the number of asset classes in the efficient portfolios range from one to seven, about half of them consist of fewer than four asset classes. By imposing weight restrictions placing a cap of 30% to the allocation of any given asset class, the efficient portfolios produced are sure to contain a minimum of four asset classes. Robo-advisors follow similar strategies to obtain proper long-term diversification of their investments. The efficient frontier calculated with weight constraints is depicted in orange in Figure 5.3, and seen relative to the true efficient frontier in grey. While the median number of asset classes in the efficient portfolios increases from four and a half to six, we observe merely slight changes in the efficient frontier.

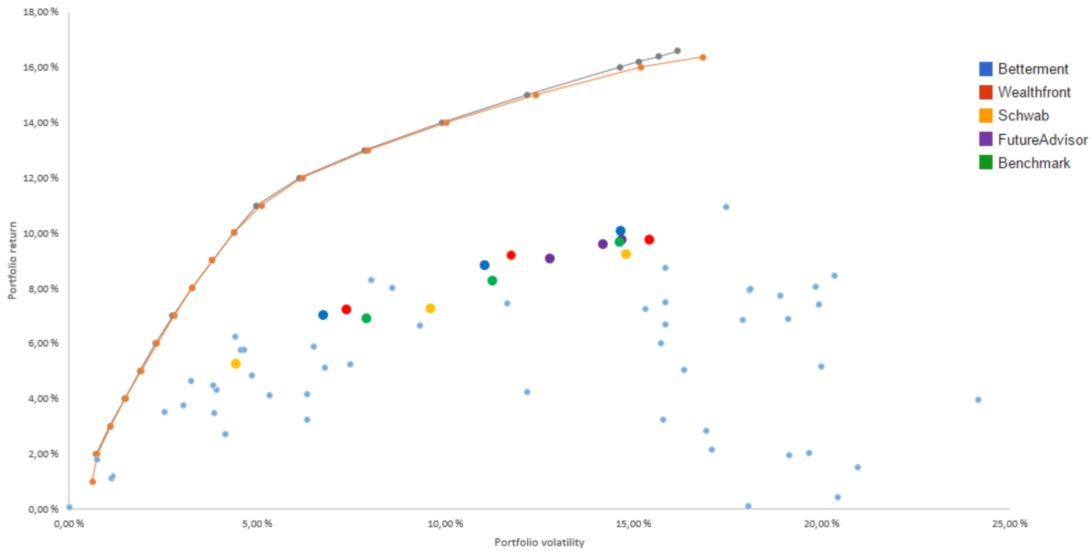


Figure 5.3: Efficient frontier based on empirical data, with time horizon from January 2009 to December 2016. The efficient frontier in orange is subject to an additional constraint on the maximum weight of any asset class.

For each robo-advisor, Figure 5.3 contains three data points representing the conservative, moderate and aggressive portfolio of that robo-advisor. As risk measured by volatility is increasing along the x-axis, the leftmost point represents the conservative portfolio and the rightmost represents the aggressive portfolio. The three benchmark portfolios with varying levels of risk are also included for comparison.

As is evident from the efficient frontier in Figure 5.3, the returns on the various robo-advisor portfolios are significantly lower than the returns on the equally volatile efficient portfolios. Comparing robo-advisor portfolios to one another and to the benchmarks, does however require that the portfolios have approximately the same level of either risk or return. As such, the relative performance of Schwab's conservative portfolio is difficult to evaluate, given that all other portfolios have both higher risk and higher return. Schwab's moderate and aggressive portfolio is inferior to Wealthfront's conservative and moderate portfolio, respectively, given that the latter yield the same return for a lower amount of risk.

Again, it is worth noting the high relative risk of not only FutureAdvisor's moderate portfolio, as we have already seen, but also its conservative portfolio. The fact that all three FutureAdvisor portfolios are so strikingly similar, and all placed at the top end of the risk spectrum, puts FutureAdvisor's ability to map customers' risk preferences onto appropriate asset allocations into question. In contrast, the three clusters of conservative, moderate and aggressive portfolios that we can observe in Figure 5.3 indicate that Wealthfront, Betterment and the AAII, whose recommendation the benchmarks are based upon, are in agreement as to the ideal type of portfolio for each type of investor risk preference. Betterment has a marginally higher Sharpe ratio at all three risk levels, and of these, the conservative portfolio comes out on top. In general, as shown in Table 5.6, of those investing with robo-advisors, conservative investors are on average rewarded with the highest risk-adjusted return, indicating that the model might be best suited for them.

Risk Level	**Average Sharpe Ratio**
Conservative	0.658
Moderate	0.562
Aggressive	0.507

Table 5.6: Average Sharpe ratios of robo-advisor portfolios with different risk levels.

Estimated Net Returns

All returns presented above are gross returns generated from income and changes in asset prices. Partaking in financial markets does however involve incurring expenses, which lower the net return on the portfolio. The net return is what the investor is ultimately left with for investing in risky securities and parting with funds for a period of time, and therefore what really matters to the investor.

To estimate the net return on the moderate portfolios of each of the four robo-advisors, we start by considering the estimated gross returns in Table 5.4, and subtract expenses. We approximate the latter to equal the sum of the advisory fee and the expense ratios of the underlying ETFs, as described in section 2.2. Thus, bid-ask spreads are omitted for the sake of simplicity. For comparison, we consider the moderate risk benchmark specified in section 5.1. The fees associated with investing in the benchmark portfolio are calculated as the sum of trading commissions ($8.95 per trade) and the average of the ETFs' expense ratios; 0.11% for VT and 0.05% for BND. In this case, we assume annual rebalancing of the benchmark portfolio, given that it is the most common rebalancing strategy and a likely choice for a passive investor seeking to minimize costs and effort. While rebalancing is offered by robo-advisors at no additional cost, rebalancing the benchmark portfolio involves commissions. Thus, increasing the rebalancing frequency will also increase the rebalancing costs of the benchmark.

Finally, due to the lack of more detailed data, we hold fixed any effect related to the desirability of robo-advisors' tax-loss harvesting algorithms, by assuming that Betterment's (2017c) estimated 0.77% in average additional after-tax return applies across the robo-advisor spectrum. This is a rough estimate, not only because the figure itself is an approximation, but also because robo-advisors' tax-loss harvesting algorithms are likely to vary in efficiency. Also, note that due to the complexities of accounting for other tax effects, which are highly dependent on the particular situation of the investor, these are omitted from the analysis. The choice of applying Betterment's estimated tax-loss harvesting benefit is based on it being the most conservative estimate, as well as the fact that Betterment provides the most extensive research on the matter.

Table 5.7 summarizes how fees and tax-loss harvesting may affect the net returns on moderate robo-advisor portfolios.

	Betterment	**Wealthfront**	**Schwab**	**FutureAdvisor**	**Benchmark**
Estimated Return	9.12%	9.15%	9.22%	9.98%	8.47%
Fees and Other Costs	(0.37%)	(0.14%)	(0.16%)	(0.65%)	(0.42%)
Tax-loss Harvesting Benefit	0.77%	0.77%	0.77%	0.77%	–
Net Return	9.52%	9.78%	9.83%	10.10%	8.05%

Table 5.7: Net annual returns for a moderate investor in the time period of between 2009 and 2016.

Remarkably, our outline of the net return calculation in table 5.7 suggests that robo-advisors' net returns are *higher* than the gross returns, due to the substantial benefit of tax-loss harvesting outweighing costs and advisory fees. Tax-loss harvesting is on the other hand not included for the benchmark portfolio, which means its net return is lower than the gross return, caused by the negative effect of costs. While any investor may in theory perform tax-loss harvesting as efficiently as a robo-advisor algorithm, doing so will involve considerable costs and effort. For instance, one popular software solution made to assist investors with tax-loss harvesting costs $20,000 a year (Traff, 2016). Moreover, the opportunities for tax-loss harvesting are much fewer for a portfolio with merely two assets.

5.3 Actual Returns

As a supplement to the estimated eight-year returns on robo-advisor portfolios produced by backtesting, we present here the actual returns for each of the robo-advisors examined in this paper. Given that many of the ETFs invested in by robo-advisors have existed for only short periods of time, we are only able to calculate actual returns for very limited investment horizons. To compare the actual returns of all four robo-advisors, we are kept to consider a one-year investment. However, by excluding Schwab Intelligent Portfolio from the group, we are able to extend the investment horizon to three years.

One-year investment

Figure 5.4 shows the actual cumulative returns on one-year investments in each of the moderate risk portfolios for the four robo-advisors, along with the cumulative benchmark return. Although the growth of the various portfolios track each other closely for the whole investment period, Schwab Intelligent Portfolios has the highest total portfolio growth, as well as a slightly higher portfolio balance for most parts of the year. Once again, all four robo-advisors have higher returns than the benchmark.

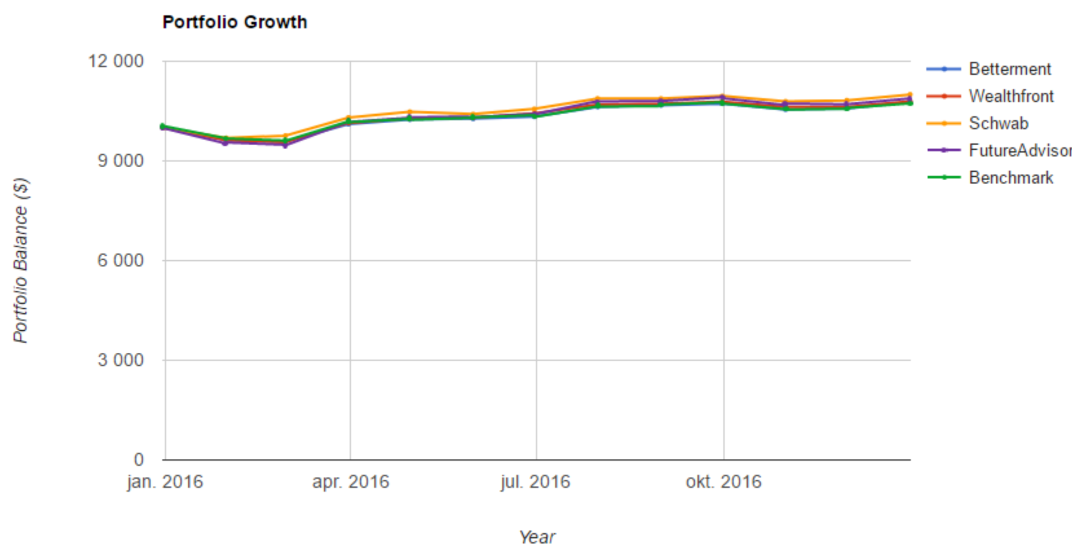


Figure 5.4: Actual performance of robo-advisor portfolios with a moderate risk level from January 2016 to December 2016

We also calculated the actual performance of one-year investments in moderate and aggressive robo-advisor portfolios. The results are provided in Table 5.8. All but one robo-advisor portfolio - the conservative Schwab portfolio - outperformed their respective benchmarks. FutureAdvisor offered the highest return among the conservative portfolios, but failed to produce substantially higher returns for their risk-taking customers. Schwab had the highest return not only among the moderate portfolios, but also among the aggressive portfolios.

Risk Level	**Betterment**	**Wealthfront**	**Schwab**	**FutureAdvisor**	**Benchmark**
Conservative	5.86%	6.14%	5.13%	8.12%	5.52%
Moderate	7.71%	7.93%	10.01%	8.66%	6.72%
Aggressive	9.33%	9.72%	12.63%	8.86%	7.92%

Table 5.8: Compound annual growth rates between January 2016 and December 2016

THREE-YEAR INVESTMENT

Next, we consider the actual performance of three-year investments in robo-advisor portfolios with conservative to aggressive risk characteristics. Schwab Intelligent Portfolios was first launched 2015, and is therefore excluded from this analysis. Figure 5.5 shows the actual cumulative returns for three-year investments in each of the moderate portfolios of the remaining robo-advisors.

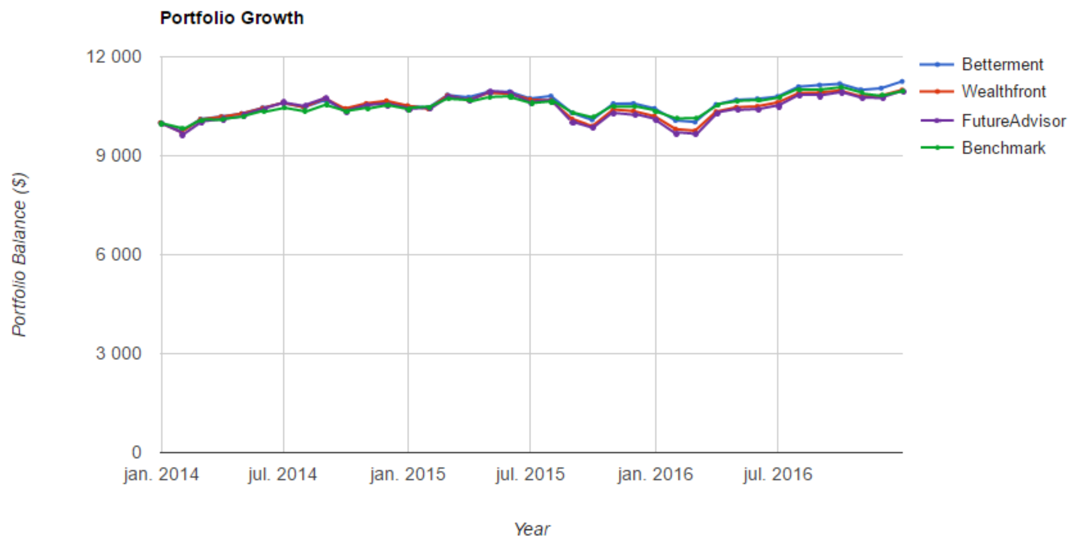


Figure 5.5: Actual performance of robo-advisor portfolios with a moderate risk level for the time period between January 2014 and December 2016

When considering the portfolio performance across all risk tolerances, as shown in Table 5.9, we observe that half of the robo-advisor portfolios are inferior to their respective benchmark, while half is superior. Betterment comes out on top for all risk levels. Remarkably, the returns on Betterment and Wealthfront portfolios are decreasing with the level of risk, which means that a conservative investor is better off than an aggressive one. As a three-year investment horizon is relatively short, the disappointing returns on the risky portfolios might only be a snapshot of a downturn caused by the volatilities of these investments. The decrease in return for riskier portfolios is however not the case for FutureAdvisor and the benchmarks.

Risk Level	Betterment	Wealthfront	Schwab	FutureAdvisor	Benchmark
Conservative	4.13%	3.66%	N/A	3.09%	3.15%
Moderate	4.01%	3.22%	N/A	3.17%	3.23%
Aggressive	3.76%	2.10%	N/A	3.20%	3.31%

Table 5.9: Compound annual growth rates between January 2014 and December 2016

6 TESTING ROBO-ADVISORY ON THE NORWEGIAN MARKET

In the following section we examine how two variations of the robo-advisor methodology can be applied to the Norwegian market. In order to backtest the hypothetical performance of these strategies using historical data, we begin by replicating their respective portfolio construction methods at the time of the initial investment. First, we identify appropriate asset classes to represent desirable subsections of the Norwegian market. Two efficient frontiers are generated using mean-variance optimization by inputting each set of values produced by the two allocation methods. Finally, portfolios with moderate risk characteristics are selected from each of the two efficient frontiers. Thus, the reader should note that the experiment is based on the investment opportunities available at the beginning of the investment period, as well as the projected desirability of each of these without any knowledge of the future. As such, the estimations are based solely on data available at the time of portfolio initiation.

Once having obtained the two hypothetical robo-advisor portfolios, we assess their performances relative to a benchmark by using data on actual returns for the investment period. This backtesting analysis, and thus necessarily also the forecasting of efficient portfolios, is performed twice for two different time horizons. We first construct the two portfolios with a ten-year investment horizon from 2007 to 2017. Next we apply the same methodology for five-year investments in the post-crisis years of 2012 to 2017.

With tax-loss harvesting being such a substantial, value-adding component of the robo-advisor model, we conclude the section with a qualitative assessment of the potential benefits of offering automated tax-loss harvesting services in Norway.

6.1 METHODOLOGY

ASSET CLASS SELECTION

The hypothetical Norwegian robo-advisor portfolios were created solely from Norwegian asset classes, in addition to some alternative asset classes such as gold, precious metals and commodities. The reason for not including international asset classes is that, according to the passive investment philosophy adopted by robo-advisors, doing so would have produced negligible allocations to Norwegian asset classes, since these constitute a very small part of the global market. Thus, given that non-US robo-advisors tend to invest primarily in asset classes local to the country in which they operate, and in order to fully assess the potential of the robo-advisor model in the Norwegian market, we therefore apply the robo-advisor methodology to the Norwegian market in isolation. For the investor who wishes to be globally diversified, these purely Norwegian portfolios could regardless be scaled down and included in a world portfolio.

The gold, precious metals and commodities asset classes are represented by well-established ETFs, the daily price data on which was sourced from Yahoo Finance (2017). Due to the lack of sufficiently granulated ETFs representing Norwegian asset classes, these were defined based on aggregated market data, as well as individual stock and bond indices. We used Netfonds (2017) as a source of daily data on Norwegian indices, as well as most of the individual asset data. Given that the Oslo Stock Exchange (OSE) does not have a distinctive large cap index, and that only two years of data are available on the OBX Mid Cap Index, index proxies tracking these subclasses had to be created from individual large cap and mid cap stock data. For the OBX Large Cap asset class, over 30 years of data on the largest companies on Oslo Stock Exchange, collected by Bernt Arne Ødegaard (2017), were used to represent the large cap index. Similarly, we backtracked data on the stocks currently constituting the OBX Mid Cap Index, OSEMX, and aggregated it to represent the mid cap index prior to 2015.

For the asset classes representing bonds, data on government-issued bonds with different maturities were retrieved from Norges Bank (2017) and combined to represent short-term and intermediate-term treasuries, as well as long-term government bonds. To represent the corporate bonds asset class, we aggregated the corporate bond funds of DNB, Nordea and KLP, using data provided by Netfonds. Lastly, the one-week Norwegian Interbank Offered Rate (NIBOR) was chosen to represent cash, given that it is the NIBOR rate with the shortest maturity.

The resulting asset classes used in the portfolio optimization are shown in table 6.1. Data on the asset classes was sourced back to 1997.

Asset Class
OBX All Share
OBX Total Return
OBX Small Cap
OBX Mid Cap
OBX Large Cap
Short Term Treasuries
Intermediate Term Treasuries
Long Term Bonds
Money Market (Cash)
Corporate Bonds
Real Estate
Gold
Precious Metals
Commodities

Table 6.1: Asset classes included in the portfolio optimization.

Robo-Advisor Strategies

With the two various investment strategies tested in this section we aim to replicate relevant parts of the robo-advisor methodology. As discussed in section 4, robo-advisors adhere by and large to a passive investment philosophy and rely heavily on MVO. Given the significant impact expected returns have on the portfolio weights generated by MVO, we find it worthwhile to test the one robo-advisor methodology based on estimating expected returns using CAPM – as used by Wealthfront – and the other estimating expected returns using a multi-factor model – as used by Betterment and FutureAdvisor. Given the data available to us, we chose the Carhart four-factor model as our multi-factor model for describing the risk-return relationship of an asset. The same restrictions on the optimization process were used for both of these models, as well as the same definition of a moderate-risk portfolio. We discuss this in greater detail later in this section.

ESTIMATION OF EXPECTED RETURNS

The two sets of expected asset class returns calculated using CAPM and the Carhart four-factor model are presented in Tables 6.3 and 6.6. For the CAPM method, three input parameters had to be determined; the risk-free return, r_f, the expected return of the market, $E(r_{\mathrm{M}KT})$, and the beta of each asset class with respect to the market, $\beta_{\mathrm{i},MKT}$. In recognition of the three risk-factors additional to market-risk, the Carhart four-factor model also considers the return on the small-minus-big portfolio, $E(r_{\mathrm{S}MB})$, the high-minus-low portfolio, $E(r_{\mathrm{HML}})$, and the prior one-year momentum portfolio, $E(r_{\mathrm{PR1}YR})$, as well as their associated beta values, $\beta_{\mathrm{i},HML}$, $\beta_{\mathrm{i},SMB}$ and $\beta_{\mathrm{i},PR1YR}$. The expected returns on each of the factor portfolios, calculated once as of 2007 and again as of 2012, are provided in table 6.2. It is worth noting that three out of the four factor portfolios have positive expected returns. The exception is the momentum portfolio, which in both cases has a negative expected return, implying a reversal in either of two ways; stocks that formerly performed relatively well performed relatively poorly, or vice versa.

Factor Portfolio	**Expected Return, 2007**	**Expected Return, 2012**
Market Premium	7.89%	3.14%
SMB	15.83%	11.82%
HML	6.10%	4.20%
PR1YR	−13.14%	−13.49%

Table 6.2: Expected returns on factor portfolios

The Oslo Stock Exchange Benchmark Index, OSEBX, was chosen as the market proxy. The OSEBX is designed to track the overall movements of the Oslo Stock Exchange, and therefore a natural choice of benchmark index for representing the Norwegian market. In accordance with Morningstar, the NIBOR 3-month rate was chosen to represent the risk-free return. The small-minus-big portfolio, the high-minus-low portfolio, and the prior one-year momentum portfolio are unavailable as indices for the Norwegian market, and therefore had to be constructed according to the criteria in Section 3.6. To that end, we collected relevant data sets from Ødegaard (2017), in which stocks on Oslo Stock Exchange are categorized according to size, book-to-market value and momentum.

The asset class beta values were estimated using linear regression. The market beta measures the market risk of an asset class, and represents the percentage change in the return of the asset class for a one percent change in the return of the market. Plotting the historical excess returns (relative to the risk-free return) of the asset class against the excess return of the market, the slope of the best-fitting line through all the data points is the estimated asset class beta. The beta estimations were performed in Excel, using daily excess returns and the built-in Data Analysis tool for the linear regression. Figure 6.1 shows the scatterplot of daily excess returns for the OBX All Share Index versus the OSEBX. Corresponding scatterplots were made to calculate each and every one of the betas, the resulting values of which are listed in Appendix E.

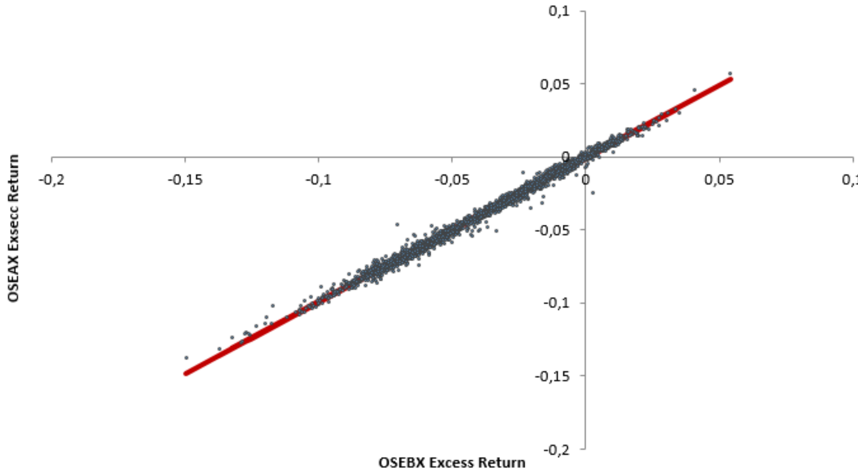


Figure 6.1: Scatterplot of daily OSEAX excess returns versus OSEBX excess returns from 1997 to 2007.

Estimation of Volatilities and Correlations

The expected volatilities and the correlation between asset classes were estimated based on historical returns for each asset class. Annual volatilities were obtained according to equation 6.1; from first calculating the standard deviation of daily returns, and then multiply by the square root of 252 – the number of trading days in a year. The resulting values are shown in Table 6.3 and Table 6.6. Asset class correlation matrices are based on historical figures, and presented in Appendix F.

$$\sigma_{annual} = \sigma_{daily}\sqrt{252} \tag{6.1}$$

Calculating the Efficient Frontiers

Using the values for volatilities, correlations and expected returns estimated by each of the robo-advisor strategies as input to MVO, two sets of efficient frontiers were generated; one forming the basis for portfolio optimization in 2007, and one forming the basis for portfolio optimization in 2012.

Consistent with Swensen (2009), Wealthfront (2017c) and Schwab Intelligent Portfolios (2017c), weight constraints were imposed to ensure sufficient diversification. Specifically, we placed an upper limit of 30% to the allocation of any asset class, as well as non-negativity constraints disallowing short positions. Alternative asset classes, such as cash, gold, precious metals and commodities, represent small parts of robo-advisor portfolios, if any at all, and did therefore demand special treatment. As such, stricter weight constraints were imposed on these asset classes. In accordance with the observed robo-advisor portfolios, the cash allocation was limited to a maximum of 15%, while each of the gold, precious metals and commodities allocations were limited to a maximum of 5%.

BACKTESTING

Next, we identified moderate risk portfolios on the resulting efficient frontiers generated using mean-variance optimization. The portfolios were identified using AAII's definition of a moderate risk portfolio as containing a 70% stock allocation, in conjunction with striking a level of volatility resembling that of the robo-advisor moderate risk portfolios.

Finally, we backtested the selected portfolios using daily asset class returns over the chosen time periods, and compared their performance relative to a benchmark. We enforced annual rebalancing in order to maintain the strategic allocations throughout the investment horizon, for both the robo-portfolios and the benchmark portfolio. The benchmark was constructed with a 70% equity allocation represented by the Oslo Stock Exchange Benchmark Index, and a 30% bond allocation comprised of three equally weighted asset classes: Short Term Treasuries, Intermediate Term Treasuries and Long Term Bonds.

6.2 RESULTS

TEN-YEAR ROBO-ADVISOR PERFORMANCE, 2007–2017

The expected asset class returns from 2007 onwards, estimated using the CAPM and the Carhart four-factor model, are presented in Table 6.3 alongside estimated asset class volatilities.

Asset Class	**Expected Return – CAPM**	**Expected Return – Carhart Four-Factor Model**	**Expected Volatility**
OBX All Share	12.73%	17.15%	19.23%
OBX Total Return	12.63%	16.84%	21.32%
OBX Small Cap	12.34%	17.27%	15.97%
OBX Mid Cap	12.23%	19.67%	16.76%
OBX Large Cap	6.43%	4.81%	21.87%
Short Term Treasuries	3.93%	4.15%	0.56%
Intermediate Term Treasuries	7.41%	9.30%	3.11%
Long Term Bonds	8.56%	11.28%	6.86%
Money Market (Cash)	4.82%	4.81%	0.03%
Corporate Bonds	11.45%	5.46%	4.02%
Real Estate	12.48%	16.61%	18.17%
Gold	10.63%	10.80%	15.25%
Precious Metals	12.46%	12.70%	21.92%
Commodities	19.37%	19.43%	19.29%

Table 6.3: Expected asset class returns and volatilities estimated as of 2007.

The efficient frontiers calculated on the basis of these sets of expected returns and volatilities are shown in Figure 6.2; one for each of the robo-advisor strategies. Given robo-advisors' reliance on mean-variance optimization, and that we have replicated other steps in their methodology, the efficient frontiers represent the range of portfolios typically recommended by robo-advisors for each level of risk preference.

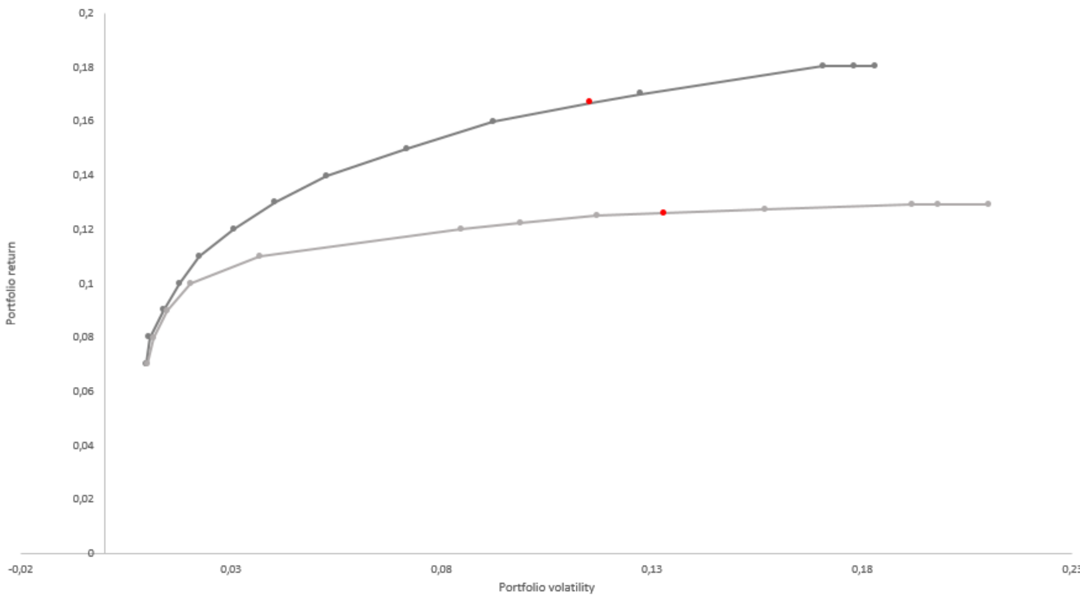


Figure 6.2: Efficient frontiers as of 2007; the light grey is based on expected returns estimated using CAPM, and the dark grey is based on expected returns estimated using the Carhart four-factor model. Both frontiers are subject to the same set of constraints.

The moderate risk portfolios selected from each of the efficient frontiers are marked in red in Figure 6.2. The asset classes constituent of each portfolio, as well as their respective portfolio weights, are provided in Table 6.4. As we can see, the portfolio created on the basis of the Carhart four-factor model is tilted in favor of stocks with lower market capitalization. In particular, while both models yield the maximum allocation to small cap stocks, the expected return on mid cap stocks is significantly higher according to the Carhart four-factor model than it is according to CAPM, manifesting in a much higher allocation to mid cap stocks. Relative to CAPM, the Carhart four-factor model also produces considerably lower expected return for the corporate bonds asset class, caused by its negative exposure to the positive size factor and positive exposure to the negative momentum factor, resulting in the complete exclusion of this asset class in the portfolio based on the Carhart model.

Asset Class	CAPM	Carhart
OBX All Share	30.00%	11.88%
OBX Small Cap	30.00%	30.00%
OBX Mid Cap	–	30.00%
Long Term Bonds	–	23.12%
Corporate Bonds	25.09%	–
Real Estate	9.91%	–
Commodities	5.00%	5.00%

Table 6.4: Moderate risk asset allocations, 2007.

The cumulative returns earned from investing $10,000 in each of the portfolios in Table 6.4 in 2007 are obtained from backtesting and shown in Figure 6.3.

Figure 6.3: Backtesting moderate risk robo-advisor portfolios between January 2007 and December 2016.

We observe that both portfolios track the benchmark relatively closely in the time period before the financial crisis of 2008. While the benchmark is initially hit harder by the financial crisis than the hypothetical robo-advisor portfolios, it is also quicker to bounce back. From 2009 onward, the benchmark portfolio achieves higher cumulative return than both robo-advisor strategies. With a volatility of 16.79%, as shown in Table 6.5, the benchmark is however also clearly the riskiest of the three investments, and is in fact outperformed by the portfolio based on the Carhart four-factor model in terms of risk-adjusted return. By outperforming the CAPM-based portfolio throughout the entire investment period, the robo-advisor strategy using the Carhart four-factor model is undoubtedly the superior one when applied to the Norwegian market. The CAPM-based portfolio produced a return barely above the risk-free rate, but at a much higher risk level, resulting in its low Sharpe ratio.

Portfolio	Final Balance	Standard Deviation	Sharpe Ratio
CAPM	$13.148	10.70%	0.006
Carhart	$14.770	7.43%	0.171
Benchmark	$16.229	16.79%	0.134

Table 6.5: Selected performance measures for two moderate risk portfolios and the benchmark portfolio between January 2007 and December 2016.

FIVE-YEAR ROBO-ADVISOR PERFORMANCE, 2012–2017

The estimated expected returns and volatilities presented in Table 6.6 portray the market outlook in 2012 according to CAPM and the Carhart four-factor model. When compared to Table 6.3, we observe that the expected returns are predominantly lower and the expected volatilities higher than in 2007. This can largely be attributed to the Great Recession of 2008 being included in the data sets that the more recent estimations are based upon.

Asset Class	Expected Return – CAPM	Expected Return – Carhart Four-Factor Model	Expected Volatility
OBX All Share	7.35%	11.04%	23.59%
OBX Total Return	7.39%	9.09%	26.45%
OBX Small Cap	7.01%	8.53%	17.51%
OBX Mid Cap	6.86%	8.94%	17.45%
OBX Large Cap	7.41%	9.17%	26.00%
Short Term Treasuries	3.12%	3.12%	0.66%
Intermediate Term Treasuries	5.02%	5.89%	3.22%
Long Term Bonds	5.43%	6.75%	7.09%
Money Market (Cash)	4.27%	4.24%	0.03%
Corporate Bonds	3.31%	5.56%	4.34%
Real Estate	6.65%	8.56%	21.75%
Gold	6.34%	8.60%	17.95%
Precious Metals	6.34%	8.37%	29.97%
Commodities	7.81%	7.11%	23.51%

Table 6.6: Expected returns and volatilities for each asset class estimated in year 2012.

The efficient frontiers reflecting market views in 2012 are shown in Figure 6.4. These frontiers are lower than the ones generated in 2007, caused by expected returns being lower and volatilites higher.

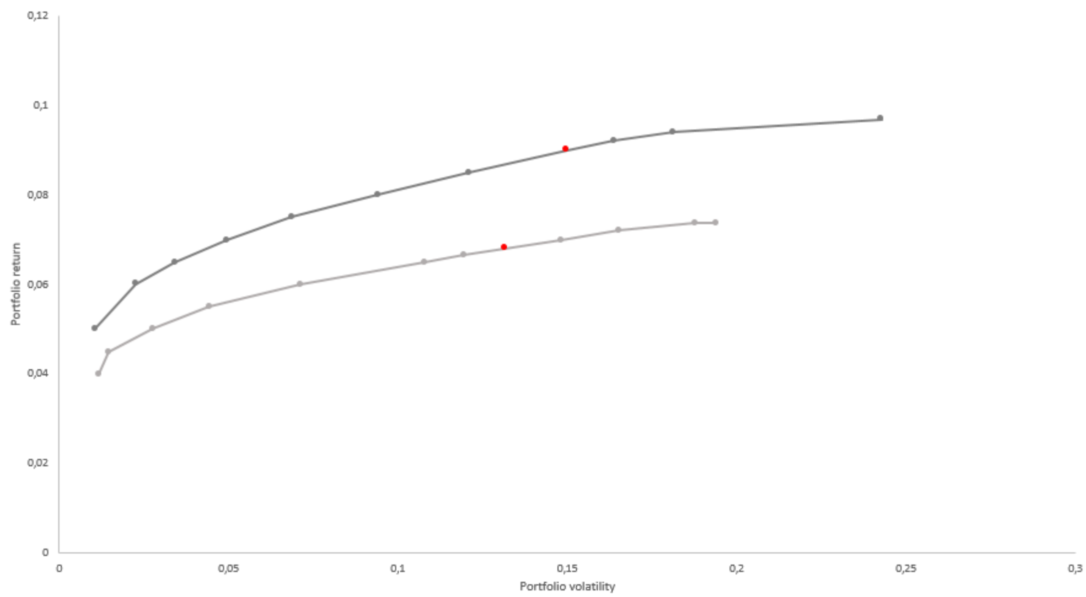


Figure 6.4: Efficient frontiers as of 2012; the light grey is based on expected returns estimated using CAPM, and the dark grey is based on expected returns estimated using the Carhart four-factor model. Both frontiers are subject to the same set of constraints.

The moderate risk portfolios selected are once again marked in red on the efficient frontiers, and their constituent asset classes provided in Table 6.7.

Asset Class	**CAPM**	**Carhart**
OBX All Share	30.00%	30.00%
OBX Small Cap	30.00%	10.96%
OBX Mid Cap	10.98%	30.00%
Long Term Bonds	19.02%	19.04%
Gold	5.00%	5.00%
Commodities	5.00%	5.00%

Table 6.7: Moderate risk asset allocations, 2012.

In comparison to the hypothetical robo-advisor portfolios originating in 2007, these portfolios contain one less asset class. The relative attractiveness of gold has increased since prior to the financial crisis, which is unsurprising given the tendency of investors to turn to gold as a stable investment in times of market turmoil. Compared to other precious metals used for investment, gold has the most effective safe haven and hedging properties (Low et al., 2016). While both robo-advisor portfolios originating in 2012 omit any allocation to the precious metals asset class, both include a maximum allocation to gold. The Carhart four-factor model yields a tilt in favor of mid cap stocks at the expense of small cap stocks, as a result of the former having higher exposure to the positive profitability-factor, as well as lower exposure to the severely negative momentum factor.

Figure 6.5: Backtesting moderate risk robo-advisor portfolios between January 2012 and December 2016.

From Figure 6.5 we observe that the benchmark still produces the highest cumulative return at the end of the investment period. However, both robo-advisor strategies have greater Sharpe ratios. Of those, the portfolio based on the Carhart four-factor model once again yields the highest risk-adjusted return. It is worth noting that the relative risk-adjusted performance of both robo-advisor portfolios increase in comparison to the ten-year case. The robo-advisor portfolios are seemingly performing better under normal market conditions.

Portfolio	**Final Balance**	**Standard Deviation**	**Sharpe Ratio**
CAPM	$14,035	6.26%	0.854
Carhart	$14,691	5.56%	1.140
Benchmark	$15,392	11.57%	0.624

Table 6.8: Selected performance measures for two moderate risk portfolios and benchmark between January 2012 and December 2016.

6.3 Tax-Loss Harvesting in Norway

Motivated by the fact that tax-loss harvesting, as described in section 4.7, represents such a defining and beneficial feature of the robo-advisor model, we conclude this section with a qualitative assessment of its potential for implementation in the Norwegian market.

Under Norwegian law, capital losses normally give entitlement to a tax deduction, conditional on the loss being realized; i.e., sold for less than its cost basis. Notably, whereas US law limits the use of capital losses to offset capital gains and up to $3,000 of ordinary income per year, Norwegian tax law permits capital losses to be offset against any kind of income (Skatteetaten, 2016a; Aarbakke, 2017). Thus, should capital losses cover total taxable income in a year, the investor pays no taxes. To the extent that capital losses exceed taxable income and result in a tax deficit, or should the investor have unutilized capital losses for some other reason, this is carried forward indefinitely (Skatteetaten, 2016b).

These regulatory conditions clearly provide fertile grounds for tax-loss harvesting. In addition, Norway has adopted the Nordic dual income tax. Under this system, labor and pensions are taxed at progressive rates as high as 46.9%, while capital income is taxed at a flat rate, currently set to 24% (Finansdepartementet, 2016). The fact that labor tax rates for most investors are higher than the capital gains tax rate, only increases the benefits of harvesting capital losses to offset taxable income.

The flat capital income tax rate also enhances the attractiveness of tax-loss harvesting to young investors, relative to investors in countries with progressive capital gains tax rates. Whereas US investors with high income growth potential face the issue of harvesting losses today and potentially end up paying more taxes whenever they reach a higher tax bracket in the future; their Norwegian counterparts do not. On the other hand, the flat rate also lowers the relative benefit Norwegian investors can reap from tax-loss harvesting, in terms of not being able to convert from short term capital gains to long term capital gains. As such, the value of tax-loss harvesting under Norwegian market conditions is attributed to the compounding of reinvested tax savings, and from pushing income into the capital gains tax rate.

Finally, there is currently no clear legislation on wash sales in Norway (Nordnet, personal communication, May 16, 2017). While this may represent increased opportunities for tax-loss harvesting with minimum risk-taking, it may also be a way of providing Norwegian tax authorities with maximum flexibility in their assessments of whether loss sales are substantial or not.

7 Conclusion

Our estimations of the performances of notable robo-advisors – Betterment, Wealthfront, Schwab Intelligent Portfolios and FutureAdvisor – show that all four of them outperformed the benchmark in terms of cumulative return over the eight-year post-financial crisis investment horizon from 2009 to 2017. More importantly, three out of four robo-advisors produced higher risk-adjusted return than the benchmark. We also find that the robo-advisor model is seemingly benefiting conservative investors the most.

For all investor risk preferences, the returns on the corresponding robo-advisor portfolios do however fall short of the returns on the equally volatile efficient portfolios. This is nonetheless unsurprising, given that the true efficient frontier is calculated using the realized values for asset class returns, correlations and standard deviations as input, whereas investment managers use expected values in order to project the future efficient frontier. In other words, robo-advisors base their recommendations on the estimated frontier, which is always placed below the true frontier. Consequently, one cannot expect an investment manager basing asset allocations on mean-variance optimization to obtain a truly efficient portfolio.

In our assessment the robo-advisor model's potential in the Norwegian market, we find that the robo-advisor strategy using expected returns estimated by a multifactor model, specifically the Carhart four-factor model, produced higher risk-adjusted returns than the benchmark for both investment horizons considered. While the CAPM-based portfolio also achieves higher Sharpe ratio than the benchmark over the five-year investment horizon from 2012 to 2017, it is consistently outperformed by the portfolio based on the Carhart four-factor model, indicating that the latter better describes the Norwegian market. We also find that current Norwegian laws make tax-loss harvesting, a pivotal feature of the robo-advisor model, particularly lucrative.

Given that robo-advisors conform by and large to passive investment, we find the flocking to robo-advisors in the wake of the recent Great Recession to be somewhat paradoxical, when it is this very "overconfidence in markets" that has largely been blamed for setting up the crisis in the first place. The financial crisis of 2008 represented a mighty blow to the efficient-market hypothesis, strongly indicating that markets do get it wrong sometimes. Perhaps it is the lessons of the Great Recessions that have led robo-advisors to stray somewhat from a true passive investment philosophy, as we observe in their varying asset allocations, and consequently also in their performances. These discrepancies can in turn be traced back to variations in their investment methodologies. While all four robo-advisors rely on the Black-Litterman model for estimating expected asset class returns, and on mean-variance optimization to produce the efficient frontier, some combine it with methods to overcome the limitations of MVO, and others add Fama-French tilts. These deviations from a truly passive portfolio represents active choices that investors should be particularly aware of.

We do however find that the portfolios of the two largest and most prominent robo-advisors, Betterment and Wealthfront, deviate the least from the world market composition, and both are similar to the benchmark in terms of risk and return characteristics. This indicates agreement among them as to what type of portfolio is ideal for each given level of investor risk preference. To arrive at these portfolios, the robo-advisors rely strongly on modern portfolio theory.

There is indeed some sort of reciprocal relationship between modern portfolio theory and robo-advisors in terms of suitability; the one lends itself especially well to the other. Mean-variance optimization was never particularly human friendly, relying on mathematical formulations and cumbersome calculations in order to produce recommended portfolios. Thus, reaping its benefits would have been impossible without the vast increase in computational power that has taken place over the past decades. Moreover, computers are to a larger extent able to uncompromisingly stick to modern portfolio theory, avoiding deviations caused by irrational human behavior. Robo-advisors are for their part unable to apply the insight of a human investment manager to any particular situation, relying solely on algorithms to manage customers' portfolios. In that sense, robo-advisors benefit immensely from the complete, well-studied and technical framework that mean-variance optimization represents.

Thus, we find the true innovation of the robo-advisor model not to lie in its component parts, which have existed for quiet some time, but rather in the consequences it may bring about. Financial markets, and the stock market in particular, is not only well-studied and highly efficient, but has brought prosperity and gratification to a selected few for decades, yet failed to reach the wider public with its benefits. Given its simplicity, cost-efficiency, transparency and accessibility, the robo-advisor model holds the promise of finally including the masses into the realm of investing.

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Appendices

A ETFs used by Betterment

Asset Class	Primary and Secondary ETF	Vendor	Underlying Index	Expense Ratio	Correlation Between ETFs
US Total Stock Market	VTI	Vanguard	CRSP US Total Market	0.05%	99%
	SCHB	Schwab	Dow Jones Broad US Market	0.03%	
US Large-Cap Value Stocks	VTV	Vanguard	CRSP US Large Cap Value	0.08%	99%
	IVE	iShares	S&P 500(R) Value	0.18%	
US Mid-Cap Value Stocks	VOE	Vanguard	CRSP US Mid Cap Value	0.08%	99%
	IWS	iShares	Russel Mid Cap Value	0.25%	
US Small-Cap Value Stocks	VBR	Vanguard	CRSP US Small Cap Value	0.08%	98%
	IWN	iShares	Russel 2000 Value	0.25%	
International Developed Stocks	VEA	Vanguard	FTSE Developed Markets Ex-North America	0.09%	99%
	SCHF	Schwab	FTSE Developed Markets Ex-US	0.08%	
Emerging Markets	VWO	Vanguard	FTSE Emerging Markets	0.15%	99%
	IEMG	iShares	MSCI Emerging Markets	0.16%	

B ETFs USED BY WEALTHFRONT

Asset Class	Primary and Secondary ETF	Vendor	Underlying Index	Expense Ratio	Correlation Between ETFs
US Total Stock Market	VTI	Vanguard	CRSP US Total Market	0.05%	99%
	SCHB	Schwab	Dow Jones Broad US Market	0.03%	
International Developed Stocks	VEA	Vanguard	FTSE Developed Markets Ex-North America	0.09%	99%
	SCHF	Schwab	FTSE Developed Markets Ex-US	0.08%	
Emerging Markets	VWO	Vanguard	FTSE Emerging Markets	0.15%	99%
	IEMG	iShares	MSCI Emerging Markets	0.18%	
Dividend Stocks	VIG	Vanguard	NASDAQ US Dividend Achievers Select Index	0.10%	97%
	SCHD	Schwab	Dow Jones US Dividend 100	0.07%	
Natural Resources	XLE	State Street	S&P Energy Select Sector Index	0.15%	73%
	VDE	Vanguard	MSCI Energy	0.14%	
TIPS	SCHP	Schwab	Barclays Capital US TIPS	0.07%	83%
	VTIP	Vanguard	Barclays Capital US TIPS 0-5 Years	0.10%	
Municipal Bonds	MUB	iShares	S&P National Municipal	0.25%	83%
	TFI	State Street	Barclays Capital Municipal	0.23%	

C ETFs used by Schwab Intelligent Portfolios

Asset Class	Primary and Secondary ETF	Vendor	Underlying Index	Expense Ratio	Correlation Between ETFs
US Large Cap Stocks	SCHX	Schwab	Dow Jones Total Stock Market Large Cap	0.03%	100%
	VOO	Vanguard	S&P 500	0.05%	
US Large Cap Stocks - Fundamental	FNDX	Schwab	Russell RAFI US Large Company	0.25%	99%
	PRF	PowerShares	Russell RAFI US Large Company	0.39%	
US Small Cap Stocks	SCHA	Schwab	Dow Jones US Small Cap Total Stock Market	0.05%	100%
	VB	Vanguard	CRSP US Small Cap	0.06%	
US Small Cap Stocks - Fundamental	FNDA	Schwab	Russell RAFI US Small Company	0.25%	98%
	PRFZ	PowerShares	FTSE RAFI US 1500 Small-Mid	0.39%	
International Developed Large Cap Stocks	SCHF	Schwab	FTSE Developed Markets Ex-US	0.08%	99%
	VEA	Vanguard	FTSE Developed Markets Ex-North America	0.09%	
International Developed Large Cap Stocks - Fundamental	FNDF	Schwab	Russell RAFI Developed ex-US Large Company	0.32%	98%
	PXF	PowerShares	FTSE RAFI Developed Markets ex-US	0.45%	

International Developed Small Cap Stocks	SCHC	Schwab	FTSE Developed Small Cap ex-US Liquid	0.12%	97%
	VSS	Vanguard	FTSE Global ex US Small Cap Index Net Tax	0.13%	
International Developed Small Cap Stocks - Fundamental	FNDC	Schwab	Russell RAFl Developed ex-US Small Company	0.39%	89%
	PDN	PowerShares	FTSE RAFI Developed Markets ex-US Mid-Small 1500	0.49%	
Emerging Market Stocks	SCHE	Schwab	FTSE Emerging	0.13%	98%
	IEMG	iShares	MSCI Emerging Markets	0.14%	
Emerging Market Stocks - Fundamental	FNDE	Schwab	Russell RAFI Emerging Markets Large Company	0.40%	97%
	PXH	PowerShares	FTSE RAFI Emerging Markets	0.49%	
US Exchange-Traded REITs	SCHH	Schwab	Dow Jones US Select REIT	0.07%	99%
	VNQ	Vanguard	MSCI US REIT	0.12%	
International Exchange-Traded REITs	VNQI	Vanguard	S&P Global ex-US Property	0.15%	97%
	IFGL	iShares	FTSE EPRA/NAREIT Developed Real Estate ex-US	0.48%	
US Treasuries	SCHR	Schwab	Bloomberg Barclays US 3-10 Year Treasury Bond	0.06%	94%
	VGIT	Vanguard	Bloomberg Barclays US 3-10 Year Government Float Adjusted	0.07%	

US Investment Grade Corporate Bonds	ITR	State Street	Bloomberg Barclays Intermediate US Corporate Bond	0.12%	70%
	VCIT	Vanguard	Bloomberg Barclays US 5-10 Year Corporate Bond	0.07%	
US Securitized Bonds	VMBS	Vanguard	Bloomberg Barclays US MBS Float Adjusted	0.07%	37%
	MBG	State Street	Bloomberg Barclays US MBS	0.20%	
US Inflation Protected Bonds	SCHP	Schwab	Bloomberg Barclays US Treasury Inflation Protected Securities	0.07%	74%
	STIP	iShares	Bloomberg Barclays US Treasury Inflation-Protected Securities (TIPS) 0-5 Year	0.10%	
International Developed Country Bonds	BNDX	Vanguard	Bloomberg Barclays Global Aggregate ex-USD Float Adjusted RIC Capped	0.12%	36%
	IGOV	iShares	S&P/Citigroup International Treasury Bond Ex-US	0.35%	
US Corporate High Yield Bonds	SHYG	iShares	Markit iBoxx USD Liquid High Yield 0-5	0.30%	84%
	JNK	State Street	Bloomberg Barclays High Yield Very Liquid	0.40%	

International Emerging Market Bonds	EMLC	VanEck	J.P. Morgan Government Bond Index - Emerging Markets Global Core	0.44%	61%
	VWOB	Vanguard	Bloomberg Barclays USD Emerging Markets Government RIC Capped	0.32%	
Gold and Other Precious Metals	IAU	iShares	Gold Spot	0.25%	92%
	GLTR	ETFS	1.1 oz Silver, .03 oz Gold, .004 oz Platinum, .006 oz and Palladium Spot	0.60%	

D ETFs USED BY FUTUREADVISOR

Asset Class	Primary and Secondary ETF	Vendor	Underlying Index	Expense Ratio	Correlation Between ETFs
Domestic Total	IVV	iShares	S&P 500	0.04%	99%
	SCHB	Schwab	Dow Jones Broad US Market	0.03%	
Domestic Value	SCHV	Schwab	Dow Jones US Large Cap Value Total Stock Market	0.04%	99%
	VTV	Vanguard	CRSP US Large Cap Value	0.08%	
Domestic Small Cap	SCHA	Schwab	Dow Jones US Small Cap Total Stock Market	0.05%	100%
	VB	Vanguard	CRSP US Small Cap	0.06%	
Developed Total	SCHF	Schwab	FTSE Developed Markets Ex-US	0.08%	99%
	VEA	Vanguard	FTSE Developed Markets Ex-North America	0.09%	
Developed Value	FNDF	Schwab	S&P Russell RAFI Developed ex-US Large Company	0.25%	99%
	EFV	iShares	MSCI EAFE Value	0.40%	
Developed Small Cap	SCHC	Schwab	FTSE Developed Small Cap ex-US Liquid	0.12%	97%
	VSS	Vanguard	FTSE Global ex US Small Cap Index Net Tax	0.13%	

Emerging Markets	SCHE	Schwab	FTSE Emerging	0.13%	99%
	VWO	Vanguard	FTSE Emerging Markets	0.15%	
Domestic REITs	SCHH	Schwab	Dow Jones US Select REIT	0.07%	99%
	VNQ	Vanguard	MSCI US REIT	0.12%	
International REITs	VNQI	Vanguard	S&P Global ex-US Property	0.15%	96%
	RWX	State Street	Dow Jones Global ex-US Select Real Estate Securities	0.59%	
Domestic Bonds	SCHZ	Schwab	Barclays US Aggregate	0.04%	85%
	BND	Vanguard	Bloomberg Barclays US Aggregate Float Adjusted	0.05%	
International Bonds	BNDX	Vanguard	Bloomberg Barclays Global Aggregate ex-USD Float Adjusted RIC Capped	0.12%	36%
	IGOV	iShares	S&P/Citigroup International Treasury Bond Ex-US	0.35%	
TIPS	VTIP	Vanguard	Barclays Capital US TIPS 0-5 Years	0.10%	87%
	STIP	iShares	Bloomberg Barclays US Treasury Inflation-Protected Securities (TIPS) 0-5 Year	0.10%	

E ESTIMATED BETA VALUES

Asset Class	β_{MKT}	β_{SMB}	β_{HML}	β_{PR1YR}
OBX All Share	0.98751	0.56490	0.54256	0.59595
OBX Total Return	0.97444	0.54096	0.51785	0.57169
OBX Small Cap	0.93719	0.59342	0.55631	0.59765
OBX Mid Cap	0.92306	0.88310	0.79013	0.86401
OBX Large Cap	0.18917	−0.07080	−0.04295	0.01813
Short Term Treasuries	0.02799	0.04374	0.03590	0.03535
Intermediate Term Treasuries	0.31250	0.21275	0.19548	0.20325
Long Term Treasuries	0.45811	0.30745	0.28272	0.29394
Money Market (Cash)	−0.01518	−0.00146	−0.00133	−0.00142
Corporate Bonds	0.26380	−0.48001	0.68173	0.39148
Real Estate	0.95469	0.48438	0.45207	0.47838
Gold	0.72154	0.02016	0.02052	0.02113
Precious Metals	0.95218	0.01590	0.01703	0.00879
Commodities	0.66758	0.01260	0.01312	0.01380

Table E.1: Estimated beta values for each asset class as of 2007.

Asset Class	β_{MKT}	β_{SMB}	β_{HML}	β_{PR1YR}
OBX All Share	0.97855	0.80356	0.85860	0.69717
OBX Total Return	0.99183	0.78830	0.85499	0.83031
OBX Small Cap	0.87160	0.79931	0.69196	0.80258
OBX Mid Cap	0.82385	0.77108	0.78004	0.76404
OBX Large Cap	0.99791	0.79818	0.86056	0.83610
Short Term Treasuries	0.00301	0.00427	0.00436	0.00281
Intermediate Term Treasuries	0.23912	0.25215	0.22779	0.22710
Long Term Bonds	0.36769	0.38722	0.35072	0.35037
Money Market (Cash)	−0.00099	−0.00109	−0.00056	0.00067
Corporate Bonds	0.06811	−0.0896	0.24388	−0.12417
Real Estate	0.75594	0.71120	0.70531	0.70054
Gold	0.65698	0.66711	0.61340	0.60751
Precious Metals	0.65869	0.65202	0.62555	0.61549
Commodities	0.62349	0.63415	0.61443	0.56224

Table E.2: Estimated beta values for each asset class as of 2012.

F Correlation Matrices

	Large cap value	Large cap blend	Large cap growth	Mid cap value	Mid cap blend	Mid cap growth	Small cap value	Small cap blend	Small cap growth	International Stock Market	Emerging Markets	International Value	International Small Cap	US Stock Market	Micro Cap	Intl Developed Markets	Intl Pacific Region	Intl Europe	REIT	Total bond	Global bond	TIPS	Long term treasuries	10-year treasuries	Intermediate term treasuries	Short term treasuries	Cash/money market	Corporate bonds	Long term corporate bonds	High yield bond	Short term inv grade	Long term tax exempt	Intermediate term tax exempt	Short term tax exempt	Gold	Precious metals	Commodities
Large cap value	1	0,99	0,95	0,97	0,96	0,92	0,94	0,93	0,91	0,89	0,85	0,89	0,82	0,99	0,87	0,89	0,82	0,87	0,81	-0,35	-0,16	-0,25	-0,42	-0,44	-0,45	-0,31	0,01	-0,38	-0,46	0,22	-0,26	-0,16	-0,17	-0,08	0,03	0,68	0,3
Large cap blend	0,99	1	0,98	0,97	0,97	0,95	0,94	0,94	0,93	0,9	0,86	0,9	0,83	1	0,88	0,9	0,84	0,88	0,79	-0,36	-0,16	-0,26	-0,43	-0,45	-0,45	-0,31	0	-0,38	-0,46	0,22	-0,27	-0,16	-0,17	-0,08	0,03	0,69	0,31
Large cap growth	0,95	0,98	1	0,95	0,97	0,97	0,92	0,94	0,94	0,89	0,86	0,89	0,82	0,99	0,88	0,89	0,83	0,87	0,75	-0,36	-0,15	-0,25	-0,43	-0,44	-0,43	-0,29	0	-0,36	-0,44	0,22	-0,26	-0,15	-0,16	-0,07	0,04	0,69	0,32
Mid cap value	0,97	0,97	0,95	1	0,99	0,95	0,97	0,97	0,95	0,87	0,84	0,88	0,83	0,98	0,91	0,87	0,8	0,85	0,85	-0,34	-0,15	-0,23	-0,41	-0,43	-0,42	-0,29	0,01	-0,36	-0,44	0,24	-0,25	-0,15	-0,15	-0,06	0,04	0,69	0,32
Mid cap blend	0,96	0,97	0,97	0,99	1	0,99	0,96	0,97	0,97	0,88	0,86	0,88	0,83	0,98	0,91	0,88	0,82	0,86	0,81	-0,35	-0,14	-0,24	-0,42	-0,44	-0,43	-0,29	0,01	-0,36	-0,44	0,25	-0,25	-0,15	-0,15	-0,06	0,05	0,71	0,35
Mid cap growth	0,92	0,95	0,97	0,95	0,99	1	0,92	0,95	0,96	0,87	0,86	0,87	0,82	0,97	0,89	0,87	0,82	0,85	0,75	-0,35	-0,13	-0,23	-0,42	-0,43	-0,42	-0,28	0,01	-0,36	-0,43	0,25	-0,25	-0,15	-0,15	-0,07	0,07	0,71	0,38
Small cap value	0,94	0,94	0,92	0,97	0,96	0,92	1	0,99	0,97	0,83	0,8	0,84	0,8	0,95	0,95	0,83	0,77	0,82	0,86	-0,33	-0,15	-0,24	-0,41	-0,42	-0,42	-0,28	0,01	-0,36	-0,43	0,21	-0,25	-0,15	-0,15	-0,06	0,02	0,66	0,28
Small cap blend	0,93	0,94	0,94	0,97	0,97	0,95	0,99	1	0,99	0,84	0,82	0,85	0,8	0,96	0,96	0,84	0,78	0,83	0,83	-0,34	-0,14	-0,24	-0,42	-0,43	-0,42	-0,28	0,01	-0,35	-0,43	0,23	-0,24	-0,15	-0,15	-0,06	0,06	0,61	0,42
Small cap growth	0,91	0,93	0,94	0,95	0,97	0,96	0,97	0,99	1	0,84	0,82	0,84	0,8	0,95	0,95	0,82	0,76	0,79	0,74	-0,34	-0,13	-0,23	-0,42	-0,43	-0,4	-0,27	0,01	-0,34	-0,42	0,24	-0,23	-0,15	-0,15	-0,06	0,06	0,6	0,41
International Stock Market	0,9	0,91	0,9	0,88	0,9	0,89	0,84	0,86	0,85	1	0,92	0,97	0,95	0,89	0,79	0,99	0,91	0,97	0,68	-0,3	-0,02	-0,18	-0,4	-0,41	-0,38	-0,24	0,02	-0,3	-0,4	0,3	-0,18	-0,16	-0,16	-0,06	0,15	0,69	0,54
Emerging Markets	0,85	0,86	0,86	0,84	0,86	0,86	0,8	0,82	0,82	0,92	1	0,86	0,88	0,84	0,75	0,87	0,83	0,83	0,66	-0,3	0,01	-0,16	-0,41	-0,39	-0,35	-0,23	0,04	-0,27	-0,35	0,38	-0,1	-0,13	-0,15	-0,03	0,14	0,66	0,52
International Value	0,89	0,9	0,89	0,88	0,88	0,87	0,84	0,85	0,84	0,97	0,86	1	0,91	0,89	0,78	0,98	0,88	0,96	0,68	-0,36	0,08	-0,21	-0,46	-0,43	-0,4	-0,27	0,01	-0,33	-0,42	0,3	-0,14	-0,2	-0,21	-0,08	0,11	0,65	0,49
International Small Cap	0,82	0,83	0,82	0,83	0,83	0,82	0,8	0,8	0,8	0,95	0,88	0,91	1	0,84	0,75	0,94	0,85	0,92	0,62	-0,29	0,13	-0,14	-0,42	-0,38	-0,34	-0,21	0,03	-0,26	-0,36	0,39	-0,06	-0,16	-0,17	-0,04	0,2	0,71	0,56
US Stock Market	0,99	1	0,99	0,98	0,98	0,97	0,95	0,96	0,95	0,89	0,84	0,89	0,84	1	0,89	0,88	0,81	0,86	0,77	-0,4	-0,07	-0,25	-0,48	-0,46	-0,44	-0,3	0	-0,38	-0,45	0,27	-0,2	-0,19	-0,2	-0,09	0,04	0,61	0,42
Micro Cap	0,87	0,88	0,88	0,91	0,91	0,89	0,95	0,96	0,95	0,79	0,75	0,78	0,75	0,89	1	0,77	0,71	0,75	0,71	-0,36	-0,06	-0,23	-0,43	-0,42	-0,4	-0,27	0,01	-0,35	-0,4	0,22	-0,18	-0,17	-0,17	-0,07	0,04	0,64	0,3
Intl Developed Markets	0,89	0,9	0,89	0,87	0,88	0,87	0,83	0,84	0,82	0,99	0,87	0,98	0,94	0,88	0,77	1	0,9	0,98	0,67	-0,29	0	-0,18	-0,4	-0,4	-0,38	-0,25	0,01	-0,3	-0,4	0,29	-0,18	-0,16	-0,16	-0,07	0,14	0,76	0,42
Intl Pacific Region	0,82	0,84	0,83	0,8	0,82	0,82	0,77	0,78	0,76	0,91	0,83	0,88	0,85	0,81	0,71	0,9	1	0,8	0,61	-0,24	-0,02	-0,15	-0,33	-0,34	-0,32	-0,19	0,02	-0,25	-0,34	0,27	-0,15	-0,13	-0,13	-0,06	0,12	0,7	0,37
Intl Europe	0,87	0,88	0,87	0,85	0,86	0,85	0,82	0,83	0,79	0,97	0,83	0,96	0,92	0,86	0,75	0,98	0,8	1	0,67	-0,3	0,02	-0,19	-0,41	-0,4	-0,38	-0,25	0,01	-0,3	-0,4	0,29	-0,18	-0,17	-0,17	-0,07	0,14	0,74	0,43
REIT	0,81	0,79	0,75	0,85	0,81	0,75	0,86	0,83	0,74	0,68	0,66	0,68	0,62	0,77	0,71	0,67	0,61	0,67	1	-0,2	-0,11	-0,17	-0,25	-0,28	-0,21	-0,12	0,01	-0,15	-0,24	0,12	-0,19	-0,1	-0,09	-0,02	0,02	0,54	0,16
Total bond	-0,35	-0,36	-0,36	-0,34	-0,35	-0,35	-0,33	-0,34	-0,34	-0,3	-0,3	-0,36	-0,29	-0,4	-0,36	-0,29	-0,24	-0,3	-0,2	1	0,59	0,77	0,89	0,9	0,94	0,77	0,01	0,93	0,92	0,17	0,77	0,47	0,47	0,35	0,14	-0,16	0,01
Global bond	-0,16	-0,16	-0,15	-0,15	-0,14	-0,13	-0,15	-0,14	-0,13	-0,02	0,01	0,08	0,13	-0,07	-0,06	0	-0,02	0,02	-0,11	0,59	1	0,54	0,48	0,55	0,58	0,5	0,05	0,54	0,47	0,15	0,58	0,36	0,36	0,22	0,35	0,11	0,26
TIPS	-0,25	-0,26	-0,25	-0,23	-0,24	-0,23	-0,24	-0,24	-0,23	-0,18	-0,16	-0,21	-0,14	-0,25	-0,23	-0,18	-0,15	-0,19	-0,17	0,77	0,54	1	0,75	0,78	0,81	0,63	0,03	0,84	0,79	0,13	0,59	0,34	0,33	0,14	0,28	0,02	0,15
Long term treasuries	-0,42	-0,43	-0,43	-0,41	-0,42	-0,42	-0,41	-0,42	-0,42	-0,4	-0,41	-0,46	-0,42	-0,48	-0,43	-0,4	-0,33	-0,41	-0,25	0,89	0,48	0,75	1	0,92	0,87	0,63	0	0,87	0,96	-0,03	0,56	0,4	0,4	0,2	0,16	-0,19	-0,14
10-year treasuries	-0,44	-0,45	-0,44	-0,43	-0,44	-0,43	-0,42	-0,43	-0,43	-0,41	-0,39	-0,43	-0,38	-0,46	-0,42	-0,4	-0,34	-0,4	-0,28	0,9	0,55	0,78	0,92	1	0,96	0,75	0,01	0,94	0,89	-0,04	0,62	0,38	0,39	0,17	0,21	-0,16	-0,08
Intermediate term treasuries	-0,45	-0,45	-0,43	-0,42	-0,43	-0,42	-0,42	-0,42	-0,4	-0,38	-0,35	-0,4	-0,34	-0,44	-0,4	-0,38	-0,32	-0,38	-0,21	0,94	0,58	0,81	0,87	0,96	1	0,82	0,01	0,92	0,84	0,04	0,74	0,42	0,43	0,29	0,23	-0,12	-0,04
Short term treasuries	-0,31	-0,31	-0,29	-0,29	-0,29	-0,28	-0,28	-0,28	-0,27	-0,24	-0,23	-0,27	-0,21	-0,3	-0,27	-0,25	-0,19	-0,25	-0,12	0,77	0,5	0,63	0,63	0,75	0,82	1	0,02	0,74	0,62	0,02	0,75	0,42	0,43	0,26	0,21	-0,02	0,01
Cash/money market	0,01	0	0	0,01	0,01	0,01	0,01	0,01	0,01	0,02	0,04	0,01	0,03	0	0,01	0,01	0,02	0,01	0,01	0,01	0,05	0,03	0	0,01	0,01	0,02	1	0,02	0,01	0,04	0,03	-0,04	-0,03	-0,01	0,05	0,07	0,06
Corporate bonds	-0,38	-0,38	-0,36	-0,36	-0,36	-0,36	-0,36	-0,35	-0,34	-0,3	-0,27	-0,33	-0,26	-0,38	-0,35	-0,3	-0,25	-0,3	-0,15	0,93	0,54	0,84	0,87	0,94	0,92	0,74	0,02	1	0,9	0,12	0,72	0,45	0,45	0,18	0,22	-0,08	0,01
Long term corporate bonds	-0,46	-0,46	-0,44	-0,44	-0,44	-0,43	-0,43	-0,43	-0,42	-0,4	-0,35	-0,42	-0,36	-0,45	-0,4	-0,4	-0,34	-0,4	-0,24	0,92	0,47	0,79	0,96	0,89	0,84	0,62	0,01	0,9	1	0,09	0,64	0,48	0,48	0,23	0,11	-0,19	-0,02
High yield bond	0,22	0,22	0,22	0,24	0,25	0,25	0,21	0,23	0,24	0,3	0,38	0,3	0,39	0,27	0,22	0,29	0,27	0,29	0,12	0,17	0,15	0,13	-0,03	-0,04	0,04	0,02	0,04	0,12	0,09	1	0,36	0,29	0,26	0,31	0,03	0,22	0,29
Short term inv grade	-0,26	-0,27	-0,26	-0,25	-0,25	-0,25	-0,25	-0,24	-0,23	-0,18	-0,1	-0,14	-0,06	-0,2	-0,18	-0,18	-0,15	-0,18	-0,19	0,77	0,58	0,59	0,56	0,62	0,74	0,75	0,03	0,72	0,64	0,36	1	0,46	0,46	0,44	0,13	-0,09	0,1
Long term tax exempt	-0,16	-0,16	-0,15	-0,15	-0,15	-0,15	-0,15	-0,15	-0,15	-0,16	-0,13	-0,2	-0,16	-0,19	-0,17	-0,16	-0,13	-0,17	-0,1	0,47	0,36	0,34	0,4	0,38	0,42	0,42	-0,04	0,45	0,48	0,29	0,46	1	0,95	0,59	0,04	-0,08	0,05
Intermediate term tax exempt	-0,17	-0,17	-0,16	-0,15	-0,15	-0,15	-0,15	-0,15	-0,15	-0,16	-0,15	-0,21	-0,17	-0,2	-0,17	-0,16	-0,13	-0,17	-0,09	0,47	0,36	0,33	0,4	0,39	0,43	0,43	-0,03	0,45	0,48	0,26	0,46	0,95	1	0,63	0,05	-0,07	0,04
Short term tax exempt	-0,08	-0,08	-0,07	-0,06	-0,06	-0,07	-0,06	-0,06	-0,06	-0,06	-0,03	-0,08	-0,04	-0,09	-0,07	-0,07	-0,06	-0,07	-0,02	0,35	0,22	0,14	0,2	0,17	0,29	0,26	-0,01	0,18	0,23	0,31	0,44	0,59	0,63	1	0,02	-0,02	0,06
Gold	0,03	0,03	0,04	0,04	0,05	0,07	0,02	0,06	0,06	0,15	0,14	0,11	0,2	0,04	0,04	0,14	0,12	0,14	0,02	0,14	0,35	0,28	0,16	0,21	0,23	0,21	0,05	0,22	0,11	0,03	0,13	0,04	0,05	0,02	1	0,46	0,46
Precious metals	0,68	0,69	0,69	0,69	0,71	0,71	0,66	0,61	0,6	0,69	0,66	0,65	0,71	0,61	0,64	0,76	0,7	0,74	0,54	-0,16	0,11	0,02	-0,19	-0,16	-0,12	-0,02	0,07	-0,08	-0,19	0,22	-0,09	-0,08	-0,07	-0,02	0,46	1	0,54
Commodities	0,3	0,31	0,32	0,32	0,35	0,38	0,28	0,42	0,41	0,54	0,52	0,49	0,56	0,42	0,3	0,42	0,37	0,43	0,16	0,01	0,26	0,15	-0,14	-0,08	-0,04	0,01	0,06	0,01	-0,02	0,29	0,1	0,05	0,04	0,06	0,46	0,54	1

Figure F.1: Correlation between US asset classes between 2009 and 2016.

	OBX All Share	OBX	OBX Small Cap	OBX Mid Cap	OBX Large Cap	US Stock Market	Total International Stock	International Developed	International Small Cap	International Value	European Stock	Pacific Region Stocks	Emerging Markets	Short term treasuries	Intermediate term treasur	Long term government	Money market (cash)	Global bonds (unhedged)	Global bonds (USD hedge	Corporate bonds	Real Estate	Gold	Precious metals	Commodities
OBX All Share	1	0,998	0,995	0,968	0,998	0,936	0,981	0,979	0,899	-0,185	0,973	0,972	0,989	0,069	-0,45	-0,686	-0,495	-0,3	-0,406	0,356	0,988	-0,706	0,973	0,582
OBX	0,998	1	0,992	0,958	0,998	0,948	0,987	0,984	0,911	-0,189	0,98	0,977	0,99	0,04	-0,477	-0,704	-0,523	-0,278	-0,395	0,325	0,985	-0,722	0,968	0,602
OBX Small Cap	0,995	0,992	1	0,979	0,991	0,918	0,973	0,97	0,873	-0,187	0,966	0,961	0,982	0,116	-0,397	-0,646	-0,443	-0,334	-0,426	0,293	0,993	-0,675	0,973	0,537
OBX Mid Cap	0,968	0,958	0,979	1	0,959	0,853	0,93	0,924	0,812	0,814	0,92	0,917	0,951	0,193	-0,29	-0,562	-0,327	-0,37	-0,453	-0,037	0,976	0,947	0,966	0,428
OBX Large Cap	0,998	0,998	0,991	0,959	1	0,949	0,987	0,985	0,915	-0,197	0,981	0,976	0,99	0,014	-0,487	-0,712	-0,529	-0,265	-0,386	0,383	0,986	-0,73	0,97	0,61
US Stock Market	0,936	0,948	0,918	0,853	0,949	1	0,977	0,978	0,959	-0,256	0,978	0,963	0,963	-0,234	-0,678	-0,825	-0,718	-0,056	-0,26	0,223	0,911	-0,831	0,911	0,722
Total International Stock	0,981	0,987	0,973	0,93	0,987	0,977	1	1	0,931	-0,231	0,997	0,989	0,993	-0,064	-0,541	-0,738	-0,59	-0,182	-0,381	0,248	0,969	-0,774	0,966	0,637
International Developed Markets	0,979	0,984	0,97	0,924	0,985	0,978	1	1	0,93	-0,237	0,998	0,988	0,99	-0,072	-0,545	-0,739	-0,595	-0,176	-0,381	0,253	0,967	-0,777	0,962	0,64
International Small Cap	0,899	0,911	0,873	0,812	0,915	0,959	0,931	0,93	1	-0,073	0,93	0,917	0,927	-0,27	-0,779	-0,902	-0,812	0,024	-0,077	0,212	0,864	-0,827	0,861	0,808
International Value	-0,185	-0,189	-0,187	0,814	-0,197	-0,256	-0,231	-0,237	-0,073	1	-0,239	-0,235	-0,199	0,309	0,159	0,071	0,243	-0,14	0,163	-0,111	-0,185	0,764	-0,229	-0,081
European Stock	0,973	0,98	0,966	0,92	0,981	0,978	0,997	0,998	0,93	-0,239	1	0,975	0,984	-0,096	-0,551	-0,746	-0,591	-0,155	-0,357	0,249	0,967	-0,78	0,954	0,64
Pacific Region Stocks	0,972	0,977	0,961	0,917	0,976	0,963	0,989	0,988	0,917	-0,235	0,975	1	0,986	-0,028	-0,523	-0,712	-0,601	-0,211	-0,422	0,254	0,949	-0,76	0,964	0,634
Emerging Markets	0,989	0,99	0,982	0,951	0,99	0,963	0,993	0,99	0,927	-0,199	0,984	0,986	1	-0,02	-0,514	-0,726	-0,557	-0,22	-0,38	0,22	0,974	-0,752	0,977	0,614
Short term treasuries	0,069	0,04	0,116	0,193	0,014	-0,234	-0,064	-0,072	-0,27	0,309	-0,096	-0,028	-0,02	1	0,707	0,496	0,663	-0,807	-0,586	-0,188	0,088	0,555	0,026	-0,475
Intermediate term treasuries	-0,45	-0,477	-0,397	-0,29	-0,487	-0,678	-0,541	-0,545	-0,779	0,159	-0,551	-0,523	-0,514	0,707	1	0,936	0,951	-0,528	-0,38	-0,244	-0,396	0,787	-0,406	-0,867
Long term government bonds	-0,686	-0,704	-0,646	-0,562	-0,712	-0,825	-0,738	-0,739	-0,902	0,071	-0,746	-0,712	-0,726	0,496	0,936	1	0,884	-0,296	-0,248	-0,25	-0,643	0,826	-0,623	-0,89
Money market (cash)	-0,495	-0,523	-0,443	-0,327	-0,529	-0,718	-0,59	-0,595	-0,812	0,243	-0,591	-0,601	-0,557	0,663	0,951	0,884	1	-0,445	-0,195	-0,263	-0,434	0,795	-0,464	-0,857
Global bonds (unhedged)	-0,3	-0,278	-0,334	-0,37	-0,265	-0,056	-0,182	-0,176	0,024	-0,14	-0,155	-0,211	-0,22	-0,807	-0,528	-0,296	-0,445	1	0,636	-0,02	-0,307	-0,291	-0,241	0,284
Global bonds (USD hedged)	-0,406	-0,395	-0,426	-0,453	-0,386	-0,26	-0,381	-0,381	-0,077	0,163	-0,357	-0,422	-0,38	-0,586	-0,38	-0,248	-0,195	0,636	1	0,088	-0,426	0,018	-0,464	0,28
Corporate bonds	0,356	0,325	0,293	-0,037	0,383	0,223	0,248	0,253	0,212	-0,111	0,249	0,254	0,22	-0,188	-0,244	-0,25	-0,263	-0,02	0,088	1	0,402	-0,173	0,002	0,407
Real Estate	0,988	0,985	0,993	0,976	0,986	0,911	0,969	0,967	0,864	-0,185	0,967	0,949	0,974	0,088	-0,396	-0,643	-0,434	-0,307	-0,426	0,402	1	-0,691	0,968	0,525
Gold	-0,706	-0,722	-0,675	0,947	-0,73	-0,831	-0,774	-0,777	-0,827	0,764	-0,78	-0,76	-0,752	0,555	0,787	0,826	0,795	-0,291	0,018	-0,173	-0,691	1	-0,712	-0,712
Precious metals	0,973	0,968	0,973	0,966	0,97	0,911	0,966	0,962	0,861	-0,229	0,954	0,964	0,977	0,026	-0,406	-0,623	-0,464	-0,241	-0,464	0,002	0,968	-0,712	1	0,526
Commodities	0,582	0,602	0,537	0,428	0,61	0,722	0,637	0,64	0,808	-0,081	0,64	0,634	0,614	-0,475	-0,867	-0,89	-0,857	0,284	0,28	0,407	0,525	-0,712	0,526	1

Figure F.2: Correlation between Norwegian asset classes between 1997 and 2007.

	OBX All Share	OBX	OBX Small Cap	OBX Mid Cap	OBX Large Cap	US Stock Market	Total International Stock	International Developed	International Small Cap	International Value	European Stock	Pacific Region Stocks	Emerging Markets	Short term treasuries	Intermediate term treasur	Long term government	Money market (cash)	Global bonds (unhedged)	Global bonds (USD hedge	Corporate bonds	Real Estate	Gold	Precious metals	Commodities
OBX All Share	1	0,994	0,983	0,955	0,952	0,68	0,898	0,157	0,061	0,842	0,782	0,743	0,953	0,236	0,497	0,631	0,441	0,145	-0,302	-0,515	0,909	0,732	0,935	-0,551
OBX	0,994	1	0,965	0,947	0,967	0,683	0,886	0,144	0,139	0,808	0,754	0,725	0,962	0,157	0,508	0,641	0,448	0,111	-0,337	-0,483	0,869	0,775	0,914	-0,538
OBX Small Cap	0,983	0,965	1	0,955	0,903	0,678	0,9	0,133	0,091	0,87	0,814	0,756	0,914	0,152	0,422	0,557	0,395	0,108	-0,303	-0,4	0,946	0,631	0,93	-0,525
OBX Mid Cap	0,955	0,947	0,955	1	0,979	0,507	0,792	0,204	-0,108	0,774	0,723	0,624	0,923	0,325	0,488	0,641	0,405	0,09	-0,231	0,548	0,895	0,876	0,889	-0,605
OBX Large Cap	0,952	0,967	0,903	0,979	1	0,56	0,779	0,25	-0,006	0,669	0,608	0,583	0,957	0,222	0,634	0,745	0,507	0,108	-0,408	0,368	0,775	0,884	0,879	-0,622
US Stock Market	0,68	0,683	0,678	0,507	0,56	1	0,866	-0,205	0,672	0,785	0,894	0,805	0,616	0,185	0,038	0,24	0,028	0,07	-0,146	-0,47	0,687	0,305	0,616	-0,057
Total International Stock	0,898	0,886	0,9	0,792	0,779	0,866	1	-0,045	0,331	0,945	0,948	0,917	0,873	0,277	0,248	0,408	0,232	0,174	-0,143	-0,671	0,881	0,51	0,852	-0,264
International Developed Markets	0,157	0,144	0,133	0,204	0,25	-0,205	-0,045	1	-0,555	0,082	-0,087	-0,129	0,177	0,319	0,709	0,599	0,676	0,538	-0,021	0,489	0,179	0,243	0,284	-0,692
International Small Cap	0,061	0,139	0,091	-0,108	-0,006	0,672	0,331	-0,555	1	0,012	0,505	0,187	-0,18	-0,332	-0,537	-0,326	-0,528	-0,541	-0,338	-0,172	-0,089	-0,423	-0,305	0,469
International Value	0,842	0,808	0,87	0,774	0,669	0,785	0,945	0,082	0,012	1	0,926	0,908	0,776	0,294	0,224	0,353	0,283	0,272	-0,021	-0,589	0,918	0,335	0,849	-0,487
European Stock	0,782	0,754	0,814	0,723	0,608	0,894	0,948	-0,087	0,505	0,926	1	0,854	0,7	0,33	0,053	0,244	0,042	0,14	-0,032	-0,585	0,86	0,267	0,759	-0,161
Pacific Region Stocks	0,743	0,725	0,756	0,624	0,583	0,805	0,917	-0,129	0,187	0,908	0,854	1	0,722	0,081	0,13	0,238	0,224	0,226	-0,075	-0,635	0,752	0,331	0,695	-0,187
Emerging Markets	0,953	0,962	0,914	0,923	0,957	0,616	0,873	0,177	-0,18	0,776	0,7	0,722	1	0,262	0,55	0,65	0,464	0,167	-0,327	-0,368	0,808	0,819	0,907	-0,603
Short term treasuries	0,236	0,157	0,152	0,325	0,222	0,185	0,277	0,319	-0,332	0,294	0,33	0,081	0,262	1	0,581	0,649	0,392	0,633	0,486	-0,133	0,365	0,377	0,233	-0,36
Intermediate term treasuries	0,497	0,508	0,422	0,488	0,634	0,038	0,248	0,709	-0,537	0,224	0,053	0,13	0,55	0,581	1	0,951	0,845	0,448	-0,156	0,203	0,325	0,718	0,462	-0,833
Long term government bonds	0,631	0,641	0,557	0,641	0,745	0,24	0,408	0,599	-0,326	0,353	0,244	0,238	0,65	0,649	0,951	1	0,727	0,395	-0,174	0,145	0,469	0,78	0,582	-0,738
Money market (cash)	0,441	0,448	0,395	0,405	0,507	0,028	0,232	0,676	-0,528	0,283	0,042	0,224	0,464	0,392	0,845	0,727	1	0,379	-0,094	-0,005	0,32	0,513	0,405	-0,84
Global bonds (unhedged)	0,145	0,111	0,108	0,09	0,108	0,07	0,174	0,538	-0,541	0,272	0,14	0,226	0,167	0,633	0,448	0,395	0,379	1	0,333	-0,246	0,185	0,099	0,213	-0,623
Global bonds (USD hedged)	-0,302	-0,337	-0,303	-0,231	-0,408	-0,146	-0,143	-0,021	-0,338	-0,021	-0,032	-0,075	-0,327	0,486	-0,156	-0,174	-0,094	0,333	1	-0,254	-0,149	-0,354	-0,265	-0,004
Corporate bonds	-0,515	-0,483	-0,4	0,548	0,368	-0,47	-0,671	0,489	-0,172	-0,589	-0,585	-0,635	-0,368	-0,133	0,203	0,145	-0,005	-0,246	-0,254	1	-0,02	0,442	0,569	-0,025
Real Estate	0,909	0,869	0,946	0,895	0,775	0,687	0,881	0,179	-0,089	0,918	0,86	0,752	0,808	0,365	0,325	0,469	0,32	0,185	-0,149	-0,02	1	0,441	0,924	-0,586
Gold	0,732	0,775	0,631	0,876	0,884	0,305	0,51	0,243	-0,423	0,335	0,267	0,331	0,819	0,377	0,718	0,78	0,513	0,099	-0,354	0,442	0,441	1	0,627	-0,703
Precious metals	0,935	0,914	0,93	0,889	0,879	0,616	0,852	0,284	-0,305	0,849	0,759	0,695	0,907	0,233	0,462	0,582	0,405	0,213	-0,265	0,569	0,924	0,627	1	-0,686
Commodities	-0,551	-0,538	-0,525	-0,605	-0,622	-0,057	-0,264	-0,692	0,469	-0,487	-0,161	-0,187	-0,603	-0,36	-0,833	-0,738	-0,84	-0,623	-0,004	-0,025	-0,586	-0,703	-0,686	1

Figure F.3: Correlation between Norwegian asset classes between 2012 and 2007.